Riemann Surfaces: Vector Bundles, Physics, and Dynamics

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To my mother, Rehana Sattar Sikander.

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CHAPTER 1

Introduction

The author has attempted to apply classical results from Teichmüller theory to the two dimensional part of the Witten-Reshetikhin-Turaev Topological Quantum Field Theory. It should be mentioned that this TQFT has been used very effectively to solve problems in Teichmüller theory, for example, the asymptotic faithfulness of the quantum representations of the mapping class groups [**And06**], the solution of Kazhdan's Property(T) conjecture for the mapping class groups [**And**], and that all finite groups are involved in the mapping class groups [**MR12**]. Here, we propose studying quantum representations of the (orbifold) fundamental groups of Teichmüller curves which will give quantum representations of pseudo-Anosov elements. We also propose studying the cocycle, and its Lyapunov spectrum, given by the parallel transport of the Hitchin connection along the geodesic flow in the unit tangent bundle of Teichmüller curves. We carry out this plan in the case of one of the oldest and most popular Teichmüller curve, namely, the one generated by the flat surface obtained from gluing two copies of a regular pentagon together. This curve was discovered by Veech in [**Vee89**] and has been extensively studied by Lochak in [**Loc**] and by McMullen in [**McM06**]. In theorem 0.2 below we give an explicit expression for the quantum representation of the orbifold fundamental group of this Teichmüller curve obtained via monodromy of the Hitchin connection. This expression involves certain iterated integrals called Hyperlogarithms which first appeared in [**Poi84**]. We then construct a cocycle, using the parallel transport of the Hitchin connection, over the geodesic flow in the unit tangent bundle of this Teichmüller curve, and relate this cocycle (in level one) to the Kontsevich-Zorich cocycle considered in [**KZ97**] and [**BM10**].

Our constructions rely on the geometric point of view of the W-R-T TQFT. This TQFT was first proposed by Witten in [**Wit89**], where he derived it from quantization, via path integral techniques, of the 2+1 dimensional Chern-Simons theory. Shortly afterwards, Reshetikhin and Turaev gave a combinatorial construction of this theory using the representation theory of the quantum group at a fixed root of unity in [**RT90**], and [**RT91**], see also [**Tur94**]. Another construction, using skein theory, was given by Blanchet, Habegger, Masbaum, and Vogel in [**BHMV95**].

From the geometric point of view, this theory relies on the existence of a Γ_g invariant vector bundle $\mathbb{V}^{(k)} \to \mathcal{T}_g$ with a Γ_g invariant projectively flat connection $\boldsymbol{\nabla}^{(k)}$ for every integer $k > 0$. Here $\mathcal{T}_g$ denotes the Teichmüller space of genus $g \geqslant 1$ surface and Γ_g denotes the mapping class group of this surface. We recall the definition of $\mathcal{T}_g$ and Γ_g in chapter four. The construction of this vector bundle and the projectively flat connection was carried out by several authors, including, [**ADPW91**], [**Fal93**], [**Hit90**], see also [**And12**], [**AU07b**], [**AU07a**], [**AU12b**], and [**AU12a**]. This connection is now dubbed the Hitchin connection

We briefly recall the construction of $\mathbb{V}^{(k)}$ and $\boldsymbol{\nabla}^{(k)}$. Let X be a compact Riemann surface of genus $g \geqslant 1$. Let $\mathcal{M}_X$ be the moduli space of semi-stable holomorphic vector bundles of rank two with trivial determinant on X. We discuss these moduli spaces, for the general situation of vector bundles of any rank and degree, and their constructions in detail in chapter two. In [**NS65**], it is shown that $\mathcal{M}_X$ is a projective variety, non-smooth in general, of dimension $3g-3$. By a result of [**DN69**] it is known that $\operatorname{Pic}(\mathcal{M}_X) \cong \mathbb{Z}$. Let $\mathcal{L}$ be the ample generator of this Picard group. Then, for all $X \in \mathcal{T}_g$, the vector spaces $H^0(\mathcal{M}_X, \mathcal{L}_X^k)$ glue together to give the vector bundle $\mathbb{V}^{(k)} \to \mathcal{T}_g$. The dimension of $H^0(\mathcal{M}_X, \mathcal{L}_X^k)$ is given by the following formula

$$\dim H^0(\mathcal{M}_X, \mathcal{L}_X^k) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \big(\sin^2 \frac{j\pi}{k+2}\big)^{1-g},$$

which is called the *Verlinde* formula.

The action of Γ_g lifts to $\mathbb{V}^{(k)}$ and we get a quotient vector bundle (in the orbifold sense) $\overline{\mathbb{V}}^{(k)} \to \mathcal{M}_g$, where $\mathcal{M}_g \cong \mathcal{T}_g/\Gamma_g$. Moreover, the connection $\boldsymbol{\nabla}^{(k)}$ is invariant under the action of Γ_g, and descends to a projectively flat connection $\overline{\boldsymbol{\nabla}}^{(k)}$ in the bundle $\overline{\mathbb{V}}^{(k)}$. The monodromy of this connection provides us with a group homomorphism

$$\varphi_k \colon \pi_1^{orb}(\mathcal{M}_g, X) \to \operatorname{End} \mathbb{P}(\overline{V}_X^{(k)}).$$

Here, $X \in \mathcal{M}_g$ is a base point and $\overline{V}_X^{(k)}$ is the fiber of $\overline{\mathbb{V}}^{(k)}$ over X. By definition, we have that $\Gamma_g = \pi_1^{orb}(\mathcal{M}_g, X)$. We call this homomorphism the *quantum* representation of the mapping class group at level k.

Following [**vGdJ98**], we give details on the construction of this vector bundle and the Hitchin connection for $g = 2$ in chapter 2 section 4. In this case, the connection and the vector bundle are defined over

$$\mathcal{C} := \{(z_1, \dots, z_6) \in \mathbb{C}^6 | z_i \neq z_j \,\text{if}\, i \neq j\},$$

which is the space of ordered configurations of six points on $\mathbb{C}$. The corresponding family of smooth genus two curves is defined by the polynomials

$$y^2 = \prod_{i=1}^{6}(x - z_i)$$

where $(z_1, \ldots, z_6) \in \mathcal{C}$.

Given a smooth genus two curve X, [**NR69**] showed that

$$\mathcal{M}_X \cong |2\Theta_X| := \mathbb{P}H^0(J_X^1, \Theta^2).$$

Here $J_X^1 = \mathrm{Pic}^1(X)$, and $\Theta := \{L \in \mathrm{Pic}^1(X) | h^0(X, L) > 0\}$ is the canonical theta divisor. We discuss this isomorphism in detail in chapter 2 section 4.1. It is well known that the dimension of $H^0(J_X^1, \Theta^2)$ is 2^g for a genus g curve, which implies

$$\mathcal{M}_X \cong \mathbb{CP}^3.$$

Moreover, the generator of the Picard group of $\mathbb{CP}^3$ is $\mathcal{O}_{\mathbb{CP}^3}(1)$, which gives $\mathcal{L}^k \cong \mathcal{O}_{\mathbb{CP}^3}(k)$. This implies that $H^0(\mathcal{M}_X, \mathcal{L}^k) \cong H^0(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(k))$. Let V be the space of homogenous polynomials in four variables of degree one. It is well known that $H^0(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(k))$ is isomorphic to the space of homogenous polynomials in four variables of degree k, thus $H^0(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(k)) \cong S^k(V)$ where S^k denotes the k^{th} symmetric power of a vector space. By gluing the vector spaces $H^0(\mathcal{M}_X, \mathcal{L}^k)$ we get a vector bundle $\mathcal{V}^{(k)} \to \mathcal{C}$ such that the fiber of this bundle is isomorphic to $S^k(V)$.

Let $\overline{\mathcal{C}}$ be the quotient of $\mathcal{C}$ by the action of the symmetric group S_6. In [**vGdJ98**] it is shown that there exists a diagram of smooth covering spaces

$$\widetilde{\mathcal{C}} \xrightarrow{\widetilde{P}} \mathcal{C} \xrightarrow{\mathrm{P}} \overline{\mathcal{C}} \tag{1.1}$$

with the corresponding short exact sequence of deck groups

$$(\mathbb{Z}/2\mathbb{Z})^4 \to A(G) \to S_6.$$

Here $A(G)$ is a subgroup of the group of automorphisms of the (finite) Heisenberg group. This group $A(G)$ can be identified with a quotient group of $\mathrm{Sp}(4, \mathbb{Z})$, i.e. $A(G) \cong \mathrm{Sp}(4, \mathbb{Z})/\Gamma_2(2, 4)$, where

$$\Gamma_2(2,4) := \left\{ \begin{pmatrix} I + 2A & 2B \\ 2C & I + 2D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}) | \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv (0, 0) \pmod 2 \right\}.$$

In [**vGdJ98**] it is shown that $\widetilde{P}^*(\mathcal{V}^{(k)}) \cong (S^k(V) \otimes \mathcal{N}) \times \widetilde{\mathcal{C}}$, where $\mathcal{N} \to \widetilde{\mathcal{C}}$ is a line bundle. In the trivial bundle $S^{(k)}(V) \times \widetilde{\mathcal{C}}$, [**vGdJ98**], see also [**GTNB00**], give an explicit expression for a flat connection which is as follows

$$\widetilde{\boldsymbol{\nabla}}^{(k)} = d + \widetilde{P}^*(\omega^{(k)})$$

where $\omega^{(k)}$ is following $\mathrm{End}(S^k(V))$ valued holomorphic 1-form on $\mathcal{C}$

$$\omega^{(k)} = \hbar \sum_{1 \leqslant i < j \leqslant 6} \widehat{\Omega}^{i,j} \frac{dz_i - dz_j}{z_i - z_j}$$

where

$$\hbar := \frac{-1}{16(k+2)}$$

and $\widehat{\Omega}^{i,j} \in \mathrm{End}(S^k(V))$, defined in (2.19), are symbols of explicit differential operators which were calculated in [**GTNB00**].

The authors in [**vGdJ98**] write down a symplectic basis for V and construct a projective (symplectic) representation of the deck group $A(G)$ on V. This representation has a natural extension to $\mathbb{P}S^k(V)$ for all $k > 0$. We can thus consider the following projective quotient bundle on $\overline{\mathcal{C}}$

$$\mathbb{P}S^k(V) \times_{A(G)} \widetilde{\mathcal{C}} \tag{1.2}$$

where $A(G)$ acts on the first factor by the projective representation and on the second factor by deck transformations. The connection $\widetilde{\boldsymbol{\nabla}}^{(k)}$ is projectively invariant under the action of $A(G)$, see [**vGdJ98**], and descends to a well defined flat connection in (1.2) which we denote by $\overline{\boldsymbol{\nabla}}^{(k)}$.

Remark 0.1. The pull back $\widetilde{P}^*(\mathcal{V}^{(k)})$ is actually isomorphic to $S^k(V) \otimes \mathcal{N}$ where $\mathcal{N} \to \widetilde{\mathcal{C}}$ is a line bundle. But this line bundle, as remarked in [**vGdJ98**], does not interfere with the construction of the connection.

Let C be a finite area hyperbolic surface equipped with a holomorphic embedding $f \colon C \to \mathcal{M}_g$, with $g \geqslant 1$. The pair (C, f) is called a Teichmüller curve if the embedding is isometric with respect to the hyperbolic metric on C and Teichmüller metric on $\mathcal{M}_g$. It is well known that $\mathcal{M}_g$, for $g \geqslant 2$, contains infinitely many Teichmüller curves, see [**McM09**].

In chapter 3 we explain the concept of an *affine* structure on a topological surface, and discuss the action of $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ on the unit cotangent bundle of the Teichmüller space. Let (X, q) be in the unit cotangent bundle, i.e. $X \in \mathcal{T}_g$ and $q \in H^0(X, K_X^2)$ is a holomorphic quadratic differential on X of norm one. We explain and give definition of the *Veech* group $V(X, q)$ associated with the pair (X, q), and show that $V(X, q)$ can always be identified with a discrete subgroup of $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$. We then discuss why the condition of $V(X, q)$ being a lattice always leads to a Teichmüller curve $\mathbb{H}/V(X, q) \to \mathcal{M}_g$ and how elements of $V(X, q)$ are elements of the mapping class group of X.

We then study a particular Teichmüller curve $\phi\colon \chi \to \mathcal{M}_2$. This curve was discovered by Veech [**Vee89**] who showed that $\chi \cong \mathbb{H}/\triangle(2,5,\infty)$ where

$$\triangle(2,5,\infty) :=< S, T | S^2 = (S \circ T)^5 = \mathrm{Id} >$$

and

$$T = \begin{pmatrix} 1 & 2\cos\frac{\pi}{5} \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \in \mathbb{P}\mathrm{SL}(2,\mathbb{R}).$$

The group $\triangle(2,5,\infty)$ is also called the Hecke triangle group of order five. In terms of affine structures and Veech group, this can be explained as follows. Let X be the genus two Riemann surface defined by the equation $y^2 = x^5 + 1$ and let $c\frac{dx^2}{y^2}$ be a quadratic differential on it, where c is a normalizing constant such that $\|c\frac{dx^2}{y^2}\| = 1$. In [**Vee89**] it is shown that the Veech group $V(y^2 = x^5 + 1, c\frac{dx^2}{y^2}) \cong \triangle(2,5,\infty)$, which is a lattice in $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$ and thus leads to a Teichmüller curve.

In terms of mapping classes, [**Vee89**] also shows that T corresponds to a Dehn twist along two non-intersecting non-separating simple closed loops on X and $U := S \circ T$ corresponds to a diffeomorphism of order five on X. As such $T, U \in \Gamma_2$.

In [**Loc**], Pierre Lochak gave a finite covering $\pi_{\tilde{\chi}}\colon \tilde{\chi}_\infty \to \chi$, where $\tilde{\chi}_\infty := (\mathbb{CP}^1 - \mu_5)$ and μ_5 is set of fifth roots of unity, see also [**McM06**]. The deck group corresponding to this covering is the order ten Dihedral group generated by $R(t) = \zeta^2(t)$, where $\zeta = e^{\frac{i2\pi}{5}}$, and $I(t) = \frac{1}{t}$ for all $t \in \tilde{\chi}_\infty$. The points 0 and ∞ are fixed under R and interchanged by I, as such they project to an order five orbifold point in χ which we will denote by $a \in \chi$. Let $\tilde{\gamma}_0$ be the path in $\tilde{\chi}_\infty$ starting from 0 and running along the real axis to $(1-\epsilon)$. Let $\tilde{\gamma}_1$ be the semi circle starting at $(1-\epsilon)$ making a counterclockwise turn around 1 to reach the point $(1-\epsilon)^{-1}$. Let γ_0 and γ_1 be projections of $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ to χ. Then the path $\gamma := \gamma_0^{-1} \cdot \gamma_1 \cdot \gamma_0$ is closed in χ since $(1-\epsilon)$ and $(1-\epsilon)^{-1}$ are identified with each other under I.

Associated with the orbifold point $a \in \chi$, is its order five stablizer group which is isomorphic to $\langle R \rangle$. It is shown in [**Loc**] that $\langle R \rangle$ together with γ generate the orbifold fundamental group $\pi_1^{orb}(\chi, a)$, and under the isomorphism $\pi_1^{orb}(\chi, a) \cong \triangle(2,5,\infty)$, U is identified with $\langle R \rangle$, and T is identified with the closed loop γ. We will consider the point a as a constant loop in χ with the additional datum of the stabilizer group of this orbifold point.

Let $\mathcal{C}_\infty := \{(z_1, \ldots, z_6) \in (\mathbb{CP}^1)^6 | z_i \neq z_j \,\mathrm{if}\, i \neq j\}$ be the space of ordered configurations of six points on the Riemann sphere. In [**Loc**] an embedding $\tilde{\phi}\colon \tilde{\chi} \to \mathcal{C}_\infty$ is given by

$$\tilde{\phi}(t) = (1 + t, \zeta + \zeta^{-1}t, \zeta^2 + \zeta^{-2}t, \zeta^3 + \zeta^{-3}t, \zeta^4 + \zeta^{-4}t, \infty).$$

Here, $\tilde{\chi} := (\mathbb{CP}^1 - \{\mu_5 \cup \infty\})$. Let S_n be the symmetric group on n elements and let $\nu, \nu' \in S_5 \triangleleft S_6$ be defined as follows

$$\nu'(z_1, \dots, z_6) = (z_5, z_1, z_2, z_3, z_4, z_6) \text{ and } \nu(z_1, \dots, z_6) = (z_1, z_5, z_4, z_3, z_2, z_6).$$

It is clear from the definition that $\nu^2 = \nu'^5 = \mathrm{Id}$ and it can be easily checked that

$$\zeta\nu'(\phi(t)) = \phi(R(t)) \text{ and } \frac{1}{t}\nu(\phi(t)) = \phi(I(t)).$$

Let $\bar{\mathcal{C}}_\infty$ be the quotient of $\mathcal{C}_\infty$ by the symmetric group S_6 and let $\mathcal{M}_2$ be the quotient of $\bar{\mathcal{C}}_\infty$ by $\mathbb{PSL}(2,\mathbb{C})$. Likewise, we can consider $\mathcal{M}_2$ to be the quotient of $\mathcal{C}_\infty$ by the product group $(\mathbb{PSL}(2,\mathbb{C}) \times S_6)$. Since multiplication by $\frac{1}{t}$ and ζ corresponds to a dilation and a rotation in $\mathbb{PSL}(2,\mathbb{C})$ we get that the map $\tilde{\phi}$ is equivariant with respect to the D action on $\tilde{\chi}$ and $(\mathbb{PSL}(2,\mathbb{C}) \times S_6)$ action on $\mathcal{C}_\infty$. This implies that $\tilde{\phi}$ descends to give a well defined map $\phi\colon \chi \to \mathcal{M}_2$. In fact we get the following commutative diagram,

$$\begin{tikzcd}
\tilde{\chi} \arrow[r, "\tilde{\phi}"] \arrow[dd, "\pi_{\tilde{\chi}}"'] & \mathcal{C}_\infty \arrow[d, "P_\infty"] \\
& \bar{\mathcal{C}}_\infty \arrow[d, "\overline{P}_\infty"] \\
\chi \arrow[r, "\phi"'] & \mathcal{M}_2
\end{tikzcd}$$

Notice that the 1-form $\omega^{(k)}$ is not defined on the entire $\mathcal{C}_\infty$. Let

$$D_m^\infty := \{(z_1, \dots, z_6) \in \mathcal{C}_\infty | z_m = \infty\}$$

where $1 \leqslant m \leqslant 6$. Notice that D_m^∞ is a divisor in $\mathcal{C}_\infty$ and isomorphic to the space of ordered configurations of five points on $\mathbb{C}$. We have the disjoint union

$$\mathcal{C}_\infty = \mathcal{C} \sqcup_{m=1}^{6} D_m^\infty.$$

The 1-form $\omega^{(k)}$ is defined only on $\mathcal{C}$.

In this case, to construct the quantum representation of $\pi_1^{orb}(\mathcal{M}_2, X)$, we first choose a lift of any $\gamma \in \pi_1^{orb}(\mathcal{M}_2, X)$ to $\mathcal{C}_\infty$ such that this lift lies in $\mathcal{C} \subset \mathcal{C}_\infty$. This is possible, since if our chosen lift lies in D_m^∞ for any m, we can use the action of $\mathbb{PSL}(2,\mathbb{C})$ to move this lift into $\mathcal{C}$. We then further lift this to the covering $\widetilde{\mathcal{C}}$ and use the parallel transport of the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}$ in the trivial bundle $S^k(V) \times \widetilde{\mathcal{C}}$, along with the projective action of $A(G)$, to give an element in $\mathrm{End}(\mathbb{P}S^k(V))$. The result is independent of our choice of the lift, since in chapter 2 section 4.5 we show that the 1-form $\omega^{(k)}$ which defines the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}$ is projectively invariant under the action of $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$. This then provides us with the homomorphism

$$\varphi_k\colon \pi_1^{orb}(\mathcal{M}_2, X) \to \operatorname{End}\mathbb{P}S^k(V).$$

Notice that this process is a little different than the general situation outlined in the beginning.

We calculate the quantum representation of $\pi_1^{orb}(\phi(\chi), \phi(a))$. A natural lift to consider of $\phi(\gamma)$ is $\tilde{\phi}(\tilde{\gamma})$, unfortunately $\tilde{\phi}(\tilde{\gamma}) \subset D_6^{\infty}$. Thus, before lifting $\tilde{\phi}(\tilde{\gamma})$ to $\widetilde{\mathcal{C}}$ we must first move it into $\mathcal{C}$. Let $Z \in \mathbb{P}\mathrm{SL}(2,\mathbb{C})$ be the inversion in the unit circle, and let $\tilde{\chi}_0 := (\mathbb{CP}^1 - (\mu_5 \cup \infty \cup -\mu_5))$. Consider the map

$$\psi\colon \tilde{\chi}_0 \to \mathcal{C},$$

explicitly given as

$$\psi(t) = (\frac{1}{\zeta+\zeta^{-1}t}, \frac{1}{\zeta^2+\zeta^{-2}t}, \frac{1}{\zeta^3+\zeta^{-3}t}, \frac{1}{\zeta^4+\zeta^{-4}t}, \frac{1}{1+t}, 0), \quad \text{for all}\, t \in \tilde{\chi}_0.$$

This map is the composition of the Lochak map $\tilde{\phi}$ with the inversion Z followed by permuting the first and the fifth coordinate. Since we have used the composition of a Möbius transformation Z with a permutation in S_6 to move the image $\tilde{\phi}(\tilde{\chi})$ into $\mathcal{C}$, it follows that $\psi(\tilde{\chi}_0)$ projects onto $\phi(\chi)$. In particular, the loop $\psi(\tilde{\gamma})$ projects to the loop $\phi(\gamma)$.

Let $\langle x_1, \ldots, x_4\rangle$ denote the basis for V considered in [**vGdJ98**]. Let $M \in A(G)$, then M is an equivalence class of 4×4 (symplectic) matrices which act on $\mathbb{P}V$. Denote also by M a choice of element in this class, then we know the action of M on every basis vector x_i. Given a monomial $x_1^a x_2^b x_3^c x_4^d$, the matrix M acts as

$$M(x_1^a x_2^b x_3^c x_4^d) = M(x_1)^a \cdot M(x_2)^b \cdot M(x_3)^c \cdot M(x_4)^d.$$

Since a basis for $S^k(V)$ is given by monomials of the form $x_1^a x_2^b x_3^c x_4^d$ with $a+b+c+d = k$ we see how M induces an action on $\mathbb{P}S^k(V)$. We will denote this induced action of M by $M^{(k)} \in \operatorname{End}\mathbb{P}S^k(V)$. This is only a projective representation since M is an element of $A(G)$ which only has a projective action on $S^k(V)$.

The differential operators $\widehat{\Omega}^{i,j}$ act on $S^k(V)$ by the Liebniz rule. We give an expression for the following differential operator which will be needed later.

$$\widehat{\Omega}^{1,4} + \widehat{\Omega}^{2,3} =$$

$$2((x_4\partial_{x_1})^2 + (x_3\partial_{x_2})^2 + (x_2\partial_{x_3})^2 + (x_1\partial_{x_4})^2 - 2x_1x_4\partial_{x_1}\partial_{x_4} - 2x_2x_3\partial_{x_2}\partial_{x_3})$$

We can now state the main theorem.

THEOREM 0.2. *Let $k > 0$ be an integer. Let $\varphi^{(k)} \colon \pi_1^{orb}(\mathcal{M}_2, \phi(a)) \to \operatorname{End}\mathbb{P}(S^k(V))$ be the quantum representation based at the orbifold point $\phi(a)$. Then, under the isomorphism $\pi_1^{orb}(\chi, a) \cong \triangle(2,5,\infty)$, we have that*
(1) $\varphi^{(k)}(U) = M_0^{(k)}$
(2) $\varphi^{(k)}(T) = (\Phi^{(k)})^{-1} \cdot \Big(M_1^{(k)} \cdot exp\big(\hbar\pi i A_1\big)\Big).\Phi^{(k)}$,
where,

$$M_0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -2 & -1 & -3 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad M_1 = \begin{bmatrix} -2 & 1 & 0 & 2 \\ 1 & -2 & 2 & 0 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

and

$$\Phi^{(k)} = \mathrm{Id} + \sum_{r=1}^{\infty} \hbar^r \sum_{\substack{\zeta^{i_1},\ldots,\zeta^{i_r} \in (\mu_5)^r \\ \zeta^{i_r} \neq 1}} L_0(\zeta^{i_1}, \ldots, \zeta^{i_r}|1) A_{i_1} \ldots A_{i_r} \in \operatorname{End}(S^{(k)}(V))$$

where, for $1 \leqslant i \leqslant 5$, the complex number $L_0(\zeta^{i_1}, \ldots, \zeta^{i_r}|1) =$

$$\int_0^1 \frac{1}{s_{i_r} - \zeta^{i_r}} \Bigg(\int_0^{s_{i_r}} \frac{1}{s_{i_{r-1}} - \zeta^{i_{r-1}}} \cdots \Bigg(\int_0^{s_{i_2}} \frac{1}{s_{i_1} - \zeta^{i_1}} ds_{i_1} \Bigg) \ldots ds_{i_{r-1}} \Bigg) ds_{i_r}$$

is an iterated integral along the interval $[0,1]$ and $A_i = \widehat{\Omega}^{a,b} + \widehat{\Omega}^{c,d}$ such that $[a+b] = [c+d] = [i]$ $\big([x] := x(mod 5)\big)$.

The iterated integrals $L_0(\zeta^{i_1}, \ldots, \zeta^{i_r}|1)$ also appear in the study of the motivic fundamental group of $(\mathbb{C}^* - \mu_n)$ (when specialized to n=5) carried out in [**DG**], see equation (5.16.1) and proposition 5.17 in the mentioned reference. In section 5.19 of the same reference, the authors discuss "motivic" properties satisfied by these integrals. These iterated integrals also appear in the construction of the cyclotomic Drinfel'd associator carried out in [**Enr**], see section 2 and Appendix A in the mentioned reference.

Since the operators $\widehat{\Omega}^{i,j}$ are homogeneous of degree two, it follows that they act by zero on V. This implies that for $k = 1$ we get

$$\varphi^{(1)}(U) = M_0 \qquad \varphi^{(1)}(T) = M_1.$$

Lastly, we construct a cocycle in the unit tangent bundle of the Teichmüller curve using the parallel transport of the Hitchin connection. Let $U\chi$ be the unit tangent bundle of the Teichmüller curve and let μ be the canonical Louiville measure on $U\chi$. It is well known that the geodesic flow in the unit tangent bundle of any finite area hyperbolic surface is ergodic with respect to the Louiville measure, [**CFS82**], which implies the geodesic flow $T_s\colon U\chi \to U\chi$, for $s \in \mathbb{R}_+$, is ergodic with respect to μ. We have the degree ten branched covering map $\pi_{\tilde{\chi}}\colon \tilde{\chi} \to \chi$. Using this map we can pull back the unit tangent bundle $\pi_{\tilde{\chi}}^* U\chi \to \tilde{\chi}$ which we denote by $U\tilde{\chi} \to \tilde{\chi}$. Likewise we can lift the geodesic flow $\pi_{\tilde{\chi}}^* T_s\colon \pi_{\tilde{\chi}}^* U\chi \to \pi_{\tilde{\chi}}^* U\chi$, which we denote by $\tilde{T}_s\colon U\tilde{\chi} \to U\tilde{\chi}$. Let $\pi_{\tilde{\chi}}^*(\mu) := \tilde{\mu}$ be the lift of the Louiville measure. Since $\tilde{\chi} \to \chi$ is a finite cover, it follows that $\tilde{\mu}(U\tilde{\chi}) < \infty$, and $\tilde{T}_s$ is ergodic with respect to $\tilde{\mu}$.

We have the embedding $\psi\colon \tilde{\chi}_0 \to \mathcal{C}$. In proposition 3.1 (chapter 5) we compute that

$$\psi^*(\omega^{(k)}) = \hbar \sum_{1 \leqslant i \leqslant 5} \frac{A_i\, dt}{t - \zeta^i}$$

where $\zeta^i = e^{2\pi \mathbf{i}\left(\frac{i}{5}\right)}$, $A_i = \widehat{\Omega}^{a,b} + \widehat{\Omega}^{c,d}$ for $1 \leqslant a < b, c < d \leqslant 5$ such that $[a+b] = [c+d] = [i]$ $\big([x] := x(mod 5)\big)$. This equation should be compared with equation (4) and (64) in [**Enr**].

The pull back $\psi^*(\omega^{(k)})$ is a meromorphic 1-form on $\mathbb{C}$ with logarithmic singularities at μ_5. Since it is regular at $-\mu_5$, it is a holomorphic 1-form when restricted to $\tilde{\chi}$. We consider $\psi^*(\omega^{(k)})$ as a 1-form on $\tilde{\chi}$ and denote it by $\omega_{\tilde{\chi}}^{(k)}$. This 1-form defines a flat connection

$$\boldsymbol{\nabla}_{\tilde{\chi}}^{(k)} := d + \omega_{\tilde{\chi}}^{(k)}$$

in the trivial bundle $S^k(V) \times \tilde{\chi}$. Both the bundle and the connection can be pulled back to $U\tilde{\chi}$, and we denote these pull backs by $S^k(V) \times U\tilde{\chi}$ and $\boldsymbol{\nabla}_{U\tilde{\chi}}^{(k)}$.

Let $\|.\|$ be the operator norm on the vector space $S^k(V)$. We now have a finite measure space $(U\tilde{\chi}, \widetilde{\mu})$, an ergodic flow $\widetilde{T}_s\colon U\tilde{\chi} \to U\tilde{\chi}$, and a flat normed vector bundle $S^k(V) \times U\tilde{\chi}$. We are thus in a position to apply Oseledets multiplicative ergodic theorem, see theorem 2.1 in the last section of chapter 6.

For any $(x, v) \in U\tilde{\chi}$ we have the map

$$PT_{U\tilde{\chi}}^{(k)}(\widetilde{T}_s(x,v))\colon \mathbb{R}_{\geqslant 0} \to \mathrm{End}(S^k(V)), \qquad s \in \mathbb{R}_{\geqslant 0}$$

where $PT_{U\tilde{\chi}}^{(k)}$ denotes the parallel transport of $\boldsymbol{\nabla}_{U\tilde{\chi}}^{(k)}$ along paths in $U\tilde{\chi}$. This induces the cocycle

$$\theta_{(x,v)}^{(k)}(s)\colon S^k(V) \to S^k(V), \qquad s \in \mathbb{R}_{\geqslant 0},\ (x,v) \in U\tilde{\chi}$$

which is equivalent to the function

$$\theta^{(k)}\colon (\mathbb{R}_{\geqslant 0} \times U\tilde{\chi}) \to \mathrm{End}(S^k(V)).$$

Let us suppose that the logarithm of the norm of $\theta^{(k)}$ is integrable. That is, for any $s \in \mathbb{R}_{\geqslant 0}$, we have that

$$\int_{U\tilde{\chi}} \log(1 + \|\theta^{(k)}(s, \cdot)\|) \, d\tilde{\mu} < \infty$$

where the norm on $\mathrm{End}(S^k(V))$ is induced from the norm on $S^k(V)$. Then Oseledets multiplicative ergodic theorem guarantees a filtration,

$$S^k(V) = F_1^{(k)} \supset F_2^{(k)} \supset \cdots \supset F_n^{(k)} \supset 0$$

where $F_j^{(k)}$ are sub-bundles, and constants

$$\lambda_1^{(k)} > \cdots > \lambda_n^{(k)}$$

such that

$$\|\theta^{(k)}_{(x,v)}(s) \cdot (f)\| = e^{(\lambda_j^{(k)} t + O(s))}, \qquad s \to \infty$$

where j is the maximal value for which $f \in F_j^{(k)}$. Moreover, the filtration is preserved by the cocycle, the number n depends on the dimension of $S^k(V)$, and due to the ergodicity of the flow, the numbers $\lambda_j^{(k)}$ do not depend on the initial point $(x, v) \in U\tilde{\chi}$. Thus for every integer $k > 0$, the numbers $\lambda_j^{(k)}$ are invariants of the cocycle $\theta^{(k)}$, and thus are called characteristic or *Lyapunov* exponents of the cocycle in question.

In [**KZ97**] a cocycle, called the *Kontsevich-Zorich cocycle*, is constructed in unit tangent bundles of Teichmüller curves, see also [**BM10**]. This cocycle, like $\theta^{(k)}$, is given by the parallel transport of a flat connection, namely the Gauss-Mannin connection. Let $\mathcal{H}_1 \to \tilde{\chi}$ be the vector bundle with fiber over $t \in \tilde{\chi}$ the first homology (with real coefficients) of the compact genus two Riemann surface given by $t \in \tilde{\chi}$. This bundle carries the flat Gauss-Mannin connection, a symplectic structure, and a norm induced from the Hodge metric. The bundle, along with the flat connection, can be pulled back to the unit co-tangent bundle $U\tilde{\chi}$, and parallel transport of the connection along the geodesic flow gives a cocycle θ^{KZ}: this is the Kontsevich-Zorich cocycle.

Remark 0.3. The phase space of the cocycle in [**KZ97**] is more general, but it specializes to the situation of unit tangent bundles of Teichmüller curves, as remarked in [**KZ97**], see also [**BM10**].

At this point, one invokes the multiplicative ergodic theorem and obtains the Lyapunov exponents

$$\lambda_1^{KZ} > \lambda_2^{KZ} > \lambda_3^{KZ} > \lambda_4^{KZ}$$

associated with the cocycle θ^{KZ}. Due to the symplectic structure in the vector bundle, one has that, see [**KZ97**] or [**BM10**],

$$1 = \lambda_1^{KZ} = -\lambda_4^{KZ} \quad \text{and} \quad \lambda_2^{KZ} = -\lambda_3^{KZ}.$$

Notice that while $\theta^{(k)}$ was associated with a trivial bundle, θ^{KZ} is *not* associated with a trivial bundle, in fact, in [**KZ97**], see also [**BM10**], there is a formula for the sum $\lambda_1^{KZ} + \lambda_2^{KZ}$ in terms of the degree of the bundle $\mathcal{H}_1 \to \tilde{\chi}$.

We have the coincidence,

$$\dim(V) = \dim(H_1(X_t, \mathbb{C}))$$

where the vector space on the right hand is the complexification of the fiber $H_1(X_t, \mathbb{R})$ of the vector bundle $\mathcal{H}_1 \to \tilde{\chi}$ over any $t \in \tilde{\chi}$. Moreover, in [**vGdJ98**], the authors introduce a symplectic structure in V. We make the following

Conjecture 0.4. *The cocycle $\theta^{(1)}$ and θ^{KZ} coincide. i.e.*

$$\lambda_i^{KZ} = \lambda_i^{(1)} \qquad \textit{where } 1 \leqslant i \leqslant 4.$$

This conjecture would be true if the monodromy of $\boldsymbol{\nabla}_{\tilde{\chi}}^{(1)}$ was conjugate to the monodromy of the Gauss-Mannin connection. That this is probable follows from the fact that monodromy of $\boldsymbol{\nabla}_{\tilde{\chi}}^{(1)}$ is given by the two symplectic matrices M_0 and M_1 given in theorem 0.2.

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CHAPTER 2

Moduli of Vector Bundles and Hitchin Connection

1. Jacobian and the Picard Group

As a motivating example we study the moduli space of line bundles on a compact Riemann surface X of $g \geqslant 1$. Associated to X is its canonical bundle $K_X \to X$. We consider the space of abelian differentials, namely $H^0(X, K_X)$. By the Riemann-Roch formula, we know that $\dim H^0(X, K_X) = g$ and we let $(\omega_1, \omega_2, \ldots, \omega_g)$ be a basis of this vector space. Given any curve γ on X we can define a linear form on $H^0(X, K_X)$ by integrating, that is,

$$\omega_i \mapsto \int_\gamma \omega_i.$$

We can consider cycles in $H_1(X, \mathbb{Z})$ as curves in X. Thus for all $\gamma \in H_1(X, \mathbb{Z})$ we get a linear form

$$\gamma \colon H^0(X, K_X) \to \mathbb{C}$$

which gives us the canonical map

$$Z \colon H_1(X, \mathbb{Z}) \to \mathrm{Hom}(H^0(X, K_X), \mathbb{C}).$$

Since $H_1(X, \mathbb{Z})$ is a lattice in $\mathbb{R}^{2g} = \mathbb{C}^g$ and Z is injective, see lemma 11.1.1 in [**LB92**], the image of Z is a lattice in $H^0(X, K_X)^*$. We are now ready to give the definition.

