Well posedness for a stochastic wave equation with super-linear coefficients

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A stochastic wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x),\ t\in(0,T], \\ u(0,x) = u_0(x), \quad \frac{\partial}{\partial t}u(0,x) = v_0(x), \end{cases}$$

$x \in \mathbb{R}^d$, $d \in \{1,2,3\}$.

Assumptions

$\sigma, b : \mathbb{R} \to \mathbb{R}$ locally Lipschitz and

$$|\sigma(x)| \le \sigma_1 + \sigma_2|x|\big(\ln(|x|)\big)^a, \quad |b(x)| \le \theta_1 + \theta_2|x|\big(\ln(|x|)\big)^\delta,\ |x|\to\infty,$$

$\sigma_1, \theta_1 \in \mathbb{R}_+$, $\sigma_2, \theta_2 > 0$, $\delta, a > 0$.

Example:

$$|\sigma(x) - \sigma(y)| \le \sigma_2|x-y|\left(\log_+(|x-y|)\right)^a,\ \log_+(z) = \log(z\vee e),$$
$$|\sigma(0)| \le \sigma_1.$$

- For $d=1$, $\dot{W}$ is space-time white noise.
- For $d=1,2,3$, $\dot{W}$ is a noise white in time and correlated in space.

What we want to prove

A *random field solution* is a jointly measurable, adapted process $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u(t,x) = [G(t) * v_0](x) + \frac{\partial}{\partial t}\left[G(t) * u_0\right](x) + \int_0^t ds\, [G(s) * b(u(t-s,\cdot))](x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\sigma(u(s,y))W(ds,dy), \text{ a.s.} \tag{1}$$

"Theorem"

Assume that the initial conditions have compact support, the coefficients are super-linear (as before) with b dominating over σ.
Then, there exists a global random field solution to (1). This solution is unique and satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u(t,x)| < \infty, \ a.s.$$

Case b and σ Lipschitz continuous (Walsh (1986), Dalang (1999), ...)

- For $d = 1$, $\dot{W}$ is space-time white noise.
- For $d = 1, 2, 3$, $\dot{W}$ is a noise white in time, correlated in space, with spectral covariance μ satisfying Dalang's condition

$$\int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1 + |\zeta|^2} < \infty.$$

- Nice initial conditions.

Then there exists a random field solution $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$, and for any $p \in [1, \infty)$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|u(t, x)\|_p < \infty. \tag{2}$$

Uniqueness in the class of jointly measurable, adapted processes u satisfying (2) ($p = 2$).

The motivation of our research

Reaction-diffusion equations with super-linear drift
[Dalang-Khoshnevisan-Zhang, AoP, 2019]

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x),\ t>0,\ x\in(0,1), \\ u(0,x)=u_0(x),\ x\in[0,1];\ \ u(t,0)=u(t,1)=0. \end{cases}$$

- b, σ locally Lipschitz functions such that

$$|b(x)| = O(|x|\ln|x|),\ |\sigma(x)| = o(|x|(\ln|x|)^{1/4}),\ |x|\to\infty.$$

- u_0 Hölder continuous.

There is a unique global random field solution u on $\mathcal{C}(\mathbb{R}_+\times[0,1])$, and

$$\sup_{(t,x)\in[0,T]\times[0,1]} |u(t,x)| < \infty,\ a.s.,\ \forall T>0.$$

Compare with [Bonder and Groisman, 2009]: There is blow-up in finite time if,

$$\sigma\ constant,\ |b(x)| \geq |x|(\ln|x|)^{1+\varepsilon},\ \varepsilon>0.$$

Blow-up for wave equations

Additive or multiplicative noise (σ constant or not constant)

Conditions

- Functional-valued solutions.
- Strong assumptions on the covariance of the noise.

