Limit theorems for ambit fields observed along curves

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Outline

1. 1-dim BSS process
2. 2-dim ambit field

Review

- A Brownian semistationary process (BSS) is defined as

$$X_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(ds),$$

  where $g : \mathbb{R}_+ \to \mathbb{R}$ is a weight function and σ is a stochastic intermittancy/volatility process; see Barndorff-Nielsen and Schmiegel (2009).
- If σ is stationary (and is independent of W), then X is stationary as well.
- These processes are typically not semimartingales.
- Barndorff-Nielsen, Corcuera and Podolskij (2011) studied the asymptotic theory of (multi)power variations of BSS.

CLT for BSS

The power variation of the BSS process X is defined via

$$V(X,p)_t^n = \sum_{i=1}^{[t/\Delta_n]} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p.$$

and let $g(x) = x^{\alpha} f(x)$.

Theorem

Assume that $\alpha \in (-1/2, 0) \cup (0, 1/2)$. Then, under some conditions on σ and f, we obtain that

$$\Delta_n^{-1/2} \left(\tau_n V(X,p)_t^n - m_p \int_0^t |\sigma_s|^p ds \right) \xrightarrow{st} \lambda_p \int_0^t |\sigma_s|^p dB_s,$$

where $m_p = \mathbb{E}[|\mathcal{N}(0,1)|^p]$, and B is a Brownian motion independent of $\mathcal{F}$ and the constant λ_p depends only on the correlation function ρ.

Asymptotic theory of power variations of ambit fields

Literature

- Barndorff-Nielsen and Graversen (2011)
- Pakkanen (2014)
- Chong (2019): space-time and the stochastic heat equation

Introduction

- We consider the following two dimensional ambit field:

$$X_{t_1,t_2} = \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} g(t_1 - s_1, t_2 - s_2)\sigma_{s_1,s_2} W(ds_1, ds_2),$$

  where W is the white noise, $g : \mathbb{R}_+^2 \to \mathbb{R}$ is a weight function s.t. $g \in L^2(\mathbb{R}_+^2)$, and σ is a càdlàg intermittency field.
- $g(x) = \|x\|^\alpha f(x)$ where $\alpha \in (-1, 0)$, $f : \mathbb{R}_+^2 \to \mathbb{R}$ is a smooth enough function and $\|\cdot\|$ is the Euclidean norm.
- The process X is observed discretely along the curve

$$z : [0, t] \to \mathbb{R}^2, \qquad z(s) = (z_1(s), z_2(s)).$$

  and the observed process is defined via

$$Y_s = X_{z(s)}.$$

- **Problem**: Stationarity of X does not transfer to stationarity of Y unless the function z is linear.

Power variations

We are interested in asymptotic theory of power variations of Y, which are defined via

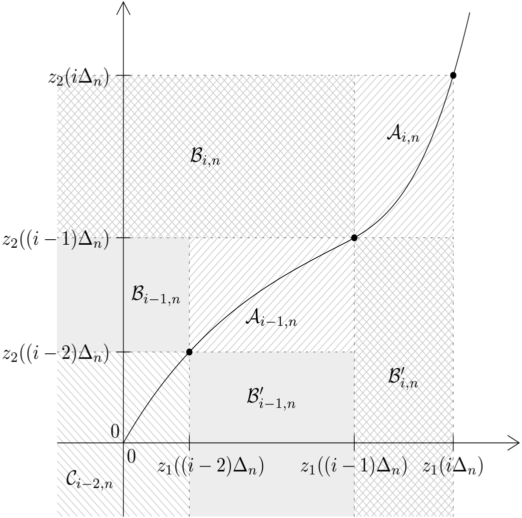
$$V(Y,p)_t^n = \sum_{i=1}^{[t/\Delta_n]} |Y_{i\Delta_n} - Y_{(i-1)\Delta_n}|^p.$$

