Generalized Itô-Nisio theorem and modes of convergence in series expansions of infinitely divisible processes

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Gaussian processes

- **Gaussian processes** play a key role in many areas

  ↙ statistical modelling   ↘ machine-learning algorithms

- The **Gaussian** assumption is often verified through the **Central Limit Theorem**:

  If (Z_k) *are i.i.d. with* $\mathbb{E}[Z_k] = 0$, $\mathbb{E}[Z_k^2] < \infty$, *then*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Z_k \xrightarrow{d} \mathcal{N}(0, \sigma^2) \qquad \text{as } n \to \infty.$$

Karhunen-Loéve representation:

- *Let* $\mathbf{X} = \{X(t) : t \in T\}$ *be a Gaussian process with continuous sample paths (*T *is a compact metric space).*
- *Let* $(e_k)_{k\in\mathbb{N}}$ *be an orthonormal basis of the reproducing Hilbert kernel space of* $H(R)$ *of* X*, and* $Z_k \sim N(0,1)$ *i.i.d.*

Then,

$$X(t) = \sum_{k=1}^{\infty} e_k(t) Z_k$$ *convergence a.s. in the sup-norm*

↙ ↘

Simulation (truncated series)

Dimension reduction (principal component analysis)

Infinitely divisible random variables

- **General Central Limit Theorem:**

  1. *If $(Z_{k,n})_{k,n\in\mathbb{N}}$ is a triangular array satisfying the UAN condition and $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that*

  $$X := \lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} Z_{k,n} \qquad \textit{exists in distribution,} \tag{1}$$

  then X is **infinitely divisible**.

  2. *Conversely, if X is* **infinitely divisible** *then there exists a triangular array $(Z_{k,n})_{k,n\in\mathbb{N}}$ such that X is given by (1).*

- **Definition:** A random variable X in $\mathbb{R}^d$ is **infinitely divisible** if for all $n \in \mathbb{N}$ there exists a random variable Z_n such that

$$\mathbb{E}[e^{i\langle\theta,X\rangle}] = \mathbb{E}[e^{i\langle\theta,Z_n\rangle}]^n \qquad \text{for all } \theta \in \mathbb{R}^d.$$

Examples of infinitely divisible distributions:

- Normal
- α-stable (e.g. Cauchy and Lévy)
- compound Poisson (e.g. Poisson)
- negative binomial (e.g. geometric)
- gamma (e.g. exponential and χ^2)
- Student's t-distribution
- Pareto
- Log-normal
- F-distribution
- Weibull
- Logistic
- Gumbel

Lévy–Khintchine representation

Any infinitely divisible random vector X in $\mathbb{R}^d$ has characteristic triplet (b, ν, Σ^2) such that

$$\mathbb{E}[e^{i\langle\theta,X\rangle}] =$$
$$\exp\left(i\langle b,\theta\rangle - \langle\Sigma\theta,\theta\rangle/2 + \int_{\mathbb{R}^d}\left\{e^{i\langle\theta,x\rangle} - 1 - i\langle\theta,x\rangle\mathbf{1}_{\{|x|\le 1\}}\right\}\nu(dx)\right).$$

- $b \in \mathbb{R}^d$ is the shift parameter
- Σ is the Gaussian component;
  a positive definite symmetric $d \times d$ matrix.
- ν is the Lévy measure;
  $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\,\nu(dx) < \infty$.

Poissonian infinitely divisible = infinitely divisible with $\Sigma = 0$.

Generalized shot noise representation:

Let $\mathbf{X} = \{X(t) : t \in T\}$ be a Poissonian infinitely divisible process

- H be a "representation of the Lévy measure" of $\mathbf{X}$
- $(\Gamma_k)_{k\in\mathbb{N}}$ be the partial sum of i.i.d. $\sim \mathrm{Exponential}(1)$
- $(V_k)_{k\in\mathbb{N}}$ be a certain i.i.d. sequence.

Then,

$$X(t) = \sum_{k=1}^{\infty} \Big(H(t, \Gamma_k, V_k) - c_k(t) \Big) \qquad \text{a.s.} \tag{2}$$

where $c_k(t)$ are random centerings.