A sample of results

- P.-L. Chow, AoAP, 2002.
- M. Ondréjat, 2004-2006.
- A. Millet - P.-L. Morien, AoAP, 2001.
- C. Marinelli - L. Quer-Sardanyons, SIAM J. Math Anal., 2012.
- ...

Approach

- Truncation of σ, b: σ_N, b_N.

  For a locally Lipschitz function $g : \mathbb{R} \to \mathbb{R}$, define the globally Lipschitz function

$$g_N(x) = g(x)1_{\{|x|\le N\}} + g(N)1_{\{x>N\}} + g(-N)1_{\{x<-N\}},\ N \ge 1.$$

- Let u_N be the solution of $\mathrm{Eq}(\sigma_N, b_N)$. Prove

$$E\left(\sup_{t\in[0,T]} \sup_{|x|\le R} |u_N(t,x)|^p\right) = o(N^p),$$

  for some $p \ge 1$.

More details on the approach

Let $u_N := \{u_N(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ be the unique random field solution to the wave equation with coefficients b_N and σ_N.
Let

$$\tau_N := \inf\left\{t > 0 \ : \ \sup_{|x| \le R} |u_N(t,x)| \ge N\right\} \wedge T.$$

We want $\sup_N \tau_N = T$ a.s., which is implied by

$$\lim_{N \to \infty} P(\tau_N < T) = 0.$$

This is, in turn, implied by

$$\lim_{N \to \infty} P\left(\sup_{t \in [0,T]} \sup_{|x| \le R} |u_N(t,x)| \ge N\right) = 0.$$

By Chebychev's inequality, it suffices to prove

$$E\left(\sup_{t \in [0,T]} \sup_{|x| \le R} |u_N(t,x)|^p\right) = o(N^p),$$

for some $p \ge 1$.

A version of Kolmogorov's continuity lemma

Theorem

$\{v(t,x),(t,x) \in I \times J\}$, random field, $I \times J \subset \mathbb{R}_+ \times \mathbb{R}^d$, bounded set of points. Set

$$\Delta(t,x;s,y) = |t-s|^{\alpha_1} + |x-y|^{\alpha_2},\ \alpha_1,\alpha_2 \in]0,1].$$

Suppose: there exist $K < \infty$, $p > \frac{1}{\alpha_1} + \frac{d}{\alpha_2}$, and for all $(t,x),(s,y) \in I \times J$,

$$E\left(|v(t,x) - v(s,y)|^p\right) \le K\left(\Delta(t,x;s,y)\right)^p.$$

Then v has a continuous version.

Moreover, if $E\left(\sup_{x\in J} |v(0,x)|^p\right) \le C_1$ then there exists $0 < C_2(p,I,J) < \infty$, such that

$$E\left(\sup_{(t,x)\in I\times J} |v(t,x)|^p\right) \le 2^{p-1}C_1 + C_2(p,I,J)K.$$

Proof of $E\left(\sup_{t\in[0,T],|x|\le R} |u_N(t,x)|^p\right) = o(N^p)$

1. A regularity theorem: Find K_N satisfying

$$E\left(|u_N(t,x) - u_N(s,y)|^p\right) \le K_N\left(\Delta(t,x;s,y)\right)^p,$$

   K_N exhibiting the dependence on b_N, σ_N.
2. Observe: $C_1 := \|u_0\|_\infty$.
3. By Kolmogorov's theorem, deduce

$$E\left(\sup_{t\in[0,T],|x|\le R} |u_N(t,x)|^p\right) \le \tilde{C}_1(p,u_0) + C_2(p,T,R)K_N.$$

4. Check

$$K_N = o(N^p).$$

Global solution in dimension 1. Space-time white noise

Theorem

- *The initial values u_0, v_0 have compact support and they are Hölder continuous functions with exponents $\gamma_1, \gamma_2 \in (0,1)$, respectively.*
- $\dot{W}(t,x)$ *is space-time white noise.*
- σ *and* b *are locally Lipschitz and satisfy*

$$|\sigma(x)| \le \sigma_1 + \sigma_2 |x| \big(\ln_+(|x|)\big)^a, \quad |b(x)| \le \theta_1 + \theta_2 |x| \big(\ln_+(|x|)\big)^\delta, \ |x| \to \infty.$$

with $\delta < 2$, *and either* $\delta > 2a$ *or* $\delta = 2a$ *and* $\theta_2 > c(\gamma_1, \gamma_2)\sigma_2^2$.

