Fractional Pearson Diffusions and Continuous Time Random Walks

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Outline

- Fractional Pearson diffusions
  - Spectral representation of the transition densities
  - Strong solutions of time-fractional Kolmogorov backward equation for fractional Fisher-Snedecor diffusion
  - Correlation structure
- Continuous time random walks (CTRWs) and fractional Pearson diffusions
  - Bernoulli-Laplace urn scheme model: fractional Ornstein-Uhlenbeck process
  - Wright-Fisher urn scheme model: fractional Cox-Ingersoll-Ross and Jacobi diffusion
  - Ehrenfest-Brillouin model: fractional Jacobi diffusion
  - Markov chain for Student diffusion

Fractional diffusion – definition

- $X_1 = (X_1(t),\ t \geq 0)$ – Markovian diffusion with transition densities $p_1(x, t; y)$
- $D = (D_t,\ t \geq 0)$ – standard stable subordinator independent of the diffusion X_1, with the Laplace transform

$$\mathbb{E}[e^{-sD_t}] = \exp\left(-ts^{\alpha}\right), \quad s \geq 0, \quad 0 < \alpha < 1$$

- $E_t = \inf\{x > 0\colon D_x > t\}$ - inverse of the α-stable subordinator D
- $(E_t,\ t \geq 0)$ – non-Markovian and non-decreasing, for every t random variable E_t has a density $f_t(\cdot)$ with the Laplace transform

$$\mathbb{E}[e^{-sE_t}] = \int_0^{\infty} e^{-sx} f_t(x)dx = \mathcal{E}_{\alpha}(-st^{\alpha}),$$

where $\mathcal{E}_{\alpha}(-st^{\alpha})$ is the Mittag-Leffler function

$$\mathcal{E}_{\alpha}(-st^{\alpha}) = \sum_{j=0}^{\infty} \frac{(-st^{\alpha})^j}{\Gamma(1+\alpha j)} \tag{1}$$

- **fractional diffusion** – non-Markovian process defined via time-change of the diffusion $X_1(t)$ by the inverse E_t of the α-stable subordinator, i.e.

$$X_{\alpha}(t) = X_1\left(E_t\right), \quad t \geq 0$$

Fractional diffusions – applications

- **hydrology** – modeling sticking and trapping of contaminant particles in a porous medium (Meerschaert et al., 2003) or a river flow (Chakraborty et al., 2009)
- **finance** – modeling delays between trades (Scalas, 2006)
- **statistical physics** – fractional time derivative appears in the equation for a continuous time random walk limit and reflects random waiting times between particle jumps (Meerschaert, 2004)

Fractional Pearson diffusion – definition

- **fractional Pearson diffusion** – non-Markovian process

$$(X_\alpha(t),\ t \geq 0) = (X_1(E_t),\ t \geq 0)\,,$$

  where $(X_1(t),\ t \geq 0)$ is the Pearson diffusion

- **Pearson diffusion** – a unique strong solution (Øksendal, Theorem 5.2.1) of the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0 \tag{2}$$

  with polynomial infinitesimal parameters

$$\mu(x) = a_0 + a_1 x, \quad \sigma(x) = \sqrt{2b(x)} = \sqrt{2(b_2x^2 + b_1x + b_0)}$$

- $\mathfrak{p}(x)$ – the stationary density of the diffusion (2) belongs to the Pearson family of continuous distributions

- $\mu(x)$ and $b(x)$ are related to the polynomials in the **Pearson differential equation**

$$\frac{\mathfrak{p}'(x)}{\mathfrak{p}(x)} = \frac{(a_1 - 2b_2)x + (a_0 - b_1)}{b_2x^2 + b_1x + b_0}$$

Pearson diffusions - classification

six subfamilies of Pearson diffusions – according to the degree of polynomial $b(x)$ and, in the quadratic case, to the sign of b_2 and the sign of its discriminant Δ:

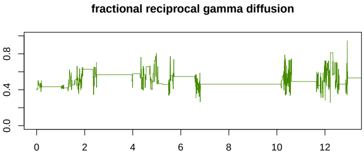
- constant $b(x)$ – OU process (Gaussian stationary distribution)
- linear $b(x)$ – CIR process (gamma stationary distribution)
- quadratic $b(x)$ with $b_2 < 0$ – Jacobi diffusion (beta stationary distribution)
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta > 0$ – Fisher-Snedecor (FS) diffusion
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta = 0$ – reciprocal gamma (RG) diffusion
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta < 0$ – Student diffusion

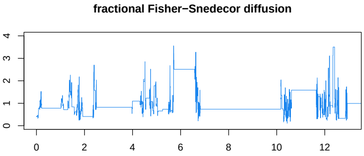
- **important references:**

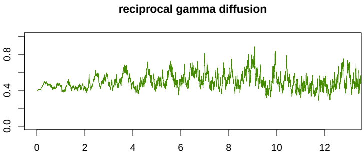
  Kolmogorov (1931), Wong (1964), Forman & Sørensen (2008), Avram et al. (2013a, 2013b)

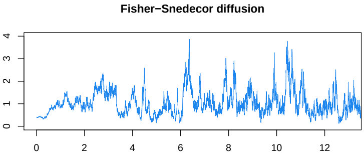
Pearson and fractional Pearson diffusions – sample paths

Sample paths of fractional and non-fractional RG and FS diffusions with parameters $\gamma = 10$, $\beta = 20$, $\theta = 0.01$ and $\alpha = 0.7$ based on 10000 points with inital state $X_0 = 0.4$









Non-heavy-tailed Pearson diffusions

- **OU, CIR and Jacobi diffusions**
- **transition densities** – $p(x,t;y) = \frac{\partial}{\partial x} P(X_t \leq x | X_0 = y)$
  - closed-form expressions

