A Minimal Contrast Estimator for the Linear Fraction Stable Motion

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Overview

1. Setup & Goal
2. Preliminary Results
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Setup

The linear fractional stable motion is a process with three parameters $\xi = (\sigma, \alpha, H)$ defined as

$$X_t = \int_{\mathbb{R}} \{(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\} \mathrm{d}L_s \quad (t \geq 0)$$

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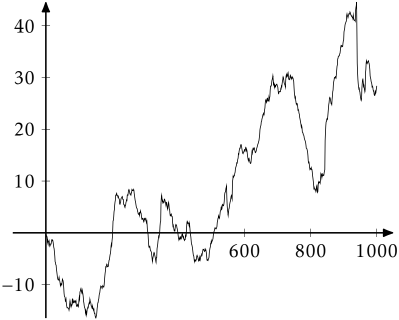
(X_t) enjoys the properties:

- Self-similar of index H
- Symmetric α-stable marginals
- Stationary increments
- $H - 1/\alpha > 0 \rightsquigarrow$ continuous
- $H - 1/\alpha < 0 \rightsquigarrow$ unbounded on bounded intervals

Sample Paths



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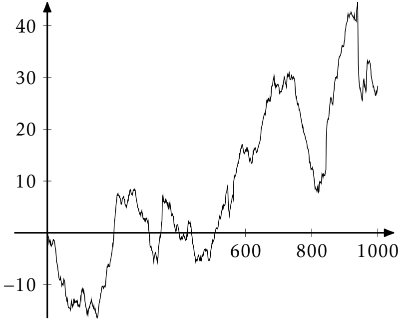

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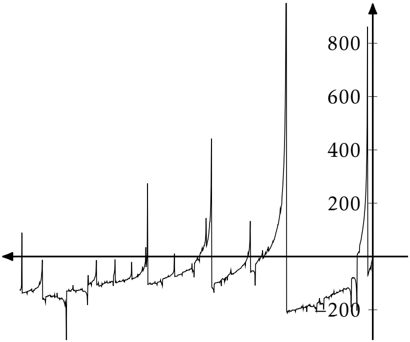

Discontinuous

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$$V_n(f,k,r) = \frac{1}{n} \sum_{i=rk}^{n} f(\Delta_{i,k}^r X).$$

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- Power variation $f(x) = f_p(x) = |x|^p$ for $p \in (-1,1)$

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- Empirical characteristic function $f(x) = \delta_t(x) = \cos(tx)$, $t \geq 0$
- Empirical distribution functions $f(x) = \mathbb{1}_{(-\infty,t]}(x)$, $t \in \mathbb{R}$

Preliminary Results

Define for $t \geq 0$, $n \in \mathbb{N}$ and $r \in \{1, 2\}$

$$\varphi_n(t) = V_n(\delta_t, k, 1) \quad \text{and} \quad \psi_n(r) = V_n(f_p, k, r).$$

and also

$$h_{k,r}(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x - rj)_+^{H-1/\alpha}.$$

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We deduce the convergences

$$\varphi_n(t) \xrightarrow{\text{a.s.}} \varphi_\xi(t) := \exp(-|t\sigma \|h_k\|_\alpha|^\alpha)$$

$$R_n(p,k) = \frac{\psi_n(2)}{\psi_n(1)} \xrightarrow{\text{a.s.}} 2^{pH},$$

in particular we obtain a consistent estimator for H:

$$H_n(p,k) = \frac{1}{p} \log_2(R_n(p,k)) \xrightarrow{\text{a.s.}} H.$$

Define the processes

$$W_n^1(r) = \sqrt{n}(\psi_n(r) - r^{pH} m_{p,k}), \quad W_n^2(t) = \sqrt{n}(\varphi_n(t) - \varphi_\xi(t))$$

with $m_{p,k} = \mathbb{E}[|\Delta_{k,k}X|^p]$, and for $\beta = 1 + \alpha(k - H)$

$$S_n^1(r) = n^{1-1/\beta}(\psi_n(r) - r^{pH} m_{p,k}), \quad S_n^2(t) = n^{1-1/\beta}(\varphi_n(t) - \varphi_\xi(t)).$$

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If $X, X_1, X_2, \ldots$ are processes index by $\mathbb{R}_+$ and $Z, Z_1, Z_2, \ldots$ is a sequence of random vectors, then

$$(X_n, Z_n) \xrightarrow{\mathcal{L}\text{-f}} (X, Z)$$

denotes finite dimensional convergence, i.e., for all $t_1, \ldots, t_d$, then

$$(X_n(t_1), \ldots, X_n(t_d), Z_n) \xrightarrow{\mathcal{L}} (X_{t_1}, \ldots, X_{t_d}, Z).$$

Theorem 2.2 in [2].
Assume either $p \in (-1/2, 0)$ or $p \in (0, 1/2)$ together with $p < \alpha/2$.