Decomposition

The increments of Y can be decomposed into 4 terms

$$
\begin{aligned}
& Y_{t+\Delta} - Y_t \\
&= \int_{z_1(t)}^{z_1(t+\Delta)} \int_{z_2(t)}^{z_2(t+\Delta)} g(z_1(t+\Delta) - s_1, z_2(t+\Delta) - s_2) \sigma_{s_1,s_2} W(ds_1, ds_2) \\
&+ \int_{-\infty}^{z_1(t)} \int_{z_2(t)}^{z_2(t+\Delta)} g(z_1(t+\Delta) - s_1, z_2(t+\Delta) - s_2) \sigma_{s_1,s_2} W(ds_1, ds_2) \\
&+ \int_{z_1(t)}^{z_1(t+\Delta)} \int_{-\infty}^{z_2(t)} g(z_1(t+\Delta) - s_1, z_2(t+\Delta) - s_2) \sigma_{s_1,s_2} W(ds_1, ds_2) \\
&+ \int_{-\infty}^{z_1(t)} \int_{-\infty}^{z_2(t)} \{g(z_1(t+\Delta) - s_1, z_2(t+\Delta) - s_2) - g(z_1(t) - s_1, z_2(t) - s_2)\} \\
&\times \sigma_{s_1,s_2} W(ds_1, ds_2).
\end{aligned}
$$

Figure


$z_2(i\Delta_n)$
$z_2((i-1)\Delta_n)$
$z_2((i-2)\Delta_n)$
0
$\mathcal{B}_{i,n}$
$\mathcal{A}_{i,n}$
$\mathcal{B}_{i-1,n}$
$\mathcal{A}_{i-1,n}$
$\mathcal{B}'_{i,n}$
$\mathcal{B}'_{i-1,n}$
$\mathcal{C}_{i-2,n}$
0
$z_1((i-2)\Delta_n)$
$z_1((i-1)\Delta_n)$
$z_1(i\Delta_n)$

Gaussian core

The Gaussian core of X is defined by

$$G_{t_1,t_2} = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} g(t_1 - s_1, t_2 - s_2)W(ds_1, ds_2),$$

and its variance is

$$v_{t,t+\Delta} := \mathbb{E}[(G_{z(t+\Delta)} - G_{z(t)})^2].$$

Assumptions

1. (A1) The curve $t \mapsto \boldsymbol{z}(t) = \big(z_1(t), z_2(t)\big)$ is C^1 and the derivatives of its both components are bounded away from zero.
2. (A2) The function f is bounded and satisfies $f \in C^1(\mathbb{R}^2_+) \cap L^2(\mathbb{R}^2_+)$ with

$$\|\nabla f(\boldsymbol{x})\| \lesssim \|\boldsymbol{x}\|^{-1}, \quad \boldsymbol{x} \in \mathbb{R}^2_+.$$

Lemma on variance

We define the function ϕ_t via

$$\phi_t^2 = z_1'(t) z_2'(t) f(\mathbf{0})^2 \left(\int_{\mathbb{R}^2 \setminus (1,\infty)^2} \|\mathbf{z}'(t) \circ \mathbf{x}\|^{2\alpha} d\mathbf{x} + \int_{\mathbb{R}_+^2} (\|\mathbf{z}'(t) \circ (\mathbf{x} + \mathbf{1})\|^{\alpha} - \|\mathbf{z}'(t) \circ \mathbf{x}\|^{\alpha})^2 d\mathbf{x} \right).$$

Lemma

Suppose that (A1)-(A2) hold, and $\alpha \in (-1, -1/2)$. Then we have for any $T > 0$,

$$\lim_{\Delta \to 0} \sup_{t \in [0,T]} |\Delta^{-(2\alpha+2)} v_{t,t+\Delta} - \phi_t^2| = 0,$$

Lemma

Let

$$\gamma_n(i,j) = \mathrm{cov}(G_{z(i\Delta_n)} - G_{z((i-1)\Delta_n)}, G_{z(j\Delta_n)} - G_{z((j-1)\Delta_n)})$$

.

Lemma

Suppose that (A1)-(A2) hold, and that $\alpha \in (-1, -1/4)$. Then, there exists a sequence $(\overline{\rho}(k))_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} \overline{\rho}(k)^2 < \infty$ and

$$|\gamma_n(i,j)| \lesssim_{\alpha,f,\mathbf{z}} \Delta^{2\alpha+2}\overline{\rho}(|i-j|)$$

for any $n \in \mathbb{N}$ and i, j..

Law of large numbers

Theorem

Suppose that (A1)-(A2) hold and $\alpha \in (-1, -1/2)$. Then, we obtain that

$$\Delta_n^{1-p(1+\alpha)} V(Y,p)_t^n \xrightarrow{u.c.p.} m_p \int_0^t |\phi_s \sigma_s|^p ds$$

with $m_p = \mathbb{E}[|\mathcal{N}(0,1)|^p]$.

