Modelling extremal clustering with trawl processes

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Based on joint work with Ragnhild Noven, Axel Gandy and Valentin Courgeau (Imperial College London)

Aim of the project and model strategy

- **Aim:** Find a time series model which can describe serial dependence in extremes.
- **Modelling strategy:** Combine ideas from
  - extreme value theory (peaks over thresholds) and
  - ambit stochastics (trawl processes)

  in a hierarchical model.
- **Main contributions:**
  - New model with characterisation of its probabilistic properties,
  - Statistical inference procedure based on a composite (pairwise) likelihood approach,
  - Detailed simulation studies to assess the inference method,
  - Empirical study: Extreme rainfall in Heathrow and air pollution in London.

A very short introduction to extreme value theory

Let $X; X_1, X_2, \ldots$ be a sequence of non-degenerate independent and identically distributed (i.i.d.) random variables (rvs) with common distribution function (df) denoted by F. (Note that we call a df non-degenerate if it is not concentrated on a single point.)

We are interested in studying *sample maxima*. Define (for $n \in \mathbb{N}$)

$$M_1 = X_1, \quad M_n = \max(X_1, \ldots, X_n), \quad n \geq 2.$$

Note that sample maxima are sometimes called *block maxima*.

Fisher-Tippett theorem

Theorem 1 (Fisher-Tippett theorem)

Let (X_n) be a sequence of i.i.d. rvs. If there exists norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non-degenerate df H such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} H,$$

then H belongs to the type of one of the following three dfs:

Fréchet*:*

$$\Phi_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ \exp(-x^{-\alpha}), & x > 0 \end{cases} \qquad \alpha > 0.$$

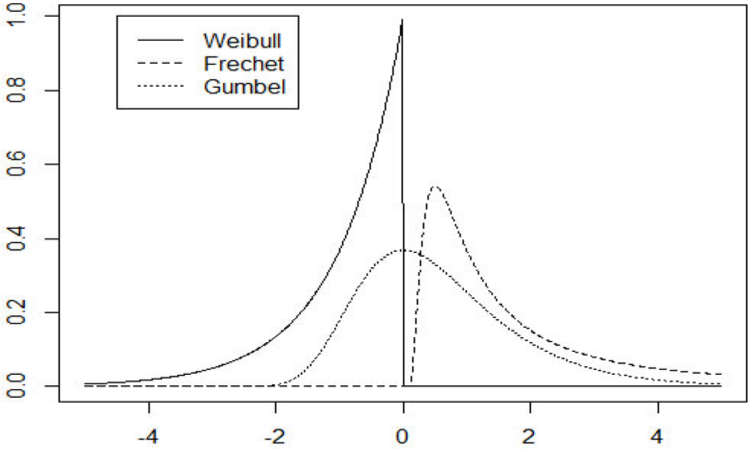
Weibull*:*

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha), & x \leq 0 \\ 1, & x > 0 \end{cases} \qquad \alpha > 0.$$

Gumbel*:*

$$\Lambda(x) = \exp(-e^{-x}), \quad x \in \mathbb{R}.$$

Fisher-Tippett theorem



Probability density functions of the Weibull ($\alpha = 1$), Fréchet ($\alpha = 1$) and Gumbel distribution.

GEV and Maximum domain of attraction

It is possible to summarise the three extreme value distributions in a one-parameter representation:

Definition 2 (Generalised extreme value (GEV) distribution)

The df H_ξ of the GEV distribution is given by

$$H_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}), & \text{if } \xi \neq 0, \\ \exp(-\exp(-x)), & \text{if } \xi = 0, \end{cases}$$

where $1 + \xi x > 0$. The latter condition implies that the support of H_ξ is given as follows: $x > -\xi^{-1}$ for $\xi > 0$; $x < -\xi^{-1}$ for $\xi < 0$; $x \in \mathbb{R}$ for $\xi = 0$.

Definition 3

The rv X belongs to the *maximum domain of attraction of the extreme value distribution* H if there exists norming constants $c_n > 0, d_n \in \mathbb{R}$ such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} H. \tag{1}$$

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e, we write $X \in MDA(H)$.

Definition 4

Let X be a rv with df F and right endpoint x_F. For a fixed $u < x_F$, define

$$F_u(x) = \mathbb{P}(X - u \leq x | X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}, \quad 0 \leq x < x_F - u,$$

which is the *excess df of the rv* X *(of the df* F*) over the threshold* u.

