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Abstract

We study the asymptotic behavior of $\mathbb{P}(X - Y > u)$ as $u \to \infty$, where X is subexponential and X, Y are positive random variables that may be dependent. We give criteria under which the subtraction of Y does not change the tail behavior of X. It is also studied under which conditions the comonotonic copula represents the worst-case scenario for the asymptotic behavior in the sense of minimizing the tail of $X - Y$ and an explicit construction of the worst-case copula is provided in the other cases.

1 Introduction

In recent years, there has been quite some progress in understanding the asymptotic effect of dependence on the tail of sums of positive subexponential random variables, see for instance Albrecher et al. [1], Mitra & Resnick [19], Ko & Tang [16], Kortschak & Albrecher [17] and Foss & Richards [13]. In this paper we are interested in the tail asymptotics of differences of positive random variables, i.e. in

$$\mathbb{P}(X - Y > u)$$

for $u \to \infty$, where X is subexponential and Y may have different forms of the tail. If X, Y are independent, this is easy (cf. [6, Lemma 3.2, p. 306]):

$$\mathbb{P}(X - Y > u) \sim \mathbb{P}(X > u) \tag{1.1}$$

without further conditions. Thus, the problem is dependence.

There are various areas in which the asymptotics of dependent differences of positive random variables are of interest, for instance random recurrence equations, queueing models and insurance risk models, each in the presence of dependence. In

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particular, in an insurance context, such a dependent difference can have a natural interpretation as the difference between a claim X and its preceding interarrival time Y, where the random walk structure of the surplus level in the portfolio after a claim occurrence is still preserved (see Albrecher & Teugels [3], Boudreault et al. [9], Asimit & Badescu [4], Li et al. [18] and also Albrecher & Boxma [2] for such and similar dependence structures). In queueing applications there are similar interpretations possible.

Asmussen & Biard [7] needed (1.1) for the case where Y is light-tailed. They showed (1.1) essentially when the tail of Y is of smaller magnitude than $\mathrm{e}^{-x^{1/2}}$ and gave a counterexample that (1.1) may not hold with lighter, still subexponential tails. The aim of this paper is to provide more general criteria on the dependence between X and Y for the insensitivity to hold and to consider more general distributions of Y. In Section 2 we give a general criterion under which the insensitivity (1.1) holds. Section 3 discusses the role of the mean excess function in this analysis. In Section 4 we discuss the case of light-tailed Y in more detail and provide a substantially simpler construction of a counterexample that $\mathrm{e}^{-x^{1/2}}$ is in fact the critical decay rate of the tail of X, if no dependence structure is specified. This rate is critical in many other contexts and is known as *square-root insensitivity* (e.g. Jelenković et al. [15]). In Section 5 we show (under some regularity conditions) that if there exists a counterexample for the insensitivity (1.1), then the comonotonic copula also provides a counter-example. Yet, the comonotonic copula may not represent the dependence structure that produces the most extreme behavior of $\mathbb{P}(X-Y>u)$. We provide criteria under which the comonotone dependence is indeed the worst case in the sense of minimizing the tail of $X-Y$ and provide an explicit construction of the worst-case copula otherwise. Finally, Section 6 deals with the case of regularly varying X and relates the present discussion to local limit laws.

2 An insensitivity result

From e.g. Foss et al. [12], if a distribution F is long-tailed, this implies that there exists a non-decreasing function δ with $\delta(u) \to \infty$ as $u \to \infty$, such that

$$\overline{F}_X\big(u \pm \delta(u)\big) \sim \overline{F}_X(u) \quad \text{as } u \to \infty\,. \tag{2.1}$$

In the following, we will be interested in choosing $\delta(u)$ as large as possible.

Proposition 2.1. *Let $X \geq 0$ be a r.v. with a long-tailed distribution F_X and $Y \geq 0$ a (not necessarily independent) r.v.. Then*

$$\mathbb{P}(X-Y>u) \sim \mathbb{P}(X>u) \tag{2.2}$$

provided $\delta(\cdot)$ in (2.1) can be chosen with

$$\mathbb{P}(Y>\delta(u), X>u+\delta(u)) = \mathrm{o}\big(\overline{F}_X(u)\big). \tag{2.3}$$

Proof. Write

$$\mathbb{P}(X-Y>u) \;=\; \mathbb{P}\big(X-Y>u,\, Y \leq \delta(u)\big) \;+\; \mathbb{P}\big(X-Y>u,\, Y>\delta(u)\big)\,.$$

Note that by (2.3)

$$\mathbb{P}\big(X - Y > u,\, Y > \delta(u)\big) \;\le\; \mathbb{P}\big(X > u + \delta(u), Y > \delta(u)\big) \;=\; \mathrm{o}\big(\overline{F}_X(u)\big)\,. \qquad (2.4)$$

Moreover,

$$\begin{aligned}
\mathbb{P}\big(X - Y > u,\, Y \le \delta(u)\big) &\le \mathbb{P}(X > u) \;=\; \overline{F}_X(u)\,,\\
\mathbb{P}\big(X - Y > u,\, Y \le \delta(u)\big) &\ge \mathbb{P}\big(X - \delta(u) > u,\, Y \le \delta(u)\big)\\
&= \mathbb{P}\big(X - \delta(u) > u\big) - \mathbb{P}\big(X - \delta(u) > u,\, Y > \delta(u)\big)\\
&\sim \overline{F}_X(u) - \mathrm{o}\big(\overline{F}_X(u)\big)\,.
\end{aligned}$$

Putting these estimates together completes the proof. □

Example 2.2. If X and Y are dependent according to a copula C that is negative quadrant dependent (i.e. $C(u,v) \le uv$ for $0 \le u,v \le 1$) and X is long-tailed, then the assumptions of Proposition 2.1 are fulfilled, in particular

$$\mathbb{P}(Y > \delta(u), X > u + \delta(u)) \le \mathbb{P}(Y > \delta(u))\mathbb{P}(X > u + \delta(u)) = \mathrm{o}\big(\overline{F}_X(u)\big).$$

Hence (2.2) holds. Note that this criterion does not involve any assumption on the distribution of Y. In terms of the survival copula, a sufficient criterion is $\widehat{C}(u,v) \le uh(v)$ with $h(v) \to 0$. In terms of distribution functions, this means that for all $x, y \ge 0$

$$\mathbb{P}(X > x, Y > y) \le \mathbb{P}(X > x)\, h(\mathbb{P}(Y > y))$$

holds. □

Example 2.3. More generally, one can formulate a criterion in terms of stochastic ordering: whenever the pair (X^1, Y^1) fulfills the condition (2.3), then every pair (X^2, Y^2) with the same marginal distributions that is dominated in concordance order (i.e. $\mathbb{P}(X^1 > x, Y^1 > y) \ge \mathbb{P}(X^2 > x, Y^2 > y)$ for all $x > x_0, y > y_0$) also fulfills (2.3). □

3 The role of the mean excess function

Assume that X is regularly varying or in the maximum domain of attraction of the Gumbel distribution with mean excess function $e(u)$ (cf. Embrechts et al. [11]). Then $\delta(u)$ in (2.1) can be any function with $\delta(u) \to \infty$ and

$$\lim_{u\to\infty} \frac{\delta(u)}{e(u)} = 0. \qquad (3.1)$$

In a more general setting assume that there exists a function $e(u)$ with

$$\liminf_{u\to\infty} \frac{\mathbb{P}(X - \varepsilon e(u) > u)}{\mathbb{P}(X > u)} < 1$$

for some $\varepsilon > 0$ and

$$\lim_{\varepsilon\to 0} \liminf_{u\to\infty} \frac{\mathbb{P}(X - \varepsilon e(u) > u)}{\mathbb{P}(X > u)} = 1.$$

Then if

$$\lim_{\varepsilon\to 0}\limsup_{u\to\infty}\frac{\mathbb{P}(Y>\varepsilon e(u))}{\mathbb{P}(X>u)}=0$$

we get by Proposition 2.1 that $\mathbb{P}(X-Y>u)\sim\mathbb{P}(X>u)$.

