Limit theory for integrated supOU processes

Danijel Grahovac

Department of Mathematics, J. J. Strossmayer University of Osijek, Croatia

Fourth Conference on Ambit Fields and Related Topics

(Joint work with: N. N. Leonenko and M. S. Taqqu)

Superpositions of OU type processes (supOU)

Ornstein-Uhlenbeck (OU) type process

- strictly stationary process

$$X(t) = \int_{\mathbb{R}} e^{-\lambda t+s} \mathbf{1}_{[0,\infty)}(\lambda t - s) dL(s)$$

  where $\{L(t)\}$ is background driving Lévy process (BDLP) and $\lambda > 0$
- for any selfdecomposable distribution $\mathcal{D}$ there corresponds a BDLP L and OU process X such that $X(t) \stackrel{d}{=} \mathcal{D}$
- acf: $r(t) = e^{-\lambda t}, \ t \geq 0$
- how to obtain a more flexible dependence structure?

supOU process

- "randomize" λ using the probability distribution π

$$X(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\xi t+s} \mathbf{1}_{[0,\infty)}(\xi t - s) dL(s) \pi(d\xi)$$

More precisely:

- let L be a Lévy process (BDLP) with the Lévy-Khintchine triplet (a, b, μ)

$$\kappa_L(\zeta) := \log \mathbb{E} e^{i\zeta L(1)} = i\zeta a - \frac{\zeta^2}{2} b + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta x \mathbf{1}_{[-1,1]}(x) \right) \mu(dx)$$

- let π be a probability measure on $\mathbb{R}_+$
- let $\Lambda = \{\Lambda(A),\ A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\}$ be a Lévy basis (homogeneous infinitely divisible independently scattered random measure) on $\mathbb{R}_+ \times \mathbb{R}$ with control measure $m = \pi \times Leb$

$$C\{\zeta \ddagger \Lambda(A)\} := \log \mathbb{E} e^{i\zeta \Lambda(A)} = m(A)\kappa_L(\zeta) = (\pi \times Leb)(A)\kappa_L(\zeta),$$

Definition 1 (Barndorff-Nielsen (2001))

A supOU process is a process $X = \{X(t),\ t \in \mathbb{R}\}$ defined by

$$X(t) = \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-\xi t + s} \mathbf{1}_{[0,\infty)}(\xi t - s) \Lambda(d\xi, ds).$$

Properties:

- X is strictly stationary
- (a, b, μ, π) – characteristic quadruple
- $X(t)$ has a self-decomposable distribution associated to BDLP L
- if $\mathbb{E}X(t)^2 < \infty$, then the acf is:

$$r(t) = \int_{\mathbb{R}_+} e^{-\xi t} \pi(d\xi), \quad t \geq 0.$$

- For ℓ slowly varying function at infinity and $\alpha > 0$

$$\pi\left((0,x]\right) \sim \ell(x^{-1})x^{\alpha}, \quad \text{as } x \to 0 \tag{1}$$

if and only if

$$r(t) \sim \Gamma(1+\alpha)\ell(t)t^{-\alpha}, \quad \text{as } t \to \infty.$$

- Example. If π is Gamma distribution with density

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \mathbf{1}_{(0,\infty)}(x),$$

and $\alpha > 0$, then

$$r(t) = (1+t)^{-\alpha}.$$

- If $\alpha \in (0,1)$ in (1), then r is not integrable – long-range dependence

Integrated process

- We will consider limiting behavior of the integrated process

$$X^*(t) = \int_0^t X(s) ds$$

- Finite variance
- Infinite variance
- Convergence of moments
- Intermittency

Outline

Limit theorems – finite variance

- Suppose X is a zero mean supOU process with characteristic quadruple (a, b, μ, π) and finite variance $\sigma^2 < \infty$
- What to expect?
- Example. Suppose $\mathbb{E}X(t)^2 < \infty$ and π is a discrete probability measure with finite support – finite superposition inherits strong mixing property of the OU type process and classical arguments show that as $T \to \infty$

$$\left\{ \frac{1}{T^{1/2}} X^*(Tt) \right\} \stackrel{fdd}{\to} \{\widetilde{\sigma} B(t)\}$$

where $\{B(t)\}$ is Brownian motion.
($\{\cdot\} \stackrel{fdd}{\to} \{\cdot\}$ denotes convergence of finite dimensional distributions)

- Example. Suppose X is a Gaussian supOU process with characteristic quadruple $(0, b, 0, \pi)$ and $\pi((0, x]) \sim \ell(x^{-1})x^{\alpha}$ as $x \to 0$, $\alpha \in (0, 1)$. Then the results for Gaussian processes imply

$$\left\{ \frac{1}{T^{1-\alpha/2}\ell(T)^{1/2}} X^*(Tt) \right\} \overset{fdd}{\to} \{\widetilde{\sigma} B_{1-\alpha/2}(t)\} \qquad (2)$$

where $\{B_{1-\alpha/2}(t)\}$ is fractional Brownian motion (FBM) with parameter $H = 1 - \alpha/2$

- Can we extend (2) to the non-Gaussian case?