Definition 5

The generalised Pareto distribution (GPD) is given by

$$G_{\xi,\sigma}(x) = 1 - \left(1 + \xi \frac{x}{\sigma}\right)^{-1/\xi}, \quad (2)$$

where $x \in [0, \infty)$ if $\xi \geq 0$, and $x \in [0, -\frac{\sigma}{\xi}]$ otherwise. Hence, the corresponding density is given by

$$f_{GPD(\xi,\sigma)} = \frac{1}{\sigma}\left(1 + \frac{\xi}{\sigma} x\right)^{-\frac{1}{\xi}-1}.$$

Pickands-Balkema-de Haan Theorem

Theorem 6 (Pickands-Balkema-de Haan Theorem)

One can find a positive, measurable function σ such that

$$\lim_{u \uparrow x_F} \sup_{0 \le x < x_F - u} |F_u(x) - G_{\xi,\sigma(u)}(x)| = 0.$$

if and only if $F \in MDA(H_\xi), \xi \in \mathbb{R}$.

- I.e. the distributions for which normalised maxima converge to a GEV distribution are the distributions for which the excess distribution converges to the GPD as the threshold rises.
- In addition, the shape parameter ξ of the GPD is the same as the shape parameter of the limiting GEV distribution for the maxima.

A latent trawl model for extreme values

- We extend a latent Markov model by Bortot and Gaetan (2014):
- Consider a time-series $Y_1, \ldots, Y_k$, assumed to be strictly stationary.
- Aim: Model the observations where the value of Y_j is considered to be extreme, meaning that $Y_j > u$ for a fixed threshold u.
- Define the exceedances X_j by

$$X_j := \max(Y_j - u; 0); \quad \text{for } j = 1, \ldots, k.$$

- From standard extreme value theory (Pickands, 1975; Davison and Smith, 1990), assuming $\{Y_j\}$ are in the domain of attraction of some extreme value distribution, the conditional distribution of the exceedances X_j given $X_j > 0$ converge to a generalised Pareto distribution (GPD) as $u \to \infty$.
- Hence assume that the conditional distribution of X_j given $X_j > 0$ can be approximated by a GPD, for a sufficiently large threshold u.

Mixture representation of the GPD

- The density of the GPD is written as

$$f_{\mathrm{GPD}}(x|\alpha,\beta) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)_{+}^{-(\alpha+1)}, \quad x \geq 0, \alpha, \beta > 0$$

  which is a reparameterisation of the standard density with shape parameter $\xi = 1/\alpha$ and scale parameter $\sigma = \beta/\alpha$.
- Observe that the GPD can be represented as a **mixture of an exponential random variable with Gamma distributed parameter** (see Reiss and Thomas (2007)), motivating a hierarchical specification for the exceedance process $\{X_j\}$.

Modelling the exceedances by a hierarchical model

- Assume that, conditional on Λ, the random variables X_j are independent with
$$X_j|(X_j > 0, \Lambda_j) \sim \mathrm{Exp}(\Lambda_j).$$

- Assume that, for $\alpha, \beta > 0$, $\Lambda_j \sim \Gamma(\alpha, \beta)$.
- Assume that, conditional on Λ_j the events $\{X_j > 0\}$ are independent, and for $\kappa > 0$,
$$P(X_j > 0|\Lambda_j) = \exp(-\kappa\Lambda_j).$$

- Then X_j conditional on $X_j > 0$ follows a $GPD(\alpha, \beta + \kappa)$ distribution and
$$P(X_j > 0) = \left(\frac{\beta}{\beta + \kappa}\right)^{\alpha}.$$

Modelling the exceedances cont'd

- We also require that the observations X_j are independent for distinct j, conditional on the corresponding values of Λ. Hence the conditional joint distribution of $(X_1, \ldots, X_k)$ can be written as a product:

$$f(x_1, \ldots, x_k | \lambda_1, \ldots, \lambda_k) = \prod_{j=1}^{k} f(x_j | \lambda_j).$$

- I.e. any dependence between observations $X_1, \ldots, X_k$ comes from the dependence between corresponding elements of the latent process Λ.
- To complete the GPD mixture construction, the latent process is required to have a Gamma marginal law, i.e.

$$\Lambda_j \sim \Gamma(\alpha, \beta), \quad \alpha, \beta > 0,$$
$$C(\theta, \Lambda_j) = -\alpha \log(1 - i\theta/\beta),$$

where $C(\theta, \Lambda_j)$ denotes the cumulant function of Λ_j with parameter θ.
- Note the restriction $\alpha > 0$ excludes the case when the data belong to the Weibull class.

