Hilbert's 16. problem

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Hilbert's Problem

Does there exist (minimal) natural numbers H_n, for $n = 2, 3, \ldots$, such that any planar autonomous system of ODE's

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P_n(x, y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = Q_n(x, y)$$

has at most H_n limit cycles. Here P_n and Q_n are polynomials in two variables of degree at most n.

The numbers H_n, if they exist, are called *Hilbert Numbers*.

Geometric Meaning I: The Equation

Let $\gamma\colon I \to \mathbb{R}^2$ be a differentiable plane curve, defined on an open interval I. Write x and y for the coordinate functions of $\gamma = (x, y)$. The left-hand side of the equation, evaluated at $t \in I$, is the velocity with which the curve passes through the point $(x(t), y(t))$.

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t}(t) \\ \frac{\mathrm{d}y}{\mathrm{d}t}(t) \end{pmatrix}.$$

The right-hand side of the equation is a prescribed velocity vector field

$$F(x, y) = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}$$

defined in the entire plane. Hilberts's problem is concerned with the special case where P and Q are polynomial expressions in x and y.

Geometric Meaning II: Solutions

A (differentiable) curve $\gamma\colon I \to \mathbb{R}^2$ is a *solution* of the equation

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = F(x, y)$$

if the curves moves through the velocity vector field, in a manner such that the velocity of the curve matches that of the prescribed vector field at any time $t \in I$.

We furthermore demand of solutions that they be inextendible.

Two Special Types of Solutions

1) The simplest possible solutions occur when the vector field $F = (P, Q)$ vanishes at some point (a, b) in the plane. Then $\gamma(t) = (a, b)$, for all $t \in I = \mathbb{R}$, is a solution; a so-called *equilibrium solution*. The point (a, b) is called an *equilibrium*.

2) Zeroing in on limit cycles, we emphasize solutions γ that return to points already visited. Such solutions are called *periodic*. They are always defined on $I = \mathbb{R}$ and satisfy: $\gamma(t + T) = \gamma(t)$, for all $t \in \mathbb{R}$, where $T > 0$ is a period of the solution.

Note: A theorem of Bendixon ensures that periodic solutions always encircle at least one equilibrium.

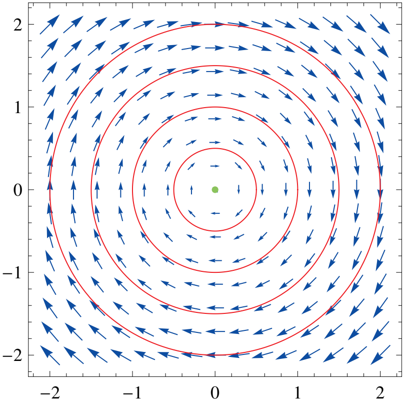
The Harmonic Oscillator

A ball in a (one-dimensional) parabolic bowl, or a mass on a spring

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x.$$

Equilibrium at $(0, 0)$ and otherwise only periodic solutions

$$\gamma(t) = R \begin{pmatrix} \cos(t+\phi) \\ \sin(t+\phi) \end{pmatrix}.$$



A *limit cycle* is first and foremost a periodic solution $\gamma\colon \mathbb{R} \to \mathbb{R}^2$.

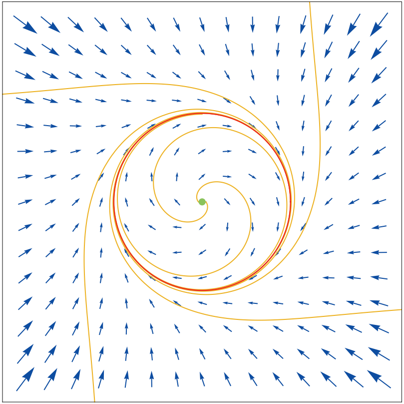
Solutions passing through the immediate vicinity of γ^* must be sucked towards γ^*, either towards the future or back towards the past.

A Deformed Harmonic Oscillator

Consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - (x^2 + y^2 - 1)x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x - (x^2 + y^2 - 1)y.$$

We retain an equilibrium at $(0, 0)$ and a single periodic solution. The remaining solutions seek towards the periodic solution (or the equilibrium). Hence, we have a limit cycle.



