Fourth Conference on Ambit Fields and Related Topics

Quasi-Infinitely Divisible (QID) Processes and Measures

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Plan of the talk

- QID distributions
- QID random measures
- QID random noises
- QID processes

QID distributions

Definition (ID distributions)

A r.v. X is said to be ID if

$$\mathbb{E}\left[\exp(i\theta X)\right] = \hat{\mathcal{L}}(X)(\theta) = \exp\left(i\theta\gamma - \frac{\theta^2}{2}\sigma^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)\rho(dx)\right)$$

where $\gamma, \sigma \in \mathbb{R}$ and ρ is such that $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\rho(dx) < \infty$.

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Equivalent definition of QID distribution

A r.v. X is QID iff there exist two ID r.v. Y and Z such that $X + Y \stackrel{d}{=} Z$ with Y independent of X.

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The characteristic triplet of a QID distribution is unique.

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Theorem [Berger, 2019+]

Let μ be a probability distribution of the form

$$\mu(dx) = \mu_d(dx) + f(x)\lambda(dx)$$

where μ_d is a non-zero discrete measure supported on a lattice of the form $r + h\mathbb{Z}$ for some $r \in \mathbb{R}$ and $h > 0$, $f \in L_1(\mathbb{R}, [0, \infty))$, and λ is the Lebesgue measure. Then μ is QID iff $\hat{\mu}(\theta) \neq 0$ for all $\theta \in \mathbb{R}$.

Theorem [Lindner, Pan and Sato, 2018]

A nondegenerate distribution μ on $\mathbb{R}$ s.t. $\exists\, k \in \mathbb{R}$ with $\mu(\{k\}) > 1/2$ is QID.

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Theorem [P., 2019b]

Let A be an interval of the real line. The class of QID distributions with finite quasi-Lévy measure, zero Gaussian variance and with support on A is dense in the class of probability distributions with support on A with respect to weak convergence.

Definition (random measure) [Kallenberg, 2017]

A *random measure* ξ on $(S, \mathbf{S})$, with underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is a function $\Omega \times \mathbf{S} \to [0, \infty]$, such that $\xi(\omega, B)$ is a $\mathcal{F}$-measurable for fixed B and a locally finite measure for fixed ω. Denote by $\hat{\mathbf{S}}$ the localising ring.

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A random measure has *independent increments* if for any disjoint sets $B_1, ..., B_n$ the r.v. $\xi(B_1), ..., \xi(B_n)$ are mutually independent. They also known as completely random measures.

We assume that S is a complete separable metric space.

Let $\mathcal{A}$ be the set of QID random measures with independent increments s.t. for $\xi \in \mathcal{A}$ we have $\xi \overset{a.s.}{=} \alpha + \sum_{j=1}^{K} \beta_j \delta_{s_j}$ with:

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Theorem [P., 2019b]

$\mathcal{A}$ is dense in the set of all random measures with independent increments under the convergence in distribution.

Theorem [P., 2019a, 2019b]

Let $\xi \in \mathcal{A}$. Then, for every $B \in \hat{\mathbf{S}}$ and $\theta \in \mathbb{R}$

$$\hat{\mathcal{L}}(\xi(B))(\theta) = \exp\left(i\theta\nu_0(B) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)F_B(dx)\right) \tag{1}$$

where ν_0 is a finite signed measure on $\mathbf{S}$ and F is a finite signed measure on $\mathbf{S} \otimes \mathcal{B}(\mathbb{R})$

Theorem [P., 2019b]

Let $\xi \in \mathcal{A}$. Then, there exists a pair (ν_0, F) s.t. (1) holds, where ν_0 and F are a finite signed measure on $\mathbf{S}$ and $\mathbf{S} \otimes \mathcal{B}(\mathbb{R})$, respectively, s.t. for every $A \in \mathbf{S}$ and $B \in \mathcal{B}(\mathbb{R})$:

(i) $\nu_0(A) = -\gamma(A) + \sum_{j=1}^{n} \delta_{s_j}(A) a_j$, for some diffuse finite measure γ on $\mathbf{S}$, $a_1, \dots, a_n \in \mathbb{R}$, and finitely many fixed atoms $s_1, \dots, s_n \in S$,

(ii) $F(A \times B) = \tilde{G}(A \times B) + \sum_{j=1}^{n} \delta_{s_j}(A) b_j(B)$, for some finite measure $\tilde{G}$ on $\mathbf{S} \otimes \mathcal{B}(\mathbb{R})$, which is the extension by zero of some measure G on $\mathbf{S} \otimes \mathcal{B}((0, \infty))$ with diffuse projections onto S, and signed measures b_j's on $\mathcal{B}(\mathbb{R})$, s.t. $\exp(b_1)$,...,$\exp(b_n)$ are measures.

Conversely, for every such pair (ν_0, F) there exists a unique r.m. $\xi \in \mathcal{A}$ s.t. (1) holds.

Let $(S, \mathcal{S})$ be s.t. S is an arbitrary non-empty set and $\mathcal{S}$ is a δ-ring with the condition that there exists an increasing sequence of sets $S_1, S_2, \dots \in \mathcal{S}$ s.t. $\bigcup_{n\in\mathbb{N}} S_n = S$.

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Definition (random noise)

Let $\Lambda = \{\Lambda(A) : \ A \in \mathcal{S}\}$ be a $\mathbb{R}$-valued stochastic process defined on some prob. space $(\Omega, \mathcal{F}, \mathbb{P})$. We call Λ to be a *random noise* if for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$

(i) the r.v. $\Lambda(A_n)$, $n = 1, 2, ...$, are independent, and,

(ii) in the case $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$, we have $\Lambda(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Lambda(A_n)$ a.s. (where the series is assumed to converge a.s.).

In addition, if $\Lambda(A)$ is a QID (ID) r.v., for every $A \in \mathcal{S}$, then we call Λ a QID (ID) random noise.

In the ID case they are called Lévy basis.

Theorem(s) [P., 2019b]

All the results of Sections II and III and some of the results of Section IV in Rajput and Rosiski's 1989 paper hold for the QID framework under the assumption that $\forall A \in \mathcal{S}$

$$\sup_{I_A} \sum_{i \in I_A} \Big| \int_B (1 \wedge x^2) F_A(dx) \Big| < \infty,$$

where the supremum is taken over all the finite families of disjoints elements of $(\mathcal{S} \cap A) \times \mathcal{B}(\mathbb{R})$.

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Observe that for every $A \in \mathcal{S}$

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However, if F is non-negative then

$$\int_{\mathbb{R}} (1 \wedge x^2) F_A(dx) = \sup_{I_A} \sum_{i \in I_A} \int_B (1 \wedge x^2) F_A(dx).$$

Definition (QID processes) [P., 2019a]

Let T be an arbitrary index set. A stochastic process $X = \{X_t;\ t \in T\}$ is said to be a *QID* process if and only if for every finite set of indices $t_1, \dots, t_k$ in the index set T

$$X_{t_1,\dots,t_k} := (X_{t_1}, \dots, X_{t_k})$$

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Lemma [P., 2019a]

All the marginal distributions of a multivariate QID distribution are QID.

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Proposition [P., 2019a]

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Theorem [P., 2019a]

The law of a generated QID process with arbitrary index set has a Lévy-Khintchine formulation and a corresponding unique characteristic triplet.

Proposition [P., 2019a]

Let X_0 be any QID r.v. and let (a, σ^2, ν) be its characteristic triplet. Then

$$\hat{\mathcal{L}}(X_t) = \exp\left(i\theta a_t - \frac{1}{2}\sigma_t^2\theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)\nu_t(dx)\right)$$

is the characteristic function of a QID r.v. for every $t \geq 0$, if $a_t \in \mathbb{R}$, $\sigma_t^2 \geq \sigma^2$, and $\nu_t(\cdot)$ is a quasi-Lévy measure with $\nu_t \geq \nu$, for every $t \geq 0$.

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THANK YOU FOR YOUR ATTENTION