Litterature on (2)**:** Khintchine (1937), Ferguson and Klass (1972), LePage (1980), Rosiński (1990).

Aim: Derive insight on $\mathbf{X}$ through a truncated series (as in the Gaussian case).

Note: Conditional on $(\Gamma_k)_{k\in\mathbb{N}}$, the series (2) consists of independent random variables.

Functional convergence: The deterministic case

Wikipedia: "Dini's theorem describes one of the few situations in mathematics where pointwise convergence implies uniform convergence."

Theorem (Dini's theorem)

Let T be a metrizable compact set, and $x_i : T \mapsto \mathbb{R}_+$ are continuous functions such that

$$\sum_{i=1}^{n} x_i(t) \to x(t) \quad \textit{for each } t \in T$$

for some continuous function $x : T \mapsto \mathbb{R}_+$. Then, as $n \to \infty$,

$$\Big\| \sum_{i=1}^{n} x_i - x \Big\|_\infty \to 0.$$

Dini's theorem says: point-wise convergence + continuous limit + positiv summas $\Rightarrow$ uniform convergence.

Functional convergence: The stochastic case

Theorem (Itô-Nisio theorem, a version)

Let $X_n = \{X_n(t)\}_{t \in T}$ be independent symmetric processes with continuous paths over a metrizable compact set T such that

$$\sum_{i=1}^{n} X_i(t) \to X(t) \quad a.s. \text{ for each } t \in T$$

for some process X with continuous paths over T. Then

$$\left\| \sum_{i=1}^{n} X_i - X \right\|_{\infty} \to 0 \quad a.s.$$

Itô-Nisio theorem says: point-wise convergence + continuous limit + symmetry summas $\Rightarrow$ uniform convergence.

Dini $\leftrightarrow$ X_i positive

Itô–Nisio $\leftrightarrow$ X_i symmetric.

Karhunen-Loéve representation:
Let $\mathbf{X} = \{X(t) : t \in T\}$ be a Gaussian process with continuous sample paths. Then,

$$X(t) = \sum_{k=1}^{\infty} e_k(t) Z_k \qquad \text{converge a.s. in the sup-norm.} \tag{3}$$

Proof:

- Marginal convergence in L^2 in (3) follows by the RHKS structure.
- The convergence in sup-norm follows by marginal convergence and the Itô–Nisio Theorem used on

$$X_k(t) = e_k(t) Z_k.$$

$\nearrow$

symmetric process

□

Theorem (Itô–Nisio Theorem; a version)

1. *Let $(F, \|\cdot\|)$ be a separable Banach space of functions $f : T \to \mathbb{R}$ such that the evaluations $\delta_t(f) = f(t)$ are continuous.*

2. *Let X_n be independent symmetric processes over T with paths in F such that for every $t \in T$,*

$$\sum_{i=1}^{n} X_i(t) \xrightarrow{\mathbb{P}} X(t),$$

   for some process X with paths in F.

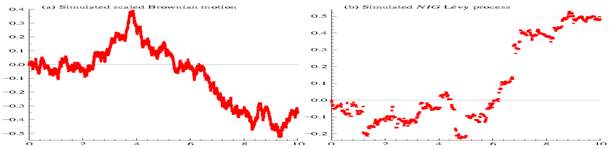
Then

$$\Big\| \sum_{i=1}^{n} X_i - X \Big\| \to 0 \qquad a.s.$$

The proof relies heavily on the fact that probability measure on separable Banach spaces are convex tight.

Lévy processes (the simplest case)

- $\mathbf{X} = \{X(t) : t \in [0,1]\}$ is called a **Lévy process** if it has stationary, independent increments, $X_0 = 0$ and càdlàg paths.
- Note: any Lévy process is infinitely divisible.
- A Gaussian Lévy process is the Brownian motion, and has continuous paths.
- Any (non-deterministic) Poissonian Lévy process is discontinous.



Brownian motion vs. Poissonian Lévy process.

Càdlàg = right-continuous with left-hand limits.