    S. Karlin and H.M. Taylor (1981) A Second Course in Stochastic Processes, Academic Press, New York
  - spectral representations of transition densities – given in terms of the pure-point spectrum of the infinitesimal generator and the corresponding eigenfunctions (Hermite, Laguerre and Jacobi polynomials, respectively)
  - spectral analysis – overview of existing results given in

    N.N. Leonenko, M.M. Meerschaert and A. Sikorskii (2013) Fractional Pearson diffusions, Journal of Mathematical Analysis and Applications, **403***(2): 532–546*

Comparing transition densities of non-heavy-tailed PD and fPD

- **non-heavy-tailed diffusion**

$$p_\alpha(x,t;y) = \mathfrak{p}(x)\sum_{n=0}^{\infty} e^{-\lambda_n t} Q_n(y) Q_n(x),\ \ x,\ y \in I,\ \ t \geq 0,$$

- **non-heavy-tailed fractional diffusion**:

$$p_\alpha(x,t;y) = \mathfrak{p}(x)\sum_{n=0}^{\infty} \mathcal{E}_\alpha(-\lambda_n t^\alpha) Q_n(y) Q_n(x),\ \ x,\ y \in I,\ \ t \geq 0,$$

where

- $\mathfrak{p}(x)$ is corresponding stationary distribution
- $\{Q_n,\ n \in \mathbb{N}\}$ are corresponding eigenfunctions
- $\{\lambda_n,\ n \in \mathbb{N}\}$ are corresponding eigenvalues

Heavy-tailed Pearson diffusions

- **reciprocal gamma, Fisher-Snedecor and Student diffusions**
- **transition densities** – representable in terms of the spectrum of the corresponding infinitesimal generator and related functions
- **infinitesimal generator** of heavy-tailed Pearson diffusion

$$\mathcal{G}f(x) = \mu(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) \tag{3}$$

$\mu(x)$ - linear; $\sigma^2(x)$ - quadratic, with positive leading coefficient

- **spectrum** of the Sturm-Liouville operator $(-\mathcal{G})$
  - discrete spectrum $\sigma_d \subset [0, \Lambda\rangle$ - finite set of eigenvalues
    eigenfunctions are finite systems of orthogonal polynomials (Bessel, Fisher-Snedecor and Romanovski polynomials, respectively)
  - absolutely continuous spectrum $\sigma_{ac}(\mathcal{G})$ in $\langle\Lambda, \infty\rangle$
    functions related to the $\sigma_{ac}(\mathcal{G})$ - confluent (RG) and Gauss (FS, Student) hypergeometric functions
- Student diffusion – heavy-tailed Pearson diffusions with $\sigma_{ac}(\mathcal{G})$ of multiplicity two (still not completely resolved)

Fisher-Snedecor (FS) diffusion

- FS diffusion **SDE**

$$dX_1(t) = -\theta\left(X_1(t) - \frac{\beta}{\beta-2}\right)dt + \sqrt{\frac{4\theta}{\gamma(\beta-2)}X_1(t)\left(\gamma X_1(t)+\beta\right)}\,dW(t), \quad (4)$$

where $t \geq 0$ and $\theta > 0$ (autocorrelation parameter)

- **stationary density**

$$\mathfrak{fs}(x) = \frac{\beta^{\frac{\beta}{2}}}{B\left(\frac{\gamma}{2}, \frac{\beta}{2}\right)} \frac{(\gamma x)^{\frac{\gamma}{2}-1}}{(\gamma x+\beta)^{\frac{\gamma}{2}+\frac{\beta}{2}}}\gamma\, \mathrm{I}_{\langle 0,\infty\rangle}(x), \quad \gamma > 0, \quad \beta > 2$$

- **transition density** – spectral representation

$$p_1(x,t;y) = p_d(x,t;y) + p_c(x,t;y) \quad (5)$$

derived in

F. Avram, N.N. Leonenko and N. Šuvak. (2013) Spectral representation of transition density of Fisher-Snedecor diffusion, Stochastics, **85***(2): 346–369*

FS diffusion – discrete part of transition density

- transition density – **discrete part**

$$p_d(x,t;y) = \mathfrak{fs}(x) \sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} e^{-\lambda_n t} F_n(y)\, F_n(x) \tag{6}$$

- **eigenvalues** of the SL operator $(-\mathcal{G})$

$$\lambda_n = \frac{\theta}{\beta - 2}\, n(\beta - 2n), \quad n \in \{0, 1, \dots, \lfloor \beta/4 \rfloor\}, \quad \beta > 2 \tag{7}$$

- **eigenfunctions** of the SL operator $(-\mathcal{G})$ – Fisher-Snedecor polynomials

$$F_n(x) = K_n\, x^{1-\frac{\gamma}{2}} (\gamma x + \beta)^{\frac{\gamma}{2}+\frac{\beta}{2}} \frac{d^n}{dx^n} \left\{ 2^n\, x^{\frac{\gamma}{2}+n-1} (\gamma x + \beta)^{n-\frac{\gamma}{2}-\frac{\beta}{2}} \right\} \tag{8}$$

FS diffusion – continuous part of transition density

- transition density – **continuous part**

$$p_c(x,t;y) = \mathfrak{s}(x)\,\frac{1}{\pi} \int\limits_{\Lambda=\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} e^{-\lambda t}\, a(\lambda)\, f_1(y,-\lambda) f_1(x,-\lambda)\, d\lambda \qquad (9)$$