Definition 1.1. *The Jacobian J_X associated to X is the quotient*

$$H^0(X, K_X)^*/Z(H_1(X, \mathbb{Z})).$$

It follows that J_X is a Complex Torus and can be identified with $\mathbb{C}^g/\mathbb{Z}^{2g}$. In fact, if we fix a basis $(\lambda_1, \ldots, \lambda_{2g})$ of $H_1(X, \mathbb{Z})$ with intersection form given by the matrix

$$\begin{pmatrix} 0 & \mathrm{Id}_g \\ \mathrm{Id}_g & 0 \end{pmatrix},$$

then we get a uniquely defined line bundle $\Theta \to J_X$ whose first chern class can be identified with the intersection form above, see [**LB92**]. Together, (J_X, Θ_X) define a principally polarized abelian variety. It is a deep theorem of Torelli that two Riemann surfaces X and Y are isomorphic if and only if (J_X, Θ_X) and (J_Y, Θ_Y) are isomorphic as principally polarized abelian varieties.

We now study holomorphic line bundles on X. A holomorphic line bundle is a complex 2 dimensional manifold L with a holomorphic projection $\pi\colon L \to X$ such that every fiber $\pi^{-1}(x) \cong \mathbb{C}$. Moreover, we require the projection to be locally trivial, that is
(1) for all $x \in X$ there exists a neighbourhood $U \subset X$ and $\Phi_U\colon \pi^{-1}(U) \to U \times \mathbb{C}$, an isomorphism, equivariant with respect to projections to U.
(2) for nonempty $V \cap U$, $\Phi_V \circ \Phi_U^{-1}$ is of the form $(m, w) \mapsto (m, f(m)w)$ where f is a non-vanishing holomorphic function. We call such f transition functions for $\pi\colon L \to X$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by $\mathcal{O}_X^*$ the sheaf of non vanishing holomorphic functions. It is obvious from the definition that a transition function defines a cocycle in cech cohomolgy group $\hat{H}^1(X, \mathcal{O}_X^*)$. Conversely, given an element $g \in \hat{H}^1(X, \mathcal{O}_X^*)$ one can construct a holomorphic line bundle $G \to X$ by declaring g to be the transition functions over the intersection of open neighbourhoods whose cech cohomology groups we are considering.

Definition 1.2. *Define Pic_X to be the set of all holomorphic lines bundles on X. From the discussion above it is clear that $Pic_X = H^1(X, \mathcal{O}_X^*)$.*

Since X is a compact complex manifold, it follows that $\hat{H}^1(X, \mathcal{O}_X)$ is isomorphic to the de Rham cohomolgy group $H^1(X, \mathcal{O}_X)$. We now associate a topological invariant to any line bundle. Consider the following fundamental *exponential sequence*

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1.$$

The map $\mathcal{O}_X \to \mathcal{O}_X^*$ is given by $f \mapsto e^{2\pi i f}$. This induces the following long exact sequence

$$\begin{aligned} 0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \\ \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^*) \to \dots \end{aligned}$$

The first part of the sequence is due to the fact that all holomorphic functions on X are constant. Since exponentiation is surjective, the map $\mathbb{C} \to \mathbb{C}^*$ is surjective, and by exactness it follows that $H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)$ is injective. We also know that $H^2(X, \mathcal{O}_X)$ must vanish. Putting all this together we get

$$0 \to H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) = \mathbb{Z} \to 0$$

Definition 1.3. *Let $c_1 \colon H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) = \mathbb{Z}$. Then the degree of $L \in H^0(X, \mathcal{O}_X^*)$ is the integer $c_1(L)$.*

By Serre Duality, $H^1(X, \mathcal{O}_X^*) = H^0(X, K_X)$, thus $\dim H^1(X, \mathcal{O}_X^*) = g$. From the short exact sequence above, it follows that the set of degree 0 line bundles, which we call Pic_X^0, can be identified with the g dimensional complex torus

$$\mathrm{Pic}_X^0 := H^1(X, \mathcal{O}_X^*)/H^1(X, \mathbb{Z}).$$

The theorem of Abel and Jacobi implies that $\mathrm{Pic}_X^0 \cong J_X$, in fact if we make the choice of a basis for the homology, then these are equivalent as principally polarized abelian varieties. In conclusion, we see that the set of degree 0 line bundles on a Riemann surface X comes naturally equipped with the rich structure of an abelian variety.

2. Moduli of Vector Bundles

In the last section we saw that the set of holomorphic line bundles of degree 0, Pic_X^0, on a Compact Riemann Surface X is isomorphic to the Complex Torus J_X, the Jacobian of X. Indeed, a choice of basis for $H_1(X, \mathbb{Z})$, yields a canonical (ample) line bundle $\Theta_X \to J_X$, the pair (J_X, Θ_X) forms an (principally polarized) abelian variety. In this section we study $\mathcal{M}_X^B(n,d)$, the set of (semi-stable) holomorphic vector bundles $E \to X$ of rank $= n \geqslant 2$ and degree $= d \in \mathbb{Z}$. This makes $\mathcal{M}_X^B(n,d)$ a generalization of the Jacobian of X. While $\mathcal{M}_X^B(n,d)$ is not a group like the Jacobian, it is still a projective variety, singular in general. The topology of this variety is well understood, in the sense that Betti numbers have been computed. We will not deal with the topology of this space, but rather concentrate on its geometric quantization, which forms an essential part of Witten's three dimensional Topological Quantum Field Theory [**Wit89**].

Let X be a compact Riemann Surface of genus $g \geqslant 2$. By a holomorphic vector bundle of rank r on X, we mean a $r+1$ dimensional complex manifold E, with a holomorphic projection map $\pi \colon E \to X$, such that π is locally trivial. The notion of locally trivial is exactly the same as in the case of line bundles, except now the fiber $\pi^{-1}(x) \cong \mathbb{C}^r$. Also, the transition functions f are now matrix valued, that is $f \in \hat{H}^1(X, \mathcal{O}_X(GL(r, \mathbb{C})))$. Here $\mathcal{O}_X(GL(r, \mathbb{C}))$ is sheaf of holomorphic functions with values in invertible $r \times r$ matrices.

Apart from the rank of the vector bundle, its degree is also one of the basic invariants. Given a vector bundle $E \to X$ we know that each fiber of the bundle is isomorphic to $\mathbb{C}^r$. We can replace each fiber with its top exterior power, that is $\wedge^r\mathbb{C}^r \cong \mathbb{C}$. This procedure yeilds a two dimensional manifold $\wedge^r E$ with holomorphic projection $\wedge^r E \to X$, thus, a line bundle. In the last section we saw that to every line bundle L is associated its chern class $c_1(L) \in \mathbb{Z}$.

Definition 2.1. *Let $E \to X$ be a holomorphic vector bundle. Then, degree of E is the integer $c_1(\wedge^r E)$.*

At this point one could try to study the set of holomorphic vector bundles of a given rank and degree. Unfortunately, this is too big a set to be parametrized by a reasonable topological space. There are examples of continuous families of holomorphic bundles $E_t \to X$ for $t \in \mathbb{D}$, where $\mathbb{D}$ is the unit disk in the complex plane, such that degree$(E_0) \neq$ degree(E_t) for any $t \in (\mathbb{D} - 0)$, and degree(E_t) =degree$(E_{t'})$ where $t, t' \in (\mathbb{D} - 0)$. This is called the jumping phenomena, that is, in a continuous family the degree of the vector bundle can jump. Jumping phenomena makes it clear that we must impose further restrictions on the set we are considering. It turns out that the correct restriction is semi stability.

Definition 2.2. *Let $E \to X$ be a holomorphic vector bundle of rank r and degree d. Then, the slope of E is the following rational number $\mu(E) = \frac{degreeE}{rankE}$.*

We are now ready to define semi-stable Vector Bundles.

Definition 2.3. *Let $E \to X$ be a holomorphic vector bundle. Then E is stable (respectively semi-stable) if $\mu(F) < \mu(E)$ (respectively $\mu(F) \leqslant \mu(E)$) for all non-trivial holomorphic sub-bundles $F \subset E$.*

Let $\mathcal{M}_X^B(n,d)^S$ be the set of stable vector bundles $E \to X$ of rank n and degree d. David Mumford, using Geometric Invariant Theory, was able to show that $\mathcal{M}_X^B(n,d)^S$ is a (smooth) quasi-projective variety. Being quasi-projective, this variety is not compact. A compactification of this variety was first given by Seshadri by using the concept of S-equivalence classes of strictly semi-stable vector bundles. We explain this concept now.

Lemma 2.4. *Let $0 \to E_1 \to E \to E_2 \to 0$ be a short exact sequence of Vector Bundles on X. If any two vector bundles in this sequence have slope μ, then the third also has slope μ.*

Proof. Since $0 \to E_1 \to E \to E_2 \to 0$ is a short exact sequence, we have that $rank(E) = rank(E_1) + rank(E_2)$, and we also have that $degree(E) = degree(E_1) + degree(E_2)$. Now a simple calculation in linear algebra shows that if slope of any two vector bundles is the same, then the third vector bundle must have the same slope as well. □

Now, suppose $E \to X$ was a strictly semi-stable vector bundle. Then, by definition of semi-stability, there exists a sub-bundle $F \subset E$ such that $\mu(F) = \mu(E)$. Now consider the short exact sequence

$$0 \to F \to E \to \frac{E}{F} \to 0.$$

Since $\mu(F) = \mu(E)$ an application of lemma 2.4 yields that $\mu(\frac{E}{F}) = \mu(E)$.

PROPOSITION 2.5. *Let $E \to X$ be a semi-stable vector bundle. Then there exists a filtration (called the Jordan-Hoelder Filtration)*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

such that $\mu(\frac{E_{j+1}}{E_j}) = \mu(E)$, and $\frac{E_{j+1}}{E_j}$ is a stable vector bundle.

PROOF. Suppose $E \to X$ is strictly semi-stable. Then there exists a sub-bundle $F \subset E$ such that $\mu(F) = \mu(E)$. We construct the sub-bundle $\frac{E}{F}$ and we know by lemma 2.4 that $\mu(\frac{E}{F}) = \mu(E)$. Now, $\frac{E}{F}$ is a semi-stable vector bundle, because if it was not then there exists some sub-bundle $W \subset \frac{E}{F}$ such $\mu(W) > \mu(\frac{E}{F}) = \mu(E)$. But since E is a semi-stable vector bundle it cannot have a sub-bundle (namely W) with slope bigger than the slope of E.
Now that we have constructed a semi-stable sub-bundle $\frac{E}{F}$, we can apply the above process again to construct further semi-stable sub-bundles. We apply the process iteratively until we hit the trivial sub-bundle, and this yeilds the desired filtration. □

REMARK 2.6. Notice that if $E \to X$ was strictly stable, that is, if all its sub-bundles had slope strictly less than the slope of E, then the corresponding Jordan-Hoelder Filtration would be empty. Thus, this concept is useful for strictly semi-stable bundles.

DEFINITION 2.7. *Let $E \to X$ be a semi-stable vector bundle. Let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

be its Jordan-Hoelder filtration. Then to $E \to X$ is associated the following graded vector bundle

$$Gr(E) := \bigoplus_{j=1}^{m} \frac{E_j}{E_{j-1}}$$

where $1 \leqslant j \leqslant m$.

While the Jordan-Hoelder filtration of $E \to X$ is not unique, the associated graded object, $Gr(E)$, is determined by E.

Definition 2.8. *Let $E \to X$ and $E' \to X$ be two semi-stable holomorphic vector bundles of rank n degree d. Then E is S-equivalent to E' if and only if $Gr(E) = Gr(E')$.*

Notice that if $E \to X$ is stricly stable, then $Gr(E) = E$.

Definition 2.9. *Let $\mathcal{M}_X^B(n,d)$ be the set of S-equivalence classes of holomorphic semi stable vector bundles of rank n and degree d.*

Now, it is a result of Seshadri [**Ses12**] that $\mathcal{M}_X^B(n,d)$ is a singular (in general) projective variety of complex dimension $n^2(g-1)+1$. The structure of a complex variety on the set $\mathcal{M}_X^B(n,d)$ is given as follows. First one shows that $\mathcal{M}_X^B(n,d)$ as a category is Artinian and Abelian. This implies that the deformation theory of $E \to X$ is unobstructed. Thus, for any $E \in \mathcal{M}_X^B(n,d)$ there exists an open ball $U \subset \mathbb{C}^{n^2(g-1)+1}$, with a fiber bundle structure, i.e. $\pi \colon \mathcal{E} \to U$ such that $\pi^{-1}(0) = E$, and $\pi^{-1}(x) = E'$ where $E' \in \mathcal{M}_X^B(n,d)$. This gives us a system of complex local charts on $\mathcal{M}_X^B(n,d)$, and it can be shown that these charts glue together to give the structure of a Complex Manifold to the set $\mathcal{M}_X^B(n,d)$. To further show that $\mathcal{M}_X^B(n,d)$ is a projective variety, one shows that there exists an ample line bundle $\mathcal{L} \to \mathcal{M}_X^B(n,d)$ and the sections of this line bundle can be used to embed $\mathcal{M}_X^B(n,d) \to \mathbb{P}^n$, making it a projective variety. We will talk more about this ample line bundle in the chapter on *Geometric Quantization*.

Since $Gr(E) = E$ if $E \to X$ is a strictly stable bundle, we get that S-equivalence class of $E \to X$ is the same as just the isomorphism class of $E \to X$ as a vector bundle. In fact, the locus of smooth points of $\mathcal{M}_X^B(n,d)$ is composed entirely of equivalence classes of strictly stable vector bundles. The locus of singular points is then entirely composed of S-equivalence classes of strictly semi-stable vector bundles. Moreover, in the generic case, it turns out the locus of singular points is strictly of co-dimension greater than 2. All of these are again results due to [**Ses12**].

We also mention another feature of stability of vector bundles.

Let E, E' be two vector bundles. We denote by $H^0(Hom(E,E'))$ the space of sections of homomorphisms from $E \to E'$. In this light we have the following lemma.

Lemma 2.10. *Let E and E' be two stable vector bundles of same rank and degree. We have that $H^0(Hom(E,E')) = 0$ if E is not isomorphic to E', and $H^0(Hom(E,E')) = \mathbb{C}^*$ if E is isomorphic to E'.*

This in particular implies that the space of automorphisms of a stable vector bundle is just one dimensional. In fact, if $E \to X$ is any vector bundle such that $H^0(Hom(E,E)) = \mathbb{C}^*$ then E is called a *simple* Vector Bundle. We in particular get that every stable holomorphic vector bundle $E \to X$ is also simple.

3. Flat Unitary Connections and a theorem of Narasimhan-Seshadri/Donaldson

In the last section we saw that to a compact Riemann Surface X is associated the space $\mathcal{M}_X^B(n,d)$ of semi-stable holomorphic vector bundles of rank n and degree d. In this section we will show that the complex variety $\mathcal{M}_X^B(n,d)$ warrants two other descriptions. In particular, we will have the following diagram

$$\begin{tikzcd}
& \mathcal{M}_X^B(n,d) \arrow[dl, leftrightarrow, "\text{Donaldson}"'] \arrow[dr, leftrightarrow, "\text{Narasimhan-Seshadri}"] & \\
\mathcal{F}_X^U(n,d) \arrow[rr, leftrightarrow, "\text{Weil}"'] & & \mathcal{R}_X^U(n,d)
\end{tikzcd}$$

where $\mathcal{F}_X^U(n,d)$ is the space of gauge equivalence classes of unitary connections (with constant central curvature) on a complex bundle $\mathbb{E} \to X$ and $\mathcal{R}_X^U(n,d)$ is the space of irreducible representations of a central extension of fundamental group of X in $U(n)$ upto conjugation. The arrows represent homeomorphisms of topological spaces. We will deal with the case d=0.

We first start with defining $\mathcal{F}_X^U(n,d)$ with $d = 0$. Let $\mathbb{E} \to X$ be a Complex Vector bundle of rank n and degree 0. By a Complex Vector bundle we mean a smooth vector bundle, not holomorphic, such that the fiber is isomorphic to $\mathbb{C}^n$. We denote by $\mathcal{E}^n$ the sheaf of functions on X with values in the vector bundle $\mathbb{E} \otimes \Omega_X^n$ where Ω_X^n is the n^{th} power of the cotangent bundle of X. Notice that since X is 1-dimensional, $\Omega_X^1 \cong K_X$ where K_X is the canonical line bundle.

Definition 3.1. *A connection on $\mathbb{E} \to X$ is a (Complex Linear) sheaf homomorphism*

$$\nabla \colon \mathcal{E}^0 \to \mathcal{E}^1$$

such that it satisfies the Liebniz rule

$$\nabla(f \cdot s) = d(f) \otimes s + f \cdot \nabla(s).$$

Here f is a function, $s \in \mathcal{E}^0$, and d stands for the exterior differential.

We will be interested in hermitian, or unitary, connections. So given a complex vector bundle $\mathbb{E} \to X$, we introduce an additional datum of a unitary inner product $h\colon \mathbb{E} \times \mathbb{E} \to \mathbb{C}$. That is, h_x is a unitary inner product in in the fiber $\mathbb{E}_x$ for all $x \in X$, and it varies smoothly as $x \in X$ varies. We call the pair $(\mathbb{E}, h) \to X$ a *unitary* vector bundle. It is an elementary fact that every complex vector bundle admits a unitary metric. We can now define the concept of unitary connections.

DEFINITION 3.2. *Let $(\mathbb{E}, h) \to X$ be a unitary vector bundle. Then a connection ∇ on $\mathbb{E} \to X$ is a unitary connection, if*

$$d(h(s_1, s_2) = h(\nabla(s_1), s_2) + h(s_1, \nabla(s_2)).$$

This definition requires additional explanation. Here $h(s_1, s_2)$ is considered as a function and $d(h(s_1, s_2)$ as the exterior differential applied to this function. We also have that $h(\alpha \otimes s_1, s_2) = \alpha h(s_1, s_2)$ and $h(s_1, \alpha \otimes s_2) = \bar{\alpha} h(s_1, s_2)$ where α is a complex valued 1-form. By definition we have that $\nabla(s) = \alpha \otimes s'$ where s, s' are sections of $\mathbb{E}$.

Given a connection $\nabla\colon \mathcal{E}^0 \to \mathcal{E}^1$ in a vector bundle, we have a natural extension

$$\nabla\colon \mathcal{E}^k \to \mathcal{E}^{k+1}$$

defined the following way. Let $\alpha \in \Omega^k_X$ be a k-form and $s \in \mathcal{E}^0$, then of course we have that $\alpha \otimes s \in \mathcal{E}^k$. The connection now acts as follows

$$\nabla(\alpha \otimes s) = d(\alpha) \otimes s + (-1)^k \alpha \wedge \nabla(s)$$

Notice that for $k = 0$ the above is just the Liebniz rule, which also ensures that the map is well defined, i.e. $\nabla(\alpha \otimes f \cdot s) = \nabla(f \cdot \alpha \otimes s)$ where f is any smooth function.

Now, of course one would like to know if a connection ∇ is actually a differential, i.e. $\nabla^2 = 0$, which will allow one to do cohomology theory on the complex $\mathcal{E}^k$. The obstruction to $\nabla^2 = 0$ is measured by the *curvature* of the connection ∇. We now define curvature.

DEFINITION 3.3. *Let $\mathbb{E} \to X$ be a vector bundle with a connection ∇. Then the curvature of the connection is defined as the following sheaf homomorphism*

$$F_\nabla = \nabla \circ \nabla\colon \mathcal{E} \to \mathcal{E}^2.$$

We modify the above to the setting we need by the following definition.

Definition 3.4. *Let $(\mathbb{E}, h) \to X$ be a unitary vector bundle. Let ∇ be a unitary connection in this bundle. Then ∇ is called a flat unitary connection if $F_\nabla = 0$.*

Let $\mathcal{G}$ be the group of automorphisms of $(\mathbb{E}, h) \to X$, which preserve the metric h. Given a connection ∇ in this vector bundle, and a $g \in \mathcal{G}$, we get a new connection $g \cdot \nabla \cdot g^{-1}$.

Definition 3.5. *Let $(\mathbb{E}, h) \to X$ be a unitary vector bundle. Then two unitary connections ∇, ∇' are said to be gauge equivalent if and only if there exists a $g \in \mathcal{G}$ such that $\nabla' = g \cdot \nabla \cdot g^{-1}$*

We now have all the requisites to define the set $\mathcal{F}^U_X(n, d)$.

Definition 3.6. *Let $(\mathbb{E}, h) \to X$ be a unitary vector bundle of rank n, such that the first chern class, or degree of $\mathbb{E}$ is $d \in \mathbb{Z}$. Then $\mathcal{F}^U_X(n, d)$ is the set of all gauge equivalent classes of unitary flat connections in $(\mathbb{E}, h) \to X$.*

By using standard techniques of gauge theory, for example slicing theorem for infinite dimensional Banach spaces, one can show that $\mathcal{F}^U_X(n, d)$ is manifold of the real dimension $2 + 2n^2(g - 1)$. For more details, see for example [**AB83**]. Now we can state the following theorem of Donaldson, please see [**Don83**].

Theorem 3.7. *For every $[\nabla] \in \mathcal{F}^U_X(n, d)$ there exists a unique $[E] \in \mathcal{M}^B_X(n, d)$.*

Here $[\nabla]$ denotes the gauge equivalent class, and $[E]$ represents S-equivalence class. The above theorem can be improved to a homeomorphism between spaces $\mathcal{M}^B_X(n, d)$ and $\mathcal{F}^U_X(n, d)$, and in fact a diffeomorphism between strictly stable bundles and irreducible flat unitary connections.

The idea behind this equivalence is following. Let ∇ be a unitary connection in $(\mathbb{E}, h) \to X$. Using the complex structure on X, we get a decomposition $\nabla = \nabla^{1,0} + \nabla^{0,1}$ where

$$\nabla^{1,0} \colon \mathcal{E}^0 \to \mathcal{E}^{1,0}, \qquad \nabla^{0,1} \colon \mathcal{E}^0 \to \mathcal{E}^{0,1},$$

and the splitting $\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ is induced by the splitting $\Omega^1_X = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$ where the later is simply induced by the holomorphic structure on X. More explicitly, the last splitting is the Hodge decomposition of complex 1-forms into holomorphic and anti-holomorphic 1-forms. On the other hand, to give a holomorphic structure on a complex vector bundle $\mathbb{E} \to X$ it suffices to provide an operator

$$\bar{\partial} \colon \mathcal{E}^0 \to \mathcal{E}^{0,1}.$$

We can already see that $\nabla^{0,1}$ behaves like the operator $\bar{\partial}$ and can thus be used to give a holomorphic structure in $\mathbb{E} \to X$. The beauty of Donaldson's theorem is that given a flat unitary connection ∇ in $(\mathbb{E}, h) \to X$, the corresponding $\nabla^{0,1}$ not only induces a holomorphic structure, but a *stable* holomorphic structure in $\mathbb{E} \to X$. It turns out that flatness of a connection and stability of vector bundles is somehow the same concept in two different guises.

We will now define a third space $\mathcal{R}_X^U(n, d)$ which has a rather topological flavor. Let $\{A_i, B_i\}$, for $1 \leqslant i \leqslant g$, be loops in X such that the fundamental group of X has the following presentation

$$\pi_1(X) = < A_i, B_i > /(\Pi_1^g [A_i, B_i]),$$

where $[A_i, B_i]$ is the commutator of the loops A_i, B_i. Let ρ be a representation of $\pi_1(X)$ on $\mathbb{C}^n$. That is,

$$\rho \colon \pi_1(X) \to \mathrm{GL}(n, \mathbb{C}).$$

Given a representation ρ, it is easy to build a flat unitary connection ∇ on X. The construction goes as follows. Consider the trivial bundle

$$\tilde{X} \times \mathbb{C}^n,$$

where $\tilde{X}$ is the universal cover of X. The fundamental group $\pi_1(X)$ has an action on both $\tilde{X}$ and $\mathbb{C}^n$, first as the deck group of transformations for X, and second as automorphisms of $\mathbb{C}^n$ given by the image of ρ, i.e.

$$(y, v) \mapsto (\gamma(y), \rho(\gamma)v).$$

Consider the quotient vector bundle

$$\mathbb{E} = \tilde{X} \times_\rho \mathbb{C}^n$$

by the action defined above. The trivial bundle is always equipped with the trivial connection. This trivial connection descends to a connection ∇ in the bundle $\mathbb{E}$. The vector bundle $\mathbb{E}$ has degree 0, or equivalently, the first chern class of $\mathbb{E}$ vanishes. Since the first chern class is the only obstruction to a connection being flat, it follows that ∇ is a flat connection. In fact, if we assume that the representation is unitary, i.e.

$$\rho \colon \pi_1(X) \to \mathrm{U}(n),$$

then ∇ is a flat unitary connection and $[\nabla] \in \mathcal{F}_X^U(n, 0)$ where $[\nabla]$ is the gauge equivalence class of ∇. Thus to every unitary representation one can associate a flat unitary connection.

On the other hand, to every flat unitary connection, a unitary representation of the fundamental group can be canonically associated. To see this, let $\gamma\colon [0,1] \to X$ be a path in X, and let $\dot{\gamma}$ be the associated vector field. Let $\mathbb{E} \to X$ be a vector bundle of degree 0, equipped with a flat connection ∇. Let $\psi \in \Gamma(X, \mathbb{E})$ be a section. Then the section ψ is parallel along γ if the following is satisfied

$$\nabla_{\dot{\gamma}(t)}(\psi(t)) = 0.$$

Above is really a differential equation in local coordinates. Since any ∇ locally has the form

$$\nabla = d + A$$

where d is the *deRahm* differential and A an endomorphism valued 1-form, the pull back to $[0,1]$ is

$$\gamma^*\nabla = dt - A(t), \qquad t \in [0,1].$$

Now the condition of ψ being parallel is that $\gamma^*\psi$ should be a solution to the following differential equation

$$\frac{d\gamma^*\psi}{dt} = A(t).\gamma^*\psi(t).$$

Let $a := \gamma(0)$ and let v be an element in the fiber over $a \in X$, i.e. $v \in \mathbb{E}_a$. Then there exists a unique section $\psi \in \Gamma(X, \mathbb{E})$ which is parallel along γ and $\psi(a) = v$. This is essentially because the above ordinary differential equation for parallel transport always has a unique solution once an initial condition is given.

Now suppose that γ is a loop in X, i.e. $\gamma(0) = \gamma(1) = a$. Then any $v \in \mathbb{E}_a$ can be parallel transported along γ and end as $v' \in \mathbb{E}_a$. Thus parallel transport along a loop based at $a \in X$ provides us with a linear map say $\rho(\gamma) \in \mathrm{GL}(n, \mathbb{C})$ such that $\rho(\gamma)(v) = v'$. Now, it turns out that if ∇ is flat, then ρ *only* depends on the homotopy class of γ. This is in particular implies that we get the following representation from the parallel transport of a flat connection

$$\rho\colon \pi_1(X, a) \to \mathrm{Aut}(\mathbb{E}_a).$$

In fact, if we start with a unitary bundle $(\mathbb{E}, h) \to X$ of degree 0, and a unitary flat connection ∇, then the parallel transport along any path in X preserves the hermitian metric h in the bundle. This in particular implies that the representation we obtain is unitary, i.e.

$$\rho\colon \pi_1(X, a) \to \mathrm{U}(n).$$

Thus given any $\nabla \in \mathcal{F}^U_X(n, 0)$, we get a unitary representation of the fundamental group which we call ρ.

In fact, given any $\nabla \in \mathcal{F}^U_X(n,d)$ we still get a representation, but of a central extension $\tilde{\pi_1}(X)$ of the fundamental group, see the appendix of [**Wel80**]. That is, we have the following short exact sequence,

$$0 \to \mathbb{Z} \to \tilde{\pi_1}(X) \to \pi_1(X) \to 1$$

and the parallel transport leads to the following representation

$$\rho_d \colon \tilde{\pi_1}(X) \to \mathrm{U}(n).$$

Notice that this representation has a topological invariant which is the integer d.

Now, fix $(\mathbb{E}, h) \to X$, a hermitian vector bundle of degree d, and consider the set $R := \{\rho_d | \rho_d \colon \tilde{\pi_1}(X) \to \mathrm{U}(n)\}$. There is an action of $\mathrm{U}(n)$ on this set which is given as follows.

$$g.\rho_d(\gamma) \mapsto g.\rho_d(\gamma).g^{-1}, \qquad g \in \mathrm{U}(n), \gamma \in \tilde{\pi_1}(X).$$

We now define the following space,

Definition 3.8. $\mathcal{R}^U_X(n,d) := \{\rho_d | \rho_d \colon \tilde{\pi_1}(X) \to \mathrm{U}(n)\}/\mathrm{U}(n)$.

It is well known that $\mathcal{R}^U_X(n,d)$ is a manifold, in fact it is a symplectic manifold. From the discussions above we have the following

Proposition 3.9. *There is a homeomorphism* $\mathcal{F}^U_X(n,d) \cong \mathcal{R}^U_X(n,d)$.

So far our discussion did not involve any holomorphic structure on $\mathbb{E} \to X$. In this light we have the theorem of Narasimhan Seshadri [**NS65**], which states that for every unitary representation, there is a semi-stable vector bundle.

Theorem 3.10. *There is a bijection between* $\mathcal{M}^B_X(n,d)$ *and* $\mathcal{R}^U_X(n,d)$.

A representation ρ is called irreducible if its action does not preserve any non-trivial sub-bundle. The result in [**NS65**] actually says that $\rho_d \in \mathcal{R}^U_X(n,d)$ is irreducible if and only if the corresponding $[E] \in \mathcal{M}^B_X(n,d)$ is stable. This theorem thus establishes a link between algebraic geometry of stable bundles and topology of the surface captured by unitary representations.

4. Geometric Quantization and Hitchin connection in genus 2

Let X be a Riemann surface of genus $g \geqslant 2$. In the last section we saw that associated to X is its moduli space of holomorphic semistable vector bundles of rank n and degree d, $\mathcal{M}^B_X(n,d)$. In this section we specialize to bundles with $n = 2$ and trivial determinant. That is, let $E \to X$ be a semistable vector bundle of rank 2. Then moduli space we consider is

$$\mathcal{M}_X \colon = \{E \to X | \wedge^2 E \cong \mathcal{O}_X\} / \sim_s$$

where $\sim_s$ means S-equivalence classes. It was shown in [**NR69**] that $\mathcal{M}_X$ is a quasi-projective variety of dimension $3g-3$ and non-smooth for $g \geqslant 3$. Under the $Narasimhan - Seshadri$ and *Donaldson* isomorphism, we have a homeomorphism between $\mathcal{M}_X$ and the moduli space of flat $SU(2)$ connections on X.

The moduli space $\mathcal{M}_X$ comes equipped with a non-abelian Theta Divisor, $\Xi \subset \mathcal{M}_X$, which is defined as follows,

$$\Xi \colon = \{[E] \in \mathcal{M}_X | h^0(X, E \otimes L) \geqslant 1\}$$

where $L \in \mathrm{Pic}^{g-1}(X)$ such that $h^0(X, L) \geqslant 1$. In [**Bea88**] it is shown that Ξ is a Cartier divisor and its linear equivalence class is independent of the choice of L. It turns out that the line bundle $\mathcal{L}_X := \mathcal{O}_{\mathcal{M}_X}(\Xi)$ is ample and generates the entire $\mathrm{Pic}(\mathcal{M}_X)$, which of course implies that $\mathrm{Pic}(\mathcal{M}_X) \cong \mathbb{Z}$ [**DN69**]. The line bundle $\mathcal{L}_X$ is called the *determinant line bundle*, and the dialyzing sheaf of $\mathcal{M}_X$ is isomorphic to $\mathcal{L}_X^{-4}$ [**DN69**]. For every positive integer k, we will denote by $V_X^{(k)}$ the vector space $H^0(\mathcal{M}_X, \mathcal{L}_X^k)$. Elements of $V_X^{(k)}$ are called the non-abelian theta functions of level k. The dimension of $V_X^{(k)}$ is given by the famous Verlinde formula,

$$\left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \big(\sin^2 \frac{j\pi}{k+2}\big)^{1-g}.$$

By a result of [**Hit90**], the vector spaces $V_X^{(k)}$ assemble to form a vector bundle $\mathbb{V}^{(k)} \to \mathcal{T}(S)$ over the Teicm̈ller space $\mathcal{T}(S)$. For the definition of $\mathcal{T}(S)$ please see section 1 of chapter 3. In [**?**], it was proposed that this vector bundle can be equipped with a projectively flat connection

$$\boldsymbol{\nabla}^{(k)} \colon \Gamma(\mathbb{V}^{(k)}) \to \Gamma(\mathbb{V}^{(k)}) \otimes \Omega^1_{\mathcal{T}(S)}.$$

That this connection exists was shown in [**And12**], [**Fal93**], [**Hit90**], and [**ADPW91**]. The connection $\boldsymbol{\nabla}^{(k)}$ is now dubbed the *Hitchin* connection. The action of the mapping class group $\Gamma(S)$ on $\mathcal{T}(S)$, please see section 3 of chapter 3, lifts to an action on the vector bundle $\mathbb{V}^{(k)}$. Moreover, the *Hitchin* connection is invariant under the action of $\Gamma(S)$. This implies that we get a quotient vector bundle $\overline{\mathbb{V}}^{(k)} \to \mathcal{M}_g$ equipped with the projectively flat connection $\overline{\boldsymbol{\nabla}}^{(k)}$, which is the descent of $\boldsymbol{\nabla}^{(k)}$. Our main aim is to study the monodromy representation of $\Gamma(S) := \pi_1^{orb}(\mathcal{M}_g)$ in $\overline{\mathbb{V}_k}$ given by the projectively flat connection $\overline{\boldsymbol{\nabla}}^{(k)}$. That is, to study the group homomorphism

$$\rho_k \colon (\mathcal{M}_g, X) \to \mathrm{End}\, \mathbb{P}(\overline{V}_X^{(k)})$$

where $X \in \mathcal{M}_g$ is an isomorphism class of a Riemann surface of genus g and $\overline{V}_X^{(k)}$ is the fiber of $\overline{\mathbb{V}^{(k)}}$ over X.

From now on we will restrict to the case that X has genus $g = 2$. In this situation, the techniques of [**Hit90**] do not apply. Nonetheless, a construction of the 'Hitchin Connection' in genus 2 was given in [**vGdJ98**], and this is the construction we will recall in the following sections. It should also be mentioned that the general technique of quantizing a family of Kaehler Manifolds developed in [**And12**] also applies in genus 2.

4.1. Moduli of Bundles and $|2\Theta|$. Let $\Theta_X := \{L \in \mathrm{Pic}^{g-1} | h^0(X, L) \geqslant 1\}$ be the canonically defined Theta divisor in the space of line bundles of degree $g-1$ on X, and let $|2\Theta_X|$ be the projective space of divisors linearly equivalent to $2\Theta_X$. It is a classical result that dimension of $|n\Theta_X|$ is $n^g - 1$, thus $|n\Theta_X| \cong \mathbb{P}^{n^g-1}$. One way of understanding the geometry of $\mathcal{M}_X$ is through the following map

$$\Delta \colon \mathcal{M}_X \to |2\Theta_X|, \tag{2.1}$$

where $\Delta(E) := \{L \in \mathrm{Pic}^{g-1} | h^0(X, E \otimes L) \geqslant 1\}$. It is shown in [**Bea88**] that for a generic $[E] \in \mathcal{M}_X$, $\Delta(E)$ is indeed in $|2\Theta_X|$ and that it only depends on the S-equivalence class of E. We record main properties of this map proved in [**Bea88**] after introducing one more piece of notation. Let $\alpha \in \mathrm{Pic}^{g-1}(X)$. Then we get a translation map $\mathrm{Pic}^{g-1}(X) \to \mathrm{Pic}^0(X)$ given by $L \mapsto L \otimes \alpha^{-1}$. The divisor $\hat{\Theta}_X \subset \mathrm{Pic}^0(X)$ is the image of $\Theta_X \subset \mathrm{Pic}^{g-1}(X)$ under the translation map with respect to some $\alpha \in \mathrm{Pic}^{g-1}(X)$.

THEOREM 4.1. *Let $\mathcal{L}_X$ be the determinant line bundle on $\mathcal{M}_X$, then $\mathbb{P}H^0(\mathcal{M}_X, \mathcal{L}_X)^* \cong |2\Theta_X| \cong |2\hat{\Theta}_X|^*$ and 2.1 coincides with the natural map $h \colon \mathcal{M}_X \to \mathbb{P}H^0(\mathcal{M}_X, \mathcal{L}_X)^*$.*

The theorem above in particular implies that $H^0(\mathcal{M}_X, \mathcal{L}_X) \cong H^0(J_X, 2\hat{\Theta}_X)$, i.e. level 1 non-abelian Theta functions are isomorphic to classical theta functions of level 2.

