Fluctuation theory for spectrally positive additive Lévy fields.

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In dimension one:

spectrally positive Lévy processes

Let $X := (X_t,\, t \geq 0)$ be a spectrally positive Lévy process (spLp), i.e. with no negative jumps. Then X admits all negative exponential moments:

$$\mathbb{E}(e^{-\lambda X_t}) = e^{t\varphi(\lambda)} < \infty, \quad \lambda \geq 0,$$

and the Laplace exponent $\varphi : [0,\infty) \to (0,\infty)$ is a convex and infinitely differentiable function with Lévy-Khintchine decomposition

$$\varphi(\lambda) = -a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbb{I}_{\{x<1\}})\, \pi(dx),$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and π is supported in $(0,\infty)$ and satisfies $\int_{(0,\infty)} (1 \wedge x^2)\, \pi(dx) < \infty$.

A remarkable feature of spectrally positive Lévy processes is that their first hitting time process,

$$T_x = \inf\{t \geq 0 : X_t = -x\}\,, \quad x \geq 0\,,$$

is a killed subordinator whose Laplace exponent $\phi : [0, \infty) \to [\phi(0), \infty)$ satisfies :

$$\varphi(\phi(\lambda)) = \lambda.$$

Moreover, $X_{T_x} = -x$ a.s. on $\{T_x < \infty\}$, the killing rate of (T_x) is $\phi(0)$ and

$$\begin{aligned} \phi(0) > 0 \quad &\Leftrightarrow \quad \lim_{t\to\infty} X_t = \infty, \quad \text{a.s.} \\ &\Leftrightarrow \quad \mathbb{E}(X_1) \in (0, \infty]. \end{aligned}$$

1. Spectrally positive Lévy processes

The law of T_x can be characterized in a more direct way through Kendall's identity:

$$\mathbb{P}(T_x \in dt)\, dx = \frac{x}{t}\mathbb{P}(-X_t \in dx)\, dt\,, \quad t, x > 0.$$

Kendall, 1965 and Borovkov, 1965.

• Many applications theory of dams (Kendall, 1965), risk theory (ruin probability).

• Long history going back to Bertrand, Barbier, André, 1887 in discrete time, (ballot theorem).

• Extensions to processes with interchangeable increments (Takács, 1965).

1. Spectrally positive Lévy processes

One of the many applications of spLp's lies in continuous state branching processes.

Let $(Z_t,\, t \geq 0)$ be any continuous state branching process, i.e. a $[0,\infty)$-valued Markov process with probabilities (P_x) such that

$$E_{x+y}(e^{-\lambda Z_t}) = E_x(e^{-\lambda Z_t})E_y(e^{-\lambda Z_t}), \quad x, y \geq 0.$$

Then

$$E_x(e^{-\lambda Z_t}) = e^{-x u_t(\lambda)}\,,$$

where the Laplace exponent u_t satisfies

$$\frac{\partial u_t}{\partial t}(\lambda) + \varphi(u_t(\lambda)) = 0$$

and φ is the Laplace exponent of a spLp.

A pathwise counterpart of this relation between spLp's and branching processes is the Lamperti representation:

$$Z_t = x + X_{\int_0^t Z_s\, ds}\,,$$

where X is a spLp with Laplace exponent φ and whose solution is given by

$$Z_t = x + X_{\tau_t \wedge T_x}\,, \quad \tau_t = \inf\left\{ s : \int_0^s \frac{du}{x + X_u} \geq t \right\}.$$

In particular,

$$T_x = \int_0^\infty Z_s\, ds, \quad a.s.$$

In higher dimension ?

Any multitype continuous state branching process $\mathrm{Z}_t = (Z_t^{(1)}, \dots, Z_t^{(d)}) \in [0,\infty)^d$ has Laplace transform

$$E_x(e^{-\langle \lambda, \mathrm{Z}_t \rangle}) = e^{-\langle x, u_t(\lambda) \rangle},$$

where $u_t(\lambda) = (u_t^{(j)}(\lambda),\, j = 1, \dots, d)$ satisfies,

$$\frac{\partial}{\partial t} u_t^{(j)}(\lambda) + \varphi_j(u_t(\lambda)) = 0\,, \quad \lambda \in [0,\infty)^d\,,$$

and φ_j is given by

$$\varphi_j(\lambda) = -\sum_{i=1}^{d} a_{i,j}\lambda_i + \frac{1}{2} q_j \lambda_j^2 + \int_{(0,\infty)^d} (e^{-\langle \lambda, x \rangle} - 1 + \langle \lambda, x \rangle 1_{\{|\mathrm{x}|<1\}})\, \pi_j(dx)$$

$a_{i,j} \geq 0$, for $i \neq j$, $q_j \geq 0$ and π_j is a measure on $(0,\infty)^d$ such that

$$\int_{(0,\infty)^d} [(1 \wedge |x|^2) + \sum_{i \neq j} (1 \wedge x_i)] \pi_j(dx) < \infty\,.$$

2. Spectrally positive additive Lévy fields

We are interested in the underlying mechanism of multitype continuous state branching processes.

