Good Filtrations and the Steinberg Square

Tobias Kildetoft

PhD Dissertation
April 2014

Supervisor: Henning Haahr Andersen

Centre for Quantum Geometry of Moduli Spaces
Science and Technology, Aarhus University

Contents

Abstract **3**

Résumé **3**

1 Introduction **4**
Acknowledgements . . . 6

2 Notation and Conventions **7**

3 Affine Schemes, Group Schemes and Representations **8**
3.1 Affine Schemes . . . 8
3.2 Group Schemes . . . 11
3.3 Representations of Group Schemes . . . 12
3.4 The Induction Functor . . . 14

4 Reductive Groups **17**
4.1 Representations of Reductive Groups . . . 17
4.2 Frobenius Kernels . . . 22
4.3 Modules with a good filtration . . . 24
4.4 The Steinberg Module . . . 25
4.5 Tilting Modules . . . 26
4.6 Representations of SL_n . . . 26

5 Finite Groups of Lie Type **28**
5.1 Composition Multiplicity of the Steinberg Module . . . 33

6 A Conjecture of Donkin **36**
6.1 Good (p, r)-filtrations . . . 36
6.2 Weights of the form $w \cdot 0$ . . . 37
6.3 Dominant weights in the root lattice . . . 38
6.4 A Cohomological Criterion . . . 38
6.5 Good filtrations for $\mathrm{St}_r \otimes L(\lambda)$: bounds on λ . . . 39
6.6 Donkin's Conjecture . . . 41
6.7 Other bounds . . . 42
6.8 Tensoring with other simple modules . . . 42
6.9 Good filtrations on $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ . . . 44
6.10 Root systems of small rank . . . 45

7 The Steinberg Square **49**
7.1 SL_2 . . . 58
7.2 SL_3 . . . 59

8 The Steinberg Square for Finite Groups of Lie type **67**
8.1 $SL_2(p^r)$ . . . 69
8.2 $SL_3(p^r)$ . . . 70

A Appendix - Dominant Weights in the Root Lattice **76**
A.1 Type A_n . . . 77
A.2 Type B_n . . . 77
A.3 Type C_n . . . 77
A.4 Type D_n . . . 78
A.5 Type E_6 . . . 78
A.6 Type E_7 . . . 78
A.7 Summary . . . 78

B Appendix - Tensor Products of Simple Modules for $GL_n(\mathbb{C})$ **79**

References **83**

Abstract

This dissertation investigates the socle of the Steinberg squares for an algebraic group in positive characteristic, and what this socle says about the decomposition of the Steinberg square into a direct sum of indecomposable modules.

A main tool in this investigation will be the fact that tensoring the Steinberg module with a simple module of restricted highest weight gives a module with a good filtration. This result was first proved by Andersen when the characteristic is large enough. In this dissertation, generalizations of those results, which are joint work with Daniel Nakano, are presented.

The main results of the dissertation provide formulas which describe how to find the multiplicities of simple modules in the socle of a Steinberg square, given information about the multiplicities of simple modules in Weyl modules. Further, it is shown that when the prime is large enough, the socle completely determines how a Steinberg square decomposes.

The dissertation also investigates the socle of the Steinberg square for a finite group of Lie type, again providing formulas which describe how to find the multiplicity of a simple module in the socle, given information about the multiplicities of simple modules in Weyl modules.

Résumé

Denne afhandling undersøger soklen for Steinberg kvadraterne for en algebraisk gruppe i positiv karakteristik, og hvad denne sokkel siger om dekompositionen af et Steinberg kvadrat som en sum af indekomposable moduler.

Et af hovedredskaberne i dette er det resultat, som siger, at når man tensorerer Steinberg modulet med et simpelt modul med restringeret højeste vægt, så får man et modul med en god filtration. Dette resultat blev først bevist af Andersen når karakteristikken er stor nok. I denne afhandling bliver genereliseringer af disse resultater, som er fælles arbejde med Daniel Nakano, præsenteret.

Hovedresultaterne i denne afhandling giver formler, der beskriver hvordan man finder multipliciteten af et simpelt modul i soklen af et Steinberg kvadrat, givet information om multipliciteterne af simple moduler i Weyl moduler. Det bliver yderligere vist at når karakteristikken er stor nok, så bestemmer soklen fuldstændigt dekompositionen af et Steinberg kvadrat.

Afhandlingen undersøger også soklen for Steinberg kvadratet for en endelig gruppe af Lie type, og giver igen formler, som beskriver hvordan man finder multipliciteten af et simpelt modul i soklen, given information om multipliciteterne af simple moduler i Weyl moduler.

1 Introduction

When studying the representation theory of a reductive algebraic group over an algebraically closed field of positive characteristic, there are several difficulties that come up, which are not present in the theory in characteristic 0.

In characteristic 0, one gets all simple modules by inducing 1-dimensional modules from a Borel subgroup, and by combining this with the Borel-Weil-Bott theorem, this implies that the category of finite dimensional modules is semisimple.

When the characteristic is positive, these induced modules are no longer generally simple. Instead, their socles are simple, and all simple modules occur as the socle of (a unique) such induced module.

A general problem in representation theory is to describe the structure of the tensor product of two simple modules. For algebraic groups in characteristic 0, this amounts to describing the composition factors of such a tensor product, and since the characters of the simple modules are given by Weyl's character formula, this is fairly well-understood.

However, when working in characteristic $p > 0$ this gets more complicated for two reasons. Firstly, since the resulting module will usually not be semisimple, it will not be sufficient to describe the composition factors. And secondly, we do not actually know the characters of the simple modules, so even describing those composition factors will in general be very hard.

There are several ways one might lessen these complications. One is to focus on a specific algebraic group of small rank, where calculations are more easily made. In [DH05], the above problem was studied for the group SL_2, in [BDM11a] it was studied for the group SL_3 when $p \leq 3$, and in [BDM11b] for $p \geq 5$.

Another way one might lessen the complications is to focus only on modules where the characters are known. One possible way to do this is to find a class of induced modules which are still simple when working in characteristic p.

An important example of such modules are the Steinberg modules. Not only are these simple, they are also self-dual, and in many cases they have the sort of properties one usually finds in characteristic 0.

Since the Steinberg modules are so well-understood, it should therefore be possible to describe the structure of the tensor product of a Steinberg module with itself (a Steinberg square).

It turns out that one of the main questions that need to be answered in order to understand this structure is what the socle of such a Steinberg square is, since (at least when p is large enough) the socle will completely determine how the module splits as a direct sum of indecomposable modules.

The Steinberg squares have not been studied much before, at least not as modules for the algebraic group. However, when viewed as modules for a finite group of Lie type (seen as a subgroup of the algebraic group), it has been shown in [HSTZ13] that (except in some special cases) all simple modules occur as composition factors of the Steinberg square.

A natural further question to ask is then what one can say about the socle of the Steinberg square as a module for the finite group of Lie type, and what this says about the splitting into a direct sum of indecomposables.

In this case, the Steinberg module is projective, so the same is true for the Steinberg square, and the decomposition will be in terms of projective covers of simple modules. The socle will completely determine which ones occur and with what multiplicities.

Knowing these multiplicities will in turn yield information about the dimensions of the projective indecomposable modules for the finite group of Lie type, since the dimension of the Steinberg square is known. Further, results due independently to Chastkofsky ([Cha81]) and Jantzen ([Jan81]) show how injective indecomposable modules for the Frobenius kernels split when viewed as modules for the finite group of Lie type (assuming that these injective indecomposable modules lift to the algebraic group), and this can potentially be used to make additional comparisons, once we know both how the Steinberg square splits for the Frobenius kernel (which will follow from the way it splits for the algebraic group when the prime is large enough) and how it splits for the finite group of Lie type.

These last two questions, of determining the socle of the Steinberg squares, both as modules for the algebraic group and as modules for a suitable finite group of Lie type, will be the focus of this dissertation.

The dissertation is structured as follows.

- In Section 3 I introduce affine schemes, group schemes and their representations. This section is based on [Jan03], though I have added some additional detailed examples.

- In Section 4 I introduce those results about representations about reductive groups which I need in later sections. This section is mainly based on [Jan03], though I have tried to include references to the original versions of the main results where possible.

- In Section 5 I describe the basics of the representation theory of finite groups of Lie type in the defining characteristic. This section is based on [Hum06], though I have in many places added a substantial amount of detail to the proofs. Also, the proof that St_r is injective and projective (Theorem 5.18) is new.

- In Section 6 I present joint work with Daniel Nakano, based on the paper [KN14], where we extend work of Andersen ([And01]) on a conjecture of Donkin (Conjecture 6.1).

The main results of this sections (Theorem 6.11 and the contents of Section 6.10) are that tensoring the r'th Steinberg module with a simple module of r-restricted highest weight gives a module with a good filtration (i.e. a filtration whose factors are isomorphic to modules induced from 1-dimensional modules of the Borel subgroup). We prove this when either $p \geq 2h-2$ (where h is the Coxeter number of the root system associated to the group), or when the root system is of type A_2, A_3, B_2 or G_2, with the possible exception of $p = 7$ in type G_2.

These results are instrumental in the later sections.

- In Section 7 I study the socle of the r'th Steinberg square and what this says about the decomposition of the Steinberg square as a direct sum of indecomposable modules.

The main results of this section are formulas (Theorem 7.8), which (assuming $p \geq 2h-2$) describe how to compute the multiplicity of a given simple module in the socle of a Steinberg square, given knowledge of the character of the simple module.

In the two subsections of Section 7 I do these calculations more explicitly for the groups SL_2 and SL_3.

- In Section 8 I once again study the socle of the Steinberg square, this time as a module for a finite group of Lie type.

The main results of this section are formulas (Theorem 8.3), which (assuming $p \geq 2h - 2$) describe how to compute the multiplicity of a given simple module in the socle of the Steinberg square as a module for the finite group of Lie type, given knowledge of the character of the simple module.

In the two subsections of Section 8 I do some more explicit calculations for the groups $SL_2(p^r)$ and $SL_3(p^r)$.

- Appendix A contains some weight calculations needed for one of the results in Section 6.

- Appendix B describes how to decompose tensor products of simple $GL_n(\mathbb{C})$- and $SL_n(\mathbb{C})$-modules, which is needed to do the explicit calculations for SL_2 and SL_3.

Acknowledgements

First of all I would like to thank my advisor, Henning Haahr Andersen, for introducing me to the subject of algebraic groups and their representation theory. I would also like to thank him for always being ready to answer questions on this and other related topics.

Further I would like to thank Daniel Nakano for his hospitality during my stay at the University of Georgia and for the many great discussions we had while I was there, which led to the results presented in Section 6. I would also like to thank the graduate students at the University of Georgia for making me feel welcome and for many good conversations, either during lunch or just when meeting in the hall.

Finally, I would like to thank the students and staff at QGM for providing a great work environment and very friendly atmosphere.

2 Notation and Conventions

In this dissertation, certain notation will be fixed at certain places. For convenience, once notation has been fixed at some point, it will be fixed in the remainder of the dissertation. This section gives a brief overview of what notation is fixed in what parts of the dissertation.

All through the dissertation:

- p is a fixed prime and r is a fixed positive integer.
- k is an algebraically closed field with $\mathrm{char}(k) = p$.

