Convergence in law of partial sum processes in p-variation norm

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Joint work with Alfredas Račkauskas

Let $X, X_1, X_2, \dots$ be independent identically distributed real-valued random variables. For each $n \geq 1$, let

$$Z_n := X_1 + \cdots + X_n, \quad \text{and} \quad V_n := (X_1^2 + \cdots + X_n^2)^{1/2}.$$

Recall that X is in the domain of attraction of the normal law (denoted by $X \in DAN$) if there exists a norming sequence $b_n \uparrow \infty$ such that

$$b_n^{-1} Z_n \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ is a standard Gaussian random variable.

Theorem (Giné, Götze and Mason (1997))

The following two statements are equivalent:

(a) $X \in DAN$ *and* $\mathrm{E}(X) = 0$*;*

(b) $V_n^{-1} Z_n \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$

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Notation for stochastic processes

Given a sequence $X_1, X_2, \dots$ of independent identically distributed real-valued random variables, for each $n \geq 1$ let $S_n = \{S_n(t)\colon\ t \in [0,1]\}$ be the n-th *partial sum process* with values

$$S_n(t) := \sum_{i=1}^{\lfloor tn \rfloor} X_i = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{n}), \\ Z_k, & \text{if } t \in [\frac{k}{n}, \frac{k+1}{n}), k \in \{1, \dots, n-1\}, \\ Z_n, & \text{if } t = 1. \end{cases}$$

For a class $\mathcal{F}$ of functions $f\colon\ [0,1] \to \mathbb{R}$ and for each $n \geq 1$, let $\mu_{n,\mathcal{F}} = \{\mu_n(f)\colon\ f \in \mathcal{F}\}$ be the n-th *random measure process* with values

$$\mu_n(f) := \sum_{i=1}^{n} X_i f\Big(\frac{i}{n}\Big) = \sum_{i=1}^{n} X_i \epsilon_{i/n}(f),$$

where ϵ_x is a point measure.
Note that $S_n(t) = \mu_n(\mathbf{1}_{[0,t]})$ for $t \in [0,1]$.

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Illustration of results

Let W be the isonormal Gaussian process on the Hilbert space $L_2([0,1],\lambda)$. This process can be obtained by the Itô integral

$$\mathsf{W}(f) = \int_0^1 f(t)dW(t), \quad f \in L_2([0,1]),$$

where $W = \{W(t)\colon\ t \in [0,1]\}$ is a standard Wiener process. Let $\mathsf{W}_{\mathcal{F}}$ be the restriction of W to a class of functions $\mathcal{F} \subset L_2([0,1])$.

Theorem

The following three statements are equivalent:

(a) $X \in DAN$ *and* $\mathsf{E}(X) = 0$;

(b) *for each (some)* $p > 2$, $V_n^{-1} S_n \xrightarrow[n\to\infty]{\mathcal{D}} W$ *in* $\mathcal{W}_p[0,1]$;

(c) *for each (some)* $q \in [1,2)$, $V_n^{-1}\mu_{n,\mathcal{F}_q} \xrightarrow[n\to\infty]{\mathcal{D}} \mathsf{W}_{\mathcal{F}_q}$ *in* $\ell^\infty(\mathcal{F}_q)$,

where $\mathcal{F}_q$ *is the unit ball of the Banach space of functions of bounded q-variation.*

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The Banach space of functions of bounded p-variation

For a function $f\colon [0,1] \to \mathbb{R}$ and for a positive number p the p-variation is

$$v_p(f) := \sup\Big\{ \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^p \colon 0 = t_0 < t_1 < \cdots < t_m = 1, \ m \in \mathbb{N}_+ \Big\}$$

If $v_p(f) < \infty$ then f has bounded p-variation and the set of all such functions is denoted by $\mathcal{W}_p[0,1]$.

1. For $0 < p < r < \infty$ $W_p[0,1] \subsetneq W_r[0,1]$. In the case $p = 1$ the elements of $\mathcal{W}_1[0,1]$ are functions with bounded total variation.
2. Each function of bounded p-variation for some $p \in (0, \infty)$ is regulated, and so it is bounded.
3. For $f \in \mathcal{W}_p[0,1]$ with $1 \le p < \infty$

$$\|f\|_{[p]} := (v_p(f))^{1/p} + \|f\|_{\sup}$$

is a norm. The set $\mathcal{W}_p[0,1]$ is a Banach space with the norm $\|\cdot\|_{[p]}$.

