Endomorphism Algebras of Tensor Powers of Modules for Quantum groups

PhD Thesis

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Contents

Abstract

The notion of *strongly multiplicity free modules* for complex semisimple Lie algebras and the corresponding quantum groups was first introduced in 2006 by G. I. Lehrer and R. B. Zhang. This new notion and the resulting properties of the modules in question led to a number of results concerning the ring structure of the endomorphism algebras of tensor powers of such modules.

Later (in 2008), Lehrer and Zhang combined these results with Kazhdan-Lusztig theory and the theory of cellular algebras. In this way they obtained more precise results on the structure of the endomorphism algebras of certain tensor spaces for $\mathfrak{sl}_2$, both for the classical and for the generic quantum case, i.e. when the quantum parameter is *not* a root of unity. The module they considered was the three-dimensional irreducible module for (quantum) $\mathfrak{sl}_2$.

The purpose of this thesis is to treat the $\mathfrak{sl}_2$ quantum case when the quantum parameter is allowed to be specialized to a root of unity. We prove that in this case there exists – under a suitable localization of our ground ring – a surjection from the group algebra of the braid group to the endomorphism algebra of any tensor power of the Weyl module for quantum $\mathfrak{sl}_2$ with highest weight 2. We also take a first step in the direction of determining the kernel of this map by reformulating well-known results on the semisimplicity of the Birman-Murakami-Wenzl algebra in a slightly new way, namely in terms of the order of the quantum parameter. Before we arrive at these main results, we investigate the structure of the endomorphism algebra of the tensor square of any Weyl module for quantum $\mathfrak{sl}_2$.

Dansk resumé (Danish abstract)

I 2006 introducerede G. I. Lehrer og R. B. Zhang begrebet *stærkt multiplicitetsfri moduler* for både komplekse semisimple Lie algebraer og de tilsvarende kvantegrupper. Dette nye begreb samt de medfølgende egenskaber ved de pågældende moduler ledte til en række resultater vedrørende ringstrukturen af endomorfialgebraer af tensorpotenser af stærkt multiplicitetsfri moduler.

Ved at kombinere disse resultater med Kazhdan-Lusztig-teori og teorien om cellulære algebraer, opnåede Lehrer og Zhang senere (i 2008) mere præcise resultater om strukturen af endomorfialgebraen af tensorpotenser af et bestemt modul for $\mathfrak{sl}_2$ i både det klassiske tilfælde og i det generiske kvantetilfælde, dvs. når kvanteparameteren ikke er en enhedsrod. Mere præcist er det det tredimensionale irreducible modul, de behandler.

Denne afhandling har til formål at behandle $\mathfrak{sl}_2$-kvantetilfældet, når kvanteparameteren tillades at være en enhedsrod. Vi viser, at der – under en passende lokalisering af vores ring – i dette tilfælde eksisterer en surjektion fra gruppealgebraen for braidgruppen til endomorfialgebraen for en vilkårlig tensorpotens af Weyl-modulet for kvante-$\mathfrak{sl}_2$ med højeste vægt 2. Vi tager også et første skridt hen imod at bestemme kernen af denne afbildning ved at formulere kendte resultater vedrørende semisimplicitet af Birman-Murakami-Wenzl algebraen på en delvist ny måde, nemlig i termer af ordenen af kvanteparameteren. Før vi når til disse hovedresultater, undersøger vi strukturen af endomorfialgebraen af et tensorkvadrat af et vilkårligt Weyl-modul for kvante-$\mathfrak{sl}_2$.

Introduction

Background

In 2006 G. I. Lehrer and R. B. Zhang introduced the notion of *strongly multiplicity free modules* for both complex semisimple Lie algebras and the corresponding quantum groups in [LZ06]. This new notion and the resulting properties of the modules in question led to a number of results concerning the ring structure of the endomorphism algebras of tensor powers of such modules. In particular, Lehrer and Zhang proved that in the generic case there exists a surjection from the group algebra of the braid group to the endomorphism algebra of a tensor power of any strongly multiplicity free module.

By combining these results with Kazhdan-Lusztig theory and the theory of cellular algebras, Lehrer and Zhang obtained later (in 2008) more precise results in [LZ08b] and [LZ08a] on the structure of the endomorphism algebras of certain tensor spaces for $\mathfrak{sl}_2$ both for the classical and for the quantum case when the quantum parameter is *not* a root of unity. The module they treated was the three-dimensional irreducible module. In the quantum case they proved that there is a surjection from the Birman-Murakami-Wenzl algebra to the endomorphism algebra just mentioned. They finished the description by providing a generator of the kernel of this surjection.

The problem of determining the ring structure of the endomorphism algebra is still open when we consider tensor powers of irreducible modules of dimension higher than three or allow the quantum parameter to be a root of unity. This fact has been the motivation and starting point of the work we present here.

Chapter overview

The purpose of this thesis is to study the $\mathfrak{sl}_2$ quantum case when the quantum parameter is allowed to be a root of unity. It consists of three parts.

The first part serves as an introduction to the subject and treats tensor squares. We identifiy the endomorphism algebra in the square case with a subalgebra of a Hecke algebra. Further, we give a surjection from the group algebra of the braid group to the endomorphism algebra under a suitable localization.

As a by-product, we produce a basis of the endomorphism algebra.

In the second part we focus our attention on any tensor power of the Weyl module for quantum $\mathfrak{sl}_2$ with highest weight 2. We prove that there is a surjection – under a suitable localization – from the group algebra of the braid group to the endomorphism algebra of such a tensor power. As a first step towards the determination of the kernel of this map, we reformulate well-known results on the semisimplicity of the Birman-Murakami-Wenzl algebra in terms of the order of the quantum parameter.

The third part is purely perspectival and contains a brief discussion of some other cases where strongly multiplicity free modules occur.

Part I Background, setup and tensor squares

Chapter 1

The first chapter presents a background to the type of problems that we seek to tackle in this thesis. We outline results by Lehrer and Zhang on endomorphism algebras of tensor powers in the generic case (i.e. when the quantum parameter q is not a root of unity). The module we treat is the finite-dimensional irreducible module for quantum $\mathfrak{sl}_2$ with highest weight 2.

Chapter 2

We introduce the setup that we will work on in this thesis and collect a few basic and well-known results. More precisely, we first define the Hecke and Temperley-Lieb algebras and subsequently introduce the integral form of the quantum group for $\mathfrak{sl}_2$ and its Weyl modules along with the universal R-matrix.

Chapter 3

This chapter contains some general results on the endomorphism algebras of tensor powers of Weyl modules for quantum $\mathfrak{sl}_2$. In particular, we treat tensor squares of such modules. The main results are Theorem 3.11 and Theorem 3.12 of which the first identifies the endomorphism algebra in the square case with a subalgebra of a Hecke algebra. The latter gives a surjection from the group algebra of the braid group to the endomorphism algebra under a suitable localization. Moreover, we give a basis of the endomorphism algebra in Proposition 3.7.

A key tool when handling the square case is a certain set of minimal double coset representatives in the symmetric group. We have collected the needed results on these in Appendix A.

Part II The Weyl module with highest weight 2

Chapter 4

Chapter 4 contains the main results of our work. We prove that under a suitable localization there exists a surjection from the group algebra of the braid group to the endomorphism algebra of any tensor power of the Weyl module with highest weight 2.

Chapter 5

As a first step in the direction of determining the kernel of the surjection from Chapter 4, we investigate the semisimplicity of the BMW-algebra. We translate results by Rui and Si ([RS09]) to fit with our setup. The result is a set of semisimplicity criteria based on the order of the quantum parameter.

Part III Discussion of similar cases

Chapter 6

The Weyl modules for quantum $\mathfrak{sl}_2$ which we treat in this thesis belong to a class of modules for quantum groups called *strongly multiplicity free* modules ([LZ06]). In this last chapter, we make some remarks about the other cases where strongly multiplicity free modules exist. More precisely, we recall to the reader results by Du, Parshall and Scott ([DPS98]) on (integral) quantum Weyl reciprocity for quantum $\mathfrak{sl}_k$, $k > 2$, along with recent results by Lehrer and Zhang ([LZ11]) on the orthogonal group where they conjecture that the situation in that case is very similar to the $\mathfrak{sl}_2$ case. In the symplectic case, we briefly discuss results and conjectures by Hu and Xiao ([HX10]).

Appendix

Appendix A

In this appendix, we collect a few results concerning minimal representatives of double cosets in symmetric groups which are needed in Chapter 3.

Appendix B

We calculate the action of the universal R-matrix on tensor squares of some low rank Weyl modules for quantum $\mathfrak{sl}_2$.

Notation

We will denote by $\mathbb{N}$ the positive integers (i.e. without 0) and by $\mathbb{Z}$ the ring of integers. Further, we will denote by $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ the rational, the real and the complex numbers, respectively. For any prime number p we denote by $\mathbb{Z}_p$ the field $\mathbb{Z}/p\mathbb{Z}$.

If a ground field, say k, is clear from the context, we will denote by $\otimes$ the tensor product $\otimes_k$ over k and by $\dim = \dim_k$ the dimension as a vector space over k.

Likewise, for any ring, say R, the tensor product $\otimes_R$ over R will be denoted $\otimes$ if R is clear from the context.

Rings and algebras are associative and unital unless explicitly stated otherwise.

Given a ring R, let R^{op} be the opposite ring of R. If M is a left (respectively, right) R-module, let M^{op} be the right (respectively, left) R^{op}-module obtained from the action of R on M.

We apply the usual notation $\mathfrak{g}$ for a simple complex Lie algebra and denote by $U(\mathfrak{g})$ the corresponding universal enveloping algebra. We denote by $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{o}_n$, $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ the general linear, special linear, orthogonal, special orthogonal and symplectic (complex) Lie algebras, respectively. By G_2 we refer to the complex Lie algebra corresponding to the root system G_2.

If S is a set, we denote by $\mathrm{id} = \mathrm{id}_S$ the identity map on S.

For any group G, we write $H \leq G$ or $H < G$ to denote that H is a subgroup or a proper subgroup of G, respectively.

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Therese Søby Andersen
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Part I

Background, setup and tensor squares

CHAPTER 1

Background: Ring structure of the endomorphism algebra in the generic case

This chapter serves as an introduction to the kind of mathematical problems that we seek to treat in this thesis. It concerns the recent work of Lehrer and Zhang on endomorphism algebras of strongly multiplicity free quantum tensor modules which has been the main inspiration and background for the work we present here.

The work by Lehrer and Zhang concerns the generic case, i.e. the case where the quantum parameter is not a root of unity. Here we outline the main results of their work. We will restrict our presentation to contain only the main results and very few proofs and we refer the interested reader to [LZ08b] and [LZ08a] for further details.

The main results of our presentation here are Theorem 1.5 and Theorem 1.6. Together these two results give a complete description of the ring structure of the endomorphism algebra of any tensor power of the three-dimensional irreducible module for quantum $\mathfrak{sl}_2$.

We begin our presentation by introducing the Birman-Murakami-Wenzl algebra which plays a prominent role in the formulation of the main results.

1.1 The BMW-algebra

Let $\mathcal{K} = \mathbb{C}(q^{\frac{1}{2}})$ and let $\mathcal{A}$ be the ring $\mathbb{C}[y^{\pm 1}, z]$ where $q^{\frac{1}{2}}, y, z$ are indeterminates over $\mathbb{C}$. The *BMW-algebra* $BMW_r(y,z)$ over $\mathcal{A}$ is the associative $\mathcal{A}$-algebra with generators $g_1^{\pm 1}, \ldots, g_{r-1}^{\pm 1}$ and $e_1, \ldots, e_{r-1}$ subject to the following relations. The braid relations for the g_i:

$$\begin{aligned} g_i g_j &= g_j g_i \text{ if } |i-j| \geq 2, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r-2. \end{aligned} \tag{1.1}$$

The Kaufmann Skein relations:

$$g_i - g_i^{-1} = z(1 - e_i) \text{ for } 1 \leq i \leq r - 1. \tag{1.2}$$

The de-looping relations:

$$\begin{aligned} g_i e_i = e_i g_i &= y e_i, \\ e_i g_{i-1}^{\pm 1} e_i &= y^{\mp 1} e_i, \\ e_i g_{i+1}^{\pm 1} e_i &= y^{\mp 1} e_i \end{aligned} \tag{1.3}$$

for all applicable i.

From the above relations we easily get the following four relations:

$$e_i e_{i\pm 1} e_i = e_i, \tag{1.4}$$

$$(g_i - y)(g_i^2 - z g_i - 1) = 0, \tag{1.5}$$

$$z e_i^2 = (z + y^{-1} - y) e_i, \tag{1.6}$$

$$-y z e_i = g_i^2 - z g_i - 1 \tag{1.7}$$

for all applicable i.

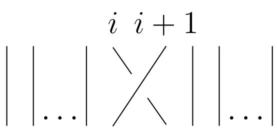
The proof of this is easy. As an example, note that by the Kaufmann Skein relations we have

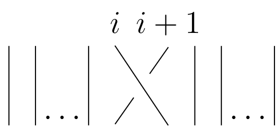
$$g_i^2 = g_i(z(1 - e_i) + g_i^{-1}) = z(g_i - y e_i) + 1 \tag{1.8}$$

for any $1 \leq i \leq r - 1$ which is (1.7).

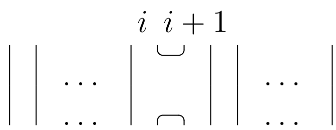
Remark 1.1 Note that the relation (1.2) is a consequence of (1.7) (just do the calculation (1.8) "backwards") which means that the BMW-algebra could have been defined using the relations (1.1), (1.3) and (1.7) instead of (1.1), (1.2) and (1.3). We shall make use of this fact in the proof of Theorem 1.5.

Note that by [MW] we have a representation of elements of $BMW_r(y, z)$ by tangle diagrams on r strands as follows:

$g_i =$ 

, $g_i^{-1} =$ 

and

$e_i =$ 

for all $i = 1, 2, \ldots, r - 1$. The elements are multiplied by composition of diagrams.

1.1.1 A specialized version of the BMW-algebra

We shall require a particular specialization of $BMW_r(y,z)$ to a subring $\mathcal{A}_q$ of $\mathcal{K}$ which is defined as follows. Write

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}]$$

for the *q-analogue* of $n \in \mathbb{Z}$. Let $\mathcal{S}$ be the multiplicative subset of $\mathbb{C}[q, q^{-1}]$ generated by $[2]_q, [3]_q$ and $[3]_q - 1$. Let

$$\mathcal{A}_q := \mathbb{C}[q, q^{-1}]_{\mathcal{S}} = \mathbb{C}[q, q^{-1}, [2]_q^{-1}, [3]_q^{-1}, (q^2 + q^{-2})^{-1}]$$

be the localization of $\mathbb{C}[q, q^{-1}]$ at $\mathcal{S}$.

Now let $\psi : \mathcal{A} \to \mathcal{A}_q$ be the homomorphism defined by

$$y \mapsto q^{-4}, \; z \mapsto q^2 - q^{-2}.$$

Then ψ makes $\mathcal{A}_q$ into an $\mathcal{A}$-module and the specialization

$$BMW_r(q) := \mathcal{A}_q \otimes_{\mathcal{A}} BMW_r(y,z)$$

is the $\mathcal{A}_q$-algebra with generators which, by abuse of notation, we denote $g_i^{\pm 1}, e_i$, for $i = 1, \dots, r-1$, and relations

$$\begin{aligned}
g_i g_j &= g_j g_i \text{ if } |i-j| \geq 2,\\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r-2,\\
g_i - g_i^{-1} &= (q^2 - q^{-2})(1 - e_i) \text{ for } 1 \leq i \leq r-1,\\
g_i e_i &= e_i g_i = q^{-4} e_i,\\
e_i g_{i-1}^{\pm 1} e_i &= q^{\pm 4} e_i,\\
e_i g_{i+1}^{\pm 1} e_i &= q^{\pm 4} e_i
\end{aligned}$$

for all applicable i.

As a consequence, we have the following relations, just like we had for $BMW_r(y,z)$:

$$\begin{aligned}
e_i e_{i\pm 1} e_i &= e_i,\\
(g_i - q^2)(g_i + q^{-2}) &= -q^{-4}(q^2 - q^{-2}) e_i, && (1.9)\\
(g_i - q^{-4})(g_i - q^2)(g_i + q^{-2}) &= 0, && (1.10)\\
e_i^2 &= (q^2 + 1 + q^{-2}) e_i && (1.11)
\end{aligned}$$

for all applicable i.

Definition 1.2 [LZ08b, Definition 4.1] Let $\phi_q : \mathcal{A}_q \to \mathcal{K}$ be the inclusion map and define the specialization $BMW_r(\mathcal{K}) := \mathcal{K} \otimes_{\phi_q} BMW_r(q)$.

1.1.2 Important elements of the BMW-algebra

We now introduce some elements of $BMW_r(q)$ which play an important role in formulating the main results. Let

$$f_i = -g_i - (1 - q^{-2})e_i + q^2 \quad \text{for } i = 1, 2, \dots, r-1,$$

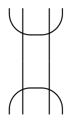
and set

$$F_q = f_1 f_3.$$

We also define

$$e_{14} = g_3^{-1} g_1 e_2 g_1^{-1} g_3 \quad \text{and} \quad e_{1234} = e_2 g_1 g_3^{-1} g_2 g_1^{-1} g_3.$$

Note that this notation is motivated by the look of the corresponding diagrams in the sense that i is in the subscript if and only if the tangle starting at the ith position of the upper row is not a vertical line:

$$e_{14} = g_3^{-1} g_1 e_2 g_1^{-1} g_3 =$$  and $$e_{1234} = e_2 g_1 g_3^{-1} g_2 g_1^{-1} g_3 =$$  .

Next we define the following important element of $BMW_4(q) \subseteq BMW_r(q)$, $r \geq 4$:

$$\Phi_q = aF_q e_2 F_q - bF_q - cF_q e_2 e_{14} F_q + dF_q e_{1234} F_q, \tag{1.12}$$

where

$$a = 1 + (1 - q^{-2})^2, \tag{1.13}$$

$$b = 1 + (1 - q^2)^2 + (1 - q^{-2})^2, \tag{1.14}$$

$$c = \frac{1 + (2 + q^{-2})(1 - q^{-2})^2 + (1 + q^2)(1 - q^{-2})^4}{([3]_q - 1)^2}, \tag{1.15}$$

$$d = (q - q^{-1})^2 = q^2(a - 1). \tag{1.16}$$

We shall leave the BMW-algebra for now and then return to it in the last section of this chapter. Meanwhile, let us take a look at the universal R-matrix which plays a key role in the formulation of Theorem 1.5.