Limit theorems – short-range dependence

- We start with the short-range dependence

$$\int_0^\infty r(\tau)d\tau = \int_0^\infty \int_0^\infty e^{-\tau\xi} d\tau \pi(d\xi) = \int_0^\infty \xi^{-1}\pi(d\xi) < \infty$$

Theorem 2 (BM case)

If

$$\int_0^\infty \xi^{-1}\pi(d\xi) < \infty,$$

then as $T \to \infty$

$$\left\{ \frac{1}{T^{1/2}} X^*(Tt) \right\} \stackrel{fdd}{\to} \{\widetilde{\sigma} B(t)\},$$

where $\{B(t)\}$ *is Brownian motion and* $\widetilde{\sigma}^2 = b\frac{\Gamma(1+\alpha)}{(2-\alpha)(1-\alpha)}$.

Limit theorems – long-range dependence

- We now consider long-range dependence scenario and assume π has a density, i.e.

$$p(x) \sim \alpha \ell(x^{-1}) x^{\alpha-1}, \quad \text{as } x \to 0,$$

for some $\alpha \in (0,1)$ and some slowly varying function ℓ

Theorem 3 (FBM case)

If $b \neq 0$, then as $T \to \infty$

$$\left\{ \frac{1}{T^{1-\alpha/2} \ell(T)^{1/2}} X^*(Tt) \right\} \overset{fdd}{\to} \left\{ \widetilde{\sigma} B_H(t) \right\},$$

where $\{B_H(t)\}$ is fractional Brownian motion with $H = 1 - \alpha/2$ and $\widetilde{\sigma}^2 = \sigma^2 \frac{\Gamma(1+\alpha)}{(2-\alpha)(1-\alpha)}$.

- If there is no Gaussian component, the type of the limit will depend on the behavior of the Lévy measure of L (or X) near origin
- Assume there exists $\beta \in [0, 2)$, $c^+, c^- \geq 0$, $c^+ + c^- > 0$ such that

$$\mu([x, \infty)) \sim c^+ x^{-\beta} \text{ and } \mu((-\infty, -x]) \sim c^- x^{-\beta} \text{ as } x \to 0. \quad (3)$$

- This implies that β is the Blumenthal-Getoor index of the Lévy measure μ

$$\beta_{BG} = \inf \left\{ \gamma \geq 0 : \int_{|x| \leq 1} |x|^\gamma \mu(dx) < \infty \right\}.$$

- Note that (3) is equivalent to

$$\mu_X([x, \infty)) \sim \beta^{-1} c^+ x^{-\beta} \text{ and } \mu_X((-\infty, -x]) \sim \beta^{-1} c^- x^{-\beta} \text{ as } x \to 0.$$

Theorem 4 (Stable Lévy process case)

If $b = 0$ and

$$\beta_{BG} < 1 + \alpha,$$

then as $T \to \infty$

$$\left\{ \frac{1}{T^{1/(1+\alpha)} \ell^{\#}\left(T\right)^{1/(1+\alpha)}} X^{*}(Tt) \right\} \xrightarrow{fdd} \{L_{1+\alpha}(t)\},$$

where $\ell^{\#}$ is de Bruijn conjugate of $1/\ell\left(x^{1/(1+\alpha)}\right)$ and $\{L_{1+\alpha}\}$ is $(1+\alpha)$-stable Lévy process such that $L_{1+\alpha}(1) \stackrel{d}{=} \mathcal{S}_{1+\alpha}(\widetilde{\sigma}_{\alpha}, \widetilde{\rho}_{\alpha}, 0)$.