As we have seen above, for regularly varying distributions or distributions in the maximum domain of attraction of the Gumbel distribution one can choose $e(u)$ as the mean excess function (the reciprocal of the hazard rate $r(u)$). The following result provides another criterion on the distribution of X such that we can still use the mean excess function in (3.1).

Lemma 3.1. *Assume that X is long-tailed with*

$$\overline{F}_X(x)=c(x)\,\mathrm{e}^{-\int_0^x r^*(t)dt},$$

where $\lim_{u\to\infty}c(x)=c$, $0<c<\infty$ and $\lim_{u\to\infty}r^*(u)=0$. Assume further that there exists an $\varepsilon_0>0$ such that, uniformly in $0<t<\varepsilon_0$,

$$\liminf_{u\to\infty}\frac{r^*\Big(u+\frac{t}{r^*(u)}\Big)}{r^*(u)}=c_l>0,\quad \liminf_{u\to\infty}\frac{r^*\Big(u+\frac{t}{r^*(u)}\Big)}{r^*(u)}=c_u<\infty.$$

Then

$$\liminf_{u\to\infty}\frac{\mathbb{P}\Big(X-\varepsilon\frac{1}{r^*(u)}>u\Big)}{\mathbb{P}(X>u)}<1,\quad \lim_{\varepsilon\to 0}\liminf_{u\to\infty}\frac{\mathbb{P}\Big(X-\varepsilon\frac{1}{r^*(u)}>u\Big)}{\mathbb{P}(X>u)}=1.$$

Remark 3.2. Note that for an X that fulfills the conditions of Lemma 3.1, the mean excess function $e(u)$ satisfies

$$\lim_{u\to\infty}r^*(u)e(u)=1.$$

Proof. We have that

$$\begin{aligned}\frac{\mathbb{P}\Big(X-\varepsilon\frac{1}{r^*(u)}>u\Big)}{\mathbb{P}(X>u)}&\sim\exp\Big(-\int_u^{u+\frac{\varepsilon}{r^*(u)}}r^*(t)dt\Big)=\exp\Big(-\int_0^{\varepsilon}\frac{r^*\Big(u+\frac{t}{r^*(u)}\Big)}{r^*(u)}dt\Big)\\&\lesssim\exp\Big(-c_l\int_0^{\varepsilon}dt\Big)=\mathrm{e}^{-c_u\varepsilon}<1.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\mathbb{P}\Big(X-\varepsilon\frac{1}{r(u)}>u\Big)}{\mathbb{P}(X>u)}&\sim\exp\Big(-\int_0^{\varepsilon}\frac{r^*\Big(u+\frac{t}{r^*(u)}\Big)}{r^*(u)}dt\Big)\\&\gtrsim\exp\Big(-c_u\int_0^{\varepsilon}dt\Big)=\mathrm{e}^{-c_u\varepsilon},\end{aligned}$$

from which the result follows. □

Remark 3.3. An example for which the conditions of Lemma 3.1 are not fulfilled is

$$\overline{F}_X(x)=\frac{1}{\log(x)}.$$

4 Light-tailed Y

It may be instructive to replace (2.3) by the stronger condition

$$\mathbb{P}(Y > \delta(u)) = \mathrm{o}\big(\overline{F}_X(u)\big), \tag{4.1}$$

which is now a criterion on the marginal distribution of Y. Clearly, if the dependence structure is not specified, there is a trade-off between how heavy the tail of X needs to be to set off a not too light tail behavior of Y. In particular, this gives rise to the following question: If Y is a light-tailed r.v. (i.e. $P(Y > u) = \mathrm{o}(e^{-gu})$ for some $g > 0$), for which long-tailed r.v. X does (2.2) hold across all dependence structures? In this case, condition (4.1) turns into

$$\mathrm{e}^{-g\delta(u)} = \mathrm{o}\big(\overline{F}_X(u)\big),$$

which holds for F_X regularly varying (take $\delta(x) = c \log x$ with c sufficiently large), the lognormal distribution ($\delta(x) = x/\log^2 x$) and the heavy-tailed Weibull with $\overline{F}_X(x) = \mathrm{e}^{-x^\beta}$ with $\beta < 1/2$ ($\delta(x) = x^{1-\beta^*}$ with $\beta < \beta^* < 1$). Thus, the condition covers most standard heavy-tailed distributions except the ones closest to the light-tailed case. Since with independent X, Y and X subexponential, X and $X-Y$ always have the same tail (as discussed in Section 1), one could believe that the condition is just technical. However, Asmussen & Biard [7] provided a counterexample that this is not the case. In the following we give a substantially simpler counterexample than the one in [7]:

Example 4.1. Assume $\mathbb{P}(X > u) \sim \mathrm{e}^{-u^\beta}$ with $0 < \beta < 1$ and let $Y = X^\beta$. Then clearly Y is exponential. Thus

$$\begin{aligned}\mathbb{P}(X - Y > u) = \mathbb{P}(X > u + X^\beta) \ &\leq\ \mathbb{P}(X > u + u^\beta) \ \sim\ \exp\{-(u + u^\beta)^\beta\} \\ = \exp\{-u^\beta(1 + u^{\beta-1})^\beta\} \ &\sim\ \exp\{-u^\beta - \beta u^{2\beta-1}\},\end{aligned}$$

and here $\exp\{-\beta u^{2\beta-1}\} = \mathrm{o}(1)$ if and only if $\beta > 1/2$. □

This counterexample (as well as the one in Asmussen & Biard [7]) is based on a comonotonic copula. It is natural to ask whether the comonotonic copula always minimizes the tail of $X - Y$. This is the topic of the next section.

5 The worst-case copula

We will now show under some regularity conditions that if there exists a counterexample for the insensitivity (1.1) to hold, then also the comonotonic copula provides a counterexample:

Lemma 5.1. *Let X and Y be two positive random variables with distribution function $F_X(x)$ and $F_Y(x)$, respectively. Define*

$$\begin{aligned}\overline{\gamma}(u) &= \sup\{x | F_Y(x-u) < F_X(x), x \geq u\} - u \quad \textit{and} \\ \underline{\gamma}(u) &= \inf\{x | F_Y(x-u) \geq F_X(x), x \geq u\} - u.\end{aligned}$$

If for some $\alpha > 0$, $c > 0$ *and all* $k > 1$, $\lim_{u\to\infty} \overline{F}_Y(ku)/\overline{F}_Y(u) \le ck^{-\alpha}$,

$$\lim_{u\to\infty} \frac{\mathbb{P}(X > u + \overline{\gamma}(u))}{\mathbb{P}(X > u)} = 1 \quad \textit{and} \quad \limsup_{u\to\infty} \frac{\mathbb{P}(Y > \overline{\gamma}(u))}{\mathbb{P}(X > u)} < \infty,$$

then

$$\lim_{u\to\infty} \frac{\mathbb{P}(X - Y > u)}{\mathbb{P}(X > u)} = 1.$$

If

$$\liminf_{u\to\infty} \frac{\mathbb{P}(X > u + \underline{\gamma}(u))}{\mathbb{P}(X > u)} < 1,$$

and X *and* Y *are comonotonic, then*

$$\liminf_{u\to\infty} \frac{\mathbb{P}(X - Y > u)}{\mathbb{P}(X > u)} < 1.$$

Proof. At first note that

$$\mathbb{P}(X - Y > u) \le \mathbb{P}(X > u).$$

We have

$$\begin{aligned}
\mathbb{P}(X - Y > u) &= \int_u^\infty \mathbb{P}(Y \le x - u | X = x) dF_X(x) \\
&= \int_u^\infty \mathbb{P}(Y \le x - u | X = x) I_{\{F_Y(x-u) < F_X(x)\}} dF_X(x) \\
&\quad + \int_u^\infty \mathbb{P}(Y \le x - u | X = x) I_{\{F_Y(x-u) \ge F_X(x)\}} dF_X(x).
\end{aligned}$$