Then, there exists a global random field solution $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ *to the stochastic wave equation* (1), *and this solution is unique and satisfies*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |u(t,x)| < \infty, \ a.s.$$

A priori estimates

Notation: for a globally Lipschitz function g, set $c(g) := g(0)$, $L(g)$ Lipschitz constant. Then,

$$|g(x)| \leq c(g) + L(g)|x|.$$

Proposition

$d = 1$, null initial conditions, space-time white noise, σ, b globally Lipschitz. Suppose that $L(b) \geq 8L(\sigma)^2$. Then for $2 \leq p \leq \frac{L(b)}{4L(\sigma)^2}$,

$$\sup_{x\in\mathbb{R}} \|u(t,x)\|_p \leq Ce^{2t\sqrt{L(b)}} \left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right], \ t \in [0,T].$$

Proof

- Use the seminorms $\mathcal{N}_{\alpha,p}(u) = \sup_{t\geq 0, |x|\leq M} e^{-\alpha t}\|u(t,x)\|_p$, $\alpha > 0$, $p \geq 1$.
- Optimize in (α, p).

Upper bounds for L^p increments $\|u(t,x) - u(\bar{t},\bar{x})\|_p$

Notation

$$\Delta(t,x;s,y) = |t-s|^{\frac{1}{2}} + |x-y|^{\frac{1}{2}},\ s,t \in [0,T],\ x,y \in [-R,R],$$

Proposition

The hypotheses are as in the previous proposition. Then, for $2 \le p \le \frac{L(b)}{4L(\sigma)^2}$,

$$E(|u(t,x) - u(s,y)|^p) \le K(\Delta(t,x;s,y))^p,$$

with

$$K = C(p,T)\Big([c(b) + \sqrt{p}\ c(\sigma)]^p + [L(b) + \sqrt{p}\ L(\sigma)]^p \times e^{2p\sqrt{L(b)}T}\Big[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\Big]\Big).$$

Upper bounds for L^p increments $\|u_N(t,x) - u_N(\bar{t},\bar{x})\|_p$

Remember,

$$|b(x)| \le \theta_1 + \theta_2|x|\big(\ln_+(|x|)\big)^\delta, \quad |\sigma(x)| \le \sigma_1 + \sigma_2|x|\big(\ln_+(|x|)\big)^a, \quad |x| \to \infty.$$

For the truncated functions, b_N, σ_N, this yields (N large),

$$|b_N(x)| \le c(b_N) + L(b_N)|x|, \; |\sigma_N(x)| \le c(\sigma_N) + L(\sigma_N)|x|,$$

with

$$c(b_N) = \theta_1, \quad c(\sigma_N) = \sigma_1, \quad L(b_N) = \theta_2(\ln(2N))^\delta, \quad L(\sigma_N) = \sigma_2(\ln(2N))^a.$$

We deduce

$$E\left(|u_N(t,x) - u_N(s,y)|^p\right) \le K_N(\Delta(t,x;s,y))^p,$$

with

$$K_N \le C(p,T)\left([\theta_1 + \sqrt{p}\,\sigma_1]^p + [L(b_N) + \sqrt{p}\,L(\sigma_N)]^p\, e^{2p\sqrt{L(b_N)}T} \times \left[\frac{\theta_1}{L(b_N)} + \frac{\sigma_1}{L(\sigma_N)}\right]^p\right).$$

Proposition

If $\delta < 2$, and either $\delta > 2a$ or $\delta = 2a$ and $\theta_2 > c\,\sigma_2^2$ then $K_N = o(N^p)$, for some $p \ge 1$.