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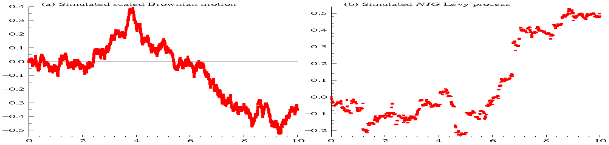

Brownian motion vs. Poissonian Lévy process.

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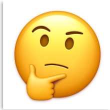
Non-separable Banach spaces

The case of non-separable Banach spaces are especially important for processes with jumps:

1. The space of càdlàg functions $D[0,1]$ is **non-separable** in the sup-norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$.
2. If $\mathbf{X} = (X(t))_{t\in[0,1]}$ is a Poisson process then
   - the law of $\mathbf{X}$ is concentrated on a **non-separable** subset of $(D[0,1], \|\cdot\|_\infty)$
   - $\Omega \ni \omega \mapsto \mathbf{X}(\omega) \in D[0,1]$ is not Borel measurable.
3. The Skorohod topology is not appropriate for studying series of jump processes since addition is not continuous (at jump times).



Generalized shot noise representation:
Let $\mathbf{X} = \{X(t) : t \in T\}$ be a Poissonian infinitely divisible process with representation

$$X(t) = \sum_{k=1}^{\infty} \Big(H(t, \Gamma_k, V_k) - c_k(t) \Big) \qquad \text{a.s.}$$

"Typically", $\mathbf{X}$ has sample paths in a *non-separable* Banach space.

Theorem (Itô–Nisio Theorem; a version)

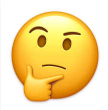
Let $(F, \|\cdot\|)$ be a separable Banach space such that for all $t \in T$,

$$\sum_{i=1}^{n} X_i(t) \xrightarrow{\mathbb{P}} X(t), \qquad \textit{with } X \in F \textit{ a.s.}$$

Then,

$$\Big\| \sum_{i=1}^{n} X_i - X \Big\| \to 0 \qquad \textit{a.s.}$$

Is *separability* important for the Itô–Nisio theorem?



Example

1. $F = \ell^\infty(\mathbb{N})$ be the space of all bounded sequences
2. $(\epsilon_i)_{i \in \mathbb{N}}$ an i.i.d. symmetric Bernoulli sequence
3. $$X_i = (0, 0, \ldots, 0, \epsilon_i, 0, \ldots) \qquad \text{and} \qquad X = (\epsilon_1, \epsilon_2, \ldots).$$

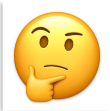
Then, $X \in \ell^\infty(\mathbb{N})$, and for every $t \in \mathbb{N}$,

$$\sum_{i=1}^{n} X_i(t) \xrightarrow{\mathbb{P}} X(t).$$

But,

$$\Big\| \sum_{j=1}^{n} X_j - X \Big\|_F = 1 \qquad \forall n \in \mathbb{N}.$$

Thus, the Itô–Nisio property does *not* hold for $\ell^\infty(\mathbb{N})$.



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Dini's theorem breaks down for $D[0,1]$.

Theorem (B. and Rosiński [1])

The Itô–Nisio theorem holds for $(D[0,1], \|\cdot\|_\infty)$.

1. Kallenberg [2] and Rosiński [3] established the a.s. uniform convergence of series representation of Lévy processes (a conjecture by Ferguson and Klass '72).
2. The above theorem implies uniform converges of general càdlàg infinitely divisible processes (beyond Lévy processes).

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[1] Basse-O'Connor, A. and J. Rosiński (2013). On the uniform convergence of random series in Skorohod space and representations of càdlàg infinitely divisible processes. *Ann. Probab.* 41.

[2] Kallenberg, O. (1974). Series of random processes without discontinuities of the second kind. *Ann. Probab.* 2.

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Bounded p-variation

- Let BV_p be set of all càdlàg functions $f : [0,1] \to \mathbb{R}$ of bounded p-variation

$$\|f\|_{BV_p} := \sup_{0 \le t_0 < \cdots < t_n \le 1,\, n \in \mathbb{N}} \left(\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|^p \right)^{1/p} < \infty.$$

- If $f \in C^1$ then

$$\|f\|_{BV_1} = \int_0^t |f'(s)|\, ds \approx \mathrm{length}(t \mapsto (t, f(t))).$$

- $$BV_1 \subsetneq BV_p \subsetneq D[0,1] \qquad (1 < p < \infty).$$

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- $$BV_1 \subsetneq BV_p \subsetneq D[0,1] \qquad (1 < p < \infty).$$

- A Poissonian Lévy process is of bounded p-variation, if and only if $\int_{|x| \le 1} |x|^p \, \nu(dx) < \infty \qquad (1 \le p < 2)$.