To prove the first statement of the Lemma note that

$$\begin{aligned}
&\int_u^\infty \mathbb{P}(Y \le x - u | X = x) I_{\{F_Y(x-u) < F_X(x)\}} dF_X(x) \\
&\quad \le \int_u^\infty I_{\{F_Y(x-u) < F_X(x)\}} dF_X(x) \\
&\quad \le \int_u^{u+\overline{\gamma}(u)} dF_X(x) = \mathbb{P}(X > u) - \mathbb{P}(X > u + \overline{\gamma}(u)) = o(\mathbb{P}(X > u)).
\end{aligned}$$

For the second integral we have

$$\begin{aligned}
&\int_u^\infty \mathbb{P}(Y \le x - u | X = x) I_{\{F_Y(x-u) \ge F_X(x)\}} dF_X(x) \\
&\quad \ge \int_{u+k\overline{\gamma}(u)}^\infty \mathbb{P}(Y \le x - u | X = x) dF_X(x) \\
&\quad \ge \int_{u+k\overline{\gamma}(u)}^\infty \mathbb{P}(Y \le k\overline{\gamma}(u) | X = x) dF_X(x) \\
&\quad = \mathbb{P}(X > u + k\overline{\gamma}(u)) - \mathbb{P}(X > u + k\overline{\gamma}(u), Y > k\overline{\gamma}(u)) \\
&\quad \ge \mathbb{P}(X > u + k\overline{\gamma}(u)) - \mathbb{P}(Y > k\overline{\gamma}(u)).
\end{aligned}$$

Hence there exists a $c_1 > 0$ that does not depend on k, with

$$\frac{\mathbb{P}(Y > k\overline{\gamma}(u))}{\mathbb{P}(X > u)} = \frac{\mathbb{P}(Y > k\overline{\gamma}(u))}{\mathbb{P}(Y > \overline{\gamma}(u))}\frac{\mathbb{P}(Y > \overline{\gamma}(u))}{\mathbb{P}(X > u)} \leq c_1 k^{-\alpha}.$$

Since for x_0 with $F_Y(x_0 - u) < F_X(x_0)$ it follows for every $\varepsilon > 0$ that $\overline{F}_{XY}\big((x_0+\varepsilon) - (u+\varepsilon)\big) < F_X(x_0+\varepsilon)$, we get that $\overline{\gamma}(u)$ is monotonically increasing. Hence the first statement follows. For the second statement note that for comonotonic X and Y one has

$$\begin{aligned}\mathbb{P}(X - Y > u) &= \int_u^\infty \mathbb{P}(Y \leq X - u | X = x) dF_X(x) \\ &\leq \int_u^\infty I_{\{F_Y(x-u) \geq F_X(x)\}} dF_X(x) \\ &\leq \int_{u+\underline{\gamma}(u)}^\infty dF_X(x) = \mathbb{P}(X > u + \underline{\gamma}(u)).\end{aligned}$$ □

Although Lemma 5.1 shows that comonotonic copulas are natural candidates for counterexamples, this does not tell whether the comonotonic copula represents the worst case. To answer that question, let us first consider the case of regularly varying X. In Proposition 6.1 below it will be shown that if $\overline{F}_Y(u)/\overline{F}_X(u) \to 0$, then all copulas provide the same asymptotic properties. On the other hand, if $F_X(x) \geq F_Y(x)$ for X, Y comonotonic, then $\mathbb{P}(X - Y > u) = 0$. Hence assume that there exists a $\hat{c} > 0$ with

$$\lim_{u\to\infty} \frac{\overline{F}_Y(u)}{\overline{F}_X(u)} = \hat{c}$$

or, equivalently, that there exists a c such that

$$\lim_{u\to\infty} \frac{\overline{F}_Y(cu)}{\overline{F}_X(u)} = 1.$$

We will study the asymptotic behavior of $X - Y$ under the additional condition that

$$\frac{\mathbb{P}(X > xu, Y > ycu)}{\mathbb{P}(X > u)} \to H(x, y),$$

where $H(x, y)$ is not degenerate. Then by extreme value theory it follows that

$$\frac{\mathbb{P}(X - Y > u)}{\mathbb{P}(X > u)} \to H(\{(x, y) | x - cy > 1\}).$$

To understand which H minimizes $H(\{(x, y)|x - cy > 1\})$, the index of regular variation α of F_X plays a role. From Resnick [21] it follows that when turning to polar coordinates, H can be written as a product of the measure on the radial and angular part. As norm we choose the sum of the components, then we get that the radial measure has density $\alpha r^{-\alpha-1}$ and the angular measure μ satisfies

$$\int_0^1 \theta^\alpha d\mu(\theta) = \int_0^1 (1-\theta)^\alpha d\mu(\theta) = 1.$$

Further note that

$$H(\{(x,y)|x-cy>1\}) = \int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^{\alpha}\, d\mu(\theta). \tag{5.1}$$

Now we can ask which μ^* minimizes equation (5.1). Consider discrete measures with $\mu(\theta=\theta_i)=p_i$ for $i=1,\dots,d$. Then there exists a $\theta_i>1/2$ ($p_i>0$) if and only if there exists a $\theta_j<1/2$ ($p_j>0$).

Lemma 5.2. *If the measure μ^* that minimizes* (5.1) *assigns positive mass p_i to a $\theta_i \leq \frac{c}{c+1}$, then*

$$\theta_i = \frac{c}{1+c}.$$

Proof. Assume that the result does not hold. Then w.l.o.g. we can assume that $\theta_1 > 1/2$ and $\theta_2 < c/(c+1)$. Define a new measure μ^{**} with $\hat{\theta}_i = \theta_i$ for $i \neq 2$ and $\hat{p}_i = p_i$ for $i > 2$, together with $\hat{\theta}_2 = c/(1+c)$. To ensure that μ is a measure we need

$$\begin{aligned} p_1\theta_1^{\alpha} + p_2\theta_2^{\alpha} &= \hat{p}_1\theta_1^{\alpha} + \hat{p}_2\left(\frac{c}{1+c}\right)^{\alpha}, \\ p_1(1-\theta_1)^{\alpha} + p_2(1-\theta_2)^{\alpha} &= \hat{p}_1(1-\theta_1)^{\alpha} + \hat{p}_2\left(\frac{1}{1+c}\right)^{\alpha}. \end{aligned}$$

It follows that

$$\hat{p}_1 = p_1 + p_2\frac{\left(\theta_2\frac{1+c}{c}\right)^{\alpha} - ((1-\theta_2)(1+c))^{\alpha}}{\left(\theta_1\frac{1+c}{c}\right)^{\alpha} - ((1-\theta_1)(1+c))^{\alpha}} < p_1,$$

where w.l.o.g. we assumed that p_2 is small enough such that $\hat{p}_1 \geq 0$. It follows that

$$\begin{aligned} &\int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^{\alpha}\, d\mu^*(\theta) - \int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^{\alpha}\, d\mu^{**}(\theta) \\ &\quad = (p_1 - \hat{p}_1)(\theta_1 - c(1-\theta_1))^{\alpha} > 0, \end{aligned}$$

which is a contradiction to μ^* minimizing (5.1) . ☐

Theorem 5.3. *Assume that $\alpha < 1$. Then μ^* is concentrated on $\theta_1 = 1$ and $\theta_2 = \frac{c}{1+c}$, with $p_1 = 1 - c^{\alpha}$ and $p_2 = (1+c)^{\alpha}$.*

