High-frequency analysis of parabolic stochastic PDEs with multiplicative noise

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Stochastic heat equation with multiplicative noise

SHE with multiplicative noise:

$$
\begin{aligned}
\partial_t u(t,x) &= \tfrac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad && (t,x) \in (0,\infty)\times\mathbb{R}^d,\\
u(0,x) &= 1, && x \in \mathbb{R}^d.
\end{aligned}
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- $\dot{W}$ Gaussian noise: white in time, white/colored in space
- σ Globally Lipschitz function

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Noise:

$$\mathbb{E}[\dot{W}(t,x)] = 0, \quad \mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)F(x-y),$$

F Riesz kernel: $F(x) = |x|^{-\alpha}$ with some $\alpha \in (0,2)\cap(0,d]$

If $\alpha = d = 1$: $F(x) = \delta_0(x)$ (space-time white noise)

Some quick facts

Theorem (Dalang 1999)

There exists a unique **mild solution** u:

$$\forall (t,x):\ \ u(t,x) = 1 + \iint_0^t G(t-s, x-y)\sigma(u(s,y))\, W(\mathrm{d}s, \mathrm{d}y) \quad \text{a.s.}$$

satisfying, for all $p > 0$ and $T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}[|u(t,x)|^p] < \infty,$$

where G is the **heat kernel**

$$G(t,x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right).$$

Some quick facts

Theorem (Sanz-Solé & Sarrà 2002)

There exists a version of u which is

- $(\frac{1}{2} - \frac{\alpha}{4} - \epsilon)$-Hölder continuous in time and
- $(1 - \frac{\alpha}{2} - \epsilon)$-Hölder continuous in space.

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Remarks:

1. Regularity in time always **worse** than Brownian motion.
2. For fixed x, $t \mapsto u(t, x)$ is **not** a semimartingale.
3. For fixed x, $t \mapsto u(t, x)$ is locally more like a fractional Brownian motion with Hurst index $H = \frac{1}{2} - \frac{\alpha}{4}$.

Problem formulation

Goal: Study normalized power variations

$$V_p^n(u,t) := \frac{1}{n}\sum_{i=1}^{[nt]}\left|\frac{u(\frac{i}{n},x) - u(\frac{i-1}{n},x)}{\tau_n}\right|^p, \qquad t \in (0,\infty)$$

Here:

- x fixed (e.g., $x = 0$) and $p > 0$;
- $\tau_n = \tau_n(\alpha)$ is a normalizing factor

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  $\to$ Direct calculation: $\tau_n = C_\alpha n^{-(\frac{1}{2}-\frac{\alpha}{4})}$.

Motivation

Application to statistics for SPDEs:

$$\partial_t u(t,x) = \tfrac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d,$$
$$u(0,x) = 1, \qquad x \in \mathbb{R}^d.$$

Case 1 (additive noise): $\sigma(x) = \sigma_0$ with $\sigma_0 > 0$

Case 2 (Parabolic Anderson model): $\sigma(x) = \sigma_0 x$ with $\sigma_0 > 0$

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Data: High-frequency observations of the solution at fixed x:

$$u(\tfrac{1}{n},x), u(\tfrac{2}{n},x), \ldots, u(\tfrac{[nT]}{n},x)$$

Question: Can we estimate σ_0 based on these observations?

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Rule of thumb:

Consistent estimator for σ_0 $\iff$ Law of large numbers for $V_p^n(u,t)$

Confidence bounds for σ_0 $\iff$ Central limit theorem for $V_p^n(u,t)$

Law of large numbers

Theorem (Law of large numbers; C. 2019a)

For every $p > 0$,

$$V_p^n(u,t) \stackrel{L^1}{\Longrightarrow} V_p(u,t) = \mu_p \int_0^t |\sigma(u(s,x))|^p \,\mathrm{d}s,$$

where $\mu_p = \mathbb{E}[|X|^p]$ for $X \sim N(0,1)$.

Special cases:

Pospíšil & Tribe 2007, Swanson 2007, Foondun & Khoshnevisan 2014, Bibinger & Trabs 2019, Cialenco & Huang 2019

Do we also have a central limit theorem?

Central limit theorem

Consider modified SHE:

$$\begin{aligned} \partial_t v(t,x) &= \tfrac{1}{2}\Delta v(t,x) + \sigma(t,x)\dot{W}(t,x), &\quad (t,x) &\in (0,\infty)\times\mathbb{R}^d, \\ v(0,x) &= 1, & x &\in \mathbb{R}^d, \end{aligned}$$

where $\sigma(t,x)$ is a general L^2-bounded predictable random field.

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where $\sigma(t,x)$ is a general L^2-bounded predictable random field.