Notice the peculiar coincidence when $g = 2$ where the dimensions of the domain and codomain of 2.1 match. It is indeed a theorem of [**NR69**] that for X of $g = 2$ we have

$$\mathcal{M}_X \cong |2\Theta_X| \cong \mathbb{CP}^3. \tag{2.2}$$

That this is an isomorphism is proved by constructing an explicit map $\rho \colon |2\Theta_X| \to \mathcal{M}_X$ and showing that this map is inverse to (2.1). This isomorphism in particular makes it possible to study the non-abelian Theta functions explicitly.

Let $[z_0, \ldots, z_n]$ be global homogenous coordinates on $\mathbb{P}^n$. Recall that $\mathcal{O}(1)$ is the dual of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^n$. Notice that any linear functional L on $\mathbb{C}^{n+1}$ induces a section σ_L of $\mathcal{O}(1)$ by restricting L to the fibers of $\mathcal{O}(-1)$. This in particular gives an injection

$$(\mathbb{C}^{n+1})^* \to H^0(\mathbb{P}^n, \mathcal{O}(1))$$

since σ_L is identically zero only if L is zero. These are in fact all the sections of $\mathcal{O}(1)$, i.e. the map above is a bijection. Now any k-linear form F on $\mathbb{C}^{n+1}$ induces a section σ_F of $\mathcal{O}(k)$ by restriction to the fiber of $\mathcal{O}(-1)$. Since σ_F is given by restriction to a line, it is zero only if it is alternating in any two factors. This gives us the map

$$Sym^k(\mathbb{C}^{n+1*}) \to H^0(\mathbb{P}^n, \mathcal{O}(k))$$

which again is bijective. We identify $Sym^k(\mathbb{C}^{n+1*})$ with the space of homogenous polynomials of degree k in $n+1$ variables.

Now if X is of genus $g = 2$, by using 2.1 we can identify $\mathcal{M}_X$ with $\mathbb{P}^3$ and the determinant line bundle $\mathcal{L}_X$ with $\mathcal{O}(1)$. Then by the discussion above, non-abelian Theta functions of level k can be identified with homogenous polynomials of degree k in four variables, $V_X^{(k)} \cong \mathbb{C}[z_0, \ldots, z_3]_k$ where $[z_0, \ldots, z_3]$ are chosen coordinates on $\mathbb{P}^3$. This in particular gives an easy representation of the Verlinde formula

$$\dim V_X^{(k)} = \binom{k+3}{3}.$$

We have that

$$H^0(\mathcal{M}_X, \mathcal{L}_X^k) \cong S^k(V) \tag{2.3}$$

4.2. Kummer Quartic Surface. We will now consider a special locus in $\mathcal{M}_X$ which will play an important role. Given X of genus $g = 2$, consider its Jacobian or equivalently $\mathrm{Pic}^0(X)$. One can define the following map [**NR69**]

$$f\colon\ \mathrm{Pic}^0(X) \to \mathcal{M}_X$$

where $f(L) := [L \oplus L^{-1}]$. If we interpret $\mathcal{M}_X$ as $|2\Theta_X|$ under (2.1), then we have an equivalent map

$$g\colon\ \mathrm{Pic}^0(X) \to |2\Theta_X|$$

given by $g(L) := L\Theta_X \cup L^{-1}\Theta_X \subset \mathrm{Pic}^1(X)$. $\mathrm{Pic}^0(X)$ acts on $\mathrm{Pic}^1(X)$ by multiplication and $L\Theta_X := \{L \otimes \xi | \xi \in \Theta_X\}$, thus $g(L) \in |2\Theta_X|$. It is obvious that these maps commute with (2.1) and we get $g(L) = \Delta(f(L))$. Recall that we denoted by $\hat{\Theta}_X \subset \mathrm{Pic}^0(X)$ some translate of the canonical Theta divisor. We know that the line bundle $\mathcal{O}_{\mathrm{Pic}^0(X)}(2\hat{\Theta}_X)$corresponding to the divisor $2\hat{\Theta}_X$ is ample and its sections give the natural map

$$h\colon \mathrm{Pic}^0 \to \mathbb{P}H^0(\mathrm{Pic}^0, \mathcal{O}_{\mathrm{Pic}^0(X)}(2\hat{\Theta}_X))^* := |2\hat{\Theta}_X|^*.$$

By theorem 4.1 we know that $|2\hat{\Theta}_X|^* \cong |2\Theta_X|$, and under this identification f can be realised as the natural map h. We thus have three different interpretations of $\mathbb{P}^3$, and three different maps of the abelian surface $\mathrm{Pic}^0 \to \mathbb{P}^3$ which can all be identified. In fact, the equivalence of f with h implies that $f^*\mathcal{L}_X \cong \mathcal{O}_{\mathrm{Pic}^0(X)}(2\hat{\Theta}_X)$.

The Kummer Surface $\mathcal{K}_X$ is defined to be the quotient $\mathrm{Pic}^0(X)/i$ where $i(L) = L^{-1}$ is an involution. Notice that L is a fixed point of the involution if and only if $L^2 = \mathcal{O}_X$, i.e. if $L \in \mathrm{Pic}_X[2]$ is a 2-torsion point. The group $\mathrm{Pic}_X[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$, thus the number of 2-torsion points for $g = 2$ is 16, which are precisely the singularities of $\mathcal{K}_X$. The map f factors over the Kummer surface and gives a closed immersion

$$\mathcal{K}_X \hookrightarrow \mathcal{M}_X.$$

The image of $\mathcal{K}_X$ is precisely the locus of strictly semi stable bundles.

Taking the tensor product of any $[E] \in \mathcal{M}_X$ with $L \in \mathrm{Pic}_X[2]$ does not effect the semi stability or the determinant and gives us $[E \otimes L] \in \mathcal{M}_X$. This gives a well defined action $\mathrm{Pic}_X[2] \times \mathcal{M}_X \to \mathcal{M}_X$. Moreover, we of course have $\mathrm{Pic}_X[2] \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ and the map f is equivariant with respect to these two actions. We also have the action $\mathrm{Pic}_X[2] \times \mathrm{Pic}^1(X) \to \mathrm{Pic}^1(X)$ by tensor product, this induces an action $\mathrm{Pic}_X[2] \times |2\Theta_X| \to |2\Theta_X|$ and the map g is also equivariant with respect to this action. Lastly, the map (2.1) is also equivariant with respect to the action of $\mathrm{Pic}_X[2]$ on its domain and codomain.

4.3. Heisenberg Group. . Now that we have an action of $\mathrm{Pic}_X[2]$ on our moduli space $\mathcal{M}_X$, we would like to lift it to the line bundle $\mathcal{L}_X$ and get an action on the space of its sections $H^0(\mathcal{M}_X, \mathcal{L}_X)$. If we are able to do that, we will get an action on non-abelian Theta functions of all levels since $V_X^{(k)} = S^k(V_X^1)$ where $S^{(k)}$ denotes the k^{th} symmetric power of a vector space. While the $\mathrm{Pic}_X[2]$ does not lift, there exits a central extension

$$1 \to \mathbb{C}^* \to \mathcal{G}_X \to \mathrm{Pic}_X[2] \to 0,$$

which has a unique action $\mathcal{G}_X \times V_{1,X} \to V_{1,X}$. In fact the group $\mathcal{G}_X$ can be defined as the group of all lifts of actions of elements of $\mathrm{Pic}_X[2]$ to the line bundle $\mathcal{L}$. The group $\mathcal{G}_X$ is called the Heisenberg group. This group also acts in a unique way on the sections of the line bundle $f^*\mathcal{L}_X \cong \mathcal{O}_{\mathrm{Pic}^0(X)}(2\hat{\Theta}_X)$ and lifts the action of $\mathrm{Pic}_X[2]$

on Pic_X^0. The Heisenberg group has a finite sub group $G(X)$ which is the extension of $\mathrm{Pic}_X[2]$ by the 4^{th} roots of unity, μ_4,

$$1 \to \mu_4 \to G(X) \to \mathrm{Pic}_X[2] \to 0.$$

This group is also referred to as the finite Heisenberg group and is actually generated by all the involutions in $\mathcal{G}_X$. As a set $G(X) := \mu_4 \times (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$. The group structure, $G(X) \times G(X) \to G(X)$, is given as follows,

$$(t, a, a')(s, b, b') \mapsto (ts(-1)^{ab'}, a + b, a' + b'),$$

where $ab' := \Sigma_{i=1}^{g} a_i b_i$. The group $G(X)$ has 64 elements, and it is non-abelian with centre $ZG(X) = \{(t, 0, 0)\}$. We are interested in the group of automorphisms of $G(X)$ which is defined as

$$A(G) := \{\alpha \in \mathrm{Aut}(G(X)) | \alpha_{|\mu_4} = \mathrm{Id}\}. \tag{2.4}$$

From [**AM99**] we have the short exact sequence

$$0 \to \mathrm{Pic}_X[2]^* \to A(G) \to \mathrm{Sp}(\mathrm{Pic}_X[2]) \to 0, \tag{2.5}$$

where $\mathrm{Sp}(\mathrm{Pic}_X[2])$ are all automorphisms of $\mathrm{Pic}_X[2]$ which preserve the Weil pairing and $\mathrm{Pic}_X[2]^* = \mathrm{Hom}(\mathrm{Pic}_X[2], \mu_4)$.

In more concrete terms, $A(G)$ can be defined as follows,

$$A(G) := \{\phi \in \mathrm{Aut}\, G(X) | \phi(t, 0, 0) = (t, 0, 0), \forall t \in \mu_4\}. \tag{2.6}$$

To understand the structure of this group better, define a symplectic form E on $(\mathbb{Z}/2\mathbb{Z})^4 \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$, by $E(x, y) = ab' + a'b$, where $x = (a, a')$ and $y = (b, b')$. Then the action of each $\phi \in A(G)$ on $(t, x) \in \mu_4 \times (\mathbb{Z}/2\mathbb{Z})^4$ can be written as

$$\phi(t, x) = (h_\phi(x)t, M_\phi(x))$$

where $h_\phi \colon (\mathbb{Z}/2\mathbb{Z})^4 \to \mathbb{C}^*$, and $M_\phi \in \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^4, E) = Sp(4, \mathbb{Z}/2\mathbb{Z})$. This gives us a group homomorphism $M \colon A(G) \to Sp(4, (\mathbb{Z}/2\mathbb{Z}))$ by $M(\phi) = M_\phi$. The kernel of this morphism is precisely $G(X)/ZG(X) \cong (\mathbb{Z}/2\mathbb{Z})^4$. This leads to the following short exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})^4 \to A(G) \to Sp(4, \mathbb{Z}/2\mathbb{Z}) \to 0 \tag{2.7}$$

which will play a crucial role in the construction of the Hitchin connection.

4.4. The Hitchin connection. In previous sections we fixed X, a compact Riemann surface of genus two, and studied the properties of the moduli space $\mathcal{M}_X$ of semi-stable holomorphic vector bundles of rank two with trivial determinant on X. The most notable feature was the isomorphism constructed in [**NR69**] between $\mathcal{M}_X$ and $\mathbb{CP}^3$, see (2.2). In this section, we will consider a family of genus two compact Riemann surfaces and study the fiber bundle over this family whose fiber is isomorphic to $\mathcal{M}_X$. We will then proceed to give the explicit expression of the Hitchin connection, obtained in [**vGdJ98**], associated with this family.

Let

$$\mathcal{C} := \{(z_1, ..., z_6) \in \mathbb{C}^6 | z_i \neq z_j \text{ for any } i, j \in 1, , 6\},$$

then $\mathcal{C}$ is the space of ordered configurations of six points on $\mathbb{C}$. Consider the family $\mathcal{X} \to \mathcal{C}$ of genus two compact Riemann surfaces where the fiber X_z over every $z \in \mathcal{C}$ is the (canonical completion) of the curve defined by the equation

$$y^2 = \prod_i^6 (x - z_i)$$

in the affine plane.

Associated to every X_z is its moduli space $\mathcal{M}_{X_z}$, which we can glue together to get a fiber bundle $\mathcal{M} \to \mathcal{C}$. By the theorem of [**NR69**], see (2.2), we get that $\mathcal{M} \to \mathcal{C}$ is a (non-trivial) $\mathbb{CP}^3$ bundle. Since the determinant line bundle of $\mathcal{M}_{X_z}$ can be identified with $\mathcal{O}(1)$ for every $z \in \mathcal{C}$, this construction also gives a line bundle $\mathcal{L} \to \mathcal{M}$ such that $\mathcal{L}_{|\mathcal{M}_{X_z}} \cong \mathcal{L}_{X_z} \cong \mathcal{O}(1)$. The vector spaces $H^0(\mathcal{M}_{X_z}, \mathcal{L}^k_{X_z})$ glue together to give the vector bundle

$$\mathbb{V}^{(k)} \to \mathcal{C} \tag{2.8}$$

for all positive integers k. Since $H^0(\mathcal{M}_{X_z}, \mathcal{L}^k_{X_z}) \cong S^k(V)$, see (2.3), where $S^k(V)$ is the vector space of homogenous polynomials of degree k in four variables, we have that the fibers of (2.8) are isomorphic to $S^k(V)$. There exists a covering $\widetilde{P}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$, such that the pull back of (2.8) to $\widetilde{\mathcal{C}}$ trivializes, see [**vGdJ98**]. We will discuss this covering since it is central to the construction of the Hitchin connection.

The symmetric group S_6 acts on $\mathcal{C}$ as

$$(z_1, \ldots, z_6) \mapsto (z_{\sigma(1)}, \ldots, z_{\sigma(6)}) \qquad \text{for all } \sigma \in S_6.$$

The resulting quotient $\mathcal{C}/S_6$ is denoted by $\overline{\mathcal{C}}$ and we get the short exact sequence

$$1 \to \pi_1(\mathcal{C}) \to \pi_1(\overline{\mathcal{C}}) \to S_6 \to 1. \tag{2.9}$$

There is a well known isomorphism $S_6 \cong \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$, see [**CF96**], thus the deck group S_6 appears as the last nontrivial group in the short exact sequence (2.7).

The group $\pi_1(\overline{\mathcal{C}})$ is isomorphic to the Braid group, B_6, of six points in $\mathbb{C}$, see [**Bir75**]. Let $T_1, \ldots, T_5$ be the standard generators of B_6. Then there exists a group homomorphism

$$H\colon B_6 \to \mathrm{Sp}(4,\mathbb{Z}) \tag{2.10}$$

which is explicitly given as, see [**CB88**],

$$T_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.11}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$T_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The group $A(G)$, see (2.12), can be given as a qoutient group of the symplectic group, i.e.

$$A(G) \cong \mathrm{Sp}(4,\mathbb{Z})/\Gamma_2(2,4)$$

where

(2.12)

$$\Gamma_2(2,4) := \left\{ \begin{pmatrix} I+2A & 2B \\ 2C & I+2D \end{pmatrix} \in Sp(4,\mathbb{Z}) | \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv (0,0) \pmod 2 \right\}.$$

This isomorphism is given in [**vGdJ98**]. Since $A(G)$ is a quotient group of $\mathrm{Sp}(4,\mathbb{Z})$, the map (2.10) induces the group homomorphism

$$\overline{H}\colon B_6 \to A(G).$$

The kernel of $\overline{H}$ defines a covering $Q\colon \widetilde{\mathcal{C}} \to \overline{\mathcal{C}}$ and gives the short exact sequence

$$1 \to \pi_1(\tilde{\mathcal{C}}) \to \pi_1(\overline{\mathcal{C}}) \to A(G) \to 1. \tag{2.13}$$

We now have two coverings $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ of $\overline{\mathcal{C}}$, with deck groups $A(G)$ and $S_6 \cong \mathrm{Sp}(4,\mathbb{Z})$. The two deck groups are related by the following short exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})^4 \to A(G) \to S_6 \to 0$$

which is just a rewriting of (2.7). This implies that we have a covering $\widetilde{P}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ and the short exact sequence

$$1 \to \pi_1(\widetilde{\mathcal{C}}) \to \pi_1(\mathcal{C}) \to (\mathbb{Z}/2\mathbb{Z})^4 \to 1.$$

In total, there is a diagram of covering spaces

$$\widetilde{\mathcal{C}} \xrightarrow{\widetilde{P}} \mathcal{C} \xrightarrow{\mathrm{P}} \overline{\mathcal{C}} \tag{2.14}$$

with the following short exact sequence of covering groups

$$(\mathbb{Z}/2\mathbb{Z})^4 \to A(G) \to S_6. \tag{2.15}$$

The following result is from [**vGdJ98**]

Lemma 4.2. *The pull back of the vector bundle $\mathbb{V}^{(k)} \to \mathcal{C}$ to the cover $\widetilde{P}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is trivial for all k, that is,*

$$\widetilde{P}^*(\mathbb{V}^{(k)}) \cong (S^k(V)) \times \widetilde{\mathcal{C}}.$$

Proof. It suffices to show the result for $k = 1$ since $\mathbb{V}^{(k)} = S^k(V)$. The argument for $k = 1$ goes as follows. Associated to the family $\mathcal{M} \to \mathcal{C}$ is the family of groups $\mathcal{G} \to \mathcal{C}$ with fiber over every $z \in \mathcal{C}$ the Heisenberg group $\mathcal{G}_{X_z}$, which, recall, is a central extension of $\mathrm{Pic}_{X_z}[2]$. Now, from the previous section we know that there exists a unique irreducible representation (the Schroedinger representation) $\mathcal{G}_{X_z} \times V^{(k)}_{X_z} \to V^{(k)}_{X_z}$. By the theory of Theta functions, we know that the pull back of $\mathcal{G}$ to $\tilde{\mathcal{C}}$ trivializes, i.e. $p^*\mathcal{G} \cong \mathcal{G}_{X_z} \times \tilde{\mathcal{C}}$, where $\mathcal{G}_{X_z}$ is a fixed group which has a unique action on every fiber of $p^*\mathcal{V}^{(1)} \cong V$. Application of Schur's lemma now gives us a canonical isomorphism between the fibers (as representations of $\mathcal{G}_{X_z}$). This gives a trivialization of $p^*\mathcal{V}^{(k)}$. $\square$

The authors in [**vGdJ98**] construct a connection in the trivial bundle $\widetilde{P}^*(\mathcal{V}^{(k)})$ as follows. For $1 \leqslant i < j \leqslant 6$, consider the following $F^{i,j} \in \mathrm{Aut}(\mathbb{C}^4)$ given in [**GTNB00**].

$$
\begin{array}{l}
+i\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\quad -i\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\quad +1\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix},\quad -1\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix},\quad +i\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\\
-1\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix},\quad -i\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\quad +i\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix},\quad +1\begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{bmatrix}\\
-i\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix},\quad -i\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}, -1\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix},\\
+1\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\quad -i\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{bmatrix}\\
+i\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}
\end{array}
$$

Here the first row represents $F^{1,2}, \dots F^{1,6}$, second row $F^{2,3} \dots F^{2,6}$, and so on. Notice that all these matrices are traceless. In fact, these matrices give a basis for $sl(4)$ and define its standard representation. Under the well known isomorphism $sl(4) \cong so(6)$, these matrices give the half spin representation of $so(6)$. Define the following elements

$$\Omega^{i,j} := F^{i,j} \otimes F^{i,j}, \qquad \in U(sl(4)).$$

Any representation $\rho\colon sl(4) \to \operatorname{End}(\mathbb{C}^4)$ lifts to a representation of the associative algebras $\tilde{\rho}\colon U(sl(4)) \to \operatorname{End}(\mathbb{C}^4)$, which thus gives us an action $\tilde{\rho}(\Omega^{i,j}) \in \operatorname{End}(\mathbb{C}^4)$. We note here the following relations

$$[\Omega^{i,j}, \Omega^{k,l}] = 0, \qquad [\Omega^{i,k}, \Omega^{i,j} + \Omega^{j,k}] = 0 \tag{2.16}$$

where i, j, k, l are distinct indices. We define $\Omega^{i,j}_k := S^k(\tilde{\rho}(\Omega^{i,j})) \in \operatorname{End} S^k(\mathbb{C}^4)$ for all $k > 0$.

Remark 4.3. The intuitive reason why $sl(4)$ shows up is that $\mathcal{M}_X \cong \mathbb{CP}^3$, and $\mathbb{P}\mathrm{SL}(4, \mathbb{C})$, which is a subgroup of finite index of $\mathrm{SL}(4, \mathbb{C})$, is the group of automorphisms of our moduli space. The representation of $sl(4)$ just defined above, is a lift of the automorphisms of $\mathbb{CP}^3$ to the line bundle $\mathcal{O}(1)$.

The $\Omega^{i,j}_k$ can be considered as second order differential operators acting on $S^k(V)$ for any k as follows. For all $1 \leqslant i < j \leqslant 6$ consider the second order differential operator

$$\rho_d(\Omega^{i,j}_1) := \left(\sum a_{kl} x_k \partial_l\right)^2 \tag{2.17}$$

where $F^{i,j} = (a_{kl})$ for $1 \leqslant k, l \leqslant 4$ and $\langle x_1, \dots, x_4 \rangle$ is a basis for V. Then $\rho_d(\Omega^{i,j}) \in \operatorname{End}(S)$ where S is the space of *all* polynomials in the four variables $x_1, \dots, x_4$. The composition law for these operators as endomorphisms of S is

$$(x_i\partial_j)\circ(x_k\partial_l) = x_i x_k \partial_j\partial_l + \delta_{jk}x_i\partial_l \tag{2.18}$$

which implies that there will be some first order terms in (2.17). Let $\sigma\rho_d(\Omega_1^{i,j})$ be the symbol of $\rho_d(\Omega_1^{i,j})$, i.e. $\sigma\rho_d(\Omega_1^{i,j})$ only contains the second order terms of $\rho_d(\Omega_1^{i,j})$. Define

$$\widehat{\Omega}^{i,j} := \sigma\rho_d(\Omega_1^{i,j}) \in \mathrm{End}(V). \tag{2.19}$$

For an integer $k>0$, let

$$\hbar := \frac{-1}{16(k+2)} \tag{2.20}$$

and define

$$\widehat{\Omega}_k^{i,j} := \hbar(\sigma\rho_d(\Omega_1^{i,j})) = \hbar\widehat{\Omega}^{i,j} \ \in \ \mathrm{End}(S^k(V)). \tag{2.21}$$

It is clear that $\widehat{\Omega}_k^{i,j}$ preserve the subspace $S^k(V)\subset S$ of homogenuous polynomials, and it is claimed in [**vGdJ98**] that $\widehat{\Omega}_k^{i,j}$ satisfy (2.16). For an integer $k>0$, consider the $\mathrm{End}(S^k(V))$ valued holomorphic 1-form on $\mathcal{C}$,

$$\omega^{(k)} := \sum_{1\leqslant i<j\leqslant 6}\widehat{\Omega}_k^{i,j}\frac{dz_i-dz_j}{z_i-z_j} \tag{2.22}$$

$$= \hbar\sum_{1\leqslant i<j\leqslant 6}\widehat{\Omega}^{i,j}\frac{dz_i-dz_j}{z_i-z_j} \tag{2.23}$$

and denote by $\widetilde{\omega}^{(k)} := \widetilde{P}^*(\omega^{(k)})$ the pull to $\widetilde{\mathcal{C}}$. Then $\widetilde{\omega}^{(k)}$ is an $\mathrm{End}(S^k(V))$ valued 1-form, and the authors in [**vGdJ98**] define a connection

$$\widetilde{\boldsymbol{\nabla}}^{(k)} := d+\widetilde{\omega}^{(k)} \tag{2.24}$$

in the trivial bundle $S^k(V)\times\widetilde{\mathcal{C}}$. That this connection is flat follows from (2.16).

We note the following relations, reminiscent of the Ward identities in two dimensional conformal field theory, see [**Koh02**], which will be used extensively later.

PROPOSITION 4.4. *For any integer $k>1$ we have that*
1) $\sum_{1\leqslant i<j\leqslant 6}\widehat{\Omega}_k^{i,j} = 3\hbar k^2\,\mathrm{Id}$
2) for all $1\leqslant j\leqslant 6$, $\sum_{\substack{i=1\\ i\neq j}}^{6}\widehat{\Omega}_k^{i,j} = \hbar k^2\,\mathrm{Id}$
3) $\sum_{1\leqslant i<j\leqslant 6}\widehat{\Omega}_k^{i,j}(z_i+z_j) = 6\hbar k^2\,\mathrm{Id}$

PROOF. Direct computations, notice that (2) implies (3). □

There exists a unique irreducible representation $U\colon G(X) \to \mathrm{Aut}(V)$ such that the centre acts by multiplication.This representation is called the *Schrödinger* representation [**vGdJ98**], and it has a natural extension to higher symmetric powers. Recall $A(G)$ is a certain subgroup of automorphisms of $G(X)$. For any $\phi \in A(G)$, we have that $U \circ \phi\colon G(X) \to \mathrm{Aut}(V)$ is still an irreducible representation. By Shur's lemma, for each $\phi \in A(G)$ we get a linear map $T_\phi\colon V \to V$, unique upto scaler multiplication, such that $T_\phi U(h) = U(\phi(h))T_\phi$. This gives us the induced representation on the projective automorphisms of V

$$T\colon A(G) \to PGL(V), \quad \phi \mapsto T_\phi, \tag{2.25}$$

which has a natural extension to projective automorphisms of higher symmetric powers of V. Since $A(G)$ acts on $\widetilde{\mathcal{C}}$ as the deck group of the covering $\widetilde{\mathcal{C}} \to \overline{\mathcal{C}}$ and acts on $\mathbb{P}S^k(V)$ by (2.25) we get a quotient bundle

$$\mathbb{P}S^k(V) \times_{A(G)} \widetilde{\mathcal{C}} \to \overline{\mathcal{C}}. \tag{2.26}$$

Let

$$\mathrm{End}_0(S^k(V)) := \mathrm{End}(S^k(V))/(\mathbb{C}^* \cdot \mathrm{Id})). \tag{2.27}$$

Two $\mathrm{End}(S^k(V))$ valued 1-forms which are equivalent as $\mathrm{End}_0(S^k(V))$ valued 1-forms are called projectively equivalent. In the next section, we give an equivalent definition of projective equivalence for $\mathrm{End}(S^k(V))$ valued 1-forms on $\mathcal{C}$ in terms of the coordinate functions. In [**vGdJ98**], the authors show that the 1-form $\widetilde{\omega}^{(k)}$, and thus the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}$, is projectively invariant under the action of $A(G)$. Projective invariance implies that the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}$ descends to a flat connection in (2.26) which we will denote by $\overline{\boldsymbol{\nabla}}^{(k)}$.

We wish to linearize the action 2.25.

Definition 4.5. *Let*

$$\widetilde{A(G)} := \{T\colon \mathbb{C}^4 \to \mathbb{C}^4 | \exists \phi \in A(G) \ s.t\ T\,U(h) = U(\phi(h))T\}. \tag{2.28}$$

Then by definition we get the following short exact sequence

$$1 \to \mathbb{C}^* \to \widetilde{A(G)} \to A(G) \to 1$$

which shows that $\widetilde{A(G)}$ is a central extension. This central extension is related to the universal central extension of the mapping class group. That is, we have the mapping class group Γ_2 of genus 2 surface, which has a universal central extension

$$1 \to \mathbb{Z} \to \widetilde{\Gamma_2} \to \Gamma_2 \to 1,$$

see for example [**MR12**]. Since $A(G)$ is isomorphic to a quotient group of $Sp(4,\mathbb{Z})$, the representation of Γ_2 on homology gives a surjective map $\Gamma_2 \to A(G)$. By the universal property, we get an induced map $\tilde{\Gamma_2} \to \tilde{A(G)}$, in fact we get a map between short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \tilde{\Gamma_2} \arrow[r] \arrow[d] & \Gamma_2 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & \tilde{A(G)} \arrow[r] & \mathrm{A(G)} \arrow[r] & 1 \end{tikzcd} \tag{2.29}$$

The main property of $\tilde{A(G)}$ is that it lifts the projective action 2.25 to a linear action

$$\tilde{T}\colon \tilde{A(G)} \to GL(\mathbb{C}^4). \tag{2.30}$$

We also define a finite version of $\tilde{A(G)}$.

Definition 4.6. *Let*

$$\tilde{A(G)}_F := \{T\colon \mathbb{C}^4 \to \mathbb{C}^4 | \exists \phi \in A(G)\ s.t\ T\,U(h) = U(\phi(h))T \text{ and } T^{order(\phi)} = Id\}.$$

In this case we also have a short exact sequence

$$1 \to F \to \tilde{A(G)}_F \to A(G) \to 1$$

where F is a finite group.

4.5. Invariance and Extension of the Hitchin Connection. In this section we show that the Hitchin connection constructed in [**vGdJ98**] on the configuration space $\overline{\mathcal{C}}$, this construction is recalled in the previous section, descends to a (projectively) flat connection on the moduli space $\mathcal{M}_2$ of genus two compact Riemann surfaces.

Recall the configuration space $\mathcal{C}$. We introduce the following equivalence relation on this set:

Definition 4.7. *Let* $\mathbf{z}$ *and* $\mathbf{z}'$ *be in* $\mathcal{C}$*, then* $\mathbf{z} \sim \mathbf{z}'$ *if and only if there exists a Möbius transformation*

$$F(z) = \frac{az+b}{cz+d} \quad \text{where } ad-bc=1 \quad \text{and } z \in \mathbb{C} \tag{2.31}$$

such that $F(\mathbf{z}) = \mathbf{z}'$*, here* F *acts on* $\mathcal{C}$ *coordinate wise.*

Lemma 4.8. *The quotient space* $C := \mathcal{C}/_\sim$ *has an action of the symmetric group* S_6*, and* C/S_6 *is isomorphic to the moduli space* $\mathcal{M}_2$*.*

PROOF. Let $\mathcal{C}_\infty$ be the configuration space of six points on $\mathbb{CP}^1$. Then the product group $(\mathbb{P}\mathrm{SL}(2,\mathbb{C}) \times S_6)$ acts on $\mathcal{C}_\infty$ and the quotient is $\mathcal{M}_2$. This gives us a map $K \colon \mathcal{C}_\infty \to \mathcal{M}_2$ and, see [**vGdJ98**], the restriction of this map to the open subset $\mathcal{C}$ of $\mathcal{C}_\infty$ is surjective. This implies that $\mathcal{C}$ contains at least one pre image of every $X \in \mathcal{M}_2$ under the map K. The pre images of K are orbits of Möbious transformations, this implies that C/S_6 is in one to one correspondence with $\mathcal{M}_2$. $\square$

DEFINITION 4.9. *Let $k > 0$ be an integer. Let A and B be holomorphic 1-forms with values in* $\mathrm{End}(S^k(V))$*, then A and B are projectively equivalent if $A - B$ is trivial as a holomorphic 1-form with values in* $\mathrm{End}_0(S^k(V))$*, see* (2.27)*.*

In the sequel, we will deal with 1-forms on $\mathcal{C}$, say A and B, with values in $\mathrm{End}(S^k(V)$ such that

$$A - B = (k^2\,\mathrm{Id}) f \sum_{i=1}^{6} dz_i$$

where f is a $\mathbb{C}^*$ valued function. By definition 4.9 it is clear that A and B are projectively equivalent.

Suppose $\boldsymbol{\nabla}_A := d + A$ and $\boldsymbol{\nabla}_B := d + B$ gave flat connections in the trivial bundle $V \times \mathcal{C}$, and that A and B were projectively equivalent. Then the holonomy of the two connections $\boldsymbol{\nabla}_A$ and $\boldsymbol{\nabla}_B$ would differ by a projective factor only. In fact, if we consider $V \times \mathcal{C}$ as an associated bundle of a principal $GL(n,\mathbb{C})$ bundle, and $\boldsymbol{\nabla}_A$ and $\boldsymbol{\nabla}_B$ as principal flat connections, then the reduction of the structure group to $\mathbb{P}GL(n,\mathbb{C})$ gives an associated bundle isomorphic to $\mathbb{P}(V) \times \mathcal{C}$ with induced flat connections $\bar{\boldsymbol{\nabla}}_A$ and $\bar{\boldsymbol{\nabla}}_B$. If A and B are projectively equivalent, then the two induced connections $\bar{\boldsymbol{\nabla}}_A$ and $\bar{\boldsymbol{\nabla}}_B$ are isomorphic. Also, if A is projectively equivalent to B, then they are isomorphic as $\mathrm{End}_0(S^k(V))$ valued 1-forms, where recall $\mathrm{End}_0(S^k(V))$ is the quotient of the lie algebra $\mathrm{End}(S^k(V))$ by its center.

Recall the $\mathrm{End}(S^k(V))$ valued holomorphic 1-form $\omega^{(k)}$ on $\mathcal{C}$ defined in the (2.22). We prove a certain invariance of this form under Möbious transformations.

PROPOSITION 4.10. *Let $\mathbf{z}$ and $\mathbf{z}'$ be in $\mathcal{C}$ with small open neighbourhoods $N(\mathbf{z})$ and $N(\mathbf{z}')$ respectively. Suppose F was a Möbious transformation such that $F(N(\mathbf{z})) =$* $\mathbf{N}(\mathbf{z}')$*. Then $F^*(\omega^{(k)}_{|N(\mathbf{z}')})$ is projectively equivalent to $\omega^{(k)}_{|N(\mathbf{z})}$.*

Proof. Since F is a Möbious transformation we know that it has a representation of the form (2.31). This implies that

$$\begin{aligned}
F^*(\omega^{(k)}_{|N(\mathbf{z}')}) &= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j} d\log\left(\frac{az_i+b}{cz_i+d} - \frac{az_j+b}{cz_j+d}\right) \\
&= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j} \Big(d\log(z_i - z_j) - d\log(cz_i+d) - d\log(cz_j+d) \Big) \\
&= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j} \frac{dz_i - dz_j}{z_i - z_j} - \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j} \left(\frac{cdz_i}{cz_i+d} + \frac{cdz_j}{cz_j+d} \right) \\
&= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j} \frac{dz_i - dz_j}{z_i - z_j} - 6k^2\hbar \,\mathrm{Id} \sum_{i=1}^{6} \frac{cdz_i}{cz_i+d}
\end{aligned}$$

where the last equality follows from (3) of proposition 4.4. This gives the proposition since

$$F^*(\omega^{(k)}_{|N(\mathbf{z}')}) - \omega^{(k)}_{|N(\mathbf{z})} = 6k^2\hbar \,\mathrm{Id} \sum_{i=1}^{6} \frac{cdz_i}{cz_i+d}.$$

□

Recall the covering $\widetilde{P}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ with the deck group $A(G)$, see (2.14). The equivalence relation in definition 4.7 is a local statement and thus lifts to an equivalence relation on $\widetilde{\mathcal{C}}$. Equivalently, we can say $\tilde{\mathbf{z}} \sim \tilde{\mathbf{z}}'$ if and only if $\widetilde{P}(\tilde{\mathbf{z}}) \sim \widetilde{\mathbf{P}}(\tilde{\mathbf{z}}')$. Let $\widetilde{C} := \widetilde{\mathcal{C}}/\sim$ be the quotient space, this has an action of $A(G)$ on it. A corollary of lemma 4.8 is

$$\widetilde{C}/A(G) \cong \mathcal{M}_2.$$

Recall the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}$, see (2.24), defined by the 1-form $\widetilde{P}^*(\omega^{(k)}) := \widetilde{\omega}^{(k)}$ in the trivial bundle $S^k(V) \times \widetilde{\mathcal{C}}$.