- If $\beta > 1 + \alpha$, the limiting process will have dependent increments

Theorem 5 ($Z_{\alpha,\beta}$ case)

Suppose that (3) *holds with* $\beta > 0$. *If* $b = 0$ *and*

$$1 + \alpha < \beta < 2,$$

then as $T \to \infty$

$$\left\{ \frac{1}{T^{1-\alpha/\beta} \ell(T)^{1/\beta}} X^*(Tt) \right\} \overset{fdd}{\to} \{Z_{\alpha,\beta}(t)\},$$

where $\{Z_{\alpha,\beta}\}$ *is a process with the stochastic integral representation*

$$Z_{\alpha,\beta}(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(\mathfrak{f}(\xi, t-s) - \mathfrak{f}(\xi, -s) \right) K(d\xi, ds),$$

$\mathfrak{f}$ *is given by* $\mathfrak{f}(x,u) = \begin{cases} 1 - e^{-xu}, & \text{if } x > 0 \text{ and } u > 0, \\ 0, & \text{otherwise,} \end{cases}$ *and* K *is a* β*-stable Lévy basis on* $\mathbb{R}_+ \times \mathbb{R}$ *with control measure* $\alpha \xi^{\alpha} d\xi ds$.

- $\{Z_{\alpha,\beta}\}$ is a β-stable process with stationary increments and $(1-\alpha/\beta)$-self-similar, first obtained in (Puplinskaitė & Surgailis 2010)

Simplified summary

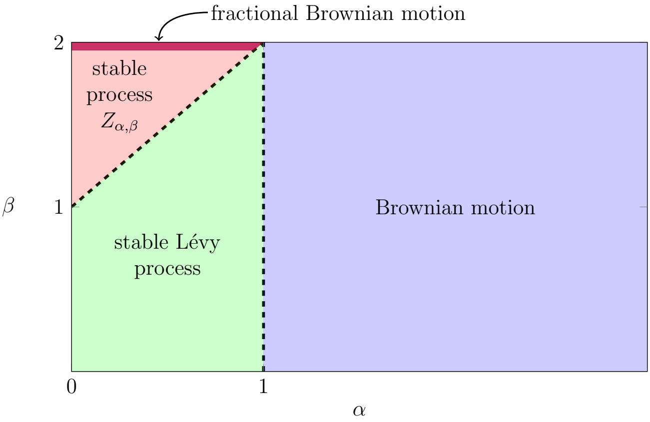


Figure: Classification of limits in the finite variance case

Functional convergence

The convergence extends to weak convergence in the space $C[0,1]$ in

- Theorem 2 (BM case) if additionally $\mathbb{E}|X(t)|^4 < \infty$ and $\int_0^\infty \xi^{-2}\pi(d\xi) < \infty$
- Theorem 3 (FBM case)
- Theorem 5 ($Z_{\alpha,\beta}$ case)

In Theorem 4 (stable Lévy process case) weak convergence in $D[0,1]$ equipped with J_1 topology cannot hold.

Outline

Limit theorems – infinite variance

- Suppose that $X(1)$ has balanced regularly varying tails:

  $$P(X(1) > x) \sim pk(x)x^{-\gamma} \quad \text{and} \quad P(X(1) \leq -x) \sim qk(x)x^{-\gamma}, \qquad \text{as } x \to \infty,$$

  for some $p, q \geq 0$, $p + q > 0$, $0 < \gamma < 2$ and some slowly varying function k (if $\gamma = 1$, we assume $p = q$)
- In particular, the variance is infinite
- These assumptions imply that $X(1)$ is in the domain of attraction of $\mathcal{S}_\gamma(\sigma, \rho, 0)$ law with

  $$\sigma = \left(\frac{\Gamma(2-\gamma)}{1-\gamma}(p+q)\cos\left(\frac{\pi\gamma}{2}\right) \right)^{1/\gamma}, \qquad \rho = \frac{p-q}{p+q}.$$
- When the mean is finite, we assume $\mathbb{E}X(1) = 0$

Further assumptions:

- there exists $\beta \in [0,2)$, $c^+, c^- \geq 0$, $c^+ + c^- > 0$ such that

$$\mu([x,\infty)) \sim c^+ x^{-\beta} \text{ and } \mu((-\infty,-x]) \sim c^- x^{-\beta} \text{ as } x \to 0.$$

- π has a density

$$p(x) \sim \alpha \ell(x^{-1}) x^{\alpha-1}, \quad \text{as } x \to 0,$$

for some $\alpha > 0$ and some slowly varying function ℓ

- π has finite mean (simplifies the presentation)

$$\int_0^\infty \xi \pi(d\xi) < \infty.$$

No Gaussian component

Theorem 6

If $b = 0$ *and*

$$\gamma < 1 + \alpha,$$

then as $T \to \infty$

$$\left\{ \frac{1}{T^{1/\gamma} k^{\#}(T)^{1/\gamma}} X^{*}(Tt) \right\} \stackrel{fdd}{\to} \{L_\gamma(t)\},$$

where $k^{\#}$ *is the de Bruijn conjugate of* $1/k\left(x^{1/\gamma}\right)$ *and the limit* $\{L_\gamma\}$ *is a* γ*-stable Lévy process such that* $L_\gamma(1) \stackrel{d}{=} \mathcal{S}_\gamma(\widetilde{\sigma}_{1,\gamma}, \rho, 0)$.