Central limit theorem

Theorem

Suppose that (A1)-(A2) hold, $\alpha \in (-1, -1/2)$ and σ is Holder continuous of order $\gamma > 1/2$. Then, we have

$$\Delta_n^{-1/2}\left(\Delta_n^{1-p(1+\alpha)}V(Y,p)_t^n - m_p\int_0^t |\phi_s\sigma_s|^p ds\right) \xrightarrow{st} \lambda_p \int_0^t |\phi_s\sigma_s|^p dB_s,$$

where B is a Brownian motion independent of $\mathcal{F}$ and the constant λ_p depends only on the correlation function ρ of fBM($1+\alpha$)

Another case

If $\alpha \in (-1/2, 0)$, the second and third terms of the decomposition dominate the variance $v_{t,t+\Delta}$, which is

$$w_t^2 = z_2'(t) \int_0^\infty |s|^{2\alpha} f^2(s,0) \sigma^2_{z_1(t)-s, z_2(t)} ds + z_1'(t) \int_0^\infty |s|^{2\alpha} f^2(0,s) \sigma^2_{z_1(t), z_2(t)-s} ds.$$

However, the CLT suffers from the bias coming from the first and fourth term of the decomposition.

Another Theorem

Theorem

Assume that $\alpha \in (-1/2, 0)$ and σ is independent of W. Under (A1)-(A2), we obtain that

$$\Delta_n^{1-p/2} V(Y,p)_t^n \xrightarrow{u.c.p.} m_p \int_0^t |w_t|^p ds \tag{1}$$

and

$$\Delta_n^{(1-p)/2} \left(V(Y,p)_t^n - \mathbb{E}[V(Y,p)_t^n]\right) \xrightarrow{st} \sqrt{m_{2p} - m_p^2} \int_0^t |w_t|^p dB_s, \tag{2}$$

where B is a Brownian motion independent of $\mathcal{F}$.

Sketch of the proof Let $\alpha \in (-1, -1/2)$.

$$
\begin{aligned}
&\Delta_n^{1-p(1+\alpha)} V(Y,p)_t^n - m_p \int_0^t |\phi_s \sigma_{z(s)}|^p ds \\
&= \Delta_n^{1-p(1+\alpha)} \sum_{i=1}^{[t/\Delta_n]} \left(|\Delta_i^n Y|^p - |\sigma_{z((i-1)\Delta_n)} \Delta_i^n G)|^p \right) \\
&\quad + \Delta_n^{1-p(1+\alpha)} \sum_{j=1}^{[\ell t]} \sum_{i \in I_n(j,\ell)} \left(|\sigma_{z((i-1)\Delta_n)}|^p - |\sigma_{z((j-1)/\ell)}|^p \right) |\Delta_i^n G|^p \\
&\quad + \sum_{j=1}^{[\ell t]} |\sigma_{z((j-1)/\ell)}|^p \left(\Delta_n^{1-p(1+\alpha)} \sum_{i \in I_n(j,\ell)} |\Delta_i^n G|^p - m_p(\phi_{j/\ell}^{p+} - \phi_{(j-1)/\ell}^{p+}) \right) \\
&\quad + m_p \left(\sum_{j=1}^{[\ell t]} |\sigma_{z((j-1)/\ell)}|^p (\phi_{j/\ell}^{p+} - \phi_{(j-1)/\ell}^{p+}) - \int_0^t |\sigma_{z(s)}|^p d\phi_s^{p+} \right) \\
&=: A_t(n) + B_t(\ell,n) + C_t(\ell,n) + D_t(\ell),
\end{aligned}
$$

where $I_n(j,\ell) = \{ i : i\Delta_n \in (\frac{j-1}{\ell}, \frac{j}{\ell}] \}$.

Four terms

- $A_t(n)$ is about the error $\Delta_i^n Y - \sigma_{z((i-1)\Delta_n)} \Delta_i^n G$ and is dealt with using $\gamma(> 1/2)$-Holder continuity of σ. We have to use new results from Corcuera, Nualart and Podolskij (2014).
- $B_t(\ell, n)$ is the error due to ℓ-blocks and has to be dealt with $0 < \varepsilon_n^1 < \ldots < \varepsilon_n^\ell < \varepsilon_n^{\ell+1} = \infty$
- $C_t(\ell, n)$ is the main term and converges in distribution to the limit. Tightness is via Nourdin and Nualart (2019).
- $D_t(\ell)$ is the Riemann-Stieltej integral error.

Thanks for your attention!