Theorem (Central limit theorem, C. 2019a):

Let $p = 2$ or $p \geq 4$. Then, under additional hypotheses on σ,

$$\sqrt{n}(V_p^n(v,t) - V_p(v,t)) \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{Z}_t = c_{p,\alpha} \int_0^t |\sigma(s,x)|^p \,\mathrm{d}B_s$$

where $c_{p,\alpha}$ is an explicit constant and B is a Brownian motion that is independent of W and σ.

Special cases ($\alpha = d = 1$, $\sigma \equiv 1$ and $p \in \{2,4\}$):

Bibinger & Trabs 2019, Cialenco & Huang 2019

Restrictions

What are the "additional hypotheses"?

Among other things, we require $\sigma(t, x)$ be (essentially)

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Unfortunately:

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Even worse:

- The $\frac{1}{2}$-Hölder in time condition of $\sigma(t,x)$ is essentially optimal!
- Such an assumption is common in high-frequency analysis of
  - Semimartingales: Jacod & Protter 2012, Aït-Sahalia & Jacod 2014
  - Moving averages: Barndorff-Nielsen, Corcuera & Podolskij 2011, Corcuera, Hedevang, Pakkanen & Podolskij 2013

Sketch of the ideas of the proof

Semimartingale: $u(t) = \int_0^t \sigma(s)\,\mathrm{d}B(s)$ (σ random process)

- $\Delta_i^n u := u(\frac{i}{n}) - u(\frac{i-1}{n})$
- $\tau_n := 1/\sqrt{n}$

$$\frac{\Delta_i^n u}{\tau_n} = \sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sigma(s)\,\mathrm{d}B(s) \approx \sigma(\tfrac{i-1}{n}) \underbrace{(\sqrt{n}\,\Delta_i^n B)}_{\overset{\text{iid}}{\sim} N(0,1)}$$

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Conclusion: Increments are conditionally independent as $n \to \infty$!

Sketch of the ideas of the proof

SHE:

$$v(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y)$$

where G is the heat kernel.

Increments:

$$\Delta_i^n v = \iint_{\frac{i-1}{n}}^{\frac{i}{n}} G(\tfrac{i}{n} - s, x-y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y)$$
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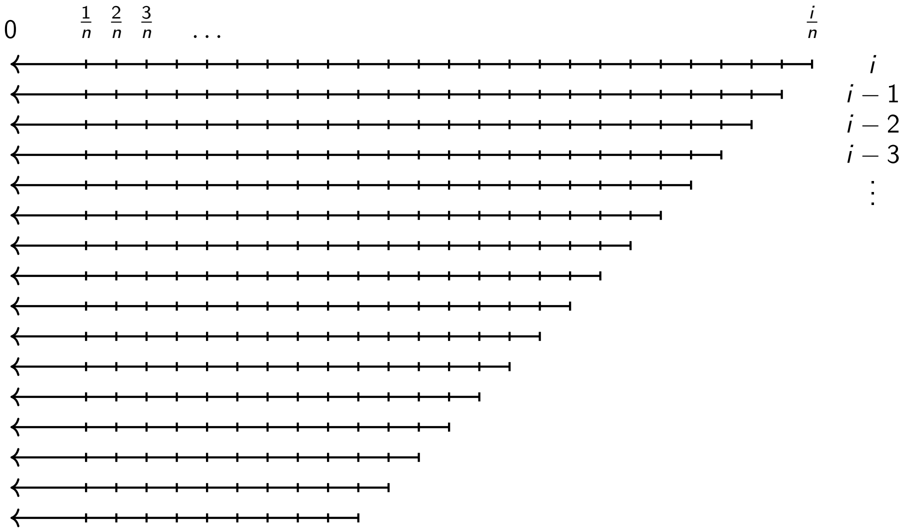
Conclusion: Increments not conditionally independent as $n \to \infty$!

First step: Restoring conditional independence



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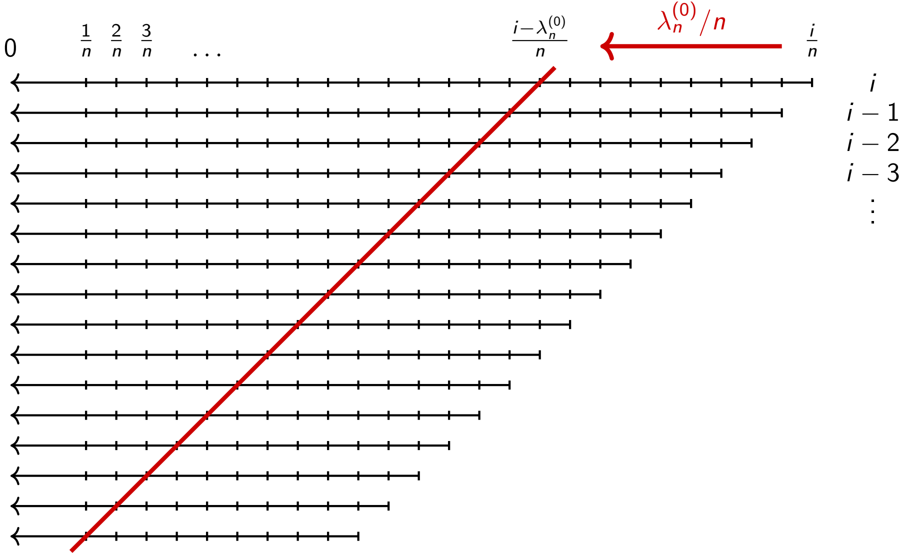