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p-variation of sample paths of Wiener process

Let $W = \{W(t)\colon\ t \in [0,1]\}$ be a standard *Wiener* process on the interval $[0,1]$. Due to results of *N. Wiener* (1923) and *P. Lévy* (1940):

$$v_p(W) < +\infty \quad \text{almost surely if and only if} \quad p > 2,$$

$$v_2(W) = +\infty \quad \text{almost surely.}$$

More precise information can be obtained in terms of ϕ-variation, defined as p-variation except that the power function $x \mapsto x^p$, $x \geq 0$, is replaced by a more general function ϕ.
Due to results of *S.J. Taylor* (1972):

$$v_{\psi_1}(W) < +\infty \quad \text{a.s. where} \quad \psi_1(x) = x^2/LL(1/x),\ 0 < x \leq e^{-e}.$$

$$v_\psi(W) = +\infty \quad \text{a.s. for any } \psi \text{ such that } \psi_1(x) = o(\psi(x)) \text{ as } x \downarrow 0.$$

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Then for any $p \in (0, \infty)$

$$v_p(S_n) = \max\Big\{ \sum_{j=1}^{m} |X_{k_{j-1}+1} + \cdots + X_{k_j}|^p \colon 0 = k_0 < \cdots < k_m = n \Big\}.$$

Theorem (J. Bretagnolle, 1972)

Let $p \in (0, 2)$ and let $X_1, X_2, \dots$ be independent, $E|X_i|^p < \infty$ and $EX_i = 0$ if $p > 1$. There exists a finite number C_p such that

$$\left(\sum_{i=1}^{n} E|X_i|^p \leq \right) E v_p(S_n) \leq C_p \sum_{i=1}^{n} E|X_i|^p.$$

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p-variation of sample paths of empirical process

Let $X_1, X_2, \ldots$ be independent uniformly in $[0,1]$ distributed real random variables. For each $n \in \mathbb{N}_+$ let F_n be the n-th empirical distribution function based on $X_1, \ldots, X_n$, i.e.

$$F_n(t) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{[0,t]}(X_i), \quad t \in [0,1].$$

Theorem (R.M. Dudley, 1992)

For $2 < p < \infty$, the convergence in law

$$\sqrt{n}(F_n - I) \Rightarrow B \quad \text{in } \mathcal{W}_p[0,1]$$

holds, where B is the Brownian bridge process and $I(t) = t$ for $t \in [0,1]$.

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Invariance principle

Let S_n be a partial sum processas as before.
The classical invariance principle says that the asymptotic distribution of a functional $F(S_n)$ does not depend on a distribution of $X_1, X_2, \ldots$ provided they are iid random variables with mean zero and finite second moment. (Erdös and Kac (1946), Doob (1949), Donsker (1951) and others)

The invariance principle can be split into two parts:

- functional limit theorem for partial sum processes in a function space E endowed with a norm $\|\cdot\|$ (say $D[0,1]$ with the uniform norm);
- continuous mapping theorem for a functional F on a function space E endowed with a norm $\|\cdot\|$.

Note that there are more continuous functionals F on a function space E with a stronger norm $\|\cdot\|$.

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Why p-variation?

Why take $E = \mathcal{W}_p[0,1]$?

- Motivation for us is the fact that a solution to a linear integral equation

$$X(t) = 1 + \int_0^t X\,df$$

(given as the limit of $\prod_{i=1}^n [1 + f(t_i) - f(t_{i-1})]$) as a mapping

$$\mathcal{W}_p[0,1] \ni f \mapsto X \in \mathcal{W}_p[0,1], \quad \text{with } 1 \le p < 2,$$

is Fréchet differentiable infinitely many times (RM Dudley and R.N, 1999).

- While with $D[0,1]$ in place of $\mathcal{W}_p[0,1]$, X is not Fréchet differentiable even once.

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Invariance principle in $\mathcal{W}_p[0,1]$

Let S_n be a partial sum process based on iid rv's and let W be a standrad Brownian motion.

Theorem (R.N. and A. Račkauskas, 2008)

The following statements are equivalent:

(a) $EX_1 = 0$ *and* $\sigma^2 = EX_1^2 < \infty$;

(b) *for every (some)* $p > 2$, $n^{-1/2}S_n \Rightarrow \sigma W$ *in* $W_p[0,1]$.

If, in the preceding theorem, S_n is replaced by a smooth version of the partial sum process and p-variation norm is replaced by $1/p$-Hölder norm, then the theorem ceases to hold.

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(a) $EX_1 = 0$ *and* $\sigma^2 = EX_1^2 < \infty$*;*

(b) *for every (some)* $p > 2$, $n^{-1/2}S_n \Rightarrow \sigma W$ *in* $W_p[0,1]$.