Taking the previous example further, we may put

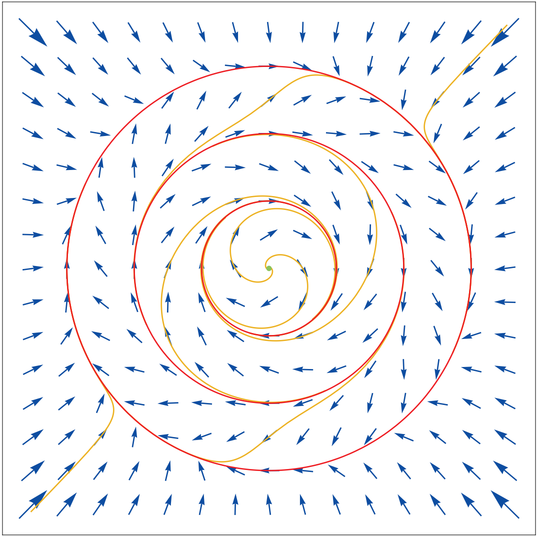
$$p_n(r) = (r-1)(r-2^2)\cdots(r-n^2)$$

and consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - p_n(x^2+y^2)x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x - p_n(x^2+y^2)y.$$

As in the previous example, we get n limit cycles at radii $1, 2, \ldots, n$, using a polynomial vector field of degree $2n+1$. Hence, $H_{2n+1} \geq n$. (provided H_{2n+1} exists).

Case with $n = 3$.



Why Polynomial Vector Fields?

Taking the previous example to its extreme, we may consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - \sin(x^2 + y^2)x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x - \sin(x^2 + y^2)y.$$

This equation has countably many limit cycles, situated at radii $1, 2^2, 3^2, \dots$.

The vector field considered here is an analytic function (of two variables), and as such falls just outside the polynomial class. Here one looses finiteness of the number of limit cycles.

In 1923, Henri Dulac published a 143 page paper with a proof of what is now called Dulac's Theorem. The theorem states that the number of limit cycles for any given planar equation system with a polynomial vector field is finite.

The proof stood for 59 years until Yulij Ilyashenko in 1982 realized that Dulac's proof was flawed.

Independent (and so far correct) proofs, were published a decade later by Ilyashenko in 1991 and by Jean Écalle in 1992. Both proofs were published in book form and are 150+ pages long.

Local Finiteness I: Poincaré's Argument

Suppose γ is a periodic solution with period T, $\gamma(0) = (1, 0)$ and $F(1, 0) = (0, -1)$ (after a change of axes).

There exist small open intervals $J \subseteq J'$ containing $x = 1$ with the property that solutions starting at $x_0 \in J$ returns to J', for the first time, at a time T', which is close to the period T. The map

$\Gamma(x_0) = x(T')$ is called Poincaré's Return Map. Periodic solutions correspond to fixed points of Γ. It turns out that $x_0 \to T'$ as well as $t \to x(t)$ are (real) analytic functions, hence; so is $x_0 \to \Gamma(x_0)$. Consequently, either γ sits in a band of periodic solutions or it is isolated amongst periodic solutions.

This in particular shows that limit cycles cannot accumulate on periodic solutions.

Lyapunov's First Method

Suppose $e = (0,0)$ is an isolated equilibrium for F.

Suppose furthermore, that we can construct a smooth positive definite (e.g. convex) function V in a neighborhood Ω of e, such that the vector field F is transversal to the level curves of V:

$$\forall (x,y) \in \Omega \backslash \{e\} : \qquad \nabla V(x,y) \cdot F(x,y) \neq 0.$$

If the sign is negative, then the vector field points inside the level sets, and solution curves are forced into the equilibrium. This is called *Asymptotic Stability*.

Conversely, If the sign is positive, then solution curves are repulsed by the equilibrium. This is an example of *Instability*.