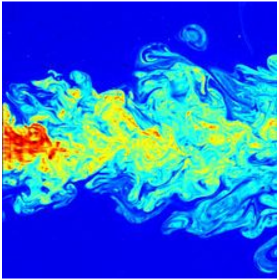
Latent trawl model

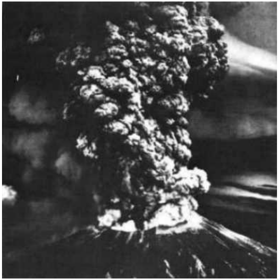
- There are various possibilities for modelling the latent gamma process Λ.
- Bortot and Gaetan (2014) consider two discrete-time Markov chains with gamma marginal law: The so-called Gaver and Lewis process (GLP) and the Warren process (WP).
- These Markov processes have the same ACF, but result in different asymptotic behaviour: The GLP is asymptotically dependent, the WP is asymptotically independent.
- We propose using a **Gamma trawl process** instead to achieve greater flexibility in modelling the serial correlation in the extremes.
- . . . what is a trawl process?

Trawl processes: Special cases of ambit processes

Ambit stochastics:

- Theory and applications of ambit fields and processes
- Probabilistic framework for spatio-temporal modelling
- Ambit fields were introduced by O. E. Barndorff-Nielsen (Aarhus) and J. Schmiegel (Aarhus) in the context of modelling turbulence in physics.





Definition of a Lévy basis

- Let S denote a Borel set in $\mathbb{R}^k$ for a $k \in \mathbb{N}$ (often we choose $S = \mathbb{R}^k$). Let $\mathcal{S} = \mathcal{B}(S)$ be the Borel σ-algebra on S.
- Let Leb denote the Lebesgue measure, and define $\mathcal{B}_b(S) = \{A \in \mathcal{S} : \mathrm{Leb}(A) < \infty\}$, which is the subset of $\mathcal{S}$ that contains sets which have bounded Lebesgue measure.

Definition 7

An $\mathbb{R}$-valued *Lévy basis* L on $(S, \mathcal{S})$ is an independently scattered, infinitely divisible random measure, i.e. it is a collection $\{L(A) : A \in \mathcal{B}_b(S)\}$ of random variables in $\mathbb{R}$ satisfying the following three conditions:

1. For any sequence $A_1, A_2, \ldots$ of disjoint elements of $\mathcal{B}_b(S)$ satisfying $\cup_{j=1}^{\infty} A_j \in \mathcal{B}_b(S)$ we have $L\left(\cup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} L\left(A_j\right)$ a.s..
2. For any sequence $A_1, A_2, \ldots$ of disjoint elements of $\mathcal{B}_b(S)$, the random variables $L(A_1), L(A_2), \ldots$ are independent.
3. For any $A \in \mathcal{B}_b(S)$, the law of $L(A)$ is infinitely divisible.

The existence of Lévy bases is due to Rajput & Rosinski (1989, Proposition

Stationary/homogeneous Lévy bases

Definition 8

A *stationary/homogeneous* Lévy basis on $(S, \mathcal{S})$ is a stationary, independently scattered, infinitely divisible random measure.

When defining trawl processes, we concentrate on *homogeneous* Lévy bases which we evaluate over Borel sets which have finite Lebesgue measure, see Barndorff-Nielsen (2011).

Definition of a trawl process

Definition 9

For $d \in \mathbb{N}$, let L be a homogeneous Lévy basis on $\mathbb{R}^d \times \mathbb{R}$ with characteristic quadruplet $(a, b, \nu(dx), \mathrm{Leb})$. Define

$$\mathcal{B}_b(\mathbb{R}^d \times \mathbb{R}) := \{A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}) : \mathrm{Leb}(A) < \infty\}.$$

and $A_t := A + (\mathbf{0}, t)$ for an $A = A_0 \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{R})$. Then

$$Y(t) = L(A_t) = \int_{\mathbb{R}^d \times \mathbb{R}} \mathbb{1}_A(\xi, s - t) L(d\xi, ds), \tag{3}$$

defines the *trawl process* associated with the Lévy basis L and the *trawl* A.

We shall assume that

$$A \subset \mathbb{R}^d \times (-\infty, 0]. \tag{4}$$

The cumulant function of a trawl process

Proposition 1

A trawl process is a strictly stationary, infinitely divisible stochastic process with cumulant function given by

$$C(\zeta; Y(t)) := \mathrm{Log}(\mathbb{E}(\exp(i\zeta Y(t)))) = \mathrm{Leb}(A)C(\zeta; L') \\ := \mathrm{Leb}(A)\mathrm{Log}(\mathbb{E}(\exp(i\zeta L'))). \quad (5)$$

- Important implication: For any infinitely divisible law π there exists a stationary trawl process having π as its one-dimensional marginal law.
- I.e. the choice of the Lévy seed L' will determine the marginal law of the trawl process.
- The shape of the trawl will determine its autocorrelation structure.