Theorem 7

If $b = 0$ and $\gamma > 1 + \alpha$, then the limit depends on the value of β, as follows.

1. *If*
$$\beta < 1 + \alpha,$$
then as $T \to \infty$
$$\left\{ \frac{1}{T^{1/(1+\alpha)} \ell^{\#}(T)^{1/(1+\alpha)}} X^*(Tt) \right\} \overset{fdd}{\to} \{ L_{1+\alpha}(t) \},$$
where the limit $\{L_{1+\alpha}\}$ is a $(1+\alpha)$-stable Lévy process such that $L_{1+\alpha}(1) \overset{d}{=} \mathcal{S}_{1+\alpha}(\widetilde{\sigma}, \widetilde{\rho}, 0)$.
2. *If*
$$1 + \alpha < \beta,$$
then as $T \to \infty$
$$\left\{ \frac{1}{T^{1-\alpha/\beta} \ell(T)^{1/\beta}} X^*(Tt) \right\} \overset{fdd}{\to} \{ Z_{\alpha,\beta}(t) \},$$
where $\{Z_{\alpha,\beta}\}$ is as in Theorem 5.

Simplified summary

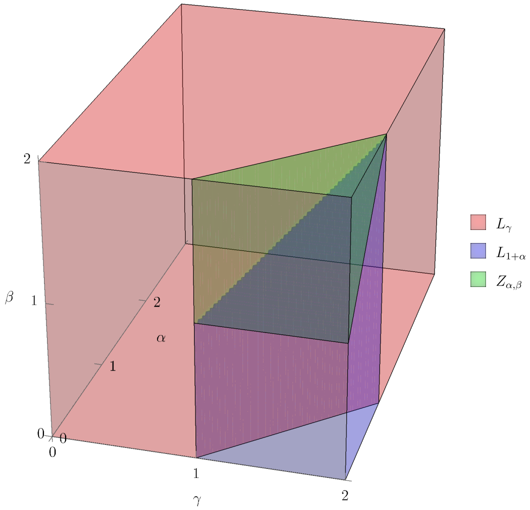


Figure: Classification of limits in the infinite variance case when $b = 0$

With Gaussian component

Theorem 8

Suppose that $b \neq 0$.

1. *If $\alpha > 1$ or if $\alpha < 1$ and $\gamma < \frac{2}{2-\alpha}$, then as $T \to \infty$*

$$\left\{ \frac{1}{T^{1/\gamma} k^{\#}(T)^{1/\gamma}} X^{*}(Tt) \right\} \xrightarrow{fdd} \{L_{\gamma}(t)\},$$

where the limit $\{L_{\gamma}\}$ is a γ-stable Lévy process defined as in Theorem 6.

2. *If $\alpha < 1$ and $\gamma > \frac{2}{2-\alpha}$, then as $T \to \infty$*

$$\left\{ \frac{1}{T^{1-\alpha/2} \ell(T)^{1/2}} X^{*}(Tt) \right\} \xrightarrow{fdd} \{\widetilde{\sigma}_{3,\alpha} B_H(t)\},$$

where $\{B_H(t)\}$ is standard fractional Brownian motion with $H = 1 - \alpha/2$ and $\widetilde{\sigma}_{3,\alpha} = \frac{b^2}{2} \frac{\Gamma(1+\alpha)}{(2-\alpha)(1-\alpha)}$.

Simplified summary

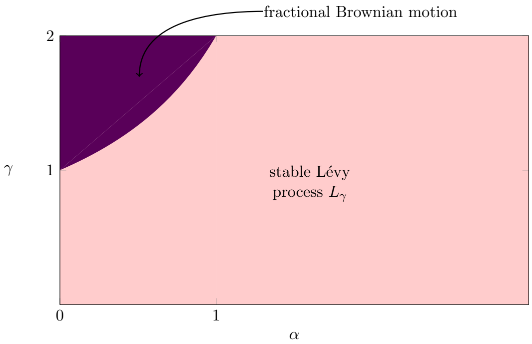


Figure: Classification of limits in the infinite variance case when $b \neq 0$

Outline

Intermittency

- Suppose we want to measure the rate of growth of moments of some process in time
- The scaling function at point $q \in \mathbb{R}$ of the process $Y = \{Y(t), t \geq 0\}$ is

$$\tau_Y(q) = \lim_{T \to \infty} \frac{\log \mathbb{E}|Y(T)|^q}{\log T},$$

  assuming the limit exists (possible equal to ∞ if $\mathbb{E}|Y(T)|^q = \infty$ for $T \geq T_0$)
- Example. If $\{Y(t), t \geq 0\}$ is H-self-similar process, then $\tau_Y(q) = Hq$.
- τ_Y is convex and

$$q \mapsto \frac{\tau_Y(q)}{q}$$

  is non-decreasing on $\mathcal{D}_\tau = \{q \in \mathbb{R} : \tau_Y(q) < \infty\}$