- function f_1 – solution of the SL equation $\mathcal{G}f(x) = -\lambda f(x)$ for $\lambda > \Lambda$

$$f_1(x,-\lambda) = {}_2F_1\left(-\frac{\beta}{4} + ik(\lambda), -\frac{\beta}{4} - ik(\lambda); \frac{\gamma}{2}; -\frac{\gamma}{\beta}x\right), \qquad (10)$$

$$k(\lambda) = -i\sqrt{\frac{\beta^2}{16} - \frac{\lambda(\beta-2)}{2\theta}}$$

- normalization constant

$$a(\lambda) = k(\lambda)\left|\frac{B^{\frac{1}{2}}\left(\frac{\gamma}{2},\frac{\beta}{2}\right)\Gamma\left(-\frac{\beta}{4} + ik(\lambda)\right)\Gamma\left(\frac{\gamma}{2} + \frac{\beta}{4} + ik(\lambda)\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(1 + 2ik(\lambda)\right)}\right|^2 \qquad (11)$$

Fractional FS diffusion – transition density

- **fractional FS diffusion** – $(X_\alpha(t),\ t \geq 0)$, where $X_\alpha(t) = X_1(E_t),\ t \geq 0$
  - $(X_1(t),\ t \geq 0)$ – FS diffusion given by the SDE (4)
  - $(E_t,\ t \geq 0)$, where $E_t = \inf\{x > 0 \colon D_x > t\}$
    inverse of the α-stable subordinator, $0 < \alpha < 1$

- **transition density** – defined as

$$P(X_\alpha(t) \in B | X_\alpha(0) = y) = \int_B p_\alpha(x, t; y) dx \tag{12}$$

for any Borel set B from $\mathcal{B}_{(0,\infty)}$

Fractional FS diffusion – transition density

Theorem

The transition density of fractional FS diffusion is given by

$$p_\alpha(x,t;y) = \mathfrak{fs}(x) \sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} F_n(y)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \tag{13}$$

$$+ \frac{\mathfrak{fs}(x)}{\pi} \int\limits_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y,-\lambda),\, f_1(x,-\lambda)\, d\lambda,$$

where F_n are FS polynomials given by (8), f_1 is the solution of the non-fractional SL problem given by (10), $a(\lambda)$ is given by (11) and $\mathcal{E}_\alpha(-\lambda t^\alpha)$ is the Mittag-Leffler function given by (1).

- detailed proof could be found in

N.N. Leonenko, I. Papić, A. Sikorskii and N. Šuvak. (2017) Heavy-tailed fractional Pearson diffusions, Stochastic Processes and their Applications, **127***(11): 3512-3535*

Fractional FS diffusion – transition density, sketch of the proof

$$
\begin{aligned}
P(X_\alpha(t) \in B | X_\alpha(0) = y) &= \int_0^\infty P(X_1(\tau) \in B | X_1(0) = y)\, f_t(\tau)\, d\tau \\
&= \int_0^\infty \int_B p_1(x, \tau; y)\, f_t(\tau)\, dx\, d\tau \\
&= \int_B \int_0^\infty (p_d(x, \tau; y) + p_c(x, \tau; y))\, f_t(\tau)\, d\tau\, dx = \qquad (14)
\end{aligned}
$$

$$
= \int_B \mathfrak{fs}(x) \left(\int_0^\infty \sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} F_n(y) F_n(x) e^{-\lambda_n \tau} f_t(\tau) d\tau + \frac{1}{\pi} \int_0^\infty \int_\Lambda^\infty e^{-\lambda \tau} f_t(\tau) a(\lambda) f_1(y, -\lambda) f_1(x, -\lambda) d\lambda d\tau \right) dx
$$

$$
= \int_B \mathfrak{fs}(x) \left(\sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} F_n(y)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{1}{\pi} \int_\Lambda^\infty \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y, -\lambda)\, f_1(x, -\lambda)\, d\lambda \right) dx \qquad (15)
$$

- change of the order of integration in (14) – follows from the non-negativity of p_1 and f_t (Fubini-Tonelli theorem)
- change of the order of integration in (15) – follows by the Fubini theorem since

$$
\int_\Lambda^\infty \int_0^\infty \left| e^{-\lambda \tau}\, f_t(\tau)\, a(\lambda)\, f_1(y, -\lambda)\, f_1(x, -\lambda) \right| d\tau\, d\lambda < \infty
$$

(for bounds regarding the Gauss hypergeometric functions we refer to Erdelyi, Equation 17, page 77)

Comparing transition densities of heavy-tailed PD anf fPD

- **FS diffusion**

$$p(x,t;x_0) = \mathfrak{fs}(x)\sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} e^{-\lambda_n t}\, F_n(x_0)\, F_n(x)$$
$$+ \frac{\mathfrak{fs}(x)}{\pi}\int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} e^{-\lambda t}\, a(\lambda)\, f_1(x_0,-\lambda) f_1(x,-\lambda)\, d\lambda$$

- **fractional FS diffusion**:

$$p_\alpha(x,t;x_0) = \mathfrak{fs}(x)\sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} \mathcal{E}_\alpha(-\lambda_n t^\alpha)\, F_n(x_0)\, F_n(x)$$
$$+ \frac{\mathfrak{fs}(x)}{\pi}\int_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(x_0,-\lambda) f_1(x,-\lambda)\, d\lambda$$

Fractional FS diffusion – transition density

- transitions density $p = p_\alpha(x, t; y)$ of the FS diffusion satisfies the following equations:
    - fractional forward (Fokker-Planck) equation

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(-\mu(x)p\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\sigma^2(x)}{2}p\right)$$

    with the point-source initial condition $p(x, 0; y) = \delta(x - y)$

    - fractional backward equation

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \mu(y)\frac{\partial p}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p}{\partial y^2}$$