Proof. Assume that μ^* assigns positive measure $p_1 > 0$ to $c/(1+c) < \theta_1 < 1$. Then we can define a new measure μ^{**} which is equivalent to μ^* except that we replace θ_1 by 1 and the corresponding probability p_1 by $\hat{p}_1$. Further we add the mass $\hat{p}_0$ to $c/(1+c)$, so that

$$\begin{aligned} \hat{p}_1 &= p_1\left(\theta_1^{\alpha} - c^{\alpha}(1-\theta_1)^{\alpha}\right) > 0 \\ \hat{p}_0 &= p_1(1-\theta_1)^{\alpha}(1+c)^{\alpha}. \end{aligned}$$

Furthermore,

$$\int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^\alpha \, d\mu^*(\theta) - \int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^\alpha \, d\mu^{**}(\theta)$$
$$= p_1 (\theta_1 - c(1-\theta_1))^\alpha - \hat{p}_1$$
$$= p_1 ((\theta_1 - c(1-\theta_1))^\alpha - (\theta_1^\alpha - c^\alpha (1-\theta_1)^\alpha)) > 0,$$

from which the result follows. $\square$

Theorem 5.4. *Assume that $\alpha > 1$. Then μ^* is concentrateed on $\theta_1 = 1/2$.*

Proof. Assume that μ^* assigns positive measure $p_1 > 0$ to $\theta_1 > 1/2$ and $p_2 > 0$ to $\theta_2 < 1/2$, where we assume w.l.o.g. that

$$p_1 \theta_1^\alpha + p_2 \theta_2^\alpha = p_1 (1-\theta_1)^\alpha + p_2 (1-\theta_2)^\alpha.$$

Define the measure μ^{**} with θ_1 and θ_2 replaced by $1/2$ with probability mass $\hat{p}_1 = 2^\alpha (p_1 \theta_1^\alpha + p_2 \theta_2^\alpha)$. We have to distinguish two cases:
a) $\theta_2 > c/(1+c)$: In this case we have to show that

$$\int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^\alpha \, d\mu^*(\theta) - \int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^\alpha \, d\mu^{**}(\theta) \geq 0.$$

The left hand side equals

$$\begin{aligned} &p_1 (\theta_1 - c(1-\theta_1))^\alpha + p_2 (\theta_2 - c(1-\theta_2))^\alpha \\ &\quad - (1-c)^\alpha (p_1 \theta_1^\alpha + p_2 \theta_2^\alpha) \\ &= p_1 (\theta_1 - c(1-\theta_1))^\alpha + p_1 \frac{\theta_1^\alpha - (1-\theta_1)^\alpha}{(1-\theta_2)^\alpha - \theta_2^\alpha} (\theta_2 - c(1-\theta_2))^\alpha \\ &\quad - p_1 (1-c)^\alpha \left(\theta_1^\alpha + \theta_2^\alpha \frac{\theta_1^\alpha - (1-\theta_1)^\alpha}{(1-\theta_2)^\alpha - \theta_2^\alpha} \right), \end{aligned}$$

so that we need to show that

$$\frac{\left(1 - c\left(\frac{1}{\theta_1} - 1\right)\right)^\alpha - (1-c)^\alpha}{1 - \left(\frac{1}{\theta_1} - 1\right)^\alpha} \geq \frac{\left(1 - c\left(\frac{1}{\theta_2} - 1\right)\right)^\alpha - (1-c)^\alpha}{1 - \left(\frac{1}{\theta_2} - 1\right)^\alpha}. \tag{5.2}$$

Since the function

$$\frac{(1-cx)^\alpha - (1-c)^\alpha}{1 - x^\alpha}$$

is decreasing for $x < 1$ and increasing for $x > 1$, we get that we only have to check (5.2) for $\theta_1 = \theta_2 = 1/2$, which holds since

$$\lim_{x \to 1} \frac{(1-cx)^\alpha - (1-c)^\alpha}{1 - x^\alpha} = (1-c)^{\alpha - 1}.$$

b) $\underline{\theta_2 = c/(1+c)}$: In this case we have to show that

$$\begin{aligned}&\int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^\alpha \, d\mu^*(\theta) - \int_{\frac{c}{1+c}}^{1} (\theta - c(1-\theta))^\alpha \, d\mu^{**}(\theta)\\ &\quad = p_1 (\theta_1 - c(1-\theta_1))^\alpha - (1-c)^\alpha \left(p_1 \theta_1^\alpha + p_2 \left(\frac{c}{1+c} \right)^\alpha \right)\\ &\quad = p_1 (\theta_1 - c(1-\theta_1))^\alpha - p_1 (1-c)^\alpha \left(\theta_1^\alpha + c^\alpha \frac{\theta_1^\alpha - (1-\theta_1)^\alpha}{1-c^\alpha} \right) \geq 0.\end{aligned}$$

This is equivalent to showing that

$$\frac{\left(1 - c\left(\frac{1}{\theta_1} - 1\right)\right)^\alpha - (1-c)^\alpha}{1 - \left(\frac{1}{\theta_1} - 1\right)^\alpha} \geq \frac{(1-c)^\alpha c^\alpha}{1-c^\alpha}.$$

Again the left side is minimized for $\theta_1 = 1/2$ and we have to show that

$$(1-c)^{\alpha-1} \geq \frac{(1-c)^\alpha c^\alpha}{1-c^\alpha},$$

which is true for $0 < c < 1$ and $\alpha > 1$. □

Lemma 5.5. *Let X be in the maximum domain of attraction of the Gumbel distribution with mean excess function $e(x)$. Further assume that there exists a $0 < c < 1$ with*

$$\lim_{u \to \infty} \frac{\mathbb{P}(Y > cu)}{\mathbb{P}(X > u)} = 1$$

and that the copula of X and Y is in the maximum domain of attraction of an extreme value copula. Then the copula that asymptotically minimizes $\mathbb{P}(X - Y > u)$ is the comonotonic copula.

Proof. From e.g. [21] we have

$$\frac{\mathbb{P}(X > u + xe(u), Y > cu + yce(u))}{\mathbb{P}(X > u)} \to H(x,y).$$

Here, $H(x,y) = H^*(\mathrm{e}^x, \mathrm{e}^y)$, where under H^*, $R = x + y$ and $\theta = x/(x+y)$ are independent, R has density r^{-2} and the measure μ of θ satisfies

$$\int_0^1 \theta d\mu(\theta) = \int_0^1 1 - \theta d\mu(\theta) = 1.$$

We get that

$$\frac{\mathbb{P}(X - Y > (1-c)u + e(u), X > u - Me(u))}{\mathbb{P}(X > u)} \to H(\{(x,y)|x - cy > 1, x > -M\})$$

with

$$\begin{aligned}&H(\{(x,y)|x - cy > 1, x > -M\})\\ &\quad = \int_0^1 \min\left(\mathrm{e}^{-\frac{1}{1-c}} (1-\theta) \left(\frac{\theta}{1-\theta} \right)^{\frac{1}{1-c}}, \mathrm{e}^M \right) d\mu(\theta).\end{aligned}$$

If $\mu(1) > 0$ and $N > 0$, then as $u \to \infty$

$$\begin{aligned}
&\frac{\mathbb{P}(X - Y > (1-c)u + e(u), X > u - Me(u))}{\mathbb{P}(X > u)} \\
&\quad \gtrsim \frac{\mathbb{P}(X > u - Ne(u)) - \mathbb{P}(X > u - Ne(u)), Y > cu - (N+2)e(u))}{\mathbb{P}(X > u)} \\
&\quad \sim \mathrm{e}^N - \int_0^1 \min(\theta \mathrm{e}^N, (1-\theta)\mathrm{e}^{c^{-1}(N+2)}))d\mu(\theta) \\
&\quad \geq \mathrm{e}^N \mu(1) \to \infty,
\end{aligned}$$

as $N \to \infty$. Hence with $M \to \infty$

$$\begin{aligned}
&\lim_{u\to\infty} \frac{\mathbb{P}(X - Y > (1-c)u + e(u))}{\mathbb{P}(X > u)} \\
&\quad \geq \mathrm{e}^{-\frac{1}{1-c}} \int_0^1 \mathrm{e}^{-\frac{1}{1-c}}(1-\theta)\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{1-c}} d\mu(\theta). \qquad (5.3)
\end{aligned}$$