1. When $k > H + 1/\alpha$

$$(W_n^1(1), W_n^1(2), W_n^2(t))_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{L}\text{-f}} (W_1^1, W_2^1, W_t)_{t\in\mathbb{R}_+}$$

where $W^1 = (W^1(1), W^1(2))$ is a centred two-dimensional normal distribution and $(W_t)_{t\in\mathbb{R}_+}$ is a centred Gaussian process. The covariance kernel between W^1 and W is known.

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$$(S_n^1(1), S_n^1(2), S_n^2(t))_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{L}\text{-f}} (S_1^1, S_2^2, S_t)_{t\in\mathbb{R}_+},$$

   where $S^1 = (S_1^1, S_1^1)$ is a β-stable vector independent of the β-stable process (S_t).

Results

Let $\theta := (\sigma, \alpha) \in (0, \infty) \times (0, 2) = \Theta$ and consider a weight function $w \in \mathcal{L}^1(\mathbb{R}_+)$. Define $F : \mathcal{L}^2_w(\mathbb{R}_+) \times (0, 1) \times \Theta \to \mathbb{R}$ as

$$F(\varphi, H, \theta) = \|\varphi - \varphi_{\theta,H}\|_w^2 = \int_0^\infty (\varphi(t) - \varphi_{\theta,H}(t))^2 w(t)\, \mathrm{d}t.$$

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Suppose $\xi_0 = (\theta_0, H_0)$ is the true parameter of the model (X_t). Recalling the estimator $H_n(p,k)$ of H_0 we define the minimal contrast estimator of θ_0 as

$$\theta_n \in \operatorname*{argmin}_{\theta \in \Theta} F(\varphi_n, H_n(p,k), \theta) = \operatorname*{argmin}_{\theta \in \Theta} \|\varphi_n - \varphi_{\theta, H_n(p,k)}\|_w^2.$$

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We wish to derive the asymptotics of the joint estimator: $\xi_n = (\theta_n, H_n(p,k))$.

By Theorem 2.2 and the Delta method there exists random variables M_1, M_2 such that

$$\begin{cases} n^{1/2}(H_n(p,k) - H_0) \xrightarrow{\mathcal{L}} M_1, & \text{if } k > H + 1/\alpha, \\ n^{1/\beta}(H_n(p,k) - H_0) \xrightarrow{\mathcal{L}} M_2, & \text{if } k < H + 1/\alpha. \end{cases}$$

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Theorem

Assume w is continuous and let ∇_θ denote the 2-dimensional gradient w.r.t. θ.

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1. ξ_n is strongly consistent, i.e, $\xi_n \xrightarrow{\text{a.s.}} \xi_0$.
2. If $k > H + 1/\alpha$ and $w \in C^1$ then

$$\sqrt{n}(\xi_n - \xi_0) \xrightarrow{\mathcal{L}} \Big[2\nabla_\theta^2 F(\varphi_{\xi_0}, \xi_0)^{-1} \Big(\int_0^\infty W_t \nabla_\theta \varphi_{\xi_0}(t) w(t)\,\mathrm{d}t + \frac{\partial}{\partial H} \nabla_\theta F(\varphi_{\xi_0}, \xi_0) M_1 \Big), M_1 \Big]^\top.$$

Theorem (continued)

3. If $k < H + 1/\alpha$ then

$$n^{1/\beta}(\xi_n - \xi_0) \xrightarrow{\mathcal{L}} \Big[-2\nabla_\theta^2 F(\varphi_{\xi_0}, \xi_0)^{-1} \Big(\int_0^\infty S_t \nabla_\theta \varphi_{\xi_0}(t) w(t)\,\mathrm{d}t + \frac{\partial}{\partial H} \nabla_\theta F(\varphi_{\xi_0}, \xi_0) M_2 \Big), M_2 \Big]^\top$$

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Remarks

- W is a continuous modification of the Gaussian process from Theorem 2.2.

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Remarks

- W is a continuous modification of the Gaussian process from Theorem 2.2.
- Here $(S_t)_{t\geq 0}$ has the form $(\kappa(t)\tilde{L})_{t\geq 0}$ where $\tilde{L}$ is a totally skewed β-stable random variable and κ is a negative-valued continuous function.