Definition 4.11. *Let $(v, \tilde{\mathbf{z}})$ and $(v', \tilde{\mathbf{z}}')$ be in $\mathbb{P}S^k(V) \times \widetilde{\mathcal{C}}$. Then $(v, \tilde{\mathbf{z}}) \sim (v', \tilde{\mathbf{z}}')$ if and only if $\tilde{\mathbf{z}} \sim \tilde{\mathbf{z}}'$ and $\widetilde{\omega}^{(k)}(\tilde{\mathbf{z}}')(v') = \widetilde{\omega}^{(k)}(\tilde{\mathbf{z}})(v)$.*

Due to proposition 4.10 we know that $\widetilde{\omega}^{(k)}$ is projectively invariant under (the lift of) Möbious transformations. This implies that $(v, \tilde{\mathbf{z}}) \sim (v', \tilde{\mathbf{z}}')$ if and only if $\tilde{\mathbf{z}} \sim \tilde{\mathbf{z}}'$ and $v = v'$. This gives that

$$(\mathbb{P}S^k(V) \times \widetilde{\mathcal{C}})/_\sim = \mathbb{P}S^k(V) \times \widetilde{C}, \tag{2.32}$$

and the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}$ restricts to a well defined connection in (2.32). We can now construct a quotient vector bundle

$$\mathbb{P}S^k(V) \times_{A(G)} \widetilde{C} \tag{2.33}$$

where $A(G)$ acts on $\mathbb{P}S^k(V)$ by (2.25). In [**vGdJ98**] it is shown that $\widetilde{\boldsymbol{\nabla}}^{(k)}$ is projectively invariant under the action of $A(G)$, and thus descends to a flat connection $\widehat{\boldsymbol{\nabla}}^{(k)}$ in the bundle (2.33). We denote this vector bundle with a flat connection by

$$(\mathbb{P}\mathcal{V}^{(k)}, \widehat{\boldsymbol{\nabla}}^{(k)}) \to \mathcal{M}_2. \tag{2.34}$$

Rest of this section is not directly related to the main results of this thesis. Let

$$\mathcal{C}_\infty := \{(z_1, \dots, z_6) \in (\mathbb{CP}^1)^6 | z_i \neq z_j \text{ for all } i \neq j\},$$

this is the space of ordered configurations of six points on $\mathbb{CP}^1$. Notice that $\mathcal{C}$ is contained in $\mathcal{C}_\infty$. We wish to describe an extension of $\omega^{(k)}$ to the entire space $\mathcal{C}_\infty$. It turns out that $\omega^{(k)}$ does admit an extension, but as a meromorphic 1-form with logarithmic singularities over certain subspaces of $\mathcal{C}_\infty$.

We now lay the ground work and recall logarithmic 1-forms. We describe $\mathcal{C}_\infty$ as a disjoint union. For all $1 \leqslant m \leqslant 6$, let

$$D_m^\infty := \{(z_1, \dots, \infty, \dots, z_6) \in \mathcal{C}_\infty \ \mid \ z_m = \infty\}. \tag{2.35}$$

Notice that D_m^∞ is a divisor in $\mathcal{C}_\infty$, and isomorphic to the space of ordered configurations of five points on $\mathbb{C}$. The space $\mathcal{C}_\infty$ can now be written as a disjoint union,

$$\mathcal{C}_\infty = \mathcal{C} \sqcup_{i=1}^6 D_m^\infty.$$

For $1 \leqslant l \neq m \leqslant 6$, let

$$D_{m,l}^\infty := \{(z_1, \dots, z_6) \in D_m^\infty \ \mid \ z_l = 0\}. \tag{2.36}$$

Notice that $D_{m,l}^\infty$ is a divisor in D_m^∞ and a co-dimension two sub-manifold of $\mathcal{C}_\infty$. For $1 \leqslant m \leqslant 6$ let

$$\mathcal{D}_m^\infty := D_m^\infty - \bigsqcup_{\substack{l=1,\\ l\neq m}}^{6} D_{m,l}^\infty. \tag{2.37}$$

Lastly, for $1 \leqslant m \leqslant 6$ let

$$D_m^0 := \{(z_1, \dots, z_6) \in \mathcal{C} \ \mid \ z_m = 0\} \tag{2.38}$$

which is a divisor in $\mathcal{C}$.

Lemma 4.12. *For all $1 \leqslant m \leqslant 6$, let*

$$I_m \colon \mathcal{D}_m^\infty \to D_m^0$$

be the restriction of inversion $I \colon \mathcal{C}^* \to \mathcal{C}$ to $\mathcal{D}_m^\infty$. Then I_m is a diffeomorphism.

PROOF. It is clear that I_m is a bijection with inverse given by the restriction of the involution I to D_m^0. Smoothness follows since inversion is an automorphism of $\mathcal{C}_\infty$. $\square$

Denote by $\omega_m^{(k)}$ the restriction of the 1-form $\omega^{(k)}$ to D_m^0. We have the following.

LEMMA 4.13. *Let* $1 \leqslant m \leqslant 6$. *Then*

$$I_m^*(\omega_m^{(k)}) = \hbar\Bigg(\sum_{\substack{1\leqslant i<j\leqslant 6 \\ i,j\neq m}} \widehat{\Omega}^{i,j} \frac{dz_i - dz_j}{z_i - z_j} + 5k^2 \,\mathrm{Id} \sum_{\substack{i=1 \\ i\neq m}}^{6} \frac{dz_i}{z_i} \Bigg) \tag{2.39}$$

PROOF. Since the coordinate $z_m = 0$ on D_m^0, we have that

$$\omega_m^{(k)} = \hbar\Bigg(\sum_{\substack{1\leqslant i<j\leqslant 6 \\ i,j\neq m}} \widehat{\Omega}^{i,j} \frac{dz_i - dz_j}{z_i - z_j} + \sum_{\substack{i=1 \\ i\neq m}}^{6} \widehat{\Omega}^{i,m} \frac{dz_i}{z_i} \Bigg)$$

Here, in the last sum $\sum_{\substack{i=1 \\ i\neq m}} \widehat{\Omega}^{i,m} \frac{dz_m}{z_m}$, if $i > m$ we replace $\widehat{\Omega}^{i,m}$ with $\widehat{\Omega}^{m,i}$ since $\widehat{\Omega}^{i,m} = \widehat{\Omega}^{m,i}$. Now,

$$I_m^*(\omega_m^{(k)}) = \hbar\Bigg(\sum_{\substack{1\leqslant i<j\leqslant 6 \\ i,j\neq m}} \widehat{\Omega}^{i,j} \; I_m^*\bigg(\frac{dz_i - dz_j}{z_i - z_j}\bigg) + \sum_{\substack{i=1 \\ i\neq m}}^{6} \widehat{\Omega}^{i,m} \; I_m^*\bigg(\frac{dz_i}{z_i}\bigg)\Bigg).$$

We have that

$$\begin{aligned} I_m^*\bigg(\frac{dz_i - dz_j}{z_i - z_j}\bigg) &= I_m^*(d\log(z_i - z_j)) \\ &= d\log(\frac{1}{z_i} - \frac{1}{z_j}) \\ &= d\log(z_i - z_j) - d\log(z_i) - d\log(z_j) \end{aligned}$$

and similarly

$$I_m^*\bigg(\frac{dz_i}{z_i}\bigg) = -d\log(z_i).$$

This gives

$$\begin{aligned} I_m^*(\omega_m^{(k)}) = \frac{-1}{16(k+2)}\Bigg(&\sum_{\substack{1\leqslant i<j\leqslant 6 \\ i,j\neq m}} \widehat{\Omega}^{i,j} \frac{dz_i - dz_j}{z_i - z_j} - \sum_{\substack{1\leqslant i<j\leqslant 6 \\ i,j\neq m}} \widehat{\Omega}^{i,j} d\log(z_i) \\ &- \sum_{\substack{1\leqslant i<j\leqslant 6 \\ i,j\neq m}} \widehat{\Omega}^{i,j} d\log(z_j) - \sum_{\substack{i=1 \\ i\neq m}}^{6} \widehat{\Omega}^{i,m} d\log(z_i)\Bigg) \end{aligned}$$

Now,

$$\sum_{\substack{1\leqslant i<j\leqslant 6\\ i,j\neq m}} \widehat{\Omega}^{i,j} d\log(z_i) + \sum_{\substack{1\leqslant i<j\leqslant 6\\ i,j\neq m}} \widehat{\Omega}^{i,j} d\log(z_j) + \sum_{\substack{i=1\\ i\neq m}}^{6} \widehat{\Omega}^{i,m} d\log(z_i) =$$

$$\sum_{\substack{1\leqslant i<j\leqslant 6\\ i\neq m}} \widehat{\Omega}^{i,j} d\log(z_i) + \sum_{\substack{1\leqslant i<j\leqslant 6\\ j\neq m}} \widehat{\Omega}^{i,j} d\log(z_j) = 5k^2 \operatorname{Id} \sum_{\substack{i=1\\ i\neq m}}^{6} d\log(z_i)$$

Here, the last equality follows from (3) of proposition 4.4. This gives (2.39). □

If we wish to consider the extension of (2.39) as a 1-form on the entire $D_m^\infty = \mathcal{D}_m^\infty \bigsqcup_{\substack{l=1,\\ l\neq m}}^6 D_{m,l}^\infty$, then the term $5k^2 \operatorname{Id} \frac{dz_l}{z_l}$ in (2.39) has a logarithmic singularity along $D_{m,l}$. Meromorphic 1-forms with logarithmic singularities along normal crossing divisors on a complex manifold are well studied, see [**GH94**], [**Voi07a**], and [**Voi07b**], and lie algebra valued meromorphic 1-forms with logarithmic singularities along normal crossing divisors, which leads to logarithmic connections, were first studied in [**Del70**].

We briefly recall this concept for the special case of smooth hypersurfaces. Let M be a complex manifold and let $N \hookrightarrow M$ be a smooth hypersurface. That is, for any $n \in N$ there exists an open subset $U \subset M$ with local holomorphic coordinates $(z_1, \dots, z_k)$ such that

$$N \cap U = \{z_1, \dots, z_n = 0, \dots, z_k\}.$$

Denote by $\Omega_M^p(rN)$, meromorphic forms of degree p with pole of order r along N. That is, $\alpha \in \Omega_M^p(rN)$ if α is holomorphic on $M^* := M - N$, and for any $n \in N$ with neighborhood $U \subset M$ as above, $z_n^r \alpha$ is holomorphic on U.

Denote by $\Omega_M^p(\log N)$ the subspace of $\Omega_M^p(rN)$, where $\alpha \in \Omega_M^p(\log N)$ if for every $n \in N$ with neighborhood $U \subset M$ as above, α can be represented as follows,

$$\alpha = g_1 \frac{dz_{n_1}}{z_{n_1}} \wedge \cdots \wedge g_l \frac{dz_{n_l}}{z_{n_l}} \wedge g_{l+1} dz_{i_1} \wedge \cdots \wedge g_m dz_{i_m}$$

where $l + m = p$, $z_i \neq z_n$, and g_i are holomorphic functions on U. In particular, if $\alpha \in \Omega_M^p(\log N)$ then α has a pole of order one along N. Also, $\Omega_M^1(\log N)_{|U}$ is a vector space over holomorphic functions on U, spanned by $\langle dz_1, \dots, \widehat{dz_n}, \dots, dz_k, \frac{dz_n}{z_n}\rangle$, and

$$\Omega_M^p(\log N)_{|U} = \bigwedge^p \Omega_M^1(\log N)_{|U}.$$

Logarithmic forms with the exterior differential form a complex usually denoted $(\Omega_M^*(\log N), d)$, for general (co)-homological results concerning this complex we refer the reader to aforementioned literature. The residue of a meromorphic function/form also has generalization to the case of higher dimensional complex manifolds. The following result shed lights on the nature of residue, which can be found in [**Voi07a**] and [**GH94**].

LEMMA 4.14. *Let $n \in N$ and $U \subset M$ its neighborhood as above, then any $\alpha \in \Omega^p_M(\log N)_{|U}$ can be represented as*

$$\alpha = \theta \wedge \frac{dz_n}{z_n} + \eta \tag{2.40}$$

where θ and η are holomorphic $p-1$ and p forms on U respectively, and the restriction of θ to $U \cap N$ is unique. Infact, one gets the following Residue map

$$Res\colon \Omega^p_M(\log N) \to \Omega^{p-1}_N$$

where Ω^{p-1}_N denotes holomorphic $p-1$ forms on N.

Since $\alpha \in \Omega^p_M(\log N)_{|U}$ can be represented as (2.40), $Res(\alpha) = \theta$.

Let $E \times M$ be a trivial vector bundle on M, and $\mathrm{End}(E) \times M$ the induced bundle. Let $N \hookrightarrow M$ be a smooth hyper surface. One similarly defines the following vector spaces as before

$$(\Omega^p_M(\log N) \otimes \mathrm{End}(E)).$$

Let $\alpha \in (\Omega^p_M(\log N) \otimes \mathrm{End}(E))$, then around every $n \in N$, α has a representation like (2.40), where now $\theta \in \Omega^p_N \otimes \mathrm{End}(E)$, that is θ is a holomorphic $p-1$ form on N with values in $\mathrm{End}(E)$. Likewise, one gets the following map, see [**Del70**] for details,

$$Res\colon (\Omega^p_M(\log N)) \otimes \mathrm{End}(E) \to (\Omega^{p-1}_N \otimes \mathrm{End}(E)). \tag{2.41}$$

Following is the logarithmic analogue of a flat connection.

DEFINITION 4.15. *Let $\alpha \in (\Omega^1_M(\log N) \otimes \mathrm{End}(E))$, then $\boldsymbol{\nabla} = d + \alpha$ is called a flat logarithmic connection with singularities along N, if $\boldsymbol{\nabla}_{|M^*}$, where $M^* = M - N$, is a flat (holomorphic) connection.*

The residue of the flat logarithmic connection $\boldsymbol{\nabla} = d + \alpha$ is defined to be

$$Res(\alpha) \in \Omega^0_N \otimes \mathrm{End}(E)$$

where Res is the map (2.41)*. Thus, residue of a logarithmic connection is a holomorphic* $\mathrm{End}(E)$ *valued function along the locus of singularity of the connection.*

The following result is essential in the theory of logarithmic flat connections, and shows why such connections are well behaved. The proof of this result can be found on page 79 of [**Del70**].

THEOREM 4.16. *Let $N \hookrightarrow M$ be a smooth hyper surface. If $\boldsymbol{\nabla} = d + \alpha$, for $\alpha \in (\Omega^1_M(\log N) \otimes \mathrm{End}(E))$, is a logarithmic flat connection on M with N as the locus of singularity, then the conjugacy class of $Res(\alpha)$ in $\mathrm{End}(E)$ is constant.*

We can now characterize the 1-form (2.39) as a logarithmic 1-form on the entire divisor D^∞_m.

COROLLARY 4.17. *We have that*

$$I^*_m(\omega^{(k)}_m) \in \Omega^1_{D^\infty_m}\big(\log(\bigsqcup_{\substack{l=1,\\ l\neq m}}^{6} D^\infty_{m,l})\big)$$

that is, the pulled back 1-form $I^*_m(\omega^{(k)}_m)$ extends to D^∞_m as a logarithmic 1-form with singularities along the disjoint union of smooth hyper surfaces $\bigsqcup^6_{\substack{l=1,\\ l\neq m}} D^\infty_{m,l}$. Moreover,

$$Res(I^*_m(\omega^{(k)}_m)) = 5k^2\,\mathrm{Id}\,.$$

PROOF. Recall from (2.37) that $D^\infty_m = \mathcal{D}^\infty_m \bigsqcup^6_{\substack{l=1,\\ l\neq m}} D^\infty_{m,l}$. The 1-form $I^*_m(\omega^{(k)}_m)$ is holomorphic on $\mathcal{D}^\infty_m$, but on each $D^\infty_{m,l}$ the term

$$k^2\,\mathrm{Id}\,\frac{dz_m}{z_m}$$

in $I^*_m(\omega^{(k)}_m)$ has a simple pole with residue $k^2\,\mathrm{Id}$. □

The group of Möbius transformations, which we denote by $\mathbb{M}$, is the group of (complex) automorphisms of $\mathbb{CP}^1$ and thus acts on $\mathcal{C}_\infty$. It is also well known, see proposition C.4 of [**Mas88**], that $\mathbb{M}$ is isomorphic to $\mathbb{PSL}(2,\mathbb{C})$ and generated by the following three one dimensional subgroups; namely
translations

$$T(z) = z + a, \quad a \in \mathbb{C} \tag{2.42}$$

dilations

$$D(z) = az, \quad a \in \mathbb{C}^*$$

and special conformal transformations

$$S(z) = \frac{z}{az+1}, \quad a \in \mathbb{C}$$

where $z \in \mathbb{CP}^1$. We show that $\omega^{(k)}_\infty$ in invariant under the action of $\mathbb{M}$ and that it descends to the quotient space $\mathcal{C}_\infty/(\mathbb{M}\times S_6)$, which is the moduli space of six points on $\mathbb{CP}^1$. We begin with the following definition.

THEOREM 4.18. *The 1-form* $\omega_\infty^{(k)}$ *is projectively invariant under the action of the group of Möbius transformations* $\mathbb{M}$ *on* $\mathcal{C}_\infty$.

PROOF. Recall from (2.42) the three groups T, D, and S which generate $\mathbb{M}$. Let $s(a)$, $t(a)$, and $d(a)$ be smooth paths contained in S, T, and D passing through the identity of the respective groups. We have the action map

$$A\colon \mathbb{M}\times\mathcal{C}_\infty \to \mathcal{C}_\infty.$$

Fix $\mathbf{z}$=$(z_1,\ldots,z_6)\in\mathcal{C}$ and denote by $\gamma_s(a)$, $\gamma_t(a)$, and $\gamma_d(a)$ the image $A(s(a),\mathbf{z})$, $A(t(a),\mathbf{z})$, and $A(d(a),\mathbf{z})$. Then $\gamma_s(a)$ is a path in $\mathcal{C}$ such that $\gamma_s(0)=\mathbf{z}$, and likewise for $\gamma_t(0)$ and $\gamma_d(1)$. Corresponding to these paths, we also have the vector fields $\dot\gamma_s(a)$, $\dot\gamma_t(a)$, and $\dot\gamma_d(a)$. Let $\mathcal{L}_X\omega$ denote the Lie derivative along a smooth vector field X of a differential form ω. We have the following computation of Lie derivatives

$$\mathcal{L}_{\dot\gamma_t}\omega^{(k)}=\mathcal{L}_{\dot\gamma_d}\omega^{(k)}=0 \tag{2.43}$$

and

$$\mathcal{L}_{\dot\gamma_s}\omega^{(k)}=6k^2\,\mathrm{Id}\sum_{i=1}^{6}dz_i. \tag{2.44}$$

Notice that these computations give the proof.

We now carry out the above mentioned computations. Let X be any vector field on $\mathcal{C}$. Then we have that

$$\begin{aligned}\mathcal{L}_X\omega^{(k)} &= \mathcal{L}_X\left(\hbar\sum_{1\leqslant i<j\leqslant 6}\widehat{\Omega}^{i,j}\,d\log(z_i-z_j)\right)\\ &= d\mathcal{L}_X\left(\hbar\sum_{1\leqslant i<j\leqslant 6}\widehat{\Omega}^{i,j}\log(z_i-z_j)\right)\end{aligned}$$

since Lie derivation commutes with exterior differential.

We compute the following

$$\begin{aligned}
\mathcal{L}_{\dot{\gamma}_s}\big(\hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j}\log(z_i - z_j)\big) &= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j}\left(\frac{d}{da}\bigg|_{a=0} \log\left(\frac{z_i}{az_i+1} - \frac{z_j}{az_j+1}\right)\right)\\
&= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j}\left(\frac{d}{da}\bigg|_{a=0} \log\left(\frac{z_i - z_j}{(az_i+1)(az_j+1)}\right)\right)\\
&= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j}\left(\frac{d}{da}\bigg|_{a=0} \log(z_i - z_j) - \log(az_i+1) - \log(az_i+1)\right)\\
&= \hbar \sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j}\big(z_i + z_j\big)\\
&= 6k^2\,\mathrm{Id}\sum_{i=1}^{6} z_i
\end{aligned}$$

where the last equality follows from (3) of proposition 4.4. This then gives us (2.44). The calculation for (2.43) is similar and we leave it to the reader.

□

We now show that the Hitchin connection constructed on $\overline{\mathcal{C}}$ in the last section descends to the moduli space $\mathcal{M}_2$ of genus 2 compact Riemann surfaces. Recall $\overline{\mathcal{C}}$, the space of unordered configurations of six points on the complex plane. We have that $P\colon \mathcal{C} \to \bar{\mathcal{C}}$ is a covering with the deck group S_6, and $\pi_1(\bar{\mathcal{C}}) \cong B_6$. We also have the explicit homomorphism $B_6 \to \mathrm{Sp}(4,\mathbb{Z})$ defined in (2.11) and the quotient group $A(G) = \mathrm{Sp}(4,\mathbb{Z})/\Gamma_2(2,4)$ where $\Gamma_2(2,4)$ is defined in (2.12). The kernel of the induced homomorphism $\pi_1(\bar{\mathcal{C}}) \to A(G)$ defines a covering $\widetilde{\mathcal{C}} \to \bar{\mathcal{C}}$. These three spaces give the following diagram of covering spaces

$$\widetilde{\mathcal{C}} \overset{\widetilde{P}}{\to} \mathcal{C} \overset{\mathrm{P}}{\to} \overline{\mathcal{C}}.$$

with the following short exact sequence of covering groups

$$(\mathbb{Z}/2\mathbb{Z})^4 \to A(G) \to S_6 \tag{2.45}$$

Let $\overline{\mathcal{C}}_\infty$ be the quotient of $\mathcal{C}_\infty$ by the symmetric group S_6. Then $\pi_1(\overline{\mathcal{C}}_\infty) \cong SB_6$, where SB_6 is the spherical braid. The group SB_6 has a presentation in terms of $T_1, \ldots, T_5$, the generators of B_6, such that T_i satisfy the usual braid relations along with additional relation, see [**Bir75**],

$$T_1 \ldots T_4 T_5^2 T_4 \ldots T_1 = 1. \tag{2.46}$$

Direct computation shows that the image of (2.46) is in the kernel of $B_6 \to A(G)$, which implies that the representation (2.11) induces a representation

$$\pi_1(\overline{\mathcal{C}}_\infty) \cong SB_6 \to A(G). \tag{2.47}$$

The kernel of (2.47) defines a covering $\widetilde{\mathcal{C}}_\infty \to \overline{\mathcal{C}}_\infty$. We again get a diagram of covering spaces

$$\widetilde{\mathcal{C}}_\infty \overset{\widetilde{P}_\infty}{\to} \mathcal{C}_\infty \overset{\mathrm{P}_\infty}{\to} \overline{\mathcal{C}}_\infty$$

with (2.45) as the sequence of deck groups.

Proposition 4.19. *The action of* $\mathbb{M}$ *on* $\mathcal{C}_\infty$ *lifts to an action on* $\widetilde{\mathcal{C}}_\infty$.

Proof. For any $\tilde{z} \in \tilde{\mathcal{C}}$ let $z := P(\tilde{z}) \in \mathcal{C}$. Recall that $\mathbb{M}$ is generated by the three one dimensional subgroups S, T, and D. For any $a \in D$ let $l_a \subset D$ be any path connecting the identity of D with a. Then $l_a \cdot z \subset \mathcal{C}$ is a path contained in $\mathcal{C}$ with endpoints z and $S(a)(z)$. Let $\widetilde{l_a \cdot z}$ be the *unique* lift of $l_a \cdot z$ such that it starts at $\tilde{z}$. Let $\widetilde{a(z)}$ be the end point of $\widetilde{l_a \cdot z}$. If $\widetilde{l_a \cdot z}$ was independent of the choice of the homotopy class of l_α then we get a well defined lift of the action of D to $\widetilde{\mathcal{C}}$. Since S and T are simply connected, by the same argument as above, we get that the action of S and T lifts.

To prove that the action of D lifts, we show that the end point $\widetilde{l_a \cdot z}$ is independent of the choice of the homotopy class of $l_a \subset D$. Let $\mathbb{S}^1_z \subset D$ be the orbit of the unit circle in D. Then $\langle \mathbb{S}^1_z \rangle \cong \pi_1(D) \cong \mathbb{Z}$. Since $\mathbb{S}^1_z \subset \mathcal{C}_\infty$ is a closed loop, we have that $\mathbb{S}^1_z \in \pi_1(\mathcal{C}_\infty)$. Let $\widetilde{\mathbb{S}}^1_z$ be the unique lift of $\mathbb{S}^1_z$ such that the identity of $\widetilde{\mathbb{S}}^1_z$ is $\tilde{z}$. Since $\widetilde{P}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is a smooth covering with deck group $(\mathbb{Z}/2\mathbb{Z})^4$ we have the short exact sequence

$$\pi_1(\widetilde{\mathcal{C}}) \to \pi_1(\mathcal{C}) \to (\mathbb{Z}/2\mathbb{Z})^4.$$

If $\mathbb{S}^1_z$ was in the kernel of the second homomorphism of the short exact sequence above, we would have that $\widetilde{\mathbb{S}}^1_z \in \pi_1(\widetilde{\mathcal{C}})$, and thus is a closed loop. If this was true, then the lemma would follow, since the product of two non homotopic loops with same end points will be some power of $\mathbb{S}^1_z$ and thus the lift of this product will be a closed loop, which implies that their end points will be same, and thus independent of the homotopy class.

We now show that $\mathbb{S}^1_z$ is indeed in the kernel of the second homomorphism of the short exact sequence above. There are two cases to consider. One, where z is not in D_i^0 for any i. In this case $\mathbb{S}^1_z$ is homotopic to the pure braid

$$(T_1T_2T_3T_4T_5)^6 \tag{2.48}$$

where T_i are generators of the braid group B_6. We have the explicit representation $B_6 \to \mathrm{Sp}(4,\mathbb{Z})$, see (2.11), and by (5.29) we have that $(\mathbb{Z}/2\mathbb{Z})^4$ is a normal subgroup of $A(G)$. Recall that $A(G)$ is the quotient group $\mathrm{Sp}(4,\mathbb{Z})/\Gamma_2(2,4)$ and we get the map

$$B_6 \to \mathrm{Sp}(4,\mathbb{Z}) \to A(G).$$

It is easily checked, using (5.29), that the image of (2.48) in $A(G)$ is trivial, since $(\mathbb{Z}/2\mathbb{Z})^4$ is a normal subgroup of $A(G)$, it follows that the image of (2.48) in $(\mathbb{Z}/2\mathbb{Z})^4$ is trivial.

Now suppose that $z \in D_i^0$ for some i. In this case $\mathbb{S}^1_z$ is homotopic to a pure braid whose fifth power is homotopic to (2.48), which implies that its image is also trivial in $(\mathbb{Z}/2\mathbb{Z})^4$. $\square$

The product group $(A(G) \times \mathbb{M})$ acts on $\widetilde{\mathcal{C}}_\infty$, where $\mathbb{M}$ also denotes the lift of the action on $\mathcal{C}_\infty$, and the moduli space $\mathcal{M}_2$ is identified with the quotient

$$\widetilde{\mathcal{C}}_\infty/(A(G) \times \mathbb{M}).$$

Let $\widetilde{\omega}^{(k)}_\infty$ denote the pull back $\widetilde{P}^*_\infty(\omega^{(k)}_\infty)$. Consider the trivial vector bundle

$$\mathbb{P}(S^k(V)) \times \widetilde{\mathcal{C}}_\infty$$

where $S^k(V)$ denotes the space of homogenous polynomials of degree k in four variables. The pulled back form defines a connection $\widetilde{\boldsymbol{\nabla}}^{(k)}_\infty := d + \widetilde{\omega}^{(k)}_\infty$. The form $\widetilde{\omega}^{(k)}_\infty$ is projectively invariant under the lifted action of $\mathbb{M}$ since it is the lift of a projectively invariant form under the action of $\mathbb{M}$. This implies that the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}_\infty$ is projectively invariant under the lifted action $\mathbb{M}$. Moreover, by the same argument used in [**vGdJ98**] to show the projective invariance of $\widetilde{P}^*_\infty(\omega^{(k)})$ under the action of the group $A(G)$, we get that the form $\widetilde{\omega}^{(k)}_\infty$, and thus the connection $\widetilde{\boldsymbol{\nabla}}^{(k)}_\infty$, is projectively invariant under the action of $A(G)$.

Consider now the quotient bundle

$$\mathbb{P}(S^{(k)}(V)) \times_{(A(G)\times\mathbb{M})} \widetilde{\mathcal{C}}_\infty \tag{2.49}$$

where $A(G)$ acts on $\widetilde{\mathcal{C}}_\infty$ by deck transformations and acts on $\mathbb{P}(S^{(k)}(V))$ by (2.25), and $\mathbb{M}$ acts on $\widetilde{\mathcal{C}}_\infty$ by its lift (proposition 4.15), and acts on $\mathbb{P}(S^{(k)}(V))$ by parallel transport of $\widetilde{\boldsymbol{\nabla}}^{(k)}_\infty$ along its orbits. Since $\widetilde{\boldsymbol{\nabla}}^{(k)}_\infty$ is projectively invariant under $\mathbb{M}$, its action defined on $\mathbb{P}(S^{(k)}(V))$ by parallel transport is trivial. The connection $\widetilde{\boldsymbol{\nabla}}^{(k)}_\infty$ descends to a flat connection in the quotient bundle (2.49), since it is projectively invariant under the action of the product group $(A(G) \times \mathbb{M})$. We thus obtain a projective vector bundle with a flat connection on the moduli space of genus two compact Riemann surfaces, which is equivalent to (2.34).

CHAPTER 3

Ergodic Theory

In this chapter, we recall the well known result, first proved by Heinz Hopf, that the geodesic flow in the unit tangent bundle of a finite area hyperbolic surface is ergodic with respect to the Louiville measure. We start with basic definitions in measure theory with the aim of introducing measure preserving transformations and measure preserving flows on a measurable space. We then give some examples of such transformations.

A foundational result in the theory of measurable transformations is the Poincare recurrence theorem, we state this theorem and discuss its consequences in section 3. We then proceed to give the definition of an ergodic transformation. Given a measurable transformation which is also ergodic, Birkhoff proved that the space average and the time average of any function (which is integrable) on this space are equal. We state and discuss this result in section 4.

Of particular interest is a measurable space not only equipped with an ergodic transformation, but also with a (normed) vector bundle with a flat connection. In the usual jargon of dynamical systems, this is the situation similar to a measurable cocycle on a space with an ergodic transformation. In this regard, there is the celebrated theorem of Oseledet called the multiplicative ergodic theorem. We recall a statement of this theorem as it appears in [**KZ97**] and discuss its consequences in section 5.

Lastly, we turn to the case of geodesic flow on a hyperbolic surface which we be used later in this thesis.

Most of the material in this chapter is from [**CFS82**], [**Wal00**], and [**BKS91**].

1. Fundamentals of Measure Theory

We recall some basic facts from measure theory which are necessary in the study of dynamical systems. Let X be a set.

Definition 1.1. *A σ-algebra of subsets of X is a collection $\mathcal{B}$ of subsets of X satisfying*

a) $X \in \mathcal{B}$,
b) if $B \in \mathcal{B} \implies X \backslash B \in \mathcal{B}$,
c) if $B_n \in \mathcal{B}$ *for all* $n > 0 \implies \bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.

The main property of a σ-algebra is that it is closed under countable unions, and by property (b) of the definition, also closed under countable intersections. If a σ-algebra has been chosen, then the pair $(X, \mathcal{B})$ is called a *measurable space*, and can be equipped with a measure. We know define what a measure is on $(X, \mathcal{B})$.

DEFINITION 1.2. *Let* $(X, \mathcal{B})$ *be a measurable space. Then a function* $\mu \colon \mathcal{B} \to \mathbb{R}_{\geqslant 0}$ *is called a* measure *if*

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

when $\{B_n\}$ *is a pairwise disjoint sequence of elements in* $\mathcal{B}$.

A measure μ is thus a non-negative set function on a σ-algebra which is also countably additive. If a σ-algebra and a measure μ has been chosen, then the triple $(X, \mathcal{B}, \mu)$ is called a measure space. The measure space $(X, \mathcal{B}, \mu)$ is called a probability space if the measure is normalized, i.e. $\mu(X) = 1$. The measure space $(X, \mathcal{B}, \mu)$ is called complete if all subsets of X of measure zero are contained in $\mathcal{B}$. From now one we will solely restrict to complete measure spaces $(X, \mathcal{B}, \mu)$ such that $\mu(X) < \infty$.

Given $(X, \mathcal{B}, \mu)$ we must now define a nice enough class of functions $f \colon X \to \mathbb{R}$ such that they preserve the measurable structure on X. To this end, we first recall a canonical σ-algebra associated to any topological space X.

DEFINITION 1.3. *Let* X *be a topological space. The* σ*-algebra generated by the closed subsets of* X *is called the Borel* σ*-algebra.*

By property (c) of the definition of a σ-algebra, it follows that the algebra generated by the open subsets of a topological space X is also the *Borel* σ-algebra. For $\mathbb{R}$, the algebra generated by subsets of the form $(-\infty, a]$ is the *Borel* algebra. We denote this σ-algebra by $\mathcal{D}$.

DEFINITION 1.4. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space. A function* $f \colon X \to \mathbb{R}$ *is called measurable if for all Borel subsets* $D \subset \mathbb{R}$*, i.e.* $D \in \mathcal{D}$*,* $f^{-1}(D) \in \mathcal{B}$.

We will now study maps between measure spaces which preserve the measurable structure.

DEFINITION 1.5. *Let* $(X_1, \mathcal{B}_1, \mu_1)$ *and* $(X_2, \mathcal{B}_2, \mu_2)$ *be measure spaces. Then*
a) $T\colon X_1 \to X_2$ *is measurable if* $T^{-1}(B_2) \in \mathcal{B}_1$ *for all* $B_2 \in \mathcal{B}_2$*.*
b)$T\colon X_1 \to X_2$ *is measure preserving if* T *is measurable and* $\mu_1(T^{-1}(B_2)) = \mu_2(B_2)$ *for all* $B_2 \in \mathcal{B}_2$*.*

A measure preserving transformation $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an automorphism if T is invertible and T^{-1} is also measure preserving. From now on we will assume measure preserving transformations are automorphims. We will also be interested in measure preserving flows on a measure space which we define below.

DEFINITION 1.6. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space. Suppose* $\{T_t\}$*, where* $t \in \mathbb{R}$*, is a one-parameter group of measure preserving transformations of* $(X, \mathcal{B}, \mu)$*, i.e* $T_{s+t}(x) = T_s(T_t(x))$ *for all* $s, t \in \mathbb{R}$ *and* $x \in X$*. Then* $\{T_t\}$ *is a* flow *if for every measurable function* $f(x)$ *on* X *the induced function* $f(T_t(x))$*on the Cartesian product* $X \times \mathbb{R}$ *is measurable.*

We can define two types of $Dynamical Systems$.

DEFINITION 1.7. *A discrete dynamical system is a measure space* $(X, \mathcal{B}, \mu)$ *equipped with a measure preserving transformation* T*.*
A continous dynamical system is a measure space $(X, \mathcal{B}, \mu)$ *equipped with a flow* $\{T_t\}$*.*

The basic problem in dynamical systems is to understand the long terms behavior of the measure preserving transformations. For a discrete dynamical system, one would like to understand

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n=1} f(T^k(x)$$

, for all $x \in X$. That is the long term evolution of a point under the measure preserving transformation. Likewise, for continuous dynamical system one would like to understand

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(T_\tau(x))d\tau.$$

An answer to these questions is provided by basic ergodic theorems which we shall cover in the following section.

We now study some examples of measure preserving transformations.

2. Examples of Dynamical Systems

Let $Tor^n := S^1 \times \cdots \times S^1$ be the n dimensional torus. Let $(x_1, \ldots, x_n)$ be local coordinates on this torus. Then the differential of the measure in local coordinates can be written as $d\mu = dx_1 \cdots dx_n$. In fact, $Tor^n \cong \mathbb{R}^n/\mathbb{Z}^n$, and as such induces a measure ρ from the canonical lebesgue measure on $\mathbb{R}^n$. Notice that this measure is normalized, that is, $\rho(Tor^n) = 1$.

Now consider the following transformation $T\colon Tor^n \to Tor^n$ given as

$$T(x_1, \ldots, x_n) := (x_1 + \alpha_1(mod 1), \ldots, x_n + \alpha_n(mod 1)), \qquad \alpha_i \in \mathbb{R}$$

This transformation obviously preserves the lebesgue measure ρ. Thus T is a measure preserving transformation, it is called a *translation* on the torus. In particular, for $n = 1$ we get the *rotation* of a circle.

Let M be a compact smooth manifold. Let $(x_1, \ldots, x_n)$ be a choice of local coordinates, and let $p(x_1, \ldots, x_n) \in \mathbb{R}_{\geqslant 0}$ be an infinitely differentiable function. Then the differential of a measure in local coordinates is given as $d\mu = p(x)dx_1 \cdots dx_n$. In fact, $p(x)$ is called the density of the measure μ. Notice, that a choice of any smooth positively valued function provides us with a measure on M. The situation is a little different if we impose the structure of a Riemannian manifold, in this we get a canonical measure.

Let (M, g) be a compact Riemannian manifold. In local coordinates, the Riemannian metric takes the following form

$$ds^2 = \sum_{i,j=1}^{n} g_{i,j} dx_i dx_j,$$

where the invertible matrix $(g_{i,j})$ is the metric tensor. In this light one has the following.

Definition 2.1. *Let (M, g) be a Riemannian manifold. Then the differential of canonical measure, μ_g, induced by the Riemannian metric g is given in local coordinates as follows*

$$d\mu_g = \sqrt{|\det(g_{i,j})|} \times dx_1, \ldots, dx_n.$$

So we see that any (M, g) is canonically a measure space.