Note that for X, Y comonotone we can replace $\geq$ by $=$. Finally we have to find the μ that minimizes (5.3). Again, we only consider μ discrete. W.l.o.g we assume that $\theta_1 > 1/2$ and $\theta_2 < 1/2$ with

$$p_1\theta_1 + p_2\theta_2 = p_1(1-\theta_1) + p_2(1-\theta_2) = \frac{p_1 + p_2}{2}$$

and we replace θ_1 and θ_2 with $\theta = 1/2$ and $p = p_1 + p_2$. We have to show that

$$p_1(1-\theta_1)\left(\frac{\theta_1}{1-\theta_1}\right)^{\frac{1}{1-c}} + p_2(1-\theta_2)\left(\frac{\theta_2}{1-\theta_2}\right)^{\frac{1}{1-c}} \geq p_1(1-\theta_1) + p_2(1-\theta_2).$$

Since

$$p_2 = p_1 \frac{2\theta_1 - 1}{1 - 2\theta_2},$$

we need to establish that

$$\frac{1-\theta_1}{2\theta_1 - 1}\left(\left(1 + \frac{2\theta_1 - 1}{1-\theta_1}\right)^{\frac{1}{1-c}}\right) \geq \frac{1-\theta_2}{2\theta_2 - 1}\left(\left(1 + \frac{2\theta_2 - 1}{1-\theta_2}\right)^{\frac{1}{1-c}}\right)$$

or for $x_i = \frac{2\theta_i - 1}{1-\theta_i}$

$$\frac{(1+x_1)^{\frac{1}{1-c}} - 1}{x_1} \geq \frac{(1+x_2)^{\frac{1}{1-c}} - 1}{x_2},$$

which holds due to $\frac{1}{1-c} > 1$ and $-1 < x_2 < 0 < x_1$. □

Theorem 5.3 shows that when $X \in \mathcal{R}_{-\alpha}$ with index $\alpha < 1$, then comonotonicity does not minimize $\mathbb{P}(X - Y > u)$. On the other hand, Theorem 5.4 suggests that for $\alpha > 1$ comonotonicity does minimize $\mathbb{P}(X - Y > u)$. However, we now show that this is not the case.

Let $\mathbb{P}_C$ denote the probability measure associated with a copula C and let $M(u,v) = \min(u,v)$ be the comonotonic copula. Then an equivalent formulation for a comonotonic copula minimizing that probability is that for every copula C

$$\liminf_{u\to\infty} \frac{\mathbb{P}_C(X-Y>u)}{\mathbb{P}_M(X-Y>u)} \geq 1. \tag{5.4}$$

In view of Proposition 6.1 shown in the next section, one can assume that for regularly varying X there exists a counterexample for (5.4) if $\overline{F}_X(x) \approx c\overline{F}_Y(x)$ for some $0<c<1$. Therefore we will choose $F_Y(x) = F_X(2x)$, i.e. $2Y \stackrel{d}{=} X$. Further, let X be in the maximum domain of attraction of an extreme value distribution. We will use the following dependence structure.

Definition 5.6. *For a random variable X with distribution function F_X and mean excess function $e(u)$, define $u_n = u_{n-1} + 2e(2u_{n-1})$ for a $u_1 > 0$ with $F(u_1) > 0$, together with a corresponding partition $(J_i)_{n\geq 1}$ of the interval $[0,1]$ $(n \geq 1)$*

$$\begin{aligned} J_1 &= [0, F(2u_1)) \\ J_{2n} &= [F(2u_n), F(2(u_n + e(2u_n)))) \\ J_{2n+1} &= [F(2(u_n + e(2u_n))), F(2u_{n+1}))\,. \end{aligned}$$

Moreover, define a series $(C_n)_{n\geq 1}$ of copulas with

$$C_{2n}(u,v) = uv \quad \textit{and} \quad C_{2n+1}(u,v) = \min(u,v).$$

Finally, define the copula $\overline{C}$ as the ordinal sum (cf. [20, 3.2.2]) of the copulas $(C_n)_{n\geq 1}$ with respect to the partition $(J_i)_{n\geq 1}$.

Remark 5.7. If $2Y \stackrel{d}{=} X$ and X, Y are dependent according to the copula in Definition 5.6, then for $0 \leq Y < u_1$ and $u_n + e(2u) \leq Y < u_{n+1}$, we have that $2Y = X$. Furthermore, for $n \geq 1$

$$\mathbb{P}(X \leq x | u_n \leq Y < u_n + e(2u_n)) = \mathbb{P}(X \leq x | 2u_n \leq X < 2u_n + 2e(2u_n)).$$

Proposition 5.8. *Let X be in the maximum domain of attraction of an extreme value distribution and let its density f_X satisfy*

$$\lim_{u\to\infty} \frac{f_X(u + xe(u))}{f_X(u)} = g(x) = \begin{cases} (1+x)^{-\alpha} & \overline{F}_X(x) \in \mathcal{R}_{-\alpha},\ \alpha > 0 \\ \mathrm{e}^{-x} & X \in MDA(\Lambda) \end{cases}.$$

Further assume that $2Y \stackrel{d}{=} X$ and that X and Y are dependent according to the copula of Definition 5.6. Then

$$\liminf_{u\to\infty} \frac{\mathbb{P}_{\overline{C}}(X-Y>u)}{\mathbb{P}_M(X-Y>u)} < 1.$$

Proof. W.l.o.g we assume that $e(x)$ is monotone. For every n we have

$$\begin{aligned} \mathbb{P}(X-Y>u_n) = {} & \mathbb{P}(X-Y>u_n, Y \leq u_n) \\ & + \mathbb{P}(X-Y>u_n, u_n < Y \leq u_n + e(2u_n)) \\ & + \mathbb{P}(X-Y>u_n, u_n + e(2u_n) < Y). \end{aligned}$$

Now one can easily check that

$$\mathbb{P}(X - Y > u_n, Y \le u_n) = 0$$

and

$$\mathbb{P}(X - Y > u_n, u_n + e(2u_n) < Y) \le \mathbb{P}(Y > u_n + e(2u_n)).$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}(X - Y > u_n, u_n < Y \le u_n + e(2u_n)) \\
&\quad = \int_{u_n}^{u_n+e(2u_n)} \mathbb{P}\left(X > u_n + y | 2u_n < X \le 2(u_n + e(2u_n))\right) f_Y(y)dy \\
&\quad = e(2(u_n)) \int_0^1 \mathbb{P}\big(X > 2u_n + ye(2u_n) | 2u_n < X \le 2(u_n + e(2u_n))\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f_Y(u_n + ye(2u_n))dy.
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{P}\left(X > 2u_n + ye(2u_n) | 2u_n < X \le 2(u_n + e(2u_n))\right) \\
&\quad = \frac{\mathbb{P}(X > 2u_n + ye(2u_n)) - \mathbb{P}(X > 2u_n + e(2u_n))}{\mathbb{P}(X > 2u_n) - \mathbb{P}(X > 2u_n + e(2u_n))} \\
&\quad \to \frac{g(y) - g(1)}{g(0) - g(1)} < 1, \quad y > 0
\end{aligned}$$

for $n \to \infty$. It follows from

$$\frac{f_Y(u_n + ye(2u_n))}{f_Y(u_n)} = \frac{f_X(2u_n + 2ye(2u_n))}{f_X(2u_n)} \to g(2y)$$

that

$$\lim_{n\to\infty} \frac{\mathbb{P}(X - Y > u_n, u_n < Y \le u_n + e(2u_n))}{\mathbb{P}(u_n < Y \le u_n + e(2u_n))} < 1$$

and hence

$$\lim_{n\to\infty} \frac{\mathbb{P}(X - Y > u_n)}{\mathbb{P}(Y > u_n)} < 1. \qquad \square$$

Example 5.9. As an illustration, consider $\mathbb{P}(X > x) = \mathbb{P}(2Y > x) = 1/x$ with $e(x) = x$ and $u_n = 5^n$. Figure 1 depicts the plot of

$$\mathbb{P}_{\overline{C}}\left(X - Y > \tfrac{1}{2}10^x\right) / \, \mathbb{P}_M\left(X - Y > \tfrac{1}{2}10^x\right).$$

Having seen now that the worst case is not always given by the comonotonic copula, we are now interested in identifying the worst case (given a specific u). For that purpose, we will use (straight) shuffles of M (cf. Nelsen [20, Th.3.2.3]).