From 4.1 and onwards:

- $\mathbf{G}$ is a reductive connected algebraic group over k, defined and split over $\mathbb{Z}$.
- $\mathbf{T}$ is a fixed maximal torus in $\mathbf{G}$.
- R is the set of roots of $\mathbf{G}$ with respect to the action of $\mathbf{T}$, S is a fixed basis of R, and R^+ (resp. R^-) is the corresponding set of positive (resp. negative) roots.
- For each $\alpha \in R$, $\mathbf{U}_\alpha$ is the root subgroup of $\mathbf{G}$ corresponding to α.
- $\mathbf{U}$ (resp. $\mathbf{U}^+$) is the subgroup of $\mathbf{G}$ generated by the $\mathbf{U}_\alpha$ for $\alpha \in R^-$ (resp. $\alpha \in R^+$).
- $\mathbf{B} = \mathbf{UT}$ and $\mathbf{B}^+ = \mathbf{U}^+\mathbf{T}$.
- $X = X(\mathbf{T})$ is the group of characters of $\mathbf{T}$.
- $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is the Weyl group of $\mathbf{G}$.
- $w_0 \in W$ is the longest element in W.
- α_0 is the highest short root of R.
- $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$.
- h is the Coxeter number of R.

From 4.2 and onwards:

- $F = F_{\mathbf{G}}^r$ is the r'th iterate of the Frobenius morphism on $\mathbf{G}$.
- $G = \mathbf{G}^F$ is the subgroup of fixed points of F in $\mathbf{G}$ and similarly, $T = \mathbf{T}^F$, $U = \mathbf{U}^F$, $U^+ = (\mathbf{U}^+)^F$, $B = \mathbf{B}^F$, $B^+ = (\mathbf{B}^+)^F$.

3 Affine Schemes, Group Schemes and Representations

For many purposes, it is sufficient to regard algebraic groups as group objects in the category of affine varieties, and certainly the main results in this dissertation can be formulated using that language. However, in order to prove some of the deeper theorems about the relations between representations of connected reductive algebraic groups and their finite subgroups of Lie type, it will be necessary to regard them as affine group schemes.

This section will introduce the basic definitions and notation needed for this purpose, though it should be noted that in order to understand the full proofs of the later results, one needs not just affine schemes but arbitrary schemes. However, we have decided to only introduce the affine schemes here in order to keep the length of the section down. For a full account of the theory, one should consult [Jan03].

In the following, let K be a commutative ring, $\{K\text{-alg}\}$ the category of commutative K-algebras and $\{\text{Sets}\}$ the category of sets (here K-algebra will always mean commutative K-algebra unless otherwise noted).

3.1 Affine Schemes

Definition 3.1 (K-functor)**.** A K-functor is a functor from $\{K\text{-alg}\}$ to $\{\text{Sets}\}$. A subfunctor of a K-functor X is a K-functor Y such that for all K-algebras A, $Y(A) \subseteq X(A)$ and such that for any morphism of K-algebras $\varphi : A \to B$, $Y(\varphi) = X(\varphi)|_{Y(A)}$. If Y is a subfunctor of X we write $Y \subseteq X$.

Note that if $Y \subseteq X$ is a subfunctor, then Y is uniquely determined by X and by what Y does to K-algebras. Thus, whenever we specify something to be a subfunctor of a given functor, we will not need to mention what it does to morphisms.

If R is a K-algebra, we define the K-functor $\mathrm{Sp}_K R$ by $\mathrm{Sp}_K R(A) = \mathrm{Hom}_{K\text{-alg}}(R, A)$ for any K-algebra A. For any morphism of K-algebras $\varphi : A \to B$ we set $\mathrm{Sp}_K R(\varphi)$ to be the morphism of sets $\mathrm{Hom}_{K\text{-alg}}(R, A) \to \mathrm{Hom}_{K\text{-alg}}(R, B)$ given by $f \mapsto \varphi \circ f$.

Definition 3.2 (Affine K-scheme)**.** A K-functor X is said to be an affine K-scheme if X is isomorphic as a functor to $\mathrm{Sp}_K R$ for some K-algebra R.

Whenever the ring K is implicit, an affine K-scheme will simply be referred to as an affine scheme.

An important example of a K-functor is the functor $\mathbb{A}^n$ (for a positive integer n), which to any K-algebra A associates the set A^n and to any morphism of K-algebras $\varphi : A \to B$ associates the map $\varphi^n : A^n \to B^n$ given by $\varphi^n(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n))$.

In fact, $\mathbb{A}^n$ is an affine K-scheme, isomorphic to $\mathrm{Sp}_K K[T_1, \ldots, T_n]$. The isomorphism is given by the maps $\psi_A : \mathrm{Sp}_K K[T_1, \ldots, T_n](A) \to \mathbb{A}^n(A)$ with $\psi_A(\varphi) = (\varphi(T_1), \ldots, \varphi(T_n))$ (for any K-algebra A and any $\varphi \in \mathrm{Sp}_K K[T_1, \ldots, T_n](A) = \mathrm{Hom}_{K\text{-alg}}(K[T_1, \ldots, T_n], A)$). Each of these maps is a bijection, since any K-algebra homomorphism from $K[T_1, \ldots, T_n]$ is uniquely determined by its value on the T_i, and any such choice gives a valid homomorphism.

To show that it is an isomorphism of functors, we just need to verify that it is compatible with the morphisms. For ease of notation, let $X = \mathrm{Sp}_K K[T_1, \ldots, T_n]$ and $Y = \mathbb{A}^n$. In order to show that the maps are compatible with morphisms, we then need to show that if $\varphi : A \to B$ is a homomorphism of K-algebras then $\psi_B \circ X(\varphi) = Y(\varphi) \circ \psi_A$, and this is straightforward from the definitions.

If $X = \mathrm{Sp}_K R$ is an affine scheme, we write $R = K[X]$, the ring of regular functions on X (as explained in [Jan03] I.1.3, given an affine scheme X, we can recover this ring as $\mathrm{Mor}(X, \mathbb{A}^1)$).

If $I \trianglelefteq K[X]$ is an ideal, we define a subfunctor $V(I)$ of X by

$$V(I)(A) = \{\varphi \in \mathrm{Hom}_{K\text{-alg}}(K[X], A) \mid I \subseteq \ker(\varphi)\}$$

(we could have done this for an arbitrary subset rather than an ideal, but it is clear that it only depends on the ideal generated by such a subset anyway).

Definition 3.3 (Closed subfunctor)**.** A subfunctor of an affine scheme X is said to be closed if it has the form $V(I)$ for an ideal $I \trianglelefteq K[X]$.

Proposition 3.4. *Let $X = \mathrm{Sp}_K R$ be an affine scheme and $I, J \trianglelefteq R$ be ideals. Then*

$$I \subseteq J \Leftrightarrow V(J) \subseteq V(I)$$

Proof. Clearly, if $I \subseteq J$ then any homomorphism containing J in its kernel will also contain I, so $V(J) \subseteq V(I)$.

To show the other implication, it is enough to show that if $I \not\subseteq J$ then there exists a K-algebra A and a homomorphism $\varphi : R \to A$ such that $J \subseteq \ker(\varphi)$ but $I \not\subseteq \ker(\varphi)$. So let $A = R/J$ and φ be the canonical map from R to R/J. This homomorphism clearly has J in its kernel, but since $I \not\subseteq J$ and $\ker(\varphi) = J$, it does not have I in its kernel, and the proof is complete. □

If Y and Z are subfunctors of a K-functor X, we define the intersection $Y \cap Z$ as the subfunctor of Y (and of Z) given by $(Y \cap Z)(A) = Y(A) \cap Z(A)$ for any K-algebra A.

Proposition 3.5. *Let $X = \mathrm{Sp}_K R$ be an affine scheme. If I_α is a family of ideals of R then*

$$\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$$

Proof. Let A be a K-algebra and $\varphi \in \mathrm{Sp}_K R(A)$. If $\varphi \in V(\sum_\alpha I_\alpha)(A)$ then $\varphi \in V(I_\alpha)(A)$ for all α since all the I_α are contained in $\sum_\alpha I_\alpha$, and thus $\varphi \in \bigcap_\alpha V(I_\alpha)$.

On the other hand, if $\varphi \in \bigcap_\alpha V(I_\alpha)$ and $x = \sum_\alpha x_\alpha$ is an arbitrary element in $\sum_\alpha I_\alpha$ (so only finitely many non-zero terms), then $\varphi(x) = \varphi(\sum_\alpha x_\alpha) = \sum_\alpha \varphi(x_\alpha) = 0$ so $\varphi \in V(\sum_\alpha I_\alpha)(A)$. □

From above, we see that the intersection of any family of closed subfunctors is again a closed subfunctor, which means that any subfunctor Y is contained in a unique smallest closed subfunctor (the intersection of all closed subfunctors containing it), called the closure of Y and denoted $\overline{Y}$.

The terminology suggests that the closed subfunctors should in some sense define a topology (though it is not clear on what set), so we should also be able to take finite unions of closed subfunctors and again get a closed subfunctor. The problem is that if we define the union of subfunctors in the obvious way (namely $(Y \cup Z)(A) = Y(A) \cup Z(A)$), we would not in general get a closed subfunctor. Instead, we have the following proposition to guide the definition:

Proposition 3.6. *Let $X = \mathrm{Sp}_K R$ be an affine scheme and $I, J \trianglelefteq R$ be ideals. Define a subfunctor $Y \subseteq X$ by $Y(A) = V(I)(A) \cup V(J)(A)$. Then $\overline{Y} = V(I \cap J)$.*

Proof. If $\varphi \in Y(A)$ and $x \in I \cap J$ then clearly $\varphi(x) = 0$, so $Y \subseteq V(I \cap J)$ and since $V(I \cap J)$ is closed, we also get $\overline{Y} \subseteq V(I \cap J)$.

On the other hand, if L is an ideal of R such that $Y \subseteq V(L)$ then we must have $V(I) \subseteq V(L)$ and $V(J) \subseteq V(L)$ so by Proposition 3.4 we get that $L \subseteq I \cap J$ and thus that $V(I \cap J) \subseteq V(L)$, and since this was for any $V(L)$ containing Y, we must have $\overline{Y} = V(I \cap J)$. $\square$

In light of the above, we define the union of two closed subfunctors as the subfunctor given by $(V(I) \cup V(J))(A) = V(I \cap J)(A)$. Note that this means we have not defined arbitrary unions, but this is sufficient for our needs.

To illustrate the problem with the naive definition of the union of closed subfunctors, let us consider the example $K = \mathbb{Z}$ (so we are just looking at rings and ring homomorphisms) and the affine scheme $\mathrm{Sp}_{\mathbb{Z}}\,\mathbb{Z}$. Let $I = (2)$ and $J = (3)$ be ideals of $\mathbb{Z}$. To show that $A \mapsto V(I)(A) \cup V(J)(A)$ does not define a closed subfunctor, we just need to find an A such that $V(I)(A) \cup V(J)(A) \neq V(I \cap J)$. If $A = \mathbb{Z}/6\mathbb{Z}$ then both $V(I)(A)$ and $V(J)(A)$ are empty, as the unique homomorphism from $\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$ vanishes at neither 2 nor 3. On the other hand, $I \cap J = (6)$, and the unique homomorphism from $\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$ does vanish at 6, so $V(I \cap J)(A)$ contains this homomorphism and is thus strictly larger than $V(I)(A) \cup V(J)(A)$.

We now have sufficient machinery to define what it means for an affine scheme to be connected, since we can define this in the usual way from topology as the non-existence of disjoint non-empty closed sets whose union is the entire set. Translated to a property of $K[X]$ this says that if I and J are non-zero ideals of $K[X]$ such that $I \cap J = (0)$ then $I + J \neq K[X]$.