Suppose one can choose V with concentric level curves encircling the equilibrium $e = (0,0)$, such that

$$\forall (x,y) \in \Omega : \qquad \nabla V(x,y) \cdot F(x,y) = 0.$$

The vector field is tangent to the level curves of V, and the level curves themselves become periodic solution curves. That is, in a neighborhood of e, all solutions are periodic solutions encircling the equilibrium. Such an equilibrium is called a *center*.

The prototypical examples of centers come from Hamilton's equations (in one dimension) with a potential energy convex near an equilibrium position. Like the harmonic oscillator, where the Hamiltonian is $V(x,y) = y^2/2 + x^2/2$.

It turns out that one may construct a function V (inductively) as a power series

$$V(x,y) = \sum_{i+j\geq 2} v_{ij} x^i y^j,$$

such that

$$\nabla V(x,y) \cdot F(x,y) = \sum_{\ell=1}^{\infty} \eta_\ell (x^2+y^2)^\ell.$$

The η_k's are called *Lyapunov Focus Quantities*. If they are all 0, we have a center.

Otherwise, by Lyapunov's First Method, the first non-zero focus quantity determines the stability properties of the equilibrium. In either case, limit cycles cannot accumulate at an equilibrium!

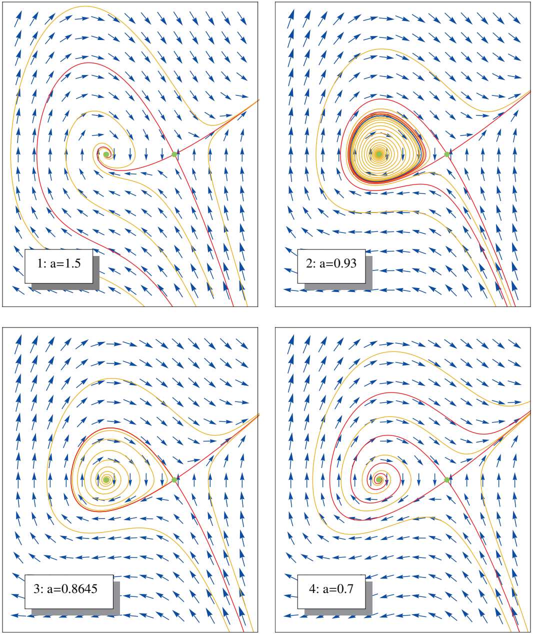
Local Finiteness III: What Remains?

The hard part of Dulac's Theorem, is to show that limit cycles cannot accumulate on more exotic structures either.

The simplest such non-trivial structure is that of a *separatrix cycle*, consisting of one equilibrium point and one loop.

Proving that limit cycles cannot accumulate on such structures is a subtle problem.

Hopf Bifurcation



Compactification

Assume P, Q are polynomials of degree n. We transform first the ODE into a new ODE by radial contraction of the plane onto the unit disc, using the map $r \to s = r/(1+r)$.

Secondly, we reparametrize the solutions, implemented by multiplying the vector field by the positive function $(1-s)^{n-1}$. This leaves the image curves intact.

The resulting vector field extends analytically across the unit circle $s = 1$ to a new analytic vector field in the plane, which is tangent to $s = 1$, possibly with (hyperbolic) equilibria on the unit circle $s = 1$.

Global Finiteness

The local finiteness arguments presented earlier only used analyticity of the vector field.

Applying those results to the compactified problem, one may argue that there are only finitely many limit cycles in the unit disc for the compactified ODE.

Since the compactified limit cycles in the unit disc are in one-one correspondence with limit cycles for the original problem, we can conclude Dulac's Theorem.

When one wants to construct limit cycles by hand, the Poincaré–Bendixon Theorem is very useful.

It classifies the possible limiting sets of accumulation points of solutions of planar ODE's.

The limiting set is a union of equilibria, joined by solution arcs. If there are no equilibria, the limit set must be a limit cycle. (Unless we start with a periodic solution).

Engineering Small Limit Cycles I

One popular approach to Hilbert's problem is to analyze the number of limit cycles that can bifurcate out of a given structure when one perturbs the vector field, while keeping the polynomial degree fixed.