1.2 The R-matrix

Keep the notation from above. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra with universal enveloping algebra $U(\mathfrak{g})$. Let $U_q(\mathfrak{g})$ denote the corresponding Drinfel'd-Jimbo quantization over the field $\mathcal{K}$. If V is an irreducible finite-dimensional $\mathfrak{g}$-module, let V_q denote the corresponding $U_q(\mathfrak{g})$-module. In particular, we denote by $V_q(d)$ the quantum analogue of the irreducible $\mathfrak{sl}_2$-module $V(d)$ with highest weight $d \geq 0$.

1.2.1 An expression of the R-matrix

Recall that by the Clebsch-Gordan formula we have

$$V_q(n) \otimes V_q(n) = \bigoplus_{l=0}^{n} V_q(2l) \text{ for any } n \geq 0.$$

Now, if V_q is a strongly multiplicity free $U_q(\mathfrak{g})$-module (see [LZ06, Definition 7.1]) then $V_q \otimes V_q$ is multiplicity free as a $U_q(\mathfrak{g})$-module, cf. [LZ06, p. 28]. Further, according to the definition of strongly multiplicity free modules, V_q is an irreducible $U_q(\mathfrak{g})$-module with highest weight, say, λ_0.

Write

$$V_q \otimes V_q \cong \bigoplus_{\mu \in \mathcal{P}_{\lambda_0}} L_q(\mu)$$

where $\mathcal{P}_{\lambda_0}$ is the relevant set of dominant weights of $\mathfrak{g}$ and $L_q(\mu)$ is the irreducible $U_q(\mathfrak{g})$-module with highest weight μ.

For $\mu \in \mathcal{P}_{\lambda_0}$, denote by $P(\mu)$ the projection $V_q \otimes V_q \to L_q(\mu)$. These projections span $\mathrm{End}_{U_q(\mathfrak{g})}(V_q \otimes V_q)$ and we have the following formula for the R-matrix $\check{R}$ acting on $V_q \otimes V_q$.

$$\check{R} = \sum_{\mu \in \mathcal{P}_{\lambda_0}} \varepsilon(\mu) q^{\frac{1}{2}(\chi_\mu(C) - 2\chi_{\lambda_0}(C))} P(\mu) \tag{1.17}$$

where C is the classical quadratic Casimir element and $\chi_\lambda(C)$ is the scalar through which it acts on the classical irreducible module with highest weight λ. The $\varepsilon(\mu) \in \{\pm 1\}$ are the eigenvalues of the permutation $s : w \otimes v \mapsto v \otimes w$ of the classical limit $V \otimes V$ of $V_q \otimes V_q$ when $q \to 1$ (on the respective irreducible modules $L(\mu)$). See [LZ08b, §3.1].

For later use, we introduce the notation

$$R_i := \mathrm{id}_{V_q}^{\otimes i-1} \otimes \check{R} \otimes \mathrm{id}_{V_q}^{\otimes r-i-1} \in \mathrm{End}_{U_q(\mathfrak{g})}(V_q^{\otimes r}) \text{ for } 1 \leq i \leq r-1.$$

Now, the point is that since the $U_q(\mathfrak{sl}_2)$-module $V_q(n)$ is strongly multiplicity free ([LZ06, Lemma 7.2 and Theorem 3.4 combined]), we can use the formula (1.17) to calculate the R-matrix $\check{R}$ in this case as follows. For the remaining part of this section, let n be any non-negative integer.

Lemma 1.3 *[LZ08b, 3.1] The endomorphism $s : v \otimes w \mapsto w \otimes v$ of $V(n) \otimes V(n)$ acts on $V(2l)$, for $l = 0, 1, \ldots, n$, as scalar multiplication by $(-1)^{n+l}$.*

Proof. From [LZ06, Corollary 3.9] we know that s belongs to the centralizer of $U(\mathfrak{g})$. Thus it acts as a scalar on each of the irreducible $U(\mathfrak{g})$-modules $V(2l)$. (Since $s^2 = 1$, we even know that each of these scalars belongs to $\{\pm 1\}$). Therefore, it is enough to check that for each $l \in \{0, \ldots, n\}$, the endomorphism s acts on a highest weight vector in $V(2l)$ as $(-1)^{n+l}$. The highest weight vectors can be calculated directly from a basis $\{v_n, v_{n-2}, \ldots, v_{-n+2}, v_{-n}\}$ of weight vectors in $V(n)$ and it is then easy to check that s acts as stated. $\square$

LEMMA 1.4 *[LZ08b, 3.2] The quadratic Casimir element C acts on $V(n)$ as multiplication by $\chi_n(C) = \frac{1}{2}n(n+2)$.*

PROOF. Let e, f, h denote the standard basis of $\mathfrak{sl}_2$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

According to [Hum72, p. 28], the Casimir of $\mathfrak{sl}_2$ is given by

$$C = ef + \tfrac{1}{2}h^2 + fe = h + \tfrac{1}{2}h^2 + 2fe$$

where in the last equality we used the fact that $ef - fe = h$. Since C is in the centre of $U(\mathfrak{g})$, it acts as a scalar on $V(n)$. Thus, the action of C on $V(n)$ is determined by the action of C on a highest weight vector in $V(n)$ and here it acts evidently as

$$n + \tfrac{1}{2}n^2 + 0 = \tfrac{1}{2}n(n+2)$$

which proves the lemma. □

Now, in view of the above results, the expression for the R-matrix $\check{R}$ acting on $V_q(n) \otimes V_q(n)$ is

$$\check{R} = \sum_{l=0}^{n} (-1)^{n+l} q^{\frac{1}{2}(l(2l+2)-n(n+2))} P(2l) \tag{1.18}$$

where $P(2l)$ is the projection to the summand $V_q(2l)$, cf. formula (1.17). From this result we obtain immediately

$$\prod_{l=0}^{n} \left(R_i - (-1)^{n+l} q^{\frac{1}{2}(l(2l+2)-n(n+2))} \right) = 0 \text{ for all } i = 1, \ldots, r-1. \tag{1.19}$$

1.2.2 The three-dimensional irreducible module

Write $E_q(2, r) := \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V_q(2)^{\otimes r})$. Note that

$$V_q(2) \otimes V_q(2) \cong V_q(0) \oplus V_q(2) \oplus V_q(4)$$

by the Clebsch-Gordan formula.

We therefore have operators R_i and $P_i(j)$ for all $i = 1, 2, \ldots, r-1$ and $j = 0, 1, 2$ on $V_q(2)^{\otimes r}$. Here $P_i(j)$ is the projection $V_q(2) \otimes V_q(2) \to V_q(2j)$ applied to the factors $(i, i+1)$ of $V_q(2)^{\otimes r}$ appropriately tensored with the identity on $V_q(2)$. The relation (1.19) reads in this case

$$(R_i - q^{-4})(R_i + q^{-2})(R_i - q^2) = 0 \quad \text{for all} \quad i = 1, \ldots, r-1. \tag{1.20}$$

1.2.3 Relations between R-matrices and projections

Let L_q be a strongly multiplicity free $U_q(\mathfrak{sl}_2)$-module, such that the trivial module $L_q(0)$ is a direct summand of $L_q \otimes L_q$ and let $f \in \operatorname{End}_{U_q(\mathfrak{sl}_2)}(L_q \otimes L_q)$. Then, by writing $P_i(0)$ for the projection $L_q \otimes L_q \to L_q(0)$ applied to the $(i, i+1)$ components of $L_q^{\otimes r}$, we have

$$P_i(0) f_{i\pm 1} P_i(0) = \frac{1}{(\dim_q(L_q))^2} \tau_{q, L_q \otimes L_q}(f) P_i(0) \tag{1.21}$$

for all applicable i. Here $\tau_{q,M}$ denotes the quantum trace of an endomorphism of the $U_q(\mathfrak{sl}_2)$-module M, the endomorphism f_i is f applied to the $(i, i+1)$ components of $L_q^{\otimes r}$ and $\dim_q(L_q)$ denotes the quantum dimension of L_q. This result can be found in [LZ08b, equation (4.2)]. For the proof in the case $f = \check{R}$ (which is all we need in this work), consult [LZ06, Lemma 6.7].

We would like to apply the formula (1.21) to the case $f = \check{R}$ with $L_q = V_q(2)$ so we calculate first the quantum trace $\tau_{q, V_q(n)^{\otimes 2}}(\check{R})$ for any $n \geq 0$ using the explicit expression (1.18) of $\check{R}$:

$$\begin{aligned}
\tau_{q, V_q(n)^{\otimes 2}}(\check{R}) &= (-1)^n q^{-\frac{1}{2}n(n+2)} \sum_{l=0}^{n} (-1)^l q^{l(l+1)} \tau_{q, V_q(n)^{\otimes 2}}(P(2l)) \\
&= (-1)^n q^{-\frac{1}{2}n(n+2)} \sum_{l=0}^{n} (-1)^l q^{l(l+1)} \dim_q(V_q(2l)) \\
&= (-1)^n q^{-\frac{1}{2}n(n+2)} \sum_{l=0}^{n} (-1)^l q^{l(l+1)} \frac{q^{2l+1} - q^{-2l-1}}{q - q^{-1}}.
\end{aligned} \tag{1.22}$$

Note that this sum is telescoping; we mark the cancellations with different underlinings:

$$\begin{aligned}
\sum_{l=0}^{n} (-1)^l q^{l(l+1)} (q^{2l+1} - q^{-2l-1}) &= \sum_{l=0}^{n} (-1)^l q^{l^2} (q^{3l+1} - q^{-l-1}) \\
= \underline{q} - \underline{\underline{q^{-1}}} - q(\underline{\underline{\underline{q^4}}} - \underline{\underline{q^{-2}}}) &+ q^4(q^7 - \underline{q^{-3}}) + q^9(q^{10} - \underline{\underline{\underline{q^{-4}}}}) \\
+ \cdots + (-1)^{(n-1)} q^{(n-1)^2} &(q^{3n-2} - q^{-n}) + (-1)^n q^{n^2}(q^{3n+1} - q^{-n-1}).
\end{aligned}$$

It reduces to

$$(-1)^n (q^{n^2+3n+1} - q^{(n-1)^2+3n-2})$$

which is

$$(-1)^n q^{n^2+2n} (q^{n+1} - q^{-n-1}).$$

Inserting this result into (1.22), we obtain

$$\tau_{q, V_q(n)^{\otimes 2}}(\check{R}) = q^{\frac{1}{2}n(n+2)} \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}.$$

Applying (1.21) to $f = \check{R}$ with $L_q = V_q(2)$, we get then

$$P_i(0) R_{i\pm 1} P_i(0) = q^4 [3]_q^{-1} P_i(0). \tag{1.23}$$

In the equation (1.23) the applicable range of values for i is understood to be such that $R_{k(i)}$ makes sense for the relevant functions $k(i)$ of i. Since $\check{R}$ acts on V_0, V_1, V_2 as $q^{-4}, -q^{-2}, q^2$, respectively, we have

$$P_i(0) = \frac{q^8(R_i + q^{-2})(R_i - q^2)}{(1+q^2)(1-q^6)}. \tag{1.24}$$

With the introduction of the BMW-algebra and the R-matrix and its properties out of the way, we are ready to formulate the main results. We will prove Theorem 1.5 in detail whereas we will restrict ourselves to merely stating Theorem 1.6 since we consider the proof of this result to lie beyond the scope of this thesis.

1.3 Results for the generic case

Keep the notation from the previous sections.

Theorem 1.5 *[LZ08b, Theorem 4.4] There is a surjection $\eta_q : BMW_r(\mathcal{K}) \to E_q(2,r)$ which takes e_i to $[3]_q P_i(0)$ and g_i to R_i for all $i = 1, 2, \ldots, r-1$.*

Proof. Since $E_q(2,r)$ is generated by the R_i (cf. [LZ06, Theorem 7.5]), we only need to establish that η_q preserves the relations of $BMW_r(\mathcal{K})$. Further, in view of Remark 1.1 it suffices to check that the endomorphisms $[3]_q P_i(0)$ and R_i satisfy the relations (1.1), (1.3) and (1.7) for the appropriate y and z.

It is well-known that the braid relations (1.1) are always satisfied by the R_i.

Note that $[3]_q = z^{-1}(z + y^{-1} - y)$ in our specialization and so (1.7) may be written

$$\begin{aligned}
[3]_q^{-1} e_i &= \frac{1}{[3]_q yz}(g_i - q^2)(g_i + q^{-2}) \\
&= \frac{-1}{[3]_q q^{-4}(q^2 - q^{-2})}(g_i - q^2)(g_i + q^{-2}) \\
&= \frac{-(q - q^{-1})q^4}{(q^3 - q^{-3})(q^2 - q^{-2})}(g_i - q^2)(g_i + q^{-2}) \\
&= \frac{q^8}{(1+q^2)(1-q^6)}(g_i - q^2)(g_i + q^{-2})
\end{aligned}$$

for all $i = 1, 2, \ldots, r-1$. But this relation is just (1.24) if we replace e_i with $[3]_q P_i(0)$.

Finally, the first de-looping relation follows from (1.7) while the other two follow from the result (1.23). $\square$

Let $\mathcal{N}_{\mathcal{K}}$ denote the kernel of the surjection $\eta_q : BMW_r(\mathcal{K}) \to E_q(2,r)$. The main result of Lehrer and Zhang in connection with the work we present here is the following.

Theorem 1.6 *[LZ08a, Theorem 2.6] If $r \leq 3$ then η_q is an isomorphism. If $r \geq 4$ then Φ_q generates $\mathcal{N}_{\mathcal{K}}$ as a two-sided ideal of $BMW_r(\mathcal{K})$.*

The proof of Theorem 1.6 is quite lengthy and complicated. It involves specialization methods combined with the theory of cellular algebras and Kazhdan-Lusztig theory. We refer the reader to [LZ08b, Sections 5-7] and [LZ08a] for the proof.

1.4 Where to go from here?

Having established the above results, it is quite obvious that further study of the subject can move in at least two directions.

First of all, as far as we know, no similar results for $\mathrm{End}_{U_q(\mathfrak{sl}_2)}(V_q(d)^{\otimes r})$ with $d > 2$ are known. Thus one way to move would be in the direction of trying to work out similar results for $d = 3$ or, if possible, for any $d > 2$ while restricting oneself to work within the bounds of the generic case.

Secondly, it would be interesting to study whether similar work can be done in the root of unity case. Again, one should probably start with $d = 2$ and then try to attack $d = 3$ or even $d > 2$ in general afterwards.

The fact that these two problems are still open has been the starting point and motivation for the work we present here and will be the main focus in what follows.

Chapter 2

The setup: Algebras, the quantum group and its modules

For the moment, we forget the setup from the previous chapter, but we keep in mind the type of problems we tackled. In the following chapters, we shall treat similar problems but in a different setup. In contrast to the situation in Chapter 1 this allows us to work with a quantum parameter which is a root of unity. This chapter is devoted to the introduction of this new setup.

2.1 The Hecke and Temperley-Lieb algebras

Let $A := \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials over the integers in an indeterminate q. For a positive integer m, denote by S_m the symmetric group on the set $\{1, 2, \ldots, m\}$.

Denote by $H_m(A)$ the corresponding Hecke algebra over A. The reflections $s_i = (i\ i+1)$, $i = 1, 2, \ldots, m-1$, generate S_m and the corresponding generators of $H_m(A)$ are denoted T_{s_i} or sometimes just T_i. When $x \in S_m$, the corresponding basis element of $H_m(A)$ is denoted T_x. The relations in $H_m(A)$ are

$$T_i T_x = T_{s_i x} \ \text{ if } \ l(s_i x) > l(x)$$

and

$$(T_i - q)(T_i + q^{-1}) = 0$$

for $x \in S_m$ and $i = 1, 2, \ldots, m-1$. Here l denotes the length of an element in S_m in the usual Coxeter group sense.

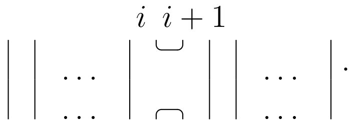
Note that in the literature, some authors prefer to define the Hecke algebra over the polynomial ring $\mathbb{Z}[q]$ using the relation $(T_i - q)(T_i + 1) = 0$ instead of the relation $(T_i - q)(T_i + q^{-1}) = 0$. However, we would like our work to fit with the definitions in [LZ08b] as presented in Chapter 1 which is the main reason for our choice.

For later use, we introduce also the Temperley-Lieb algebra $TL_r(A)$ on r strands with parameter $[2]$ over A. (For notational convenience we write here $[n]$ for the quantum analogue $[n]_q$ of $n \in \mathbb{Z}$ introduced on p. 5). It is the A-algebra generated by elements $U_1, U_2, \dots, U_{r-1}$ and relations

$$\begin{aligned} &U_i U_{i\pm 1} U_i = U_i, \\ &U_i U_j = U_j U_i, \ |i-j| > 1, \\ &U_i^2 = [2] U_i \end{aligned}$$

for all applicable indices. (See e.g. [Jon85]).

It is well-known that the Temperley-Lieb algebra can be represented as a *planar diagram algebra* (see e.g. [Kau90]). Each generator U_i can be represented as a diagram



Multiplication corresponds to composition of diagrams where every occurring loop is replaced by a factor $[2]$.

2.2 The quantum group and its modules

The quantum group for $\mathfrak{sl}_2$ over A is the A-algebra $U_A = U_A(\mathfrak{sl}_2)$ with generators $E^{(n)}$, $F^{(n)}$, K, K^{-1}, $n \in \mathbb{N}$, and relations

$$\begin{aligned} KK^{-1} = K^{-1}K &= 1, \\ KE^{(n)}K^{-1} &= q^{2n}E^{(n)}, \\ KF^{(n)}K^{-1} &= q^{-2n}F^{(n)}, \\ (q-q^{-1})(EF-FE) &= K - K^{-1}, \\ [n]!E^{(n)} &= E^n, \\ [n]!F^{(n)} &= F^n. \end{aligned}$$

Here we denote by $[m]!$ the product

$$[m]! = [m][m-1]\cdots[2][1]$$

for $m \in \mathbb{N}$. This algebra is called Lusztig's integral form and was introduced by Lusztig in [Lus88]. A good reference to the details is e.g. [CP94, §9.3].