Caveat

It is tempting to try the minimal contrast estimator for all parameters $\xi = (\sigma, \alpha, H)$ simultaneously and one would expect that

$$\sqrt{n}(\xi_n - \xi_0) \xrightarrow{\mathcal{L}} 2\nabla_\xi^2 F(\varphi_{\xi_0}, \xi_0)^{-1} \int_0^\infty W_t \nabla_\xi \varphi_{\xi_0}(t) w(t)\, \mathrm{d}t$$

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in the case $k > H + 1/\alpha$. But the 3×3 Hessian $\nabla_\xi^2 F$ is not invertible! The problem stems from the fact that ξ that ξ is not identifiable from the characteristic function φ_ξ.

Partial Extensions

1-dimensional parametric Lévy driven moving averages are

$$X_t = \int_{-\infty}^{t} g_\theta(t-s)\,\mathrm{d}L_s \quad (t \geq 0).$$

- $L \sim \mathrm{S}\alpha\mathrm{S}$ with scale 1
- Θ open subset of $\mathbb{R}$
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Assume that

- $\theta \mapsto \|g_\theta\|_\alpha$ is injective
- $(\theta,\alpha) \mapsto \|g_\theta\|_\alpha$ is C^2
- The series

$$\sum_{j=1}^{\infty}\left(\int_{j-1}^{j} |g_\theta(s)|^\alpha\,\mathrm{d}s\right)^{1/2} \quad \text{and} \quad \underbrace{\sum_{l=1}^{\infty}\int_{0}^{\infty} |g_\theta(s+l)|^\alpha\,\mathrm{d}s}_{:=\mu_l}$$

are finite

Theorem

Let $\xi_0 \in (0,2) \times \Theta$ denote the true parameter. Then

$$\xi_n := \operatorname*{argmin}_{\xi \in (0,2)\times\Theta} \|\varphi_n - \varphi_\xi\|_w^2 \xrightarrow{\text{a.s.}} \xi_0.$$

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Additionally

$$\sqrt{n}(\xi_n - \xi_0) \xrightarrow{\mathcal{L}} (\langle \partial_{\xi_i} \varphi_{\xi_0}, \partial_{\xi_j} \varphi_{\xi_0} \rangle_w)_{i,j=1,2}^{-1} (\langle \partial_{\xi_i} \varphi_{\xi_0}, G \rangle_w),$$

where G is a continuous zero mean Gaussian process with “known” covariance.

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Remark

As $n \to \infty$

$$\left(\langle \partial_{\xi_i}\varphi_{\xi_n}, \partial_{\xi_j}\varphi_{\xi_n}\rangle_w\right)_{i,j=1,2}^{-1} \xrightarrow{\text{a.s.}} \left(\langle \partial_{\xi_i}\varphi_{\xi_0}, \partial_{\xi_j}\varphi_{\xi_0}\rangle_w\right)_{i,j=1,2}^{-1}$$

and similarly for the covariance of G, however the latter is numerically very difficult.

Proofs

Let $F(\psi,\xi) = \|\psi - \varphi_\xi\|_w^2$. Then $\xi_n \in \operatorname{argmin}_\xi F(\varphi_n, \xi)$ can be obtained using the criteria: $\nabla_\xi F(\psi,\xi) = 0$, which holds at (φ_{ξ_0}, ξ_0). The implicit function theorem gives a bijective Φ such that

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where $R(\varphi_n - \varphi_{\xi_0}) \xrightarrow{\text{a.s.}} 0$ if $\|\varphi_n - \varphi_{\xi_0}\|_w \xrightarrow{\text{a.s.}} 0$.

It is then enough to prove

$$(\langle \partial_{\xi_i} \varphi_{\xi_0}, \sqrt{n}(\varphi_n - \varphi_{\xi_0}) \rangle_w)_{i=1,2} \xrightarrow{\mathcal{L}} (\langle \partial_{\xi_i} \varphi_{\xi_0}, G \rangle_w)_{i=1,2}$$

$$\sqrt{n} \|\varphi_n - \varphi_{\xi_0}\|_w \xrightarrow{\mathcal{L}} \|G\|_w.$$

These are integral functionals and we have finite dimensional convergence.