Suppose F is the vector field of a deformed Harmonic Oscillator

$$F(x,y) = \begin{pmatrix} y + P(x,y) \\ -x + Q(x,y) \end{pmatrix},$$

where P, Q are polynomials of degree at most n with no terms of orders 0 and 1.

The associated Focus Quantities $\eta_1, \eta_2, \ldots$ are polynomial expressions in the coefficients of P and Q.

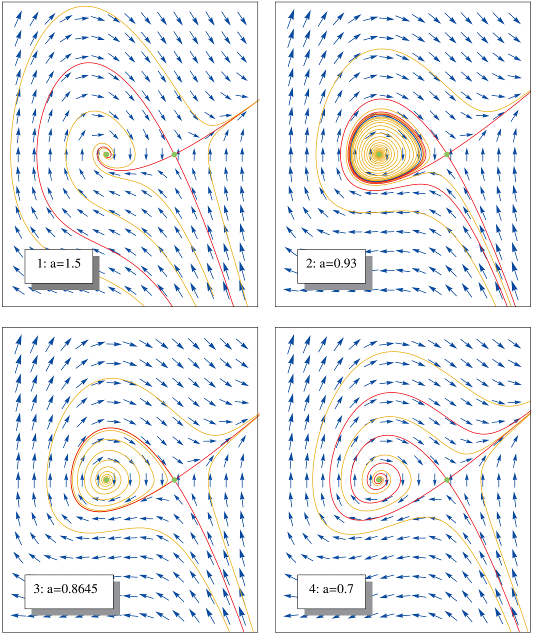
Engineering Small Limit Cycles II

It is a consequence of *Hilbert's Basis Theorem*, that there exists a number B_n, depending on the maximal degree n of P, Q, such that either all focus quantities are zero, or there is an $\ell \leq B_n$ such that $\eta_\ell \neq 0$.

It is a theorem of Bautin (1952) that $B_2 = 3$. For $n > 2$, the number B_n is unknown, but $B_3 \geq 11$ was established by Zoladek (1995).

Starting from a vector field of degree n with $\eta_1 = \ldots = \eta_{B_n - 1} = 0$ and $\eta_{B_n} \neq 0$, one may – by carefully perturbing (tweaking) the vector field, while preserving its order – turn on the focus quantities $\eta_{B_n - 1}, \cdots, \eta_1$ one by one in such a way that they have alternating signs and that $F(x, y) \cdot \nabla V(x, y)$ changes sign B_n times. The resulting ODE will now have B_n small limit cycles encircling $(0, 0)$.

Hopf Bifurcation Redux



Landis and Petrovskii

Spurred on by Bautin's result, Landis and Petrovskii set out to prove that $H_2 = 3$, that is; Bautin's 3 small limit cycles is the best you can do for a quadratic vector field. In 1955 they published a paper containing exactly this result.

Two years later they topped themselves with a new paper establishing the existence of all Hilbert numbers!!

In the beginning of the 1960'ies, Yulij Ilyashenko and Sergei Novikov found a mistake in the proofs of Landis and Petrovskii. Resulting in 1967 in a letter to the editor of the journal carrying their two papers, that the flaw was not fixable and asking for the papers to be retracted.

Shi Songlin's Equation

In 1982 Shi Songlin published a paper where he studied the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y - 10x^2 + 5xy + y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + x^2 + 25xy.$$

There are equilibria at $(0,0)$ and at $(0,1)$. The equilibrium at $(0,0)$ is of Bautin's type permitting the engineering of three limit cycles by perturbation.
The equilibrium at $(0,1)$ is unstable (preserved by perturbations).

The circle at infinity has a hyperbolic equilibrium, and there is a large limit cycle encircling the unstable equilibrium. See drawing.

$H_2 \geq 4$

This implies that

$$H_2 \geq 4.$$

In particular, one may conclude that Landis and Petrovskii gave not just a wrong proof, but a wrong proof of a wrong theorem.

Hilbert's problem remains wide open, but progress has been made for a restriction to a smaller class of so-called Liénard Equations originally proposed by Stephen Smale in 1998.