The natural module V_A for U_A is the free A-module of rank 2 with basis $\{v_0, v_1\}$ and action given by

$$\begin{aligned} Fv_0 &= v_1, \\ Ev_1 &= v_0, \\ K^{\pm 1}v_0 &= q^{\pm 1}v_0, \\ K^{\pm 1}v_1 &= q^{\mp 1}v_1 \end{aligned}$$

and for all other cases

$$Xv_i = 0 \quad (X \text{ generator for } U_A).$$

The co-multiplication $\Delta : U_A \to U_A \otimes U_A$ is given by

$$\begin{aligned}\Delta(K) &= K \otimes K,\\ \Delta(E) &= E \otimes K + 1 \otimes E,\\ \Delta(F) &= F \otimes 1 + K^{-1} \otimes F.\end{aligned}$$

(See [CP94, §9.3]).

Then the universal R-matrix can be expressed as follows. Suppose V and V' are two integrable U_A-modules of type 1 (see e.g. [Lus93, §3.5.1] respectively [Jan96, Sections 3.19 and 5.2] for these definitions) and let $v \in V$, $v' \in V'$ have weights m and m' respectively. Then by [KR90, equation (8)]

$$R(v \otimes v') = \sum_{n \geq 0} q^{\frac{1}{2}(mm'-3n^2-n)+n(m'-m)}(q-q^{-1})^n [n]! F^{(n)}v' \otimes E^{(n)}v. \tag{2.1}$$

Note that this requires replacing A by $A' = \mathbb{Z}\big[q^{\frac{1}{2}}, q^{-\frac{1}{2}}\big]$. We will *keep this replacement for the remaining part of this chapter.*

If $V = V' = V_A$ then we get

$$\begin{aligned}R : v_0 \otimes v_0 &\mapsto q^{\frac{1}{2}} v_0 \otimes v_0,\\ v_0 \otimes v_1 &\mapsto q^{-\frac{1}{2}} v_1 \otimes v_0,\\ v_1 \otimes v_0 &\mapsto q^{-\frac{1}{2}} v_0 \otimes v_1 + \big(q^{\frac{1}{2}} - q^{-\frac{3}{2}}\big)(v_1 \otimes v_0),\\ v_1 \otimes v_1 &\mapsto q^{\frac{1}{2}} v_1 \otimes v_1.\end{aligned}$$

We see that $T := q^{\frac{1}{2}} R$ belongs to $\mathrm{End}_{U_A}(V_A^{\otimes 2})$ and satisfies $(T-q)(T+q^{-1}) = 0$. This reflects the well-known fact that

$$\mathrm{End}_{U_A}(V_A^{\otimes 2}) \cong H_2(A).$$

More generally, if $r \geq 2$ and we set $E_r(A) = \mathrm{End}_{U_A}(V_A^{\otimes r})$ then the natural homomorphism $\Phi_r : H_r(A) \to E_r(A)$ given by

$$\Phi_r(T_i) = R_i := 1 \otimes \cdots \otimes 1 \otimes q^{\frac{1}{2}} R \otimes 1 \otimes \cdots \otimes 1$$

for all i (with R placed at the $(i, i+1)$ spots) is a surjection. The kernel of Φ_r is generated by the element $\sum_{x \in S_3} (-q)^{l(x)} T_x$, i.e. according to [GL04, (4.17)] we have the following theorem.

Theorem 2.1 *For any $r \geq 2$, the algebra $E_r(A)$ is isomorphic to the Temperley-Lieb algebra $TL_r(A)$ $\big($of rank $\frac{1}{r+1}\binom{2r}{r}\big)$.*

PROOF. We argue as in the proof of [LZ08b, Theorem 3.5] where the authors work over the field $\mathcal{K} = \mathbb{C}(q^{\frac{1}{2}})$ and use results from [GL04]. Everything carries over to the present case since the results of [GL04] are formulated for algebras over an arbitrary commutative ring with 1 and hence hold over A. $\square$

Above we have introduced the natural module V_A for U_A. More generally, for any non-negative integer d, we denote by $V_A(d)$ the Weyl module for U_A with highest weight d. This is the free A-module with basis $\{w_0, w_1, \dots, w_d\}$ and U_A-action determined by

$$\begin{aligned} Kw_j &= q^{d-2j} w_j, \\ F^{(j)} w_0 &= w_j \ (w_j = 0 \text{ for } j > d), \\ Ew_j &= [d+1-j] w_{j-1} \ (w_{-1} = 0) \end{aligned}$$

for all applicable j.

In the next chapter, we will return to these modules and their tensor squares.

As a last comment and for the sake of completeness, we mention that the action of the universal R-matrix on $V_A(d) \otimes V_A(d)$ for any $d \geq 1$ can – at least in theory – be calculated using the formula (2.1). We have already treated the case $d = 1$ above. For $d = 2, 3, 4$, the results are displayed in Appendix B. It should, however, be obvious from those results that even though these cases in low rank are fairly manageable, it seems quite time-consuming to handle high rank cases. We will return to this at the end of the next chapter.

Chapter 3

Endomorphism algebras of tensor modules

In this chapter we will introduce the general notation and setup concerning endomorphism algebras which we will work with throughout the rest of this thesis, except the perspectival Chapter 6. Our main object of interest will be the endomorphism algebra $\mathrm{End}_{U_A}(V_A(d)^{\otimes r})$, and our aim is to determine its ring structure. In this chapter we will focus on the case $r = 2$. The results obtained for $r = 2$ are interesting per se, but they will also be used in the following chapters since even when we treat an r-fold, $r > 2$, tensor power of a module then occasionally it suffices to focus on only two of the r factors at a time.

3.1 General notation and setup

Keep the notation S_m for the symmetric group on $\{1, 2, \ldots, m\}$ and $H_m(A)$ for the corresponding Hecke algebra over $A = \mathbb{Z}[q, q^{-1}]$ from the previous chapter.

3.1.1 Minimal double coset representatives

Let $d \in \mathbb{N}$ and embed S_d in S_m, $m \geq d$, by letting S_d be the permutations in S_m which fix all $i > d$. Consider also the "opposite" embedding of S_d in S_{2d} which identifies S_d with the permutations in S_{2d} that fix all $i \leq d$. To distinguish this copy from the above, we denote it $S_d' \leq S_{2d}$. We let H denote the Young subgroup $S_d \times S_d' \leq S_{2d}$ and define $x_i \in S_{2d}$, $i = 0, 1, \ldots, d$, recursively as follows

$$\begin{aligned}
&x_0 = 1,\ x_1 = s_d,\\
&x_i = x_{i-1} s_{d+i-1} s_{d+i-2} \cdots s_{d+1} s_{d-i+1} s_{d-i+2} \cdots s_d,\ i = 2, 3, \ldots, d.
\end{aligned}$$

Note that if we denote by $(j\ k)$ the transposition interchanging j and k, then we can write

$$x_i = (d-i+1\ d+1)(d-i+2\ d+2)\cdots(d\ d+i)$$

for all i.

For later use, we would like to think of the elements x_i, $i = 0, 1, \ldots, d$, as elements of S_{rd} for any $r \geq 2$. Also, we will need the elements which are defined like the x_i but "shifted $(j-1)d$ places to the right" where $j = 1, 2, \ldots, r$. For fixed d, we shall denote such an element by x_i^{jd}. One way to remember what this notation refers to is to notice that x_i^{jd} has length i^2 (cf. Lemma 3.1 below) and starts and ends with the reflection s_{jd} if we write it as the reduced expression suggested by the recursive definition of the x_i from above. We encourage the reader to keep this in mind in what follows since we will employ this notation several times without further notice.

Maintain the notation $l(x)$ for the length of an element $x \in S_m$.

LEMMA 3.1

(i) *The number of double cosets of H in S_{2d} is $d+1$.*

In fact, the disjoint union of S_{2d} into double cosets for H is

$$S_{2d} = \bigcup_{i=0}^{d} Hx_iH.$$

(ii) $l(x_i) = i^2$.

(iii) *The element x_i is the unique element of minimal length in Hx_iH.*

Consequently, for all $x \in S_{2d}$ there exists an $i \in \{0, 1, \ldots, d\}$ and elements $h, h' \in H$ such that $x = hx_ih'$ and $l(x) = l(h) + l(x_i) + l(h')$.

In particular,

$$T_{hx_ih'} = T_hT_{x_i}T_{h'}.$$

PROOF.

(i) Let HxH be a double coset of H in S_{2d}. By multiplying x from the right and the left by appropriate elements of H we can assume that x satisfies

(a) if $x(j) \leq d$ for some $j \leq d$ then $x(j) = j$,

(b) if $x(j) > d$ for some $j > d$ then $x(j) = j$,

(c) $x(j) = j$ for $j = 1, \ldots, i$ while $x(j) > d$ for $i < j \leq d$,

(d) $x(j) = j$ for $j = 2d-i+1, \ldots, 2d$ while $x(j) \leq d$ for $d < j \leq d+i$,

(e) $x(d-j+1) = d+j$ for $j = 1, \ldots, i$,

(f) $x(d+j) = d-j+1$ for $j = 1, \ldots, i$.

But then $x = x_i$. It is clear that Hx_iH and $Hx_{i'}H$ are disjoint for $i \neq i'$.

(ii) A set of positive roots for the root system of type A_{2d-1} is

$$\{\varepsilon_a - \varepsilon_b \mid 1 \leq a < b \leq 2d\}$$

where, for any $1 \leq i \leq 2d$, we denote by ε_i the ith standard basis vector of $\mathbb{R}^{2d}$.

Note that x_i takes $\varepsilon_a - \varepsilon_b$ into a negative root precisely when $d-i+1 \leq a \leq d$ and $d < b \leq d+i$. There are exactly i^2 such positive roots. It is well-known (see e.g. [Hum90]) that this number equals the length of x_i.

(iii) The arguments in this part of the proof rely heavily on some general results concerning minimal representatives of double cosets in symmetric groups which we have collected in Appendix A. We encourage the reader to read or at least browse through this short appendix now before reading the proof.

In order to prove the first statement, we must check that the x_i satisfy the conditions (1) and (2) of Lemma A.2 in Appendix A:

Let $t^{(d,d)}_{(d,d)}$ denote the (d,d)-tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \hline 2 & 2 & \cdot & \cdot & \cdot & 2 \\ \hline \end{array}.$$

Let S_{2d} act from the right on the set of (d,d)-tableaux by place permutation. (These definitions are consistent with the definitions of Appendix A. See the appendix for details). We must check that for all $i = 0, 1, \ldots, d$,

(1) $t^{(d,d)}_{(d,d)} x_i$ has increasing rows,

(2) if $j < k$ and the jth and kth positions of $t^{(d,d)}_{(d,d)}$ are equal then $jx_i < kx_i$.

To this end, note that for any $i = 0, 1, \ldots, d$, the relevant (d,d)-tableau is

$$t^{(d,d)}_{(d,d)} x_i = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdot & \cdot & \cdot & 1 & 2 & 2 & \cdot & 2 \\ \hline 1 & 1 & \cdot & 1 & 2 & 2 & \cdot & \cdot & \cdot & 2 \\ \hline \end{array},$$

where the figure 2 occurs i times at the end of the first row and the figure 1 occurs i times at the beginning of the second row. Obviously, both rows of this tableau are increasing and thus (1) holds.

To see that (2) is satisfied, simply observe that x_i fixes the places 1, 2, $\ldots$, $d-i$ and $d+i+1$, $d+i+2$, $\ldots$, $2d$ of the tableau and recall how x_i permutes the places $d-i+1$, $d-i+2$, $\ldots$, $d+i$.

The statement on the lengths is a direct consequence of the fact that x_i is the minimal double coset representative of Hx_iH, cf. Theorem A.3. $\square$

Remark 3.2 Note that in the proof of Lemma 3.1 everything works equally well if we replace x_i by x_i^{jd} and "shift everything $(j-1)d$ places to the right".

Let
$$y_d := x_d^{2d} = (d+1\ 2d+1)(d+2\ 2d+2)\cdots(2d\ 3d) \in S_{3d}.$$
Then y_d is the element analogues to x_d but "shifted d places to the right", cf. the notation introduced above. By Lemma 3.1 (ii), we have $l(y_d) = d^2$.

Lemma 3.3
(i) In S_{3d} we have $x_d y_d x_d = y_d x_d y_d$ and $l(x_d y_d x_d) = 3d^2$.

(ii) The braid relation $T_{x_d} T_{y_d} T_{x_d} = T_{y_d} T_{x_d} T_{y_d}$ holds in $H_{3d}(A)$.

Proof.
(i) The first statement is an easy exercise.

In order to prove the statement on the length, we argue as in the proof of Lemma 3.1 (ii) as follows. A set of positive roots for the root system of type A_{3d-1} is
$$\{\varepsilon_a - \varepsilon_b \mid 1 \le a < b \le 3d\}$$
where, for any $1 \le i \le 3d$, we denote by ε_i the ith standard basis vector of $\mathbb{R}^{3d}$.

Note that $x_d y_d x_d$ takes $\varepsilon_a - \varepsilon_b$ into a negative root precisely when we have one of the following two situations:

(1) $1 \le a \le d$ and $d < b \le 3d$. There are exactly $2d^2$ such positive roots.

(2) $d < a \le 2d$ and $2d < b \le 3d$. There are exactly d^2 such positive roots.

In all, we see that $x_d y_d x_d$ takes precisely $3d^2$ positive roots to a negative root which proves the statement.

(ii) This statement follows from (i) since $T_{x_d} T_{y_d} T_{x_d} = T_{x_d y_d x_d} = T_{y_d x_d y_d} = T_{y_d} T_{x_d} T_{y_d}$. □

Remark 3.4 Of course, Lemma 3.3 still holds if we replace x_d by x_d^{jd} and y_d by $x_d^{(j+1)d}$ for any $j = 1, 2, \ldots, r-1$ and work in the appropriate symmetric group and the corresponding Hecke algebra. We just considered x_d and y_d in order to simplify notation.

3.1.2 Idempotents in the Hecke algebra

Consider the element
$$h = h_m = \sum_{x \in S_m} q^{l(x)} T_x \in H_m(A).$$
We have
$$T_i h = qh = hT_i \text{ for all } i = 1, 2, \ldots, m-1.$$

Hence also

$$T_x h = q^{l(x)} h, \; x \in S_m.$$

It follows that

$$h^2 = \left(\sum_{x \in S_m} q^{2l(x)} \right) h.$$

Note that

$$\sum_{x \in S_m} q^{2l(x)} = q^{m(m-1)/2} [m]!.$$

DEFINITION 3.5 Denote by A_m the localization of A in the multiplicative set generated by $[i]$, $i = 1, 2, \ldots, m$.

The above considerations give

LEMMA 3.6 *The Hecke algebra $H_m(A_m)$ over A_m contains an idempotent p_m which satisfies*

$$T_x p_m = q^{l(x)} p_m \text{ for all } x \in S_m.$$

PROOF. In the above notation we let

$$p_m = \frac{q^{-m(m-1)/2}}{[m]!} h.$$

The claim then follows from the considerations above. □

Consider now p_d as an element of $H_{2d}(A_d)$ by the natural embedding of S_d into S_{2d} and let p'_d be the corresponding element when the embedding is now $S_d \equiv S'_d \leq S_{2d}$ (see the first paragraph of section 3.1.1). Set

$$p = p_d \cdot p'_d \in H_{2d}(A_d).$$

3.1.3 A basis induced from double coset representatives

Recall the elements $x_i \in S_{2d}$ from Lemma 3.1.

PROPOSITION 3.7 *The A_d-subalgebra $pH_{2d}(A_d)p$ of $H_{2d}(A_d)$ is free as an A_d-module with basis $\{pT_{x_i}p \mid i = 0, 1, \ldots, d\}$.*

PROOF. Let $x \in S_{2d}$. According to Lemma 3.1, there exists an $i \in \{0, 1, \ldots, d\}$ and two elements $h, h' \in H$ such that $x = hx_ih'$ and $l(x) = l(h) + l(x_i) + l(h')$. Then $T_x = T_h T_{x_i} T_{h'}$. Hence by Lemma 3.6 we have

$$pT_x p = q^n pT_{x_i} p$$

for some $n \in \mathbb{N}$. (In fact $n = l(h) + l(h')$). It follows that the $pT_{x_i}p$ span $pH_{2d}(A_d)p$.

It will follow from the proof of Theorem 3.11 that this generating set is in fact a basis. □

3.2 Main results for tensor squares

Fix now $d \in \mathbb{N}$ and recall the notation V_A and $V_A(d)$ for the natural module respectively the Weyl module with highest weight d of U_A introduced in the previous chapter. Recall, furthermore, the notation $w_0, w_1, \ldots, w_d$ for the basis elements of $V_A(d)$ introduced on p. 16.

3.2.1 Inclusion and retraction

We have an inclusion

$$i : V_A(d) \to V_A^{\otimes d}$$

which is defined as the U_A-homomorphism sending the highest weight vector w_0 of $V_A(d)$ into the element $v_0 \otimes \cdots \otimes v_0$ where v_0 denotes the highest weight vector of the natural module V_A as introduced on p. 14.

The inclusion i can be described as follows. Let $\underline{j} = (j_1, j_2, \ldots, j_m)$ be an m-tuple with $1 \leq j_1 < j_2 < \cdots < j_m \leq d$. Then

$$i(w_m) = \sum q^{m(m+1)/2 - l(\underline{j})} v_{\underline{j}}$$

where the sum runs over all such m-tuples, $l(\underline{j}) = \sum j_k$ and $v_{\underline{j}} = v_{s_1} \otimes \cdots \otimes v_{s_d}$ with $s_r = 1$ if $r \in \{j_1, \ldots, j_m\}$ and $s_r = 0$ otherwise.