Autocorrelation function of a trawl process

Proposition 2

Let $h > 0$, then

$$\rho(h) := \mathrm{Cov}(Y(t), Y(t+h)) = \mathrm{Leb}(A \cap A_h) Var(L'). \quad (6)$$

For the autocorrelation, we get

$$r(h) = \mathrm{Cor}(Y(t), Y(t+h)) = \frac{\mathrm{Leb}(A \cap A_h)}{\mathrm{Leb}(A)}. \quad (7)$$

Example 10

As a working example, we will choose a so-called *exponential trawl* which is defined as $A_t = A + (0, t)$, where

$$A = \{(x, s) : s \leq 0,\ 0 \leq x \leq d(s)\} \subset (-\infty, 0] \times [0, 1], \quad (8)$$

and $d(s) = \exp(\rho s)$ for $\rho > 0$. Here $r(h) = \exp(-\rho|h|)$ for $h \in \mathbb{R}$.

Autocorrelation of the exceedances

Proposition 3

The mean of the exceedance process (X_j) *is for all* $j \in \mathbb{N} \cup \{0\}$ *given by*

$$E[X] := E[X_j] = (1 + \kappa/\beta)^{-\alpha}(\beta + \kappa)/(\alpha - 1), \quad \alpha > 1.$$

As X *is a stationary process, it has autocovariance function* $\varphi(h) = E[X_0 X_h] - E^2[X]$, *where for* $h \in \mathbb{N}$

$$E[X_0 X_h] = \int_\kappa^\infty \int_\kappa^\infty \left(1 + \frac{u_0}{\beta}\right)^{b_{0\setminus h}} \left(1 + \frac{u_0 + u_h}{\beta}\right)^{b_{0,h}} \left(1 + \frac{u_h}{\beta}\right)^{b_{h\setminus 0}} du_0\, du_h,$$

where $b_i = -\alpha \mathrm{Leb}(B_i)/\mathrm{Leb}(A)$ *for* $i \in \{(0 \setminus h), (0, h), (h \setminus 0)\}$ *with* $B_{0\setminus h} = A_0 \setminus A_h$, $B_{0,h} = A_0 \cap A_h$, $B_{h\setminus 0} = A_h \setminus A_0$. *Note that* $b_{0\setminus h} = b_{h\setminus 0}$.

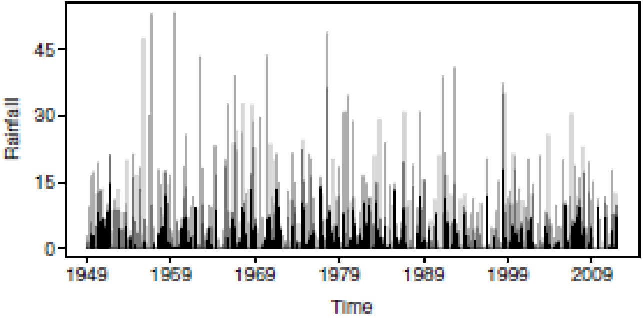
Note that in the hierarchical model the parameters α, β and κ from the marginal distribution also influence the autocovariance structure of the process.

Model estimation

- The latent trawl model can be estimated using pairwise likelihood (using explicit formulas which rely on the joint characteristic function for trawl processes).
- Confidence bounds for the parameters can be obtained either from the asymptotic results of the composite likelihood or from a parametric bootstrap approach.

Rainfall data from Heathrow

- The data set consists of hourly accumulated rainfall amounts at Heathrow (UK) over the years 1980-2012, provided by UK Meteorological Office (2012).
- We set the threshold *u* at the 95 percentile of the original data (8.9mm), resulting in the time series of exceedance values:



Model comparison

- Compare the fit of the latent trawl model versus the latent Markov chain model of Bortot and Gaetan (2014), using both the GLP and WP for the Markov chain.
- Estimates:

	α	β	ρ	κ
Latent trawl	6.33	20.12	0.27	12.18
G-LP	6.43	20.64	0.70	12.25
WP	6.30	19.94	0.78	12.15

- The marginal parameters are similar across the models, which is reasonable as the models all have a marginal GPD(α; $\beta + \kappa$) distribution and κ controls the marginal exceedance probability.
- The parameter ρ controls the latent dependence structure of the models. For the latent trawl process it is the decay parameter of the exponential trawl function, whereas for the latent Markov chains it enters in the autocorrelation function $r(h) = \rho^h = exp(h \log(\rho))$. [$\log(0.7) \approx -0.356$, $\log(0.78) \approx -0.248$.]
- All three latent processes have similar autocorrelation functions, where the GLP and WP processes have the fastest and slowest decay, respectively.

