Power variation of some infinitely divisible random fields

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joint work in progress with
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Let $X := \{X(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbb{R}^d}$ be a random field with integral representation

$$\{X(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbb{R}^d} \overset{\text{fdd}}{=} \left\{ \int_{\mathbb{R}^d} g(\boldsymbol{t},\boldsymbol{u}) L(\mathrm{d}\boldsymbol{u}) \right\}_{\boldsymbol{t}\in\mathbb{R}^d},$$

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where

- $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is measurable,
- L is an infinitely divisible random measure on $\mathbb{R}^d$ such that for every Borel subset B of $\mathbb{R}^d$ with finite Lebesgue measure $\lambda_d(B) < \infty$,

$$L(B) \sim (0, 0, \lambda_d(B)\nu),$$

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where for some $\theta \in (0,2]$,

- $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is measurable with $g(\boldsymbol{t},\cdot) \in L^\theta(\mathbb{R}^d)$ bounded, $\boldsymbol{t} \in \mathbb{R}^d$.
- L is an infinitely divisible random measure on $\mathbb{R}^d$ such that for every Borel subset B of $\mathbb{R}^d$ with finite Lebesgue measure $\lambda_d(B) < \infty$,

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  Moreover, $\limsup_{y\to\infty} \nu((-y,y)^c) y^\theta < \infty$ if $\theta < 2$, and $\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) < \infty$ if $\theta = 2$.

In (1) let

$$g(\boldsymbol{t},\boldsymbol{u}) := \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^d} (-1)^{d+\|\boldsymbol{\varepsilon}\|_1} g_{\boldsymbol{\varepsilon}}(\varepsilon_1 t_1 - u_1, \ldots, \varepsilon_d t_d - u_d), \quad \boldsymbol{t},\boldsymbol{u} \in \mathbb{R}^d,$$

where every $g_{\boldsymbol{\varepsilon}} : \mathbb{R}^d \to \mathbb{R}$ is a measurable function, $\boldsymbol{\varepsilon} \in \{0,1\}^d$.

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- X has stationary increments, i.e. for any fixed $\boldsymbol{s} \in \mathbb{R}^d$,

  $$\{X([\boldsymbol{s},\boldsymbol{t}))\}_{\boldsymbol{s}<\boldsymbol{t}} \overset{\text{fdd}}{=} \{X([\boldsymbol{0},\boldsymbol{t}-\boldsymbol{s}))\}_{\boldsymbol{s}<\boldsymbol{t}},$$

  where the increment of X over $[\boldsymbol{s},\boldsymbol{t}) = [s_1,t_1) \times \cdots \times [s_d,t_d) \subset \mathbb{R}^d$ is defined by

  $$X([\boldsymbol{s},\boldsymbol{t})) := \Delta^{(1)}_{t_1-s_1} \ldots \Delta^{(d)}_{t_d-s_d} X(\boldsymbol{s})$$

  with $\Delta^{(i)}_{t_i-s_i} X(\boldsymbol{s}) := X(\boldsymbol{s} + (t_i - s_i)\boldsymbol{e}_i) - X(\boldsymbol{s})$, $i = 1,\ldots,d$, and $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d\}$ being the standard basis for $\mathbb{R}^d$.

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Set $g := g_{(1,\dots,1)}$.

Let

$$g(\boldsymbol{u}) \sim h(\boldsymbol{u}), \quad \boldsymbol{u} \to \boldsymbol{0},$$

where

- $h(\boldsymbol{u}) := \|\boldsymbol{u}\|^{d\alpha}$ for some $\alpha > 0$ with $\|\boldsymbol{u}\| := (u_1^2 + \cdots + u_d^2)^{1/2}$,
- $h(\boldsymbol{u}) := \prod_{i=1}^{d} |u_i|^{\alpha_i}$ for some $\alpha_1, \ldots, \alpha_d > 0$.

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For $p > 0$, we define the power variation of $\{X(\boldsymbol{t})\}_{\boldsymbol{t} \in [0,1]^d}$ by

$$V_n^{(p)} := \sum_{0 \le i_1, \ldots, i_d < n} \left| X\left([\tfrac{i_1}{n}, \tfrac{i_1+1}{n}) \times \cdots \times [\tfrac{i_d}{n}, \tfrac{i_d+1}{n}) \right) \right|^p.$$

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In the following we will study the asymptotic behavior of $V_n^{(p)}$ as $n \to \infty$.

We will see that it depends on the interplay between p, β and the behavior of g at $\boldsymbol{0}$.