If we consider the localization A_d of A defined in Definition 3.5 and denote by U_{A_d} and $V_{A_d}(d)$ the quantum group for $\mathfrak{sl}_2$ over A_d respectively the Weyl module for U_{A_d} with highest weight d, then we have a retraction

$$r : V_{A_d}^{\otimes d} := V_{A_d}(1)^{\otimes d} \to V_{A_d}(d).$$

Indeed, the map r is given by the following. For any $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ we define the Gaussian binomial coefficients

$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a][a-1]\cdots[a-n+1]}{[1][2]\cdots[n]}$$

and $\begin{bmatrix} a \\ 0 \end{bmatrix} = 1.$

Define $r : V_{A_d}^{\otimes d} \to V_{A_d}(d)$ by ($\underline{j}$ being an m-tuple as above)

$$r(v_{\underline{j}}) = f_m(\underline{j}) w_m$$

where

$$f_m(\underline{j}) = \begin{bmatrix} d \\ m \end{bmatrix}^{-1} q^{m(d+1) - m(m+1)/2 - l(\underline{j})} \in A_d.$$

LEMMA 3.8 *The map r is a U_{A_d}-homomorphism.*

PROOF. It is clear from the definition that r preserves weights. Therefore, it commutes with the action of K.

To see that r commutes with the action of $F^{(j)}$ it is enough to prove that the action of F commutes with r and then use induction.

Moreover, if we have already established the commutativity for the action of E, we can treat F in an analogous way. Therefore, we only need to consider E.

Now, if $m = 0$, it is obvious that r commutes with the action of E, so let us assume that $m \geq 1$.

Let $\underline{j}$ be an element as above. We have

$$\begin{aligned}
r(E(v_{\underline{j}})) &= r\left(\sum_{i=1}^{d} v_{s_1} \otimes \cdots \otimes v_{s_{i-1}} \otimes Ev_{s_i} \otimes Kv_{s_{i+1}} \otimes \cdots \otimes Kv_{s_d}\right) \\
&= \sum_{k=1}^{m} q^{d-2m+2k-j_k} r\left(v_{s_1} \otimes \cdots \otimes v_{s_{j_k-1}} \otimes v_0 \otimes v_{s_{j_k+1}} \otimes \cdots \otimes v_{s_d}\right) \\
&= \sum_{k=1}^{m} q^{d-2m+2k-j_k} f_{m-1}\left(j_1, j_2, \ldots, j_{k-1}, j_{k+1}, \ldots, j_m\right) w_{m-1}.
\end{aligned}$$

On the other hand,

$$E(r(v_{\underline{j}})) = f_m(\underline{j})[d+1-m]w_{m-1}.$$

Inserting the expression for f_m respectively f_{m-1}, we must prove that

$$\begin{aligned}
&\sum_{k=1}^{m} q^{d-2m+2k-j_k} \begin{bmatrix} d \\ m-1 \end{bmatrix}^{-1} q^{(m-1)(d+1)-(m-1)m/2-l(\underline{j})+j_k)} \\
&\qquad = \begin{bmatrix} d \\ m \end{bmatrix}^{-1} q^{m(d+1)-m(m+1)/2-l(\underline{j})}[d+1-m],
\end{aligned}$$

which reduces to

$$\sum_{k=1}^{m} q^{-m+2k-1} = [m].$$

Since this equality holds, we have proved that r commutes with the action of E and hence is a U_{A_d}-homomorphism. □

Lemma 3.9 *The map r is a retraction of the inclusion i.*

Proof. The statement follows from the formula

$$\sum_{\underline{j}} q^{m(d+1)-2l(\underline{j})} = \begin{bmatrix} d \\ m \end{bmatrix}.$$

This result can be proved by induction on d by use of the formula

$$\begin{bmatrix} d+1 \\ m \end{bmatrix} = q^m \begin{bmatrix} d \\ m \end{bmatrix} + q^{-d-1+m} \begin{bmatrix} d \\ m-1 \end{bmatrix}$$

from [Jan96, 0.2(2)]. □

3.2.2 Idempotents of the endomorphism algebra

In analogy with previous definitions over A, we denote by $E_r(A_d)$ the endomorphism algebra $\mathrm{End}_{U_{A_d}}(V_{A_d}^{\otimes r})$ for any $r, d \in \mathbb{N}$. Set

$$\bar{p}_d = i \circ r \in E_d(A_d).$$

Recall from p. 15 the definition of the homomorphism $\Phi_d : H_d(A_d) \to E_d(A_d)$.

LEMMA 3.10 *The idempotent $p_d \in H_d(A_d)$ maps by Φ_d into $\bar{p}_d$.*

PROOF. We claim that it suffices to prove that the image of the endomorphism $\Phi_d(h_d)$ is contained in the summand $V_{A_d}(d)$ of $V_{A_d}^{\otimes d}$.

Indeed, since $V_{A_d}^{\otimes d} \cong V_{A_d}(d) \oplus M$, where $M = \ker r$ has highest weight $d-2$, we can write

$$E_d(A_d) \cong \mathrm{End}_{U_{A_d}}(V_{A_d}(d)) \oplus \mathrm{End}_{U_{A_d}}(M),$$

because (if we denote by k the field of fractions of A_d and by U_k the quantum group $U_{A_d} \otimes_{A_d} k$ over k)

$$\mathrm{Hom}_{U_{A_d}}(V_{A_d}(d), M) \subseteq \mathrm{Hom}_{U_k}(V_{A_d}(d) \otimes_{A_d} k, M \otimes_{A_d} k) = 0,$$

and the same for $\mathrm{Hom}_{U_{A_d}}(M, V_{A_d}(d))$.

Now,

$$\mathrm{Hom}_{U_{A_d}}(V_{A_d}^{\otimes d}, V_{A_d}(d)) = A_d \cdot r,$$

so if we can prove that the image of $\Phi_d(h_d)$ is contained in $V_{A_d}(d)$, i.e.

$$\Phi_d(h_d) = i \circ \phi$$

for some $\phi \in \mathrm{Hom}_{U_{A_d}}(V_{A_d}^{\otimes d}, V_{A_d}(d)) = A_d \cdot r$, then $\Phi_d(h_d) = a(i \circ r)$ for some $a \in A_d$.

Thus in order to prove the claim, it suffices to establish that

$$a = q^{d(d-1)/2}[d]!.$$

I.e. if we denote by v_0 the highest weight vector of V_{A_d}, we must prove that

$$\Phi_d(h_d)(v_0 \otimes v_0 \otimes \cdots \otimes v_0) = q^{d(d-1)/2}[d]!(i \circ r)(v_0 \otimes v_0 \otimes \cdots \otimes v_0)$$

(where both tensor products are d-fold). To this end, note that

$$\begin{aligned}
\Phi_d(h_d)(v_0 \otimes v_0 \otimes \cdots \otimes v_0) &= \sum_{x \in S_d} q^{l(x)} \Phi_d(T_x)(v_0 \otimes v_0 \otimes \cdots \otimes v_0) \\
&= \sum_{\substack{x = s_{i_1} s_{i_2} \cdots s_{i_{l(x)}} \\ \in S_d}} q^{\frac{3}{2}l(x)} R_{i_1} R_{i_2} \cdots R_{i_{l(x)}} (v_0 \otimes v_0 \otimes \cdots \otimes v_0) \\
&= \sum_{x \in S_d} q^{2l(x)} v_0 \otimes v_0 \otimes \cdots \otimes v_0 \\
&= q^{d(d-1)/2}[d]! v_0 \otimes v_0 \otimes \cdots \otimes v_0.
\end{aligned}$$

(To avoid any confusion, we point out that in the third expression above we mean to sum over all $x \in S_d$ only once; i.e. $s_{i_1}s_{i_2}\dots s_{i_{l(x)}}$ is just an arbitrary reduced expression of x).

Since $(i \circ r)(v_0 \otimes v_0 \otimes \cdots \otimes v_0) = v_0 \otimes v_0 \otimes \cdots \otimes v_0$, this proves the claim.

To prove that the image of the endomorphism $\Phi_d(h_d)$ *is* in fact contained in $V_{A_d}(d)$, note that the element

$$h_d = \sum_{x \in S_d} q^{l(x)} T_x$$

can be written in the following two different ways

$$h_d = h_{d-1}g = h'_{d-1}g',$$

where

$$g = \sum_{i=1}^{d-1} q^{d-i} T_{d-1}T_{d-2}\cdots T_i,$$
$$g' = \sum_{i=1}^{d-1} q^{i} T_1T_2\cdots T_i,$$

and h'_{d-1} is the same as h_{d-1} but with respect to $T_2,\dots,T_{d-1}$. Thus, by induction, the image of $\Phi_d(h_d)$ is contained in both of the two subspaces $V_{A_d}(d-1) \otimes V_{A_d}$ and $V_{A_d} \otimes V_{A_d}(d-1)$ of $V_{A_d}^{\otimes d}$.

Note that

$$(V_{A_d}(d-1) \otimes V_{A_d}) \cap (V_{A_d} \otimes V_{A_d}(d-1)) = V_{A_d}(d).$$

(They both split as $V_{A_d}(d) \oplus V_{A_d}(d-2)$, and thus their intersection is $V_{A_d}(d)$. Indeed, there is no vector of weight $d-2$ which is annihilated by E and contained in both subspaces. In $V_{A_d}(d-1) \otimes V_{A_d}$, the only such vector is

$$-q[d-1]v_{\underline{0}} \otimes v_1 + Fv_{\underline{0}} \otimes v_0$$

and in $V_{A_d} \otimes V_{A_d}(d-1)$ the only such vector is $v_0 \otimes Fv_{\underline{0}} - q^{1-d}[d-1]v_1 \otimes v_{\underline{0}}$). This proves the lemma. □

3.2.3 Description of the endomorphism algebra

Consider now the idempotent $\bar{p} \in E_{2d}(A_d)$ obtained as the product of $\bar{p}_d \otimes 1 \otimes \cdots \otimes 1$ and $1 \otimes \cdots \otimes 1 \otimes \bar{p}_d$. Recall from Lemma 3.6 that we have a corresponding element $p \in H_{2d}(A_d)$.

Theorem 3.11 *The homomorphism $\Phi_{2d} : H_{2d}(A_d) \to E_{2d}(A_d)$ induces an isomorphism between the subalgebras $pH_{2d}(A_d)p$ and*

$$\bar{p}E_{2d}(A_d)\bar{p} \cong \mathrm{End}_{U_{A_d}}(V_{A_d}(d)^{\otimes 2}).$$

PROOF. The right-hand side is a free module over A_d of rank $d+1$. In fact, $V_{A_d}(d)^{\otimes 2}$ is a tilting module for U_{A_d} and it has $d+1$ Weyl factors, namely $V_{A_d}(0)$, $V_{A_d}(2)$, ..., $V_{A_d}(2d)$, each occuring once.

Now, the surjectivity of Φ_{2d} implies that the induced map is also surjective. However, Proposition 3.7 says that the left hand side is spanned by $d+1$ elements. Hence the induced map is an isomorphism and the generators of Proposition 3.7 form a basis. □

3.2.4 Surjection from the braid group

Above we have described the endomorphism algebra $\mathrm{End}_{U_{A_d}}(V_{A_d}(d)^{\otimes 2})$ as a subalgebra of the Hecke algebra $H_{2d}(A_d)$ which is free as an A_d-module with a basis induced by minimal double coset representatives in the symmetric group. In this section, we describe it in a slightly different way, namely by showing that – at least under a certain localization – there exists a surjection from the group algebra of the 2-string braid group to the endomorphism algebra in question.

Denote by B_r the braid group on r strands (or strings), i.e. the group with generators $X_1, X_2, \dots, X_{r-1}$ and relations

$$X_i X_j = X_j X_i \text{ for } |i-j| > 1,$$
$$X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1} \text{ for } i = 1, 2, \dots, r-2.$$

Let $A_{2d(d+1)}[B_2]$ denote the group algebra of the braid group B_2 on two strands over the ring $A_{2d(d+1)}$. (Recall from Definition 3.5 the definition of the ring A_m for any $m \in \mathbb{N}$).

THEOREM 3.12 *We have a surjection*

$$A_{2d(d+1)}[B_2] \to \mathrm{End}_{U_{A_{2d(d+1)}}}(V_{A_{2d(d+1)}}(d)^{\otimes 2}).$$

PROOF. Note that $A_{2d(d+1)}[B_2]$ is the polynomial ring $A_{2d(d+1)}[X]$, X an indeterminate.

Let

$$A_{2d(d+1)}[X] \to \mathrm{End}_{U_{A_{2d(d+1)}}}(V_{A_{2d(d+1)}}(d)^{\otimes 2})$$

be the homomorphism sending X to the universal R-matrix R. Since R satisfies a monic polynomial equation $f(R) = 0$ of degree $d+1$ over $A_{2d(d+1)}$ determined by the eigenvalues of R, (see (1.19) above), the map factorizes through

$$A_{2d(d+1)}[X]/(f) \cong A_{2d(d+1)}^{d+1},$$

where (f) denotes the ideal in $A_{2d(d+1)}[X]$ generated by f.

The induced homomorphism

$$A_{2d(d+1)}[X]/(f) \to \mathrm{End}_{U_{A_{2d(d+1)}}}(V_{A_{2d(d+1)}}(d)^{\otimes 2})$$

is injective. We would like it to be surjective as well.

Due to Nakayama's Lemma it is enough to test this surjectivity after having tensored with all residue fields k of $A_{2d(d+1)}$.

Let k be any such residue field and denote by $\bar{f}$ the image of f under the canonical map $A_{2d(d+1)}[X] \to k[X]$. Let $U_k = U_{A_{2d(d+1)}} \otimes_{A_{2d(d+1)}} k$ and $V_k = V_{A_{2d(d+1)}} \otimes_{A_{2d(d+1)}} k$ and note that

$$\mathrm{End}_{U_k}\left(V_k(d)^{\otimes 2}\right) \cong \mathrm{End}_{U_{A_{2d(d+1)}}}\left(V_{A_{2d(d+1)}}(d)^{\otimes 2}\right) \otimes_{A_{2d(d+1)}} k. \tag{3.1}$$

(We can argue for this in two steps using methods as in [APW91, Section 3]. First, if we choose a maximal ideal $\mathfrak{m} \subseteq A_{2d(d+1)}$ which contains the kernel of the map $A_{2d(d+1)} \to k$, then by flat base change in [APW91, 3.3], we have

$$\begin{aligned}\mathrm{End}_{U_{A_{2d(d+1)}}}&\left(V_{A_{2d(d+1)}}(d)^{\otimes 2}\right) \otimes_{A_{2d(d+1)}} \left(A_{2d(d+1)}\right)_{\mathfrak{m}} \\ &\cong \mathrm{End}_{U_{(A_{2d(d+1)})_{\mathfrak{m}}}}\left(V_{A_{2d(d+1)}}(d)^{\otimes 2} \otimes_{A_{2d(d+1)}} \left(A_{2d(d+1)}\right)_{\mathfrak{m}}\right),\end{aligned}$$

where $\left(A_{2d(d+1)}\right)_{\mathfrak{m}}$ denotes the localization of the ring $A_{2d(d+1)}$ in the ideal $\mathfrak{m}$.

To take the step from $(A_{2d(d+1)})_{\mathfrak{m}}$ to k, we can then use [APW91, 3.6(8)]. Here we need the fact (from e.g. [And, Corollary 2.6]) that

$$\mathrm{Ext}^1_{U_{(A_{2d(d+1)})_{\mathfrak{m}}}}\left(V_{(A_{2d(d+1)})_{\mathfrak{m}}}(d)^{\otimes 2}, V_{(A_{2d(d+1)})_{\mathfrak{m}}}(d)^{\otimes 2}\right) = 0$$

since the module $V_{(A_{2d(d+1)})_{\mathfrak{m}}}(d)^{\otimes 2} = V_{A_{2d(d+1)}}(d)^{\otimes 2} \otimes_{A_{2d(d+1)}} \left(A_{2d(d+1)}\right)_{\mathfrak{m}}$ is a tilting module).

Over k the map

$$(A[X]/(f)) \otimes_{A_{2d(d+1)}} k \cong k[X]/(\bar{f}) \to \mathrm{End}_{U_k}(V_k(d)^{\otimes 2})$$

is still injective, if the eigenvalues for R stay different in k. (Then the image $\bar{R} \in \mathrm{End}_{U_k}(V_k(d)^{\otimes 2})$ of R under the specialization to k cannot be a root of a polynomial of degree less than $d+1$).

In that case, since the dimension of $\mathrm{End}_{U_k}(V_k(d)^{\otimes 2})$ is also $d+1$ (in fact $V_k(d)^{\otimes 2}$ is a tilting module for U_k and it has $d+1$ Weyl factors, namely the Weyl modules $V_k(0), V_k(2), \dots, V_k(2d)$ over k, each occuring once), surjectivity follows.

The fact that the eigenvalues of R *are* actually different in k is secured by the chosen localization $A_{2d(d+1)}$ of A. Indeed, two eigenvalues are equal in k if and only if there exists $m, n = 0, 1, \dots, d$ with $n > m$, such that (if we denote by ζ the image in k of q under the natural map $A_{2d(d+1)} \to k$)

$$(-1)^{d+n}\zeta^{\frac{1}{2}(n(2n+2)-d(d+2))} = (-1)^{d+m}\zeta^{\frac{1}{2}(m(2m+2)-d(d+2))},$$

which is equivalent to $\pm\zeta^{(n(n+1)-m(m+1))} = 1$ or

$$\pm\zeta^{(n+m+1)(n-m)} = 1, \quad \text{cf. (1.19)}.$$

Since the maximal value of $(n+m+1)(n-m)$ for $n, m = 0, 1, \ldots, d$ with $n > m$ is $d(d+1)$, we conclude that the eigenvalues of R are different in k, if ζ is not a root of unity of order $l \leq 2d(d+1)$. This is obtained by working over $A_{2d(d+1)}$ instead of A. $\square$

Remark 3.13 Note that in Theorem 3.12 we have found a *sufficiently* "large" localization of A in order to ensure surjectivity. We do not say whether this localization is in fact necessary.