Marginal transformation model and extremal index

- So far, our model was restricted to positive shape parameters (Fréchet-type), since the shape parameter is linked to one of the (positive) parameters in the Gamma distribution.
- Use the transformation trick to get rid off the restriction:
  - First, fix the parameters α, β of the latent Gamma distribution such that only the parameters associated with the trawl set will influence the trawl process, and thus also the dependence of the exceedances. In the following we will work with $\alpha = \beta = 1$, such that the exceedances have marginal law given by GPD$(1, 1+\kappa)$.
  - Second, use a standard probability integral transform to give the marginals a GPD(ξ, σ), specifically assume that $\widetilde{X}_j \sim GPD(1, 1+\kappa)$. Then model the exceedances by

$$X_j = F^{-1}_{GPD(\xi,\sigma)}(F_{GPD(1,1+\kappa)}(\widetilde{X}_j)), \quad \xi \in \mathbb{R}, \sigma > 0.$$

- Empirical finding (for negative shape parameter): Our model outperformed the two competitors clearly for the London air pollution data with the WP process underestimating and the GLP process overestimating the extremal dependence in the data.

Main contributions and outlook

- Proposed a hierarchical model for peaks over thresholds where the latent process is modelled by a Gamma trawl process.
- Using a transformation trick we can extend the model for all possible choices of the shape parameter ξ.
- Compared new model with Markov-chain based competitors: Similar performance for extreme rainfall data, but trawl-based model performed much better for London air pollution data (the case of negative shape parameter ξ).
- More details available in *Noven, R. C., Veraart, A. E. D. and Gandy, A., A latent trawl process model for extreme values, Journal of Energy Marktes, 2018, Volume 11 (3), 1-24.*
- Ongoing extensions and refinements (joint with Valentin Courgeau)
  - Improvements to the inference procedure in the univariate case:*Inference, simulation and application of a latent trawl model for extreme values*.
  - Extensions to a multivariate setting with extremal clustering both in time and between components: *The X-Vine model for serial and cross-sectional extreme propagation*.

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Lévy-Khintchine representation

A Lévy basis naturally extends the concepts of a Lévy process to (multi-parameter) random measures. Also, the important concepts of the Lévy-Khintchine formula and the Lévy-Itô decomposition extend to the case of Lévy bases.
Since a Lévy basis L is ID, it has a Léevy-Khintchine representation, see Rajput & Rosiński (1989, Proposition 2.1 (a)).

Proposition 4

Let L denote a Lévy basis, and let $\theta \in \mathbb{R}$ and $A \in \mathcal{B}_b(S)$, then

$$\begin{aligned} C(\theta; L(A)) &= \mathrm{Log}\left(\mathbb{E}(\exp(i\theta L(A)))\right) \\ &= i\theta a^*(A) - \frac{1}{2}\theta^2 b^*(A) + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbb{I}_{[-1,1]}(x)\right) n(dx, A), \end{aligned} \tag{9}$$

where a^* is a signed measure on $\mathcal{B}_b(S)$, b^* is a measure on $\mathcal{B}_b(S)$, and $n(\cdot, \cdot)$ is the generalised Lévy measure, i.e. $n(dx, A)$ is a Lévy measure on $\mathbb{R}$ for fixed $A \in \mathcal{B}_b(S)$ and a measure on $\mathcal{B}_b(S)$ for fixed dx.

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Log denotes the *distinguished logarithm*, see Sato (1999, p. 33).

The control measure

Next, we define the so-called *control measure* as introduced in Rajput & Rosinski (1989, Proposition 2.1 (c), Definition 2.2).

Definition 11

Let L be a Lévy basis with Lévy-Khintchine representation (9). Then the measure c is defined by

$$c(B) = |a^*|(B) + b^*(B) + \int_{\mathbb{R}} \min(1, x^2) n(dx, B), \qquad B \in \mathcal{B}_b(S), \tag{10}$$

where $|\cdot|$ denotes the total variation. The extension of the measure c to a σ-finite measure on $(S, \mathcal{S})$ is called the *control measure* of L.

Note: When working with homogeneous Lévy bases we will take c to be the Lebesgue measure.

Radon-Nikodym derivatives

Define the Radon-Nikodym derivatives of the three components of c:

$$a(\mathbf{z}) = \frac{da^*}{dc}(\mathbf{z}), \qquad b(\mathbf{z}) = \frac{db^*}{dc}(\mathbf{z}), \qquad \nu(dx, \mathbf{z}) = \frac{n(dx, \cdot)}{dc}(\mathbf{z}). \tag{11}$$

In particular this implies that $n(dx, d\mathbf{z}) = \nu(dx, \mathbf{z})c(d\mathbf{z})$. W.l.o.g. we assume that $\nu(dx, \mathbf{z})$ is a Lévy measure for each fixed $\mathbf{z}$.