For $p > 0$, consider

$$V_n^{(p)} := \sum_{0 \le i < n} |X((i+1)/n) - X(i/n)|^p$$

of

$$X(t) := \int_{\mathbb{R}} (g(t-u) - g_0(-u)) \mathrm{d}L(u), \quad t \in [0,1],$$

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Theorem (Basse-O'Connor, Lachièze-Rey and Podolskij 2017)

(i) If $\alpha + 1/\max(\beta,p) < 1$, $p > \beta$, then $n^{\alpha p} V_n^{(p)} \overset{\mathcal{F}\text{-d}}{\to} \sum_{m:T_m \in [0,1]} |\Delta L(T_m)|^p \sum_{j \in \mathbb{Z}} |(j+1-U_m)_+^\alpha - (j-U_m)_+^\alpha|^p$, where $\{T_m\}_{m \geq 1}$ is a sequence of stopping times exhausting the jumps of L and $\{U_m\}_{m \geq 1}$ is a sequence of independent, $\mathcal{U}([0,1])$-distributed random variables, defined on an extension of the original $(\Omega, \mathcal{F}, \mathbb{P})$, independent of L.

(ii) If $\alpha + 1/\max(\beta,p) < 1$, $p < \beta$ and L is symmetric β-stable, $\theta = \beta$, then $n^{p(\alpha+1/\beta)-1} V_n^{(p)} \overset{\mathbb{P}}{\to} m_p$, where $m_p > 0$ is a finite constant.

(iii) If $\alpha + 1/\max(\beta,p) > 1$, $p \geq 1$, then $n^{p-1} V_n^{(p)} \overset{\mathbb{P}}{\to} \int_0^1 |X'(t)|^p \mathrm{d}t$ with $X'(t) := \int_{-\infty}^t g'(t-u) \mathrm{d}L(u)$.

The linear fractional stable motion

$$X(t) := \int_{\mathbb{R}} \left((t-u)_+^{H-1/\beta} - (-u)_+^{H-1/\beta} \right) \mathrm{d}L(u), \quad t \in \mathbb{R},$$

where L is symmetric β-stable with scale parameter $\sigma > 0$, $\beta \in (0,2)$, $H \in (0,1)$.

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- Its extension is the linear fractional stable sheet, where L is symmetric β-stable random measure on $\mathbb{R}^d$ with control measure λ_d and

$$g(\boldsymbol{t},\boldsymbol{u}) = \prod_{i=1}^{d} \Big((t_i - u_i)_+^{H_i-1/\beta} - (-u_i)_+^{H_i-1/\beta}\Big), \quad \boldsymbol{t},\boldsymbol{u} \in \mathbb{R}^d.$$

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Asymptotics of $V_n^{(p)}$ for $d > 1$: Benassi, Cohen and Istas 2004, Pakkanen 2014, Pakkanen and Réveillac 2016, Diu Tran 2019+, ...

(A1) Let $g(\boldsymbol{u}) = f(\boldsymbol{u})h(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{R}^d$, where

$$h(\boldsymbol{u}) := \|\boldsymbol{u}\|^{d\alpha} \quad \text{for some } \alpha > 0$$

and $f(\mathbf{0}) = 1$, $f \in C^d(\mathbb{R}^d)$. There exists $\rho > 0$ such that $\partial^d g := \frac{\partial^d g}{\partial u_1 \ldots \partial u_d}$ satisfies $\int_{\|\boldsymbol{u}\| \geq \rho} |\partial^d g(\boldsymbol{u})|^\theta \mathrm{d}\boldsymbol{u} < \infty$ and $|\partial^d g(\boldsymbol{u})| \geq |\partial^d g(\boldsymbol{v})|$ if $\rho \leq \|\boldsymbol{u}\| \leq \|\boldsymbol{v}\|$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$.

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Theorem 1 (Basse-O'Connor, P. and Podolskij 2019+)

Assume (A1). *Set* $\Delta_1 h(\boldsymbol{u}) := \Delta_1^{(1)} \ldots \Delta_1^{(d)} h(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{R}^d$.

(i) *Let* $p > \beta$ *and* $\alpha + 1/p < 1$. *Then*

$$n^{d\alpha p} V_n^{(p)} \stackrel{\mathcal{F}\text{-d}}{\to} \int_{[0,1]^d \times \mathbb{R}} \left(|y|^p \sum_{\boldsymbol{j} \in \mathbb{Z}^d} |\Delta_1 h(\boldsymbol{j} - \boldsymbol{u})|^p \right) N(\mathrm{d}\boldsymbol{u}, \mathrm{d}y)$$

where N *is a Poisson random measure on* $[0,1]^d \times \mathbb{R}$ *having intensity measure* $\lambda_d \otimes \nu$.