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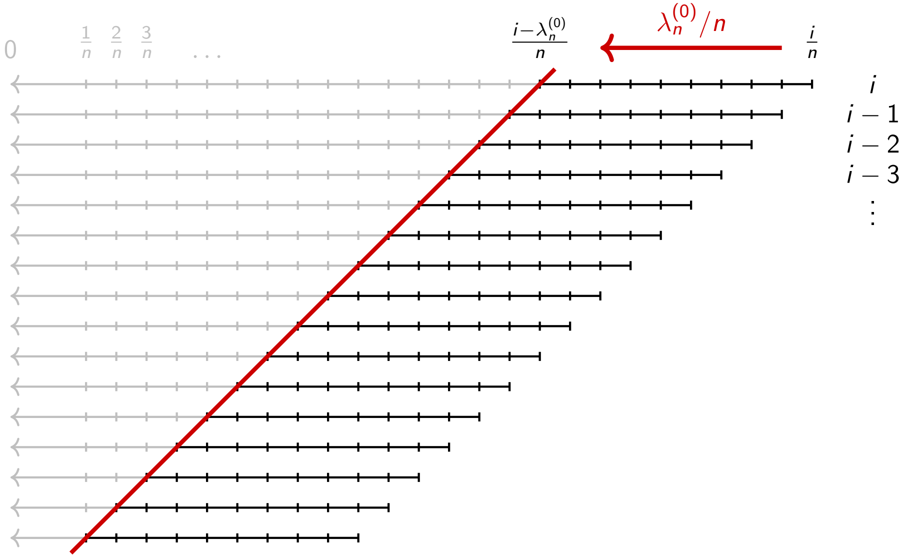

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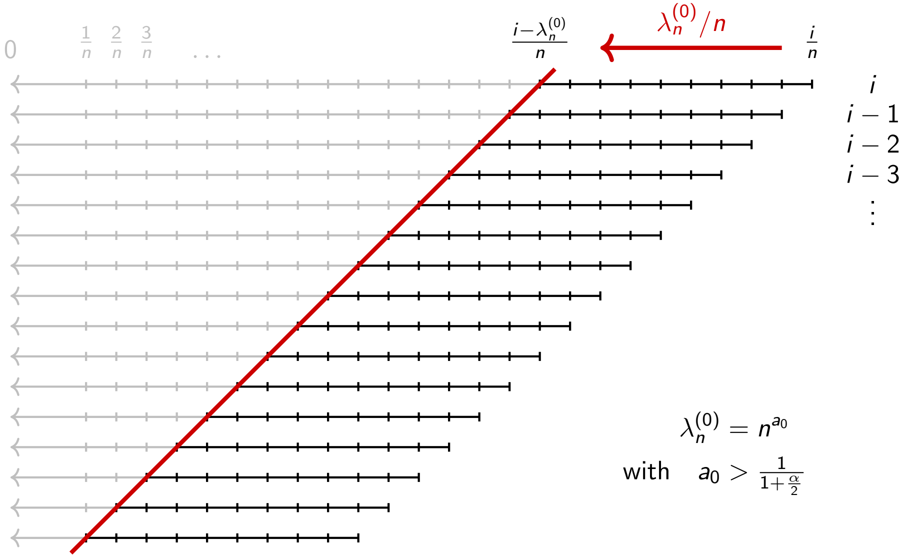

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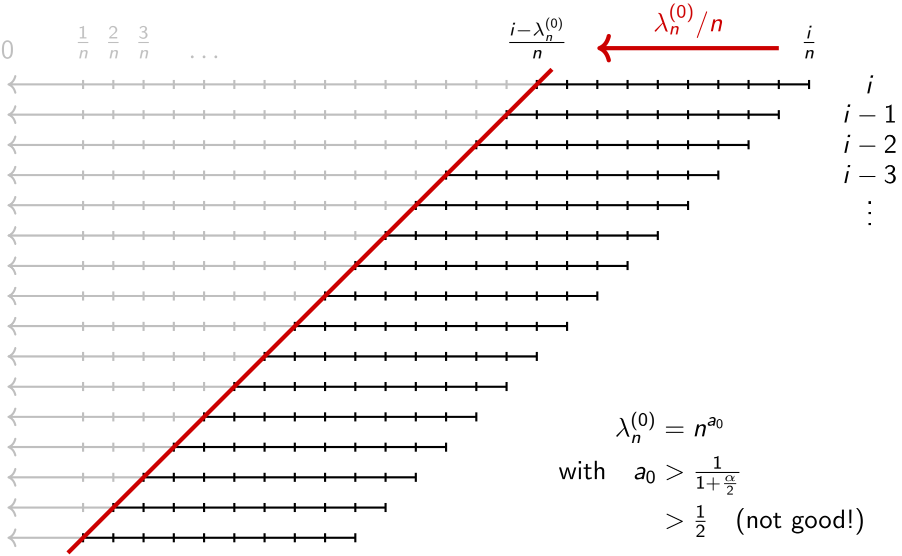