Definition 12

We call $(a, b, \nu(dx, \cdot), c) = (a(\mathbf{z}), b(\mathbf{z}), \nu(dx, \mathbf{z}), c(d\mathbf{z}))_{\mathbf{z}\in S}$ a *characteristic quadruplet* (CQ) associated with a Lévy basis L on $(S, \mathcal{S})$ provided the following conditions hold:

1. Both a and b are functions on S, where b is restricted to be non-negative.
2. For fixed $\mathbf{z}$, $\nu(dx, \mathbf{z})$ is a Lévy measure on $\mathbb{R}$, and, for fixed dx, it is a measurable function on S.
3. The control measure c is a measure on $(S, \mathcal{S})$ such that $\int_B a(\mathbf{z})c(d\mathbf{z})$ is a (possibly signed) measure on $(S, \mathcal{S})$, $\int_B b(\mathbf{z})c(d\mathbf{z})$ is a measure on $(S, \mathcal{S})$ and $\int_B \nu(dx, \mathbf{z})c(d\mathbf{z})$ is a Lévy measure on $\mathbb{R}$ for fixed $B \in \mathcal{S}$.

Characteristic quadruplet

Summary:
Every Lévy basis on $(S, \mathcal{S})$ determines a CQ of the form
$(a, b, \nu(dx, \cdot), c) = (a(\mathbf{z}), b(\mathbf{z}), \nu(dx, \mathbf{z}), c(d\mathbf{z}))_{\mathbf{z} \in S}$.

Conversely, every CQ satisfying the conditions in Definition 12 determines, in law, a Lévy basis on $(S, \mathcal{S})$.

(See Rajput & Rosinski (1989, Lemma 2.3, Proposition 2.4).)

Lévy-Itô-decomposition

See Pedersen (2003, Section 4).

Definition 13

Suppose that $S = \mathbb{R}^k$, for $k \in \mathbb{N}$. A Lévy basis on $(S, \mathcal{S})$ with CQ $(a, b, \nu(dx, \cdot), c)$ is said to be *dispersive* if

$$c(\{\mathbf{z}\}) = 0 \text{ for all } \mathbf{z} \in S. \tag{12}$$

When working with a Lévy basis whose control measure is proportional to the Lebesgue measure, then the dispersiveness condition (12) is trivially satisfied.

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Lévy-Itô-decomposition cont'd

Proposition 5

For a dispersive Lévy basis L on $(S, \mathcal{S})$ with characteristic quadruplet $(a, b, \nu(dx, \cdot), c)$ there is a modification L^* with the same characteristic quadruplet which has the following Lévy-Itô decomposition:

$$L^*(A) = a^*(A) + W(A) + \int_{\{|y|\leq 1\}} y(N-n)(dy, A) + \int_{\{|y|>1\}} yN(dy, A), \tag{13}$$

for $A \in \mathcal{B}_b(S)$ and for a Gaussian basis W (with characteristic quadruplet $(0, b, 0, c)$, i.e. $W(A) \sim N(0, \int_A b(\mathbf{z})c(d\mathbf{z}))$), and a Poisson basis N (independent of W) with compensator $n(dy; A) = \mathbb{E}\{N(dy; A)\}$ where $n(dx, d\mathbf{z}) = \nu(dx, \mathbf{z})c(d\mathbf{z})$.

Lévy-Itô-decomposition cont'd

The Lévy-Itô decomposition (13) can be written in infinitesimal form

$$L^*(d\mathbf{z}) = a^*(d\mathbf{z}) + W(d\mathbf{z}) + \int_{\{|x|>1\}} xN(dx, d\mathbf{z}) + \int_{\{|x|\leq 1\}} x(N-n)(dx, d\mathbf{z}). \tag{14}$$

Similarly, the corresponding Lévy-Khintchine representation is given by

$$\begin{aligned} C(\theta; L(d\mathbf{z})) &= \mathrm{Log}\left(\mathbb{E}(\exp(i\theta L(d\mathbf{z})))\right) \\ &= i\theta a^*(d\mathbf{z}) - \frac{1}{2}\theta^2 b^*(d\mathbf{z}) + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbb{I}_{[-1,1]}(x)\right) n(dx, d\mathbf{z}) \\ &= \left(i\theta a(\mathbf{z}) - \frac{1}{2}\theta^2 b(\mathbf{z}) + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbb{I}_{[-1,1]}(x)\right) \nu(dx, \mathbf{z})\right) c(d\mathbf{z} \\ &= C(\theta; L'(\mathbf{z}))c(d\mathbf{z}), \qquad \theta \in \mathbb{R}, \end{aligned} \tag{15}$$

where $L'(\mathbf{z})$ denotes the so-called *Levy seed* of L at $\mathbf{z}$, which is defined to be an infinitely divisible random variable having Lévy-Khintchine representation

$$C(\theta; L'(\mathbf{z})) = i\theta a(\mathbf{z}) - \frac{1}{2}\theta^2 b(\mathbf{z}) + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbb{I}_{[-1,1]}(x)\right) \nu(dx, \mathbf{z}). \tag{16}$$