Definition 5.10 (Shuffles). *Let $\mathcal{J} = \{J_1, \dots, J_n\}$ be a partition of $[0,1]$ into n closed subintervals and π a permutation of $\{1, \dots, n\}$. Then the copula $M_s(\mathcal{J}, \pi)$ is a shuffle, if the stripes $J_i \times [0,1]$ of M are reordered according to π.*

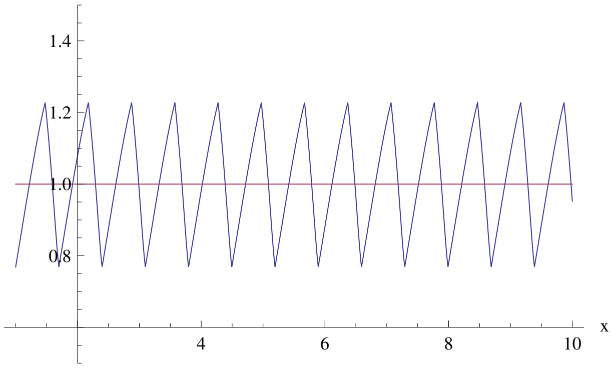

Figure 1: Plot of $\mathbb{P}_{\overline{C}}\left(X - Y > \frac{1}{2}10^x\right)/\mathbb{P}_M\left(X - Y > \frac{1}{2}10^x\right)$

From the discussion after Th.3.2.3 in [20] it follows that every copula can be approximated arbitrary closely by a shuffle. Hence we want to find the shuffle that minimizes $\mathbb{P}(X - Y > u)$. For a given F_X, F_Y and u, define

$$g_u(x) = \begin{cases} \inf\{t : F_Y^{-1}(t) \geq F_X^{-1}(x) - u\} & \text{if } F_X^{-1}(x) > u, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

For uniformly distributed (U_1, U_2) with the same copula C as (X, Y) it is valid that

$$\mathbb{P}(U_2 < g_u(U_1)) = \mathbb{P}(X - Y > u).$$

Lemma 5.11. *Let $g(x)$ be a monotone increasing function, such that for all $c \in [-1, 1]$ the number of times $g(x) - x - c$ changes sign is finite. Then the shuffle M_s^* that minimizes*

$$\mathbb{P}_{M_s}(U_2 < g(U_1))$$

is of the form $\mathcal{J} = \{[0, x_0], [x_0, 1]\}$ and $\pi = (2, 1)$ for some $0 < x_0 < 1$.

Proof. Let M_s be a shuffle with finite partition $\mathcal{J}$ and permutation π. For $J \in \mathcal{J}$ and $x \in J$, denote by J^π and x^π the interval J (point to which x, respectively) is mapped by the permutation. W.l.o.g we assume that for every $J \in \mathcal{J}$

$$\mathbb{P}(U_1 \in \{x^\pi : x \in J \;\&\; x < g(x^\pi)\}) \in \{0, 1\}.$$

Denote with $x_0 = \mathbb{P}_{M_s}(U_2 < g(U_1))$. W.l.o.g we can assume that for every $J \in \mathcal{J}$, $(J \cap [0, x_0]) \in \{\emptyset, J\}$. Further we can split the intervals in the partition $\mathcal{J}$, such that to every interval $J \in \mathcal{J}$ with $\mathbb{P}(U_1 \in \{x^\pi : x \in J \;\&\; x < g(x^\pi)\}) = 1$ we can assign a unique interval $\hat{J}$ with $\hat{J} \cap [0, x_0] = \hat{J}$ and $|J| = |\hat{J}|$. If we change the position of J and $\hat{J}$ in the permutation then $\mathbb{P}(U_2 < g(U_1))$ is the same for both shuffles. Hence we can assume that if $\mathbb{P}(U_1 \in \{x^\pi : x \in J \;\&\; x < g(x^\pi)\}) = 1$, then $J \subset [0, x_0]$. Since $g(x)$ is monotone increasing we can reorder the partitions such that we get the form of M_s^* from which the Lemma follows. $\square$

The worst copula is not unique, as can be seen by the following straight-forward result.

Lemma 5.12. *Let $g(x)$ be a monotone increasing function. Let $x_1 = \inf\{x : x \geq g(x)\}$. If $x_1 < 1 - x_0$ for some x_0, then the shuffles $M_s(\{[0,x_0],[x_0,1]\}, (2,1))$ and $\hat{M}_s(\{[0,x_1],[x_1,x_1+x_0],[x_1+x_0,1]\},(1,3,2))$ fulfill*

$$\mathbb{P}_{M_s}(U_2 < g(U_1)) \geq \mathbb{P}_{\hat{M}_s}(U_2 < g(U_1)).$$

If $x_1 \geq 1 - x_0$, then

$$\mathbb{P}_{M_s}(U_2 < g(U_1)) \geq \mathbb{P}_{M}(U_2 < g(U_1)).$$

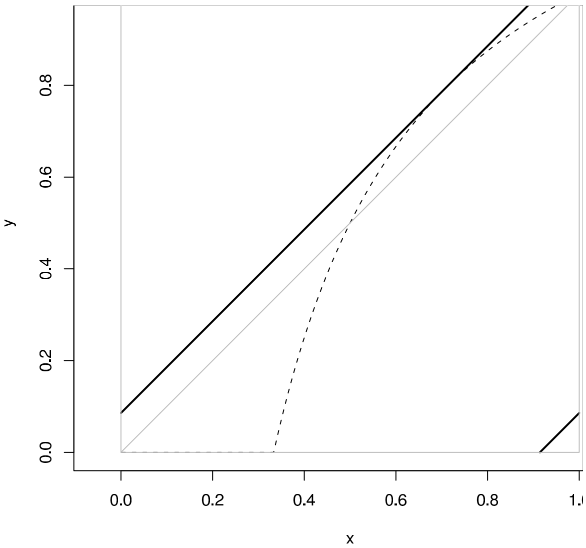

Figure 2: A worst-case copula

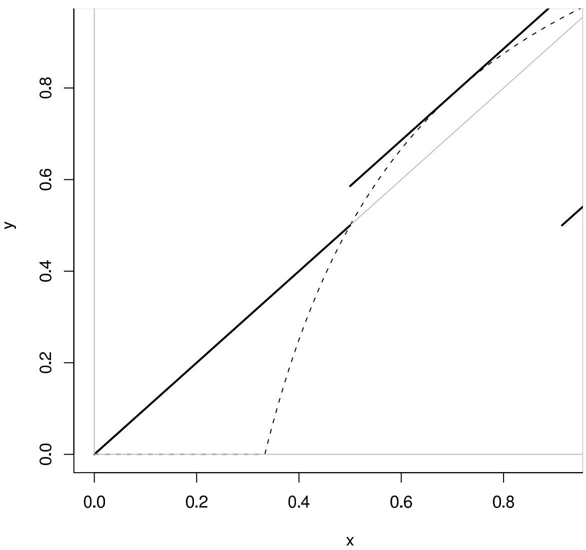

Figure 3: Another worst-case copula

Example 5.13. Let $F_X(x) = 1 - 1/x$, $F_Y(x) = 1 - 1/(2x)$ and $u = 1$. For this case, Figure 2 shows the support of the copula in Lemma 5.11 (bold line), where $x_0 \approx 0.086$. In Figure 3, the bold line depicts the support of the copula in Lemma