If, in the preceding theorem, S_n is replaced by a smooth version of the partial sum process and p-variation norm is replaced by $1/p$-Hölder norm, then the theorem ceases to hold.

α-Hölder norm ($\alpha = \frac{1}{p}$)

Given i.i.d. r.v.'s $X, X_1, X_2, \ldots$, let W_n be the *polygonal line process* with values

$$W_n(t) :== \begin{cases} X_1 + \cdots + X_{[nt]} + (nt - [nt])X_{[nt]+1}, & \text{if } t \in [0,1), \\ X_1 + \cdots + X_n, & \text{if } t = 1. \end{cases}$$

Given $\alpha \in (0,1]$, let $\mathrm{H}^o_\alpha[0,1]$ be the set of functions $f\colon [0,1] \to \mathbb{R}$ such that

$$|f(t+h) - f(t)| = o(h^\alpha) \quad \text{uniformly in } t.$$

Endowed with the norm $f(0) + \sup_{0<t-s<1} |f(t) - f(s)|/(t-s)^\alpha$ the set $\mathrm{H}^o_\alpha[0,1]$ is separable Banach space.

Theorem (A. Račkauskas and C. Suquet, 2004)

Let $\alpha \in (0, 1/2)$. The following statements are equivalent:

(a) $\mathrm{E}X = 0$ *and* $\mathrm{P}\{|X| > t\} = o(t^{-\frac{1}{1/2-\alpha}})$ *as* $t \to \infty$

(b) $n^{-1/2} W_n \xrightarrow[n\to\infty]{\mathcal{D}} W$ *in* $\mathrm{H}^o_\alpha[0,1]$.

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Back to new results

- Given a sequence $X, X_1, X_2, \ldots$ of i.i.d. r.v.'s, for each $n \geq 1$, let

$$\mu_n = \sum_{j=1}^{n} X_j \epsilon_{j/n}, \quad \text{and} \quad M_n := \sum_{j=1}^{n} X_j \lambda_{n,j}$$

where ϵ_x is the point measure and λ_{nj} is the uniform probability on $[(j-1)/n, j/n]$.

- Note that values of the partial sum process values are obtained by

$$S_n(t) = \mu_n(\mathbf{1}_{[0,t]}) \qquad \text{for } t \in [0,1]$$

and values of the polygonal line process are obtained by

$$W_n(t) = M_n(\mathbf{1}_{[0,t]}) \qquad \text{for } t \in [0,1].$$

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Back to new results (cont.)

- For a class $\mathcal{F}$ of functions $f\colon [0,1] \to \mathbb{R}$, let ρ_2 be a pseudo-metric with values
$$\rho_2(f,g) := \Big(\int_0^1 (f-g)^2 \, d\lambda \Big)^{1/2}.$$
Then $M_n \in UC(\mathcal{F}, \rho_2)$. While $\mu_n \in \ell^\infty(\mathcal{F})$.
- When $\mathcal{F} = \mathcal{F}_q$ is the unit ball in $\mathcal{W}_q[0,1]$ and $1 \le q < 2$ then $(\mathcal{F}_q, \rho_2)$ is totally bounded. So, $UC(\mathcal{F}, \rho_2)$ equipped with the sup norm can be imbedded into $C(K)$, which is a separable Banach space.

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Back to our results (cont.)

Theorem

Let $1 \leq q < 2$, and let $\mathcal{F}_q$ be the unit ball of $\mathcal{W}_q[0,1]$. The following three statements are equivalent:

(a) $X \in DAN$ *and* $\mathsf{E}(X) = 0$;

(b) $V_n^{-1} M_{n,\mathcal{F}_q} \xrightarrow[n\to\infty]{\mathcal{D}} \mathsf{W}_{\mathcal{F}_q}$ *in* $\mathrm{UC}(\mathcal{F}_q, \rho_2)$;

(c) $V_n^{-1} \mu_{n,\mathcal{F}_q} \xrightarrow[n\to\infty]{\mathcal{D}} \mathsf{W}_{\mathcal{F}_q}$ *in* $\ell^\infty(\mathcal{F}_q, \rho_2)$.