In particular, we see that if R is an integral domain then the affine scheme $\mathrm{Sp}_K\, R$ is connected (in fact irreducible, as the intersection of any two non-zero ideals contains a non-zero element).

One of the reasons we will not need to deal with general schemes is the following:

Proposition 3.7. *Let $X = \mathrm{Sp}_K\, R$ be an affine scheme and $I \trianglelefteq R$ be an ideal. Then $V(I) \cong \mathrm{Sp}_K\, R/I$.*

Proof. For each K-algebra A define the map $\psi_A : V(I)(A) \to \mathrm{Sp}_K\, R/I(A)$ by $\varphi \mapsto \varphi \circ \pi$ where $\pi : R \to R/I$ is the canonical projection.

This is obviously a bijective map for each K-algebra A, so we just need to check that it is compatible with the morphisms. Thus, we need to show that for any homomorphism $\varphi : A \to B$ we have $\psi_B \circ V(I)(\varphi) = \mathrm{Sp}_K\, R/I(\varphi) \circ \psi_A$, but from the definitions, this is simply the associativity of function composition. $\square$

From the above we see that a closed subfunctor of an affine scheme is again an affine scheme.

The affine schemes we will be most interested in are the ones satisfying the following:

Definition 3.8 (Algebraic and reduced schemes)**.** An affine scheme X is said to be algebraic if $K[X]$ is a finitely presented K-algebra.

It is called reduced if 0 is the only nilpotent element in $K[X]$.

If A is a K-algebra and X is a K-functor, we define the A-functor X_A as the restriction of X to those K-algebras that are A-algebras (this gives an A-functor as any A-algebra is also a K-algebra since A is a K-algebra).

If $X = \mathrm{Sp}_K\, R$ is an affine scheme, then X_A is also an affine scheme, isomorphic to $\mathrm{Sp}_A(R \otimes_K A)$.

If Y is an A-functor, then Y is said to be defined over K if $Y = X_A$ for some K-functor X.

3.2 Group Schemes

Definition 3.9 (K-group functor)**.** A K-group functor is a functor from $\{K\text{-alg}\}$ to the category of groups.

Since we can compose a K-group functor with the forgetful functor from the category of groups to {Sets}, we can regard any K-group functor as a K-functor, so all the concepts introduced in the previous section will also be applicable to K-group functors.

Definition 3.10 (K-group scheme)**.** A K-group scheme is a K-group functor which, when viewed as a K-functor, is an affine scheme.

As with affine K-schemes, whenever the algebra K is implicit, we will simply refer to a K-group scheme as a group scheme (note that we have also dropped the word "affine", since we will not be dealing with non-affine schemes).

Example 3.11 (Additive group scheme $\mathbb{G}_a$)**.** The additive group scheme $\mathbb{G}_a$ is given by the functor $\mathbb{G}_a(A) = (A, +)$ for any K-algebra A, and $\mathbb{G}_a(\varphi) = \varphi$ for any homomorphism of K-algebras. It is a group scheme since the corresponding functor to {Sets} is just $\mathbb{A}^1 = \mathrm{Sp}_K\, K[T]$.

Example 3.12 (Multiplicative group scheme $\mathbb{G}_m$)**.** The multiplicative group scheme $\mathbb{G}_m$ is given by the functor $\mathbb{G}_m(A) = (A^\times, \cdot)$ (i.e. the units in A under multiplication) for any K-algebra A, and $\mathbb{G}_m(\varphi) = \varphi$ for any homomorphism of K-algebras. This is a group scheme as the corresponding functor to {Sets} is isomorphic to $\mathrm{Sp}_K\, K[T, T^{-1}]$.

Example 3.13 (GL_n)**.** The group scheme GL_n is given by the functor

$$GL_n(A) = \{M \in M_n(A) \mid \det(M) \in A^\times\}$$

for any K-algebra A, where $M_n(A)$ is the ring of $n \times n$ matrices with coefficients in A. For any homomorphism $\varphi : A \to B$ of K-algebras, $GL_n(\varphi)$ is the map that applies φ to each entry of a matrix. This is a group scheme as the corresponding functor to {Sets} is isomorphic to $\mathrm{Sp}_K\, R$ where

$$R = K[T_{1,1}, T_{1,2}, \ldots, T_{1,n}, T_{2,1}, T_{2,2}, \ldots, T_{n,n}, \det{}^{-1}]$$

and

$$\det = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} T_{i,\sigma(i)}$$

is the determinant polynomial ($\mathrm{sign} : S_n \to \{-1, 1\}$ is the sign homomorphism).

If K is an integral domain, then all the algebras in the above examples are integral domains, so the examples are all connected and reduced. Since the algebras are finitely presented, all the groups in the examples are algebraic.

A subgroup functor of a K-group functor is defined in the obvious way, and it is said to be closed if the corresponding functor to {Sets} is. In that case, it is also called a closed subgroup scheme.

A subgroup functor H of G is said to be normal if $H(A)$ is normal in $G(A)$ for each K-algebra A.

From now on, if G is a group scheme, a subgroup of G will mean a subgroup functor of G.

If G and H are group schemes, $\mathrm{Mor}(G, H)$ will denote the set of morphisms from G to H as K-functors, and $\mathrm{Hom}(G, H)$ will denote the set of morphisms from G to H as K-group functors (so $\mathrm{Hom}(G, H) \subseteq \mathrm{Mor}(G, H)$).

Example 3.14 (SL_n)**.** The group scheme SL_n is given as the subgroup functor of GL_n with

$$SL_n(A) = \{M \in M_n(A) \mid \det(M) = 1\}$$

for any K-algebra A. It is a closed subgroup scheme of GL_n, given by the ideal $(\det -1)$, and hence it is a group scheme isomorphic to $\mathrm{Sp}_K R$ with

$$R = K[T_{1,1}, T_{1,2}, \dots, T_{1,n}, T_{2,1}, T_{2,2}, \dots, T_{n,n}]/(\det -1)$$

A final example that will be important for the definition of representations is the following:

Example 3.15 (M_a)**.** If M is a K-module, we will define the K-group functor M_a by letting $M_a(A) = (M \otimes_K A, +)$ for any K-algebra A, and $M_a(\varphi) = \mathrm{id}_M \otimes \varphi$ for any homomorphism of K-algebras φ.

If $M = K^n$ then we see that we can identify $M_a(A) = (K^n \otimes_K A, +)$ with $(A^n, +)$, so in this case M_a is a group scheme and the corresponding functor to $\{\text{Sets}\}$ is isomorphic to $\mathbb{A}^n$.

3.3 Representations of Group Schemes

If X and Y are K-functors, we define the K-functor $X \times Y$ by $(X \times Y)(A) = X(A) \times Y(A)$ for any K-algebra A, and for any homomorphism of K-algebras $\varphi : A \to B$ and any $(x, y) \in X(A) \times Y(A)$ we set $(X \times Y)(\varphi)(x, y) = (X(\varphi)(x), Y(\varphi)(y)) \in X(B) \times Y(B)$.

If G is a K-group functor and X is any K-functor, a left operation of G on X is a morphism of functors from $G \times X$ to X such that for each K-algebra A the induced map from $G(A) \times X(A)$ to $X(A)$ is an action of $G(A)$ on $X(A)$ in the usual sense from group theory (a right operation is defined similarly).

If M is a K-module then we define a representation of G on M to be an operation of G on the K-functor M_a such that for each K-algebra A, the action of $G(A)$ on $M_a(A) = M \otimes_K A$ is via A-linear maps. We also call such an operation a structure as a G-module on M, and we simply call M a G-module.

Example 3.16. Let $G = GL_n$ and $M = K^n$. Define for each K-algebra A a map ψ_A from $G(A) \times M_a(A)$ to $M_a(A)$ by $(g, x) \mapsto gx$ where we identify $M_a(A) = K^n \otimes_K A$ with A^n in the usual way, and where gx is the regular multiplication of matrices. Each of these maps is clearly an action of $G(A)$ on $M_a(A)$ via A-linear maps, so we just need to check that they define a morphism of the functors.

So let $\varphi : A \to B$ be a homomorphism of K-algebras. What we then need to check is that $\psi_B \circ (G \times M_a)(\varphi) = M_a(\varphi) \circ \psi_A$, so let $(g, x) \in G(A) \times M_a(A)$. We compute

$$\psi_B((G \times M_a)(\varphi)(g, x)) = \psi_B((G(\varphi)(g), M_a(\varphi)(x))) = G(\varphi)(g)M_a(\varphi)(x)$$

and

$$M_a(\varphi)(\psi_A(g, x)) = M_a(\varphi)(gx)$$

But under our identification of $M_a(A)$ with A^n we see that $M_a(\varphi)(x)$ simply applies φ to each entry in the vector x, and since $G(\varphi)$ does the same to g, it is clear that

$$G(\varphi)(g)M_a(\varphi)(x) = M_a(\varphi)(gx)$$

so the maps do indeed define a morphism of functors.

Hence, we have now defined a representation of G on M.

If G is a group scheme and M is a G-module, we have an action of $G(K[G]) = \mathrm{End}_{K\text{-alg}}(K[G])$ on $M_a(K[G]) = M \otimes_K K[G]$. In particular, we get an action of the element $\mathrm{id}_{K[G]} \in \mathrm{End}_{K\text{-alg}}(K[G])$, giving us a K-linear map $\Delta_M : M \to M \otimes_K K[G]$ with $\Delta_M(m) = \mathrm{id}_{K[G]}(m \otimes 1)$ for each $m \in M$. The map Δ_M is called the comodule map of the G-module M. For the properties of this map, one should refer to [Jan03] I.2.8.

If M and M' are G-modules and $\varphi : M \to M'$ is a homomorphism of K-modules, then we call φ a homomorphism of G-modules if $\Delta_{M'} \circ \varphi = (\varphi \otimes \mathrm{id}_{K[G]}) \circ \Delta_M$ (again, we refer to [Jan03] I.2.8 for the details).

Example 3.17. Let $G = GL_n$ and $M = K^n$ and define a G-module structure on M as in 3.16. We wish to compute the comodule map $\Delta_M : M \to M \otimes_K K[G]$. For this, we first note that $\mathrm{id}_{K[G]} \in \mathrm{End}_{K\text{-alg}}(K[G])$ corresponds to the matrix in $M_n(K[G])$ which on the (i,j)-entry has $T_{i,j}$. Call this matrix Δ. In order to see what $\Delta(m \otimes 1)$ looks like, we need to remember that the action is given by matrix multiplication, so we need to identify $m \otimes 1 \in M \otimes_K K[G]$ with a vector in $K[G]^n$. The way this identification works is that we simply use that the structure homomorphism from K to $K[G]$ is injective in this case, so we can identify any element in K with its image in $K[G]$, and thus we identify the element $(m_1, \dots, m_n) \otimes 1 \in K^n \otimes_K K[G]$ with $(m_1, \dots, m_n) \in K[G]^n$.

We now see that if $m = (m_1, \dots, m_n) \in M$ then the i'th entry in $\Delta_M(m)$ equals $\sum_{j=1}^n m_j T_{i,j}$. When we then revert the identification to get an element in $M \otimes_K K[G]$ we get (here e_i is the i'th standard basis vector of K^n)

$$\Delta_M(m) = \sum_{i=1}^n \sum_{j=1}^n m_j (e_i \otimes T_{i,j})$$

Usually, given a G-module, it would be natural to ask about whether it has any non-trivial submodules and what these are. But due to the way a G-module has been defined, it is not clear how to define submodules at all, unless we at least assume a bit more about G.