5.12, where $x_0 \approx 0.086$ and $x_1 = 0.5$. In both plots the dashed line corresponds to the function $g_u(x)$. Here

$$x_0 = x_0^* = \sup_{0 \le x \le 1} g_u(x) - x. \tag{5.6}$$

□

In fact, the choice of $x_0 = x_0^*$ in (5.6) is optimal in general, as can be verified by the following arguments: If $x_0 > x_0^*$, then the line $x + x_0$ corresponding to the interval $[x_0, 1]$ lies above the line $g_u(x)$. Hence we can decrease x_0 to x_0^* until it touches the line $g_u(x)$; certainly $\mathbb{P}_{M_s}(U_2 < g_u(U_1))$ then does not increase. If on the other hand $x_0 < x_0^*$ and x^* is a point with $x_0^* = g_u(x^*) - x^*$, then the monotonicity of $g_u(x)$ implies that the line segment of $x + x_0$ from x^* to $g_u(x^*) - x_0$ lies below $g_u(x)$. Since this line segment has length $g_u(x^*) - x_0 - x^* = x_0^* - x_0$ we see that by using x_0^* instead of x_0 we do not increase the probability of $\mathbb{P}_{M_s}(U_2 < g_u(U_1))$. Further if $x^* > 1/2$ then the line corresponding to the interval $[0, x_0]$ lies below $g_u(x)$. Thus we have proved the following:

Proposition 5.14. *Assume that the conditions of Lemma 5.11 hold and that u is large enough such that x^* with*

$$g_u(x^*) - x^* = \sup_{0 \le x \le 1} g_u(x) - x$$

fulfills $x^* > 1/2$. Then

$$\inf_C \mathbb{P}_C(X - Y > u) = \sup_{0 \le x \le 1} g_u(x) - x.$$

Let us compare this result to the comonotonic copula. To that end, assume that there exists a unique point γ_u such that $g_u(x) - x \le 0$ for $x < \gamma_u$ and $g_u(x) - x > 0$ for $x > \gamma_u$, then $\mathbb{P}_M(X - Y > u) = 1 - \gamma_u$ and

$$\begin{aligned}\inf_C \mathbb{P}_C(X - Y > u) &= \mathbb{P}_M(X - Y > u) \sup_{0 \le x \le 1} \frac{g_u\left(\gamma_u + x(1-\gamma_u)\right) - \gamma_u - x(1-\gamma_u)}{1-\gamma_u} \\ &= \sup_{0 \le x \le 1} \left(g_u\left(\gamma_u + x(1-\gamma(u))\right) - \gamma_u - x(1-\gamma_u)\right).\end{aligned}$$

If the function

$$h_u(x) = \frac{g_u\left(\gamma_u + x(1-\gamma_u)\right) - \gamma_u - x(1-\gamma_u)}{1-\gamma_u}$$

converges for $u \to \infty$ to a function $h_\infty(x)$ with $\sup_{0<x<1} h_\infty(x) = 1$ (i.e. $h_\infty(x) = 1 - x$), then for every copula C (5.4) holds. On the other hand if there exists a sequence u_n with $\lim_{n\to\infty} u_n = \infty$ and $\limsup_{n\to\infty} \sup_{0<x<1} h_{u_n}(x) < 1$ then we can analougously to Proposition 5.8 construct a copula where (5.4) does not hold. The following example shows such a situation where X is Weibull and Y is light tailed.

Example 5.15. Let $F_X(x) = 1 - \mathrm{e}^{-x^\beta}$ $(1/2 < \beta < 1)$ and $F_Z(x) = 1 - \mathrm{e}^{-\frac{(1+\varepsilon)\beta^2}{2\beta-1}x^{2-1/\beta}}$. Define $u_0 = 0$, $u_n = 2^n$ and

$$F_Y(x) = 1 - \mathrm{e}^{-u_n} + \frac{F_Z(x) - F_Z(u_n)}{F_Z(u_{n+1}) - F_Z(u_n)}\left(\mathrm{e}^{-u_n} - \mathrm{e}^{-u_{n+1}}\right), \quad u_n \le x < u_{n+1}.$$

Since for $x > 2$

$$\frac{\overline{F}_Y(x)}{\mathrm{e}^{-x/2}} \leq \frac{\overline{F}_y(u_n)}{\mathrm{e}^{-u_{n+1}/2}} = 1$$

we get that Y is light tailed. Further for $u = u_n^{1/\beta} - u_n$ we get that $\gamma_u = (1 - e^{-u_n})$ and since $F_Y(x) \leq 1 - \mathrm{e}^{-x}$ there are no roots of $F_Y(F_X^{-1}(x) - u) = x$ to the left of γ_u. We get that

$$h_u(x) = 1 - x - \frac{\overline{F}_Y((u_n - \log(1-x))^{1/\beta} - u^{1/\beta} + u_n)}{\mathrm{e}^{-u_n}}$$

since for $n \to \infty$

$$(u_n - \log(1-x))^{1/\beta} - u^{1/\beta} + u_n = u_n + (1+o(1))\frac{(-\log(1-x))}{\beta}u_n^{1/\beta-1} \leq 2u_n = u_{n+1}.$$

We get that

$$\begin{aligned}
&\frac{\overline{F}_Y((u_n - \log(1-x))^{1/\beta} - u^{1/\beta} + u_n)}{\mathrm{e}^{-u_n}} \\
&\quad = 1 - \frac{\overline{F}_Z\left(u_n + (1+o(1))\frac{(-\log(1-x))}{\beta}u_n^{1/\beta-1}\right) - \overline{F}_Z(u_n)}{\overline{F}_Z(u_{n+1}) - \overline{F}_Z(u_n)}\left(1 - e^{-u_n}\right) \\
&\quad \sim \frac{\overline{F}_Z\left(u_n + \frac{(-\log(1-x))}{\beta}u_n^{1/\beta-1}\right)}{\overline{F}_Z(u_n)} \\
&\quad \sim (1-x)^{1+\varepsilon}
\end{aligned}$$

Hence as $n \to \infty$

$$h_{u_n}(x) \to (1-x)(1-(1-x)^{\varepsilon}).$$

6 Regularly varying X

Proposition 6.1. *If $X \in \mathcal{R}_{-\alpha}$ and $\mathbb{P}(Y > u) = o(\mathbb{P}(X > u))$, then* (2.2) *holds.*

Proof. For every n there exists an $\hat{u}_n$ such that for all $u > \hat{u}_n$

$$\frac{\mathbb{P}(Y > u)}{\mathbb{P}(X > nu)} \leq \frac{1}{n}.$$

Define $u_0 = 0$ and $u_n = \max(n\hat{u}_n, u_{n-1}) + 1$ for $n > 0$. Then for all $u > u_n$

$$\frac{\mathbb{P}(Y > u/n)}{\mathbb{P}(X > u)} \leq \frac{1}{n}.$$

Define

$$\varepsilon(u) = \begin{cases} 1, & u < u_1, \\ \frac{1}{n}, & u_n < u < u_{n+1}. \end{cases}$$

Then for $\delta(u) = \varepsilon(u)u$ we have

$$\lim_{u\to\infty} \frac{\mathbb{P}(Y > \delta(u))}{\mathbb{P}(X > u)} = 0.$$

The result then follows from

$$\frac{\mathbb{P}(X > u + \delta(u))}{\mathbb{P}(X > u)} = \frac{\mathbb{P}(X > u + \varepsilon(u)u)}{\mathbb{P}(X > u)} \sim (1 + \varepsilon(u))^{-\alpha} \to 1. \qquad \square$$

Remark 6.2. Note that Proposition 6.1 is still valid if X is of consistent variation (cf. Robert and Segers [22] and Cline [10]), i.e.

$$\lim_{\varepsilon\to 0} \liminf_{u\to\infty} \frac{\mathbb{P}(X > (1 + \varepsilon)u)}{\mathbb{P}(X > u)} = 1. \qquad \square$$

6.1 Approach with local limit laws

Let us now use local limit laws as in Heffernan and Resnick [14] to find the asymptotic behavior of $\mathbb{P}(X - Y > u)$. For that purpose, let either $E = [-\infty, \infty] \times (-\infty, \infty])$ $(e(u)/u \to 0)$ or $E = [-\infty, \infty] \times (-1, \infty])$ $(e(u) = u)$. Further we assume that there exists a measure μ (not equal to zero) for which for every fixed y in $\mathbb{E}$

- $\mu([-\infty, x], (y, \infty])$ is a non-degenerate distribution function in x,
- $\mu([-\infty, x], (y, \infty]) < \infty$, and
- $$\lim_{u\to\infty} \frac{\mathbb{P}(Y \leq \beta(u) + x\alpha(u), X > u + ye(u))}{\mathbb{P}(X > u)} = \mu([-\infty, x], (y, \infty])$$

  at continuity points (x, y) of the limit.