    - $\partial^\alpha / \partial t^\alpha$ – **Caputo fractional derivative** of order $0 < \alpha < 1$

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{d}{dx} f(x-y) y^{-\alpha}\, dy$$

Fractional FS diffusion - strong solutions of time-fractional Kolmogorov backward equation

Theorem

For any g from the domain of the generator $\mathcal{G}$, a strong solution of the fractional Cauchy problem

$$\frac{\partial^\alpha q(y,t)}{\partial t^\alpha} = \mathcal{G} q(y,t), \quad q(y,0) = g(y), \tag{16}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ *is the Caputo fractional derivative of order* $0 < \alpha < 1$, *is given by*

$$q(t;y) = \int_0^\infty p_\alpha(x,t;y) g(x)\, dx, \tag{17}$$

where the transition density $p_\alpha(x,t;y)$ *of the fractional FS diffusion is given by* (13).

- detailed proof - Leonenko et al. (2017)

Correlation structure of the fractional Pearson diffusion

Theorem

Let us assume that $X_\alpha(0)$ has the probability density m, where $m(\cdot)$ is the stationary density of the corresponding Pearson diffusion. Then

$$\mathrm{Corr}\left[X_\alpha(t),\, X_\alpha(s)\right] = \mathcal{E}_\alpha(-\theta t^\alpha) + \frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)} \int\limits_0^{s/t} \frac{\mathcal{E}_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}} dz \qquad (18)$$

for $t \geq s > 0$.

N.N. Leonenko, M.M. Meerschaert and A. Sikorskii (2013) Correlation structure of fractional Pearson diffusions, Computers and Mathematics with Applications, **66***(5): 737–745*

- **fractional diffusion:**

$$\mathrm{Corr}[X_\alpha(t), X_\alpha(s)] = \frac{1}{t^\alpha \Gamma(1-\alpha)} \left(\frac{1}{\theta} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right) (1 + o(1)), \quad t \to \infty$$

- **diffusion**

$$\mathrm{Corr}[X_1(t), X_1(s)] = e^{-\theta(t-s)}$$

General approach to diffusion approximation via Markov chains

- **starting Markov chain** with state space $S_n \subseteq \mathbb{N}_0$ and transition probabilities p_{ij}, $i,j \in S_n$:

$$\{N^{(n)}(r),\ r \in \mathbb{N}\}$$

- **diffusion process** $\{X(t),\ t \geq 0\}$ with state space S:

$$dX(t) = \mu(X(t))\,dt + \sqrt{\sigma^2(X(t))}\,dW(t), \quad t \geq 0, \quad x \in S$$

- connection between starting points $N^{(n)}(0) = i \in S_n$ and $X(0) = x \in S$

$$i = \lfloor g_n(x) \rfloor ,$$

for n large enough, where $g_n : S \to \mathbb{R}$, is strictly monotonic function such that

$$\lim_{n\to\infty} \left\| g_n^{-1}(i+1) - g_n^{-1}(i) \right\|_\infty = 0.$$

General approach to diffusion approximation via Markov chains

- **new** Markov chain with state space $g_n^{-1}(S_n)$:

$$H^{(n)}(r) = g_n^{-1}\left(N^{(n)}(r)\right) \tag{19}$$

- time-changed stochastic process $\{X^{(n)}(t),\ t \geq 0\}$:

$$X^{(n)}(t) := H^{(n)}\left(\lfloor h_n^{-1} t \rfloor\right), \tag{20}$$

where $(h_n,\ n \in \mathbb{N})$ is sequence of positive reals tending to zero as $n \to \infty$.

General approach to diffusion approximation via Markov chains

Theorem

Let $\{H^{(n)}(r),\ r \in \mathbb{N}_0\}$, for each $n \in \mathbb{N}$, be the Markov chain defined by (19). Let $X^n = \{X^{(n)}(t),\ t \geq 0\}$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time change (20). If

$$\mu_n(x) := h_n^{-1} \sum_{j=0}^{n} p_{ij} \left(g_n^{-1}(j) - g_n^{-1}(i) \right), \quad \sigma_n^2(x) := h_n^{-1} \sum_{j=0}^{n} p_{ij} \left(g_n^{-1}(j) - g_n^{-1}(i) \right)^2,$$

$$R_n(x) := h_n^{-1} \sum_{j=0}^{n} p_{ij} \frac{\left(g_n^{-1}(j) - g_n^{-1}(i) \right)^3}{3!} f'''(\zeta), \quad |\zeta - g_n^{-1}(i)| < |g_n^{-1}(j) - g_n^{-1}(i)| \tag{21}$$

have uniform limits

$$\lim_{n\to\infty} \|\mu_n - \mu\|_\infty = \lim_{n\to\infty} \left\| \sigma_n^2 - \sigma^2 \right\|_\infty = \lim_{n\to\infty} \|R_n\|_\infty = 0, \tag{22}$$

where μ and σ^2 are infinitesimal parameters of the coresponding diffusion process $X = \{X(t),\ t \geq 0\}$ with state space S, then

$$X^n \Rightarrow X \text{ in } \mathbb{D}([0, +\infty\rangle ; S).$$

CTRW – definition

- $S(n) = Y_1 + Y_2 + \dots Y_n$ - random walk with iid particle jumps
- $T(n) = J_1 + J_2 + \dots J_n$, $T(0) = 0$ - random walk where $J_n \geq 0$ are iid waiting times between particle jumps
- particle arrives at location $S(n)$ at time $T(n)$
- we assume that Y_n is independent of J_n.
- $$N(t) = \max\{n \geq 0 \colon T_n \leq t\} \tag{23}$$
  is the number of jumps up to time $t \geq 0$
- CTRW $S(N(t))$ - represents particle location at time $t \geq 0$