Definition 3.18 (Flat group scheme)**.** A K-group scheme G is said to be flat if $K[G]$ is a flat K-module.

Example 3.19. If $G = GL_n$ then

$$K[G] \cong K[T_{1,1}, T_{1,2}, \dots, T_{1,n}, T_{2,1}, T_{2,2}, \dots, T_{n,n}, \det^{-1}]$$

If K is a field or if $K = \mathbb{Z}$ then this is certainly a flat K-module, so in these cases (which will be the important ones later on), G is a flat group scheme.

We can now define what we mean by a G-submodule of a G-module M.

Definition 3.20. If G is a flat group scheme and M is a G-module, then a G-submodule of M is a K-submodule N of M such that $\Delta_M(N) \subseteq N \otimes_K K[G]$.

For more information about how submodules can be defined without the assumption that G is flat and for proofs that this definition gives the usual properties of submodules, we refer to [Jan03] I.2.9.

If G is a flat group scheme and M is a G-module, we see that the K-submodules 0 and M of M are G-submodules. If these are the only G-submodules of M (and $M \neq 0$), we call M simple (when referring to M as a representation of G rather than as a G-module, we will call M irreducible instead).

Example 3.21. Let $G = GL_n$ and assume that K is a field (so G is automatically flat). Let $M = K^n \otimes_K K^n$ and identify $M_a(A) = K^n \otimes_K K^n \otimes_K A$ with $A^n \otimes_A A^n$. Define a structure as a G-module on M by letting $G(A)$ act on $M_a(A)$ via $v \otimes w \mapsto gv \otimes gw$ for any $v, w \in A^n$ and $g \in G(A)$ where the multiplication is the usual matrix multiplication (and extend this linearly to all of $A^n \otimes_A A^n$). That this gives a G-module is essentially the same arguments as in 3.16. Now let us compute the comodule map for this module. Let Δ be as in 3.16 and let a_i be the i'th standard basis vector of $K[G]^n$ and e_i the i'th standard basis vector of K^n. We have

$$\Delta_M(e_i \otimes e_j) = \Delta(a_i \otimes a_j) = \Delta a_i \otimes \Delta a_j = \Delta_i \otimes \Delta_j$$

where Δ_i is the i'column of Δ. Under our identification, this is the element

$$\sum_{l=1}^{n} \sum_{m=1}^{n} (e_l \otimes e_m \otimes T_{l,i} T_{m,j})$$

so we get

$$\Delta_M(e_i \otimes e_j + e_j \otimes e_i) = \sum_{l=1}^{n} \sum_{m=1}^{n} ((e_l \otimes e_m + e_m \otimes e_l) \otimes T_{l,i} T_{m,j})$$

and hence we see that the K-submodule of M spanned by $e_i \otimes e_j + e_j \otimes e_i$ (for $1 \leq i, j \leq n$) is a G-submodule of M. Call this submodule $S^2(M)$.

Similarly, we can see that the K-submodule spanned by $e_i \otimes e_j - e_j \otimes e_i$ for $1 \leq i, j \leq n$ is a G-submodule. Call this submodule $\Lambda^2(M)$. If the characteristic of K is not 2, then any vector in $K^n \otimes_K K^n$ can be written as a linear combination of vectors in $S^2(M)$ and $\Lambda^2(M)$, and since their dimensions add up to n^2 ($S^2(M)$ has as a basis those $e_i \otimes e_j + e_j \otimes e_i$ with $i \leq j$, so it has dimension $\frac{n(n+1)}{2}$ and $\Lambda^2(M)$ has as a basis those $e_i \otimes e_j - e_j \otimes e_i$ with $i < j$ so it has dimension $\frac{n(n-1)}{2}$), we see that $M = S^2(M) \oplus \Lambda^2(M)$.

If G is a group scheme and M is a representation of G, we define the set of fixed points of G on M as

$$M^G = \{m \in M \mid \Delta_M(m) = m \otimes 1\}$$

This is obviously a K-submodule of M. See [Jan03] I.2.10 for further details.

3.4 The Induction Functor

Let G be a group scheme and H a subgroup scheme of G. In this section the functor ind_H^G from the category of H-modules to the category of G-modules will be defined, and its main properties will be recalled.

Let G be a flat group scheme and H a subgroup scheme of G. Define a functor ind_H^G from the category of H-modules to the category of G-modules by

$$\operatorname{ind}_H^G(M) = \{f \in \operatorname{Mor}(G, M_a) \mid f_A(gh) = h^{-1} f_A(g) \text{ for all } A \in \{K\text{-alg}\},\ g \in G(A) \text{ and } h \in H(A)\}$$

for any H-module M. At first, this is a functor from the category of H-modules to the category of K-modules (with the obvious structure of a K-module on $\operatorname{Mor}(G, M_a)$ and treating morphisms in the usual way). To define a structure as a G-module on this K-module, we need to define an action of $G(A)$ on $\operatorname{Mor}(G, M_a) \otimes_K A$, and we can identify the latter with $\operatorname{Mor}(G_A, (M \otimes_K A)_a)$. For $g \in G(A)$ and $f \in \operatorname{Mor}(G_A, (M \otimes_K A)_a)$ we define $gf \in \operatorname{Mor}(G_A, (M \otimes_K A)_a)$ by $(gf)_B(g') = f_B(g^{-1}g')$ for any A-algebra B.

It is now easy (though the notation can get tedious) to check that this does indeed define a functor from H-modules to G-modules. The details can be found in [Jan03] I.3.3.

The main property of the induction functor we will need is the following (res_H^G is the restriction functor from G-modules to H-modules). This result is known as Frobenius Reciprocity.

Proposition 3.22 ([Jan03, Proposition I.3.4])**.** *Let H be a flat subgroup scheme of G. For any H-module M and any G-module N, there is a natural isomorphism*

$$\operatorname{Hom}_G(N, \operatorname{ind}_H^G(M)) \cong \operatorname{Hom}_H(\operatorname{res}_H^G(N), M)$$

We will also need the following property of the induction functor, known as the Tensor Identity.

Proposition 3.23 ([Jan03, Proposition I.3.6])**.** *Let H be a flat subgroup scheme of G. For any G-module N, which is flat over K, and any H-module M, there is a canonical isomorphism of G-modules*

$$\operatorname{ind}_H^G(M \otimes \operatorname{res}_H^G(N)) \cong \operatorname{ind}_H^G(M) \otimes N$$

Since the induction functor is left exact but not generally right exact, we will also be interested in the right derived functors $R^i \operatorname{ind}_H^G$ for $i > 0$.

In certain cases, the induction functor has a relation to quotients of group schemes, which we will need a consequence of. To avoid introducing such quotients in general, we will only look at a special case:

Assume that K is an algebraically closed field, let G be a K-group scheme, H a closed subgroup scheme and X be an affine K-scheme. Let $\pi : G \to X$ be a surjective morphism of affine schemes (which means that it is surjective on each K-algebra that is a field), and which is constant on left cosets of H. Then we can identify X with the quotient G/H, and if we follow the constructions in [Jan03] I.5, we get an associated sheaf $\mathcal{O}_{G/H}$ which has $K[G/H]$ as its global sections. We then have the following:

Proposition 3.24. *There is an isomorphism*

$$\operatorname{ind}_H^G(K) \cong K[G/H]$$

Proof. This is a special case of [Jan03] I.5.12(a) with $n = 0$ and $M = k$ since the 0'th cohomology of a sheaf is just the global sections of the sheaf, which by construction is $K[G/H]$. □

Now we note that since $\operatorname{ind}_H^G(K) = \{f \in K[G] \mid f(gh) = f(g) \text{ for all } h \in H, g \in G\}$, we can identify $K[G/H]$ with $K[G]^H$ by the above proposition. This identification will be needed later on.

We will also need the following.

Proposition 3.25 ([Jan03, Corollary I.5.13])**.** *In the above situation, where G/H is affine, the functor ind_H^G is exact.*

If the functor ind_H^G is exact, then we say that H is exact in G. Later, we will need the following consequence of this property.

Proposition 3.26 ([Jan03, Corollary I.4.6(a)])**.** *Assume that H is exact in G and let M be an H-module. For any G-module N and any $i \geq 0$ there is an isomorphism*

$$\mathrm{Ext}^i_G(N, \mathrm{ind}_H^G(M)) \cong \mathrm{Ext}^i_H(N, M)$$

4 Reductive Groups

In this section we present the results about reductive groups and their representations needed in the later parts.

4.1 Representations of Reductive Groups

In this section, we will give a brief overview of how the irreducible representations of a connected reductive algebraic group are obtained.

Assume from now on that K is an algebraically closed field.

If n is a positive integer, we define the K-group scheme $\mathbb{T}_n$ to be the functor that assigns to any K-algebra A the group of diagonal matrices with entries in A, and with the obvious morphisms. This is a closed subgroup scheme of GL_n.

A K-group scheme is said to be a torus if it is isomorphic to $\mathbb{T}_n$ for some n.

Definition 4.1. Let G be an algebraic and reduced K-group scheme and let H be a maximal solvable, normal, connected, closed subgroup of G. G is said to be reductive if H is a torus.

G is said to be semisimple if H is trivial.

For the details of the following, see [Jan03] II.1.

From now on, let $\mathbf{G}$ be a connected, reductive K-group scheme, which is defined over $\mathbb{Z}$, and $\mathbf{T}$ a maximal torus in $\mathbf{G}$.

Let $X = X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ be the character group of $\mathbf{T}$ and $Y = Y(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ be the group of cocharacters of $\mathbf{T}$.

There is a natural bilinear pairing, denoted $\langle \cdot, \cdot \rangle$, on $X \times Y$.

Associated to $\mathbf{G}$ (and $\mathbf{T}$) is a root system, from now on denoted R (identified with a subset of X). Let S be a fixed basis of R and R^+ the corresponding set of positive roots.

The number $|S|$ is called the semisimple rank of $\mathbf{G}$.

Each root α determines a unique coroot $\alpha^\vee \in Y$. The set of coroots will be denoted $R^\vee$.

For each root α, we have a homomorphism $x_\alpha : \mathbb{G}_a \to \mathbf{G}$ such that for any $t \in \mathbf{T}$ and any $a \in \mathbb{G}_a$, $t x_\alpha(a) t^{-1} = x_\alpha(\alpha(t)a)$.

This homomorphism gives us, for each root α, a root subgroup $\mathbf{U}_\alpha$ of $\mathbf{G}$ (the image of x_α), and we set $\mathbf{U}$ to be the subgroup of G generated by $\mathbf{U}_\alpha$ for all $\alpha \in -R^+$. Similarly, we set $\mathbf{U}^+$ to be the subgroup generated by the $\mathbf{U}_\alpha$ for $\alpha \in R^+$.

There is also for each $\alpha \in R$ a homomorphism $\varphi_\alpha : SL_2 \to \mathbf{G}$. (This is the homomorphism that lets us define the coroots). Unless $\alpha^\vee$ is twice a cocharacter, then this homomorphism is injective, and it has image generated by U_α and $U_{-\alpha}$, which intersects $\mathbf{T}$ exactly in those elements of the form $\alpha^\vee(\mathbb{G}_m)$.

The multiplication in $\mathbf{G}$ induces an isomorphism of schemes from the product of the $\mathbf{U}_\alpha$ with $\alpha \in -R^+$ to $\mathbf{U}$ and similarly one from the product of the $\mathbf{U}_\alpha$ with $\alpha \in R^+$ to $\mathbf{U}^+$.