Assume that $\alpha(u)/e(u) \to c$ for some constant c, then we have that

$$\begin{aligned}\lim_{u\to\infty} \frac{\mathbb{P}(X - Y > u - \beta(u))}{\mathbb{P}(X > u)} &= \lim_{u\to\infty} \frac{\mathbb{P}\left(\frac{X-u}{e(u)} - \frac{\alpha(u)}{e(u)} \cdot \frac{Y-\beta(u)}{\alpha(u)} > 0, \frac{X-u}{e(u)} > 0\right)}{\mathbb{P}(X > u)} \\ &= \mu\left(\{(y, x) | x - cy > 0, x > 0\}\right) \leq 1\end{aligned}$$

at least if μ is sufficiently continuous. The area we have to measure is depicted in Figure 4.

It follows that

$$\frac{\mathbb{P}(X - Y > u)}{\mathbb{P}(X > u)} \sim \frac{\mathbb{P}(X > u)}{\mathbb{P}(X > u - \beta(u))} \mu\left(\{(y, x) | x - cy > 0, x > 0\}\right).$$

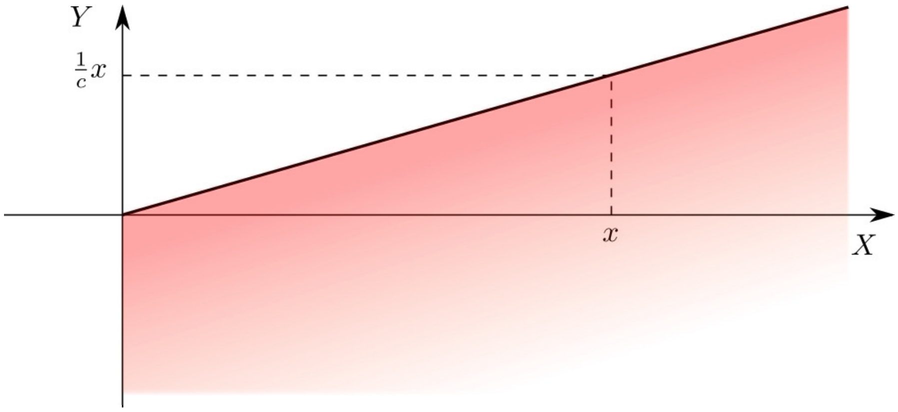

Figure 4: Area to be measured (shaded)

If (1.1) is valid, then we have to assume that $\beta(u)/e(u) \to 0$ and $c = 0$ (i.e. $\alpha(u)/e(u) \to 0$). However note that for every $\varepsilon > 0$

$$\begin{aligned}
&\lim_{u\to\infty} \frac{\mathbb{P}(Y \le \varepsilon e(u), X > u)}{\mathbb{P}(X > u)} \\
&= \lim_{u\to\infty} \frac{\mathbb{P}\left(Y \le \beta(u) + \frac{\varepsilon e(u) - \beta(u)}{\alpha(u)} \alpha(u), X > u\right)}{\mathbb{P}(X > u)} \\
&\ge \lim_{u\to\infty} \frac{\mathbb{P}\left(Y \le \beta(u) + M\alpha(u), X > u\right)}{\mathbb{P}(X > u)} \\
&= \mu([-\infty, M) \times \mu(0, \infty]) \to 1
\end{aligned}$$

as $M \to \infty$. Hence the conditions of Proposition 2.1 are fulfilled, so that we do not need to use local limit law for establishing (1.1).

References

[1] H. Albrecher, S. Asmussen & D. Kortschak (2006) Tail asymptotics for the sum of two heavy-tailed dependent risks. *Extremes* **9**, 107–130.

[2] H. Albrecher & O. J. Boxma (2004) A ruin model with dependence between claim sizes and claim intervals. *Insurance Math. Econom.* **35**, 245–254.

[3] H. Albrecher & J. Teugels (2006) Exponential behavior in the presence of dependence in risk theory. *J. Appl. Prob.* **43**, 257–273.

[4] A. V. Asimit & A. L. Badescu (2010) Extremes on the discounted aggregate claims in a time dependent risk model. *Scand. Actuar. J.*, 93–104.

[5] S. Asmussen (2003) *Applied Probability and Queues* (2nd ed.). Springer-Verlag, New York.

[6] S. Asmussen & H. Albrecher (2010) *Ruin Probabilities* (2nd ed.). World Scientific, Singapore.

[7] S. Asmussen & R. Biard (2011) Ruin probabilities for a regenerative Poisson gap generated risk process. *Europ. Act. J.* **1**, 3–22.

[8] G. Balkema & L. de Haan (1974) Residual life-time at great age. *Ann. Probab.* **2**, 792–804.

[9] M. Boudreault, H. Cossette, D. Landriault & E. Marceau (2006) a risk model with dependence between interclaim arrivals and claim sizes. *Scand. Actuar. J.* 265–285.

[10] D. B. H. Cline (1994) Intermediate regular and Π variation. *Proc. London Math. Soc. (3)* **68**, 594–616.

[11] P. Embrechts, C. Klüppelberg & T. Mikosch (1997) *Modelling Extreme Events for Insurance and Finance*. Springer-Verlag, Berlin.

[12] S. Foss, D. Korshunov & S. Zachary (2011) *An Introduction to Heavy-tailed and Subexponential Distributions*. Springer-Verlag, New York.

[13] S. Foss & A. Richards (2010) On sums of conditionally independent subexponential random variables. *Math. Oper. Res.* **35**, 102–119.

[14] P. Heffernan & S. Resnick (2007) Limit laws for random vectors with an extreme component. *Ann. Appl. Probab.* **17**, 537–571.

[15] P. Jelenković, P. Momcilović & B. Zwart (2004) Reduced load equivalence under subexponentiality. *QUESTA* **46**, 97–112.

[16] B. Ko & Q. Tang (2008) Sums of dependent nonnegative random variables with subexponential tails. *J. Appl. Prob.* **45**, 85–94.

[17] D. Kortschak & H. Albrecher (2009) Asymptotic results for the sum of dependent non-identically distributed random variables. *Methodol. Comput. Appl. Probab.* **11**, 279–306.

[18] J. Li, Q. Tang & R. Wu (2010) Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model. *Adv. in Appl. Probab.*, **42**, 1126–1146.

[19] A. Mitra & S. Resnick (2009) Aggregation of rapidly varying risks and asymptotic independence. *Adv. in Appl. Probab.* **41**, 797–828.

[20] R. B. Nelsen (1999) *An Introduction to Copulas*, Springer-Verlag, New York.

[21] S. I. Resnick (1987) *Extreme Values, Regular Variation, and Point Processes*, Springer-Verlag, New York.

[22] C. Y. Robert & J. Segers (2008) Tails of random sums of a heavy-tailed number of light-tailed terms. *Insurance Math. Econom.* **43**, 85–92.