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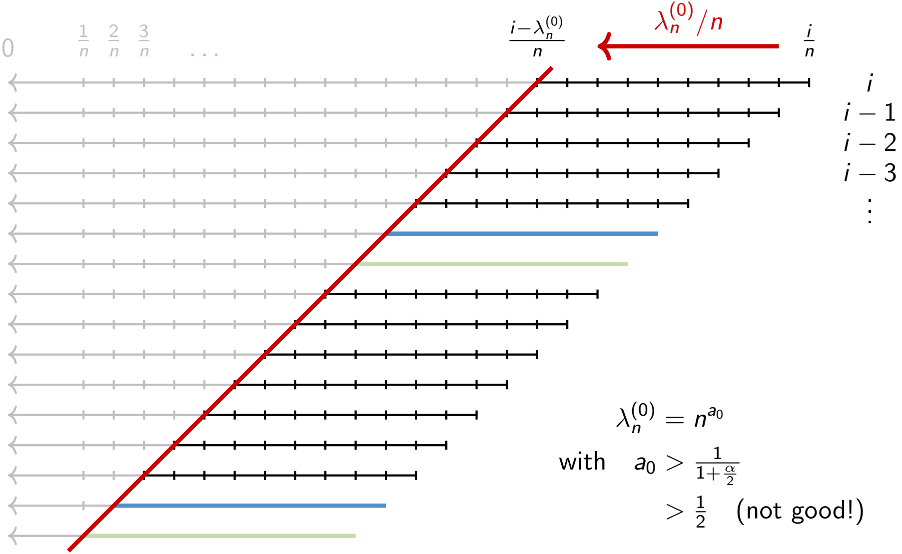

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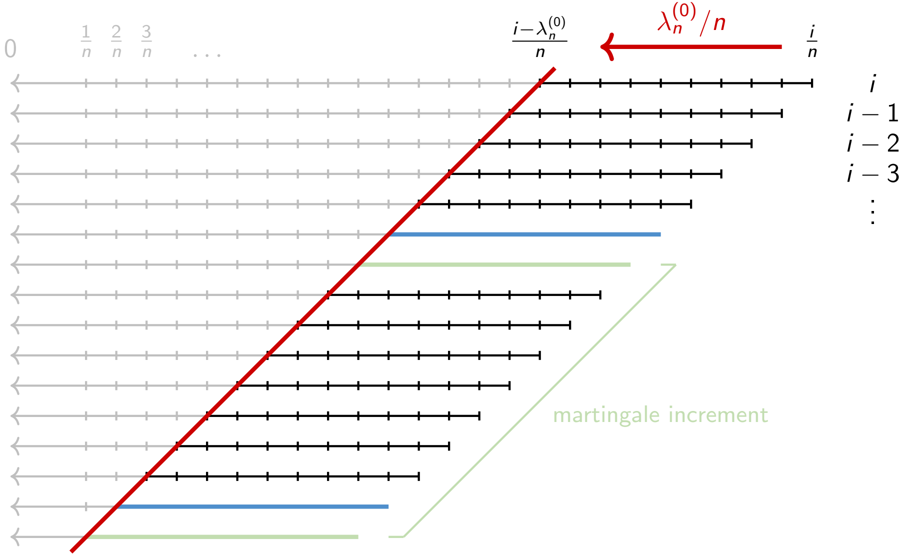

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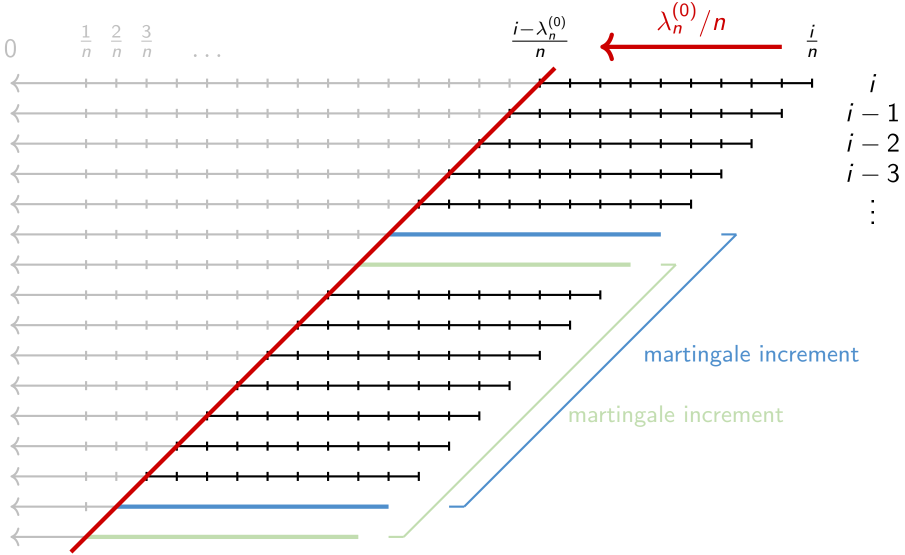