Let $\mathbf{B}$ be the subgroup of G generated by $\mathbf{U}$ and $\mathbf{T}$ and similarly $\mathbf{B}^+$ be the subgroup generated by $\mathbf{U}^+$ and $\mathbf{T}$.

Each $\lambda \in X$ defines a representation of $\mathbf{T}$ on K, and this extends uniquely to a representation of $\mathbf{B}$ on K, which will also be denoted λ for simplicity (if confusion is possible, it will be denoted K_λ).

Define $X_+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}$. If $\lambda \in X_+$ then λ is called dominant.

For $\lambda \in X$, define $\nabla(\lambda) = \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda)$ (though note that because of Proposition 4.8 these will only be of interest when $\lambda \in X_+$. Also note that these modules are denoted $H^0(\lambda)$ in [Jan03]).

Let W be the Weyl group of G, identified with a group of automorphisms of $X \otimes_{\mathbb{Z}} \mathbb{R}$ and let w_0 be the longest element of W.

Denote by α_0 the highest short root of R.

Define $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$.

Denote by $h = \langle \rho, \alpha_0^\vee \rangle + 1$ the Coxeter number of R.

If M is any $\mathbf{T}$-module and $\lambda \in X$, then we will consider the submodule

$$M_\lambda = \{m \in M \mid t.m = \lambda(t)m \text{ for all } t \in \mathbf{T}(K)\}$$

and we then have $M = \bigoplus_{\lambda \in X} M_\lambda$.

The "dot" action is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $w \in W$ and $\lambda \in X$.

With all this notation defined, let us consider two examples.

Example 4.2. Let $\mathbf{G} = GL_n$, which is reductive and connected. $\mathbf{T} = \mathbb{T}_n$ is a maximal torus in $\mathbf{G}$ (identified with the diagonal matrices). The group of characters $X = X(\mathbf{T})$ is isomorphic to $\mathbb{Z}^n$, where we identify the element $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ with the homomorphism that sends $\mathrm{diag}(t_1, \ldots, t_n)$ to $t_1^{x_1} t_2^{x_2} \cdots t_n^{x_n}$.

The root system R of $\mathbf{G}$ then consists of those $\lambda \in X$ of the form $\lambda_{i,j} = e_i - e_j$ for $1 \leq i, j \leq n$ and $i \neq j$, where e_i is the element in $\mathbb{Z}^n$ which has a 1 on the i'th position and 0's elsewhere. We choose our basis of R to be the roots of the form $\lambda_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$, so our positive roots are the ones of the form $\lambda_{i,j}$ with $i < j$.

We have

$$X_+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_i \geq \lambda_{i+1} \text{ for } 1 \leq i \leq n-1\}$$

The root subgroup $\mathbf{U}_{\lambda_{i,j}}$ consists of the matrices with 1's on the diagonal and an arbitrary element in the (i, j) position. We then have the subgroups $\mathbf{U}$ and $\mathbf{U}^+$ as the subgroups consisting of respectively lower and upper triangular matrices with 1's on the diagonal.

The subgroups $\mathbf{B}$ and $\mathbf{B}^+$ and thus the groups consisting of respectively lower and upper triangular matrices.

The Weyl group of $\mathbf{G}$ is S_n, the symmetric group on n symbols.

The modules $\nabla(\lambda)$ will be described in more detail later, but for now, let us consider a few examples.

If $\lambda = (1, 0, \ldots, 0)$ then $\nabla(\lambda)$ is the representation of $\mathbf{G}$ described in 3.16 (the standard representation).

If $\lambda = (2, 0, \ldots, 0)$ then $\nabla(\lambda)$ is the first of the two submodules described in 3.21 (the second symmetric power of the standard representation).

If $\lambda = (1, 1, 0, \ldots, 0)$ (and $n \geq 2$) then $\nabla(\lambda)$ is the second of the two submodules described in 3.21 (the second exterior power of the standard representation).

If $\lambda = (1, 1, \ldots, 1)$ then $\nabla(\lambda)$ is the 1-dimensional determinant module Det where each matrix A acts by the scalar $\det(A)$.

Example 4.3. Let $\mathbf{G} = SL_n$ (with $n \geq 2$) which is connected and semisimple. The diagonal matrices $\mathbf{T}$ are still a maximal torus in $\mathbf{G}$ but this time we have them isomorphic to $\mathbb{T}_{n-1}$ as we require the determinant to be 1. The group of characters $X = X(\mathbf{T})$ is then isomorphic to $\mathbb{Z}^{n-1}$ where the element $(x_1, \dots, x_{n-1}) \in \mathbb{Z}^{n-1}$ corresponds to the homomorphism

$$\operatorname{diag}(t_1, \dots, t_{n-1}, (t_1 t_2 \dots t_{n-1})^{-1}) \mapsto t_1^{x_1} \dots t_{n-1}^{x_{n-1}}$$

The root system R consists of the elements $\lambda_{i,j}$ with $1 \leq i, j \leq n-1$ and $i \neq j$ (defined as above), along with the elements

$$\pm\lambda_{i,n} = \pm \left(e_i + \sum_{j=1}^{n-1} e_j \right)$$

for $1 \leq i \leq n-1$.

Here we have that

$$X_+ = \{(\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{Z}^{n-1} \mid 0 \leq \lambda_i \leq \lambda_{i-1} \text{ for } 2 \leq i \leq n-1\}$$

so the dominant weights correspond to partitions of length at most $n-1$.

The subgroups $\mathbf{U}_\alpha$, $\mathbf{U}$, $\mathbf{U}^+$, $\mathbf{B}$, and $\mathbf{B}^+$ have the same description as for GL_n, and the Weyl group is also the same.

If $\lambda = (1, 0, \dots, 0)$, $\lambda = (2, 0, \dots, 0)$ or $\lambda = (1, 1, 0, \dots, 0)$ (and $n \geq 3$), then the modules $\nabla(\lambda)$ are just the ones described in the case of GL_n viewed as $\mathbf{G}$-modules.

If $\mathbf{G}$ is semisimple, S is a basis for $X \otimes_{\mathbb{Z}} \mathbb{Q}$ and the set $\{\alpha^\vee \mid \alpha \in S\}$ is a basis for $Y \otimes_{\mathbb{Z}} \mathbb{Q}$, so there are elements $\omega_\alpha \in X \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in S$. These are called the fundamental weights.

If all the ω_α are in X (rather than just in $X \otimes_{\mathbb{Z}} \mathbb{Q}$) then $\mathbf{G}$ is called simply connected. In this case, $\rho = \sum_{\alpha \in S} \omega_\alpha$, so in particular, $\rho \in X$ (and in fact, $\rho \in X_+$ since $\langle \rho, \beta^\vee \rangle = 1$ for all $\beta \in S$).

Example 4.4. Let $\mathbf{G} = SL_n$.

Since $\mathbf{G}$ is semisimple, let us compute the fundamental weights for $\mathbf{G}$. In order to do this, we need to look closer at how the pairing between X and Y looks, and what the coroots look like.

To make this simpler, we will instead of identifying X with $\mathbb{Z}^{n-1}$, identify it with a quotient of $\mathbb{Z}^n$ (we have the surjective homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ given by

$$(\lambda_1, \lambda_2, \dots, \lambda_n) \mapsto (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n)$$

so to each element in $\mathbb{Z}^n$ we associate a character (though two different element can give the same character).

With this identification, the roots, positive roots and simple roots have representatives in $\mathbb{Z}^n$ given by exactly those elements described for GL_n.

We will also identify Y with the subset of $\mathbb{Z}^n$ consisting of those tuples that sum to 0, by assigning to a tuple $(\lambda_1, \dots, \lambda_n)$ summing to 0 the cocharacter $x \mapsto \operatorname{diag}(x^{\lambda_1}, \dots, x^{\lambda_n})$.

With these identifications, the pairing between X and Y is simply the usual dot product on $\mathbb{Z}^n$

$$\langle (\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_n) \rangle = \sum_{i=1}^{n} \lambda_i \mu_i$$

and for each root α, we have $\alpha^\vee = \alpha$ (note that all the roots described for GL_n are tuples that sum to 0).

Now define $\omega_i = \sum_{j=1}^{i} e_j$ for $1 \leq i \leq n-1$. We easily check that for each $1 \leq i, j \leq n-1$ we have $\left\langle \omega_i, \lambda_j^\vee \right\rangle = \delta_{i,j}$, so the images of the ω_i in $\mathbb{Z}^{n-1}$ are the fundamental weights for $\mathbf{G}$ (if we also denote the image of ω_i by ω_i we have $\omega_i = \omega_{\lambda_i}$).

Thus, we see that SL_n is simply connected, and we can now easily compute

$$\rho = \sum_{\alpha \in S} \omega_\alpha = \sum_{i=1}^{n-1} \omega_i = (n-1, n-2, \dots, 2, 1)$$

(we could of course also have done this directly from the definition, but this calculation is slightly simpler).

Example 4.5. If we look at $\mathbf{G} = GL_n$, we see that

$$\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{-n+3}{2}, \frac{-n+1}{2})$$

so unless n is odd, this is not a character of $\mathbf{T}$.

On the other hand, if we look at the character $\widetilde{\rho} = (n-1, n-2, \dots, 1, 0)$ then this defines as described previously a character for $\mathbf{T} \cap SL_n$, and for $\mathbf{T} \cap SL_n$ the character is exactly ρ.

We also see that for any $\alpha \in S$, $\langle \widetilde{\rho}, \alpha^\vee \rangle = 1$, just like what holds for ρ (for a semisimple group this property would have determined the character uniquely, but this is not the case in general for reductive groups).

In the following, if some statement is made where GL_n would have satisfied the criteria of the statement if ρ had been a weight for GL_n, then that statement should be read as "...in the case of GL_n, replace ρ by $\widetilde{\rho}$" unless otherwise noted. The reason for this is that whenever one can show something about a module where the highest weight involves ρ, then what one really needs is the above property about the pairing with the coroots of the simple roots.

The following results describe what $\nabla(\lambda)$ looks like when it is not 0, and also tell us exactly when this happens.

Proposition 4.6 ([Jan03, Proposition II.2.2(a)]). *Let $\lambda \in X$ with $\nabla(\lambda) \neq 0$. Then* $\dim(\nabla(\lambda)^{\mathbf{U}^+}) =$ 1 *and* $\nabla(\lambda)^{\mathbf{U}^+} = \nabla(\lambda)_\lambda$.

Corollary 4.7 ([Jan03, Corollary II.2.3]). *If $\nabla(\lambda) \neq 0$ then $\mathrm{soc}_{\mathbf{G}}(\nabla(\lambda))$ is simple.*

Proposition 4.8 ([Jan03, Proposition II.2.6]). *Let $\lambda \in X$. The following are equivalent:*

1. *λ is dominant*
2. *$\nabla(\lambda) \neq 0$*
3. *There is a $\mathbf{G}$-module V with $(V^{\mathbf{U}^+})_\lambda \neq 0$*

If $\lambda \in X_+$, then by the above $\nabla(\lambda)$ is a non-zero $\mathbf{G}$-module with a unique simple submodule. We will call this simple submodule $L(\lambda)$.

We will also write $\Delta(\lambda) = \nabla(-w_0(\lambda))^*$ (this is called $V(\lambda)$ in [Jan03]).

Proposition 4.9 ([Jan03, Proposition II.2.4]).