Definition 9

A stochastic process Y is intermittent if there exist $p, r \in \mathcal{D}_\tau$ such that

$$\frac{\tau_Y(p)}{p} < \frac{\tau_Y(r)}{r}.$$

Theorem 10

Let $Y = \{Y(t),\ t \geq 0\}$ and $Z = \{Z(t),\ t \geq 0\}$ be two processes such that $Z(t)$ is nondegenerate for every $t > 0$ and suppose that for a sequence (A_T), $A_T > 0$, $\lim_{T\to\infty} A_T = \infty$, one has

$$\left\{ \frac{Y(Tt)}{A_T} \right\} \overset{fdd}{\to} \{Z(t)\}\,, \tag{4}$$

as $T \to \infty$. Then there exists a constant $H > 0$ such that for every $q \in \mathbb{R}$ satisfying

$$\frac{\mathbb{E}|Y(Tt)|^q}{A_T^q} \to \mathbb{E}|Z(t)|^q, \quad \forall t \geq 0, \tag{5}$$

the scaling function of Y at q is

$$\tau_Y(q) = Hq.$$

- Therefore, in the intermittent case if (4) holds, then (5) must fail to hold

Finite variance case

- Assume that X has zero mean and the cumulant function κ_X is analytic in the neighborhood of the origin (in particular, all moments are finite) and π has a density $p(x) \sim \alpha \ell(x^{-1}) x^{\alpha-1}$ for some $\alpha > 0$

Theorem 11 (FBM case)

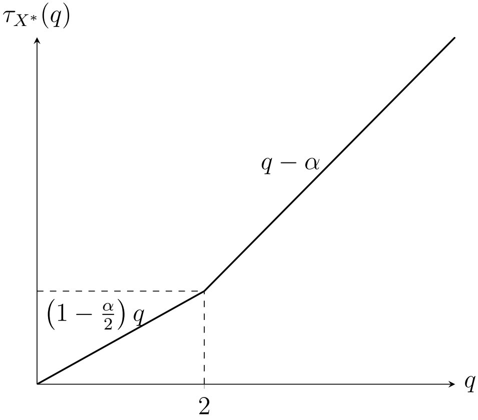
If $\alpha \in (0,1)$, $b \neq 0$ and $\mu \not\equiv 0$, then

$$\tau_{X^*}(q) = \begin{cases} \left(1 - \frac{\alpha}{2}\right) q, & 0 < q \leq 2, \\ q - \alpha, & q \geq 2. \end{cases}$$

In particular,

$$q \mapsto \frac{\tau_{X^*}(q)}{q},$$

is strictly increasing on $[2, \infty)$, hence X^* is intermittent.
If $\mu \equiv 0$, then X^* is Gaussian and $\tau_{X^*}(q) = \left(1 - \frac{\alpha}{2}\right) q$ for every $q > 0$.


$\tau_{X^*}(q)$
$q-\alpha$
$\left(1-\frac{\alpha}{2}\right)q$
q
2

- For FBM $\mathbb{E}|B_H(T)|^q = \mathbb{E}|X(1)|^q T^{Hq}$, in particular

$$\mathbb{E}\left(B_{1-\alpha/2}(T)\right)^2 = \mathbb{E}|X(1)|^2 T^{2-\alpha}$$
$$\mathbb{E}\left(B_{1-\alpha/2}(T)\right)^4 = \mathbb{E}|X(1)|^4 T^{4-2\alpha}$$

- But for the non-Gaussian supOU

$$\mathbb{E}\left(X^*(T)\right)^2 \sim \widetilde{\ell}_2(T) T^{2-\alpha}$$
$$\mathbb{E}\left(X^*(T)\right)^4 \sim \widetilde{\ell}_4(T) T^{4-\alpha}$$

for some slowly varying $\widetilde{\ell}_2$, $\widetilde{\ell}_4$

What happens?