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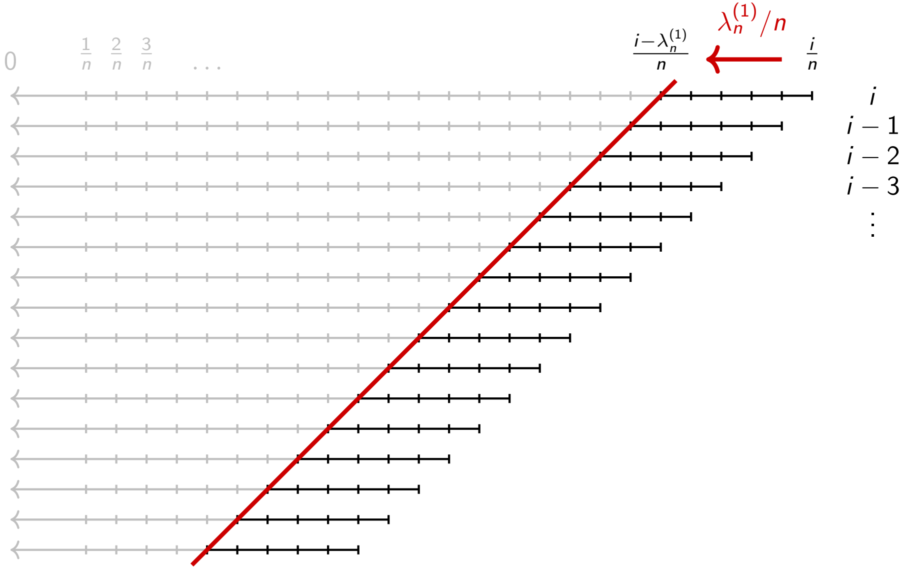

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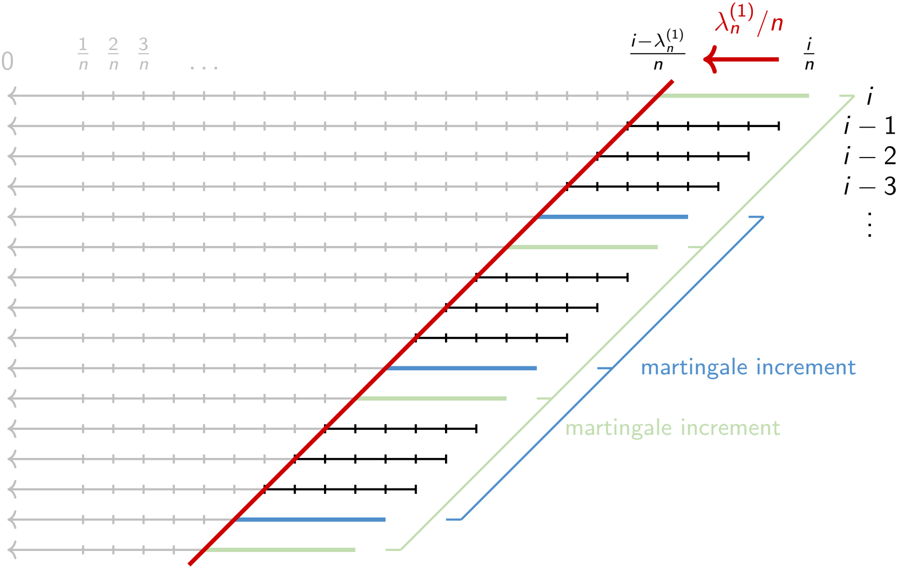