- BV_p is a non-separable Banach space.

Bounded p-variation and the Itô–Nisio theorem

BV_p is a non-separable Banach space:

$$\|f\|_{BV_p} := \sup_{0 \le t_0 < \cdots < t_n \le 1,\, n \in \mathbb{N}} \Big(\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|^p \Big)^{1/p} < \infty.$$

Theorem (Jain and Monrad [1])

The Itô–Nisio theorem holds for BV_1.

Theorem (Jain and Monrad [2])

*For $1 < p < \infty$ the Itô–Nisio theorem does **not** hold for BV_p.*

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[1] Jain, N, and D. Monrad (1982). Gaussian quasimartingales. *Z. Wahrsch. Verw. Gebiete* 59.

[2] Jain, N, and D. Monrad (1983). Gaussian measures in B_p. *Ann. Probab.* 11.

Towards a general Itô–Nisio theorem

$$BV_1 \subsetneq BV_p \subsetneq D[0,1] \qquad (\simeq BV_\infty)$$

✓ % ✓

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1. Let $(F, \|\cdot\|)$ be a Banach space of functions $f : T \mapsto \mathbb{R}$ such that the evaluations $\delta_t(f) = f(t)$ are continuous.

2. Suppose that $\|\cdot\|_F$ is $\mathcal{C}(T) = \sigma(\delta_t : t \in T)$ measurable and $\mathrm{linspan}\{\delta_t : t \in T\}$ contains a countable norming set Ξ:

$$c\|f\| \leq \sup\{\xi(f) : \xi \in \Xi\} \leq C\|f\|$$

for all $f \in F$ and some $c, C > 0$.

3. Each stochastic process $X = (X(t))_{t \in T}$ can be regarded as a random variable in $(F, \mathcal{C}(T))$. However, it will usually not be (strongly) Borel measurable, when F is non-separable.

Towards a general Itô–Nisio theorem

- We will say that a random element X in F is a Rademacher random element if

$$\xi(X) = \sum_{j=1}^{\infty} \epsilon_j \xi(f_j) \qquad a.s., \forall \xi \in \Xi,$$

  where $(f_j) \subseteq F$ and ϵ_j are i.i.d. random signs.

- The distribution of any such X will be called a Rademacher measure on F.

Theorem

Let $(F, \|\cdot\|)$ be a Banach space of countable type relative to Ξ. Then the following two conditions are equivalent:

(I-N) *For every sequence $(X_n)_{n\in\mathbb{N}}$ of independent symmetric random elements in F and a random element X in F, the convergence*

$$\sum_{j=1}^{n} \xi(X_j) \xrightarrow{\mathbb{P}} \xi(X) \qquad \forall \xi \in \Xi,$$

implies

$$\Big\| \sum_{j=1}^{n} X_j - X \Big\| \to 0 \qquad a.s.$$

(R) *For every sequence $(\xi_k)_{k\in\mathbb{N}} \subseteq \Xi$ and a Rademacher measure μ on F there exists a subsequence $(\xi_{k_l})_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} \xi_{k_l}(f)$ exists for μ-almost all $f \in F$.*

Theorem (continued)

If (R) holds and X_n are only independent in (I-N), then the implication

$$\sum_{j=1}^{n} \xi(X_j) \xrightarrow{\mathbb{P}} \xi(X) \quad \forall \xi \in \Xi \qquad \Rightarrow \qquad \Big\| \sum_{j=1}^{n} X_j - X \Big\| \to 0$$

holds if and only if for every $\delta > 0$

$$\limsup_{n>m\to\infty} \mathbb{P}\Big(\sum_{k=m}^{n} \xi(X_k) > \delta \Big) < 1. \tag{4}$$

In particular, if

- $\{\xi(X) : \xi \in \Xi\}$ *is uniformly integrable*
- $\mathbb{E}[\xi(X_n)] = 0$ *for each* $\xi \in \Xi$ *and* $n \in \mathbb{N}$

then condition (4) holds.