1. *Any simple $\mathbf{G}$-module is isomorphic to exactly one $L(\lambda)$ with $\lambda \in X_+$.*

2. Let $\lambda \in X_+$. Then $L(\lambda)^{\mathbf{U}^+} = L(\lambda)_\lambda$, and $\dim(L(\lambda)^{\mathbf{U}^+}) = 1$. Any weight μ of $\nabla(\lambda)$ satisfies $w_0(\lambda) \leq \mu \leq \lambda$. The multiplicity of $L(\lambda)$ as a composition factor of $\nabla(\lambda)$ is equal to one.

What the above result tells us is that in order to understand the simple $\mathbf{G}$-modules, we just need to study the $L(\lambda)$ (for λ dominant) and the previous results tell us that in order to understand these, we need to understand the $\nabla(\lambda)$.

Further, to understand all the $L(\lambda)$, it turns out that it is actually sufficient to understand those where $\lambda \in X_1$, due to Steinberg's tensor product theorem:

Theorem 4.10 ([Ste63, Theorem 1.1],[Jan03, Proposition II.3.16])**.** *Let λ be a dominant weight and write $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$.*

Then $L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(r)}$.

Since we would like to understand $\nabla(\lambda)$, this means we should also try to understand the functor $\mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}$, and since this functor is unfortunately not exact, this again means we would like to know something about $R^i \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}$. The following three results will be important for us. The first shows that there are only finitely many i for which we need to study this.

Theorem 4.11 ([Jan03, II.4.2(3)])**.** *$R^i \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}} = 0$ for all $i > |R^+|$.*

The next tells us that when inducing dominant weights, we do not need to consider any higher derived functors. This result is known as Kempf's vanishing theorem.

Theorem 4.12 ([Kem76, Theorem 1],[Jan03, Proposition II.4.5])**.** *If $\lambda \in X_+$ then $R^i \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda) = 0$ for all $i \geq 1$.*

The third result then tells us that for certain weights, all the derived functors vanish.

Proposition 4.13 ([Jan03, Proposition II.5.4(a)])**.** *Let $\lambda \in X$ and assume that $\langle \lambda, \alpha^\vee \rangle = -1$ for some $\alpha \in S$. Then $R^i \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda) = 0$ for all $i \geq 0$.*

For dealing with specific cases, it will be useful to know for which dominant weights λ we have $L(\lambda) \cong \nabla(\lambda)$ when R is of type A. The following formulation is taken from the last part of II.8.21 in [Jan03] with the difference that we have the requirement $\alpha - \beta_0 \in R \cup \{0\}$ instead of $\alpha - \beta_0 \in R$ (without this change the formulation is not correct).

Theorem 4.14 ([Jan73, Satz 9])**.** *Assume that R is of type A_n and let $\lambda \in X_+$. For each $\alpha \in R^+$ write $\langle \lambda + \rho, \alpha^\vee \rangle = a_\alpha p^{s_\alpha} + b_\alpha p^{s_\alpha+1}$ for natural numbers $a_\alpha, b_\alpha, s_\alpha$ with $0 < a_\alpha < p$.*

Then $L(\lambda) \cong \nabla(\lambda)$ if and only if for all $\alpha \in R^+$ there are positive roots $\beta_0, \beta_1, \dots, \beta_{b_\alpha}$ such that $\langle \lambda + \rho, \beta_0^\vee \rangle = a_\alpha p^{s_\alpha}$ and for all $1 \leq i \leq b_\alpha$ we have $\langle \lambda + \rho, \beta_i^\vee \rangle = p^{s_\alpha+1}$ and such that further $\alpha = \sum_{i=0}^{b_\alpha} \beta_i$ and $\alpha - \beta_0 \in R \cup \{0\}$.

Note that when applying the above theorem to determine whether $L(\lambda) \cong \nabla(\lambda)$ for some $\lambda \in X_+$, we only need to consider those $\alpha \in R^+$ with $\langle \lambda + \rho, \alpha^\vee \rangle > p$, since the condition is trivially satisfied for all other positive roots (by picking $\beta_0 = \alpha$ since in that case we have $b_\alpha = 0$).

Note that a special case of the above shows that if $\lambda \in X_+$ with $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p$ then $L(\lambda) \cong \nabla(\lambda)$. This actually holds without restricting to type A, which will be useful later.

Proposition 4.15 ([Jan03, Corollary II.5.6])**.** *If $\lambda \in X_+$ with $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p$ then $L(\lambda) \cong \nabla(\lambda)$.*

We will also need to know something about when simple $\mathbf{G}$-modules can extend each other. This is given in the following.

Proposition 4.16 ([Jan03, II.2.14]). *Let λ and μ be dominant weights with $\lambda \not< \mu$ and assume that* $\mathrm{Ext}^1_{\mathbf{G}}(L(\lambda), L(\mu)) \neq 0$*. Then $L(\lambda)$ is a composition factor of $\Delta(\mu)$.*

Another result, which tells us something about when it is possible to have a non-trivial extension between simple modules is the linkage principle. The original reference for this is [And80b] where a stronger statement is proved. The following is an immediate consequence due to the above.

Theorem 4.17 ([And80b, Theorem 1],[Jan03, Proposition II.6.13]). *Let $\lambda, \mu \in X_+$ and assume that* $\mathrm{Ext}^1_{\mathbf{G}}(L(\lambda), L(\mu)) \neq 0$*. Then $\lambda = w \cdot \mu + p\beta$ for some $w \in W$ and some $\beta \in \mathbb{Z}R$.*

From the above we see that many questions will require knowing the composition factors of $\nabla(\lambda)$. Unfortunately, these are not completely known, but a partial answer is given by the Jantzen sum formula. In order to formulate this, we will need a bit more notation.

For a $\mathbf{G}$-module M, we define $\operatorname{ch} M \in \mathbb{Z}[X]$ by

$$\operatorname{ch} M = \sum_{\lambda \in X} \dim(M_\lambda) e(\lambda)$$

where $e(\lambda)$ is the element in $\mathbb{Z}[X]$ corresponding to $\lambda \in X$.

For $\lambda \in X$ we then define

$$\chi(\lambda) = \sum_{i \geq 0} (-1)^i \operatorname{ch} R^i \operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda).$$

Note that the sum is in fact finite due to Theorem 4.11, and that we for $\lambda \in X_+$ get $\chi(\lambda) = \operatorname{ch} \nabla(\lambda)$ by Theorem 4.12. The following was originally proved for $p \geq h$ in [Jan80] and then for all p and in more generality in [And83]

Theorem 4.18 ([Jan03, Proposition II.8.19]). *For any $\lambda \in X_+$ there is a filtration*

$$\Delta(\lambda) \supset V^1 \supset V^2 \supset \cdots$$

such that $\Delta(\lambda)/V^1 \cong L(\lambda)$ and such that

$$\sum_{i \geq 1} \operatorname{ch} V^i = \sum_{\alpha \in R^+} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} \nu_p(mp) \chi(s_\alpha \cdot \lambda + mp\alpha)$$

where ν_p denotes the p-valuation.

In applying the above formula, the following is generally useful.

Proposition 4.19 ([Jan03, II.5.9]). *For any $\lambda \in X$ and any $\alpha \in S$ we have $\chi(\lambda) = -\chi(s_\alpha \cdot \lambda)$.*

4.2 Frobenius Kernels

In this section, we will be working over k, which is an algebraically closed field of characteristic $p > 0$.

We then have the map $F : k \mapsto k$ given by $F(x) = x^p$ which is an automorphism of k. It thus makes sense to talk about x^{p^m} for any $x \in k$ and any integer m.

For any k-algebra A and any integer m, we define the k-algebra $A^{(m)}$ to be the k-algebra that coincides with A as a ring, but where each $x \in k$ acts as $x^{p^{-m}}$ would on A.

It is easy to see that $A \mapsto A^{(m)}$ defines an autoequivalence of $\{K\text{-alg}\}$ for any integer m (the inverse being given by $A \mapsto A^{(-m)}$).

For each k-algebra A, each integer m and each positive integer r, the map $\gamma_r : A^{(m)} \to A^{(m-r)}$ given by $a \mapsto a^{p^r}$ is clearly a homomorphism.

Given any k-functor X and positive integer r, we can then define a new k-functor $X^{(r)}$ by $X^{(r)}(A) = X(A^{(-r)})$ (so we just compose the two functors).

We also get a morphism of functors $F_X^r : X \to X^{(r)}$ given by

$$(F_X^r)_A = X(\gamma_r) : X(A) \to X(A^{(-r)}) = X^{(r)}(A)$$

The morphism F_X^r will be called the r'th Frobenius morphism on X.

Note that if X is defined over $\mathbb{F}_p$ then we can identify X with $X^{(r)}$ for any positive integer r.

If G is a group scheme, we see that $G^{(r)}$ is again a group scheme and the functor F_G^r is a morphism of group schemes. We define the r'th Frobenius kernel of G, written G_r, to be the kernel of F_G^r. This is then a normal subgroup scheme of G.

Let us now return to looking at the reductive group $\mathbf{G}$, this time over k (so we retain all the notation introduced for this group). Let r be a fixed positive integer. Since $\mathbf{G}$ is defined over $\mathbb{Z}$, it is also defined over $\mathbb{F}_p$, so we will identify $\mathbf{G}$ and $\mathbf{G}^{(r)}$ and view $F_{\mathbf{G}}^r$ as an endomorphism of $\mathbf{G}$. We note that the subgroups $\mathbf{T}$, $\mathbf{U}$, $\mathbf{U}^+$, $\mathbf{B}$ and $\mathbf{B}^+$ are all stable under $F_{\mathbf{G}}^r$ and the restriction of the map to these subgroups is just the corresponding Frobenius map for the corresponding k-functors. We therefore also get the subgroups $\mathbf{T}_r$, $\mathbf{U}_r$, $\mathbf{U}_r^+$, $\mathbf{B}_r$ and $\mathbf{B}_r^+$.

Any $\lambda \in X$ defines a character for $\mathbf{T}_r$ by restriction, which we will again call λ as long as no confusion is possible. In the same way as for $\mathbf{G}$, we can extend this to a module for $\mathbf{B}_r$ by having $\mathbf{U}_r$ act trivially, which will also be called λ.

We now define

$$Z'_r(\lambda) = \operatorname{ind}_{\mathbf{B}_r}^{\mathbf{G}_r}(\lambda)$$

The following results show that the representation theory of these Frobenius kernels look much like that of $\mathbf{G}$ itself.

Proposition 4.20 ([Jan03, II.3.9(1)])**.** *For any $\lambda \in X$, $Z'_r(\lambda)$ has a simple socle as a $\mathbf{G}_r$-module.*

Since the above tells us that each $Z'_r(\lambda)$ has a unique simple submodule, we set $L_r(\lambda)$ equal to this submodule, analogously to what we did for $\mathbf{G}$ (though now we do not need to assume that λ is dominant for this).

Proposition 4.21 ([Jan03, Proposition II.3.10])**.** *For all $\lambda \in X$, we have $L_r(\lambda)^{\mathbf{U}_r^+} = \lambda$.*

If Λ is a set of representatives in X for X/p^rX then each simple $\mathbf{G}_r$-module is isomorphic to exactly one $L_r(\lambda)$ with $\lambda \in \Lambda$.

Define

$$X_r = \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in S\}$$

The $r = 1$ case of the following was originally proven in [Cur60] under some additional restrictions.

Proposition 4.22 ([Jan03, Proposition II.3.15])**.** *If $\lambda \in X_r$ then $L(\lambda)$ remains simple when viewed as a $\mathbf{G}_r$-module, and is isomorphic to $L_r(\lambda)$.*

If we look at the example $\mathbf{G} = GL_n$ then we see that the map $F^r_{\mathbf{G}}$ sends a matrix with (i,j)-entry $a_{i,j}$ to the matrix with (i,j)-entry $a_{i,j}^{p^r}$, so if we only look at the points of $\mathbf{G}$ over k, the Frobenius kernel is trivial. On the other hand, as we can see from the above results, seen as affine schemes, these kernels have a very rich representation theory, which will be of enormous use later on.