Fractional diffusions as the correlated CTRWs limits

- $T_0 = 0,\ T(r) = G_1 + \dots G_r$, where $G_r \geq 0$ are iid waiting times between particle jumps that are independent of the Markov chain $(H_r^{(n)},\ r \in \mathbb{N}_0)$
- G_1 is in the domain of attraction of the α-stable distribution with index $0 < \alpha < 1$
- the waiting time of the Markov chain $(H_r^{(n)},\ r \in \mathbb{N}_0)$ until its r-th move is described by G_r
- $$N(t) = \max\{r \geq 0 \colon\ T_r \leq t\} \qquad (24)$$
  is the number of jumps up to time $t \geq 0$
- $$\left(H^{(n)}(N(t)),\ t \geq 0\right)$$
  is the correlated continuous time random walks (CTRW) and describes the state of the Markov chain at time $t \geq 0$
- $$n^{-\frac{1}{\alpha}} T(\lceil nt \rceil) \Rightarrow D_t, \quad n \to \infty \qquad (25)$$
  in Skorohod space $\mathbb{D}(\mathbb{R}^+)$ with J_1 topology

Fractional diffusions as the correlated CTRWs limits

Theorem

Let $\{A(t),\, t \geq 0\}$ be the weak limit of $\{A^{(n)}(t),\, t \geq 0\}$, where both processes are càdlàg, i.e. let

$$A^{(n)} \Rightarrow A \text{ in } \mathbb{D}([0,\, +\infty\rangle\, ; S)$$

with J_1 topology, where S is the state space for the process A. Let $\{N(t),\, t \geq 0\}$ be the renewal process defined in (24), and $\{E(t),\, t \geq 0\}$ be the inverse of the standard α-stable subordinator $\{D(t),\, t \geq 0\}$ with $0 < \alpha < 1$. Then

$$A^{(n)}\left(n^{-1} N\left(n^{1/\alpha} t\right)\right) \Rightarrow A(E(t)), \quad n \to \infty \tag{26}$$

in the Skorokhod space $\mathbb{D}([0,\, +\infty\rangle\, ; S)$ with J_1 topology.

Fractional Pearson diffusions as the correlated CTRWs limits

- **Bernoulli-Laplace urn-scheme model:** Ornstein-Uhlenbeck process
- **Wright-Fisher urn-scheme model:** Cox-Ingersoll-Ross and Jacobi diffusion
- **Ehrenfest-Brillouin model:** Jacobi diffusion
- **without specific model:** heavy-tailed Pearson diffusions (Fisher-Snedecor, reciprocal gamma and Student diffusion)

Bernoulli-Laplace urn scheme - historical roots

Bernoulli-Laplace urn scheme

P. Laplace (1812) Théorie Analytique des Probabilités, Ve. Courcier, Paris

- two urns, A and B, each contains n of total $2n$ balls
- Of total $2n$ balls, n balls are black and n are white
- from each urn we randomly choose one ball which is then placed in the opposite urn
- number of white balls in urn A after r draws?

$(Z_r^{(n)}, r \in \mathbb{N}_0)$, for each $n \in \mathbb{N}$ - Markov chain with state space $\{0, 1, 2, \ldots, n\}$ transition probabilities:

$$p_{x,x+1} = \left(1 - \frac{x}{n}\right)^2, \quad p_{x,x} = 2\frac{x}{n}\left(1 - \frac{x}{n}\right), \quad p_{x,x-1} = \left(\frac{x}{n}\right)^2, \quad 0 \text{ otherwise} \tag{27}$$

Bernoulli-Laplace urn scheme - historical roots

- Laplace was interested in finding heat kernel
- $z_{x,r}$ - probability that urn A contains x white balls after r draws.
- $$z_{x,\,r+1} = \left(\frac{x+1}{n}\right)^2 z_{x+1,\,r} + 2\frac{x}{n}\left(1-\frac{x}{n}\right) z_{x,\,r} + \left(1-\frac{x-1}{n}\right) z_{x-1,\,r} \qquad (28)$$
- in order to approximate the solution of the equation (28), Laplace introduced the following transformations:

$$x = \frac{1}{2}(n + \mu\sqrt{n}), \quad r = nr'$$

Bernoulli-Laplace urn scheme - historical roots

- $$\begin{aligned} z_{x+1,\,r} &\approx z_{x,\,r} + \frac{\partial z_{x,\,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,\,r}}{\partial x^2} \\ z_{x-1,\,r} &\approx z_{x,\,r} - \frac{\partial z_{x,\,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,\,r}}{\partial x^2} \\ z_{x,\,r+1} &\approx z_{x,\,r} + \frac{\partial z_{x,\,r}}{\partial r} \end{aligned}$$

- $\Delta\mu = \frac{2}{\sqrt{n}},\ z_{x,\,r} = U(\mu,\, r')$

$$\begin{aligned} z_{x+1,\,r} &= U(\mu + \Delta\mu,\, r') \approx U + \Delta\mu\frac{\partial U}{\partial \mu} + \frac{1}{2}(\Delta\mu)^2\frac{\partial^2 U}{\partial \mu^2} \\ z_{x-1,\,r} &= U(\mu - \Delta\mu,\, r') \approx U - \Delta\mu\frac{\partial U}{\partial \mu} + \frac{1}{2}(\Delta\mu)^2\frac{\partial^2 U}{\partial \mu^2} \\ z_{x,\,r+1} &= U(\mu,\, r' + \frac{1}{n}) \approx U + \frac{1}{n}\frac{\partial U}{\partial r'} \end{aligned}$$

Bernoulli-Laplace urn scheme - historical roots

- $$\frac{\partial U}{\partial r'} = -\frac{\partial}{\partial \mu}(-2\mu U) + \frac{1}{2}\frac{\partial^2}{\partial \mu^2}(2U) = 2U + 2\mu\frac{\partial U}{\partial \mu} + \frac{\partial^2 U}{\partial \mu^2} \tag{29}$$
- special case of Kolmogorov forward equation (Fokker-Planck equation)

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(\mu(x)p(x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)p(x,t)\right)$$

for

$$\mu(x) = -2x,\ \sigma^2(x) = 2$$

which are the infinitesimal parameters of specifically parametrized Ornstein-Uhlenbeck process.