Global solution $d = 1, 2, 3$. Noise white in time and coloured in space

$\{W(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{1+d})\}$ is Gaussian, centered,

$$E\left(W(\varphi)W(\psi)\right) = \int_0^\infty dt \int_{\mathbb{R}^d} \Lambda(dx)\left(\varphi(t) * \tilde{\psi}(t)\right)(x).$$

Theorem

- *The initial values* u_0, v_0 *satisfy* $[H_I]$.
- $\dot{W}(t,x)$ *is white in time, colored in space satisfying hypotheses* $[H_\Lambda]$.
- σ *and* b *are locally Lipschitz and satisfy*

$$|\sigma(x)| \le \sigma_1 + \sigma_2 |x| \left(\ln_+(|x|)\right)^a, \quad |b(x)| \le \theta_1 + \theta_2 |x| \left(\ln_+(|x|)\right)^\delta, \ |x| \to \infty,$$

with $\delta < 2$ *and either* $\delta > 8a$ *or* $\delta = 8a$ *and* $\theta_2 > c(\gamma_1, \gamma_2, \mu)\ \sigma_2^4$.

Then, there exists a global random field solution $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ *to the stochastic wave equation* (1). *This solution is unique and satisfies*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |u(t,x)| < \infty, \ a.s.$$

Hypotheses on the initial contitions

$[H_I]$

- u_0, v_0 have compact support.
- v_0 is Hölder continuous, exponent γ_2.
- $d = 1$: u_0 is Hölder continuous, exponent γ_1.
- $d = 2$: u_0 is $\mathcal{C}^1$, ∇u_0 is Hölder continuous, exponent γ_1.
- $d = 3$: u_0 is $\mathcal{C}^2$, Δu_0 is Hölder continuous, exponent γ_1.

Hypotheses on the noise

Hypotheses $[H_\Lambda]$

1. $[H_P]$ $\Lambda(dx) = f(x)dx$ is such that

$$\int_{\mathbb{R}^d} G(t,dy) \int_{\mathbb{R}^d} G(s,dz) f(y-z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}G(t,\cdot)(\xi) \overline{\mathcal{F}G(s,\cdot)(\xi)}.$$

2. $[H_D(\eta)]$ There exists $\eta \in (0,1)$ such that $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty$.
3. $[H_D(1,t)]$ There exists $\alpha \in (0,1)$ such that $\int_{\mathbb{R}^d} |\mathcal{F}G(t)(\xi)|^2 \mu(d\xi) \le Ct^\alpha$.
4. $[H_{\mathrm{t}-incr}]$ Control on integrals of first and second order increments of f in terms of the size of the increment:

$$\int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1,dy)G(1,dz) D_h^1 f(y,z) \le Ch^b,$$

$$\int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1,dy)G(1,dz) D_h^2 f(y,z) \le Ch^{\bar{b}}.$$

Examples of covariances

A sufficient condition for $[H_\Lambda]$ ([Khoshnevisan-Xiao, 2009]):

f symmetric, and either

- $f(x) < \infty \iff x \neq 0$;
  or
- $\mathcal{F}f \in L^\infty(\mathbb{R}^d)$ and $f(x) < \infty$ when $x \neq 0$.

Examples

- Riesz kernel

$$f_\beta(x) = |x|^{-\beta},\ \beta \in (0, d \wedge 2).$$

- Bessel kernel

$$f_\beta(x) = \int_0^\infty w^{\frac{\beta-d-2}{2}} e^{-w} e^{-\frac{|x|^2}{4w}}\ dw,\ \beta \in (d-2, d).$$

Open problems

- Qualitative study of blow-up. Is there same critical growth exponent?
- More general space covariances (e.g. remove the absolute continuity property on μ).

Thanks for your attention!