Being able to use these Frobenius kernels is the main reason why this dissertation has been dealing with group schemes, rather than just Zariski-closed subgroups of some $GL_n(k)$ such as might often be the outset of a study of algebraic groups and their representations. And indeed much of what is contained in the previous chapters could as well have been done in that setting.

The main results presented in this dissertation can also be formulated in terms of just such subgroups of $GL_n(k)$, but in order to prove these, we will need access to the Frobenius kernels and their representations.

Finally, let us take a look at what a Frobenius kernel can look like when it is not trivial. Let $r = 1$ and let $A = k[T]/(T^p)$. Now we see that on A, the map sending x to x^p just returns the p'th power of the constant term of a polynomial, so $\mathbf{G}_1(A)$ consists of those matrices where the diagonal entries have constant term 1, and the remaining entries have constant term 0.

Later we will also need to be able to compare extensions of $\mathbf{G}$-modules to extensions of $\mathbf{G}_r$-modules. The way to do this is to apply the Lyndon-Hochschild-Serre spectral sequence. We only state the special case of it that we will need.

Theorem 4.23 ([Jan03, Proposition I.6.6])**.** *For any pair of $\mathbf{G}$-modules M and N, there is a spectral sequence with terms $E_2^{p,q} = \operatorname{Ext}^p_{\mathbf{G}/\mathbf{G}_r}(k, \operatorname{Ext}^q_{\mathbf{G}_r}(M,N)) \implies \operatorname{Ext}^{p+q}_{\mathbf{G}}(M,N)$*

4.3 Modules with a good filtration

We once again consider the reductive group $\mathbf{G}$ over k. A $\mathbf{G}$-module M is said to have a good filtration if it has a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

such that $\bigcup_{i \geq 0} M_i = M$ and for all $i \geq 1$, the quotient M_i/M_{i-1} is isomorphic to $\nabla(\lambda_i)$ for suitable $\lambda_i \in X_+$.

For modules with a good filtration, such a filtration will turn out to be able to play the same role as a composition series in many situations. First of all, the quotients appearing are uniquely determined (with multiplicity), though the order can of course vary:

Theorem 4.24 ([Don81, Corollary 1.3],[Jan03, Proposition II.4.16])**.** *Let V be a $\mathbf{G}$-module.*

1. *If V admits a good filtration, then for each $\lambda \in X_+$, the number of factors in the filtration isomorphic to $\nabla(\lambda)$ is equal to* $\dim(\operatorname{Hom}_{\mathbf{G}}(\Delta(\lambda), V))$.
2. *Suppose that* $\dim(\operatorname{Hom}_G(\Delta(\lambda), V)) < \infty$ *for all $\lambda \in X_+$. Then the following are equivalent:*
   (a) *V admits a good filtration.*
   (b) $\operatorname{Ext}^i_G(\Delta(\lambda), V) = 0$ *for all $\lambda \in X_+$ and all $i \geq 1$.*
   (c) $\operatorname{Ext}^1_G(\Delta(\lambda), V) = 0$ *for all $\lambda \in X_+$.*

If M admits a good filtration and $\lambda \in X_+$ we will denote the number of times $\nabla(\lambda)$ occurs in a good filtration of M by $[M : \nabla(\lambda)]_\nabla$. We use the subscript ∇ to distinguish this from the composition multiplicity of $\nabla(\lambda)$ in M (in case $\nabla(\lambda)$ is simple). By the above proposition, this is a well-defined number.

Theorem 4.25 ([Mat90, Theorem 1],[Jan03, Proposition II.4.19])**.** *Let V and V' be $\mathbf{G}$-modules admitting good filtrations. Then so does $V \otimes V'$.*

Another way to find the factors occurring in a good filtration is looking at the character of the module. The reason this works is due to the following (recall that if $\lambda \in X_+$ then $\chi(\lambda) = \operatorname{ch} \nabla(\lambda)$).

Proposition 4.26 ([Jan03, Remark to II.5.8])**.** *The $\chi(\lambda)$ for $\lambda \in X_+$ are linearly independent elements of $\mathbb{Z}[X]$.*

To use this we need to be able to write the character of a module with a good filtration in terms of the $\chi(\lambda)$. Fortunately, we know the $\chi(\lambda)$ explicitly by Weyl's character formula.

Theorem 4.27 ([Jan03, Proposition II.5.10])**.** *For all $\lambda \in X_+$*

$$\chi(\lambda) = \frac{A(\lambda + \rho)}{A(\rho)}$$

where for any $\mu \in X_+$

$$A(\mu) = \sum_{w \in W} \det(w) e(w(\mu)) \in \mathbb{Z}[X \otimes_{\mathbb{Z}} \mathbb{Q}]$$

In particular, it should be noted that the above formula does not depend on the field k, so if we denote by $\mathbf{G}_{\mathbb{C}}$ the group corresponding to $\mathbf{G}$ over $\mathbb{C}$ (recall that $\mathbf{G}$ was required to be defined over $\mathbb{Z}$, so this makes sense), we get that the character of $\nabla(\lambda)$ as a $\mathbf{G}$-module is the same as the character of the simple $\mathbf{G}_{\mathbb{C}}$-module with highest weight λ. This will be especially important when we work with SL_n.

Proposition 4.28 ([Don88, 1.4(17)],[Jan03, Proposition II.4.20])**.** *Consider $k[\mathbf{G}]$ as a $(\mathbf{G} \times \mathbf{G})$-module with the first factor acting via the left regular representation and the second factor acting via the right regular representation. Then $k[\mathbf{G}]$ admits a good filtration. The factors are of the form $\nabla(\lambda) \otimes \nabla(-w_0(\lambda))$ for $\lambda \in X_+$ and each such factor occurs exactly once.*

4.4 The Steinberg Module

In this section, we will look at a specific family of modules for the reductive group $\mathbf{G}$ over k.

Let r be a positive integer.

Throughout the section, we will assume that $(p^r - 1)\rho \in X$. This is automatic if p is odd or if $\mathbf{G}$ is semisimple and simply connected (and recall that if $\mathbf{G} = GL_n$ then ρ has a different meaning, such that it is also guaranteed to hold in that case).

Note also that in this case, we in fact have $(p^r - 1)\rho \in X_+$ and even $(p^r - 1)\rho \in X_r$.

We define $\mathrm{St}_r = L((p^r - 1)\rho) \cong \nabla((p^r - 1)\rho) \cong \Delta((p^r - 1)\rho)$ (see the remark at the end of II.3.19 in [Jan03]).

By Proposition 4.22 we also get that as a $\mathbf{G}_r$-module, $\mathrm{St}_r \cong L_r((p^r - 1)\rho)$.

For any $\mathbf{G}$-module M, we can compose the action of $\mathbf{G}$ with the Frobenius map $F^r_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}$ to get a new $\mathbf{G}$-module, which will be denoted $M^{(r)}$.

We have the following nice property of St_r.

Theorem 4.29 ([And80a, Theorem 2.5],[Jan03, Proposition II.3.19])**.** *For each $\lambda \in X_+$ there is an isomorphism of $\mathbf{G}$-modules*

$$\nabla((p^r - 1)\rho + p^r\lambda) \cong \mathrm{St}_r \otimes \nabla(\lambda)^{(r)}$$

Recall that as mentioned, the above result still holds for GL_n, even though ρ is defined differently in that case.

As a $\mathbf{G}_r$-module, we also have the following.

Theorem 4.30 ([Hum76, Proposition 10.1],[Jan03, Proposition II.10.2]). *St_r is both projective and injective as a $\mathbf{G}_r$-module.*

4.5 Tilting Modules

If M is a finite-dimensional $\mathbf{G}$-module such that both M and M^* have good filtrations, then M is called a tilting module (or M is said to be tilting).

It turns out that the indecomposable tilting modules have a similar classification to that of the simple modules, and all tilting modules will be a direct sum of indecomposable tilting modules in a unique way. The following was originally proved by Ringel in [Rin91] in a more general setting. In [Don93] Donkin then translated this general setting to our setting in the following way.

Theorem 4.31 ([Don93, Theorem 1.1],[Jan03, Proposition E.6]). *For any $\lambda \in X_+$ there is a unique (up to isomorphism) indecomposable tilting module $T(\lambda)$ with $\dim(T(\lambda)_\lambda) = 1$ and such that for all $\mu \in X$ we have $T(\lambda)_\mu \neq 0 \implies \mu \leq \lambda$.*

If Q is a tilting module, then there are uniquely determined natural numbers $n(\nu)$ such that $Q \cong \bigoplus_{\nu \in X_+} n(\nu) T(\nu)$.

Indecomposable tilting modules have many properties in common with the irreducible modules. Some that we will need are.

Proposition 4.32 ([Jan03, Remark to E.6]). *Let $\lambda, \mu \in X_+$. Then $T(\lambda)^* \cong T(-w_0(\lambda))$ and $\mathrm{Hom}_{\mathbf{G}}(L(\mu), T(\lambda)) \cong \mathrm{Hom}_{\mathbf{G}}(T(\lambda), L(\mu))$.*

Later we will need to know when indecomposable tilting modules are injective as $\mathbf{G}_r$-modules. This is completely described by the following.

Proposition 4.33 ([Don93, Proposition 2.4],[Jan03, Lemma E.8]). *If $\lambda \in X_+$ then $T(\lambda)$ is injective as a $\mathbf{G}_r$-module if and only if $\langle \lambda, \alpha^\vee \rangle \geq p^r - 1$ for all $\alpha \in S$.*

Proposition 4.34 ([Don93, Proposition 2.1],[Jan03, Lemma E.9]). *Let $\mu = (p^r - 1)\rho + \lambda$ with $\lambda \in X_r$ and let $\nu \in X_+$. Then $T(\mu) \otimes T(\nu)^{(r)}$ is tilting, and if $T(\mu)$ is indecomposable as a $\mathbf{G}_r$-module, then $T(\mu) \otimes T(\nu)^{(r)} \cong T(\mu + p^r \nu)$.*

The condition that $T(\mu)$ is indecomposable as a $\mathbf{G}_r$-module when μ is as in the above proposition is in fact conjectured to always hold (see [Don93, Conjecture 2.2]). It is known to hold if $p \geq 2h - 2$.

Theorem 4.35 ([Jan03, E.9]). *If $\lambda \in X_r$ and $p \geq 2h - 2$ then*

$$\mathrm{soc}_{\mathbf{G}}(T((p^r - 1)\rho + \lambda)) = \mathrm{soc}_{\mathbf{G}_r}(T((p^r - 1)\rho + \lambda)) = L((p^r - 1)\rho + w_0(\lambda))$$

4.6 Representations of SL_n

In this section we will take a closer look at the representations of SL_n. We will in particular be interested in finding filtrations of tensor products.

First, we need to setup some notation for this special case, as we will be denoting the weights in two different ways.

Usually, we will write all weights in terms of the fundamental weights (since SL_n is semisimple and simply connected), so this way we identify X with $\mathbb{Z}^{n-1}$, X_+ is the set of elements in $\mathbb{Z}^{n-1}$ with non-negative entries, and X_r is the set of elements in $\mathbb{Z}^{n-1}$ with all entries x_i satisfying $0 \leq x_i \leq p^r - 1$.