Bernoulli-Laplace urn scheme - OU diffusion

- **space variable transformation**

$$H_r^{(n)} = \frac{1}{a\sqrt{n}}\left(2Z_r^{(n)} - n - b\sqrt{n}\right), \quad a,\ b \in \mathbb{R},\ a \neq 0 \tag{30}$$

- **time variable transformation**

$$X_t^{(n)} := H^{(n)}_{\lfloor \frac{\theta}{2} nt \rfloor},\ \ \theta > 0 \tag{31}$$

- Let $X = (X_t,\ t \geq 0)$ be the OU diffusion, i.e. the solution of the SDE

$$dX_t = -\theta\left(X_t + \frac{b}{a}\right)dt + \sqrt{\frac{\theta}{a^2}}dW(t), \quad t \geq 0, \tag{32}$$

  where $(W(t), t \geq 0)$ is the standard Brownian motion.

- infinitesimal generator:

$$\mathcal{A}f(x) = -\theta\left(x + \frac{b}{a}\right)f'(x) + \frac{1}{2}\frac{\theta}{a^2}f''(x)$$

- core of the generator: $C_c^3(\mathbb{R})$

Bernoulli-Laplace urn scheme - OU diffusion

Theorem

Let $(H_r^{(n)},\ r \in \mathbb{N}_0)$*, for each* $n \in \mathbb{N}$*, be the Markov chain defined by* (30)*. Let* $(X_t^{(n)},\ t \geq 0)$ *for each* $n \in \mathbb{N}$*, be the time-changed process* (31)*. Then*

$$X^n \Rightarrow X,\ \ n \to \infty$$

in Skorohod space $\mathbb{D}(\mathbb{R})$*, where* $X = (X_t,\ t \geq 0)$ *is the Ornstein-Uhlenbeck process defined by* (32)*.*

Theorem

Let $\{N(t),\ t \geq 0\}$ *be the renewal process defined in* (24)*, and* $\{E(t),\ t \geq 0\}$ *be the inverse of the standard* α*-stable subordinator* $\{D(t),\ t \geq 0\}$ *with* $0 < \alpha < 1$*. Then*

$$X^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}(\mathbb{R})$ *with* J_1 *topology.*

Wright-Fisher urn scheme - Jacobi and CIR diffusion

Wright-Fisher urn scheme

S. Karlin and H.M. Taylor (1981) A Second Course in Stochastic Processes, Academic Press, New York

- describes gene mutations (in some genetic pool) over time, strongly influencing selection in the corresponding population
- population has n **individuals**, where in the current generation, i individuals are of type A and $n - i$ are of type a
- Once born, individual of A-type can mutate in a-type with probability α and individual of a-type can mutate in A-type with probability β
- survival ability of each type is modeled by parameter **parametar** s: the ratio of A-types over a-types is equal to $1 + s$

fraction of mature A-types in population before reproduction is

$$p_i = \frac{(1+s)\left[i(1-\alpha) + (n-i)\beta\right]}{(1+s)\left[i(1-\alpha) + (n-i)\beta\right] + \left[i\alpha + (n-i)(1-\beta)\right]} \tag{33}$$

Wright-Fisher urn scheme - Jacobi and CIR diffusion

Wright-Fisher urn scheme

- **Assumption of the model:** the composition of the next generation is determined through n binomial trials, where the probability of producing an A-type in each trial is p_i, where p_i is given via (33)
- the number of A-types in population over time is described by Markov chain $(G_r^{(n)},\ r \in \mathbb{N}_0)$ with state space $\{0, 1, 2, \ldots, n\}$ and transition probabilities

$$p_{ij} = \binom{n}{j} p_i^j (1 - p_i)^{n-j} \tag{34}$$

- **parameters of the model:** α, β i s

Wright-Fisher urn scheme - Jacobi diffusion

- **parameters of the model:**

$$\alpha = \frac{a}{n}, \quad \beta = \frac{b}{n}, \quad s = 0, \quad a, b > 0$$

- **space variable transformation**

$$H_r^{(n)} = \frac{1}{n} G_r^{(n)} \tag{35}$$

- **time variable transformation**

$$Y_t^{(n)} := H_{\lfloor \theta n t \rfloor}^{(n)}, \quad \theta > 0 \tag{36}$$

- Let $Y = (Y_t,\, t \geq 0)$ be the Jacobi diffusion, i.e. solution of the SDE

$$dY_t = -\theta(a+b)\left(Y_t - \frac{b}{a+b}\right) dt + \sqrt{\theta Y_t(1-Y_t)}dW(t), \quad t \geq 0, \tag{37}$$

  where $(W(t), t \geq 0)$ is standard Brownian motion
- infinitesimal generator:

$$\mathcal{A}f(y) = \theta(-y(a+b)+b)f'(y) + \frac{1}{2}\theta y(1-y)f''(y)$$

- core of the generator: $C_c^3([0,1])$

Wright-Fisher urn scheme - Jacobi diffusion

Theorem

Let $(H_r^{(n)},\ r \in \mathbb{N}_0)$, *for each* $n \in \mathbb{N}$, *be the Markov chain defined by* (35). *Let* $(Y_t^{(n)},\ t \geq 0)$ *for each* $n \in \mathbb{N}$, *be the time-changed process* (36). *Then*

$$Y^n \Rightarrow Y,\ \ n \to \infty$$

in Skorohod space $\mathbb{D}([0,1])$, *where* $Y = (Y_t,\ t \geq 0)$ *is the Jacobi diffusion defined by* (37).