3.3 Notation for idempotents

Until now, we have only considered the case $r = 2$. However, the element $p = p_d \cdot p_d' \in H_{2d}(A_d)$ introduced on p. 21 can be generalized in the following way which will be essential for our discussion of the case with a general $r \geq 2$ in the next chapter.

Let p_d^{jd} denote the analogue of p_d in $H_{rd}(A_d)$, but "shifted $(j-1)d$ places to the right". Then in general, we let

$$p = p_d^d \cdot p_d^{2d} \cdots p_d^{rd},$$

whenever it is clear from the context which d and r we consider.

We shall also use the following notation for each of the factors in the above idempotent p,

$$p_{(j-1)d+1,(j-1)d+2,\ldots,jd-1} = p_d^{jd},$$

indicating that this element is contained in the A_d-subalgebra of $H_{rd}(A_d)$ generated by the elements $T_{(j-1)d+1}$, $T_{(j-1)d+2}$, $\ldots$, T_{jd-1}.

In order to distinguish this latter notation from our original notation p_d for the element p_d^d, we will, from now on, always denote the "old" p_d and its "translations" by p_d^{jd} (for the appropriate j) as introduced above.

Likewise, the element $\bar{p} \in E_{2d}(A_d)$ from p. 25 can be generalized as follows: In general, we let $\bar{p}$ denote the product of the r elements $\bar{p}_d \otimes 1 \otimes \cdots \otimes 1$, $1 \otimes \bar{p}_d \otimes 1 \otimes \cdots \otimes 1$, $\ldots$, $1 \otimes \cdots \otimes 1 \otimes \bar{p}_d$ in $E_{rd}(A_d)$ whenever r and d are clear from the context.

Notice that with the new notation the isomorphism

$$\bar{p} E_{rd}(A_d) \bar{p} \cong \mathrm{End}_{U_{A_d}}(V_{A_d}(d)^{\otimes r})$$

mentioned for $r = 2$ in Theorem 3.11 holds for arbitrary r.

3.4 A link to R-matrices

Above we have given two seemingly different descriptions of the endomorphism algebra $\mathrm{End}_{U_{A_d}}(V_{A_d}(d)^{\otimes 2})$. (Though, to be precise, in one case we needed further localization of the ring A_d). However, there is a link between the two in the following way.

The first description involves the basis elements $pT_{x_i}p$ in the A_d-module $pH_{2d}(A_d)p \cong \mathrm{End}_{U_{A_d}}(V_{A_d}(d)^{\otimes 2})$ whereas the other description involves the R-matrix of the endomorphism algebra $\mathrm{End}_{U_{A_d}}(V_{A_d}(d)^{\otimes 2})$. But, as we shall see below, the element T_{x_d} of $H_{2d}(A_d)$ actually maps to the R-matrix (up to a power of $q^{\frac{1}{2}}$) on $V_A^{\otimes 2d}$ under the map $\Phi_{2d} : H_{2d}(A_d) \to \mathrm{End}_{U_{A_d}}(V_A^{\otimes 2d})$, thus providing a link between the world of the elements $pT_{x_i}p$ and the world of R-matrices. We prove this rigorously below, but note as a start that if you calculate the action of $\Phi_{2d}(T_{x_d})$ on $V_A^{\otimes 2d}$ for, say, $d = 1, 2, 3, 4$, you get the exact same results (up to a power of $q^{\frac{1}{2}}$) as we have calculated for the R-matrix in section 2.2 and Appendix B. Also, we showed in Lemma 3.3 that all of the $T_{x_d^{jd}}$ satisfy the braid relations – just like the R-matrices do. The proof of the statement goes as follows.

Let $V = V_A$ be the natural U_A-module (as in Chapter 2). Again, we actually need $A' = \mathbb{Z}\left[q^{\frac{1}{2}}, q^{-\frac{1}{2}}\right]$ instead of A, so *replace A by A' for the rest of this chapter.*

For $\underline{r} = (r_1, r_2, \ldots, r_d)$ with $r_i \in \{0, 1\}$ for all i, we set

$$v_{\underline{r}} = v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_d} \in V^{\otimes d}.$$

We let

$$l(\underline{r}) = \#\{i \mid r_i = 0\} - \#\{i \mid r_i = 1\}$$

where $\#$ denotes the number of elements of the relevant set. Then the general formula (2.1) for the R-matrix gives

$$R_d(v_{\underline{r}} \otimes v_{\underline{s}}) = \sum_{n \geq 0} a_n(\underline{r}, \underline{s}) F^{(n)} v_{\underline{s}} \otimes E^{(n)} v_{\underline{r}}$$

with

$$a_n(\underline{r}, \underline{s}) = q^{\frac{1}{2}(l(\underline{r})l(\underline{s}) - 3n^2 - n) + n(l(\underline{s}) - l(\underline{r}))} (q - q^{-1})^n [n]!.$$

Here R_d denotes the R-matrix on $V^{\otimes d} \otimes V^{\otimes d}$. This notation is not to be confused with the notation R_i from Chapter 2. Recall that $R_1 = R$ is given by the recipe in Chapter 2.

Set

$$T_d = \Phi_{2d}(T_{x_d}),$$

see Chapter 2. It should cause no confusion that we keep the notation T_i for the generators of the Hecke algebra. It will be clear from the context which of the two notations we employ.

PROPOSITION 3.14 $q^{\frac{d^2}{2}} R_d = T_d$ *as elements of* $\mathrm{End}_{U_A}(V^{\otimes d})$.

In order to prove this result, we need a lemma.

LEMMA 3.15 *For all $\phi \in E_d(A) (= \mathrm{End}_{U_A}(V^{\otimes d}))$, we have*

$$R_d \circ (\phi \otimes \mathrm{id}_{V^{\otimes d}}) = (\mathrm{id}_{V^{\otimes d}} \otimes \phi) \circ R_d.$$

Proof. We have

$$\begin{aligned}R_d \circ (\phi \otimes \mathrm{id})(v_{\underline{r}} \otimes v_{\underline{s}}) &= \sum_{n \geq 0} a_n(\underline{r}, \underline{s}) F^{(n)} v_{\underline{s}} \otimes E^{(n)} \phi(v_{\underline{r}}) \\ &= \sum_{n \geq 0} a_n(\underline{r}, \underline{s}) F^{(n)} v_{\underline{s}} \otimes \phi(E^{(n)} v_{\underline{r}}) \\ &= (\mathrm{id} \otimes \phi) \circ R_d(v_{\underline{r}} \otimes v_{\underline{s}}).\end{aligned}$$

Here the first equality uses the fact that $\phi(v_{\underline{r}}) = \sum_{\underline{t}} c_{\underline{t},\underline{r}} v_{\underline{t}}$ for some $c_{\underline{t},\underline{r}} \in A$ with $c_{\underline{t},\underline{r}} = 0$ unless $l(\underline{t}) = l(\underline{r})$. (Weight spaces are preserved by ϕ). □

Note that $\Phi_d(T_i) \in E_d(A)$ for $i = 1, 2, \ldots, d-1$. Hence the lemma gives the following identities

(1) $R_d \circ \Phi_{2d}(T_i) = \Phi_{2d}(T_{d+i}) \circ R_d$, $i = 1, 2, \ldots, d-1$.

A similar argument gives

(2) $R_d \circ \Phi_{2d}(T_j) = \Phi_{2d}(T_{j-d}) \circ R_d$, $j = d+1, d+2, \ldots, 2d-1$.

Now

$$x_d s_i = s_{d+i} x_d \text{ for } i = 1, 2, \ldots, d-1$$

and

$$x_d s_j = s_{j-d} x_d \text{ for } j = d+1, d+2, \ldots, 2d-1.$$

Therefore, we have similar identities in $H_{2d}(A)$

(3) $T_{x_d} T_i = T_{d+i} T_{x_d}$ and $T_{x_d} T_j = T_{j-d} T_{x_d}$.

Proof of the proposition. We proceed by induction on d. If $d = 1$, the claim of the proposition is obviously true so we assume $d > 1$.

We divide the investigation into a few different cases. Our two main (exhaustive) cases are $\underline{r} \neq (1, \ldots, 1)$ and $\underline{r} = (1, \ldots, 1)$, respectively.

Note that in the case $\underline{r} \neq (1, \ldots, 1)$, it suffices to treat the case $\underline{s} \neq (0, \ldots, 0)$ since, after having completed the case $\underline{r} = (1, \ldots, 1)$, the case $\underline{s} = (0, \ldots, 0)$ can be treated analogously.

Before we treat the two main cases, we consider the special case $r_d = 0$, $s_1 = 1$, which will be used to prove the statement when $\underline{r} \neq (1, \ldots, 1)$.

Case 1: $r_d = 0$, $s_1 = 1$. Set $\underline{r}' = (r_1, \ldots, r_{d-1})$ and $\underline{s}' = (s_2, \ldots, s_d)$. Then

$$\begin{aligned}E^{(n)} v_{\underline{r}} &= q^n E^{(n)} v_{\underline{r}'} \otimes v_0, \\ F^{(n)} v_{\underline{s}} &= q^n v_1 \otimes F^{(n)} v_{\underline{s}'}\end{aligned}$$

which implies

$$R_d(v_{\underline{r}} \otimes v_{\underline{s}}) = \sum_{n \geq 0} a_n(\underline{r}, \underline{s}) q^{2n} v_1 \otimes (F^{(n)} v_{\underline{s}'} \otimes E^{(n)} v_{\underline{r}'}) \otimes v_0.$$

Note that

$$a_n(\underline{r},\underline{s})q^{2n} = q^{\frac{1}{2}(l(\underline{s}')-l(\underline{r}')-1)}a_n(\underline{r}',\underline{s}').$$

Hence

$$R_d(v_{\underline{r}} \otimes v_{\underline{s}}) = q^{\frac{1}{2}(l(\underline{s}')-l(\underline{r}')-1)}v_1 \otimes R_{d-1}(v_{\underline{r}'} \otimes v_{\underline{s}'}) \otimes v_0.$$

On the other hand,

$$\begin{aligned}
T_d(v_{\underline{r}} \otimes v_{\underline{s}}) &= \Phi_{2d}(T_{x_{d-1}})\Phi_{2d}(T_{2d-1}\cdots T_{d+1}T_1\cdots T_d)(v_{\underline{r}} \otimes v_{\underline{s}}) \\
&= \Phi_{2d}(T_{x_{d-1}})\Phi_{2d}(T_{2d-1}\cdots T_{d+1}T_1\cdots T_{d-1})(v_{\underline{r}'1} \otimes v_{0\underline{s}'}) \\
&= q^{d-1}\Phi_{2d}(T_{x_{d-1}})\Phi_{2d}(R_{2d-1}\cdots R_{d+1}R_1\cdots R_{d-1})(v_{\underline{r}'1} \otimes v_{0\underline{s}'}) \\
&= q^{d-1-\frac{1}{2}(l(\underline{r}')-l(\underline{s}'))}\Phi_{2d}(T_{x_{d-1}})(v_{1\underline{r}'} \otimes v_{\underline{s}'0}) \\
&= q^{d-1-\frac{1}{2}(l(\underline{r}')-l(\underline{s}'))}v_1 \otimes T_{d-1}(v_{\underline{r}'} \otimes v_s') \otimes v_0.
\end{aligned}$$

So our induction hypothesis now shows that $q^{\frac{d^2}{2}}R_d = T_d$ in this case.

Case 2: $\underline{r} \neq (1,1,\ldots,1)$, $\underline{s} \neq (0,0,\ldots,0)$. Choose j maximal such that $r_j = 0$ and i minimal such that $s_i = 1$. If $j = d$ and $i = 1$, we are in case 1.

Set $\underline{r}' = (r_1,\ldots,r_{j-1},1,\ldots,1)$ and $\underline{s}' = (0,\ldots,0,s_{i+1},\ldots,s_d)$, each with $d-1$ elements. Then

$$\Phi_{2d}(T_{d-1}\cdots T_j)(v_{\underline{r}}) = v_{\underline{r}'0}$$

and

$$\Phi_{2d}(T_{d+1}\cdots T_{d+i-1})(v_{\underline{s}}) = v_{1\underline{s}'}.$$

Hence

$$\begin{aligned}
R_d(v_{\underline{r}} \otimes v_{\underline{s}}) &= R_d(\Phi_{2d}(T_{d-1}\cdots T_jT_{d+1}\cdots T_{d+i-1})^{-1}(v_{\underline{r}'0} \otimes v_{1\underline{s}'})) \\
&= \Phi_{2d}(T_{2d-1}\cdots T_{d+j}T_1\cdots T_{i-1})^{-1}R_d(v_{\underline{r}'0} \otimes v_{1\underline{s}'}) \\
&= q^{-\frac{d^2}{2}}\Phi_{2d}(T_{2d-1}\cdots T_{d+j}T_1\cdots T_{i-1})^{-1}T_d(v_{\underline{r}'0} \otimes v_{1\underline{s}'}) \\
&= q^{-\frac{d^2}{2}}T_d\Phi_{2d}(T_{d-1}\cdots T_jT_{d+1}\cdots T_{d+i-1})(v_{\underline{r}'0} \otimes v_{1\underline{s}'}) \\
&= q^{-\frac{d^2}{2}}T_d(v_{\underline{r}} \otimes v_{\underline{s}}),
\end{aligned}$$

where we have applied first formula (1) and (2), then the result obtained in Case 1 and at the end formula (3).

Case 3: $\underline{r} = (1,\ldots,1)$. In this case $v_{\underline{r}} \otimes v_{\underline{s}} = F^{(d)}v_{\underline{0}} \otimes v_{\underline{s}}$. On the other hand,

$$F^{(d)}(v_{\underline{0}} \otimes v_{\underline{s}}) = F^{(d)}v_{\underline{0}} \otimes v_{\underline{s}} + \sum_{\underline{p},\underline{q}} c_{\underline{p},\underline{q}}v_{\underline{p}} \otimes v_{\underline{q}}$$

where $\underline{p} \neq (1,\ldots,1)$, $\underline{q} \neq (0,\ldots,0)$.

Applying R_d to this equation yields

$$\begin{aligned}
R_d(v_{\underline{r}} \otimes v_{\underline{s}}) &= R_d(F^{(d)}(v_{\underline{0}} \otimes v_{\underline{s}})) - \sum_{\underline{p},\underline{q}} c_{\underline{p},\underline{q}} R_d(v_{\underline{p}} \otimes v_{\underline{q}}) \\
&= F^{(d)} R_d(v_{\underline{0}} \otimes v_{\underline{s}}) - q^{\frac{d^2}{2}} \sum_{\underline{p},\underline{q}} c_{\underline{p},\underline{q}} T_d(v_{\underline{p}} \otimes v_{\underline{q}}) \\
&= q^{\frac{d^2}{2}} F^{(d)} T_d(v_{\underline{0}} \otimes v_{\underline{s}}) - q^{\frac{d^2}{2}} \sum_{\underline{p},\underline{q}} c_{\underline{p},\underline{q}} T_d(v_{\underline{p}} \otimes v_{\underline{q}}) \\
&= q^{\frac{d^2}{2}} T_d(v_{\underline{r}} \otimes v_{\underline{s}}).
\end{aligned}$$

by Case 1 and 2 and the trivial case $\underline{r} = \underline{0} = \underline{s}$. $\square$

Part II

The Weyl module with highest weight 2

Chapter 4

Surjection from the braid group

This is the main chapter of this thesis. It contains the results concerning the structure of the endomorphism algebra $\mathrm{End}_{U_{A_d}}(V_{A_d}(d)^{\otimes r})$ in the case $d = 2$. The main result is Corollary 4.3 where we establish surjectivity from the group algebra of the braid group to the endomorphism algebra under a suitable localization. The crucial part of the work is done in the proofs of the Propositions 4.1 and especially 4.2.

Maintain the notation introduced previously. In particular, let T_i, $i = 1, 2, \dots, r-1$, denote the generators of the Hecke algebra $H_r(A)$. Let $u := q - q^{-1}$ and fix the highest weight $d = 2$.

4.1 A set of generators for the Temperley-Lieb algebra

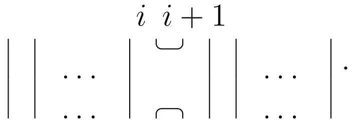
Inspired by [Här99, section 4], we consider the following elements

$$U_i := q - T_i,$$

$i = 1, 2, \dots, r-1$, in $H_r(A)$. By abuse of notation, we denote by U_i also the image of U_i in the quotient $TL_r(A)$ of $H_r(A)$. Note that the U_i satisfy the following relations:

$$U_i U_{i\pm 1} U_i = U_i, \tag{4.1}$$
$$U_i U_j = U_j U_i, \ |i-j| > 1, \tag{4.2}$$
$$U_i^2 = [2] U_i. \tag{4.3}$$

This implies that the U_i generate $TL_r(A)$, (see section 2.1). Moreover, the element U_i corresponds to the diagram



These elements are multiplied by composition of diagrams where each occurring loop is replaced by a factor [2].

Denote by $U_{x_2^{2i+2}}$ the element $U_{2i+2}U_{2i+1}U_{2i+3}U_{2i+2}$ for any $i \geq 0$. In this regard, we should point out that – in contrast to the case of the Hecke algebra – for a general x in a symmetric group, it is not possible to define $U_x = U_{i_1}U_{i_2}\cdots U_{i_k}$ where $x = s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced expression. For example, $s_1s_2s_1 = s_2s_1s_2$ are two reduced expressions for the same element of S_3, but $U_1U_2U_1 = U_1$ does not usually equal $U_2U_1U_2 = U_2$. Thus one has to be very careful when introducing and working with such notation.

Since $TL_r(A)$ is a module over $H_r(A)$, we will allow ourselves to *multiply elements of* $TL_r(A)$ *by elements of* $H_r(A)$ *without further notice.*

4.2 Results

In this section, we work over $A_d = A_2$ (see Definition 3.5) – and localizations of this ring – instead of A. We denote by $TL_r(A_2)$ the Temperley-Lieb algebra over A_2 and maintain the notation $H_r(A_2)$ for the Hecke algebra over A_2. We will apply similar notation for the Temperley-Lieb and Hecke algebras over localizations of A_2 below.