- Decompose $X^* = X_1^* + X_2^*$ into
  - X_1^* corresponding to Gaussian part of the underlying Lévy basis
  - X_2^* corresponding to pure jump part of the underlying Lévy basis
- With normalization $A_T = T^{1-\alpha/2}\ell(T)^{1/2}$

$$A_T^{-1} X_1^*(Tt) \xrightarrow{d} \widetilde{\sigma} B_{1-\alpha/2}(t),$$

  and

$$\mathbb{E}\left|A_T^{-1} X_1^*(Tt)\right|^q \to \mathbb{E}\left|\widetilde{\sigma} B_{1-\alpha/2}(t)\right|^q, \quad \forall q > 0.$$

- On the other hand

$$A_T^{-1} X_2^*(Tt) \xrightarrow{P} 0,$$

  but

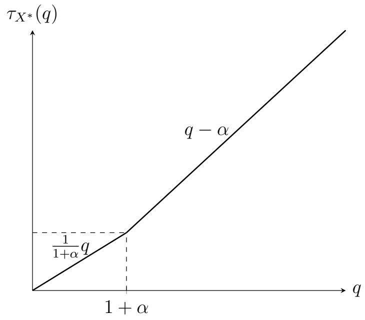
$$\mathbb{E}\left|A_T^{-1} X_2^*(Tt)\right|^q \to \infty, \quad \forall q > 2.$$

Other scenarios:

Theorem 12 (Stable Lévy process case)

If $\alpha \in (0,1)$, $b = 0$ *and* $\beta_{BG} < 1 + \alpha$, *then*

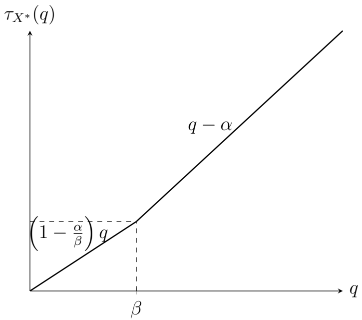
$$\tau_{X^*}(q) = \begin{cases} \frac{1}{1+\alpha} q, & 0 < q \leq 1 + \alpha, \\ q - \alpha, & q \geq 1 + \alpha. \end{cases}$$



Theorem 13 ($Z_{\alpha,\beta}$ case)

If $\alpha \in (0,1)$, $b = 0$ *and* (3) *holds with* $1 + \alpha < \beta < 2$, *then*

$$\tau_{X^*}(q) = \begin{cases} \left(1 - \frac{\alpha}{\beta}\right) q, & 0 < q \leq \beta, \\ q - \alpha, & q \geq \beta. \end{cases}$$



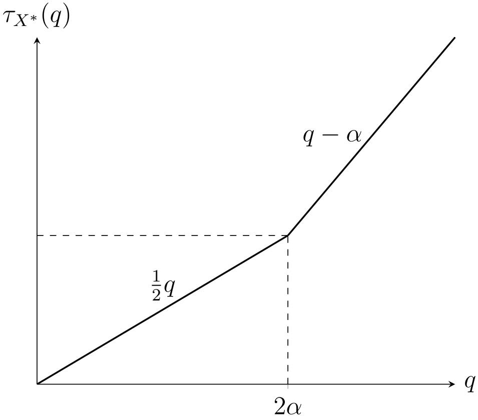
- In the short-range dependence setting, intermittency may not occur
- Example. For finite superpositions uniform integrability may be shown for any $q > 0$ so that $\tau_{X^*}(q) = q/2$ for any $q > 0$.
- However, when π is regularly varying at zero, intermittency is present

Theorem 14 (BM case)

Suppose that π satisfies (1) with integer $\alpha > 1$ and some slowly varying function L. If $\mu \not\equiv 0$, then

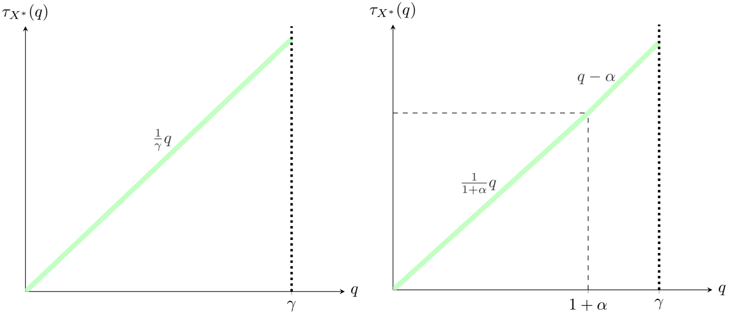
$$\tau_{X^*}(q) = \begin{cases} \frac{1}{2}q, & 0 < q \leq 2\alpha, \\ q - \alpha, & q \geq 2\alpha. \end{cases}$$

If $\mu \equiv 0$, then X^* is Gaussian and $\tau_{X^*}(q) = \frac{1}{2}q$ for every $q > 0$.


