MSDDEs and CAR representations of CARMA processes

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05 August 2019

Introduction

CARMA process:

$$P(D)X(t) = Q(D)DL(t), \quad t \in \mathbb{R},$$

where:

- L is a Lévy process.
- P is a polynomial of degree p.
- Q is a polynomial of degree q, $p > q$.
- D denotes differentiation with respect to t.

$$P(D)X(t) = Q(D)DL(t)$$

Taking Fourier transform and dividing by P leads to

$$X(t) = g * DL = \int_{\mathbb{R}} g(t-u)dL(u),$$

where $\mathcal{F}[g](y) = Q(iy)/P(iy)$.

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- Polynomial R is added and subtracted to get something in L^2.

Result:

$$R(D)X(t) + f * X(t) = DL(t),$$

where R is a polynomial of degree $p - q$ such that

$$\mathcal{F}[f](y) = \frac{P(iy) - R(iy)Q(iy)}{Q(iy)} \in L^2.$$

Conclusion (without technical assumptions)

A CARMA process $(X(t))_{t\in\mathbb{R}}$ satisfies the **stochastic delay differential equation** (SDDE)

$$\sum_{i=0}^{p-q} r_i D^{i-1} \mathrm{d}X(t) + \int_0^\infty X(t-u)f(u)\mathrm{d}u\mathrm{d}t = \mathrm{d}L(t)$$

where r_i are the coefficients of R and $\mathcal{F}[f](y) = \frac{P(iy)-R(iy)Q(iy)}{Q(iy)}$.

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4. can be used to model co-integrated time series (more on that in the next talk).
5. have motivated a fractional model with long memory but no change to the roughness (as opposed to models driven by fractional noise), see [2].
6. have intuitive condition for non-negativity giving new condition for non-negativity of (M)CARMA processes (paper under preparation).

SDDEs have been studied earlier in the literature in the univariate case with finite delay.

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- infinite delay
- the multivariate case

Both are key to the connection to CARMA processes.

Multivariate stochastic delay differential equation

MSDDE

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- η is an finite $n \times n$ matrix-valued signed measure concentrated on $[0, \infty)$ with $\int_{[0,\infty)} v^2 |\eta_{ij}|(dv) < \infty$, $i, j = 1, \ldots, n$.

MSDDE

A stationary n-dimensional process $X(t) = (X_1(t), \ldots, X_n(t))^\top$ solves the associated MSDDE if

$$\mathrm{d}X(t) = \eta * X(t)\,\mathrm{d}t + dZ(t)$$

or, written out,

$$X_i(t) - X_i(s) = \sum_{j=1}^{n} \int_s^t \int_{[0,\infty)} X_j(u-v)\eta_{ij}(dv)\,\mathrm{d}u + Z_i(t) - Z_i(s),$$

for $i = 1, \ldots, n$.

Let $\mathcal{L}[\eta] = (\mathcal{L}[\eta]_{ij})$ be given by

$$\mathcal{L}[\eta]_{ij}(z) = \int_{[0,\infty)} e^{-zv} \eta_{ij}(\mathrm{d}v)$$

for $z \in \mathbb{C}$ such that $\mathcal{L}[\eta]_{ij}(z)$ exists and $i, j = 1, \ldots, n$.

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for $z \in \mathbb{C}$ such that $\mathcal{L}[\eta]_{ij}(z)$ exists and $i, j = 1, \ldots, n$. Define

$$h(z) = zI_n - \mathcal{L}[\eta](z).$$

Theorem (Existence)

Assume $\det(h(iy)) \neq 0$ for all $y \in \mathbb{R}$. Then there exists a function $g : \mathbb{R} \to \mathbb{R}^{n\times n}$ in L^2 characterized by

$$\mathcal{F}[g](y) = h(iy)^{-1},$$

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$$\mathrm{d}X(t) = \eta * X(t)\, \mathrm{d}t + \mathrm{d}Z(t).$$

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If $\det(h(z)) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ then the solution is casual, that is, $(X(t))_{t\in\mathbb{R}}$ is adapted to $\sigma(Z(t) - Z(s) : s < t)_{t\in\mathbb{R}}$.

Fix $s < t$ and let

$$\hat{Z}(u) = \mathbb{E}[Z(u) - Z(s) \mid Z(s) - Z(r), r < s], \quad u > s.$$

In particular, if $(Z(t))_{t \in \mathbb{R}}$ is a Lévy process,

$$\hat{Z}(u) = (u - s)\mathbb{E}[Z(1)].$$

Theorem (Prediction)

Assume $\det(h(z)) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$. Then

$$\begin{aligned}\mathbb{E}[X(t)\,|X(u), u \leq s] \\ &= g(t-s)X(s) + \int_s^t g(t-u)\eta * \{\mathbb{1}_{(-\infty,s]}X\}(u)\,\mathrm{d}u \\ &+ g * \{\mathbb{1}_{(s,\infty)}\hat{Z}\}(t)\end{aligned}$$

where

$$\left(\eta * \{\mathbb{1}_{(-\infty,s]}X\}(u)\right)_j := \sum_{k=1}^n \int_{[u-s,\infty)} X_k(u-v)\eta_{jk}(\mathrm{d}v)$$

$$\left(g * \{\mathbb{1}_{(s,\infty)}\hat{Z}\}(u)\right)_j := \sum_{k=1}^n \int_{[0,u-s)} \hat{Z}_k(u-v)g_{jk}(\mathrm{d}v).$$

Note: General noise (e.g. fractional).