Recall from p. 21 and section 3.3 that in this situation, the element $p \in H_{2r}(A_2)$ is

$$\begin{aligned} p &= p_1p_3\cdots p_{2r-1} \\ &= \frac{1}{[2]^r}\left(q^{-1}+T_1\right)\left(q^{-1}+T_3\right)\cdots\left(q^{-1}+T_{2r-1}\right) \\ &= \left(1-\frac{U_1}{[2]}\right)\left(1-\frac{U_3}{[2]}\right)\cdots\left(1-\frac{U_{2r-1}}{[2]}\right). \end{aligned}$$

Recall also from Lemma 3.6 that

$$pT_x = T_xp = q^{l(x)}p \text{ for any } x \in \langle s_1, s_3, \ldots, s_{2r-1}\rangle\,,$$

where $\langle\ \rangle$ denotes "the subgroup of S_{2r} generated by".

Note also that

$$T_{x_2^i}p = pT_{x_2^i} \tag{4.4}$$

for all $i = 2, 4, \ldots, 2r-2$ since

$$T_{x_2^i}T_{i\pm1} = T_{i\mp1}T_{x_2^i}$$

for the applicable i.

In the proof of the following proposition, we need $[2]-\frac{3}{[2]}$ to be a unit. We therefore work over the ring A_2^* which we define as A_2 localized in $[2]-\frac{3}{[2]}$. We write $\langle\ \ \rangle = \langle\ \ \rangle_{A_2^*}$ for "the A_2^*-algebra generated by".

Proposition 4.1 *For $r \geq 2$ arbitrary, the element $pU_2U_4\cdots U_{2r}p$ is contained in the algebra $\langle pU_2p, pU_4p, \dots, pU_{2r}p\rangle \subseteq pTL_{2r+1}(A_2^*)p$.*

Proof. We proceed by induction on r.

For any $i \geq 1$, we have the following calculation in $pTL_{2i+3}(A_2^*)p$

$$\begin{aligned}
&pU_{2i+2}pU_2U_4\cdots U_{2i}pU_{2i+2}p\\
&\quad = pU_{2i+2}p_{2i+1}p_{2i+3}U_2U_4\cdots U_{2i}p_{2i+1}U_{2i+2}p\\
&\quad = pU_{2i+2}\left(1-\tfrac{U_{2i+1}}{[2]}\right)\left(1-\tfrac{U_{2i+3}}{[2]}\right)U_2U_4\cdots U_{2i}\left(1-\tfrac{U_{2i+1}}{[2]}\right)U_{2i+2}p\\
&\quad = pU_{2i+2}U_2U_4\cdots U_{2i}U_{2i+2}p - pU_{2i+2}\tfrac{U_{2i+1}}{[2]}U_2U_4\cdots U_{2i}U_{2i+2}p\\
&\qquad - pU_{2i+2}\tfrac{U_{2i+3}}{[2]}U_2U_4\cdots U_{2i}U_{2i+2}p + pU_{2i+2}\tfrac{U_{2i+1}U_{2i+3}}{[2]^2}U_2U_4\cdots U_{2i}U_{2i+2}p\\
&\qquad - pU_{2i+2}U_2U_4\cdots U_{2i}\tfrac{U_{2i+1}}{[2]}U_{2i+2}p + pU_{2i+2}\tfrac{U_{2i+3}}{[2]}U_2U_4\cdots U_{2i}\tfrac{U_{2i+1}}{[2]}U_{2i+2}p\\
&\qquad + pU_{2i+2}\tfrac{U_{2i+1}}{[2]}U_2U_4\cdots U_{2i}\tfrac{U_{2i+1}}{[2]}U_{2i+2}p\\
&\qquad - pU_{2i+2}\tfrac{U_{2i+1}U_{2i+3}}{[2]^2}U_2U_4\cdots U_{2i}\tfrac{U_{2i+1}}{[2]}U_{2i+2}p.
\end{aligned}$$

Now, by use of the relations (4.1),(4.2) and (4.3), this reduces to

$$\begin{aligned}
&pU_{2i+2}pU_2U_4\cdots U_{2i}pU_{2i+2}p\\
&\quad = [2]pU_2U_4\cdots U_{2i+2}p - \tfrac{3}{[2]}pU_2U_4\cdots U_{2i+2}p + \tfrac{1}{[2]^2}pU_{x_2^{2i+2}}U_2U_4\cdots U_{2i}p\\
&\qquad + \tfrac{1}{[2]^2}pU_2U_4\cdots U_{2i}U_{x_2^{2i+2}}p + \tfrac{1}{[2]^2}pU_2U_4\cdots U_{2i-2}U_{2i+2}p\\
&\qquad - \tfrac{1}{[2]^3}pU_2U_4\cdots U_{2i-2}U_{x_2^{2i+2}}p\\
&\quad = \left([2]-\tfrac{3}{[2]}\right)pU_2U_4\cdots U_{2i+2}p + \tfrac{1}{[2]^2}(pU_{x_2^{2i+2}}U_2U_4\cdots U_{2i}p\\
&\qquad + pU_2U_4\cdots U_{2i}U_{x_2^{2i+2}}p + pU_2U_4\cdots U_{2i-2}pU_{2i+2}p)\\
&\qquad - \tfrac{1}{[2]^3}pU_2U_4\cdots U_{2i-2}pU_{x_2^{2i+2}}p.
\end{aligned}$$

Note that

$$(pU_{2i+2}p)^2 = pU_{2i+2}p_{2i+1}p_{2i+3}U_{2i+2}p = [2]pU_{2i+2}p - \tfrac{2}{[2]}pU_{2i+2}p + \tfrac{1}{[2]^2}pU_{x_2^{2i+2}}p$$

which means that $pU_{x_2^{2i+2}}p \in \langle pU_2p, pU_4p, \dots, pU_{2i+2}p\rangle$.

Further, we have

$$\begin{aligned}
pU_2U_4\cdots U_{2i}U_{x_2^{2i+2}}p &= pU_2U_4\cdots U_{2i}p_1p_3\cdots p_{2i-1}p_{2i+3}U_{x_2^{2i+2}}p\\
&= pU_2U_4\cdots U_{2i}p_1p_3\cdots p_{2i-1}\left(p_{2i+1}+\tfrac{U_{2i+1}}{[2]}\right)p_{2i+3}U_{x_2^{2i+2}}p\\
&= pU_2U_4\cdots U_{2i}pU_{x_2^{2i+2}}p + 0,
\end{aligned}$$

and, of course, the analogous result for $pU_{x_2^{2i+2}}U_2U_4\cdots U_{2i}p$.

This implies

$$pU_2U_4\cdots U_{2i}U_{x_2^{2i+2}}p,\ pU_{x_2^{2i+2}}U_2U_4\cdots U_{2i}p \\ \in \langle pU_2p, pU_4p, \ldots, pU_{2i+2}p, pU_2U_4\cdots U_{2i}p\rangle .$$

Now the induction goes as follows. For $i=1$, the above calculations yield $pU_2U_4p \in \langle pU_2p, pU_4p\rangle$ which starts the induction.

Then assume that

$$pU_2U_4\cdots U_{2j}p \in \langle pU_2p, pU_4p, \ldots, pU_{2j}p\rangle$$

for all $j \leq i$, $1 \leq i$. We must prove that

$$pU_2U_4\cdots U_{2i+2}p \in \langle pU_2p, pU_4p, \ldots, pU_{2i+2}p\rangle .$$

Again, this is a result of the above calculations. Thus we have proved the proposition. □

Let π denote the map $H_{2r}(A_2^*) \to TL_{2r}(A_2^*)$ given by $\pi(T_i) = q - U_i$. (We will continue to omit π from the notation just like we have done so far. We only introduce notation for the map here, because we want to discuss its properties). Since π is a surjection, the induced map $\pi : pH_{2r}(A_2^*)p \to pTL_{2r}(A_2^*)p$ is also a surjection. Thus, in order to prove that

$$pTL_{2r}(A_2^*)p = \langle pU_2p, pU_4p, \ldots, pU_{2r-2}p\rangle ,$$

it suffices to prove that

$$\pi(pH_{2r}(A_2^*)p) \subseteq \langle pU_2p, pU_4p, \ldots, pU_{2r-2}p\rangle .$$

In the proof of this result, we will alternate between calculations involving the generators T_i of the Hecke algebra and the generators U_i of the Temperley-Lieb algebra, respectively. In this way we use the advantages of each set of generators. For instance, only the T_i satisfy the relation (4.4) whereas the relations (4.1) and (4.3) among the U_i turn out to be very useful in other calculations.

PROPOSITION 4.2 *We have* $\pi(pH_{2r}(A_2^*)p) \subseteq \langle pU_2p, pU_4p, \ldots, pU_{2r-2}p\rangle$.

PROOF. It suffices to prove that $\pi(pT_xp) \in \langle pU_2p, pU_4p, \ldots, pU_{2r-2}p\rangle$ for all $x \in S_{2r}$. We proceed by induction on r, but let us first make some general observations.

For all natural n, we have a natural inclusion $S_n < S_{n+1}$, (S_n consists of the permutations in S_{n+1} which keep $n+1$ fixed). The following is a set of minimal right coset representatives of S_n in S_{n+1} (cf. e.g. [BB05, Cor. 2.4.5 and Lemma 2.4.7]):

$$1,\ s_n,\ s_ns_{n-1},\ s_ns_{n-1}s_{n-2},\ \ldots,\ s_ns_{n-1}s_{n-2}\cdots s_1.$$

This implies that for all $r \geq 2$, any $x \in S_{2r}$ can be written on the form

$$x = ys_{2r-1}s_{2r-2}\cdots s_i,$$

where $y \in S_{2r-1}$ and $i \in \{1, 2, \ldots, 2r\}$. Here we let $i = 2r$ mean $x = y \in S_{2r-1} < S_{2r}$. As $T_i p = qp$ when i is odd, we can assume that i is an even number.

Now, if $i < 2r$, then we can rewrite

$$x = y' s_{2r-1}s_{2r-2}\cdots s_{i+2}(s_i s_{i-1} s_{i+1} s_i) = y' s_{2r-1}s_{2r-2}\cdots s_{i+2}x_2^i,$$

$y' \in S_{2r-1}$. Recall from p. 36 that $T_{x_2^i}p = pT_{x_2^i}$. Hence,

$$pT_xp = pT_{y'}T_{2r-1}T_{2r-2}\cdots T_{i+2}T_{x_2^i}p = pT_{y'}T_{2r-1}T_{2r-2}\cdots T_{i+2}pT_{x_2^i}p.$$

Repeating this argument, we see that we can assume $x \in S_{2r-1}$. We shall make use of this fact now.

For the induction start $r = 2$, we must prove that $pT_xp \in \langle pU_2p\rangle$ for all $x \in S_4$. Now, using the above, we can assume that $x \in S_3$.

Write

$$x = zs_2\cdots s_i,$$

$z \in S_2$. Since we may assume that i is even, we have either $x = z$ or $x = zs_2$. In the first case, pT_xp is p (or qp), so we are done. In the second case, pT_xp is pT_2p (or qpT_2p), and we are also done.

Now let $r > 2$ and $x \in S_{2r}$. Using the above, we may assume that $x \in S_{2r-1}$. Then we write

$$x = ys_{2r-2}s_{2r-3}\cdots s_i,$$

where $y \in S_{2r-2}$ and $i \in \{1, 2, \ldots, 2r-1\}$. Again, we may assume that i is even. By the above argument, we reduce to the cases $x \in S_{2r-2}$ (in which case we are done by induction) or $x = ys_{2r-2}$.

In the last case we write

$$y = zs_{2r-3}s_{2r-4}\cdots s_i,$$

where $z \in S_{2r-3}$.

If $i = 2r-3$, we have

$$\begin{aligned} pT_xp &= pT_zT_{2r-3}T_{2r-2}p \\ &= pT_z([2]p_{2r-3} - q^{-1})T_{2r-2}p \\ &= [2]pT_zpT_{2r-2}p - q^{-1}pT_zT_{2r-2}p. \end{aligned}$$

Hence, in this case, we may assume $x = zs_{2r-2}$ with $z \in S_{2r-3}$.

If $i < 2r-3$, we may assume that i is even (since otherwise $T_iT_{2r-2}p = qT_{2r-2}p$), and we rewrite

$$x = z' s_{2r-3}s_{2r-4}\cdots s_{i+2}s_{2r-2}x_2^i$$

with $z' \in S_{2r-3}$ and argue as above. This reduces to the case

$$x = zs_{2r-3}s_{2r-4}s_{2r-2}$$

with $z \in S_{2r-3}$.

In this case, consider

$$T_{2r-3}T_{2r-4}T_{2r-2} = (q - U_{2r-3})(q - U_{2r-4})(q - U_{2r-2}).$$

Hence, the element pT_xp is a linear combination of terms of the form pT_zUp, where

$$\begin{aligned} U \in \{&1, U_{2r-3}, U_{2r-4}, U_{2r-2}, U_{2r-3}U_{2r-4}, \\ &U_{2r-3}U_{2r-2}, U_{2r-4}U_{2r-2}, U_{2r-3}U_{2r-4}U_{2r-2}\}. \end{aligned}$$

We may assume $U = U_{2r-3}U_{2r-4}U_{2r-2}$ or $U = U_{2r-2}$. In fact, if U does not contain U_{2r-2}, we are in H_{2r-2} and hence done by induction. If $U = U_{2r-4}U_{2r-2}$ then

$$T_zU \in H_{2r-3}U_{2r-2}$$

and if $U = U_{2r-3}U_{2r-2}$ then

$$Up = [2]U_{2r-2}p - [2]p_{2r-3}U_{2r-2}p$$

such that

$$pT_zUp = [2]pT_zU_{2r-2}p - [2]pT_zpU_{2r-2}p.$$

In the case $U = U_{2r-3}U_{2r-4}U_{2r-2}$, we use the identity

$$U_{2r-3}U_{2r-4} = U_{2r-3}U_{2r-4}p_{2r-3} + 1 - p_{2r-3}.$$

This gives

$$pT_zUp = pT_zU_{2r-3}U_{2r-4}pU_{2r-2}p + pT_zU_{2r-2}p - pT_zpU_{2r-2}p.$$

We can conclude that we have reduced to elements of the form $pT_zU_{2r-2}p$ with $z \in S_{2r-3}$.

In this case, we write

$$z = vs_{2r-4}s_{2r-5}\cdots s_i$$

with $i \in \{1, 2, \ldots, 2r-3\}$ and $v \in S_{2r-4}$. We may assume that i is even and rewrite

$$z = v's_{2r-4}s_{2r-5}\cdots s_{i+2}x_2^i$$

with $v' \in S_{2r-4}$.

Then

$$\begin{aligned} pT_zU_{2r-2}p &= pT_{v'}T_{2r-4}T_{2r-5}\cdots T_{i+2}T_{x_2^i}U_{2r-2}p \\ &= pT_{v'}T_{2r-4}T_{2r-5}\cdots T_{i+2}U_{2r-2}pT_{x_2^i}p, \end{aligned}$$

so, by the argument used several times above, it is enough to check that

$$pT_{v'}T_{2r-4}U_{2r-2}p \in \langle pU_{2j}p \mid j = 1, 2, \dots, r-1 \rangle$$

with $v' \in S_{2r-4}$.

Now, note that $T_{2r-4} = q - U_{2r-4}$, and we have reduced to elements of the form $pT_v U_{2r-4}U_{2r-2}p$ with $v \in S_{2r-4}$. Continuing like this we reduce to elements of the form

$$pU_2U_4 \cdots U_{2r-2}p.$$

Now use Proposition 4.1 to finish the proof. □

Let B_r denote the braid group on r strands with generators $X_1, X_2, \dots, X_{r-1}$, see section 3.2.4. Let A_2' denote A_2^* localized in $u = q - q^{-1}$. Denote by $\langle \ \rangle_{A_2'}$ "the A_2'-algebra generated by". We are now ready to formulate the main result of this section.

Corollary 4.3 *The map* $A_2'[B_r] \to pTL_{2r}(A_2')p$ $(\cong \mathrm{End}_{U_{A_2'}(\mathfrak{sl}_2)}(V_{A_2'}(2)^{\otimes r}))$, $X_i \mapsto q^{-2}pT_{x_2^{2i}}p$ *is surjective.*

Proof. Having established in Proposition 4.2 that

$$\mathrm{End}_{U_{A_2'}(\mathfrak{sl}_2)}(V_{A_2'}(2)^{\otimes r}) \cong pTL_{2r}(A_2')p = \langle pU_2p, pU_4p, \dots, pU_{2r-2}p \rangle_{A_2'},$$

we must prove that

$$\langle pU_2p, pU_4p, \dots, pU_{2r-2}p \rangle_{A_2'} \subseteq \left\langle pT_{x_2^{2i}}p \mid i = 1, 2, \dots, r-1 \right\rangle_{A_2'}.$$

It suffices to prove that $pU_ip \in \left\langle pT_{x_2^{2i}}p \right\rangle_{A_2'}$ or, equivalently, $pT_ip \in \left\langle pT_{x_2^{2i}}p \right\rangle_{A_2'}$ for any $i = 2, 4, \dots, 2r-2$. To this end, note that

$$(pT_{x_2^{2i}}p)^2 = u^2q[2]pT_{x_2^{2i}}p + uq^2[2]^2pT_ip + p.$$

Since we work over A_2', we are done. □

Chapter 5

Semisimplicity of the BMW-algebra

In the previous chapter, we have established surjectivity from the group algebra of the r-string braid group to the endomorphism algebra $\mathrm{End}_{U_{A_2'}(\mathfrak{sl}_2)}(V_{A_2'}(2)^{\otimes r})$ where A_2' is a specific localization of A_2. Even though this surjectivity is a nice result in itself, we would like to search for the kernel of the surjection and thus arrive at a more complete description of the endomorphism algebra. If we compare our current results with our point of departure, the work by Lehrer and Zhang presented in Chapter 1, we get a clue that the next fruitful step will probably be to try to establish a surjection from the BMW-algebra which is a quotient of the group algebra of the braid group. From there, one can then try to find the full kernel of the map.

Since the surjection from the BMW-algebra exists in the generic case (cf. Theorem 1.5), simple arguments from commutative algebra tell us that it will also exist for all but finitely many specializations of the quantum group over A. We would like to make this observation more precise by finding these exceptions if they exist.