Theorem

Let $\{N(t),\ t \geq 0\}$ *be the renewal process defined in* (24), *and* $\{E(t),\ t \geq 0\}$ *be the inverse of the standard* α*-stable subordinator* $\{D(t),\ t \geq 0\}$ *with* $0 < \alpha < 1$. *Then*

$$Y^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow Y(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}([0,1])$ *with* J_1 *topology.*

Wright-Fisher urn scheme - CIR diffusion

- **parameters of the model:**

$$\alpha = \frac{a}{n^d}, \quad \beta = \frac{b}{n}, \quad 0 < d < 1, \quad a,\ b > 0, \quad s = 0$$

- **space variable transformation**

$$H_r^{(n)} = \frac{G_r^{(n)}}{n^d} \tag{38}$$

- **time variable transformation**

$$Z_t^{(n)} := H^{(n)}_{\lfloor \frac{\theta}{a} n^d t \rfloor}, \quad \theta > 0 \tag{39}$$

- Let $Z = (Z_t,\ t \geq 0)$ be the CIR diffusion, i.e. the solution of the SDE

$$dZ_t = -\theta \left(Z_t - \frac{b}{a} \right) dt + \sqrt{\frac{\theta}{a} Z_t dW_t}, \quad t \geq 0, \quad \theta > 0, \quad a > 0, \quad b > 0, \tag{40}$$

  where $(W(t), t \geq 0)$ is standard Brownian motion
- infinitesimal generator:

$$\mathcal{A}f(z) = -\theta \left(z - \frac{b}{a} \right) f'(z) + \frac{1}{2} \frac{\theta}{a} z f''(z)$$

- core of the generator: $C_c^3([0, \infty))$

Wright-Fisher urn scheme - CIR diffusion

Theorem

Let $(H_r^{(n)},\ r \in \mathbb{N}_0)$*, for each* $n \in \mathbb{N}$*, be the Markov chain defined by* (38)*. Let* $(Z_t^{(n)},\ t \geq 0)$ *for each* $n \in \mathbb{N}$*, be the time-changed process* (39)*. Then*

$$Z^n \Rightarrow Z,\ n \to \infty$$

in Skorohod space $\mathbb{D}(\mathbb{R}^+)$*, where* $Z = (Z_t,\ t \geq 0)$ *is the CIR diffusion defined by* (40)*.*

Theorem

Let $\{N(t),\ t \geq 0\}$ *be the renewal process defined in* (24)*, and* $\{E(t),\ t \geq 0\}$ *be the inverse of the standard* α*-stable subordinator* $\{D(t),\ t \geq 0\}$ *with* $0 < \alpha < 1$*. Then*

$$Z^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow Z(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}(\mathbb{R}^+)$ *with* J_1 *topology.*

Ehrenfest-Brillouin model - Jacobi diffusion

Ehrenfest-Brillouin model

Garibaldi & Scalas (2010) Finitary Probabilistic Methods in Econophysics, Cambridge University Press

- consider a population of n objects that could be interpreted as particles in a physical system, genes in applications in genetics or agents in economics models
- state of the system:

$$\mathbf{n} = (n_1, \dots, n_i, \dots, n_N), \quad n_k \geq 0, \quad \forall k \in \{1, \dots N\}, \quad \sum_{k=1}^{N} n_k = n.$$

- **dynamics of the system:** from initial state $\mathbf{n} = (n_1, \dots, n_i, \dots, n_k, \dots, n_N)$ to the final state $\mathbf{n}_i^k = (n_1, \dots, n_i - 1, \dots, n_k + 1, \dots, n_N)$

Ehrenfest-Brillouin model - Jacobi diffusion

- the destruction of the object on the ith coordinate (category) in the initial state $\mathbf{n}$ (the "Ehrenfest's term"), resulting in the state vector

  $$\mathbf{n}_i = (n_1, \ldots, n_i - 1, \ldots, n_k, \ldots, n_N),$$

  which happens with probability

  $$P(\mathbf{n}_i|\mathbf{n}) = \frac{n_i}{n}$$

- the creation of the object in the kth coordinate (category) given the state vector $\mathbf{n}_i$, resulting in the final state vector $\mathbf{n}_i^k$, with probability

  $$P(\mathbf{n}_i^k|\mathbf{n}_i) = \frac{\alpha_k + n_k - \delta_{k,i}}{\alpha + n - 1},$$

  where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is the vector of parameters such that $\sum_{k=1}^{N} \alpha_k = \alpha$ and $\delta_{k,i}$ is the usual Kronecker's delta symbol, taking value 1 when $k = i$ and zero otherwise.