Now, let M be a smooth manifold and let T be a smooth diffeomorphism. Let μ be a measure on M with density $p\colon M \to \mathbb{R}_{\geqslant 0}$. We will state a condition for the diffeomorphism T to be measure preserving. Let $(x_1, \ldots, x_n)$ be local coordinates around $x \in M$ and let $(y_1, \ldots, y_2)$ be local coordinates around $T(x) \in M$. T is locally determined by smooth functions $f_1, \ldots, f_n$ such that $y_i = f_i(x_1, \ldots, x_n)$. Then T preserves the measure μ if and only if

$$|\det\big(\frac{\partial f_k}{\partial x_l}\big)|_x = \frac{p(T(x))}{p(x)}.$$

If the above is true for all $x \in X$, then T is a measure preserving transformation of M with measure μ.

We now define a continuous dynamical system on a smooth manifold. Let X be a smooth vector field on M. Then, for all $x \in M$ we get a system of differential equations $\frac{dx}{dt} = X(x)$. Let $x_1, \dots, x_n)$ be local coordinates around x, the system of differential equations can be written as

$$\frac{dx_1}{dt} = X_1(x_1, \dots, x_n), \dots, \frac{dx_n}{dt} = X_n(x_1, \dots, x_n).$$

DEFINITION 2.2. *Let X be a vector on a smooth manifold M. For all $x \in M$, define $T_t(x) \in M$ for $t \in \mathbb{R}$ to be the point of M given by the solution to above differential equation at time t with initial condition $T_0(x) = x$.*

It is clear that this defines for us the a flow $\{T_t\}$. Weather this flow is measure preserving or not is given by the following theorem of *Louiville*.

THEOREM 2.3. *Let M be a smooth manifold equipped with measure μ of density p. Then μ is invariant with respect to $\{T_t\}$ if and only if*

$$\sum_{k=1}^{m} \frac{\partial}{\partial x_k}(pX_k) = 0.$$

Thus, any vector field on a smooth manifold with measure which satisfies the above induces a continuous dynamical system, or a measure preserving flow on M.

3. Recurrence and Ergodicity

Let $(X, \mathcal{B}, \mu)$ be a measure space equipped with a measure preserving transformation $T\colon X \to X$. Corner stone of dynamical systems is the *Poincare Recurrence Theorem* which states that almost all $x \in X$ under iterations of T will come arbitrarily close to $x \in X$. That is given enough time, the point x will return to any neighborhood to x. We will make this more explicit.

DEFINITION 3.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $T\colon X \to X$ a measure preserving transformation. Let $B \in \mathcal{B}$. A point $x \in B$ is said to be recurrent with respect to B if there exists at least one $n \geqslant 1$ such that $T^n(x) \in B$.*

The statement of *Poincare Recurrence Theorem* is as follows,

THEOREM 3.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $T\colon X \to X$ a measure preserving transformation. Then for each $B \in \mathcal{B}$ almost every point $x \in B$ is recurrent with respect to B.*

Here, by almost every $x \in B$ we mean that $\mu\{x \in B|\text{there exists no } n \geqslant 1\ T^n(x) \in B\} = 0$, i.e. the set of points in B which are not recurrent is of measure zero.

PROOF. Denote by N the subset of B consisting of points which are not recurrent. This then implies that $N \in \mathcal{B}$ since

$$N = B \cap \left(\cup_{n=1}^{\infty} T^{-n}(X \backslash B) \right).$$

If $x \in N$, then all points of the form $T^n(x)$, $n = 1, 2, \ldots$ do not belong to B, and hence $T^n(x) \notin N$. Therefore, $N \cap T^{-n} = \varnothing$, which means that all the sets, $N, T(N), T^2(N), \ldots$ are disjoint. Therefore

$$1 \geqslant \mu\left(\cup_{n=0}^{\infty} T^{-n}(N) \right) = \sum_{n=0}^{\infty} \mu(T^{-n}(N)) = \sum_{n=0}^{\infty} \mu(N)$$

But the last inequality follows if and only if $\mu(N) = 0$. $\square$

Given a measure space $(X, \mathcal{B}, \mu)$ and a measure preserving transformation $T \colon X \to X$, it is possible that there exists some $B \in \mathcal{B}$ such that $T^{-1}(B) = B$. In this case the transformation can be split into two pieces and studied separately. That is, since $T^{-1}(B) = B$ we get that $T^{-1}(X \backslash B) = X \backslash B$, and one can study $T_{|B}$ and $T_{|X \backslash B}$ individually. We thus need a concept of irreducibility of the transformation, and this is provided by ergodicity. In other words, if a measure preserving transformation is also ergodic, then it can not be reduced. We give the definition now.

DEFINITION 3.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $T \colon X \to X$ a measure preserving transformation. Then $T \colon X \to X$ is ergodic if for $B \in \mathcal{B}$, $T^{-1}(B) = B \implies \mu(B) = 0$ or $\mu(B) = \mu(X)$.*

The ergodicity condition can be restated as: the only subsets of X which are invariant under $T \colon X \to X$ are of measure 0 or of full measure. This in particular implies that the dynamical is irreducible. We also have the following equivalent definitions of ergodicity.

LEMMA 3.4. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $T \colon X \to X$ a measure preserving transformation. Then the following are equivalent*
a) T is ergodic.
b) $\mu(T^{-1}(B) \triangle B) = 0$, $B \in \mathcal{B} \implies \quad \mu(B) = 0$ or $\mu(B) = \mu(X)$.
c) $\forall A, B \in \mathcal{B}, \quad \mu(A), \mu(B) > 0 \quad \exists n > 0$ s.t. $\mu(T^{-n}(A) \cap B) > 0$.

Here $(A \triangle B)$ denotes the symmetric difference between the sets A and B, i.e. $(A \triangle B) := (A \cup B) \backslash (A \cap B)$. To understand further properties of ergodicity we will need the notion of invariant functions, which we define below.

DEFINITION 3.5. *Let $(X, \mathcal{B}, \mu)$ be a measure space equipped with $T \colon X \to X$, a measure preserving transformation, or $\{T_t\}$, a measure preserving flow. Then a function f is invariant with respect to T or $\{T_t\}$ if for all $x \in X$ we have*

$$g(T(x)) = g(x) = g(T^{-1}(x))$$

or

$$g(T_t(x)) = g(x), \quad \forall t \in \mathbb{R}.$$

Ergodicity has strong implications for invariant functions. Namely,

LEMMA 3.6. *Let $(X, \mathcal{B}, \mu)$ be a measure space equipped with $T \colon X \to X$, an ergodic transformation, or $\{T_t\}$, an ergodic flow. Then every invariant function f with respect to T or T_t is constant on any set of full measure.*

PROOF. If $f(x)$ is an invariant function, then for any a the set $C_a = \{x \in X | f(x) < a\}$ is invariant. Therefore $\mu(C_a)$ equals 0 or $\mu(X)$. This proves the lemma. $\square$

4. Space Average, Time Average, and Birkhoff's Ergodic Theorem

One of the most useful property of ergodic transformations is that the space average and the time average of functions (of class L^1) is equal. This property has found numerous applications in mathematics, and is essential in the study of Statistical Mechanics. Let us begin by defining these objects.

DEFINITION 4.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space equipped with $T \colon X \to X$, a measure preserving transformation. Let $f \in L^1(X, \mathcal{B}, \mu)$. Then the time average of f for all $x \in X$ is*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n=1} f(T^k(x)) := \bar{f}(x). \tag{3.1}$$

Let $(X, \mathcal{B}, \mu)$ be a measure space equipped with $\{T_t\}$, a measure preserving flow. Let $f \in L^1(X, \mathcal{B}, \mu)$. Then the time average of f for all $x \in X$ is

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(T_\tau(x)) d\tau := \bar{f}(x). \tag{3.2}$$

At this point the question of existence of the limits defined above, or the existence of time averages, arises. This is answered by the following *Birkhoff Ergodic Theorem.*

THEOREM 4.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space equipped with $T\colon X \to X$, a measure preserving transformation, or $\{T_t\}$, a measure preserving flow. In either case, the time averages 3.1 and 3.2 exist for almost μ every $x \in X$. Moreover, the time average $\bar{f}(x) \in L^1(X, \mathcal{B}, \mu)$, and it is invariant under T or $\{T_t\}$, and*

$$\int_X \bar{f}(x) d\mu = \int_X f(x) d\mu.$$

The integral $\int_X f(x) d\mu$ is called the space average of the function. If we assume that our dynamical system is ergodic then we get the following corollary.

COROLLARY 4.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space equipped with $T\colon X \to X$, an ergodic transformation, or $\{T_t\}$, an ergodic flow. Then for any $f \in L^1(X, \mathcal{B}, \mu)$, the space average and the time average coincide, i.e. for almost μ every $x \in X$, we have*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n=1} f(T^k(x)) = \int_X f(x) d\mu$$

or

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(T_\tau(x)) d\tau = \int_X f(x) d\mu.$$

Ofcourse the RHS of the equalities above are independent of $x \in X$. This shows that for an ergodic dynamical system the time averages are the same for almost any $x \in X$, in other words, they are an invariant of the ergodic transformation.

PROOF. Since Birkhoff ergodic theorem states that the time average is invariant under the transformation or flow, and since any invariant function of an ergodic transformation or ergodic flow is constant, we it follows that the time average is constant. $\square$

The fact that the time average and the space average is equal means that the trajectory of a measure preserving transformation goes to every set in the measure space, except possibly sets of measure zero. If the transformation is further assumed to be ergodic, then the trajectories not only go to every set, but they are equidistributed. In this light we can state one more lemma.

LEMMA 4.4. *Let $(X, \mathcal{B}, \mu)$ be a measure space with T an ergodic transformation. Let $B \in \mathcal{B}$ be such that $\mu(B) > 0$, then $\mu(\cup_{n>0} T^{-n}(B)) = 1$.*

5. Oseledet's Multiplicative Ergodic Theorem and Lyapunov Exponents

We recall the setting in the last section. Let $(X, \mathcal{B}, \mu)$ be a measure space with a measure preserving transformation T. In this section we will study the situation where the dynamical system is equipped with a measurable function

$$\varphi \colon X \to GL(n, \mathbb{R}).$$

Such a function induces the map

$$\Phi \colon \mathbb{Z} \times X \to GL(n, \mathbb{R})$$

defined by

$$\Phi(n, x) := \varphi^{n-1}(x) \cdot \dots \cdot \varphi^2(x) \cdot \varphi(x)$$

for all $x \in X$ and $n \in \mathbb{Z}$.

DEFINITION 5.1. *Given a discrete dynamical system $(X, \mathcal{B}, \mu)$ with T, any measurable function $\Phi \colon \mathbb{Z} \times X \to GL(n, \mathbb{R})$ is called a measurable co-cycle over T.*

Notice that $\Phi(x, 0) = \mathrm{Id}$. Also the map φ is said to be the generator of the cocycle. The cocyle Φ also induces the extension of φ which we call

$$F \colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n.$$

This extension is defined as follows

$$F(x, v) := (T(x), \phi(x) \cdot v).$$

In this case the m^{th} iterate of F is defined as

$$F^m(x, v) = (T^n(x), \Phi(m, x) \cdot v).$$

This extension F effectively translates the dynamics, or behavior of any point $x \in X$ under iterations of T, into linear dynamics, that is the associated behavior of $(x, v) \in X \times \mathbb{R}^n$ under iterations F^m. In the case of dynamical system without co cycles, we had the $Birkhoff$ Ergodic theorem which stated the existence of time averages. One wonders time averages can be defined for the extension F and if they exist or not. As it turns out, there is a natural notion of time averages in this setting, and there is a generalization of the $Birkhoff$ Ergodic theorem, called the $Oseledets$ Multiplicative Ergodic theorem.

THEOREM 5.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space and T a measure preserving transformation. Let $\varphi \colon X \to GL(n, \mathbb{R})$ be a measurable function, and $\Phi \colon X \times \mathbb{Z} \to GL(n, \mathbb{R})$ the associated co cycle. If*

$$\int_X \log \|\varphi(x)\| < \infty,$$

here $\|.\|$ is any norm, then

a) for almost all $x \in X$ there exist real numbers $\lambda_1(x) < \cdots < \lambda_n(x)$ and a filtration

$$V_0(x) \subset V_1(x) \subset \cdots \subset V_n(x) = \mathbb{R}^n$$

which is kept invariant under the action of the extension $F\colon X \times \mathbb{R}^n \to X \times \mathbb{R}^n$.

b) for all $v \in V_i \backslash V_{i-1}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log\|\Phi(n,x) \cdot v\| = \lambda_i.$$

The numbers λ_i are called the Lyapunov exponents and the filtration is called the Lyapunov filtration.

A stated in the theorem, the choice of the norm does not effect the filtration or the exponents. The geometric meaning of Lyapunov exponents is to indicate how the length of any given vector changes under successive iterations of F.

If in the theorem we had also assumed that the measure preserving transformation T is also ergodic, then the Lyapunov exponents $\lambda_i(x)$ are constant for all $x \in X$. In other words, if T is ergodic, then the exponents are *invariants* of the ergodic dynamical system $(X, \mathcal{B}, \mu)$ with T.

There is also a version of the Multiplicative Ergodic theorem for a continuous dynamical system. We will state this theorem as it appears in [**KZ97**].

Theorem 5.3. *Let $T_t\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, for $t \in \mathbb{R}_+$, be an ergodic flow on measurable space such that $\mu(X) < \infty$. Let $V \to X$ be an $\mathbb{R}_+$ equivariant finite dimensional vector bundle. Assume that a (non-equivaraint) norm $\|.\|$ on V is chosen such that for all $t \in \mathbb{R}_+$,*

$$\int_X \log(1 + \|T_t\colon V_x \to V_{T_t(x)}\|)\, \mu < \infty. \tag{3.3}$$

Then there are constants $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$ and an invariant filtration of V

$$V = V_{\lambda_1} \supset \cdots \supset V_{\lambda_k} \supset 0$$

such that, for μ-almost every $x \in X$ and all $v \in V_x$ one has

$$\|T_t(v)\| = e^{(\lambda_j t + O(t))}, \qquad t \to \infty$$

where j is the maximal value for which $v \in (V_{\lambda_j})_x$. Moreover, the filtration and the constants do not depend on the choice of the norm.

Notice that this theorem is stated for a *semi* flow, i.e. $\{T_t\}$ for $t \in \mathbb{R}_+$. If we consider a flow, i.e. $t \in \mathbb{R}$, then the dynamical system is reversible and the one gets a filtration similar to above for $t \in \mathbb{R}_-$. The filtrations in the forward and reverse direction are compatible, and one gets an invariant decomposition of the vector bundle.

We now give a few examples of cocycles on dynamical systems taken from [**KH95**].

Let $X = \{x\}$ be a single element, and $\phi(x) = A \in GL(n, \mathbb{R})$. Then the Lyapunov exponents are logarithms of absolute values of the eigenvalues of A. In this case the limits of course exist.

In the previous section we saw that a compact Riemannian manifold (M, g) is a measure space, and a diffeomorphism $f\colon M \to M$ acts as a measure preserving transformation. The derivative of the diffeomorphism

$$D_x f\colon T_x M \to T_{f(x)} M$$

acts on the tangent bundle of M and induces a cocycle as follows.

Represent M as a finite union of diffeomorphic copies of the n-simplex, that is $M = \cup \triangle_i$, such that in each $\triangle_i$ there exist local coordinates so that $TM \supset T\triangle_i \cong \triangle_i \times \mathbb{R}^n$ and all the non-empty intersections $\triangle_i \cap \triangle_j$ are $(n-1)$ dimensional manifolds. By slightly perturbing the boundaries $\partial\triangle_i$ of the $\triangle_i$ if necessary one can always obtain a decomposition $\{\triangle\}_{i=1}^{r}$such that the measure of all the boundaries $\partial\triangle_i$ is zero. Thus we obtain the following decomposition

$$M = \cup_i Int\triangle_i$$

and on each $\triangle_i$ the tangent bundle is trivial. Therefore the derivative Df can be interpreted as a linear cocycle

$$Df\colon M = \cup_i Int\triangle_i \to \mathbb{R}^n$$

with $Df(x)$ being the matrix representing the derivative at x in local coordinates, so the cocycle Df depends on the choices of decomposition of M and local coordinates. However, for another choice of local coordinates the coordinate change sending one representation into another in uniformly bounded together with its inverse, and we get that the spectrum of the derivative of the cocycle does not depend on the coordinate representation.

6. Geodesic Flow on Hyperbolic Surfaces

Let $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 | y > 0\}$ be the upper half-plane. Equip $\mathbb{H}$ with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The hyperbolic metric defines a topology on $\mathbb{H}$. Once we have a topology, we can take the *Borel* σ-algebra which is generated by all the open subsets of $\mathbb{H}$. Denoting by $\mathcal{B}$ this σ-algebra, we get a measurable space $(\mathbb{H}, \mathcal{B})$.

The hyperbolic metric provides us with the area form; $\frac{dxdy}{y^2}$. That is, given any subset $B \subseteq \mathbb{H}$, the area of B is defined as

$$Area(B) = \int_B \frac{dxdy}{y^2}.$$

This area form provides us with a non-negative function $Area\colon \mathcal{B} \to \mathbb{R}_{\geqslant 0}$, which is also countably additive, i.e.

$$Area\big(\cup_{n=1}^{\infty} B_n\big) = \sum_{n=1}^{\infty} Area(B_n)$$

where all $B_n \in \mathcal{B}$ are disjoint. Being a non-negative countably additive function on the set $\mathcal{B}$, *Area* provides us with a measure on $(\mathbb{H}, \mathcal{B})$. Notice that $(\mathbb{H}, \mathcal{B})$ with the *Area* is *not* a finite measure space, i.e. $Area(\mathbb{H}) = \infty$.

We now study the geodesic flow on $\mathbb{H}$ with respect to the hyperbolic metric. To study the geodesic flow, we need to consider $U\mathbb{H}$, the unit tangent bundle of $\mathbb{H}$. As a set

$$U\mathbb{H} := \{(x, y, \vartheta) | (x, y) \in \mathbb{H},\ \vartheta \in (0, 2\pi)\}$$

That is, the tuple $(x, y, \vartheta) \in U\mathbb{H}$ represents the unit tangent vector based at $(x, y) \in \mathbb{H}$ which makes the angle ϑ with the x-axis. It is clear that $U\mathbb{H}$ is a circle bundle over $\mathbb{H}$, i.e. every fiber of U$\mathbb{H}$ can be identified with S^1. The circle has a natural lebesgue measure, infact if we choose $S^1 = e^{i\vartheta}$ for $\vartheta \in (0, 2\pi)$ then the differential of this measure is simply $d\vartheta$. The form $\frac{dxdyd\vartheta}{y^2}$ now induces a measure, which we denote by μ, on $U\mathbb{H}$. That is, the measure of any subset $B \subseteq U\mathbb{H}$ is now given as

$$\mu(B) = \int_B \frac{dxdyd\vartheta}{y^2}.$$

Again, notice that $\mu(U\mathbb{H}) = \infty$.

Definition 6.1. *The measure μ on the unit tangent bundle of the hyperbolic plane is called the* Louiville *measure.*

To describe the geodesic flow on $U\mathbb{H}$ the following identifications are useful.

Lemma 6.2. $SL(2, \mathbb{R}) = U\mathbb{H}$ *and* $SL(2, \mathbb{R})/SO(2) = \mathbb{H}$.

Proof. $SL(2,\mathbb{R})$ acts transitively on $\mathbb{H}$. Consider the map

$$\phi\colon SL(2,\mathbb{R}) \to \mathbb{H},$$

where $\mathrm{Id} \mapsto (0,1) = i \in \mathbb{H}$, i.e. the identity element of the group is identified with i, and for any $A \in SL(2,\mathbb{R})$, we have that

$$A \mapsto Ai \in \mathbb{H}.$$

Since $SL(2,\mathbb{R})$ acts transitively, we get that the map ϕ is surjective. Now, the subgroup $SO(2)$ acts by rotations, which means it has a fixed point or stabilizer. This fixed point is $i \in \mathbb{H}$. Thus, if $S \in SO(2)$ then $Si = i$ and $SO(2)$ forms the kernel of the map ϕ which implies $SL(2,\mathbb{R})/SO(2) = \mathbb{H}$.

Since the fiber over i of $U\mathbb{H}$ is a circle, $SO(2)$ acts transitively on this fiber, and thus can be identified with it. Now, if $z \in \mathbb{H}$ then there exists $A \in SL(2,\mathbb{R})$ such that $Ai = z$, and we can identify the fiber of $U\mathbb{H}$ over z with the set $A \cdot SO(2)$. This gives us that $SL(2,\mathbb{R}) = U\mathbb{H}$. $\square$

$SL(2,\mathbb{R})$, being a locally compact lie group, can be equipped with the *Haar* measure. Since we have identified $SL(2,\mathbb{R})$ with $U\mathbb{H}$, we will consider both these spaces with the aforementioned Louiville measure, μ.

We now describe the geodesic flow on $U\mathbb{H} \cong SL(2,\mathbb{R})$.

Definition 6.3. *The geodesic flow $T_t\colon SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is given by*

$$A \mapsto A \cdot \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix}$$

for all $t \in \mathbb{R}$.

The fact that $\{T_t\}$ is a flow, i.e. $T_{t+s} = T_t \circ T_s$, is obvious. The fact that $\{T_t\}$ is measure preserving follows because the action of $SL(2,\mathbb{R})$ on $\mathbb{H}$ is by isometries, thus $\begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix} \in SL(2,\mathbb{R})$ preserves the hyperbolic metric. Since the measure μ is essentially the area form of the metric, if the metric is preserved then the area form is preserved, thus the measure μ is preserved.

In the jargon of dynamical systems, we have a measure space, $(U\mathbb{H}, \mathcal{B}, \mu)$ with a measure preserving flow $\{T_t\}$. We still have the problem that $\mu(U\mathbb{H}) = \infty$. To overcome this problem, we will pass to quotients of $U\mathbb{H}$ by discrete subgroups of $SL(2,\mathbb{R})$.

Let Γ be discrete subgroup of $SL(2,\mathbb{R})$ such that $\mathbb{H}/\Gamma := X$ is a finite area not necessarily compact surface. Since Γ is a subgroup of isometries of the hyperbolic metric, we get a well defined metric on the quotient X. Let $\mathcal{F} \subset \mathbb{H}$ be a fundamental domain for Γ. Then we have

$$Area(X) = \int_{\mathcal{F}} \frac{dxdy}{y^2}.$$

For the unit tangent bundle $UX = U\mathbb{H}/\Gamma$, we have the identification with the homogenous space

$$UX = SL(2,\mathbb{R})/\Gamma.$$

In fact, the surface X can be identified with the double coset space

$$SO(2)\backslash SL(2,\mathbb{R})/\Gamma.$$

Remark 6.4. Notice now we are considering the left action of $SO(2)$ no $SL(2,\mathbb{R})$ where as before we were considering the right action.

The Liouville measure μ on $U\mathbb{H}$ descends to a measure $\bar{\mu}$ on UX. The total measure $\bar{\mu}(UX) < \infty$ if and only if $Area(X) < \infty$, where $Area(X)$, as mentioned before, is just the hyperbolic are of the fundamental domain $\mathcal{F}$.

Since the flow $T_t \colon U\mathbb{H} \to U\mathbb{H}$ is an isometry for all $t \in \mathbb{R}$, it also descends to a flow on UX. This is explicitly given as follows. We have that

$$UX = SL(2,\mathbb{R})/\Gamma := \{B \cdot \Gamma | B \in SL(2,\mathbb{R})\}$$

We use the same notation for the flow $T_t \colon UX \to UX$ and define it as

$$B \cdot \Gamma \mapsto B \cdot \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix} \cdot \Gamma$$

for all $t \in \mathbb{R}$.
Assume that $X := SL(2,\mathbb{R})/\Gamma$ has finite area. Then we get a continuous dynamical system.

Definition 6.5. *Let X be as above. Then we have a measure space $(UX, \mathcal{B}, \bar{\mu})$ with a measure preserving flow $\{T_t\}$ which is described above. This dynamical system is called the geodesic flow on the hyperbolic surface X*

Of course now the question of ergodicity of $\{T_t\}$ arises. In this regard we have the following theorem which was first proved by *Heinz Hopf*.

Theorem 6.6. *Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ such that $\mathbb{H}/\Gamma := X$ has finite area. Then the geodesic flow $\{T_t\}$ on the space $(UX, \mathcal{B}, \bar{\mu})$ is ergodic.*

Remark 6.7. This theorem has been generalized to hyperbolic manifolds of higher dimension.

CHAPTER 4

Teichmüller Theory

Given a topological surface S of genus $g \geqslant 1$, one wonders 'how many' different complex structures S can be endowed with and whether the set of all complex structures on S itself is a nice space, and if yes, whether it is an object in the category of smooth manifolds, complex manifolds, etc. If one considers the set of complex structures, up to the equivalence of being biholomorphic, then Riemann showed that this set $\mathcal{M}_g$, now called Riemann's Moduli space, is a non-compact complex manifold.

The Teichmüller space $\mathcal{T}(S)$ of S, is the set of equivalence classes of *marked* complex structures on S. The significance of this space is that it can be thought of as the universal covering space of $\mathcal{M}_g$ with the deck group of transformations being the mapping class group, Γ_g, of S. The space $\mathcal{T}(S)$ is rich with geometric structures. To name a couple, it has a (non-complete) Kähler metric, called the Weil-Petersson metric, and it has a Finsler metric, called the Teichmüller metric.

In this chapter, after defining $\mathcal{T}(S)$, we take a look at its unit cotangent $U^*\mathcal{T}(S)$ and in particular the geometry of the pair $(X, q) \in U^*\mathcal{T}(S)$. Such a pair defines a singular flat structure on X, and provides a way to study X in terms of euclidean geometry. In this respect, we study the $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$ action on any pair (X, q) and define the Veech group $V(X, q)$ associated with any pair (X, q). The Veech group $V(X, q)$ has a map to $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$, and if the image of this map is a lattice, then the $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$ orbit of (X, q) descends to a unit tangent bundle of a hyperbolic surface (non-compact) in the unit tangent bundle of the moduli space: $U^*\mathcal{M}_g$. Such a hyperbolic surface is called the Teichmüller curve given by (X, q).

We end this section by studying a particular Teichmüller curve which was discovered in [**Vee89**]. We will use a degree ten branched cover of this curve which was explicitly constructed in [**Loc**], see also [**McM06**].

Most of the material in this chapter is from [**ACG11**], [**FM11**], and [**Pen12**]. We wish to thank Anton Zorich for allowing us to use the figures appearing in his beautiful treatise [**Zor**] on Flat surfaces.

1. Teichmüller Space and its cotangent bundle

Let S be a smooth compact surface of genus $g \geqslant 2$. Let X be a compact Riemann surface with an orientation preserving diffeomorphism $f\colon S \to X$. The pair (X, f) is called a *marked* Riemann surface, and f a marking. Two marked Riemann surfaces (X, f) and (X', f') are said to be equivalent if $f' \circ f^{-1}$ is isotopic to an isomorphism between X and X'.

Definition 1.1. *Given S as above, its Teichmüller space $\mathcal{T}(S)$ is the set of equivalence classes of marked Riemann surfaces (X, f).*

The space $\mathcal{T}(S)$ admits the structure of a complex analytic manifold of dimension $3g-3$, see [**ACG11**], and [**Abi80**]. Topologically, $\mathcal{T}(S)$ is homeomorphic to the unit ball in $\mathbb{C}^{3g-3}$, this is a result of Fricke. Let $t = (X, f) \in \mathcal{T}(S)$, then the fiber $T_t\mathcal{T}(S)$ of the tangent bundle $T\mathcal{T}(S)$ is the space of deformations of the complex structure on X. By $Kodaira-Spencer$ theory, this space of deformations is the first cohomology of the sheaf of holomorphic vector fields on X, see [**ACG11**]. More explicitly, let K_X be the canonical bundle of X, then sections of K_X^* are holomorphic vector fields on X, thus $H^1(X, K_X^*)$ is the deformation space of X. Moreover, by $Serre$ duality we obtain

$$H^1(X, K_X^*)^* = H^0(X, K_X^2)$$

where the right hand side is referred to as the space of holomorphic quadratic differentials on X. By Riemann-Roch formula, we know that the dimension of $H^0(X, K_X^2)$ is $3g-3$. Notice that this provides us with an identification of the fiber of the cotangent bundle $T_t^*\mathcal{T}(S)$ with $H^0(X, K_X^2)$.

Let $(X, f) \in \mathcal{T}(S)$ and $q \in H^0(X, K_X^2)$. We will recall some facts about the geometry of a quadratic differential q on a Riemann surface X. Let $\{z_\alpha\colon U_\alpha \to \mathbb{C}\}$ be an atlas of charts for X. With respect to this atlas, any $q \in H^0(X, K_X^2)$ is specified by a collection of expressions $\{\phi_\alpha(z_\alpha)dz_\alpha^2\}$, where $dz_\alpha^2 = dz_\alpha \otimes dz_\alpha$, such that

1) Each $\phi_\alpha\colon z_\alpha(U_\alpha) \to \mathbb{C}$ is a holomorphic function with at most a finite set of zeroes.
2) For any two overlapping charts z_α and z_β, we have

$$\phi_\beta(z_\beta)\Big(\frac{dz_\beta}{dz_\alpha}\Big)^2 = \phi_\alpha(z_\alpha).$$

The last condition says if ψ is the change of coordinate map between z_α and z_β then

$$\phi_\beta(z)(d\psi)^2 = \phi_\alpha(z).$$

This also implies that the order of the zeros of a quadratic differential are independent of the coordinate representation.

A quadratic differential $q \in H^0(X, K_X^2)$ induces a Hermitian metric on the punctured surface $X \backslash Z(q)$ where $Z(q)$ is the finite set of zeroes of q. If $\phi(z)dz^2$ is a local expression for q, then the metric in these coordinates is given by

$$|\phi(z)|dzd\bar{z}.$$

Notice that this metric is flat, or euclidean, as one can introduce a coordinate ζ such that $\phi(z)dz^2 = d\zeta^2$, and the metric takes the form

$$d\zeta d\bar{\zeta}.$$

Thus, $X \backslash Z(q)$ can be equipped with a metric which locally looks like the euclidean metric. In fact, this metric extends to a singular euclidean metric on X with cone type singularities precisely at $Z(q)$. Let $\gamma\colon [0,1] \to X \backslash Z(q)$ be a path, then γ is a geodesic of this metric if and only if at each point $\gamma(s)$, it is a straight line in the coordinate ζ. An intrinsic characterization of geodesics is that

$$\arg(q)_{|\gamma} = constant,$$

where, by definition, we set $\arg(q)_{|\gamma} = \arg(\phi(\gamma(s)) \cdot \dot{\gamma}^2)$ if $q(z) = \phi(z)dz^2$.

Definition 1.2. *Let X be a Riemann surface with a holomorphic quadratic differential q. Then a path $\gamma\colon [0,1] \to X \backslash Z(q)$ is a*
a) horizontal geodesic if $\arg(q)_{|\gamma} = 0$
b) vertical geodesic if $\arg(q)_{|\gamma} = \pi$
in the singular euclidean metric induced by q on X.

Let us analyze what happens at zeroes of q. If $p \in Z(q)$ is a zero of q of order n, then locally around p there exist a coordinate ζ such that in this coordinate q can be written as

$$q(\zeta) = \left(d\zeta^{\frac{n+2}{2}}\right)^2.$$

This coordinate neighborhood is the same as gluing $n+2$ half disks together. In other words, p will have $n+2$ horizontal geodesics emanating from it, likewise, $n+2$ vertical geodesics. The point p can be thought of as an $n+2$ pronged singularity, or a conic singularity of angle $\pi(n+2)$.

This metric also induces an area form. If $\phi(z)dz^2$ is a local expression for q, then the area form is defined as

$$dA_q(z) = \frac{i}{2}|\phi(z)|d\bar{z} \wedge dz = |\phi(z)|dx \wedge dy.$$

In fact, using the area form one can define a norm in $H^0(X, K_X^2)$, which is given as follows

$$\|q\| = \int_X dA_q \qquad \forall q \in H^0(X, K_X^2).$$

Any $q \in H^0(X, K_X^2)$ also determines a foliation on X. The recipe is as follows, any q can be thought of as a map $q\colon TX \to \mathbb{C}$, where TX is the holomorphic tangent bundle of X. In particular given a section of TX, i.e. a holomorphic vector field, q acts as an evaluation on it. If $\dot{\gamma}$ is a holomorphic vector field such that it evaluates under q to positive real numbers, then the integral curve γ of this vector field will be part of the *horizontal* foliation. The set of all integral curves γ such that $\dot{\gamma}$ evaluates to a positive real number, together with the set of zeroes of the differential q provide foliation of X which we call the horizontal foliation. Likewise, the set of all integral curves whose vector fields evaluate to negative real numbers provide the *vertical* foliation of X. Both these foliations, as the name suggests, are transverse to each other.

In fact, these foliations coming from q can also be equipped with a measure. Again, suppose in some chart $U \subset X$, q has the representation $\phi(z)dz^2$. Then the following function

$$\mu(z) = \int_U \left| Im\left(\sqrt{\phi(z)}dz\right)\right|$$

provides a measure for the horizontal foliation. If we replace the imaginary value by the real value, then we obtain a measure for the vertical foliation. Thus, any $q \in H^0(X, K_X^2)$ provides a measure foliation of X. In fact, it is a deep theorem of Hubbard and Masur [**HM79**], that *any* measured foliation (upto a certain equivalence) on X can be obtained from some $q \in H^0(X, K_X^2)$.

2. Flat structures and $\mathbb{P}SL(2,\mathbb{R})$ action on the (unit)cotangent bundle

Let S be a smooth surface of genus $g \geqslant 2$, and let Σ be a finite set of marked points on S. The following definition if due to [**Vee89**].

DEFINITION 2.1. *Let $\{V_i, \phi_i\}$ where $\phi_i\colon V_i \to \mathbb{C}$ be an atlas for $S\backslash\Sigma$ such that on non-trivial intersection $V_i \cap V_j$ the transition function*

$$\phi_{ij} = \phi_j \circ \phi_i^{-1}\colon \mathbb{C} \to \mathbb{C}$$

is a translation, i.e. $\phi_{ij}(z_i) = \pm z_j + c_{ij}$ where $c_{ij} \in \mathbb{C}$ is a constant. Then a Flat structure on $S\backslash\Sigma$ is the obvious equivalence class of such atlases. We denote by (S, Σ, u) a choice of Flat structure.

The above defined transition maps are of course biholomorphic, thus a Flat structure on $S\backslash\Sigma$ is in particular a complex structure. In fact this complex structure extends over the entire S. Outside of Σ, Flat structure endows S with a euclidean metric which is simply the pull back of the euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$ by the functions ϕ_i. In particular this metric induces an area form, and the area of $S\backslash\Sigma$ with respect to this metric is called the euclidean area associated to the Flat structure.

At the points Σ a singularity of this euclidean metric is developed. That this should be the case is obvious, since S has genus $g \geqslant 2$ and by uniformization theorem can not admit a metric of zero curvature everywhere. Thus, all the negative curvature of S is concentrated at Σ. In fact, the singularity of the euclidean metric at Σ is of cone type. That is, around each $p \in \Sigma$ the total angle is not 2π, but rather $\pi(n+2)$ for $n \geqslant 1$.

As an example, consider a polygon P with even number of edges, such that for each edge e_i there exist another edge $e_{i'}$ of same length and parallel to e_i. Let $P \subset \mathbb{C}$. Then P admits a Flat structure, where the interior of P is covered by one chart, and each edge e_i is identified with the corresponding parallel edge $e_{i'}$ by translation. This endows an atlas on $P\backslash V$ where $V \subset P$ are the vertices of the polygon. Moreover, the transition functions of this atlas are translations, thus $P\backslash V$ is endowed with a Flat structure.