To this end, we notice that a necessary condition for the existence of a surjection from the BMW-algebra to the endomorphism algebra is the following: If the BMW-algebra is semisimple then also the endomorphism algebra must be semisimple. As a first step towards our goal, it is therefore relevant to investigate when the BMW-algebra is semisimple and then compare the result with the results on semisimplicity of the endomorphism algebra. As a by-product, the investigation of the semisimplicity might reveal more precisely how the non-generic case is different from the generic case. This chapter is devoted to the investigation of the semisimplicity.

5.1 The BMW-algebra

Maintain the notation $A = \mathbb{Z}[q, q^{-1}]$ for the ring of Laurent polynomials in q. Let $BMW_r(A)$ be the A-algebra with generators which, by abuse of notation, we denote $g_i^{\pm 1}, e_i$, for $i = 1, \ldots, r-1$, and relations

(a) $g_i g_j = g_j g_i$ if $|i-j| \geq 2$,

(b) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $1 \leq i \leq r-2$,

(c) $g_i - g_i^{-1} = (q^2 - q^{-2})(1 - e_i)$ for $1 \leq i \leq r-1$,

(d) $g_i e_i = e_i g_i = q^{-4} e_i$ for $1 \leq i \leq r-1$,

(e) $e_i g_{i+1}^{\pm 1} e_i = q^{\pm 4} e_i$ for $1 \leq i \leq r-2$

(f) $e_i g_{i-1}^{\pm 1} e_i = q^{\pm 4} e_i$ for $2 \leq i \leq r-1$.

Let us first look at an example where the endomorphism algebra is non-semisimple and isomorphic to the BMW-algebra.

Example 5.1
Keep $d = 2$ and fix $r = 2$. Assume that q has been specialized to a complex root of unity ϵ of order 3; i.e. obtain via the map $A \ni q \mapsto \epsilon \in \mathbb{C}$ specializations $U_\epsilon = U_A \otimes_A \mathbb{C}$ and $V_\epsilon(n) = V_A(n) \otimes_A \mathbb{C}$ for any $n \geq 0$. Denote by $T_\epsilon(4)$ the unique indecomposable tilting module for U_ϵ with highest weight 4. It is well-known (see e.g. [And, §2.3]) that the tensor product $V_\epsilon(2) \otimes V_\epsilon(2)$ splits up as $T_\epsilon(4) \oplus V_\epsilon(0)$. This implies that ($X$ an indeterminate)

$$\mathrm{End}_{U_\epsilon}(V_\epsilon(2)^{\otimes 2}) \cong \mathbb{C} \oplus \mathbb{C}[X]/(X^2)$$

which is non-semisimple.

Denote by $BMW_2(\epsilon)$ the BMW-algebra $BMW_2(A) \otimes_A \mathbb{C}$ over $\mathbb{C}$ obtained via the map $q \mapsto \epsilon$. We will construct an isomorphism from $BMW_2(\epsilon)$ to $\mathrm{End}_{U_\epsilon}(V_\epsilon(2)^{\otimes 2})$. Consequently, the algebra $BMW_2(\epsilon)$ is non-semisimple in contrast to the situation we have in the generic setting where $BMW_r(q)$ is semisimple for $r \leq 4$ ([LZ08b]). This result is well-known from [RS09, Thm. 5.9(b1)], see Theorem 5.4 (ii)(a) below.

The isomorphism is defined as follows:

$$\begin{aligned} 1 &\mapsto (1, 1), \\ e &\mapsto (0, X), \\ g &\mapsto (-\epsilon^{-2}, (\epsilon - 1)X + \epsilon^2). \end{aligned}$$

It is easy to check that the relations of $BMW_2(\epsilon)$ are preserved under the map. Also, since the three basis elements 1, e, g of $BMW_2(\epsilon)$ are mapped to three linearly independent elements over $\mathbb{C}$ in $\mathrm{End}_{U_\epsilon}(V_\epsilon(2)^{\otimes 2})$ which is known to have rank three, the map is an isomorphism.

As a remark on the uniqueness of the above map, note that we can also define an isomorphism from $BMW_2(A)$ to $\mathrm{End}_{U_A}(V_A(2)^{\otimes 2})$ – up to a suitable localization – by using the fact that $\mathrm{End}_{U_A}(V_A(2)^{\otimes 2}) \cong pH_4(A)p = pTL_4(A)p$ as follows:

$$\begin{aligned} 1 &\mapsto p \\ e &\mapsto pT_{x_2^2}p - q^2[2]pT_2p + q^2p \\ g &\mapsto q^{-2}pT_{x_2^2}p. \end{aligned}$$

It is easy to check that the map defines an isomorphism; we only need to localize A in the two elements $u = q - q^{-1}$ and $[2]$. △

5.2 Semisimplicity criteria for the BMW-algebra

In [RS09], Rui and Si give necessary and sufficient conditions for the BMW-algebra to be semisimple over a field. We give here a reformulation of their results in our setting. For convenience, we will first restate the original result of [RS09].

5.2.1 Semisimplicity criteria by Rui and Si

Rui and Si work over the ground ring $R = \mathbb{Z}[r^{\pm 1}, t^{\pm 1}, u^{-1}]$ where r, t and $u = t - t^{-1}$ are indeterminates. They define the Birman-Wenzl algebra $\mathcal{B}_n$ as follows (where we have changed the notation slightly to fit better with our current notation).

Definition 5.2 [RS09, Definition 1.1] The Birman-Wenzl algebra $\mathcal{B}_n$ is the unital associative R-algebra with generators g_i, $1 \leq i \leq n-1$ and relations

(a) $(g_i - t)(g_i + t^{-1})(g_i - r^{-1}) = 0$ for $1 \leq i \leq n-1$,

(b) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $1 \leq i \leq n-2$,

(c) $g_i g_j = g_j g_i$ if $|i - j| \geq 2$,

(d) $e_i g_{i+1}^{\pm 1} e_i = r^{\pm 1} e_i$ for $1 \leq i \leq n-2$,

(e) $e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i$ for $2 \leq i \leq n-1$,

(f) $g_i e_i = e_i g_i = r^{-1} e_i$ for $1 \leq i \leq n-1$
where $e_i = 1 - u^{-1}(g_i - g_i^{-1})$ for $1 \leq i \leq n-1$.

Let F be a field which contains non-zero elements $\mathbf{t}$, $\mathbf{r}$ and $\mathbf{u} = \mathbf{t} - \mathbf{t}^{-1}$. Then the Birman-Wenzl algebra $\mathcal{B}_{n,F}$ over F is isomorphic to $\mathcal{B}_n \otimes_R F$, cf. [RS09, §1]. Here we consider F as an R-module by letting r, t, u act on F as $\mathbf{r}$, $\mathbf{t}$ and $\mathbf{u}$, respectively. We will use the notation $\mathcal{B}_n$ for $\mathcal{B}_{n,F}$ where it causes no confusion.

Note that $\mathcal{B}_{1,F} = F$ is always semisimple. For this reason, we will only consider $\mathcal{B}_n = \mathcal{B}_{n,F}$ for $n \geq 2$ below. Denote by $o(v)$ the multiplicative order of an element $v \in F$.

Theorem 5.3 *[RS09, Theorem 5.9] Let $\mathcal{B}_n$ be the Birman-Wenzl algebra over a field F containing non-zero parameters q, r and $q - q^{-1}$ which play the role of $\boldsymbol{t}$, $\boldsymbol{r}$ and $\boldsymbol{u} = \boldsymbol{t} - \boldsymbol{t}^{-1}$, respecively.*

(a) Suppose $r \notin \{q^{-1}, -q\}$.

(a1) If $n \geq 3$ then $\mathcal{B}_n$ is semisimple if and only if

$$o(q^2) > n \quad \textit{and} \quad r \notin \bigcup_{k=3}^{n} \{q^{3-2k}, \pm q^{3-k}, -q^{2k-3}, \pm q^{k-3}\}.$$

(a2) $\mathcal{B}_2$ is semisimple if and only if $o(q^2) > 2$.

(b) Suppose $r \in \{q^{-1}, -q\}$.

(b1) $\mathcal{B}_n$ is not semisimple if n is odd with $n \geq 7$ or even.

(b2) $\mathcal{B}_3$ is semisimple if and only if $o(q^2) > 3$ and $q^4 + 1 \neq 0$.

(b3) $\mathcal{B}_5$ is semisimple if and only if $o(q^2) > 5$, $q^6 + 1 \neq 0$, $q^8 + 1 \neq 0$ and $\mathrm{char}(F) \neq 2$.

5.2.2 Semisimplicity criteria in our setting

Now, if we compare the Birman-Wenzl algebra $\mathcal{B}_n$ introduced above with the BMW-algebra that we introduced in section 5.1, we realise that we want to look at the following situation.

Let $A'' = \mathbb{Z}[q, q^{-1}, (q^2 - q^{-2})^{-1}]$ be the ring $A = \mathbb{Z}[q, q^{-1}]$ localized in the element $q^2 - q^{-2}$ and denote by $BMW_n(A'')$ the A''-algebra with generators $g_i^{\pm 1}, e_i$, for $i = 1, 2, \ldots, r-1$, and relations (a)-(f) from section 5.1. Note that A'' is an R-algebra via $t \mapsto q^2$, $r \mapsto q^4$, and we have $\mathcal{B}_n \otimes_R A'' \cong BMW_n(A'')$. Denote by $U_{A''}$ the quantum group for $\mathfrak{sl}_2$ over A'' (defined as U_A in section 2.2 but over the ring A'' instead of A) and let $V_{A''}(2)$ be the Weyl module with highest weight 2 for $U_{A''}$ (see section 2.2).

Let F be a field and $\zeta \in F$ a non-zero element satisfying $\zeta^4 \neq 1$. Then F becomes an A''-algebra via $q \mapsto \zeta$. Hence F is an R-algebra via the composite $R \to A'' \to F$ given by $t \mapsto \zeta^2$, $r \mapsto \zeta^4$. We get

$$\mathcal{B}_n \otimes_R F \cong \mathcal{B}_n \otimes_R A'' \otimes_{A''} F \cong BMW_n(A'') \otimes_{A''} F.$$

According to [RS09], the algebra $\mathcal{B}_n \otimes_R F$ is the Birman-Wenzl algebra over F with parameters $q = \zeta^2$, $r = \zeta^4$ and $q - q^{-1} = \zeta^2 - \zeta^{-2}$ in F. We shall denote it $BMW_n(F)$.

Theorem 5.4

(i) If $\zeta^l \neq 1$ for all l, then $BMW_n(F)$ is semisimple if and only if $n \leq 4$.

(ii) Suppose ζ is a root of unity and let $o(\zeta) = l$. Since $\zeta^4 \neq 1$ we have $l \neq 1, 2, 4$.

(a) If $l \in \{3, 6\}$, then $BMW_n(F)$ is non-semisimple for all $n \geq 2$.

(b) If $l \notin \{3, 6\}$, then

(b1) $BMW_2(F)$ is semisimple if and only if $l \neq 8$.

(b2) $BMW_3(F)$ is semisimple if and only if $l \notin \{5, 8, 10, 12\}$.

(b3) $BMW_4(F)$ is semisimple if and only if $l \notin \{5, 7, 8, 10, 12, 14, 16\}$

(b4) For $n \geq 5$ the algebra $BMW_n(F)$ is non-semisimple.

Proof. We prove the statements by use of Theorem 5.3.

(i) Suppose $\zeta^l \neq 1$ for all l. Then we are in case (a) of Theorem 5.3. Note that $o(q^2) = o(\zeta^4) > n$ for all n. In particular, the algebra $BMW_2(F)$ is semisimple according to (a2).

For $n \geq 3$, by (a1) we must check whether the condition $q^2 = \zeta^4 = r \notin \cup_{k=3}^{n} \{q^{3-2k}, \pm q^{3-k}, -q^{2k-3}, \pm q^{k-3}\}$ is true or false. For $n = 3, 4$ it is true. For $n > 4$, we have $r = q^{k-3}$ for $k = 5$ which makes the condition false.

Hence, the algebra $BMW_n(F)$ is semisimple if and only if $n \leq 4$.

(ii) Suppose ζ is a root of unity and $o(\zeta) = l$.

If $l \in \{3, 6\}$, then we are in case (b) of Theorem 5.3. Note that $o(q^2) = o(\zeta^4) = 3$ in this case which implies that $BMW_3(F)$ and $BMW_5(F)$ are both non-semisimple, cf. (b2) and (b3), respectively. Hence, by (b1), the algebra $BMW_n(F)$ is non-semisimple for all $n \geq 2$.

If $l \notin \{3, 6\}$, then we are in case (a) of Theorem 5.3. From (a2) we see that $BMW_2(F)$ is semisimple if and only if $l \neq 8$. From (a1) we get the results for $n \geq 3$. □

Remark 5.5 Note that the criteria for the semisimplicity of the Birman-Murakami-Wenzl algebra $BMW_n(F)$ in Theorem 5.4 agree with the semisimplicity conditions on the endomorphism algebra

$$\mathrm{End}_{U_F}(V_F(2)^{\otimes n}) \cong \mathrm{End}_{U_{A''}}(V_{A''}(2)^{\otimes n}) \otimes_{A''} F$$

in the sense that if $BMW_n(F)$ is semisimple, then also $\mathrm{End}_{U_F}(V_F(2)^{\otimes n})$ is semisimple. (Here $U_F = U_{A''} \otimes_{A''} F$ and $V_F(2) = V_{A''}(2) \otimes_{A''} F$ as usual and the isomorphism holds by a base change argument like the one from below equation (3.1) in section 3.2.4).

Indeed, if q is *not* a root of unity, the endomorphism algebra is always semisimple. If q is a root of unity of odd order l or even order $2l$, then the endomorphism algebra is semisimple if and only if $2n < l$.

Hence, to conclude, our investigation of the semisimplicity did *not* in itself lead directly to any exceptions where a surjection is impossible. However, we still feel that it does indicate for which roots of unity the non-generic case is likely to differ from the generic case. In this light the above results on the semisimplicity might prove valueable in connection with future investigations of the subject.

5.3 An element of the kernel

As a last comment in connection with the search for the kernel of the surjection from Chapter 4, we point out that some elements of the kernel are already known. This fact has been stated in [LZ08a] and we adopt their notation here.

Recall from Chapter 1 the element Φ_q which generates the kernel in the generic case (cf. Theorem 1.6). Define

$$\tilde{\Phi}_q = (q^2 + q^{-2})\Phi_q \in BMW_r(A).$$

Then $\tilde{\Phi}_q$ acts trivially on $V_A(2)^{\otimes r}$ since $V_A(2)^{\otimes r}$ is an A-lattice in $V_q(2)^{\otimes r}$ on which Φ_q acts trivially. Hence, the ideal generated by $\tilde{\Phi}_q$ is contained in the kernel. The obvious question to ask is then of course whether this ideal is the whole kernel.

Part III

Discussion of similar cases

Chapter 6

Strongly multiplicity free modules of other types

In Chapter 1 concerning the results by Lehrer and Zhang on the generic case, we used to a great extent the fact that the $U_q(\mathfrak{sl}_2)$-modules $V_q(n)$ are strongly multiplicity free. More precisely, we used the fact that for these modules, we have a very nice formula for the universal R-matrix (see equation (1.17)) which allowed us to construct a surjection from the BMW-algebra to the endomorphism algebra that we were investigating. However, there are other strongly multiplicity free modules which we have not yet treated in this thesis. In order to make our presentation as complete as possible, we will consider the remaining cases in this chapter.

6.1 Classification of strongly multiplicity free modules

Before proceeding any further, let us stop to have a brief look at strongly multiplicity free (smf) modules. From [LZ06] we have the following results concerning the classification of these modules.

Theorem 6.1 *[LZ06, Theorem 3.4] Let $\mathfrak{g}$ be a simple complex Lie algebra. The following is a complete list of the strongly multiplicity free irreducible $\mathfrak{g}$-modules:*

(i) the natural $\mathfrak{sl}_k$-module $\mathbb{C}^k$ and its dual for $k > 2$,

(ii) the natural $\mathfrak{so}_{2k+1}$-module $\mathbb{C}^{2k+1}$ for $k \geq 2$,

(iii) the natural $\mathfrak{sp}_{2k}$-module $\mathbb{C}^{2k}$ for $k > 1$,

(iv) the 7-dimensional irreducible G_2-module,

(v) all irreducible $\mathfrak{sl}_2$-modules of dimension greater than 1.

Let $U_q(\mathfrak{g})$ denote the Drinfel'd-Jimbo quantization corresponding to $\mathfrak{g}$ over the field $\mathcal{K} = \mathbb{C}(q^{\frac{1}{2}})$ as in Chapter 1.

Lemma 6.2 *[LZ06, Lemma 7.2] An irreducible $U_q(\mathfrak{g})$-module is smf if and only if the irreducible $U(\mathfrak{g})$-module with the same highest weight is smf.*

Remark 6.3 In view of Theorem 6.1, the lemma provides a classification of the smf modules for the generic quantum case.

6.2 Status for strongly multiplicity free modules of other types

From the above section we see that the only infinite class of strongly multiplicity free modules is the class consisting of the irreducible (quantum) $\mathfrak{sl}_2$-modules of dimension greater than 1. Hence, in some sense the main problem when dealing with smf modules is to tackle this class. A fact that has been the underlying motivation in this thesis. However, as mentioned above, we would like to give a short account of each of the other cases. While one of these cases has been investigated completely, the rest are (at least partially) unsolved.

The natural $U_q(\mathfrak{sl}_k)$-module and its dual for $k > 2$: This case has been treated not only generically, but integrally by Du, Parshall and Scott in [DPS98]. More precisely, they establish quantum Weyl reciprocity over the ring $A = \mathbb{Z}[q, q^{-1}]$ of Laurent polynomials over the integers in a variable q. We give here a brief summary of their results.

Let $U_A = U_A(\mathfrak{gl}_n)$ be Lusztig's A-form of the quantum group of $\mathfrak{gl}_n$ and let H denote the Hecke algebra over A corresponding to the symmetric group S_r on $\{1, 2, \dots, r\}$ with generators T_i, $i = 1, 2, \dots, r-1$, and relations

$$\begin{aligned} T_i T_w &= T_{s_i w} \text{ if } l(s_i w) > l(w), \\ 0 &= (T_i - q)(T_i - 1). \end{aligned}$$

Here T_w denotes, as usual, the basis element of H corresponding to the element $w \in S_r$, and the reflection $s_i = (i\ i+1)$ is a generator of S_r.