For $j = 1, \dots, d$,

$$\varphi_j(\lambda) = -\sum_{i=1}^{d} a_{i,j}\lambda_i + \frac{1}{2}q_j\lambda_j^2 - \int_{(0,\infty)^d} (1 - e^{-\langle \lambda, x\rangle} - \langle \lambda, x\rangle 1_{\{|\mathrm{x}|<1\}})\,\pi_j(dx)$$

is the Laplace exponent of a d-dimensional Lévy process $X^{(j)} = {}^t(X^{1,j}, \dots, X^{d,j})$, such that

$X^{i,j}$, $i \neq j$ are subordinators and

$X^{j,j}$ is a spectrally positive Lévy process.

$\mapsto$ Lamperti representation?

Theorem (Chaumont, 2015; Caballero, Pérez, Uribe, 2017)

Any multitype branching process Z*, issued from* $x = (x_1, \ldots, x_d)$ *admits the following representation,*

$$(Z_t^{(1)}, \ldots, Z_t^{(d)}) = x + \left(\sum_{j=1}^{d} X^{1,j}_{\int_0^t Z_s^{(j)}\, ds}, \ldots, \sum_{j=1}^{d} X^{d,j}_{\int_0^t Z_s^{(j)}\, ds} \right),$$

where the processes,

$$X^{(j)} = {}^t(X^{1,j}, \ldots, X^{d,j}), \quad j = 1, \ldots, d,$$

are independent d*-dimensional Lévy processes, such that*

$X^{i,j}$, $i \neq j$ *are subordinators and*

$X^{j,j}$ *is a spectrally positive Lévy process.*

2. Spectrally positive additive Lévy fields

Define the matrix valued stochastic field,

$$\mathrm{t} = (t_1, \dots, t_d) \mapsto \mathbb{X}_{\mathrm{t}} = \begin{pmatrix} X_{t_1}^{1,1} & \dots & X_{t_d}^{1,d} \\ \vdots & \vdots & \vdots \\ X_{t_1}^{d,1} & \dots & X_{t_d}^{d,d} \end{pmatrix}$$

and the additive Lévy process

$$\begin{aligned} \mathbf{X}_{\mathrm{t}} &= \mathbb{X}_{\mathrm{t}} \cdot \mathbf{1}\,, \quad \mathbf{1} = (1, 1, \dots, 1) \\ &= \sum_{j=1}^{d} X_{t_j}^{i,j}, \quad i = 1, \dots, d \end{aligned}$$

$(\mathbf{X}_{\mathrm{t}},\, \mathrm{t} \in [0, \infty)^d)$ is called a spectrally positive additive Lévy field (spaLf).

$\mapsto$ Fluctuation theory for spaLf's?

Then the multivariate Lamperti representation can be written as

$$\begin{aligned}
(Z_t^{(1)},\dots,Z_t^{(d)}) &= x+\left(\sum_{j=1}^{d} X^{1,j}_{\int_0^t Z_s^{(j)}\,ds},\dots,\sum_{j=1}^{d} X^{d,j}_{\int_0^t Z_s^{(j)}\,ds}\right),\\
&= x+\mathbf{X}_{\int_0^t \mathrm{Z}_s\,ds},
\end{aligned}$$

where we denote $\int_0^t \mathrm{Z}_s\,ds = (\int_0^t Z_s^{(j)}\,ds, j=1,\dots,d)$.

Question: how to define the 'first hitting time' $\mathbf{T}_x$ of $x=(x_1,\dots,x_d)$ by the spaLf $(\mathbf{X}_{\mathrm{t}},\, \mathrm{t}\in[0,\infty)^d)$, so that

$$\mathbf{T}_x=\int_0^\infty \mathrm{Z}_s\,ds?$$

Theorem (Chaumont, Marolleau, 2019)

For each $x \in [0,\infty)^d$ *there is a multivariate random time* $\mathbf{T}_x = (T_x^{(1)}, \dots, T_x^{(d)})$ *such that*

$$\mathbf{X}_{\mathbf{T}_x} = \left(\sum_{j=1}^{d} X^{i,j}_{T_x^{(j)}}, i = 1, \dots, d \right) = -x, \quad \text{a.s. on } \{\mathbf{T}_x < \infty\}$$

and $\mathbf{T}_x \leq \mathrm{t}$, *for all* t *such that* $\mathbf{X}_{\mathrm{t}} = -x$.

Moreover, for each $x, x' \in [0,\infty)^d$,

$$\mathbf{T}_{x+x'} \stackrel{(d)}{=} \mathbf{T}_x + \tilde{\mathbf{T}}_{x'} \,,$$

where $\tilde{\mathbf{T}}_{x'}$ *is an independent copy of* $\mathbf{T}_{x'}$.