Connection to MCARMA

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2. If μ is a finite Borel measure on $\mathbb{R}$ with $\int |x|\mu(dx) < \infty$ then

$$I_k\left(\int_{\mathbb{R}} f_r(t - \cdot)\mu(dr)\right) = \int_{\mathbb{R}} I_k\left(f_r(t - \cdot)\right)\mu(dr)$$

almost surely for all $t \in \mathbb{R}$ where $f_r = \mathbb{1}_{[0,\infty)}(\cdot - r) - \mathbb{1}_{[0,\infty)}$ for $r \in \mathbb{R}$.

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Examples include Lévy and fractional Lévy processes with first moment.

Consider matrix polynomials

$$P(z) = I_n z^p + A_1 z^{p-1} + \cdots + A_p$$
$$Q(z) = B_0 + B_1 z + \ldots B_q z^q \quad z \in \mathbb{C}$$

for $A_1, \ldots, A_p, B_0, \ldots, B_q \in \mathbb{R}^{n \times n}$ ($p > q$).

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for $A_1, \ldots, A_p, B_0, \ldots, B_q \in \mathbb{R}^{n \times n}$ ($p > q$). Assume $\det(P(z)) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ and consider the process

$$X(t) = \int_{-\infty}^{t} \tilde{g}(t-u) \mathrm{d}Z(u)$$

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1. The kernel $\tilde{g}$ is characterized by $\mathcal{F}[\tilde{g}](y) = P(iy)^{-1} Q(iy)$.

Introduction to MCARMA

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2. $(Z(t))_{t\in\mathbb{R}}$ is an n-dimensional regular integrator.

We call $(X(t))_{t\in\mathbb{R}}$ an $(Z(t))_{t\in\mathbb{R}}$-driven MCARMA process.

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- If $(Z(t))_{t\in\mathbb{R}}$ is an n-dimensional Lévy process then $(X(t))_{t\in\mathbb{R}}$ is an MCARMA.
- If $Z(t) = (Z_1(t), \ldots, Z_n(t))^\top$ is given by

$$Z_i(t) - Z_i(s) = \frac{1}{\Gamma(1+\beta_i)} \int_{\mathbb{R}} \left[(t-u)_+^{\beta_i} - (-u)_+^{\beta_i}\right] \mathrm{d}L_i(u)$$

for $\beta_j \in (0, 1/2)$, $i = 1, \ldots, n$, and an n-dimensional Lévy process $L(t) = (L_1(t), \ldots, L_n(t))^\top$ with second moment, then $(X(t))_{t\in\mathbb{R}}$ is an MFICARMA.

Let $R(z) = I_n z^{p-q} + C_{p-q-1} z^{p-q-1} + \cdots C_0$ be such that

$$z \mapsto Q(z)R(z) - P(z), \quad z \in \mathbb{C},$$

is a polynomial of order $q - 1$.

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Theorem (MCARMA is MSDDE)

Suppose $\det(P(z)) \neq 0$ and $\det(Q(z)) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$. Then the $(Z(t))_{t\in\mathbb{R}}$-driven MCARMA $(X(t))_{t\in\mathbb{R}}$ is the unique solution to

$$\begin{aligned} \mathrm{d}X^{(m-1)}(t) = &- \sum_{j=0}^{m-1} C_j X^{(j)}(t)\mathrm{d}t \\ &+ \int_0^\infty f(u)X(t-u)\,\mathrm{d}u\,\mathrm{d}t + dZ(t) \end{aligned}$$

where $\mathcal{F}[f](y) = R(iy) - Q(iy)^{-1}P(iy)$ and $X^{(j)}$ is the j'th derivative of $X(t)$.

Since $\mathcal{F}[\tilde{g}](y) = P(iy)^{-1}Q(iy)$,

$$\tilde{g}(t) = \mathbb{1}_{[0,\infty)}(t) \left(e_1^p \otimes I_n\right)^\top e^{At} E$$

where $\otimes$ denotes the Kronecker product, e_1^p is the first standard basis vector of $\mathbb{R}^p$, and

$$A = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n \\ -A_p & -A_{p-1} & \cdots & -A_2 & -A_1 \end{pmatrix} \text{ and } E = \begin{pmatrix} E_1 \\ \vdots \\ E_p \end{pmatrix}$$

with $E(z) = E_1 z^{p-1} + \cdots + E_p$ chosen such that

$$z \mapsto P(z)E(z) - Q(z)z^p$$

is at most of degree $p - 1$.

Let

$$\tilde{g}^j(t) = \left(e_1^p \otimes I_n\right)^\top e^{At} \sum_{k=j}^{p-q} A^{k-j} E C_k, \quad j = 1, \ldots, p-q$$

and recall that $\hat{Z}(u) = \mathbb{E}[Z(u) - Z(s) \mid Z(s) - Z(r), r < s]$.

Corollary (Prediction)

Suppose $\det(P(z)) \neq 0$ and $\det(Q(z)) \neq 0$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$. Fix $s < t$. Then

$$\mathbb{E}[X(t) \mid X(u), u \leq s] = \sum_{j=1}^{p-q} \tilde{g}^j(t-s)X^{(j-1)}(s) + \int_{-\infty}^{s} \int_{s}^{t} \tilde{g}(t-u)f(u-v)\mathrm{d}uX(v)\mathrm{d}v + \tilde{g} * \{\hat{Z}\mathbb{1}_{(s,\infty)}\}(t).$$

where

$$\tilde{g}*\{\hat{Z}\mathbb{1}_{(s,\infty)}\}(t) = \mathbb{1}_{\{p=q+1\}}\hat{Z}(u) + (e_1^p \otimes I_n)^\top A e^{At} \int_s^t e^{-Av} E\hat{Z}(v)\,\mathrm{d}v.$$

Thank you for your attention!

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