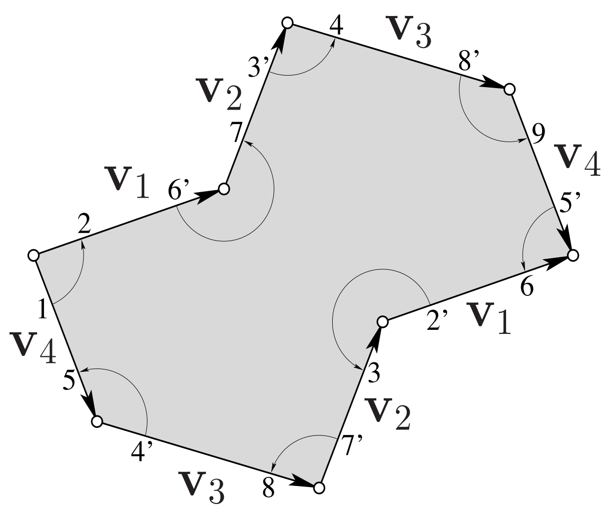

FIGURE 1. The polygon P (After A. Zorich)

We will now study deformations of Flat structures. Given a Flat structure (S, Σ, u), let $\{V_i, \phi_i\}$ be the collection of charts corresponding to u. Let $A \in \mathbb{P}SL(2,\mathbb{R})$. Then we get a new Flat structure (S, Σ, Au) which is defined as follows. Recall $\phi_i \colon V_i \to \mathbb{C} \cong \mathbb{R}^2$. Any matrix $A \in \mathbb{P}SL(2,\mathbb{R})$ acts on the $\mathbb{R}^2$. We post compose the chart function ϕ_i with this action. That is, we define new holomorphic functions $\phi_{i'} := A \circ \phi_i$, and get

$$\phi_{i'} \colon V_i \to \mathbb{C}.$$

LEMMA 2.2. *The atlas $\{V_i, \phi_{i'}\}$ defines a Flat structure on $S \backslash \Sigma$.*

PROOF. For $\{V_i, \phi_{i'}\}$ to define a Flat structure we must check that the transition functions are translations. Since $\phi_{i'} := A \circ \phi_i$, we get that the transition function $\phi_{(ij)}(z_i) = \pm z_j + c_{ij}$ changes to $\phi_{(ij)'}(z_i) = \pm z_j + Ac_{ij}$. Since Ac_{ij} is again constant, we get that $\phi_{(ij)'}$ is a translation. □

However, if A is a rotation. That is, if $A = e^{i\vartheta}$, then the local change of coordinates is simply $z' = e^{i\vartheta} z$, and there is no change in the Flat structure. More precisely, if $A \in \mathbb{P}SO(2, \mathbb{R})$ then $(S, \Sigma, u) = (S, \Sigma, Au)$ as Flat structure. In fact, $(S, \Sigma, u) = (S, \Sigma, Au)$ only if $A \in \mathbb{P}SO(2, \mathbb{R})$.

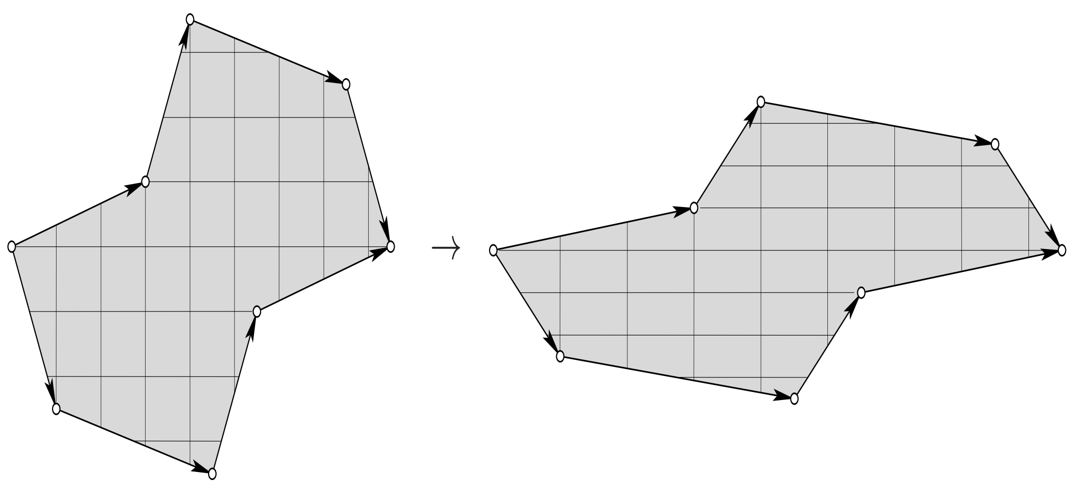

FIGURE 2. Deformation of a Flat structure. (After A. Zorich)

Since the action of any $A \in \mathbb{P}SL(2, \mathbb{R})$ is area preserving, we get that the euclidean area of the Flat structure (S, Σ, Au) is the same as the euclidean area of (S, Σ, u).

The most important point of this construction is the fact that this deformation of the Flat structure also provides a deformation of the complex structure. That is, (S, Σ, Au) has the structure of a Riemann surface, and this complex structure is not the same as the complex structure on (S, Σ, u), provided $A \notin \mathbb{P}SO(2, \mathbb{R})$.

Let $\mathbb{H}$ be the upper half plane and let (S,Σ,u) be a flat structure. Since $\mathbb{H} \cong \mathbb{P}SL(2,\mathbb{R})/\mathbb{P}SO(2,\mathbb{R})$ (see the section "Geodesic Flow on Hyperbolic Surfaces"), we get a family of Flat structures parametrized by $\mathbb{H}$, where the fiber over i corresponds to (S,Σ,u) and fiber over any $A \in \mathbb{H}$ corresponds to (S,Σ,Au).

As each Flat structure (S,Σ,u) is also a complex structure on S, we also get a family of complex structures parametrized by $\mathbb{H}$. We wish to make this more explicit and give embeddings

$$\mathbb{H} \hookrightarrow \mathcal{T}(S).$$

The trick to this construction is the fact that the pair (X,q), where X is a Riemann surface and q is a holomorphic quadratic differential, induces canonically a Flat structure $(S,\Sigma,u(q))$, where S is the underlying topological surface of X and Σ are the zeroes of q.

Lemma 2.3. *Let (X,q) be a Riemann surface with a quadratic differential q. Then there exists a canonical Flat structure $(S,Z(q),u(q))$ where S is the underlying topological surface of X and $Z(q) \subset X$ is the set of zeroes of q.*

Proof. We need to define an atlas on $\bar{X} := X\backslash Z(q)$ satisfying the desired properties of a Flat structure. This is done by the help of the local expression of q outside of $Z(q)$. At any point of $\bar{X}$, there exist *natural* coordinates, say ζ such that $q(\zeta) = d\zeta^2$, and moreover these coordinates are defined upto an addition of a constant. Thus, these coordinates provide an atlas for $\bar{X}$ with the property that transition functions are all translations, giving us the desired Flat structure. □

Let $A \in \mathbb{P}SL(2,\mathbb{R})/\mathbb{P}SO(2,\mathbb{R})$ and consider the Flat structure $(S,Z(q),Au(q))$. Corresponding to this Flat structure is a Riemann surface with a quadratic differential (X',q'), where X' is the complex structure induced by the atlas of $Au(q)$ on X', and the quadratic differential $q' \in H^0(X',K_{X'})$ is the corresponding deformation of q. Notice that the number and the order of zeros of q and q' is the same. Also, since A preserves the euclidean area, which is the norm of the quadratic differential in complex analytic language, we get that

$$\|q\| = \|q'\|.$$

We use the notation $A(X,q) := (X')$ to talk about the complex structure coming from the Flat structure $(S,Z(q),Au(q))$. Let $t = (X,f) \in \mathcal{T}(S)$ and let $q \in H^0(X,K_X^2)$ such that $\|q\| = 1$. All the constructions above work if one considers (X,f,q) that is a *marked* Riemann surface with a holomorphic quadratic differential.

We then have an embedding

$$\mathbb{H}(X, f, q) \hookrightarrow \mathcal{T}(S),$$

given by $A \mapsto A(X, f, q) = (X', f)$ based at the point $(X, f, q) \in U^*\mathcal{T}(S)$ in the unit cotangent bundle of $\mathcal{T}(S)$. That is under the embedding the point $(i, i) \in U\mathbb{H}$ is identified with (X, f, q). Notice that we have a family of complex structures parametrized by $\mathbb{H}$, but the marking f remains the same. Thus, we get a family of marked Riemann surfaces.

Definition 2.4. *The embedding* $\mathbb{H}(X, f, q) \hookrightarrow \mathcal{T}(S)$ *is called the Teichmüller disk based at* (X, f, q).

Notice that if $A \in \mathbb{P}SO(2, \mathbb{R})$ then $A(X, f, q) = (X, f)$. That is, A stabilizes the marked complex structure. On the other hand, the action of A on q is by rotation, that is $Aq = e^{i\vartheta} q$. In particular, the orbit $\mathbb{P}SO(2, \mathbb{R}) \cdot (X, f, q) \subset U^*_{(X,f)}\mathcal{T}(S)$ is a copy of S^1. In particular we have an action of $\mathbb{P}SL(2, \mathbb{R})$ on $U^*\mathcal{T}(S)$.

3. Veech Groups and Teichmüller curves

Let S be a smooth surface of genus $g \geqslant 2$. Let $\Gamma(S)$ be the group of all isotopy classes of orientation preserving diffeomorphisms of S, in other words $\Gamma(S) = \pi_0(Diff^+(S))$. The group $\Gamma(S)$ is called the mapping class group of S. This group acts on $\mathcal{T}(S)$ as follows, if $g \in \Gamma(S)$ and $(X, f) \in \mathcal{T}(S)$, then

$$g \cdot (X, f) = (X, f \circ g^{-1}).$$

If we consider $\mathcal{T}(S)$ as a complex manifold, then the action of $\Gamma(S)$ is by holomorphic transformations. Moreover, Γ acts properly and discontinuously but not effectively. The resulting orbifold quotient is called the moduli space of genus g curves,

$$\mathcal{M}_g := \mathcal{T}(S)/\Gamma(S).$$

In the last section we saw that every $(X, f, q) \in U^*_{(X,f)}\mathcal{T}(S)$ generated a Teichmüller disk which was an embedding of the upper half plane,

$$\mathbb{H}(X, f, q) \hookrightarrow \mathcal{T}(S)$$

based at (X, f, q).

Definition 3.1. *Let* $(X, f, q) \in U^*_{(X,f)}\mathcal{T}(S)$. *Then the Veech group of* (X, f, q), *denoted by* $V(X, f, q) \subset \Gamma(S)$, *is the group of mapping classes which stabilize the Teichmüller disk* $\mathbb{H}(X, f, q)$.

Here by stabilize we mean a mapping class which preserves the disk globally, and not point wise. Although, there may exist some mapping class which fixes the disk point wise. We have the following result on generic Veech groups, see Möller's article in [**Pap09**].

PROPOSITION 3.2. *For a generic (X, f, q) the group $V(X, f, q)$ is trivial.*

We will especially be interested in the case $V(X, f, q)$ is large. To make this more precise, we first observe that there exists a group homomorphism

$$V(X, f, q) \to \mathbb{P}SL(2, \mathbb{R})$$

which might have a non-trivial kernel. This is given as follows. Let (S, Σ, u) be a Flat structure. Let $\Gamma(S, \Sigma)$ be the group of orientation preserving diffeomorphisms of S which preserve the set Σ point wise. Then $g \in \Gamma(S, \Sigma)$ acts on (S, Σ, u) by precomposition with charts. More precisely, let $\{V_i, \phi_i\}$ be the atlas giving the Flat structure, consider a new atlas defined by $\{g(V_i), \phi_i \circ g^{-1}\}$. That is, if $\phi_i \colon V_i \to \mathbb{C}$ is a chart for u then $\phi_i \circ g^{-1} \colon g(V_i) \to \mathbb{C}$ is a chart for $g \cdot u$.

Now consider the Flat structure $(S, Z(q), u(q))$ induced by (X, f, q). Let $g \in V(X, f, q)$, and let g act on the Flat structure $(S, Z(q), u(q))$ as explained above. Now, by the very definition of $\mathbb{H}(X, f, q)$ as an orbit of $\mathbb{P}SL(2, \mathbb{R})$, it follows that $\mathbb{H}(X, f, q)$ is stabilized by g if and only if there exists an $A_g \in \mathbb{P}SL(2, \mathbb{R})$ such that $g \cdot u(q) = A_g(u)$. On a given chart (V, ϕ) this condition means that $\phi \circ g^{-1} = A_g \circ \phi$. In other words, the action of g on (X, f, q) is $affine$, the result again is a Flat structure which is a deformation by A_g. This provides us with the aforementioned group homomorphism

$$V(X, f, q) \to \mathbb{P}SL(2, \mathbb{R}) \qquad g \mapsto A_g.$$

We state the following main characteristics of the image of $V(X, f, q)$ obtained in [**Vee89**].

THEOREM 3.3. *The image of $V(X, f, q)$ in $\mathbb{P}SL(2, \mathbb{R})$ is always discrete and never co-compact.*

A discrete subgroup $\Lambda \triangleleft \mathbb{P}SL(2, \mathbb{R})$ is called a *lattice* if a fundamental domain of Λ has finite hyperbolic area, or equipvalenty, if the quotient surface $\mathbb{H}/\Lambda$ has finite hyperbolic area.

DEFINITION 3.4. *(X, f, q) is said to be a Veech surface if the image of $V(X, f, q)$ in $\mathbb{P}SL(2, \mathbb{R})$ is a lattice.*

Although we are not concerned with dynamics of Flat structures, we state the following remarkable result dubbed the *Veech* dichotomy.

THEOREM 3.5. *If (X, f, q) is a Veech surface, then for any given direction in S^1 either all foliations are ergodic or all foliations are periodic. In particular, there exist at least one periodic direction.*

It is the existence of this periodic direction which implies the non co-compactness of $V(X, f, q)$. If $V(X, f, q)$ is a lattice then $\mathbb{H}(X, f, q)/V(X, f, q)$ is a finite area non-compact hyperbolic surface.

DEFINITION 3.6. *Let $(X, f, q) \in U^*\mathcal{T}(S)$ be such that the Veech group $V(X, f, q)$ is a lattice. Then the surface $\chi := \mathbb{H}(X, f, q)/V(X, f, q)$ is called the Teichmüller curve generated by (X, f, q).*

Notice that this curve descends to an embedding in the moduli space of curves

$$\begin{tikzcd} \mathbb{H}(X,f,q) \arrow[r, hook] \arrow[d] & \mathcal{T}(S) \arrow[d] \\ \chi \arrow[r, hook] & \mathcal{M}_g \end{tikzcd}$$

We of course have that $\pi_1^{orb}(\chi) = V(X, f, q)$. Teichmüller curves also provide a large supply of pseudo-Anosov elements. By a theorem of Thurston, see [**Thu88**], under the identification of π_1^{orb} as a subgroup of $\mathbb{P}SL(2, \mathbb{R})$, any $\gamma \in \pi_1^{orb}(\chi)$ whose corresponding matrix $A_\gamma \in \mathbb{P}SL(2, \mathbb{R})$ is such that $|TrA_\gamma| > 2$ corresponds to a pseudo-Anosov mapping class in $\Gamma(S)$. Moreover, any matrix with absolute value of trace bigger than 2 is conjugate in $\mathbb{P}SL(2, \mathbb{R})$ to a unique diagonal matrix with entries λ and λ^{-1} where λ is a positive real number. The stretch factor of A_γ is bigger of two values λ and λ^{-1}.

4. Example

Let X be the genus 2 algebraic curve defined by the equation $y^2 = x^5 + 1$. Let q be the quadratic differential on X defined as $c\frac{dx^2}{y^2}$. Here c is a constant such that $\|q\| = 1$. Thus, $(X, f, q) \in U^*\mathcal{T}(S)$ where $f\colon X \to S$ is any arbitrary marking. In [**Vee89**] it is shown that the image of $V(X, f, q)$ in $\mathbb{P}SL(2, \mathbb{R})$ is a lattice, in fact it is shown to be conjugate to the Hecke triangle group $\triangle(2, 5, \infty)$, which has the following presentation

$$\triangle(2,5,\infty) :=< S, T | S^2 = (S \circ T)^5 = \mathrm{Id} > . \tag{4.1}$$

where

$$T = \begin{pmatrix} 1 & 2\cos\frac{\pi}{5} \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Flat structure $(S, \Sigma, u(q))$ corresponding to (X, f, q) can be described as follows. Take a regular pentagon P in $\mathbb{C}$. Let P_1 be a reflection of P along one of its edges. The union $P \cup P_1$ is a non-convex octagon, call it O.

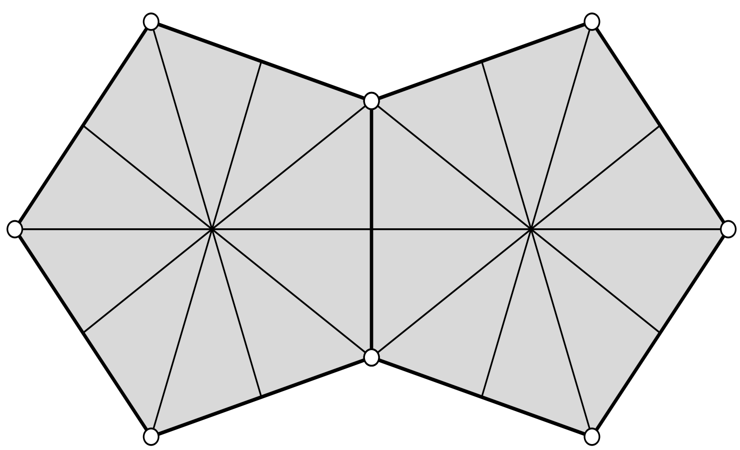

FIGURE 3. The octagon O (After A. Zorich)

For each edge $e_i \subset P$, where e_i is not the edge along which we reflected P, there exists an edge $e_{i'}$ in P_1 which is parallel to e_i. We identify these parallel sides to obtain a closed compact surface S. It can be checked, by triangulating O and applying the Euler characteristic formula, that the genus of S is two.

Endow S with the complex structure X', such that the quadratic differential dz^2 extends to a quadratic differential ψ on X. Scale O such that $\|\psi\| = 1$. All the edges of O are identified to a single point $\Sigma \in S$. This point is the only zero of ψ and it has order four. It is shown in [**Vee89**] that (X, q) is isomorphic to (X', ψ), see also the article by Earle and Gardiner in [**QS97**].

Since the image of $V(X,f,q)$ in $\mathbb{P}SL(2,\mathbb{R})$ is a lattice, it follows that $\chi := \mathbb{H}(X,f,q,)/V(X,f,q)$ is a Techmüller curve, and we have the embedding,

$$\chi \hookrightarrow \mathcal{M}_2.$$

We will study a degree ten ramified covering $\pi_{\tilde{\chi}}\colon \tilde{\chi} \to \chi$ which was introduced in [**Loc**]. The advantage of this covering is that it has an embedding in the configuration space of six points over which the Hitchin connection $\boldsymbol{\nabla}$ was defined in chapter 1 section 4. We make this more precise now.

Let $\mathcal{C}_\infty := \{(z_1,\dots,z_6) \in (\mathbb{CP}^1)^6 | z_i \neq z_j\}$, be the space of ordered configurations of six points and $\bar{\mathcal{C}} \cong \mathcal{C}/\mathcal{S}_6$, the space of unordered configurations defined in section 4 of chapter 1. Since $\mathrm{PSL}(2,\mathbb{C})$ is the group of complex automorphisms of $\mathbb{CP}^1$, it also acts on $\mathcal{C}_\infty$. We consider the action of the product of two groups

$$\mathrm{PSL}(2,\mathbb{C}) \times \mathcal{S}_6 \curvearrowright \mathcal{C}_\infty$$

and the qoutient $\mathcal{M}_2$ is the moduli space of six points on the Riemann sphere, or equivalently, the moduli space of genus 2 curves. These moduli spaces are equivalent since any $(z_1,\dots,z_6) \in \mathcal{C}$ defines a genus two curve given as the zero set of the equation $y^2 = (x-z_1)\dots(x-z_6)$. On the other hand given any genus 2 Riemann surface X, a choice of basis for the homology induces a basis $<\alpha_1,\alpha_2>$ for $H^1(X,K_X)$. Then the following map

$$\Phi\colon X \to \mathbb{CP}^1, \qquad \Phi(x) := \frac{\alpha_1(x)}{\alpha_2(x)}$$

is of degree two and simply branched over six points in $\mathbb{CP}^1$. Thus, X is a hyper elliptic curve. The branching locus of X and X' is the same if and only if X and X' are isomorphic as Riemann surfaces.

We now study the aforementioned covering of the Teichmüller curve discovered in [**Loc**].

Define $\tilde{\chi} := (\mathbb{P}^1 - \mu_5)$ where μ_5 are the fifth roots of unity. We have the embedding $\phi\colon \tilde{\chi}_\infty \to \mathcal{C}$, where $\tilde{\chi}_\infty := (\mathbb{P}^1 - (\mu_5 \cup \infty))$ given by the following explicit map introduced in [**Loc**] and [**McM06**]

$$\tilde{\phi}(t) = (1+t, \zeta+\zeta^{-1}t, \zeta^2+\zeta^{-2}t, \zeta^3+\zeta^{-3}t, \zeta^4+\zeta^{-4}t, \infty), \tag{4.2}$$

where $\zeta = e^{\frac{2\pi i}{5}}$. Notice that $\phi(t)$ belongs to $\mathcal{C}_\infty$ for all $t \in \tilde{\chi}$. This follows from the following computation: Suppose $\zeta^a \neq \zeta^b$. Then $\zeta^a + \zeta^{-a}t = \zeta^b + \zeta^{-b}t$ implies $t = -\frac{\zeta^a - \zeta^b}{\zeta^{-b}+\zeta^{-a}} = \zeta^{a+b} \in \mu_5$.

Dihedral group.

We will now consider the action of Dihedral group of order 10 on $\tilde{\chi}$. Consider the transformations $R, I\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by $R(x) = \zeta^2 x$ and $I(x) = \frac{1}{x}$. Notice that R generates a cyclic group of order five and I is an involution. We define $D := < R, I >$. Explicitly, D has the following presentation

$$D = < R, I | R^5 = I^2 = (R \circ I)^2 = \mathrm{Id} > . \tag{4.3}$$

Notice that $D(\mu_5) = \mu_5$, i.e. the set of fifth roots of unity is preserved under the action of D. This gives us a well defined action $D \curvearrowright \tilde{\chi}$. We define the quotient $\chi := \tilde{\chi}/D$ and let $\pi_{\tilde{\chi}} : \tilde{\chi} \to \chi$ be the quotient map. Following is proposition 4.11 from [**Loc**].

Proposition 4.1. *$\pi_1^{orb}(\chi) \cong \triangle(2, 5, \infty)$ where $\triangle(2, 5, \infty)$ is the Hecke triangle subgroup of $\mathbb{P}SL(2, \mathbb{R})$.*

Notice that this proposition implies that χ is a hyperbolic orbifold, namely a sphere with two orbifold points, say $a, b \in \chi$ and one puncture. Let us fix the notation, such that the order of ramification at a, b is two and five respectively.

We consider now the following two elements $\nu, \nu' \in S_5 \triangleleft S_6$.

Definition 4.2. $\nu(z_1, \dots, z_6) = (z_5, z_1, z_2, z_3, z_4, z_6)$
and $\nu'(z_1, \dots, z_6) = (z_1, z_5, z_4, z_3, z_2, z_6)$

It is clear from definition that $\nu^5 = \nu'^2 = \mathrm{Id}$. We have the following relationship between ν and R and between ν' and I.

Lemma 4.3. *$\nu(\tilde{\phi}(t)) = \zeta^{-1}\tilde{\phi}(R(t))$ and $\nu'(\tilde{\phi}(t)) = t\tilde{\phi}(I(t))$.*

Proof. We compute

$$\begin{aligned}\zeta^{-1}\tilde{\phi}(R(t)) &= \zeta^{-1}(1 + \zeta^2 t, \zeta + \zeta t, \zeta^2 + t, \zeta^3 + \zeta^{-1}t, \zeta^4 + \zeta^{-2}t, \infty)\\ &= (\zeta^{-1} + \zeta t, 1 + t, \zeta + \zeta^{-1}t, \zeta^2 + \zeta^{-2}t, \zeta^3 + \zeta^{-3}t, \infty).\end{aligned}$$

Now, the last tuple is the same as

$$(\zeta^4 + \zeta^{-4}t, 1 + t, \zeta + \zeta^{-1}t, \zeta^2 + \zeta^{-2}t, \zeta^3 + \zeta^{-3}t, \infty) = \nu(\tilde{\phi}(t)).$$

By a similar calculation, we obtain that

$$t\tilde{\phi}(I(t)) = (1 + t, \zeta^4 + \zeta^{-4}t, \zeta^3 + \zeta^{-3}t, \zeta^2 + \zeta^{-2}t, \zeta + \zeta^{-1}t, \infty) = \sigma(\tilde{\phi}(t)).$$

□

Since multiplication by ζ^{-1} and t are both elements of the dilation subgroup of $\mathbb{P}\mathrm{SL}(2, \mathbb{C})$ this lemma immediately shows that the map (5.1) descends to give a well defined map $\phi\colon \chi \to \mathcal{M}_2$ and that we get the following commutative diagram

$$\begin{tikzcd}
\tilde{\chi} \arrow[r, "\tilde{\phi}"] \arrow[dd, "\pi_{\tilde{\chi}}"'] & \mathcal{C}_\infty \arrow[d, "P_\infty"'] \\
 & \bar{\mathcal{C}}_\infty \arrow[d, "\overline{P}_\infty"'] \\
\chi \arrow[r, "\phi"'] & \mathcal{M}_2
\end{tikzcd}$$

Here $\overline{\mathcal{C}}_\infty$ is the quotient of $\mathcal{C}_\infty$ by the group S_6 and $\mathcal{M}_2$ is the quotient of $\overline{\mathcal{C}}_\infty$ by $\mathbb{P}\mathrm{SL}(2, \mathbb{C})$.

CHAPTER 5

Quantum representation of the (orbifold) fundamental group

1. Introduction

Recall $\mathcal{C}_\infty$, the space of ordered configurations of six points and $\bar{\mathcal{C}}_\infty \cong \mathcal{C}_\infty/\mathcal{S}_6$, the space of unordered configurations defined in section 4.4. Since $\mathrm{PSL}(2,\mathbb{C})$ is the group of complex automorphisms of $\mathbb{CP}^1$, it also acts on $\mathcal{C}_\infty$. We consider the action of the product of two groups

$$\mathrm{PSL}(2,\mathbb{C}) \times \mathcal{S}_6 \curvearrowright \mathcal{C}_\infty$$

and denote the quotient by $\mathcal{M}_2$. Hence $\mathcal{M}_2$ is the moduli space of six points on the Riemann sphere, or equivalently, the moduli space of genus 2 curves. We will study the embedding of a Teichmüller curve in this moduli space discovered in [**Vee89**] and further studied in [**Loc**].

Recall from the previous chapter, that $\tilde{\chi}_\infty := (\mathbb{CP}^1 - \mu_5)$, where μ_5 are the fifth roots of unity, and that we have an embedding $\tilde{\phi}\colon \tilde{\chi} \to \mathcal{C}_\infty$, where $\tilde{\chi} := \mathbb{CP}^1 - (\mu_5 \cup \infty)$, given by

$$\tilde{\phi}(t) = (1+t, \zeta + \zeta^{-1}t, \zeta^2 + \zeta^{-2}t, \zeta^3 + \zeta^{-3}t, \zeta^4 + \zeta^{-4}t, \infty), \tag{5.1}$$

where $\zeta = e^{\frac{2\pi \mathbf{i}}{5}}$.

We consider the action of the Dihedral group D of order 10 on $\tilde{\chi}_\infty$. This group is generated by the transformations $R, I\colon \mathbb{CP}^1 \to \mathbb{CP}^1$, defined by $R(x) = \zeta^2 x$ and $I(x) = \frac{1}{x}$, which are of order five and two, respectively.

Notice that $D(\mu_5) = \mu_5$, i.e. the set of fifth roots of unity is preserved under the action of D. This gives us a well defined action $D \curvearrowright \tilde{\chi}_\infty$, and we define the quotient $\chi := \tilde{\chi}_\infty/D$ with corresponding quotient map $\pi_{\tilde{\chi}} : \tilde{\chi}_\infty \to \chi$. Then χ is a finite volume hyperbolic surface with orbifold singularities. Topologically, it is a sphere with two orbifold points $b, a \in \chi$ and one puncture, and the order of ramification at b and a is two and five, respectively, i.e. $\pi^*_{\tilde{\chi}}(b) = \{-\mu_5\}$ and $\pi^*_{\tilde{\chi}}(b) = \{0, \infty\}$.

The orbifold fundamental group is calculated in [**Loc**],

$$\pi_1^{orb}(\chi) \cong \triangle(2,5,\infty),$$

where $\triangle(2,5,\infty)$ is the Hecke triangle subgroup of $\mathbb{P}SL(2,\mathbb{R})$, which is generated by the elements

$$T = \begin{pmatrix} 1 & 2\cos\frac{\pi}{5} \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These satisfy the relations $S^2 = (S \circ T)^5 = \mathrm{Id}$.

We consider the following two elements $\nu, \nu' \in \mathcal{S}_6$,

$$\nu(z_1, \ldots, z_6) = (z_5, z_1, z_2, z_3, z_4, z_6) \text{ and } \nu'(z_1, \ldots, z_6) = (z_1, z_5, z_4, z_3, z_2, z_6),$$

and it is clear that $\nu^5 = \nu'^2 = \mathrm{Id}$. Also, we have the following relationship between ν and R and between ν' and I,

$$\zeta\nu(\tilde{\phi}(t)) = \tilde{\phi}(R(t)) \text{ and } \frac{1}{t}\nu'(\tilde{\phi}(t)) = \tilde{\phi}(I(t)).$$

Since multiplication by $\frac{1}{t}$ and ζ corresponds to a dilation and a rotation in $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$, we get that the map $\tilde{\phi}$ is equivariant with respect to the D action on $\tilde{\chi}$ and $(\mathbb{P}\mathrm{SL}(2,\mathbb{C}) \times S_6)$ action on $\mathcal{C}_\infty$. This implies that $\tilde{\phi}$ descends to give a well defined map $\phi\colon \chi \to \mathcal{M}_2$. In fact we get the following commutative diagram,

$$\begin{tikzcd} \tilde{\chi} \arrow[r, "\tilde{\phi}"] \arrow[dd, "\pi_{\tilde{\chi}}"'] & \mathcal{C}_\infty \arrow[d, "P_\infty"] \\ & \bar{\mathcal{C}}_\infty \arrow[d, "\overline{P}_\infty"] \\ \chi \arrow[r, "\phi"'] & \mathcal{M}_2 \end{tikzcd} \tag{5.2}$$

Here $\overline{\mathcal{C}}_\infty$ is the quotient of $\mathcal{C}_\infty$ by the group S_6 and $\mathcal{M}_2$ is the quotient of $\overline{\mathcal{C}}_\infty$ by $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$.

In [**Vee89**], William Veech showed that the algebraic curve $y^2 = x^5 + 1$ equipped with the quadratic differential $c\frac{dx^2}{y^2}$ generates a Teichmuller curve $V \to \mathcal{M}_2$, and that $\pi_1^{orb}(V)$ is precisely $\triangle(2,5,\infty)$. In proposition 5.8 of [**Loc**] it is shown that $V \cong \chi$ as Riemann surfaces and their images in $\mathcal{M}_2$ coincide. Thus, $\phi(\chi)$ can be considered the Teichmüller curve generated by the algebraic curve $y^2 = x^5 + 1$ equipped with the quadratic differential $c\frac{dx^2}{y^2}$. Here c is a constant such that $\|c\frac{dx^2}{y^2}\| = 1$.

2. Generators of the (orbifold) fundamental group

Recall from the last section that $\pi_1^{orb}(\chi) \cong \triangle(2,5,\infty)$. We choose explicit paths based at the order five orbifold point of χ such that they generate the group $\pi_1^{orb}(\chi)$. Let $\tilde{\gamma}_0 \subset \tilde{\chi}$ be the path starting from 0 and running along the real axis to $1-\epsilon$, where ϵ is an arbitrarily small positive real number. Let $\tilde{\gamma}_1$ be the semi-circle starting from $1-\epsilon$ moving around 1 in an anti clockwise direction until it reaches $(1-\epsilon)^{-1}$. We will also need $\tilde{\gamma}_0^{-1}$ which is the path running from $1-\epsilon$ to 0 along the real axis.

Recall that the order five rotation R stabilizes, or fixes, the point 0. This implies that

$$\pi_{\tilde{\chi}}(0) =: a$$

is an orbifold point of order five, and the stabilizer group of this orbifold point is isomorphic to $\langle R \rangle$. The end points of the semi-circle $\tilde{\gamma}_1$ are identified with each other under the involution I, which implies that the projection

$$\pi_{\tilde{\chi}}(\tilde{\gamma}_1) =: \gamma_1 \subset \chi$$

is a homotopically non-trivial closed loop. Consider the projection of the path $\tilde{\gamma}_0$,

$$\pi_{\tilde{\chi}}(\tilde{\gamma}_0) =: \gamma_0 \subset \chi$$

then the composition

$$\gamma_0^{-1} \cdot \gamma_1 \cdot \gamma_0 =: \gamma \subset \chi \tag{5.3}$$

is a closed, connected, homotopically non-trivial loop with the property that it starts and ends at the order five orbifold point $a \in \chi$.

The following is part of proposition 4.11 from [**Loc**].

PROPOSITION 2.1. *The loop γ along with the orbifold stabilizer group $\langle R \rangle$ generates $\pi_1^{orb}(\chi, a)$, and under the isomorphism $\pi_1^{orb}(\chi, a) \cong \triangle(2,5,\infty)$ the order five stabilizer of the orbifold point a is identified with $U := S \circ T$ and γ is identified with the infinite order element T.*

By the commutativity of (5.2) and (5.3), it follows that

$$(\overline{P}_\infty \circ P_\infty)(\tilde{\phi}(\tilde{\gamma}_0)^{-1}) \cdot (\overline{P}_\infty \circ P_\infty)(\tilde{\phi}(\tilde{\gamma}_1))) \cdot (\overline{P}_\infty \circ P_\infty)(\tilde{\phi}(\tilde{\gamma}_0))) = \phi(\gamma).$$

We will construct a lift $\tilde{\gamma}$, of $\phi(\gamma)$, such that $\tilde{\gamma}$ is a connected path contained in $\mathcal{C}$. Later we will compute the holonomy of the connection given by the 1-form (2.22) along $\tilde{\gamma}$.

Recall from (2.35) the divisors $D_m^\infty := \{(z_1, \ldots, \infty, \ldots, z_6) \in \mathcal{C}_\infty\}$ where $1 \leqslant m \leqslant 6$ and m^{th} entry is fixed to ∞. Notice that D_m is a divisor in $\mathcal{C}_\infty$ and isomorphic to the space of ordered configurations of five points on $\mathbb{C}$. We have the disjoint union

$$\mathcal{C}_\infty = \mathcal{C} \sqcup_{m=1}^{6} D_m^\infty. \tag{5.4}$$

In particular, the image $\tilde{\phi}(\tilde{\chi}) \subset D_6^\infty$. Let $Z \in \mathbb{P}\mathrm{SL}(2, \mathbb{C})$ be the inversion in the unit circle, i.e.

$$Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and let

$$\tilde{\chi}_0 := (\mathbb{CP}^1 - (\mu_5 \cup \infty \cup -\mu_5)) \tag{5.5}$$

where μ_5 is the set of fifth roots of unity. Consider the map

$$\psi \colon \tilde{\chi}_0 \to \mathcal{C}, \quad \psi := Z \circ \tilde{\phi}$$

which can be written explicitly as

$$\psi(t) = (\frac{1}{\zeta + \zeta^{-1}t}, \frac{1}{\zeta^2 + \zeta^{-2}t}, \frac{1}{\zeta^3 + \zeta^{-3}t}, \frac{1}{\zeta^4 + \zeta^{-4}t}, \frac{1}{1+t}, 0), \quad \text{for all } t \in \tilde{\chi}_0. \tag{5.6}$$

The image $\psi(\tilde{\chi}_0)$ is now contained in $\mathcal{C}$. Since $Z \in \mathbb{P}\mathrm{SL}(2, \mathbb{C})$ and $\mathcal{M}_2$ is the quotient of $\mathcal{C}_\infty$ by the product $(\mathbb{P}\mathrm{SL}(2, \mathbb{C}) \times S_6)$ it follows that the map $\psi \colon \tilde{\chi}_0 \to \mathcal{C}$ covers the map $\phi \colon \chi \to \mathcal{M}_2$. In particular, $(\overline{P}_\infty \circ P_\infty)(\psi(\tilde{\chi}_0)) \subset \phi(\chi)$. Also, since $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are in the complement of $-\mu_5$, it follows that

$$(\overline{P}_\infty \circ P_\infty)(\psi(\tilde{\gamma}_0)^{-1}) \cdot (\overline{P}_\infty \circ P_\infty)(\psi(\tilde{\gamma}_1))) \cdot (\overline{P}_\infty \circ P_\infty)(\psi(\tilde{\gamma}_0))) = \phi(\gamma). \tag{5.7}$$

We will construct a path $p_I \subset \mathcal{C}$ such that

$$(\nu'\psi((\tilde{\gamma}_0)^{-1}))) \cdot p_I \cdot \psi(\tilde{\gamma}_1) \cdot \psi(\tilde{\gamma}_0) \subset \mathcal{C}$$

is a connected path and projects to $\phi(\gamma)$ in $\mathcal{M}_2$.