$\tau_{X^*}(q)$
$q - \alpha$
$\frac{1}{2}q$
q
2α

Infinite variance case

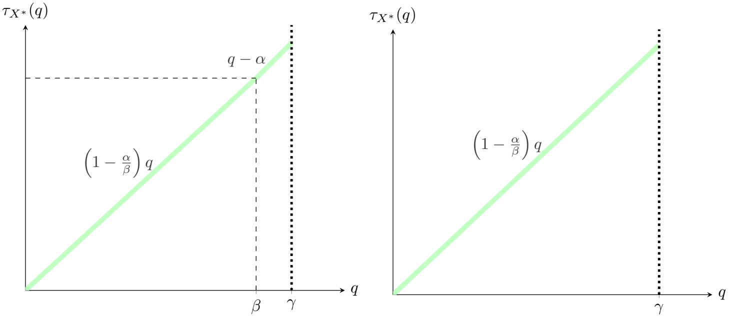
- The range of finite moments is limited – intermittency does not appear in every case



(a) $\alpha > 1$ or $\alpha \in (0,1)$ and $\gamma < 1+\alpha$

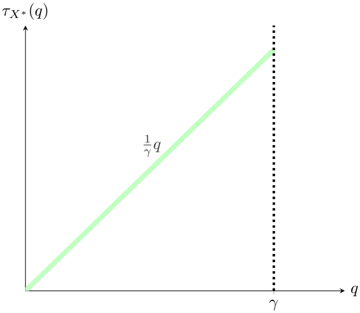
(b) $\beta < 1+\alpha < \gamma$

Figure: Scaling functions – $b = 0$



(a) $1+\alpha<\beta\leq\gamma$ (b) $1+\alpha<\gamma<\beta$

Figure: Scaling functions – $b=0$



(a) $\alpha > 1$ or $\alpha \in (0,1)$ and $\gamma < \frac{2}{2-\alpha}$

(b) $\alpha \in (0,1)$ and $\gamma > \frac{2}{2-\alpha}$

Figure: Scaling functions – $b \neq 0$

Intermittency

- What makes a process intermittent?
- We will focus on the finite variance FBM case ($\alpha \in (0,1)$, $b \neq 0$ and $\mu \neq 0$), when

$$\tau_{X^*}(q) = \begin{cases} \left(1 - \frac{\alpha}{2}\right) q, & 0 < q \leq 2, \\ q - \alpha, & q \geq 2. \end{cases}$$

It is instructive to consider simple example replicating this scaling function:

- Suppose $\{Y_T,\ T \geq 1\}$ is a sequence of random variables such that

$$Y_T = \begin{cases} T^{1-\alpha/2}, & \text{with probability } 1 - T^{-\alpha} \\ T, & \text{with probability } T^{-\alpha} \end{cases}$$

- Then

$$\mathbb{E} Y_T^q = T^{(1-\alpha/2)q}(1 - T^{-\alpha}) + T^{q-\alpha} \sim \begin{cases} T^{(1-\alpha/2)q}, & q \leq 2 \\ T^{q-\alpha}, & q > 2 \end{cases}$$

- With suitable normalization we have

$$T^{-1+\alpha/2} Y_T = \begin{cases} 1, & \text{with probability } 1 - T^{-\alpha} \\ T^{\alpha/2}, & \text{with probability } T^{-\alpha} \end{cases}$$

  hence $T^{-1+\alpha/2} Y_T \xrightarrow{d} 1$.
- However, $T^{-1+\alpha/2} Y_T$ exhibits increasingly large values albeit with decreasing probability.
- $\{Y_T\}$ has two rates of growth (scales) – multiscaling, multifractal, separation of scales

- How to show this for integrated supOU?
- How is this behavior related to moments?
- Large deviations theory (Gärtner-Ellis theorem): for the sequence $\{Z_T,\ T \in \mathbb{N}\}$

$$\limsup_{T\to\infty} \frac{1}{a_T} \log P(Z_T \in C) \leq -\inf_{x \in C} \Lambda^*(x), \quad \text{for closed set } C,$$

$$\liminf_{T\to\infty} \frac{1}{a_T} \log P(Z_T \in O) \geq -\inf_{x \in O \cap E} \Lambda^*(x), \quad \text{for open set } O,$$

where Λ^* is the Legendre-Fenchel transform

$$\Lambda^*(x) = \sup_{q \in \mathbb{R}} \{qx - \Lambda(q)\}$$

of Λ

$$\Lambda(q) = \lim_{T\to\infty} \frac{1}{a_T} \log \mathbb{E}\left[e^{q a_T Z_T}\right]$$