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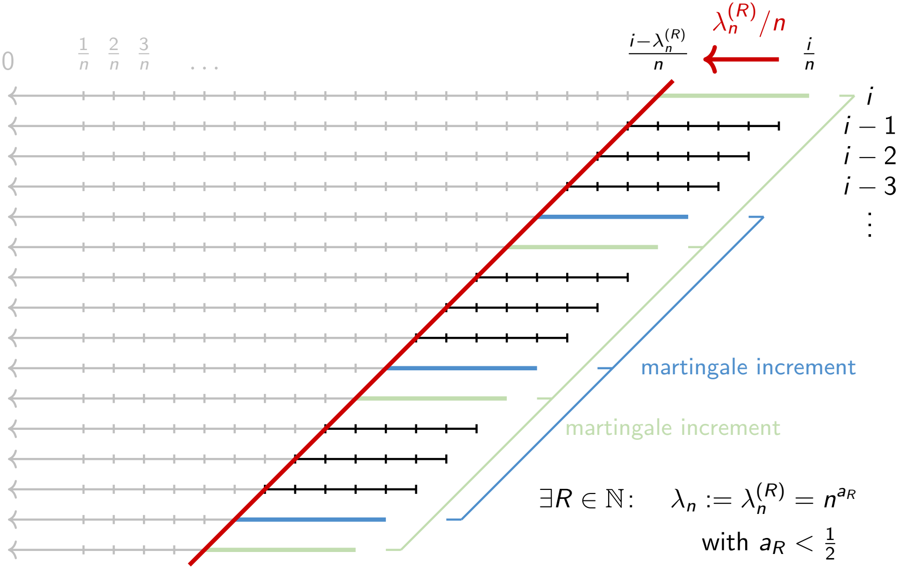
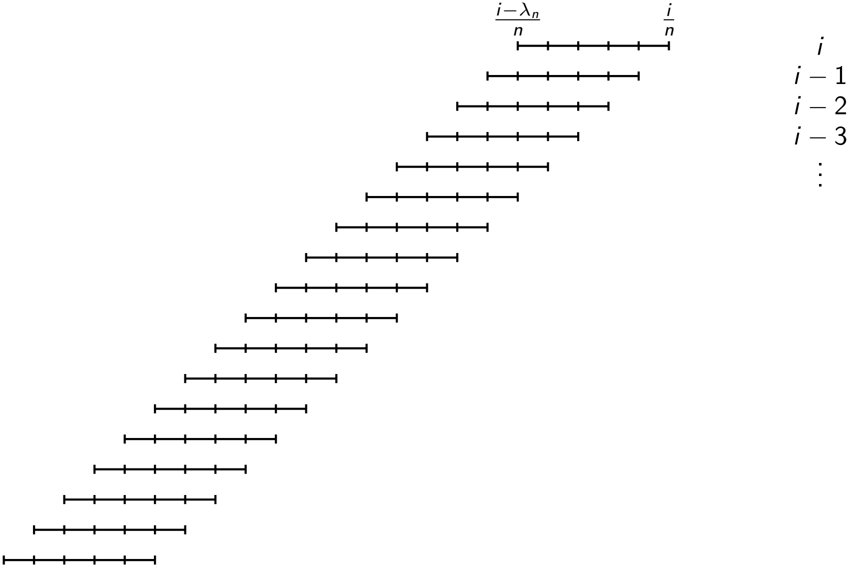


$\exists R \in \mathbb{N}: \quad \lambda_n := \lambda_n^{(R)} = n^{a_R}$

with $a_R < \frac{1}{2}$

Second step: Block splitting and CLT



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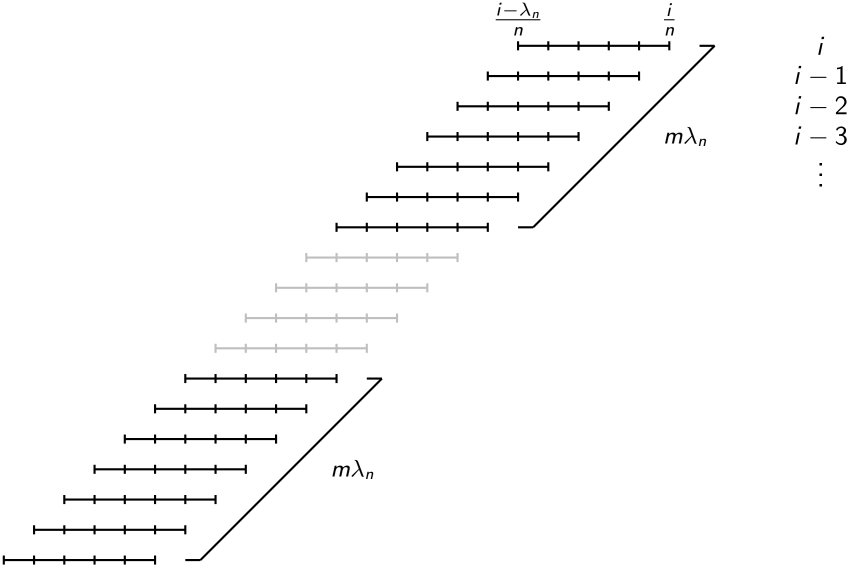