Let us first analyze the image $\psi(\tilde{\gamma}_0)$. The initial point $\psi(0)$ is shown in Figure 1. Here the six points form a symmetric configuration by taking the positions at the fifth roots of unity and zero in the specified order.

In Figure 2 the dashed lines show the trajectory of the six points as one travels along $\psi(\tilde{\gamma}_0)$. At the end point of $\psi(\tilde{\gamma}_0)$, the points z_1 and z_4 are a small distance (depends on ϵ) away from the positive real number $\frac{1}{2\mathrm{Re}(\zeta)}$ and a quick calculation shows that at the end points z_1 and z_4 are conjugate to each other, and both have their real parts bigger than $\frac{1}{2\mathrm{Re}(\zeta)}$. Moreover, z_1 lies in lower half plane of $\mathbb{C}$ and z_4 lies in the upper half plane. This local picture at $\frac{1}{2\mathrm{Re}(\zeta)}$ is shown in Figure 3. A similar story holds for the pair z_2 and z_3 around the negative real number $\frac{1}{2\mathrm{Re}(\zeta^2)}$.

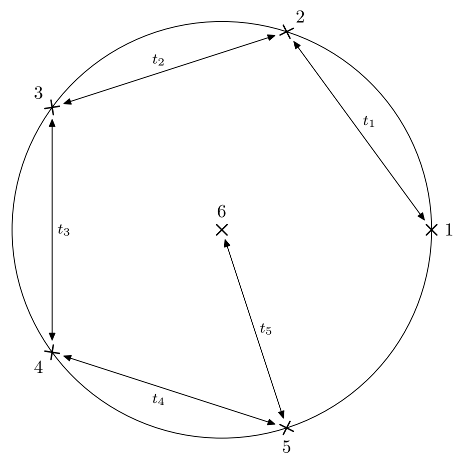

FIGURE 1. The configuration at $\psi(0)$. The crosses represent the positions of the points and t_i are the generators of the braid group

The point z_5 moves to $\frac{1}{2}$ and z_6 remains fixed at zero. Notice also that the end point of $\psi(\tilde{\gamma}_0)$ is the initial point of $\psi(\tilde{\gamma}_1)$. We now look at the configuration at the end

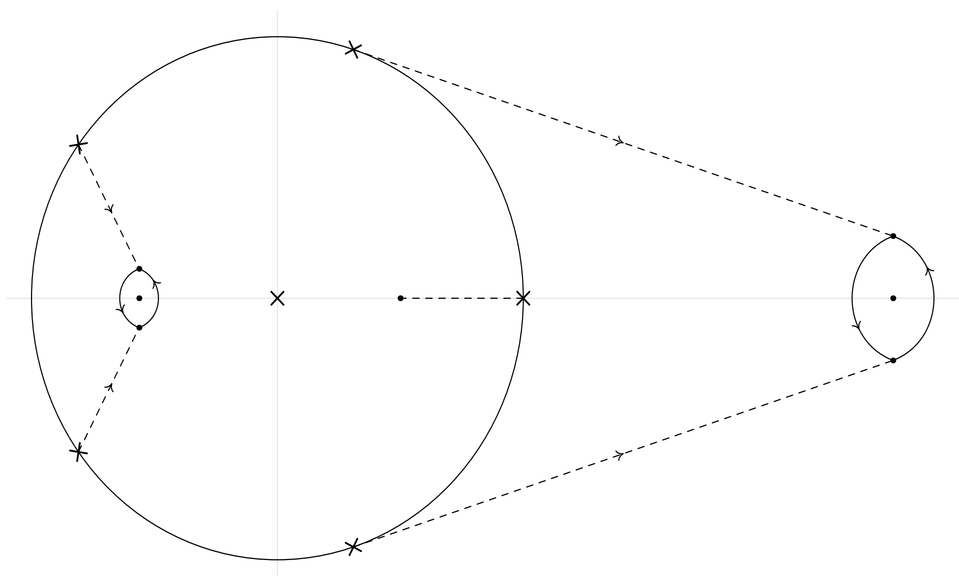

FIGURE 2

point of $\psi(\tilde{\gamma}_1)$. Recall that $\tilde{\gamma}_1$ was a semi-circle traversed anti-clockwise around one from $(1-\epsilon)$ to $(1-\epsilon)^{-1}$. At the end point of $\psi(\tilde{\gamma}_1)$, the points z_1 and z_4 are still conjugate to each other but both have real parts smaller than $\frac{1}{2\mathrm{Re}(\zeta)}$ and z_1 lies in the upper half plane and z_4 lies in the lower half plane. In effect, both z_1 and z_4 move in an anti-clockwise semi-circle around $\frac{1}{2\mathrm{Re}(\zeta)}$. This is also shown in figure 3.

A similar story holds for the pair z_2 and z_3 around the negative real number $\frac{1}{2\mathrm{Re}(\zeta^2)}$. The point z_5 moves to $\frac{1}{2}$ and z_6 remains fixed at zero. Notice also that the end point of $\psi(\tilde{\gamma}_1)$ is not the initial point of $\psi(\tilde{\gamma}_0)^{-1}$.

We now construct a path p_I which whose initial point is the end point of $\psi(\tilde{\gamma}_1)$. We calculate that

$$\psi(1-\epsilon) = \nu'(1-\epsilon)^{-1}(\psi(1-\epsilon)^{-1}). \tag{5.8}$$

Consider the following map

$$p_I\colon [1,(1-\epsilon)^{-1}] \to \mathbb{P}\mathrm{SL}(2,\mathbb{C}) \quad s \mapsto \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $p_I(\psi((1-\epsilon)^{-1}))$ be the path in $\mathcal{C}$ obtained by applying the path in Möbius transformations p_I to $\psi(1-\epsilon)^{-1}$. We will denote by p_I also the path $p_I(\psi((1-\epsilon)^{-1}))$. It follows from (5.8) that the configuration at the end point of p_I differs from the configuration at $\psi(1-\epsilon)$ only by the permutation ν'. The trajectory of z_4 and z_1 along the path p_I is shown in red in figure 3. A similar story holds for z_2 and z_3. The point z_5 moves a little to the right and z_6 is fixed at zero.

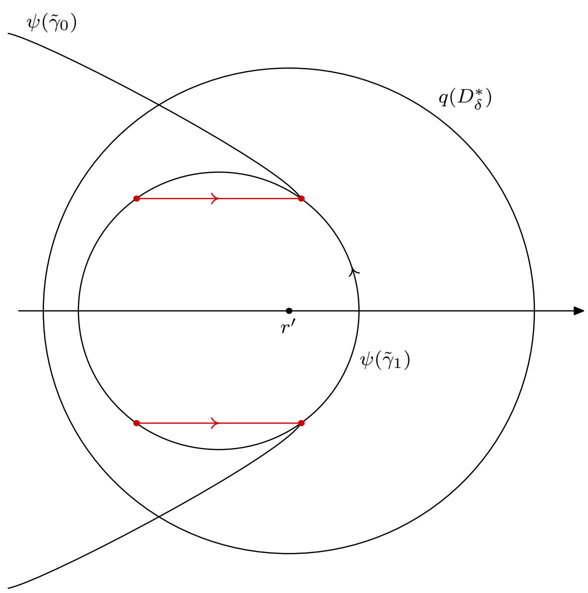

FIGURE 3

Since the end point of p_I differs from the initial point of $(\psi(\tilde{\gamma}_0)^{-1})$ by the permutation ν', it follows that $\nu'(\psi(\tilde{\gamma}_0)^{-1})) \cdot p_I$ is a connected path with initial point $\psi((1-\epsilon)^{-1})$ and end point $\nu'(\psi(0)$.

Since the initial point of $\nu'(\psi(\tilde{\gamma}_0)^{-1}))\cdot p_I$ is the end point of $\psi(\tilde{\gamma}_1)\cdot\psi(\tilde{\gamma}_0)$ it follows that

$$\tilde{\gamma} := \nu'(\psi((\tilde{\gamma}_0)^{-1})))\cdot p_I\cdot\psi(\tilde{\gamma}_1)\cdot\psi(\tilde{\gamma}_0)\subset\mathcal{C} \tag{5.9}$$

is a closed path in $\mathcal{C}$ with initial point $\psi(0)$ and end point $\nu'(\psi(0))$. Since the two endpoints of $\tilde{\gamma}$ are related by the element ν' in S_6, it follows that $P_\infty(\tilde{\gamma})\subset\overline{\mathcal{C}}$ is a closed loop in $\overline{\mathcal{C}}$. Moreover, since p_I is entirely contained in the $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$ orbit, it follows that $(\overline{P}_\infty\circ P_\infty)(p_I)$ is a constant loop in $\mathcal{M}_2$. Lastly, from (5.7) it follows that

$$(\overline{P}_\infty\circ P_\infty)(\tilde{\gamma}) = \phi(\gamma).$$

While $\mathcal{M}_2$ is an orbifold, its covering $\mathcal{C}$ is a smooth manifold, thus the stabilizer group $\langle R\rangle$ of the orbifold point $\phi(a)\in\mathcal{M}_2$ acts on $\mathcal{C}$. Since $0\in\tilde{\chi}$ is a fixed point of the action of the generator R, we analyze the action of R on $\mathcal{C}$ by computing

$$\begin{aligned}\psi(R(0)) &= \nu^{-1}\zeta^{-1}\psi(0)\\ &= \psi(0)\end{aligned}$$

where the second equality follows from the fact that $R(0) = 0$. This in particular implies that $\psi(0)$ is a fixed point of the transformation $\nu^{-1}\zeta^{-1}$ in the product group $(S_6\times\mathbb{P}\mathrm{SL}(2,\mathbb{C}))$. We represent $\nu^{-1}\zeta^{-1}$ as a path in $\mathcal{C}$ as follows. Consider the following path in $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$

$$p_R\colon [0,\frac{1}{5}]\to\mathbb{P}\mathrm{SL}(2,\mathbb{C}),\qquad s\mapsto\begin{bmatrix} e^{-\pi i s} & 0\\ 0 & e^{\pi i s}\end{bmatrix}\ s\in[0,\frac{1}{5}].$$

Denote also by p_R the path in $\mathcal{C}$ given by the action of p_R on the point $\psi(0)$. At both the initial and the end point of this path the first five points sit at the fifth roots of unity (sixth at zero) but with different ordering. The ordering differs by ν^{-1}. Since the end points of p_R are related by an element in S_6, $P_\infty(p_R)\subset\overline{\mathcal{C}}$ is a closed loop. Moreover, since p_R is entirely contained in the $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$ orbit, $(\overline{P}_\infty\cdot P_\infty)(p_R)$ is a constant path in $\mathcal{M}_2$, namely $\phi(a)$.

The paths $\tilde{\gamma}$ and p_R both start at $\psi(0)$. We now represent the image of these paths in $\overline{\mathcal{C}}$ as braids. Let

$$\overline{\psi(0)} := P(\psi(0))\qquad\overline{\gamma} := P(\tilde{\gamma})\qquad\overline{p_R} := P(p_R) \tag{5.10}$$

where P is the restriction of P_∞ to $\mathcal{C}$. Then $\overline{\gamma}$ and $\overline{p_R}$ are both closed loops based at $\overline{\psi(0)}$. We pick $\overline{\psi(0)}\in\overline{\mathcal{C}}$ as the base point for the fundamental group. It is well known that

$$\pi_1(\overline{\mathcal{C}},\overline{\psi(0)})\cong B_6$$

where B_6 is the braid group of degree 6. The choice of generators that we make for B_6 based at $\overline{\psi(0)}$ is shown in figure 1. Here $t_1, \ldots, t_5$ are the generators, where t_i corresponds to the point z_i and z_{i+1} exchanging positions by traveling around each other in anti-clockwise fashion.

It is now a straight forward exercise to represent the loops $\overline{\gamma}$ and $\overline{p_R}$ in $\pi_1(\overline{\mathcal{C}}, \overline{\psi(0)})$ as braids in terms of the generators t_i. Figure 4 shows the braid and the word in the generators corresponding to $\overline{p_R}$ and figure 5 shows the braid and the word in the generators corresponding to $\overline{\gamma}$.

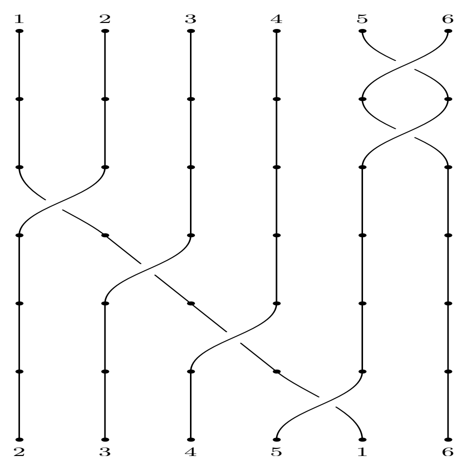

FIGURE 4. The loop $\overline{p_R}$ illustrated as the braid $t_4t_3t_2t_1t_5t_5$ from top to bottom

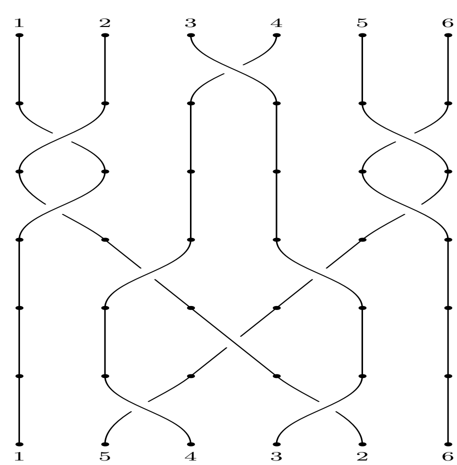

FIGURE 5. The loop $\overline{\gamma}$ illustrated as the braid $t_4t_2t_3t_4t_2t_5t_5t_1t_1t_3$ from top to bottom

Recall from (2.10) the group homomorphism

$$H\colon B_6 \to \mathrm{Sp}(4,\mathbb{Z})$$

explicitly given by (2.11) and from (2.12) the quotient group $A(G) \cong \mathrm{Sp}(4,\mathbb{Z})/\Gamma_2(2,4)$ where

$$\Gamma_2(2,4) := \left\{ \begin{pmatrix} I+2A & 2B \\ 2C & I+2D \end{pmatrix} \in Sp(4,\mathbb{Z}) | \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv (0,0) \pmod 2 \right\}.$$

Since $A(G)$ is a quotient group of $\mathrm{Sp}(4,\mathbb{Z})$, we have the short exact sequence

$$\Gamma_2(2,4) \to \mathrm{Sp}(4,\mathbb{Z}) \to A(G).$$

By composing the map $H\colon B_6 \to \mathrm{Sp}(4,\mathbb{Z})$ with the second homomorphism above, we get the homomorphism

$$\mathcal{H}\colon B_6 \to A(G).$$

The following lemma will be essential in our monodromy computations.

LEMMA 2.2. *The image of the closed loops $\overline{p_R}$ and $\overline{\gamma}$ under the homomorphism*

$$\mathcal{H}\colon B_6 \cong \pi_1(\overline{\mathcal{C}}, \overline{\psi(0)}) \to A(G)$$

is M_0 and M_1 respectively where

$$M_0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 3 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 \end{bmatrix} \quad M_1 = \begin{bmatrix} 0 & -1 & 0 & -2 \\ -1 & 0 & -2 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}.$$

PROOF. Figure 4 and figure 5 show the words in our choice of generators for B_6 for the loops $\overline{p_R}$ and $\overline{\gamma}$. We simply use the representation (2.11) on these words to arrive at M_0 and M_1. □

3. Pullback of the connection and iterated integrals on the Teichmüller curve

In the last section we defined the curve $\tilde{\chi}_0$ given by (5.5), and an embedding

$$\psi\colon \tilde{\chi}_0 \to \mathcal{C}$$

given by (5.6). Recall from section 4.4 the $\mathrm{End}(S^k(V))$ valued 1-form $\omega^{(k)}$ on $\mathcal{C}$ defined in (2.22). In this section we compute and analyze the pull back of the 1-form $\omega^{(k)}$ under ψ.

PROPOSITION 3.1. *We compute that projectively*

$$\psi^*(\omega^{(k)}) = \hbar \sum_{1\leqslant i\leqslant 5} \frac{A_i\, dt}{t-\zeta^i} \tag{5.11}$$

where $\zeta^i = e^{2\pi\imath\left(\frac{i}{5}\right)}$, $A_i = \widehat{\Omega}^{a,b} + \widehat{\Omega}^{c,d}$ *for* $1 \leqslant a < b, c < d \leqslant 5$ *such that* $[a+b] = [c+d] = [i]$ *where* $[x] := x(mod 5)$.

REMARK 3.2. For any $1 \leqslant i \leqslant 5$, there exists only one solution for A_i with the given constraints.

PROOF. The image $\psi(\tilde{\chi}_0)$ is contained in $D_6^0 \subset \mathcal{C}$, where D_m^0 for $1 \leqslant m \leqslant 6$ are defined in (2.38). The 1-form $\omega^{(k)}$ restricted to D_6^0 takes the following form

$$\begin{aligned}
\omega^{(k)}_{|D_6^0} &= \hbar\Bigg(\sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j}\frac{dz_i - dz_j}{z_i - z_j} + \sum_{i=1}^{5}\widehat{\Omega}^{i,6}\frac{dz_i}{z_i}\Bigg) \\
&= \hbar\Bigg(\sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log(z_i - z_j) + \sum_{i=1}^{5}\widehat{\Omega}^{i,6} d\log(z_i)\Bigg)
\end{aligned}$$

Thus the pull back is given by

$$\begin{aligned}
\psi^*(\omega^{(k)}) = \hbar\Bigg(&\sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log\Bigg(\frac{1}{\zeta^i + \zeta^{-i}t} - \frac{1}{\zeta^j + \zeta^{-j}t}\Bigg) \\
&- \sum_{i=1}^{5}\widehat{\Omega}^{i,6} d\log(\zeta^i + \zeta^{-i}t)\Bigg)
\end{aligned}$$

Now

$$\begin{aligned}
d\log\left(\frac{1}{\zeta^i + \zeta^{-i}t} - \frac{1}{\zeta^j + \zeta^{-j}t}\right) =& d\log(\zeta^i + \zeta^{-i}t - \zeta^j - \zeta^{-j}t) \\
&- d\log(\zeta^i + \zeta^{-i}t) - d\log(\zeta^j + \zeta^{-j})
\end{aligned}$$

which implies that

$$\begin{aligned}
&\psi^*(\omega^{(k)}) = \hbar\Bigg(\sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log(\zeta^i + \zeta^{-i}t - \zeta^j - \zeta^{-j}t) \\
&- \sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log(\zeta^i + \zeta^{-i}t) - \sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log(\zeta^j + \zeta^{-j}t)) - \sum_{i=1}^{5}\widehat{\Omega}^{i,6} d\log(\zeta^i + \zeta^{-i}t)\Bigg).
\end{aligned}$$

Now,

$$\sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log(\zeta^i + \zeta^{-i}t) + \sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log(\zeta^j + \zeta^{-j}t)) + \sum_{i=1}^{5}\widehat{\Omega}^{i,6} d\log(\zeta^i + \zeta^{-i}t) =$$

$$\sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j} d\log(\zeta^i+\zeta^{-i}t)+\sum_{1\leqslant i<j\leqslant 6} \widehat{\Omega}^{i,j} d\log(\zeta^j-\zeta^{-j}t)) = -k^2\,\mathrm{Id}\sum_{i=5}^{6} d\log(\zeta^i+\zeta^{-i}t)$$

where the last equality follows from (3) of propsition 4.4. This gives us that

$$\psi^*(\omega^{(k)}) = \hbar\Bigg(\sum_{1\leqslant i<j\leqslant 5} \widehat{\Omega}^{i,j} d\log(\zeta^i+\zeta^{-i}t-\zeta^j-\zeta^{-j}t)+k^2\,\mathrm{Id}\sum_{i=5}^{6} d\log(\zeta^i+\zeta^{-i}t)\Bigg)$$

Now,

$$\begin{aligned} d\log(\zeta^i+\zeta^{-i}t-\zeta^j-\zeta^{-j}t) &= \frac{(\zeta^{-i}-\zeta^{-j})dt}{(\zeta^i-\zeta^j)+(\zeta^{-i}-\zeta^{-j})t} \\ &= \frac{dt}{\frac{\zeta^i-\zeta^j}{\zeta^{-i}-\zeta^{-j}}+t}. \end{aligned}$$

This can be simplified by

$$-\frac{\zeta^i-\zeta^j}{\zeta^{-i}-\zeta^{-j}} = \frac{\zeta^i-\zeta^j}{\zeta^{-j}-\zeta^{-i}}.\frac{\zeta^{i+j}}{\zeta^{i+j}} = \zeta^{i+j},$$

the result being

$$d\log(\zeta^i+\zeta^{-i}t-\zeta^j-\zeta^{-j}t) = \frac{dt}{t-\zeta^{i+j}}.$$

Notice that

$$\zeta^{i+j}=\zeta^{k+l}$$

if $i+j=k+l(mod 5)$.

Similarly,

$$\begin{aligned} d\log(\zeta^i+\zeta^{-i}t) &= \frac{\zeta^{-i}dt}{\zeta^i+\zeta^{-i}t} \\ &= \frac{dt}{t+\zeta^{2i}}. \end{aligned}$$

This implies that

$$\psi^*(\omega^{(k)}) = \hbar\Bigg(\sum_{1\leqslant i\leqslant 5}\frac{A_i\,dt}{t-\zeta^i}+k^2\,\mathrm{Id}\sum_{1\leqslant i\leqslant 5}\frac{dt}{t+\zeta^{2i}}\Bigg) \tag{5.12}$$

which is projectively equivalent to (5.11).

□

Notice that while (5.12) has poles over the set of negative fifth roots of unity, the 1-form (5.11) does not and it extends as a holomorphic 1-form over $\tilde{\chi}$. Both (5.11) and (5.12) are Fuchsian differential equations with (regular) singularities at μ_5 and

μ_{10} respectively. Form now on we will consider (5.11) as an $\mathrm{End}_0(S^k(V))$ valued holomorphic 1-form over $\tilde{\chi}$ and denote by

$$\omega^{(k)}_{\tilde{\chi}} := \psi^*(\omega^{(k)}) \tag{5.13}$$

and the corresponding flat connection in the trivial bundle $\mathbb{P}(S^{(k)}(V)) \times \tilde{\chi}$ by

$$\boldsymbol{\nabla}^{(k)}_{\tilde{\chi}} := d + \omega^{(k)}_{\tilde{\chi}}. \tag{5.14}$$

Let us make the choice of line segments running from ζ^i, for $1 \leqslant i \leqslant 5$, to infinity as the branch cuts for the logarithm on $\tilde{\chi}$. With this choice (5.13) can be written as

$$\psi^*(\omega^{(k)}) = \hbar \sum_{1\leqslant i\leqslant 5} d\log(t-\zeta^i)A_i.$$

From this expression it is clear that $\omega^{(k)}_{\tilde{\chi}}$ is a logarithm 1-form on $\mathbb{C}$ with values in $\mathrm{End}_0(S^k(V))$, where the logarithmic singularities are at μ_5, and the residue at each $\zeta^i \in \mu_5$ is A_i.

The differential forms $\frac{dt}{t-\zeta^i}$ where $\zeta^i \in \mu_5$ give a basis for $H^1_{dR}(\tilde{\chi}, \mathbb{C})$. Let $\gamma\colon [a,b] \to \tilde{\chi}$ be a smooth map and for each ζ^i let us denote by

$$\gamma^*\big(\frac{dt}{t-\zeta^i}\big) = f_i(s_i)ds_i$$

the pull back of the 1-form $\frac{dt}{t-\zeta^i}$ to the interval. The ordinary line integral of this 1-form is given by

$$\int_\gamma \frac{dt}{t-\zeta^i} = \int_a^b f_i(s_i)ds_i$$

which is independent of the parametrization of γ. Now choose some r-tuple $(\zeta^{i_1}, \dots, \zeta^{i_r})$ where each $\zeta^{i_l} \in \mu_5$, and consider the recursively defined integral

$$L_a(\zeta^{i_1}, \dots, \zeta^{i_r}|b) = \int_a^b f_{i_r}(s_{i_r})\bigg(\int_0^{s_{i_r}} f_{i_{r-1}}(s_{i_{r-1}})\dots\bigg(\int_a^{s_{i_2}} f_{i_1}(s_{i_1})ds_{i_1}\bigg)\dots ds_{i_{r-1}}\bigg)ds_{i_r} \tag{5.15}$$

where

$$f_{i_r}(s_{i_r})ds_{i_r} = \gamma^*\big(\frac{dt}{t-\zeta^{i_r}}\big).$$

Given a smooth path γ and a tuple $(\zeta^{i_1}, \dots, \zeta^{i_r})$, the integral (5.15) is the iterated integral of the differential forms $\frac{dt}{t-\zeta^i_l}$ along the path γ. The integral (5.15) was extensively studied in [**LD53**] under the name of *Hyperlogarithms*. This integral is independent of the choice of the parametrization of γ and depends on the homotopy class of γ. In particular, such integrals are building blocks of the solution to the differential equation given by the parallel transport of (5.14), and thus play a crucial role in monodromy computations involving the connection (5.14). For general properties of iterated integrals, we refer the reader to [**Kas95**] and [**LD53**]. We use these integrals in the next chapter to give an explicit expression of the parallel transport of the connection (5.14) along arbitrary path in $\tilde{\chi}$.

For now, let us consider iterated integrals over a generating set of the $\pi_1(\tilde{\chi}, 0)$. Let f_m for $1 \leqslant m \leqslant 5$ be the image of

$$p_m\colon [0, 1-\epsilon] \to \tilde{\chi}, \qquad p_m(s) = s\,\zeta^m.$$

Associate to f_m the loop $g_m := f_m^{-1} \cdot \delta_m \cdot f_m \in \pi_1(\tilde{\chi}, 0)$, where δ_m is a small loop of radius ϵ traveling around ζ^m based at the end point of f_m.

The fundamental group $\pi_1(\tilde{\chi}, 0)$ is generated by g_m. Recall the automorphism $R\colon \tilde{\chi} \to \tilde{\chi}$ given by multiplication by ζ. The automorphism R acts on the generators of the fundamental group as follows,

$$R(g_m) = g_{[m]+1}$$

and it is clear that the action of R is transitive on the set of generators $\{g_m\}$, i.e. the set $\{g_m\}$ is the R orbit of any g_m.

For the path f_5, we have that

$$p_5^*\left(\frac{dt}{t-\zeta^i}\right) = \frac{ds}{s-\zeta^i},$$

thus for the choice of the r-tuple $(\zeta^{i_1}, \dots, \zeta^{i_r})$, the corresponding iterated integral according to the formula (5.15) is

(5.16)

$$L_0(\zeta^{i_1}, \dots, \zeta^{i_r}|1-\epsilon) = \int_0^{1-\epsilon} \frac{1}{s_{i_r} - \zeta^{i_r}} \left(\int_0^{s_{i_r}} \frac{1}{s_{i_{r-1}} - \zeta^{i_{r-1}}} \cdots \left(\int_0^{s_{i_2}} \frac{1}{s_{i_1} - \zeta^{i_1}} ds_{i_1} \right) \dots ds_{i_{r-1}} \right) ds_{i_r}.$$

It is clear that the integral above converges as $\epsilon \to 1$ if and only if $\zeta^{i_r} \neq 1$. For a choice of $(\zeta^{i_1}, \dots, \zeta^{i_r})$ such that $\zeta^{i_r} \neq 1$ we will denote by

$$L_0(\zeta^{i_1}, \dots, \zeta^{i_r}|1) \in \mathbb{C} \tag{5.17}$$

the limit of the iterated intergal (5.16) as $\epsilon \to 0$. For a given $(\zeta^{i_1}, \dots, \zeta^{i_r})$, the number (5.17) can be evaluated as a sum of polylogarithms of degree n where $1 \leqslant n \leqslant r$. We have that

$$\sum_{1 \leqslant i \leqslant 4} L_0(\zeta^i|1) = \log(5), \qquad \text{and} \qquad \sum_{\substack{\zeta^{i_1}, \zeta^{i_2} \\ \zeta^{i_2} \neq 1}} L_0(\zeta^{i_1}, \zeta^{i_2}|1) \in \mathbb{R}.$$

It is probable that for any $r \geqslant 1$, the sum

$$\sum_{\substack{\zeta^{i_1}, \dots, \zeta^{i_r} \in (\mu_5)^r \\ \zeta^{i_r} \neq 1}} L_0(\zeta^{i_1}, \dots, \zeta^{i_r}|1)$$

is a real number.

For an integer $k > 0$, let us define the following convergent power series in $\operatorname{End}\mathbb{P}(S^k(V))$

$$\Phi_5^{(k)} := \mathrm{Id} + \sum_{r=1}^{\infty} \hbar^r \sum_{\substack{\zeta^{i_1},\ldots,\zeta^{i_r}\in(\mu_5)^r \\ \zeta^{i_r}\neq 1}} L_0(\zeta^{i_1},\ldots,\zeta^{i_r}|1)A_{i_1}\ldots A_{i_r}. \tag{5.18}$$

We note here the striking similarity between the series $\Phi_5^{(k)}$ and the cyclotomic version of the Drinfel'd associator defined in [**Enr**]. The series, $\Phi_5^{(k)}$ also appears in [**DG**], see equation 5.16.1 and proposition 5.17 in the mentioned reference.

We construct a power series similar to (5.18) for other paths f_m. For this purpose we notice that $R(f_m) = f_{[m]+1}$ and

$$p_j^*\left(\frac{dt}{t-\zeta^i}\right) = p_{[j]+m}^*\left(\frac{dt}{t-\zeta^{i+m}}\right).$$

In the light of these relations, we define a convergent power series for $1 \leqslant m \leqslant 5$, which is a shift of (5.18) by $[m]$. Let

$$\Phi_m^{(k)} := \mathrm{Id} + \sum_{r=1}^{\infty} \hbar^r \sum_{\substack{\zeta^{i_1},\ldots,\zeta^{i_r}\in(\mu_5)^r \\ \zeta^{i_r}\neq 1}} L_0(\zeta^{i_1},\ldots,\zeta^{i_r}|1)A_{i_1+[m]}\ldots A_{i_r+[m]}. \tag{5.19}$$

Notice that for all $r \geqslant 1$ we have $A_{i_r+[m]} \neq A_{1+[m]}$ since $\zeta^{i_r} \neq 1$.

CONJECTURE 3.3. *Recall the order five matrix M_0 from lemma 2.2. This matrix has an action on $\operatorname{End}\mathbb{P}(S^k(V))$ for all $k > 0$ which is denoted by $M_0^{(k)}$. We have that*

$$(M_0^{(k)})^{-1} \cdot \hbar A_i \cdot M_0^{(k)} = A_{[i]+1}.$$

Let us now define a map from the set of generators $\{g_m\}$ to $\operatorname{End}\mathbb{P}(S^k(V))$, by

$$\rho_{\tilde{\chi}}^{(k)}(g_m) := (\Phi_m^{(k)})^{-1} \cdot e^{\hbar A_m} \cdot \Phi_m^{(k)}. \tag{5.20}$$

The permutation $\nu' \in S_5$ has a natural action on the set $\{\Phi_m\}$ given by permuting the index m. A simple calculation also gives the following equivariance property

$$\begin{aligned}\rho_{\tilde{\chi}}^{(k)}(R(g_m)) &= (\Phi_{[m]+1}^{(k)})^{-1} \cdot e^{\hbar A_{[m]+1}} \cdot \Phi_{[m]+1}^{(k)} \\ &= (\Phi_{\nu'(m)}^{(k)})^{-1} \cdot e^{\hbar A_{\nu'(m)}} \cdot \Phi_{\nu'(m)}^{(k)}\end{aligned}$$

where the second equality follows from the definition of ν'.

LEMMA 3.4. *The representation* (5.20) *extends to a group homomorphism*

$$\rho_{\tilde{\chi}}^{(k)} \colon \pi_1(\tilde{\chi}, 0) \to \operatorname{End}\mathbb{P}(S^k(V))$$

which is infact the monodromy of the connection (5.14).

PROOF. The represenation (5.20) clearly extends to the entire set of generators $\{g_m\}$ of $\pi_1(\tilde{\chi}, 0)$ thus inducing the morphism.The fact that this morphism is the monodromy of (5.14) follows from the general theory of monodromy of such connections outlined in [**Was87**], [**Kas95**] and [**LD53**]. $\square$

For monodromy computation of the orbifold fundamental group of χ, we need to analyze the asymptotic behavior of the parallel transport of $\boldsymbol{\nabla}^{(k)}_{\tilde{\chi}}$ in arbitrarily small neighborhoods of the singularities μ_5. Following results will be used in the sequel.

Fix an arbitrarily small positive real number ϵ. For each $1 \leqslant i \leqslant 5$, consider the following embedding

$$B_i \colon D_i^* \to \tilde{\chi}, \qquad b_i \mapsto \zeta^i + b_i \tag{5.21}$$

where $D_i^* := \{b_i \in \mathbb{C} | 0 < |b_i| < \epsilon\}$. An easy calculation shows

$$B_i^*(\omega^{(k)}_{\tilde{\chi}}) = \hbar \left(\frac{A_i}{b_i} + \sum_{\substack{1 \leqslant j \leqslant 5 \\ i \neq j}} \frac{A_j}{b_i - (\zeta^j - \zeta^i)} \right) db_i \tag{5.22}$$

Let $Y_i(b_i)$ be an $\operatorname{End} \mathbb{P}(S^{(k)}(V))$ valued function over B_i. Then Y_i is a solution of (5.22) if

$$Y_i' = \hbar \left(\frac{A_i}{b_i} + \sum_{\substack{1 \leqslant j \leqslant 5 \\ i \neq j}} \frac{A_j}{b_i - (\zeta^j - \zeta^i)} \right) Y_i. \tag{5.23}$$

PROPOSITION 3.5. *For all $1 \leqslant i \leqslant 5$, there exists a unique Y_i satisfying* (5.23) *such that*

$$Y_i(b_i) = Q_i(b_i)\, b_i^{\hbar A_i} \tag{5.24}$$

where $Q_i(b_i) = \sum_{r \geqslant 0} q_r^{(i)} b_i^r$, with $q_r^{(i)} \in \operatorname{End} \mathbb{P}(S^k(V))$, and $Q_i(0) = q_0^{(i)} = \mathrm{Id}$.

REMARK 3.6. For the expression $b_i^{\hbar A_i}$ to make sense on D_i^*, we must choose a branch cut for the logarithm. We choose the the positive real axis in D_i^* as the branch cut for the logarithm.