Theorem 1 (cnt)

(ii) *Let L be a symmetric β-stable random measure on $\mathbb{R}^d$ with control measure λ_d. Let $p < \beta = \theta$ and $H := \alpha + 1/\beta < 1$. Then*

$$n^{d(pH-1)} V_n^{(p)} \xrightarrow{\mathbb{P}} \mathbb{E}|L([0,1]^d)|^p \Big(\int_{\mathbb{R}^d} |\Delta_1 h(\boldsymbol{u})|^\beta \, \mathrm{d}\boldsymbol{u} \Big)^{p/\beta}.$$

(iii) *Let $p \geq 1$ and $\alpha + 1/\max(\beta, p) > 1$. Then*

$$n^{d(p-1)} V_n^{(p)} \xrightarrow{\mathbb{P}} \int_{[0,1]^d} |\partial^d X(\boldsymbol{t})|^p \mathrm{d}\boldsymbol{t},$$

where $\{\partial^d X(\boldsymbol{t})\}_{\boldsymbol{t}\in[0,1]^d}$ is a measurable random field satisfying

$$\int_{[0,1]^d} |\partial^d X(\boldsymbol{t})|^p \mathrm{d}\boldsymbol{t} < \infty \text{ a.s.}$$

and having an integral representation

$$\{\partial^d X(\boldsymbol{t})\}_{\boldsymbol{t}\in[0,1]^d} \overset{\text{fdd}}{=} \Big\{ \int_{\mathbb{R}^d} \partial^d g(\boldsymbol{t} - \boldsymbol{u}) L(\mathrm{d}\boldsymbol{u}) \Big\}_{\boldsymbol{t}\in[0,1]^d}.$$

In the following let $d = 2$.

(A2) Let $g(\boldsymbol{u}) = \prod_{i=1}^{2} g_i(u_i)$, $\boldsymbol{u} \in \mathbb{R}^2$, where $g_i(u) = f_i(u)h_i(u)$, $u \in \mathbb{R}$, with

$$h_i(u) := |u|^{\alpha_i} \quad \text{for some } \alpha_i > 0,$$

and $f_i \in C^1(\mathbb{R})$ such that $f_i(0) = 1$, $i = 1, 2$. There exists $\rho > 0$ such that $\int_{|u|\geq\rho} |g_i'(u)|^q \mathrm{d}u < \infty$ with $q := \min(\theta, \max(\beta, p))$ and $|g_i'(u)| \geq |g_i'(v)|$ if $\rho \leq |u| \leq |v|$, $u, v \in \mathbb{R}$, $i = 1, 2$.

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For $[\boldsymbol{u}, \boldsymbol{v}) \subset \mathbb{R}^2$, we have that $g([\boldsymbol{u}, \boldsymbol{v})) = \prod_{i=1}^{2}(g_i(v_i) - g_i(u_i))$.

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Theorem 2 (Basse-O'Connor, P. and Podolskij 2019+)

Assume (A2). *Set* $\Delta_1 h_i(u) := h_i(u+1) - h_i(u)$, $u \in \mathbb{R}$, $i = 1, 2$.

(i) *Let* $p > \beta$ *and* $\max(\alpha_1, \alpha_2) + 1/p < 1$. *Moreover, let* $p \neq \theta$ *if* $\theta < 2$. *Then*

$$n^{(\alpha_1+\alpha_2)p} V_n^{(p)} \xrightarrow{\mathcal{F}\text{-d}} \int_{[0,1]^2\times\mathbb{R}} \left(|y|^p \prod_{i=1}^{2} \sum_{j\in\mathbb{Z}} |\Delta_1 h_i(j - u_i)|^p \right) N(\mathrm{d}\boldsymbol{u}, \mathrm{d}y)$$

where N *is a Poisson random measure on* $[0,1]^2 \times \mathbb{R}$ *having intensity measure* $\lambda_2 \otimes \nu$.