(E is the set of exposed points of Λ^* – points $x \in \mathbb{R}$ such that for some $\lambda \in \mathbb{R}$ and all $y \neq x$, $\Lambda^*(y) - \Lambda^*(x) > \lambda(y - x)$)

Large deviations for the rate of growth

- If we denote the rate of growth of $\{X^*(t)\}$

$$Z_T = \frac{\log |X^*(T)|}{\log T}, \quad T \in \mathbb{N},$$

and put $a_T = \log T$, then

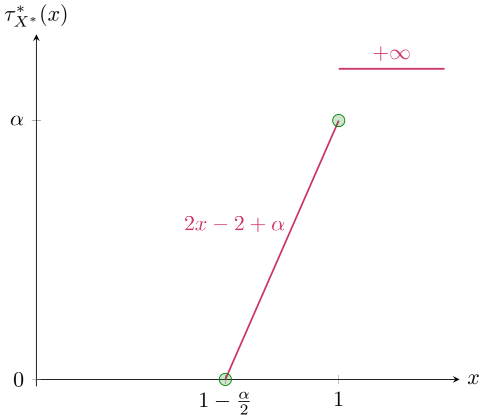
$$\Lambda(q) = \lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{E}\left[e^{q \log |X^*(T)|}\right] = \lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{E}|X^*(T)|^q = \tau_{X^*}(q).$$

- The scaling function plays the role of the function Λ in the Gärtner-Ellis theorem for the rate of growth

- For describing probabilities

$$P\left(a-\varepsilon \leq \frac{\log |X^*(T)|}{\log T} \leq a+\varepsilon\right) = P\left(T^{a-\varepsilon} \leq |X^*(T)| \leq T^{a+\varepsilon}\right)$$

Legendre transform $\tau^*_{X^*}$ of τ_{X^*} is important



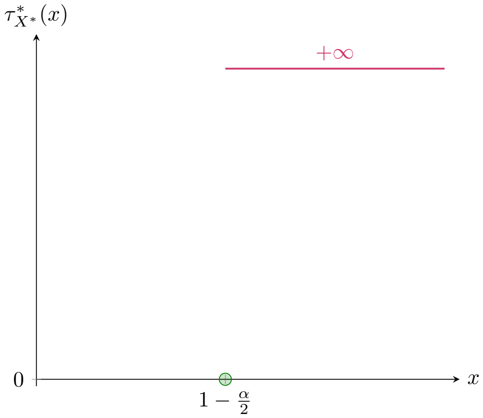
We can show for X^* that for any $\varepsilon > 0$:

- $$P\left(T^{1-\frac{\alpha}{2}-\varepsilon} \leq |X^*(T)| \leq T^{1-\frac{\alpha}{2}+\varepsilon}\right) \to 1, \quad \text{as } T \to \infty.$$
- $$\begin{aligned} -\alpha &\leq \liminf_{T\to\infty} \frac{1}{\log T} \log P\left(T^{1-\varepsilon} < |X^*(T)| < T^{1+\varepsilon}\right) \\ &\leq \limsup_{T\to\infty} \frac{1}{\log T} \log P\left(T^{1-\varepsilon} < |X^*(T)| < T^{1+\varepsilon}\right) \leq -(\alpha - 2\varepsilon), \end{aligned}$$

- In contrast, for the Gaussian supOU $\tau_{X^*}(q) = \left(1 - \frac{\alpha}{2}\right) q$ for every $q > 0$ and

$$P\left(|X^*(T)| > T^{1-\frac{\alpha}{2}+\varepsilon}\right)$$

goes to zero faster than any negative power of T – monoscale



References I

Barndorff-Nielsen, O. E. (2001), 'Superposition of Ornstein–Uhlenbeck type processes', *Theory of Probability & Its Applications* **45**(2), 175–194.

Grahovac, D., Leonenko, N. N., Sikorskii, A. & Taqqu, M. S. (2019), 'The unusual properties of aggregated superpositions of Ornstein-Uhlenbeck type processes', *Bernoulli* **25**(3), 2029–2050.

Grahovac, D., Leonenko, N. N. & Taqqu, M. S. (2019*a*), 'Limit theorems, scaling of moments and intermittency for integrated finite variance supOU processes', *Stochastic Processes and their Applications* . In press.

Grahovac, D., Leonenko, N. N. & Taqqu, M. S. (2019*b*), 'The multifaceted behavior of integrated supOU processes: The infinite variance case', *Journal of Theoretical Probability* . accepted.

Puplinskaitė, D. & Surgailis, D. (2010), 'Aggregation of a random-coefficient AR(1) process with infinite variance and idiosyncratic innovations', *Advances in Applied Probability* **42**(02), 509–527.

Thank you!