PROOF. By the theory of ordinary differential equations there exists a solution Y_i satisfying (5.23). We show that there exits a family q_r^i satisfying the above stated requirements such that Y_i is of the form (5.24). We write out

$$Y_i'(b_i) = \left(Q_i'(b_i) + \hbar \frac{Q_i(b_i) A_i}{b_i} \right) b_i^{\hbar A_i} = \hbar \left(\frac{A_i}{b_i} + \sum_{\substack{1 \leqslant j \leqslant 5 \\ i \neq j}} \frac{A_j}{b_i - (\zeta^j - \zeta^i)} \right) Q_i(b_i)\, b_i^{\hbar A_i}. \tag{5.25}$$

This can be rewritten as

$$b_i Q_i'(b_i) - \hbar[A_i, Q_i(b_i)] = -\hbar \sum_{\substack{1\leqslant j\leqslant 5\\ i\neq j}} A_j \frac{b_i\, Q_i(b_i)}{(\zeta^j - \zeta^i) - b_i}. \tag{5.26}$$

Expanding the above in powers of b_i, we get $[A_i, q_0^i] = 0$, and for $r > 0$

$$r q_r^{(i)} - \hbar[A_i, q_r^{(i)}] = -\hbar \sum_{l=1}^{r} \Bigg(\sum_{\substack{1\leqslant j\leqslant 5\\ i\neq j}} \frac{A_j}{(\zeta^j - \zeta^i)^l} \Bigg) q_{l-1}^{(i)} \tag{5.27}$$

The above equations, for every r, have a solution. Indeed, if we take $q_0^{(i)} = \mathrm{Id}$, then $q_r^{(i)}$ is uniquely determined by $q_0^{(i)}, \dots, q_{r-1}^{(i)}$ due to the fact that the operator $r\,\mathrm{Id} - \hbar\ \mathrm{ad}(A_i)$ is invertible with inverse equal to

$$\frac{1}{r} \sum_{n\geqslant 0} \frac{\hbar^n}{r^n}\, \mathrm{ad}(A_i)^n. \tag{5.28}$$

The convergence of $Q_i(b_i)$ results from the general fact that a formal solution of a regular singular equation is necessarily convergent, see [**Kas95**]. □

4. Computing Monodromy

Recall from Chapter 1 that we have the diagram of covering spaces

$$\widetilde{\mathcal{C}} \xrightarrow{\widetilde{P}} \mathcal{C} \xrightarrow{\mathrm{P}} \overline{\mathcal{C}}$$

with the corresponding sequence of deck groups

$$(\mathbb{Z}/2\mathbb{Z})^4 \to A(G) \to S_6 \tag{5.29}$$

where $A(G)$ is a quotient group of $\mathrm{Sp}(4,\mathbb{Z})$. We also have $\overline{P}\colon \overline{\mathcal{C}} \to \mathcal{M}_2$ where $\mathcal{M}_2$ is the moduli space of compact genus two Riemann surfaces. We also have the bundles $\mathbb{P}(S^k(V)) \times \tilde{\mathcal{C}}$, $\mathbb{P}(S^k(V)) \times \mathcal{C}$, and the quotient bundle (2.26) on $\overline{\mathcal{C}}$, with flat connections $\widetilde{\boldsymbol{\nabla}}^{(k)}$, $\boldsymbol{\nabla}^{(k)}$, and $\overline{\boldsymbol{\nabla}}^{(k)}$ respectively, where

$$\boldsymbol{\nabla}^{(k)} := d + \omega^{(k)}$$

and $\widetilde{\boldsymbol{\nabla}}^{(k)}$ is defined as the pull-back of $\boldsymbol{\nabla}^{(k)}$, and it ($\widetilde{\boldsymbol{\nabla}}^{(k)}$) is equivalent to the pull back of $\overline{\boldsymbol{\nabla}}^{(k)}$ by the appropriate composition of covering maps. We also have the (projective) bundle with a flat connection on $\mathcal{M}_2$, see (2.34), and the pull-back of (2.34) to $\overline{\mathcal{C}}$ coincides with the bundle (2.26) and the connection $\overline{\boldsymbol{\nabla}}^{(k)}$.

Recall from (5.10) the loops $\bar{p}_R$ and $\bar{\gamma}$ in $\pi_1(\overline{\mathcal{C}}, \overline{\psi(0)})$ such that the projection of $\overline{p}_R$ to $\mathcal{M}_2$ was the constant loop $\phi(a)$, namely the order five orbifold point of the Teichmüller curve $\phi(\chi)$, and the projection of $\bar{\gamma}$ to $\mathcal{M}_2$ was the closed loop

$\phi(\gamma) \subset \phi(\chi)$. Together, $\phi(\gamma)$ and the order five stabilizer group of the orbifold point $\phi(a)$ generate the $\pi_1^{orb}(\phi(\chi), \phi(a))$.

Recall also, that we have preferred lifts of the point $\overline{\psi(0)}$ and the paths $\bar{p}_R$ and $\bar{\gamma}$ in $\mathcal{C}$, namely $\psi(0)$, p_R and $\tilde{\gamma}$. In this section, we compute the parallel transport of the connection $\boldsymbol{\nabla}^{(k)}$ in the bundle $S^k(V) \times \mathcal{C}$ along the paths p_R and $\tilde{\gamma}$. Our strategy is to use specific parametrizations of these paths, pull back the connection to the parameter space and compute the parallel transport there.

Let $\mathbb{C}^*$ be the group of non-zero complex numbers. There exists an obvious action of $\mathbb{C}^*$ on $\mathcal{C}$ given by coordinate wise multiplication. We denote by $\mathbb{C}^*_z \subset \mathcal{C}$ the $\mathbb{C}^*$ orbit of $z \in \mathcal{C}$. Thus given any $z \in \mathcal{C}$, we get an embedding

$$\tau_z \colon \mathbb{C}^* \to \mathcal{C}$$

based at the point z.

LEMMA 4.1. *Given $z \in \mathcal{C}$, the pull-back connection $\tau_z^* \boldsymbol{\nabla}^{(k)}$ is trivial in the bundle $\mathbb{P}S^k(V) \times \mathbb{C}^*$ over $\mathbb{C}^*$.*

REMARK 4.2. This lemma is a direct corollary of proposition 4.10 and theorem 5.25 contained in chapter 2, but we give a direct proof here which is more instructive.

PROOF. Recall the 1-form, (2.22), defining the connection $\boldsymbol{\nabla}^{(k)}$ is given as a sum of holomorphic 1-forms $\hbar \widehat{\Omega}^{i,j} \frac{dz_i - dz_j}{z_i - z_j}$, which take values in the endomorphisms of $S^k(V)$. It is easy to see that

$$\tau_z^* \left(\frac{dz_i - dz_j}{z_i - z_j} \right) = \frac{z_i - z_j}{\alpha(z_i - z_j)} d\alpha = \frac{d\alpha}{\alpha},$$

where α is the global coordinate on $\mathbb{C}^*$. This implies that

$$\begin{aligned} \tau_z^* \boldsymbol{\nabla}^{(k)} &= d + \hbar \sum_{1 \leqslant i < j \leqslant 6} \widehat{\Omega}^{i,j} \frac{d\alpha}{\alpha} \\ &= d - 3\hbar \,\mathrm{Id}\, \frac{d\alpha}{\alpha} \end{aligned}$$

where the second equality follows from (4.4) of chapter 2.

Consider the following parametrization of the unit circle in $\mathbb{C}^*$,

$$\delta \colon [0, 2\pi] \to \mathbb{C}^* \quad \text{ where } \delta(s) = e^{is}.$$

Notice that $\langle \delta \rangle \cong \pi_1(\mathbb{C}^*, 1)$, i.e. δ generates the fundamental group of $\mathbb{C}^*$.

Now $\tau_z^* \boldsymbol{\nabla}^{(k)}$ restricted to δ is

$$(\tau_z \circ \delta)^* \boldsymbol{\nabla}^{(k)}(s) = d - 3\hbar \,\mathrm{Id}\, \frac{\delta'}{\delta} = d - i3\hbar \,\mathrm{Id}\,. \tag{5.30}$$

Let

$$Y(s) = exp\Big(- i3\hbar \, \mathrm{Id}(s) \Big).$$

Then

$$Y'(s) = -i3\hbar \, \mathrm{Id} \cdot Y(s),$$

which implies that $Y(s)$ gives the parallel transport of (5.30). The monodromy of (5.30) around the generator δ is then

$$Y(2\pi) = exp\Big(- 3(2\pi i)\hbar \, \mathrm{Id} \Big) = \mathrm{Id} \in \mathrm{End}_0(S^k(V)).$$

This implies that the monodromy of $\tau_z^* \boldsymbol{\nabla}^{(k)}$ along δ is trivial in $\mathbb{P}S^k(V)$, and since δ generates the entire fundamental group, it follows that $\boldsymbol{\nabla}^{(k)}$ is trivial over any orbit $\mathbb{C}_z^*$. $\square$

We also get the following.

COROLLARY 4.3. *The parallel transport of $\boldsymbol{\nabla}^{(k)}$ along p_R is projectively trivial.*

PROOF. Recall that the path $p_R \subset \mathcal{C}$ is given as the orbit of a rotation in $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$ applied to $\psi(0)$. This implies that p_R is contained in $\mathbb{C}^*_{\psi(0)}$. The result now follows from lemma 4.1 $\square$

We will need the following statement later.

PROPOSITION 4.4. *The $\mathbb{C}^*$ action on $\mathcal{C}$ lifts to $\widetilde{\mathcal{C}}$.*

PROOF. This follows from theorem 4.19 in chapter 2, where $\mathbb{M}$ is $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$, since this $\mathbb{C}^*$ action is the action of dilations in $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$. $\square$

Now we compute the parallel transport of $\boldsymbol{\nabla}^{(k)}$ along $\tilde{\gamma}$. Recall from (5.9)

$$\tilde{\gamma} := \nu'(\psi((\tilde{\gamma}_0)^{-1}))) \cdot p_I \cdot \psi(\tilde{\gamma}_1) \cdot \psi(\tilde{\gamma}_0) \subset \mathcal{C}.$$

In order to compute the parallel transport along $\tilde{\gamma}$ we must compute the parallel transport along all the paths separately appearing in the expression above.

We observe that the parallel transport of $\boldsymbol{\nabla}^{(k)}$ along p_I is trivial. This follows since p_I is contained in the $\mathbb{C}^*$ orbit of $\psi((1-\epsilon)^{-1})$ and proposition 4.1 tells us that $\boldsymbol{\nabla}^{(k)}$ is trivial along any $\mathbb{C}^*$ orbit.

We now compute the parallel transport along $\psi(\tilde{\gamma}_1)$. The semi circle $\tilde{\gamma}_1 \subset \tilde{\chi}$ is contained in the image of B_1, defined in (5.21), which is a punctured disk of radius ϵ based at $1 \in \tilde{\chi}$.

PROPOSITION 4.5. *The parallel transport along $(B_1 \circ \psi)^*(\psi(\tilde{\gamma}_1))$ with respect to $(B_1 \circ \psi)^* \boldsymbol{\nabla}^{(k)}$ in $\mathbb{P}S^k(V)$ is*

$$Q_1(-i\epsilon) \cdot exp\big(-\pi i \hbar A_1\big) \cdot Q_1(i\epsilon),$$

where $Q_1(b_1)$ is the power series from proposition 3.5 (of this chapter).

REMARK 4.6. We assume the ϵ in the definition of B_1 to be bigger than the ϵ in the definition of $\tilde{\gamma}_1$. The ϵ appearing in the proposition refers to the ϵ for $\tilde{\gamma}_1$.

PROOF. From proposition 3.5 we know that the parallel transport of $(B_1 \circ \psi)^* \boldsymbol{\nabla}^{(k)}$ is given by a solution of the form $Y(b_1)$ shown in (5.24). The pull back $(B_1 \circ \psi)^*(\psi(\tilde{\gamma}_1))$ is a semi circle in D_1 traveling in an anti-clockwise direction from $i\epsilon$ to $-i\epsilon$, avoiding the positive real axis in D_1 (which was our choice for the branch cut of the logarithm on D_1). Thus the parallel transport is given by

$$\begin{aligned}
Y(-i\epsilon)(Y(\epsilon))^{-1} &= Q_1(-i\epsilon)(-i\epsilon)^{\hbar A_1}(i\epsilon^{-\hbar A_1})Q_1(i\epsilon)^{-1} \\
&= Q_1(-i\epsilon)exp(log(-i\epsilon)\hbar A_i)exp(-log(i\epsilon)\hbar A_i)Q_1(i\epsilon)^{-1} \\
&= Q_1(-i\epsilon)exp((log(i\epsilon)+\pi i)\hbar A_1)exp(-log(i\epsilon)\hbar A_1)Q_1(i\epsilon)^{-1} \\
&= Q_1(-i\epsilon)exp(\pi i \hbar A_i)Q_1(i\epsilon)^{-1},
\end{aligned}$$

where the first equality is from (5.24) and third equality follows from our choice of the branch cut for the logarithm. $\square$

Now we compute the parallel transport of $\boldsymbol{\nabla}^{(k)}$ along $\psi(\tilde{\gamma}_0)$. Recall that $\tilde{\gamma}_0 \subset \tilde{\chi}$ is the interval $[0, 1-\epsilon]$. We take the pull back $\psi^*(\boldsymbol{\nabla}^{(k)})$, which is given in (5.11), restrict it to the interval $[0, 1-\epsilon]$, and compute the monodromy there. This monodromy was computed in section 3 of this chapter and is given by the following convergent series

$$P^{(k)}(1-\epsilon) = \mathrm{Id} + \sum_{r=1}^{\infty} \hbar^r \sum_{\zeta^{i_1},\ldots,\zeta^{i_r} \in \mu_5} L_0(\zeta^{i_1}, \ldots, \zeta^{i_r} | 1-\epsilon)) A_{i_1} \ldots A_{i_r}, \tag{5.31}$$

where the complex numbers $L_0(\zeta^{i_1}, \ldots, \zeta^{i_r}|1-\epsilon)$ are iterated integrals defined in (5.16).

THEOREM 4.7. *Let $k > 0$ be an integer. Let*

$$\rho^{(k)} \colon \pi_1^{orb}(\mathcal{M}_2, \phi(a)) \to \operatorname{End} \mathbb{P}(S^k(V))$$

be the monodromy representation of the Hitchin connection based at the orbifold point $\phi(a)$.

Recall U and T, the generators of the orbifold fundamental subgroup $\pi_1^{orb}(\phi(\chi), \phi(a))$. Then

1) $\rho^{(k)}(U) = (M_0^{(k)})^{-1}$
2) $\rho^{(k)}(T) = P^{(k)}(1-\epsilon)^{-1} \cdot (M_1^{(k)})^{-1} \cdot Q_1(-i\epsilon) \cdot exp\big(-\pi i \hbar A_1\big) \cdot Q_1(i\epsilon)^{-1} \cdot P^{(k)}(1-\epsilon)$
where $M_0^{(k)}$ *and* $M_1^{(k)}$ *are from lemma 2.2.*

Proof. Recall from Proposition 2.1, that the elements U and T of $\pi_1^{orb}(\chi, a)$ are represented by the generator of the stabilizer of the orbifold point $a \in \chi$ and the curve γ, respectively. These in turn lift to the loops $\overline{p}$ and $\overline{\gamma}$ in $\bar{\mathcal{C}}$, so we must calculate the holonomy around these loops with respect to $\overline{\boldsymbol{\nabla}}^{(k)}$.

Starting with $\bar{p}$, let $\overline{\mathrm{PT}}(\bar{p})$ denote the desired parallel transport map of $\overline{\boldsymbol{\nabla}}^{(k)}$ around $\bar{p}$, and let $\mathrm{PT}(p(\psi(0)))$ denote the parallel transport of $\boldsymbol{\nabla}^{(k)}$ around the lift $p(\psi(0))$ of $\bar{p}$ to $\mathcal{C}$. Then lemma 2.2 and corollary 4.3 imply that

$$\overline{\mathrm{PT}}(\bar{p}) = M_0^{-1} \cdot \mathrm{PT}(p(\psi(0))) = M_0^{-1}$$

as projective endomorphisms of $S^k(V)$, because $\mathrm{PT}(p(\psi(0)))$ is projectively trivial. Also, proposition 4.4 implies that the element $M_0 \in A(G)$, which is the image of $\overline{p} \in \pi_1(\overline{\mathcal{C}}, \overline{\psi(0)})$ under the map

$$\pi_1(\overline{\mathcal{C}}, \overline{\psi(0)}) \to A(G),$$

is in fact independent of the curve $p \subset \mathbb{C}^*$ connecting 1 to ζ^{-1}. This proves the first part.

For the second statement, we first get that

$$\overline{\mathrm{PT}}(\bar{\gamma}) = M_1^{-1} \cdot \mathrm{PT}(\gamma),$$

and by applying (5.9), we get that

$$\begin{aligned}\overline{\mathrm{PT}}(\bar{\gamma}) &= M_1^{-1} \cdot \mathrm{PT}(\nu'(\psi(\tilde{\gamma}_0)^{-1})) \cdot \mathrm{PT}(p'(\psi((1-\epsilon)^{-1})) \cdot \psi(\tilde{\gamma}_1)) \cdot \mathrm{PT}(\psi(\tilde{\gamma}_0)) \\ &= \mathrm{PT}(\psi(\tilde{\gamma}_0)^{-1}) \cdot M_1^{-1} \cdot \mathrm{PT}(p'(\psi((1-\epsilon)^{-1})) \cdot \psi(\tilde{\gamma}_1)) \cdot \mathrm{PT}(\psi(\tilde{\gamma}_0)),\end{aligned}$$

where we used the fact that the connection $\tilde{\boldsymbol{\nabla}}^{(k)}$ is equivariant with respect to the action of $A(G)$ to commute M_1^{-1} with the parallel transport. This commutation also had the effect of removing ν' from $\nu'(\psi(\tilde{\gamma}_0)^{-1})$ because M_1^{-1} maps to ν' under the map $A(G) \to \mathcal{S}_6$. Finally, $\mathrm{PT}(\psi(\tilde{\gamma}_0)^{-1})$ is equal to $Y(\beta)$, and therefore, the desired expression follows from proposition 4.5. □

For $k = 1$, we in particular have that

$$\rho^{(1)}(U) = (M_0^{(1)})^{-1} \qquad \rho^{(1)}(T) = (M_1^{(1)})^{-1}$$

The second equality follows since $\widehat{\Omega}^{i,j}$ are differential operators of order two and act on homogenous polynomials of degree one, thus on $\mathbb{P}(V)$, by zero. This implies that $P^{(1)}(1-\epsilon) = Q_1(i\epsilon) = exp(-\pi i \hbar A_i) = \mathrm{Id}$.

Moreover, in [**Kas95**] and [**LD53**] it is shown that

$$\lim_{\epsilon\to 0}\rho^{(k)}(T) = (\Phi_1^{(k)})^{-1}\cdot\Big((M_1^{(k)})^{-1}\cdot exp\big(-\pi i\hbar A_i\big)\Big)\cdot\Phi_1^{(k)}$$

where recall from the last section that

$$\Phi_1^{(k)} = \mathrm{Id} + \sum_{r=1}^{\infty}\hbar^r \sum_{\substack{\zeta^{i_1},\dots,\zeta^{i_r}\in(\mu_5)^r\\ \zeta^{i_r}\neq 1}} L_0(\zeta^{i_1},\dots,\zeta^{i_r}|1)A_{i_1}\dots A_{i_r}.$$

This gives the theorem 0.2 in the introduction with $\Phi^{(k)} = \Phi_1^{(k)}$.

CHAPTER 6

Geodesic Flow

1. Hyperlogarithm and Parallel Transport of Hitchin connection

Recall that $\tilde{\chi} := \mathbb{CP}^1 - \{\mu_5 \cup \infty\}$. Let us make the choice of line segments running from each element in μ_5 to infinity as the branch cuts for the logarithm. Let $p, q \in \tilde{\chi}$ be distinct and let $\gamma\colon [0,1] \to \tilde{\chi}$ be a path starting from p and ending at q and avoiding the branch locus of logarithm. and let $a \in \mu_5$. Consider the following function

$$L_p(a|q) := \int_p^q \frac{dt}{t-a} = \log \frac{q-a}{p-a},$$

where t is the global rational coordinate on $\tilde{\chi}$. Now choose some tuple $(a_1, a_2, \ldots, a_r)$, where each $a_i \in \mu_5$, and consider the recursively defined integral in [**LD53**]

$$L_p(a_1, a_2, ..., a_r|q) := \int_p^q \frac{L_p(a_1, a_2, \ldots, a_{r-1}|q)}{t - a_r}. \tag{6.1}$$

Such integrals are called *Hyperlogarithms* for $a_l \in \mu_5$.

We have the uniform bound

$$|L_p(a_1, a_2, ..., a_r|q)| < \frac{1}{r!}\left(\frac{l(\gamma)}{\sigma}\right)^r, \tag{6.2}$$

where $l(\gamma)$ is the euclidean length of γ and σ is the shortest euclidean distance from γ to μ_5. If the path γ is such that it is contained entirely in a small disk around p, then (6.1) can be expanded as the following series

$$(-1)^r \sum_{m=r}^{\infty} (q-p)^m \sum_{\nu_1+\nu_2+\ldots+\nu_r=m} \frac{1}{\nu_1(\nu_1+\nu_2)\ldots(\nu_1+\nu_2+\ldots+\nu_r)}$$
$$\times \frac{1}{(a_1-p)^{\nu_1}(a_2-p)^{\nu_2}\ldots(a_r-p)^{\nu_r}}$$

in this disk.

Recall from (5.14) the connection $\boldsymbol{\nabla}^{(k)}_{\tilde{\chi}}$ in the trivial bundle $\mathbb{P}(S^k(V)) \times \tilde{\chi}$, where V is the space of homogeneous polynomials of degree one in four variables. Let $\gamma\colon [0,1] \to \tilde{\chi}$, we denote the parallel transport of $\boldsymbol{\nabla}^{(k)}_{\tilde{\chi}}$ along this path by

$$PT^{(k)}(\gamma(s))\colon [0,1] \to \operatorname{End}\mathbb{P}(S^k(V)), \qquad s \in [0,1]$$

and define

(6.3)
$$PT^{(k)}(\gamma(s)) := \mathrm{Id} + \sum_{r\geqslant 1}^{\infty} \hbar^r \sum_{a_1,a_2,\ldots,a_r \in (\mu_5)^r} L_a(a_1,a_2,...,a_r|\gamma(s)) A_{a_1}, A_{a_2}, \ldots A_{a_r}$$

where $A_{a_i} \in \operatorname{End}\mathbb{P}(S^k(V))$ are second order differential operators defined in (5.11). We show that (6.3) is convergent with respect to a suitable norm.

Let $\|.\|$ be the operator norm on $\mathbb{P}S^k(V)$. This norm satisfies the schwartz and the triangle inequalities. For convenience of notation, we define

$$\sum_{r\geqslant 0}^{\infty} X_r(s) := \sum_{r\geqslant 0}^{\infty} \Bigg(\sum_{a_1,a_2,\ldots,a_r \in \mu_5} L_a(a_1,a_2,\ldots,a_r|\gamma(s)) A_{a_1} \ldots A_{a_{r-1}}, A_{a_r} \Bigg). \tag{6.4}$$

We rewrite the norm (6.2) as

$$|L_a(a_1,a_2,...,a_r|\gamma(s))| \leqslant \frac{D(s)^r}{r!}, \tag{6.5}$$

where $D(s) := \frac{l(s)}{\delta(s)}$ and $l(s)$ is the euclidean length of the path $\gamma(s)$, and $\delta(s)$ is the shortest length from the path $\gamma(s)$ to any $a \in \mu_5$.

Proposition 1.1. *The series $\sum_{r\geqslant 0}^{\infty} X_r(s)$ converges absolutely for $s \in [0,1]$ if $\delta(s) > 0$.*

Proof. From definition we get

$$\begin{aligned}
\sum_{r\geqslant 0}^{\infty} \|X_r(s)\| &= \sum_{r\geqslant 0}^{\infty} \Bigg(\| \sum_{a_1,a_2,\ldots,a_r \in \mu_5} L_a(a_1,a_2,...,a_r|\gamma(s)) A_{a_1,k} \ldots A_{a_{r-1},k} A_{a_r,k} \| \Bigg) \\
&\leqslant \sum_{r\geqslant 0}^{\infty} \Bigg(\sum_{a_1,a_2,\ldots,a_r \in \mu_5} |L_a(a_1,a_2,...,a_r|\gamma(s))| \; \|A_{a_1,k} \ldots A_{a_{r-1},k} A_{a_r,k}\| \Bigg) \\
&\leqslant \sum_{r\geqslant 0}^{\infty} \Bigg(\sum_{a_1,a_2,\ldots,a_r \in \mu_5} |L_a(a_1,a_2,...,a_r|\gamma(s))| \; \|A_{a_1,k}\| \ldots \|A_{a_{r-1},k}\| \|A_{a_r,k}\| \Bigg) \\
&\leqslant \sum_{r\geqslant 0}^{\infty} \Bigg(\frac{D(s)^r}{r!} \Bigg(\sum_{a_1,a_2,\ldots,a_r \in \mu_5} \|A_{a_1,k}\| \ldots \|A_{a_{r-1},k}\| \|A_{a_r,k}\| \Bigg) \Bigg)
\end{aligned}$$

Here, the first equality follows from triangle inequality, second from schwartz inequality, and third by the bound (6.5). Now, let $z_a := \|A_{a,k}\|$ for all $a \in \mu_5$, and

$z := \sum_{a\in\mu_5} z_a$. We then get that

$$\begin{aligned}\sum_{r\geqslant 0}^{\infty}\left(\sum_{a_1,\dots,a_r\in\mu_5}\|A_{a_1,k}\|\dots\|A_{a_{r-1},k}\|\|A_{a_r,k}\|\right) &= \sum_{r\geqslant 0}^{\infty}\left(\sum_{a_1,\dots,a_r\in\mu_5} z_{a_1}z_{a_2}\dots z_{a_{r-1}}z_{a_r}\right)\\ &\leqslant \sum_{r\geqslant 0}^{\infty}\left(\sum_{a\in\mu_5} z_a\right)^r = \sum_{r\geqslant 0}^{\infty} z^r\end{aligned}$$

We substitute this in the expression derived above to obtain

$$\begin{aligned}\sum_{r\geqslant 0}^{\infty}\|X_r(s)\| &\leqslant \sum_{r\geqslant 0}^{\infty}\left(\frac{D(s)^r}{r!}\left(\sum_{a_1,a_2,\dots,a_r\in\mu_5}\|A_{a_1,k}\|\dots\|A_{a_{r-1},k}\|\|A_{a_r,k}\|\right)\right)\\ &\leqslant \sum_{r\geqslant 0}^{\infty}\left(\frac{D(s)^r z^r}{r!}\right) = e^{D(s)z}\end{aligned}$$

If $D(s)z < \infty$ then $e^{D(s)z} < \infty$, and the lemma is proved. Now, $D(s)z = \infty$ if and only if $\delta(s) = 0$. This case is excluded by assumption. $\square$

We now show that (6.3) is the parallel transport of $\boldsymbol{\nabla}^{(k)}_{\tilde{\chi}}$. Recall that

$$\boldsymbol{\nabla}^{(k)}_{\tilde{\chi}} = d + \omega^{(k)}_{\tilde{\chi}}$$

where the 1-form $\omega^{(k)}_{\tilde{\chi}}$ is defined in (5.13).

THEOREM 1.2. *Let $\gamma(s)$ be a path in $\tilde{\chi}$. Then*

$$(PT^{(k)})' = PT^{(k)}\omega^{(k)}_{\tilde{\chi}}.$$

PROOF. From the definition of $\omega^{(k)}_{\tilde{\chi}}$ we must show that

$$\frac{d\,PT^{(k)}(\gamma(s))}{ds} = \hbar\sum_{a\in\mu_5} PT^{(k)}(\gamma(s))\frac{A_a}{\gamma(s)-a}. \tag{6.6}$$

Using (6.3) we have that,

$$\begin{aligned}\frac{d\,PT^{(k)}(\gamma(s))}{ds} &= \frac{d}{ds}\left(\mathrm{Id} + \sum_{r\geqslant 1}^{\infty}\hbar^r\sum_{a_1,\dots,a_r\in(\mu_5)^r} L_a(a_1,\dots,a_r|\gamma(s))A_{a_1},\dots A_{a_r}\right)\\ &= \mathrm{Id} + \sum_{r\geqslant 1}^{\infty}\hbar^r\frac{d}{ds}\left(\sum_{r\geqslant 1}\sum_{a_1,\dots,a_r\in(\mu_5)^r} L_a(a_1,\dots,a_r|\gamma(s))A_{a_1}\dots A_{a_r}\right)\end{aligned}$$

From (6.1) it follows that the hyperlogarithm satisfies the following differential equation.

$$\frac{d}{ds}L_a(a_1,\dots,a_r|\gamma(s)) = \frac{L_a(a_1,\dots,a_{r-1}|\gamma(s))}{\gamma(s)-a_r}.$$

Using this differential the right hand side above becomes

$$\mathrm{Id}+\sum_{r\geqslant 1}^{\infty}\hbar^r\sum_{a\in\mu_5}\left(\sum_{r\geqslant 1}\sum_{a_1,\dots,a_{r-1}\in(\mu_5)^{r-1}}L_a(a_1,\dots,a_{r-1}|\gamma(s))A_{a_1}\dots A_{a_{r-1}}\right)\frac{A_a}{\gamma(s)-a}.$$

Rearranging the terms we get

$$\hbar\sum_{a\in\mu_5}\left(\mathrm{Id}+\sum_{r>1}^{\infty}\hbar^r\sum_{a_1,\dots,a_{r-1}\in\mu_5}L_a(a_1,\dots,a_{r-1}|\gamma(s))A_{a_1}\dots A_{a_{r-1}}\right)\frac{A_{a,k}}{\gamma(s)-a}.$$

The term in the bracket is exactly $PT^{(k)}(\gamma(s))$, which yields the desired result (6.6). $\square$

2. An application of Oseledets multiplicative ergodic theorem

Let $U\chi$ be the unit tangent bundle of the Teichmüller curve and let μ be the canonical Louiville measure on $U\chi$. It is well known that geodesic flow on the unit tangent bundle of any finite area hyperbolic surface is ergodic with respect to the Louiville measure, [**CFS82**], which implies the geodesic flow $T_s\colon U\chi\to U\chi$, for $s\in\mathbb{R}_+$, is ergodic with respect to μ. We have the degree ten branched covering map $\pi_{\tilde\chi}\colon\tilde\chi\to\chi$. Using this map we can pull back the unit tangent bundle $\pi^*_{\tilde\chi}U\chi\to\tilde\chi$ which we denote by $U\tilde\chi\to\tilde\chi$. Likewise we can lift the geodesic flow $\pi^*_{\tilde\chi}T_s\colon\pi^*_{\tilde\chi}U\chi\to\pi^*_{\tilde\chi}U\chi$, which we denote by $\tilde T_s\colon U\tilde\chi\to U\tilde\chi$. Let $\pi^*_{\tilde\chi}(\mu):=\tilde\mu$ be the lift of the Louiville measure. Since $\tilde\chi\to\chi$ is a finite cover, it follows that $\tilde\mu(U\tilde\chi)<\infty$, and $\tilde T_s$ is ergodic with respect to $\tilde\mu$.

In the previous section we developed that for integer an $k>0$, we have the trivial vector bundle $\mathbb{P}S^k(V)\times\tilde\chi$ with the flat connection $\boldsymbol{\nabla}^{(k)}_{\tilde\chi}$, where V is the space of homogeneous polynomials of degree one in four variables. Both the bundle and the connection can be pulled back to $U\tilde\chi$, and we denote these pull backs by $\mathbb{P}S^k(V)\times U\tilde\chi$ and $\boldsymbol{\nabla}^{(k)}_{U\tilde\chi}$.

Let us choose some norm $\|.\|$ on the vector space $\mathbb{P}S^k(V)$. We now have a finite measure space $(U\tilde\chi,\widetilde\mu)$, an ergodic flow $\widetilde{T}_s\colon U\tilde\chi\to U\tilde\chi$, and a flat normed vector bundle $\mathbb{P}S^k(V)\times U\tilde\chi$. We thus in the situation of Oseledets multiplicative ergodic theorem which we recall as it appears in [**KZ97**].

THEOREM 2.1. *Let $T_s\colon(X,\mu)\to(X,\mu)$, for $s\in\mathbb{R}_+$, be an ergodic flow on measurable space such that $\mu(X)<\infty$. Let $V\to X$ be an $\mathbb{R}_+$ equivariant finite dimensional vector bundle. Assume that a (non-equivariant) norm $\|.\|$ on V is chosen such that for all $s\in\mathbb{R}_+$,*

$$\int_X\log(1+\|T_s\colon V_x\to V_{T_s(x)}\|)\,d\mu<\infty. \tag{6.7}$$

Then there are constants $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ and an invariant filtration of V

$$V = V_{\lambda_1} \supset \cdots \supset V_{\lambda_k} \supset 0$$

such that, for μ-almost every $x \in X$ and all $v \in V_x$ one has

$$\|T_s(v)\| = e^{(\lambda_j t + O(s))}, \qquad s \to \infty$$

where j is the maximal value for which $v \in (V_{\lambda_j})_x$.

For any $(x, v) \in U\tilde{\chi}$ we have the map

$$PT^{(k)}_{U\tilde{\chi}}(\widetilde{T}_s(x, v)) \colon \mathbb{R}_{\geqslant 0} \to \operatorname{End} \mathbb{P}S^k(V), \qquad s \in \mathbb{R}_{\geqslant 0}$$

where $PT^{(k)}_{U\tilde{\chi}}$ denotes the parallel transport of $\boldsymbol{\nabla}^{(k)}$ along paths in $U\tilde{\chi}$. This induces the cocycle

$$\theta^{(k)}_{(x,v)}(s) \colon \mathbb{P}S^k(V) \to \mathbb{P}S^k(V), \qquad s \in \mathbb{R}_{\geqslant 0},\ (x, v) \in U\tilde{\chi}. \tag{6.8}$$

The cocycle (6.8) is equivalent to the function

$$\theta^{(k)} \colon (\mathbb{R}_{\geqslant 0} \times U\tilde{\chi}) \to \operatorname{End} \mathbb{P}S^k(V). \tag{6.9}$$

Let us suppose that our cocycle (6.9) is integrable. That is, for any $s \in \mathbb{R}_{\geqslant 0}$, we have that

$$\int_{U\tilde{\chi}} \log(1 + \|\theta^{(k)}(s, \cdot)\|) \, d\tilde{\mu} < \infty \tag{6.10}$$

where the norm on $\operatorname{End} \mathbb{P}S^k(V)$ is induced from the norm on $\mathbb{P}S^k(V)$.

Given (6.10) holds, Oseledets multiplicative ergodic theorem, see theorem 2.1 in the introduction, guarantees a filtration,

$$\mathbb{P}S^k(V) = F^{(k)}_1 \supset F^{(k)}_2 \supset \cdots \supset F^{(k)}_n \supset 0$$

where $F^{(k)}_j$ are projective sub-bundles, and constants

$$\lambda^{(k)}_1 > \cdots > \lambda^{(k)}_n$$

such that

$$\|\theta^{(k)}_{(x,v)}(s) \cdot (f)\| = e^{(\lambda^{(k)}_j t + O(s))}, \qquad s \to \infty$$

where j is the maximal value for which $f \in F^{(k)}_j$. Moreover, the filtration is preserved by the cocycle (6.8), the number n depends on the dimension of $S^k(V)$, and due to the ergodicity of the flow, the numbers $\lambda^{(k)}_j$ do not depend on the initial point $(x, v) \in U\tilde{\chi}$. Thus for every integer $k > 0$, the numbers $\lambda^{(k)}_j$ are invariants of the cocycle (6.8), and thus are called characteristic or *Lyapunov* exponents of the cocycle in question.

Our situation is analogous to [**KZ97**]. Recall that $\tilde{\chi}$ parametrizes a family of compact genus two Riemann surfaces. Corresponding to this family, one gets a vector bundle on $\tilde{\chi}$ whose fiber is the first homology (with coefficients in $\mathbb{C}$) of the Riemann surface. This bundle comes equipped with a flat connection called the Gauss-Mannin connection. Just as in our case, one lifts this bundle and the connection to $U\tilde{\chi}$, and uses the parallel transport of this lifted connection to give a cocycle

$$\theta^{KZ} \colon (\mathbb{R}_{\geqslant 0} \times U\tilde{\chi}) \to \mathrm{End}(H) \tag{6.11}$$

where H is the first homology group (with coefficients in $\mathbb{C}$) of a genus two Riemann surface corresponding to some $x \in \tilde{\chi}$. Notice that in this case the vector bundle is *not* trivial, but the authors in [**KZ97**] claim that it is trivializable (on the complement of a set of measure zero) and thus the procedure works.

Remark 2.2. In [**KZ97**], the authors actually take H to be the first homology group with *real* coefficients. It becomes clear in conjecture 2.3 why we consider complex coefficients here.

As is well known, there exists a canonical intersection form with respect to which H is a symplectic vector space, and this form provides a norm $\|.\|_{KZ}$ on H. Moreover, the Gauss-Mannin connection preserves the symplectic form, which implies that the KZ cocycle (6.11) takes values in $\mathrm{Sp}(4,\mathbb{C})$.

Now we have the peculiar coincidence

$$\dim(V) = \dim(H).$$

Moreover, in [**vGdJ98**] authors write a symplectic form in V, and we will denote the induced norm by $\|.\|_{GJ}$ in V. Recall that the monodromy representation of $\pi_1^{orb}(\chi, a)$ calculated by the parallel transport of $\boldsymbol{\nabla}_{\tilde{\chi}}^{(k)}$ when $k = 1$ is given by the two symplectic matrices M_0 and M_1 calculated in lemma 2.2. This means that $\boldsymbol{\nabla}_{\tilde{\chi}}^{(1)}$ preserves the symplectic form in V defined in [**vGdJ98**]. In the light of this discussion, we make the following

Conjecture 2.3. *Let us equip the vector spaces H and V with norms $\|.\|_{KZ}$ and $\|.\|_{GJ}$. Then,* (6.9) *is integrable, and the Lyapunov exponents of the cocycle $\theta^{(1)}$ and θ^{KZ} coincide.*

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