Useful Lemma

Let $G, G_1, G_2, \ldots$ be processes index by $\mathbb{R}_+$ and paths in $\mathcal{L}^1_w(\mathbb{R}_+)$. We seek conditions for which

$$G_n \xrightarrow{\mathcal{L}\text{-f}} G \quad \Longrightarrow \quad \int_0^\infty G_n(t)w(t)\,\mathrm{d}t \xrightarrow{\mathcal{L}} \int_0^\infty G(t)w(t)\,\mathrm{d}t \tag{§}$$

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holds.

$$Z_{n,m} = \int_0^m G_n(t)w(t)\,\mathrm{d}t, \quad Z_{n,m,l} = \int_0^m G_n(\tfrac{\lfloor tl \rfloor}{l})w(\tfrac{\lfloor tl \rfloor}{l})\,\mathrm{d}t,$$

$$Y_{n,m} = \int_m^\infty G_n(t)w(t)\,\mathrm{d}t$$

Lemma

Suppose $w, G, G_1, G_2, \dots$ are continuous. Then

1. $\forall m \in \mathbb{N}\, \forall \varepsilon > 0 : \lim_{l\to\infty} \limsup_{n\to\infty} \mathbb{P}(|Z_{n,m,l} - Z_{n,m}| \geq \varepsilon) = 0,$
2. $\forall \varepsilon > 0 : \lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}(|Y_{n,m}| \geq \varepsilon) = 0,$

are sufficient for (§).

Gaussian Limit

Define the following dependence measure $U_{g,h}(u,v)$ between $X = \int h \,\mathrm{d}L$ and $Y = \int g \,\mathrm{d}L$ as

$$U_{g,h}(u,v) = \mathbb{E}[\exp(\mathrm{i}(uX - vY))] - \mathbb{E}[\exp(\mathrm{i}\,uX)]\mathbb{E}[\exp(-\mathrm{i}\,vY)].$$

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Lemma ((3.4)–(3.6) from [3]) For any $u, v \in \mathbb{R}$ it holds that

$$\begin{aligned}|U_{g,h}(u,v)| &\leq 2|uv|^{\alpha/2}\int_0^\infty |g(x)h(x)|^{\alpha/2}\,\mathrm{d}x \\ &\qquad \times \exp\Big(-2|uv|^{\alpha/2}\Big(\|g\|_\alpha^{\alpha/2}\|h\|_\alpha^{\alpha/2} - \int_0^\infty |g(x)h(x)|^{\alpha/2}\,\mathrm{d}x\Big)\Big), \\ |U_{g,h}(u,v)| &\leq 2|uv|^{\alpha/2}\int_0^\infty |g(x)h(x)|^{\alpha/2}\,\mathrm{d}x \exp\Big(-\Big(\|ug\|_\alpha^{\alpha/2} - \|vh\|_\alpha^{\alpha/2}\Big)^2\Big).\end{aligned}$$

If $\rho_l = \int_{\mathbb{R}} |g_\theta(x) g_\theta(x+l)|^{\alpha/2}\,\mathrm{d}x$, then $\sum_{l\in\mathbb{Z}} \rho_\ell < \infty$.

Stable case

Result
Let $X_1, X_2, \ldots$ be a sequence of i.i.d. centred α-stable variables and set

$$S_n = n^{-1/\alpha} \sum_{k=1}^{n} X_k.$$

Then

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|S_n|^p] < \infty \quad \text{for all } p \in (0, \alpha).$$

Continuous Case

Bias, standard deviation and median absolute error of ξ_n. We use $k = 2$, $p = 0.4$ and the true parameter is $(\sigma_0, \alpha_0, H_0) = (0.3, 1.8, 0.8)$

	n	σ	α	H
Bias	100	−0.045 55	−0.106 45	−0.072 89
	1000	−0.001 02	−0.011 99	−0.007 51
	10 000	−0.000 46	−0.000 95	−0.001 60
Standard deviation	100	0.126 58	0.176 43	0.157 92
	1000	0.086 32	0.072 81	0.050 09
	10 000	0.005 55	0.019 52	0.015 82
Median absolute error	100	0.058 44	0.151 69	0.156 72
	1000	0.017 63	0.059 88	0.050 45
	10 000	0.005 65	0.019 85	0.015 70

Discontinuous Case

Bias, standard deviation and median absolute error of ξ_n. We use $k = 2$, $p = -0.4$ and the true parameter is $(\sigma_0, \alpha_0, H_0) = (0.3, 0.8, 0.8)$

	n	σ	α	H
Bias	100	0.575 54	0.241 44	0.053 31
	1000	0.091 56	0.042 69	0.021 86
	10 000	0.009 44	0.002 55	0.015 93
Standard deviation	100	0.822 78	0.482 36	0.348 77
	1000	0.290 00	0.223 31	0.160 80
	10 000	0.089 52	0.041 12	0.054 47
Median absolute error	100	0.774 05	0.337 27	0.373 78
	1000	0.168 10	0.099 35	0.152 46
	10 000	0.051 54	0.030 04	0.051 20

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