Proof of $(a) \Rightarrow (b)$

- Assuming (a) we prove convergence of f.d.d.:

$$V_n^{-1} M_n(f) \xrightarrow[n\to\infty]{\mathcal{D}} \mathsf{W}(f) \quad \text{for each } f \in \mathcal{F}_q$$

and asymptotic ρ_2-equicontinuity: for each $\epsilon > 0$ and each $\delta_n \downarrow 0$

$$\lim_{n\to\infty} \mathsf{P}\Big\{ \sup_{\rho_2(f,g)<\delta_n} |M_n(f-g)| > \epsilon V_n \Big\} = 0$$

- We show that it is enough to prove the above asymptotic ρ_2-equicontinuity with V_n replaced by b_n from the definition of DAN.
- Then we show that Lindeberg-type conditions stated on the next slide hold and applied to $(Z_{n1}, \ldots, Z_{nn})$ defined by

$$Z_{nk}(f) := b_n^{-1}[X_{nk} - \mathsf{E}X_{nk}]\lambda_{nk}(f),$$

where $X_{nk} := X_k \mathbf{1}_{\{|X_k| \le \tau_n b_n\}}$ with suitable $\tau_n \downarrow 0$.

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Theorem (Van der Vaart and Wellner, (1996))

For each $n \in \mathbb{N}$ let $(Z_{n1}, \ldots, Z_{nn})$ be independent stochastic processes indexed by a totally bounded pseudometric space $(\mathcal{F}, \rho)$ and almost all sample paths in $UC(\mathcal{F}, \rho)$. Suppose that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{E}\, \|Z_{nk}\|_{\mathcal{F}}^2 \mathbf{1}_{\{\|Z_{nk}\|_{\mathcal{F}} > \epsilon\}} = 0 \quad \forall\, \epsilon > 0,$$

$$\lim_{n\to\infty} \sup_{\rho(f,g)<\delta_n} \sum_{k\in\mathbb{Z}} E[Z_{nk}(f) - Z_{nk}(g)]^2 = 0 \quad \forall\, \delta_n \downarrow 0,$$

$$\lim_{n\to\infty} \int_0^{\delta_n} \sqrt{\log N(x, \mathcal{F}, d_n)}\, dx = 0 \quad \text{in probability for every } \delta_n \downarrow 0,$$

where

$$d_n(f,g) := \Big(\sum_{k=1}^{n} [Z_{nk}(f) - Z_{nk}(g)]^2 \Big)^{1/2}, \quad f, g \in \mathcal{F}.$$

Then $\sum_{k=1}^{n} Z_{nk}$ is asymptotically ρ-equicontinuous.

Proof of $(b) \Rightarrow (c)$

Assuming (b) note that

$$V_n^{-1} M_{n,\mathcal{F}_q} \xrightarrow{\mathcal{D}} \mathsf{W}_{\mathcal{F}_q} \quad \text{in } \ell^\infty(\mathcal{F}_q, \rho_2).$$

Then (c) follows by proving that

$$V_n^{-1} \|\mu_{n,\mathcal{F}} - M_{n,\mathcal{F}}\|_{\mathcal{F}}^* \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Here ξ^* denotes a minimal measurable majorant of a function $\xi \colon \Omega \to \mathbb{R}$.

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At the beginning we stated:

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The following three statements are equivalent:

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(c) *for each (some)* $q \in [1,2)$, $V_n^{-1} \mu_{n,\mathcal{F}_q} \xrightarrow[n\to\infty]{\mathcal{D}} \mathsf{W}_{\mathcal{F}_q}$ *in* $\ell^\infty(\mathcal{F}_q)$,

where $\mathcal{F}_q$ *is the unit ball of the Banach space of functions of bounded* q*-variation.*

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From $\ell^\infty(\mathcal{F}_q)$, $1 \le q < 2$, to $\mathcal{W}_p[0,1]$, $p > 2$

Theorem

Let $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* $L\colon \mathcal{W}_q[0,1] \to \mathbb{R}$ *be a linear bounded functional and let* $F(t) := L(\mathbf{1}_{[0,t]})$, $t \in [0,1]$. *Then*

$$\|F\|_{[p]} \le 4 \sup \left\{ |L(f)|\colon\ f \in \mathcal{F}_q \right\} = 4\|L\|_{\mathcal{F}_q},$$

where $\mathcal{F}_q := \{f \in \mathcal{W}_q[0,1]\colon\ \|f\|_{[q]} \le 1\}$.

To use this fact we $L = \mu_n$. Then $F(t) = S_n(t)$ is the n-th partial sum process as follows

$$S_n(t) = \sum_{i=1}^{\lfloor tn \rfloor} X_i = \sum_{i=1}^{n} X_i \epsilon_{i/n}(\mathbf{1}_{[0,t]}) = \mu_n(\mathbf{1}_{[0,t]}), \quad t \in [0,1].$$

From $\ell^\infty(\mathcal{F}_q)$, $1 \leq q < 2$, to $\mathcal{W}_p[0,1]$, $p > 2$

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Thank you for your attention