Ehrenfest-Brillouin Markov chain- Jacobi diffusion

- $(G^{(n)}(r),\ r \in \mathbb{N}_0)$ - marginal Ehnrefest-Brillouin Markov chain with state space $\{0, 1, \ldots, n\}$
- transition probabilities:

$$p_{i,i+1} = \frac{n-i}{n} \cdot \frac{\alpha_1 + i}{\alpha_1 + \alpha_2 + n - 1}, \quad p_{i,i-1} = \frac{i}{n} \cdot \frac{\alpha_2 + n - i}{\alpha_1 + \alpha_2 + n - 1}$$
$$p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \quad 0 \ \textit{otherwise},\ a > 0,\ b > 0.$$

Ehrenfest-Brillouin Markov chain- Jacobi diffusion

- **space variable transformation**

$$H_r^{(n)} = \frac{G_r^{(n)}}{n} \tag{41}$$

- **time variable transformation**

$$Y_t^{(n)} := H_{\lfloor \theta n^2 t \rfloor}^{(n)}, \ \theta > 0 \tag{42}$$

- Let $Y = (Y_t, \ t \geq 0)$ be the Jacobi diffusion, i.e. the solution of the SDE

$$dY_t = -\theta((\alpha_1 + \alpha_2)y - \alpha_1)dt + \sqrt{(2\theta)y(1-y)}dW_t, \quad t \geq 0 \tag{43}$$

  where $(W(t), t \geq 0)$ is the standard Brownian motion.

- infinitesimal generator:

$$\mathcal{A}f(y) = -\theta((\alpha_1 + \alpha_2)y - \alpha_1)f'(y) + \frac{1}{2}(2\theta)y(1-y)f''(y)$$

- core of the generator: $C_c^3([0, 1])$

Ehrenfest-Brillouin Markov chain- Jacobi diffusion

Theorem

Let $(H_r^{(n)},\ r \in \mathbb{N}_0)$*, for each* $n \in \mathbb{N}$*, be the Markov chain defined by* (41)*. Let* $(Y_t^{(n)},\ t \geq 0)$ *for each* $n \in \mathbb{N}$*, be the time-changed process* (42)*. Then*

$$Y^n \Rightarrow Y,\ \ n \to \infty$$

in Skorohod space $\mathbb{D}([0,1])$*, where* $Y = (Y_t,\ t \geq 0)$ *is the Jacobi diffusion defined by* (43)*.*

Theorem

Let $\{N(t),\ t \geq 0\}$ *be the renewal process defined in* (24)*, and* $\{E(t),\ t \geq 0\}$ *be the inverse of the standard* α*-stable subordinator* $\{D(t),\ t \geq 0\}$ *with* $0 < \alpha < 1$*. Then*

$$Y^{(n)}\left(n^{-1}N\left(n^{1/\alpha}t\right)\right) \Rightarrow Y(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}([0,1])$ *with* J_1 *topology.*

Markov chain for Student diffusion

- $(Z^{(n)}(r),\ r \in \mathbb{N})$ - Markov chain with state space $\{0, 1, \ldots, n\}$
- transition probabilities:

$$p_{0,1} = 1, \quad p_{n,n-1} = 1,$$

$$p_{i,i+1} = \frac{1}{2c}\left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n}\left(1 - \frac{i}{n}\right)^2, \quad p_{i,i-1} = \frac{1}{2c}\left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n}\left(\frac{i}{n}\right)^2,$$
$$p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \quad 0 \quad otherwise,$$

where $i \in \{1, 2, \ldots, n-1\}$, $0 < d < 1$, $c > 1$ and n is large enough to ensure $p_{i,i+1} + p_{i,i-1} < 1$.

Markov chain for Student diffusion

- **space variable transformation**

$$H_r^{(n)} = \frac{1}{a\sqrt{n}}\left(2Z^{(n)}(r) - n - b\sqrt{n}\right), \; a > 0, \; b \in \mathbb{R} \tag{44}$$

- **time variable transformation**

$$X_t^{(n)} := H^{(n)}\left(\left\lfloor \frac{\theta}{2} n^2 t \right\rfloor\right), \; \theta > 0 \tag{45}$$

- Let $X = (X_t, \; t \geq 0)$ be the Student diffusion, i.e. the solution of the SDE

$$dX_t = -\theta\left(X_t + \frac{b}{a}\right)dt + \sqrt{2\theta\left(\frac{1}{c}\left(x + \frac{b}{a}\right)^2 + \frac{1}{2a^2}\right)}\,dW_t, \quad t \geq 0 \tag{46}$$

  where $(W(t), t \geq 0)$ is the standard Brownian motion.
- infinitesimal generator:

$$\mathcal{A}f(x) = -\theta\left(x + \frac{b}{a}\right)f'(x) + \frac{1}{2}2\theta\left(\frac{1}{c}\left(x + \frac{b}{a}\right)^2 + \frac{1}{2a^2}\right)f''(x)$$

- core of the generator: $C_c^3(\mathbb{R})$

Markov chain for Student diffusion

Theorem

Let $(H_r^{(n)},\ r \in \mathbb{N}_0)$, *for each* $n \in \mathbb{N}$, *be the Markov chain defined by* (44). *Let* $(Y_t^{(n)},\ t \geq 0)$ *for each* $n \in \mathbb{N}$, *be the time-changed process* (45). *Then*

$$X^n \Rightarrow X,\ \ n \to \infty$$

in Skorohod space $\mathbb{D}(\mathbb{R})$, *where* $X = (X_t,\ t \geq 0)$ *is the Student diffusion defined by* (46).

Theorem

Let $\{N(t),\ t \geq 0\}$ *be the renewal process defined in* (24), *and* $\{E(t),\ t \geq 0\}$ *be the inverse of the standard* α*-stable subordinator* $\{D(t),\ t \geq 0\}$ *with* $0 < \alpha < 1$. *Then*

$$X^{(n)}\left(n^{-1} N\left(n^{1/\alpha} t\right)\right) \Rightarrow X(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}(\mathbb{R})$ *with* J_1 *topology.*

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