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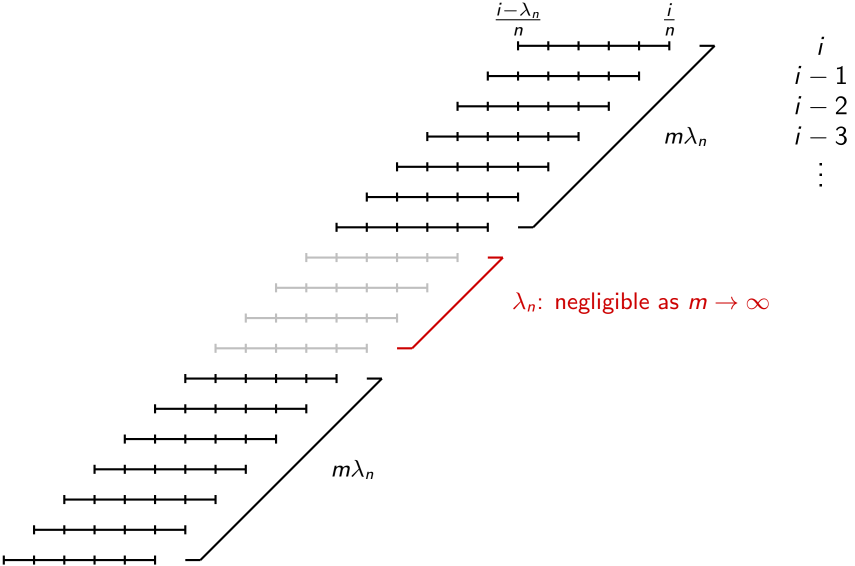

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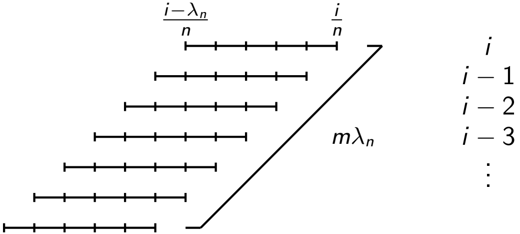
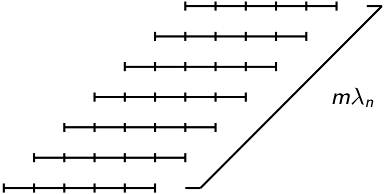
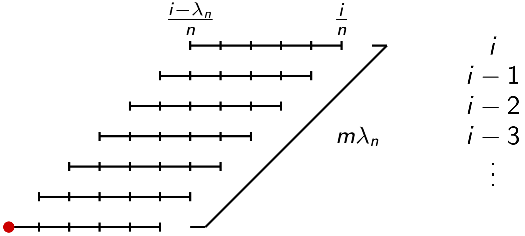


Truncated increment:

$$\iint_{\frac{i-\lambda_n}{n}}^{\frac{i}{n}} \left(G\left(\tfrac{i}{n}-s, x-y\right) - G\left(\tfrac{i-1}{n}-s, x-y\right)\right) \sigma(s,y)\, W(\mathrm{d}s,\mathrm{d}y)$$





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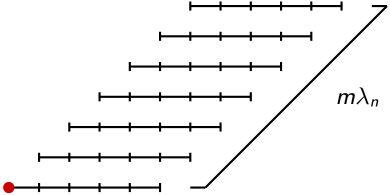
Replace by

$$\iint_{\frac{i-\lambda_n}{n}}^{\frac{i}{n}} \left(G\left(\tfrac{i}{n}-s, x-y\right) - G\left(\tfrac{i-1}{n}-s, x-y\right)\right) \sigma(\bullet,y)\, W(\mathrm{d}s, \mathrm{d}y)$$



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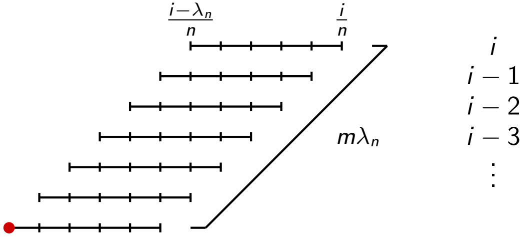

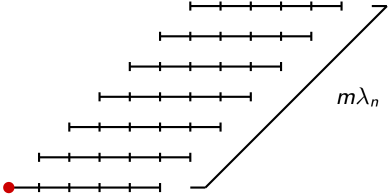

Define:

$$\widetilde{\Delta_i^n v} := \iint_{\frac{i-\lambda_n}{n}}^{\frac{i}{n}} \big(G(\tfrac{i}{n}-s, x-y) - G(\tfrac{i-1}{n}-s, x-y)\big)\sigma(\bullet, y)\, W(\mathrm{d}s, \mathrm{d}y)$$

Martingale CLT

$$\frac{1}{\sqrt{n}} \sum_i \left\{ \left| \frac{\widetilde{\Delta_i^n v}}{\tau_n} \right|^p - \mathbb{E}\left[\left| \frac{\widetilde{\Delta_i^n v}}{\tau_n} \right|^p \,\middle|\, \mathcal{F}_\bullet \right] \right\} \xrightarrow{\mathrm{d}} \mathcal{Z}$$

Third step: Remove discretization

Last step:

$$\sqrt{n}\left(\frac{1}{n}\sum_i \mathbb{E}\left[\left|\frac{\widetilde{\Delta}_i^n v}{\tau_n}\right|^p \,\middle|\, \mathcal{F}_\bullet\right] - V_p(v,t)\right) \xrightarrow{L^1} 0$$

Back to our original question

Why does $\sigma(t,x)$ have to be $\frac{1}{2}$-Hölder?