Let V be a free A-module of rank n. We have two natural algebra homomorphisms

$$\varphi : U_A \to \mathrm{End}_H(V^{\otimes r}) \quad \text{and} \quad \psi : H^{\mathrm{op}} \to \mathrm{End}_{U_A}(V^{\otimes r}),$$

cf. [DPS98, p. 19].

Theorem 6.4 *[DPS98, Theorem 6.3] Both maps φ and ψ are surjective. Therefore, for any specialization of A into a field k, we have*

$$\mathrm{Im}(\varphi_k) = \mathrm{End}_{H_k}(V_k^{\otimes r}) \quad \textit{and} \quad \mathrm{Im}(\psi_k) = \mathrm{End}_{U_k}(V_k^{\otimes r}).$$

Here $V_k = V \otimes_A k$, $H_k = H \otimes_A k$ and $U_k = U_A \otimes_A k$. The homomorphisms are defined as $\varphi_k = \varphi \otimes \mathrm{id}_k$ and $\psi_k = \psi \otimes \mathrm{id}_k$.

It should be added that when $k = \mathbb{C}$ and q is *not* a root of unity, the quantized Weyl reciprocity was established already in [Jim86]. However, the result in Theorem 6.4 is obviously much stronger.

The natural $U_q(\mathfrak{so}_{2k+1})$-module for $k \geq 2$: Lehrer and Zhang has conjectured in [LZ11] on the second fundamental theorem of invariant theory of the orthogonal group that we can expect this case to be similar to the $\mathfrak{sl}_2$ case which is a special case of the conjecture.

More precisely, they look at the following situation. Let $U_q(\mathfrak{o}_n)$ be the smash product of the quantum group corresponding to the complex Lie algebra $\mathfrak{so}_n$ with the group algebra of $\mathbb{Z}_2$ (see [LZ06, §8]). Denote by $\mathcal{C}_q$ the category of finite-dimensional type $(1, 1, ..., 1)$ representations (see e.g. [LZ06, §6.1]) of $U_q(\mathfrak{o}_n)$.

If we use Lusztig's integral form of $U_q(\mathfrak{o}_n)$ and lattices in the irreducible $U_q(\mathfrak{o}_n)$-modules, we obtain a specialization functor $S : M_q \mapsto M$ taking modules M_q in $\mathcal{C}_q$ to their "classical limit" M.

Let V_q be the "natural" representation of the quantum group $U_q(\mathfrak{o}_n)$, that is, the representation which corresponds to the natural representation V of $\mathfrak{o}_n$ under the specialization above. It is well-known that there is a surjective homomorphism $\psi : \mathbb{C}[B_r] \to \mathrm{End}_{U_q(\mathfrak{o}_n)}(V_q^{\otimes r})$ where B_r denotes the braid group on r strings acting through the generalized R-matrices. Moreover, this action factorizes through the BMW-algebra $BMW_r(q^{2(1-n)}, q^2 - q^{-2})$ over $\mathbb{C}(q)$ (cf. [LZ11]) with the indicated parameters.

Conjecture 6.5 *[LZ11, Conjecture 9.2] In the above notation there is an element*

$$\Phi_q \in BMW_r(q^{2(1-n)}, q^2 - q^{-2})$$

such that $\ker(\psi) = \langle \Phi_q \rangle$.

The natural $U_q(\mathfrak{sp}_{2k})$-module for $k > 1$: This case has been treated in [HX10].

Let $A = \mathbb{Z}[q, q^{-1}]$ and denote the symplectic Lie algebra over the complex numbers $\mathbb{C}$ by $\mathfrak{sp}_{2k}$, $k \geq 1$. Denote by $U_q(\mathfrak{sp}_{2k})$ the corresponding quantized enveloping algebra over $\mathbb{Q}(q)$. Let V be the natural module for $U_q(\mathfrak{sp}_{2k})$, $k \geq 1$.

Hu and Xiao work with a version of the BMW-algebra over the field $\mathbb{Q}(q)$ which they call the specialized BMW-algebra with parameters $-q^{2k+1}$ and q, denoted $B_{n,q}$. See [HX10, Definition 2.1 and §5] for the precise definition.

Let φ, ψ be the natural algebra homomorphisms

$$\varphi : (B_{n,q})^{\mathrm{op}} \to \mathrm{End}_{U_q(\mathfrak{sp}_{2k})}(V^{\otimes n}) \quad \text{and} \quad \psi : U_q(\mathfrak{sp}_{2k}) \to \mathrm{End}_{B_{n,q}}(V^{\otimes n}),$$

respectively. The following result, a part of so-called quantized Schur-Weyl duality for type C, is well-known from [CP94, 10.2] and [Hay92].

THEOREM 6.6 *[HX10, Theorem 1.1] Both φ and ψ are surjective. If $k \geq n$ then φ is an isomorphism.*

DEFINITION 6.7 The annihilator of $V^{\otimes n}$ in $B_{n,q}$ is defined by

$$\mathrm{Ann}_{B_{n,q}}(V^{\otimes n}) = \{x \in B_{n,q} \mid vx = 0 \ \forall\, v \in V^{\otimes n}\}.$$

In order to state the main result in Proposition 6.8 below, we need a certain element Y_{k+1} which, for $n > k$, is in $B_{n,q}$. The precise definition of Y_{k+1} can be found in [HX10, Theorem 5.4]. We omit it here since it is not really vital for our presentation and requires the introduction of several new notions and notations. The important fact is that it has the following property.

PROPOSITION 6.8 *[HX10, Proposition 5.6] Suppose that $n > k$. Then*

$$\langle Y_{k+1} \rangle = \mathrm{Ann}_{B_{n,q}}(V^{\otimes n})$$

where the left-hand side denotes the $\mathbb{Q}(q)$-subalgebra of $B_{n,q}$ generated by Y_{k+1}.

COROLLARY 6.9 *[HX10, Corollary 5.7] Let K be any field. Then Proposition 6.8 is still true with $\mathbb{Q}(q)$ replaced by $K(q)$ where q is now an indeterminate over K.*

CONJECTURE 6.10 *[HX10, Conjecture 5.8] Let K be any field which is an A-algebra. Then Proposition 6.8 is still true with $\mathbb{Q}(q)$ replaced by K.*

COROLLARY 6.11 *[HX10, Corollary 5.9] Conjecture 6.10 is true in the case where q is specialized to 1 and in the case where $n = k + 1$.*

The 7-dimensional irreducible G_2-module: As far as we know, this case is still unsolved.

Appendix

APPENDIX A

MINIMAL DOUBLE COSET REPRESENTATIVES

We collect here some definitions and results concerning minimal representatives of double cosets in symmetric groups needed in Chapter 3. The content of this appendix stems from [Wil07] to which we refer the reader for the complete proofs of the statements below.

For a non-negative integer n, let S_n denote the symmetric group with generators s_i, $i = 1, 2, \ldots, n-1$. A composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a tuple of strictly positive integers with $\sum \lambda_i = n$ and $\lambda_i \geq \lambda_j$ for $i < j$. For any composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, we let S_λ denote the Young subgroup $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$ of S_n.

THEOREM A.1 *[Wil07, Theorem 4.1] Let λ and μ be compositions of n and let $x \in S_n$. The double coset $S_\lambda x S_\mu$ contains a unique element of minimal length.*

Let μ be a composition of n. A Young diagram of shape μ is a collection of n boxes arranged in left-justified rows with μ_i boxes in the ith row when we number the rows from above. For example,

is a Young diagram of shape $\mu = (3, 2)$.

A μ-tableau of type λ is a Young diagram of shape μ filled with the numbers $i = 1, 2, \ldots, k$ such that i appears exactly λ_i times. With the above definitions,

3	1	2	1
2	1	3	

is an example of a $(4, 3)$-tableau of type $(3, 2, 2)$.

Given a tableau t, we denote by mt the entrance of the mth box of t, if we number the boxes as indicated by the following example:

1	2	3	4
5	6	7	

. (A.1)

We let S_n act from the right on the set of μ-tableaux by place permutation as follows. If t is a tableau and $x \in S_n$ we define a new tableau tx by

$$m(tx) = (mx^{-1})t \text{ for } m = 1, 2, \ldots, n.$$

Denote by t_λ^μ the unique μ-tableau of type λ with weakly increasing entries when we read the diagram in the order indicated by (A.1). The proof of the above theorem proceeds by first proving the following.

Lemma A.2 *[Wil07, The proof of Theorem 4.1] An element $y \in S_\lambda x S_\mu$ is of minimal length if and only if*

(1) $t_\lambda^\mu y$ has increasing rows

(2) if $i < j$ and the ith and jth positions of t_λ^μ are equal then $iy < jy$.

Theorem A.1 can be generalised as follows.

Theorem A.3 *[Wil07, Theorem 4.3] Let x be a minimal double coset representative for a $S_\lambda \backslash S_n / S_\mu$ double coset. If $y \in S_\lambda x S_\mu$, then there exist $h \in S_\lambda$ and $k \in S_\mu$ such that $l(y) = l(h) + l(x) + l(k)$.*

Appendix B

Actions of R-matrices

Recall the notation concerning the Weyl modules for $U_A(\mathfrak{sl}_2)$ from Chapter 2. As in Chapter 2, we replace $A = \mathbb{Z}[q, q^{-1}]$ by $A' = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ in this entire appendix. The action of the universal R-matrix on $V_A(d) \otimes V_A(d)$ for any $d \geq 1$ can – as mentioned at the end of Chapter 2 – in theory be calculated using the formula (2.1). We have already treated the case $d = 1$ in section 2.2. For $d = 2, 3, 4$, we get the following results, where we denote by u the Laurent polynomial $q - q^{-1}$.

Highest weight $d = 2$:

$$\begin{aligned} R : w_0 \otimes w_0 &\mapsto q^2 w_0 \otimes w_0 \\ w_0 \otimes w_1 &\mapsto w_1 \otimes w_0 \\ w_0 \otimes w_2 &\mapsto q^{-2} w_2 \otimes w_0 \\ w_1 \otimes w_0 &\mapsto w_0 \otimes w_1 + u[2] w_1 \otimes w_0 \\ w_1 \otimes w_1 &\mapsto w_1 \otimes w_1 + q^{-2} u[2]^2 w_2 \otimes w_0 \\ w_1 \otimes w_2 &\mapsto w_2 \otimes w_1 \\ w_2 \otimes w_0 &\mapsto q^{-2} w_0 \otimes w_2 + u w_1 \otimes w_1 + q^{-1} u^2 [2] w_2 \otimes w_0 \\ w_2 \otimes w_1 &\mapsto w_1 \otimes w_2 + u[2] w_2 \otimes w_1 \\ w_2 \otimes w_2 &\mapsto q^2 w_2 \otimes w_2. \end{aligned}$$

Highest weight $d = 3$:

$$\begin{aligned} R : w_0 \otimes w_0 &\mapsto q^{\frac{9}{2}} w_0 \otimes w_0 \\ w_0 \otimes w_1 &\mapsto q^{\frac{3}{2}} w_1 \otimes w_0 \\ w_0 \otimes w_2 &\mapsto q^{-\frac{3}{2}} w_2 \otimes w_0 \\ w_0 \otimes w_3 &\mapsto q^{-\frac{9}{2}} w_3 \otimes w_0 \end{aligned}$$

$$
\begin{aligned}
w_1 \otimes w_0 &\mapsto q^{\frac{3}{2}} w_0 \otimes w_1 + q^{\frac{3}{2}} u[3] w_1 \otimes w_0 \\
w_1 \otimes w_1 &\mapsto q^{\frac{1}{2}} w_1 \otimes w_1 + q^{-\frac{3}{2}} u[3]! w_2 \otimes w_0 \\
w_1 \otimes w_2 &\mapsto q^{-\frac{1}{2}} w_2 \otimes w_1 + q^{-\frac{9}{2}} u[3]^2 w_3 \otimes w_0 \\
w_1 \otimes w_3 &\mapsto q^{-\frac{3}{2}} w_3 \otimes w_1 \\
w_2 \otimes w_0 &\mapsto q^{-\frac{3}{2}} w_0 \otimes w_2 + q^{\frac{1}{2}} u[2] w_1 \otimes w_1 + q^{-\frac{1}{2}} u^2 [3]! w_2 \otimes w_0 \\
w_2 \otimes w_1 &\mapsto q^{-\frac{1}{2}} w_1 \otimes w_2 + q^{-\frac{1}{2}} u[2]^2 w_2 \otimes w_1 + q^{-\frac{7}{2}} u^2 [3]! [3] w_3 \otimes w_0 \\
w_2 \otimes w_2 &\mapsto q^{\frac{1}{2}} w_2 \otimes w_2 + q^{-\frac{3}{2}} u[3]! w_3 \otimes w_1 \\
w_2 \otimes w_3 &\mapsto q^{\frac{3}{2}} w_3 \otimes w_2 \\
w_3 \otimes w_0 &\mapsto q^{-\frac{9}{2}} w_0 \otimes w_3 + q^{-\frac{1}{2}} u w_1 \otimes w_2 + q^{\frac{1}{2}} u^2 [2] w_2 \otimes w_1 \\
&\quad + q^{-\frac{3}{2}} u^3 [3]! w_3 \otimes w_0 \\
w_3 \otimes w_1 &\mapsto q^{-\frac{3}{2}} w_1 \otimes w_3 + q^{\frac{1}{2}} u[2] w_2 \otimes w_2 + q^{-\frac{1}{2}} u^2 [3]! w_3 \otimes w_1 \\
w_3 \otimes w_2 &\mapsto q^{\frac{3}{2}} w_2 \otimes w_3 + q^{\frac{3}{2}} u[3] w_3 \otimes w_2 \\
w_3 \otimes w_3 &\mapsto q^{\frac{9}{2}} w_3 \otimes w_3.
\end{aligned}
$$

Highest weight $d = 4$:

$$
\begin{aligned}
R : w_0 \otimes w_0 &\mapsto q^8 w_0 \otimes w_0 \\
w_0 \otimes w_1 &\mapsto q^4 w_1 \otimes w_0 \\
w_0 \otimes w_2 &\mapsto w_2 \otimes w_0 \\
w_0 \otimes w_3 &\mapsto q^{-4} w_3 \otimes w_0 \\
w_0 \otimes w_4 &\mapsto q^{-8} w_4 \otimes w_0 \\
w_1 \otimes w_0 &\mapsto q^4 w_0 \otimes w_1 + q^4 u[4] w_1 \otimes w_0 \\
w_1 \otimes w_1 &\mapsto q^2 w_1 \otimes w_1 + u[2][4] w_2 \otimes w_0 \\
w_1 \otimes w_2 &\mapsto w_2 \otimes w_1 + q^{-4} u[3][4] w_3 \otimes w_0 \\
w_1 \otimes w_3 &\mapsto q^{-2} w_3 \otimes w_1 + q^{-8} u[4]^2 w_4 \otimes w_0 \\
w_1 \otimes w_4 &\mapsto q^{-4} w_4 \otimes w_1 \\
w_2 \otimes w_0 &\mapsto w_0 \otimes w_2 + q^2 u[3] w_1 \otimes w_1 + q u^2 [4][3] w_2 \otimes w_0 \\
w_2 \otimes w_1 &\mapsto w_1 \otimes w_2 + u[3]! w_2 \otimes w_1 + q^{-3} u^2 [4][3]^2 w_3 \otimes w_0 \\
w_2 \otimes w_2 &\mapsto w_2 \otimes w_2 + q^{-2} u[3]^2 w_3 \otimes w_1 + q^{-7} u^2 [2]^{-1} [3]^2 [4]^2 w_4 \otimes w_0 \\
w_2 \otimes w_3 &\mapsto w_3 \otimes w_2 + q^{-4} u[3][4] w_4 \otimes w_1 \\
w_2 \otimes w_4 &\mapsto w_4 \otimes w_2 \\
w_3 \otimes w_0 &\mapsto q^{-4} w_0 \otimes w_3 + u[2] w_1 \otimes w_2 + q u^2 [3]! w_2 \otimes w_1 \\
&\quad + q^{-1} u^3 [4]! w_3 \otimes w_0
\end{aligned}
$$

$$
\begin{aligned}
w_3 \otimes w_1 &\mapsto q^{-2} w_1 \otimes w_3 + u[2]^2 w_2 \otimes w_2 + q^{-1} u^2 [3]![3] w_3 \otimes w_1 \\
&\quad + q^{-5} u^3 [4]![4] w_4 \otimes w_0 \\
w_3 \otimes w_2 &\mapsto w_2 \otimes w_3 + u[3]! w_3 \otimes w_2 + q^{-3} u^2 [3]^2 [4] w_4 \otimes w_1 \\
w_3 \otimes w_3 &\mapsto q^2 w_3 \otimes w_3 + u[2][4] w_4 \otimes w_2 \\
w_3 \otimes w_4 &\mapsto q^4 w_4 \otimes w_3 \\
w_4 \otimes w_0 &\mapsto q^{-8} w_0 \otimes w_4 + q^{-2} u w_1 \otimes w_3 + q u^2 [2] w_2 \otimes w_2 \\
&\quad + q u^3 [3]! w_3 \otimes w_1 + q^{-2} u^4 [4]! w_4 \otimes w_0 \\
w_4 \otimes w_1 &\mapsto q^{-4} w_1 \otimes w_4 + u[2] w_2 \otimes w_3 + q u^2 [3]! w_3 \otimes w_2 \\
&\quad + q^{-1} u^3 [4]! w_4 \otimes w_1 \\
w_4 \otimes w_2 &\mapsto w_2 \otimes w_4 + q^2 u[3] w_3 \otimes w_3 + q u^2 [3][4] w_4 \otimes w_2 \\
w_4 \otimes w_3 &\mapsto q^4 w_3 \otimes w_4 + q^4 u[4] w_4 \otimes w_3 \\
w_4 \otimes w_4 &\mapsto q^8 w_4 \otimes w_4
\end{aligned}
$$

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