In the case when

$$\int_0^1 |x|\nu(dx, \mathbf{z}) < \infty,$$

the Lévy-Khintchine formula simplifies to

$$C(\theta; L'(\mathbf{z})) = i\theta a_0(\mathbf{z}) - \frac{1}{2}\theta^2 b(\mathbf{z}) + \int_{\mathbb{R}} \left(e^{i\theta x} - 1\right) \nu(dx, \mathbf{z}), \tag{17}$$

where

$$a_0(\mathbf{z}) = \left(a(\mathbf{z}) - \int_{\mathbb{R}} x \mathbb{I}_{[-1,1]}(x) \nu(dx, \mathbf{z})\right).$$

Homogeneous Lévy bases

In the following, we will not work with the most general Lévy bases, but will consider relevant subclasses.

Definition 14

Let L denote a Lévy basis on $(S, \mathcal{S})$ with CQ given by $(a, b, \nu(dx, \cdot), c)$. We call L *factorisable* if $\nu(dx, \mathbf{z})$ does not depend on $\mathbf{z}$. If L is factorisable and if c is proportional to the Lebesgue measure and $a(\mathbf{z})$ and $b(\mathbf{z})$ do not depend on $\mathbf{z}$, then L is called a *homogeneous* Lévy basis. In that case we write $c(d\mathbf{z}) = v\mathrm{Leb}(d\mathbf{z}) = vd\mathbf{z}$ for a positive constant $v > 0$ and where $\mathrm{Leb}(\cdot)$ denotes the Lebesgue measure.

Note that on the Euclidean space the terms *stationary* and *homogeneous* Lévy bases are equivalent.
When working with homogeneous Lévy bases we shall in the following assume that $v \equiv 1$ to simplify the exposition.

Examples

Example 15 (Lévy process)

Suppose L is a Lévy basis on $\mathbb{R}$ with control measure given by $c = \mathrm{Leb}$, then $(L_t)_{t \geq 0}$ with $L([0, t]) = L_t$, $t \geq 0$ is a Lévy process, see Pedersen (2003, Remark 3.5). On the other hand, a Lévy process L determines a Lévy basis by defining the value of the Lévy basis of an interval $(a, b]$ as $L((a, b]) = L_b - L_a$.

Example 16 (Gaussian Lévy basis)

Next consider the case when $\nu(dx, \mathbf{z}) \equiv 0$, i.e. L constitutes a Gaussian Lévy basis with $L(A) \sim N\left(\int_A a(\mathbf{z})c(d\mathbf{z}), \int_A b(\mathbf{z})c(d\mathbf{z})\right)$, for $A \in \mathcal{B}_b(\mathbb{R}^k)$. If, in addition, L is homogeneous, then $L(A) \sim N\left(a\mathrm{Leb}(A), b\mathrm{Leb}(A)\right)$.

Example 17 (Poisson Lévy basis)

We obtain a Poisson Lévy basis by setting $c(d\mathbf{z}) = d\mathbf{z}$ and $a \equiv b \equiv 0$ and $\nu(dx; \mathbf{z}) = \lambda(\mathbf{z})\delta_1(dx)$, where δ_1 denotes the Dirac measure with point mass at 1 and $\lambda(\mathbf{z}) > 0$ is the intensity function. If, in addition, L is factorisable, i.e. λ does not depend on $\mathbf{z}$, then $L(A) \sim \mathit{Poisson}\left(\lambda \mathrm{Leb}(A)\right)$, for all $A \in \mathcal{B}_b(\mathbb{R}^k)$.

Examples

Example 18 (Inverse Gaussian Lévy basis)

Next we define an *inverse Gaussian Lévy basis* L by setting $c(d\mathbf{z}) = d\mathbf{z}$, $a \equiv b \equiv 0$ and assuming that the (generalised) Lévy measure is of the form $\nu(dx; \mathbf{z}) = x^{-3/2} e^{-\frac{1}{2}\gamma^2(\mathbf{z})x} dx$, where $\gamma(\mathbf{z}) > 0$. If, in addition, L is factorisable, i.e. the function γ does not depend on the parameter $\mathbf{z}$, then $L(A)$ has an inverse Gaussian law for all $A \in \mathcal{B}_b(\mathbb{R}^k)$.