- **Step 2**:

$$\Delta_i^n v - \widetilde{\Delta}_i^n v = \iint_{\frac{i-\lambda_n}{n}}^{\frac{i}{n}} \left(G(\tfrac{i}{n}-s, x-y) - G(\tfrac{i-1}{n}-s, x-y)\right) \times \underbrace{(\sigma(s,y)-\sigma(\bullet,y))}_{=O((\lambda_n/n)^{\gamma})} W(\mathrm{d}s,\mathrm{d}y),$$

  where γ is the temporal Hölder index of $\sigma(t,x)$.

  This is $o(\frac{1}{\sqrt{n}})$ only if $\gamma > \frac{1}{2}$ (and λ_n grows very slowly)!

- **Step 3**: Same issue when we go back from $\sigma(\bullet,y)$ to $\sigma(s,y)$

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- **Step 3**: Same issue when we go back from $\sigma(\bullet, y)$ to $\sigma(s,y)$

Conclusion:

There seems to be no hope for the multiplicative case!

CLT for multiplicative case

Theorem (C. 2019b)

Assume:

- $p = 2$ or $p \geq 4$;
- $\sigma \colon \mathbb{R} \to \mathbb{R}$ is C^4 with all derivatives at most of polynomial growth;
- $\alpha \in (0, 1)$.

If u denotes the solution to

$$\begin{aligned} \partial_t u(t,x) &= \tfrac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), && (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= 1, && x \in \mathbb{R}^d. \end{aligned}$$

then

$$\sqrt{n}(V_p^n(u,t) - V_p(u,t)) \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{Z}_t = c_{p,\alpha} \int_0^t |\sigma(u(s,x))|^p \, \mathrm{d}B_s,$$

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Then

$$\sqrt{n}(V_p^n(u,t) - V_p(u,t)) \xrightarrow{\mathrm{d}} \mathcal{Z}_t = A_t + c_{p,\alpha} \int_0^t |\sigma(u(s,x))|^p \,\mathrm{d}B_s,$$

where A_t is a finite variation process, adapted to W, and

$$A \equiv 0 \quad \iff \quad p = 2 \quad \text{or} \quad \sigma \equiv \text{const. (i.e., additive noise).}$$

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$$\sqrt{n}(V_p^n(u,t) - V_p(u,t)) \xrightarrow{\mathrm{d}} \mathcal{Z}_t = A_t + c_{p,\alpha} \int_0^t |\sigma(u(s,x))|^p \, \mathrm{d}B_s,$$

where A_t is a finite variation process, adapted to W, and

$$A \equiv 0 \quad \Longleftrightarrow \quad p = 2 \quad \text{or} \quad \sigma \equiv \text{const. (i.e., additive noise).}$$

In statistical terms: If $A \not\equiv 0$, we have an asymptotic bias!

Why?

- **Step 2**:

$$\Delta_i^n u - \widetilde{\Delta}_i^n u = \iint_{\frac{i-\lambda_n}{n}}^{\frac{i}{n}} \left(G(\tfrac{i}{n} - s, x - y) - G(\tfrac{i-1}{n} - s, x - y)\right) \times \underbrace{(\sigma(s,y) - \sigma(\bullet, y))}_{=O((\lambda_n/n)^{\gamma})} W(\mathrm{d}s, \mathrm{d}y),$$

This is $o(\frac{1}{\sqrt{n}})$ only if $\gamma > \frac{1}{2}$ (and λ_n grows very slowly)!

- **Step 3**: Same issue when we go back from $\sigma(\bullet, y)$ to $\sigma(s, y)$

Crucial observation:

- If $\alpha \in (0,1)$, there is **sum** between terms from Step 2 + 3 (difference is $o(\frac{1}{\sqrt{n}})$).
- If $\alpha = 1$, there is **not (enough) cancellation**!

Summary

1. LLN valid for all α and for additive/multiplicative noise;
2. CLT in the case of additive noise: valid for all α;
3. CLT in the case of multiplicative noise:
   - valid for $\alpha \in (0, 1)$;
   - valid for $\alpha = 1$ if $p = 2$; bias term if $p \neq 2$;
   - open if $\alpha \in (1, 2)$!

Selected references

Main references:

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