2. Spectrally positive additive Lévy fields

If $\mathbb{P}(\mathbf{T}_x < \infty) > 0$, then there is $\phi : [0,\infty)^d \to (0,\infty)^d$ such that

$$\mathbb{E}(e^{-\langle \lambda, \mathbf{T}_x \rangle}) = e^{-\langle \phi(\lambda), x \rangle}, \quad \lambda \in \mathbb{R}_+^d .$$

Theorem (Chaumont, Marolleau, 2019)

The first hitting time process $\mathbf{T}_x = \inf\{\mathrm{t} : \mathbf{X}_\mathrm{t} = -x\}$ *satisfies:*

1. $\mathbf{T}_x < \infty$ *holds with positive probability for all* $x \in [0,\infty)^d$ *if and only if*

$$(H) \qquad D := \{\lambda \in \mathbb{R}_+^d : \varphi_j(\lambda) > 0,\ j \in [d]\} \neq \emptyset.$$

2. *Suppose that* (H) *holds, then the mapping* $\phi : (0,\infty)^d \to D$ *is a diffeomorphism whose inverse corresponds to the mapping* $\varphi := (\varphi_1, \ldots, \varphi_d) : D \to (0,\infty)^d$, *that is*

$$\varphi(\phi(\lambda)) = \lambda\,, \ \ \lambda \in (0,\infty)^d.$$

$\mapsto$ Long-term behaviour of the spaLf $(\mathbf{X}_{\mathrm{t}},\ \mathrm{t} \in [0,\infty)^d)$?

From

$$\mathbb{E}(e^{-\langle \lambda, \mathbf{T}_x \rangle}) = e^{-\langle \phi(\lambda), x \rangle}, \quad \lambda \in \mathbb{R}_+^d\,,$$

we derive that

$$\mathbb{P}(\mathbf{T}_x < \infty) = e^{-\langle \phi(0), x \rangle},$$

so that $(\mathbf{X}_{\mathrm{t}},\ \mathrm{t} \in [0,\infty)^d)$ hits almost surely all levels $-(x_1,\dots,x_d)$ if and only if

$$\phi(0) = 0\,.$$

Recall the assumption

$(H)\quad D := \{\lambda \in \mathbb{R}_+^d : \varphi_j(\lambda) > 0,\ j \in [d]\} \neq \emptyset$, and define $M := (\mathbb{E}(X_1^{i,j}))_{i,j\in[d]}$. Note that $\mathbb{E}(X_1^{i,j}) \in [0,\infty]$.

Theorem (Chaumont, Marolleau, 2019)

Assume that (H) holds and that M is irreducible.

1. *the values 0 and $\phi(0) := \lim_{\lambda\to 0} \phi(\lambda)$ are the only roots of the equation $\varphi(\lambda) = 0,\ \lambda \in \mathbb{R}_+^d$.*

   Furthermore, either $\phi(0)$ is equal to 0 or it belongs to $(0,\infty)^d$.

2. *If $\mathbb{E}(X_1^{i,j}) = \infty$, for some $i,j \in [d]$, then $\phi(0) > 0$. Assume that $\mathbb{E}(X_1^{i,j}) < \infty$, for all $i,j \in [d]$, then $\phi(0) = 0$ if and only if $\rho \le 0$, where ρ be the Perron-Frobenius root of M.*

Recall the matrix valued stochastic field,

$$\mathrm{t} = (t_1, \dots, t_d) \mapsto \mathbb{X}_{\mathrm{t}} = \begin{pmatrix} X^{1,1}_{t_1} & \dots & X^{1,d}_{t_d} \\ \vdots & \vdots & \vdots \\ X^{d,1}_{t_1} & \dots & X^{d,d}_{t_d} \end{pmatrix},$$

the spaLf $\mathbf{X}_{\mathrm{t}} = \sum_{j=1}^{d} X^{i,j}_{t_j}, \quad i = 1, \dots, d$, and its first hitting time $\mathbf{T}_x = \inf\{\mathrm{t} : \mathbf{X}_{\mathrm{t}} = -x\}$.

Question: what is the law of

$$(\mathbf{T}_x, \mathbb{X}_{\mathbf{T}_x})?$$

(Note that when $d = 1$, $\mathbb{X}_{\mathbf{T}_x} = -x$.)

Kendall's identity

Theorem (Chaumont, Marolleau, 2019)

The law of $(\mathbf{T}_x, \mathbb{X}_{\mathbf{T}_x})$ *for* $x \in [0,\infty)^d$ *is given by*

$$\begin{aligned} &\mathbb{P}(\mathbf{T}_x \in \mathrm{dt},\, \mathbb{X}_{\mathrm{t}} \in \mathrm{d}x_{ij}) \\ = \quad &\frac{\mathit{det}(-x_{ij})}{t_1 t_2 \ldots t_d} \mathbb{P}(\mathbb{X}_{\mathrm{t}} \in \mathrm{d}x_{ij}) \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_d \,, \end{aligned}$$

where $x_{ij} \geq 0$ *for* $i \neq j$ *and* $\sum_{j=1}^{d} x_{ij} = -x_i$ *for all* $i \in [d]$.

$\mapsto$ Applications: multivariate risk theory, generalization of dams,...?

Thank you very much for your attention.