Example 19 (Gamma Lévy basis)

Now suppose that $c(d\mathbf{z}) = d\mathbf{z}$, $a \equiv b \equiv 0$ and the (generalised) Lévy measure is of the form $\nu(dx; \mathbf{z}) = x^{-1} e^{-\alpha(\mathbf{z})x} dx$, where $\alpha(\mathbf{z}) > 0$. In that case, we call the corresponding Lévy basis L a *Gamma Lévy basis*. If, in addition, L is factorisable, i.e. the function α does not depend on the parameter $\mathbf{z}$, then $L(A)$ has a Gamma law for all $A \in \mathcal{B}_b(\mathbb{R}^k)$.

Examples

Example 20 (Normal inverse Gaussian (NIG) Lévy basis)

Finally suppose that $c(d\mathbf{z}) = d\mathbf{z}$, $b \equiv 0$ and $a(\mathbf{z}) = \mu(\mathbf{z}) + 2\pi^{-1}\delta(\mathbf{z})\alpha(\mathbf{z}) \int_0^1 \sinh(\beta(\mathbf{z})y)K_1(\alpha(\mathbf{z})y)dy$, where K_1 denotes the modified Bessel function of the third kind with index 1. The parameter functions are assumed to satisfy $\mu(\mathbf{z}) \in \mathbb{R}, \delta(\mathbf{z}) > 0, 0 \leq \beta(\mathbf{z}) < \alpha(\mathbf{z})$. Moreover the (generalised) Lévy measure is of the form $\nu(dx; \mathbf{z}) = \pi^{-1}\delta(\mathbf{z})\alpha(\mathbf{z})|x|^{-1}K_1(\alpha(\mathbf{z})|x|)e^{\beta(\mathbf{z})x}dx$. We call the corresponding Lévy basis L a *Normal inverse Gaussian (NIG) Lévy basis*. If, in addition, L is factorisable, i.e. the functions $\alpha, \beta, \mu, \delta$ do not depend on the parameter $\mathbf{z}$, then $L(A)$ has an NIG law for all $A \in \mathcal{B}_b(\mathbb{R}^k)$.

Monotonic trawls

Definition 21

A *monotonic* trawl is defined as $A_t = A + (0, t)$, where

$$A = \{(x, s) : s \leq 0,\ 0 \leq x \leq d(s)\} \subset (-\infty, 0] \times [0, 1], \tag{18}$$

where $d : (-\infty, 0] \mapsto [0, 1]$ is a continuous, non-decreasing function such that $\mathrm{Leb}(A) < \infty$ and $d(0) = 1$.

(The restriction that d takes values in $[0, 1]$ and that $d(0) = 1$ is not essential.)

Proposition 6

For all $s \leq t \in \mathbb{R}$, we have

$$\mathrm{Leb}(A_t) = \mathrm{Leb}(A) = \int_{-\infty}^{0} d(u)du,$$

$$\mathrm{Leb}(A_s \cap A_t) = \int_{-\infty}^{s-t} d(u)du,$$

$$\mathrm{Leb}(A_t \setminus A_s) = \mathrm{Leb}(A_s \setminus A_t) = \int_{s-t}^{0} d(u)du.$$

Monotonic trawls cont'd

Corollary 22

In the case of a monotonic trawl, the autocorrelation function is given by

$$r(h) = \frac{\int_h^\infty d(-x)dx}{\int_0^\infty d(-x)dx}.$$

Moreover, we find that

$$r'(h) = \frac{-d(-h)}{\int_0^\infty d(-x)dx}. \tag{19}$$

- Equation (19) demonstrates a direct link between the derivative of the autocorrelation function and the trawl function d.
- This relation can be used for model building.

Example 23

Let $d(x) = \exp(\lambda x)$ for $\lambda > 0, x \leq 0$. Then, for $h \geq 0$,

$$r(h) = \mathrm{Cor}(Y(t), Y(t+h)) = \exp(-\lambda h).$$

Examples cont'd

Example 24

We set

$$d(z) = \int_0^\infty e^{\lambda z} \pi(d\lambda), \qquad \text{for } z \leq 0,$$

for a probability measure π on $(0, \infty)$. If π is absolutely continuous with density f_π, we can write the trawl function as

$$d(z) = \int_0^\infty e^{\lambda z} f_\pi(\lambda) d\lambda.$$

We remark that $d(z)$ is increasing (non-decreasing) in z, hence, $d(s-t)$ is also increasing (non-decreasing) in s for $s \leq t$. Then

$$r(h) = \mathrm{Cor}(Y(t), Y(t+h)) = \frac{\int_0^\infty \frac{1}{\lambda} e^{-\lambda h} \pi(d\lambda)}{\int_0^\infty \frac{1}{\lambda} \pi(d\lambda)},$$

provided that $\int_0^\infty \frac{1}{\lambda} \pi(d\lambda) < \infty$.