Itô–Nisio theorem

vs.

p-variation

(BV_p % for $p \in (1, \infty)$)

We do not give up!

Definition (BV_p° – the core of BV_p)

The closure of the space of càdlàg step functions in the $\|\cdot\|_{BV_p}$ norm is denoted by BV_p°. That is,

$$BV_p^\circ = \overline{\{\text{càdlàg step functions on } [0,1]\}}_{BV_p}.$$

1. Rough paths theory: A geometric rough path of order p is a continuous element in $BV_p^\circ([0,1];G)$.
2. $(BV_p^\circ, \|\cdot\|_{BV_p})$ is a non-separable Banach space
3. $BV_\infty = BV_\infty^\circ$
4. $$\bigcup_{\epsilon>0} BV_{p-\epsilon} \subsetneq BV_p^\circ \subsetneq BV_p, \qquad 1 < p < \infty.$$

A Itô–Nisio theorem for the core of BV_p

Theorem (B., Hoffmann-Jørgensen and Rosiński)

The Itô–Nisio theorem holds for BV_p° *for all* $1 \leq p \leq \infty$.

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Theorem (B., Hoffmann-Jørgensen and Rosiński)

The Itô–Nisio theorem holds for BV_p° for all $1 \leq p \leq \infty$.

"Proof:"

- For $p > 1$, we show that BV_p° satisfies (R); for all $(\xi_n) \subseteq \Xi$ there exists a subsequence (n_k) such that $\lim_{k\to\infty} \xi_{n_k}(f)$ exists in $\mathbb{R}$ for all $f \in BV_p^\circ$.
- The proof relies heavily on Helly's selection theorem for p-variation functions. □

A Itô–Nisio theorem for the core of BV_p

Theorem (B., Hoffmann-Jørgensen and Rosiński)

The Itô–Nisio theorem holds for BV_p° for all $1 \le p \le \infty$.

"Proof:"

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  for all $(\xi_n) \subseteq \Xi$ there exists a subsequence (n_k) such that $\lim_{k\to\infty} \xi_{n_k}(f)$ exists in $\mathbb{R}$ for all $f \in BV_p^\circ$.
- The proof relies heavily on Helly's selection theorem for p-variation functions. $\square$

Hence for all $1 < q < p < \infty$

$$\color{blue}{BV_1} \subseteq \color{red}{BV_q} \subseteq \color{blue}{BV_p^\circ} \subseteq \color{red}{BV_p} \subseteq \color{blue}{D[0,1]}$$

✓ % ✓ % ✓

Theorem (B., Hoffmann-Jørgensen and Rosiński)

Let $1 \leq p < 2$ and $\mathbf{X}$ be a Poissonian infinitely divisible process of bounded p-variation. Then, the series,

$$X(t) = \sum_{k=1}^{\infty} \Big(H(t, \Gamma_k, V_k) - c_k(t) \Big)$$

converge a.s. in q-variation for any $q > p$.

Theorem

- *Let* $F : \mathbb{R}^m \to Mat_{m\times d}$ *be* C^2.
- *Let* $1 \leq p < 2$ *and* $\mathbf{X}$ *be a Poissonian infinitely divisible process of bounded p-variation.*
- *Let* $\mathbf{Y}$ *be the solution to the SDE*

$$dY(t) = F(X(t-))\,dX(t).$$

- *Let* $\mathbf{Y}_N$ *be the solution to*

$$dY_N = F(Y_N(t-))\,dX_N(t), \quad X_N(t) = \sum_{k=1}^{N} \Big(H(t,\Gamma_k,V_k) - c_k(t)\Big).$$

- *Then,*

$$\mathbf{Y}_N \to \mathbf{Y} \qquad \textit{in } q\textit{-variation for any } q > p.$$

The proof relies on $X_N \to X$ in BV_q and a continuity result. Convergence in $\|\cdot\|_\infty$ is not sufficient.

Next step: **Applications to support theorems**

Thank you for your attention!