Theorem 2 (cnt)

(ii) *Let L be a symmetric β-stable random measure on $\mathbb{R}^2$ with control measure λ_2. Let $p < \beta = \theta$ and $\max(\alpha_1, \alpha_2) + 1/\beta < 1$. Set $H_i := \alpha_i + 1/\beta$, $i = 1, 2$. Then*

$$n^{p(H_1+H_2)-2} V_n^{(p)} \xrightarrow{\mathbb{P}} \mathbb{E}|L([0,1]^2)|^p \prod_{i=1}^{2} \left(\int_{\mathbb{R}} |\Delta_1 h_i(u)|^\beta \, \mathrm{d}u \right)^{p/\beta}.$$

(iii) *Let $p \geq 1$ and $\min(\alpha_1, \alpha_2) + 1/\max(\beta, p) > 1$. Moreover, let $p \neq \theta$ if $\theta < 2$. Then*

$$n^{2(p-1)} V_n(p) \xrightarrow{\mathbb{P}} \int_{[0,1]^2} |\partial^2 X(\boldsymbol{t})|^p \mathrm{d}\boldsymbol{t},$$

where $\{\partial^2 X(\boldsymbol{t})\}_{\boldsymbol{t}\in[0,1]^2}$ is a measurable random field satisfying

$$\int_{[0,1]^2} |\partial^2 X(\boldsymbol{t})|^p \mathrm{d}\boldsymbol{t} < \infty \text{ a.s.}$$

and having an integral representation

$$\{\partial^2 X(\boldsymbol{t})\}_{\boldsymbol{t}\in[0,1]^2} \overset{\text{fdd}}{=} \left\{ \int_{\mathbb{R}^2} \prod_{i=1}^{2} g_i'(t_i - u_i) L(\mathrm{d}\boldsymbol{u}) \right\}_{\boldsymbol{t}\in[0,1]^2}.$$

Theorem (cnt)

(iv) *Let L be a symmetric β-stable random measure on $\mathbb{R}^2$ with control measure λ_2. Let $p < \beta = \theta$ and $H_1 := \alpha_1 + 1/\beta < 1 < \alpha_2 + 1/\beta$. Then*

$$n^{(pH_1-1)+(p-1)} V_n^{(p)} \xrightarrow{\mathbb{P}} \mathbb{E}|L([0,1]^2)|^p \left(\int_{\mathbb{R}} |\Delta_1 h_1(u_1)|^\beta \mathrm{d}u_1 \int_{\mathbb{R}} |g_2'(u_2)|^\beta \mathrm{d}u_2 \right)^{p/\beta}.$$

Theorem (cnt)

(iv) *Let L be a symmetric β-stable random measure on $\mathbb{R}^2$ with control measure λ_2. Let $p < \beta = \theta$ and $H_1 := \alpha_1 + 1/\beta < 1 < \alpha_2 + 1/\beta$. Then*

$$n^{(pH_1-1)+(p-1)} V_n^{(p)} \xrightarrow{\mathbb{P}} \mathbb{E}|L([0,1]^2)|^p \left(\int_{\mathbb{R}} |\Delta_1 h_1(u_1)|^\beta \mathrm{d}u_1 \int_{\mathbb{R}} |g_2'(u_2)|^\beta \mathrm{d}u_2 \right)^{p/\beta}.$$

(v) *Conjecture: Let $p > \beta$ and $\alpha_1 + 1/p < 1 < \alpha_2 + 1/p$. Moreover, let $p \neq \theta$ if $\theta < 2$. Then*

$$n^{\alpha_1 p+(p-1)} V_n^{(p)} \xrightarrow{\mathcal{F}\text{-d}} \int_{[0,1]\times\mathbb{R}\times\mathbb{R}} \left(|y|^p \sum_{j\in\mathbb{Z}} |\Delta_1 h_1(j-u)|^p \int_0^1 |g_2'(t-x_2)|^p \mathrm{d}t \right) N(\mathrm{d}u, \mathrm{d}x_2, \mathrm{d}y),$$

where N is a Poisson random measure on $[0,1]\times\mathbb{R}\times\mathbb{R}$ having intensity measure $\lambda_1 \otimes \lambda_1 \otimes \nu$.

Other open questions:

- $\alpha < 0$ or $\alpha_i < 0$ for some $i = 1, \ldots, d$,
- tangent limits of X under (A1) if $d = 2$,
- 2nd order asymptotics of $V_n^{(p)}$ in some cases,
- 1st order asymptotics of $V_n^{(p)}$ for volatility modulated fields,
- ... statistical inference!

References

- Basse-O'Connor, A., Lachièze-Rey, R. and Podolskij, M. (2017). Power variation for a class of stationary increments Lévy driven moving averages. *Ann. Probab.* **45**, 4477–4528.
- Braverman, M. and Samorodnitsky, G. (1998). Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes. *Stochastic Process. Appl.* **78** 1–26.
- Rajput, B.S. and Rosinski, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82**, 451–487.
- Wang, Y. and Roy, P. and Stoev, S. (2013). Ergodic properties of sum- and max-stable stationary random fields via null and positive group actions, *Ann. Probab.* **41**, 206–228.

Thank you for your attention