On Lévy Bases in Free Probability

6. august 2019

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b. If $E_1, E_2, E_3, \ldots$ are disjoint sets from $\mathcal{E}$, and $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{E}$, then

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c. For all E in $\mathcal{E}$ the distribution $L\{N(E)\}$ of $N(E)$ is an inifinitely divisible probability measure.

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$$\mathbb{E}\{(f(X) - \mathbb{E}\{f(X)\})(g(Y) - \mathbb{E}\{f(Y)\})\} = 0,$$

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Except for trivial cases, free independence entails that

$$ab \neq ba.$$

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Moreover, there exists a unique probability measure $L\{a\}$ on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} f(t)\, L\{a\}(\mathrm{d}t) = \langle f(a)\xi, \xi\rangle$$

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A probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible, if

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By $\mathcal{ID}(\boxplus)$ we denote the class of all $\boxplus$-infinitely divisible probability measures on $\mathbb{R}$.

Free cumulant transform and Lévy-Khintchine repr.

Let μ be a prob. measure on $\mathbb{R}$, and consider its Cauchy transform:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(\mathrm{d}t), \qquad (z \in \mathbb{C}^+).$$

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$$\mathcal{C}_\mu(z) = zG_\mu^{\langle -1\rangle}(z) - 1, \qquad (z \in \mathcal{D}_\mu \subseteq \mathbb{C}^-).$$

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For any μ in $\mathcal{ID}(\boxplus)$ there exist unique a in $[0,\infty)$, η in $\mathbb{R}$ and a Lévy measure ρ on $\mathbb{R}$, such that

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The triplet (a, ρ, η) is called the *free characteristic triplet* for μ.

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Existence of Free Lévy Bases

Theorem

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$$\nu(E_1 \cup \dots \cup E_n, \cdot) \sim \sum_{k=1}^{n} m(E_j)(a, \rho, \eta) \sim \nu(E_1, \cdot) \boxplus \dots \boxplus \nu(E_n, \cdot).$$

By the previous theorem there exists a free Lévy basis $M_{m,\mu}$ s.t.

$$L\{M_{m,\mu}(E)\} = \nu(E, \cdot) \sim m(E)(a, \rho, \eta) \quad \text{for all } E \text{ in } \mathcal{E}.$$

Example: Factorizable Free Lévy Bases (continued)

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(II) If μ is the Marchenko-Pastur distribution, then $M_{m,\mu}$ is a free Poisson random measure:

$$L\{M_{m,\mu}(E)\} = \begin{cases} (1-m(E))\delta_0 + \frac{1}{2\pi t}\sqrt{(t-a)(b-t)}1_{[s,u]}(t)\,\mathrm{d}t, & \text{if } m(E) \leq 1, \\ \frac{1}{2\pi t}\sqrt{(t-a)(b-t)}1_{[s,u]}(t)\,\mathrm{d}t, & \text{if } m(E) > 1, \end{cases}$$

where $s = (1-m(E))^2$ and $u = (1+m(E))^2$.

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The *control measure* κ of M is subsequently defined by

$$\kappa(E) = |\Theta|(E) + \Sigma(E) + \int_{\mathbb{R}} \min\{1, x^2\} F_E(\mathrm{d}x), \qquad (E \in \mathcal{E}).$$

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The class of functions $f\colon X \to \mathbb{R}$ satisfying (a) and (b) is denoted by $\mathcal{L}^1(M)$.

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Proposition

For any function f from $\mathcal{L}^1(M)$, we have the representation:

$$\mathcal{C}_{\int_X f \, \mathrm{d}M}(z) = \int_X R(f(x)z, x) \, \kappa(\mathrm{d}x), \qquad (z \in \mathbb{C}^-),$$

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Lévy measure for each x!

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- G_Σ *and P_F are freely independent.*

Future Work

Realize a free Lévy basis $M = \{M(E) \mid E \in \mathcal{E}\}$ is the limit (in law) of a sequence of matrix-valued random measures:

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where $\lambda_1(\omega) \leq \lambda_2(\omega) \leq \cdots \leq \lambda_n(\omega)$ are the eigenvalues of $A(\omega)$ (counted with multiplicity).

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Then (under some additional assumptions!)

$$L\{A_n(\omega) + B_n(\omega)\} \xrightarrow{\text{w}} \mu \boxplus \nu \quad \text{for almost all } \omega.$$