Given a $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \in X_+$ (written in the basis as mentioned above), we get a partition $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{n-1})$ where $\tilde{\lambda}_i = \sum_{j=i}^{n-1} \lambda_j$. In this way we have $\lambda_i = \tilde{\lambda}_i - \tilde{\lambda}_{i+1}$ (where for convenience we will set $\tilde{\lambda}_n = 0$).

From now on, any element $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in X_+$ will be assumed to be written in terms of the fundamental weights, whereas elements in X_+ written as partition will be decorated with a $\tilde{}$ (so whenever we write $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}) \in X_+$ this means that when written in terms of the fundamental weights this is the element $\lambda = (\tilde{\lambda}_1 - \tilde{\lambda}_2, \dots, \tilde{\lambda}_1 - \tilde{\lambda}_0)$).

As will be seen later, we need to be able to compute $[\nabla(\lambda) \otimes \nabla(\mu) : \nabla(\nu)]_\nabla$ for any $\lambda, \mu, \nu \in X_+$ (this is well-defined by Theorem 4.25). In order to do this, we first use that by Proposition 4.26 this number can be determined by computing $\chi(\lambda)$ and $\chi(\mu)$, multiplying these and writing the result as a linear combination of $\chi(\sigma)$'s for various $\sigma \in X_+$. $[\nabla(\lambda) \otimes \nabla(\mu) : \nabla(\nu)]_\nabla$ is then the number of times $\chi(\nu)$ appears in this.

To compute these characters, we use that by Theorem 4.27 these characters only depend on the root system, so we can do the same calculation working with $SL_n(\mathbb{C})$ instead.

Now, for $SL_n(\mathbb{C})$ we have that $\nabla(\lambda)$ is simple for all $\lambda \in X_+$ by [Jan03, Corollary II.5.6], so what we need to do is decompose a tensor product of irreducible modules for $SL_n(\mathbb{C})$, and the way to do this is described in Appendix B.

Summarizing the above we get (we use the notation from Appendix B, so for a dominant weight $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ we have $|\tilde{\lambda}| = \sum_{i=1}^{n-1} i\lambda_i$).

Proposition 4.36. *Let $\mathbf{G} = SL_n$ and $\lambda, \mu, \nu \in X_+$. Set $m = \frac{|\tilde{\lambda}| + |\tilde{\mu}| - |\tilde{\nu}|}{n}$ and $\tilde{\omega} = \overbrace{(1, 1, \dots, 1)}^{n}$. Then*

$$[\nabla(\lambda) \otimes \nabla(\mu) : \nabla(\nu)]_\nabla = \begin{cases} c_{\tilde{\lambda}, \tilde{\mu}}^{\tilde{\nu} + m\tilde{\omega}} & \text{if } m \text{ is a non-negative integer} \\ 0 & \text{else} \end{cases}$$

The same arguments also show.

Proposition 4.37. *Let $\lambda_1, \dots, \lambda_s, \mu \in X_+$ and set*

$$m = \frac{\sum_{i=1}^{s} |\tilde{\lambda}_i| - |\tilde{\mu}|}{n}$$

Then

$$[\bigotimes_{i=1}^{n} \nabla(\lambda_i) : \nabla(\mu)]_\nabla = 0$$

unless m is a non-negative integer.

5 Finite Groups of Lie Type

We consider again the reductive group $\mathbf{G}$ over k and let r be a fixed positive integer. Denote by F the r'th Frobenius morphism $F^r_{\mathbf{G}}$.

Let $G = \mathbf{G}^F$ be the subgroup of $\mathbf{G}$ consisting of those elements that are fixed by F (since we identify $\mathbf{G}$ with $\mathbf{G}^{(r)}$ this makes sense). Similarly, we define the subgroups T, U, U^+, B and B^+.

In this dissertation, a finite group of Lie type will be a group of the form $G(k)$ (so we take the points over k of a group as above). The proofs in this section are generally the same as those in chapter 2 of [Hum06], though with some more details.

Example 5.1. Let $\mathbf{G} = GL_n$, so the map F sends the matrix with (i,j)-entry $a_{i,j}$ to the one with (i,j)-entry $a_{i,j}^{p^r}$.

We then get that $G(k)$ consists of invertible $n \times n$ matrices with coefficients that are fixed by the map $x \mapsto x^{p^r}$, so we can identify this with the set of matrices with coefficients in the finite field $\mathbb{F}_{p^r}$, $GL_n(\mathbb{F}_{p^r})$.

It turns out that the representation theory of $\mathbf{G}$, G and of $\mathbf{G}_r$ have a lot in common. One of the tools in studying the connection between them is the Lang map $L : \mathbf{G} \to \mathbf{G}$ given by $L(x) = F(x)x^{-1}$. The following is known as Lang's Theorem.

Theorem 5.2 ([Spr09, Theorem 4.4.17])**.** *The map $L(x) = F(x)x^{-1}$ is a surjective morphism of $\mathbf{G}$.*

If $\mathbf{H}$ is a subgroup of $\mathbf{G}$ which is invariant under F, then $L(x) \in \mathbf{H}$ if and only if $x \in \mathbf{H}$, so the restriction of L to any such $\mathbf{H}$ gives a surjective morphism $L_{\mathbf{H}} : \mathbf{H} \to \mathbf{H}$.

We denote by L^* the comorphism of L from $k[\mathbf{G}]$ to itself.

The properties of L (and L^*) that we now need are the following:

Lemma 5.3. *$k[\mathbf{U}^+]^{U^+}$ is the image of L^* restricted to $k[\mathbf{U}^+]$, i.e. the elements in $k[\mathbf{U}^+]$ of the form $f \circ L$ for $f \in k[\mathbf{U}^+]$.*

Similarly, $k[\mathbf{U}^+]^{\mathbf{U}^+_r}$ is the image of F^* restricted to $k[\mathbf{U}^+]$, i.e. the elements of the form $f \circ F$ for $f \in k[\mathbf{U}^+]$.

Proof. Since L is surjective, it is a quotient map, which gives an isomorphism of affine schemes $\mathbf{U}^+/U^+ \to \mathbf{U}^+$, so the comorphism gives an isomorphism of algebras $L^* : k[\mathbf{U}^+] \to k[\mathbf{U}^+/U^+]$. But since we have $k[\mathbf{U}^+/U^+] = k[\mathbf{U}^+]^{U^+}$, this means that $k[\mathbf{U}^+]^{U^+}$ is the image of L^* which exactly consists of the elements of the form $f \circ L$ for $f \in k[\mathbf{U}^+]$.

The second statement follows by the same argument. □

We can regard $k[\mathbf{U}^+]$ as a $\mathbf{B}^+$-module by letting $\mathbf{U}^+$ act by translation from the right and $\mathbf{T}$ act by conjugation. This gives us a decomposition into weight spaces as usual for $\mathbf{T}$-modules.

Lemma 5.4. *If $f \in k[\mathbf{U}^+]_\mu$ for some $\mu \in X$ then $f \circ L$ is the sum of $f \circ F$ (which is in $k[\mathbf{U}^+]_{p^r\mu}$) and other elements f' which lie in certain $k[\mathbf{U}^+]_{p^r\mu-\alpha}$ where α is in $\mathbb{N}R^+$.*

Proof. We will view $k[\mathbf{U}^+]$ as morphisms from $\mathbf{U}^+$ to $\mathbb{A}^1$, but we will simplify the notation a bit by letting A be a fixed k-algebra and writing f instead of f_A for $f \in k[\mathbf{U}^+]$, so we are really looking at maps. Let $\mu \in X$ and $f \in k[\mathbf{U}^+]_\mu$ be fixed.

For any positive root α, we can compose f with x_α to get a morphism $\mathbb{A}^1 \to \mathbb{A}^1$, and since any such map is given by a polynomial, for any $a \in \mathbb{G}_a$, we can write $f(x_\alpha(a)) = \sum_{i \geq 0} c_i a^i$ for suitable c_i in k, only finitely many of which are non-zero.

Since f has weight μ, we have for any $t \in \mathbf{T}$ and any $a \in \mathbb{G}_a$ that

$$\sum_{i \geq 0} \mu(t) c_i a^i = \mu(t) f(x_\alpha(a)) = f(t x_\alpha(a) t^{-1}) = f(x_\alpha(\alpha(t)a)) = \sum_{i \geq 0} \alpha(t)^i c_i a^i$$

so we get that for some $i \geq 0$, $\mu(t) = \alpha(t)^i$, so $\mu = i\alpha$, and for all $j \neq i$ we have $c_j = 0$.

This means that $f(x_\alpha(a)) = c_i a^i$.

On the other hand, if $\beta \neq \alpha$ is a positive root, then we can again write $f(x_\beta(a)) = \sum_{j \geq 0} d_j a^j$ and we get that

$$\sum_{j \geq 0} \alpha(t)^i d_j a^j = (i\alpha)(t) f(x_\beta(a)) = f(t x_\beta(a) t^{-1}) = f(x_\beta(\beta(t)a)) = \sum_{j \geq 0} \beta(t)^j d_j a^j$$

so since we picked $\beta \neq \alpha$ this means that all $d_j = 0$ so $f(x_\beta(a)) = 0$ for all $a \in \mathbb{G}_a$. We also get that $(f \circ L)(x_\beta(a)) = 0$, so we now only need to check that the statement holds on elements of the form $x_\alpha(a)$.

First, let us show that the weight of $f \circ F$ is indeed $p^r\mu$. This is easily done by a direct computation.

$$\begin{aligned} t.(f \circ F)(u) &= f(F(tut^{-1})) = f(F(t)F(u)F(t)^{-1}) \\ &= f(t^{p^r} F(u) t^{-p^r}) = \mu(t)^{p^r} f(F(u)) = (p^r\mu)(t)(f \circ F)(u) \end{aligned}$$

We will also need to note that $F(x_\alpha(a)) = x_\alpha(a^{p^r})$ (this is clear from the way F has been defined).

Let us now compute $(f \circ L)(x_\alpha(a))$ using the above. We get

$$\begin{aligned} (f \circ L)(x_\alpha(a)) &= f(L(x_\alpha(a))) = f(F(x_\alpha(a)) x_\alpha(a)^{-1}) \\ &= f(x_\alpha(a^{p^r}) x_\alpha(-a)) = f(x_\alpha(a^{p^r} - a)) = c_i (a^{p^r} - a)^i = c_i a^{p^r i} + b \end{aligned}$$

where b is some polynomial of degree less than $p^r i$.

From the above we see now that since $f \circ L$ is uniquely determined by what it does to elements of the form $x_\alpha(a)$, that the decomposition of $f \circ L$ into a sum of elements in various weight spaces, corresponds exactly to the above decomposition of $f \circ L \circ x_\alpha$ as a decomposition into monomials, and some $f' \in k[\mathbf{U}^+]$ has weight $j\alpha$ exactly if $f(x_\alpha(a)) = ca^j$ for some $c \in k$.

The summands we get are thus elements in weight spaces of weight $p^r\mu - j\alpha$ for suitable $j \geq 0$, so for $j \neq 0$ these are strictly smaller than $p^r\mu$. All we then need to show is that the summand with weight $p^r\mu$ is indeed $f \circ F$.

But if we calculate $(f \circ F)(x_\alpha(a)) = f(F(x_\alpha(a))) = f(x_\alpha(a^{p^r})) = c_i a^{p^r i}$, we see that this is the case.

$\square$

We will also need the following:

Lemma 5.5 ([Jan03, II.8.17]). *For any $\lambda \in X_+$ there is a symmetric, non-degenerate, bilinear form $(\cdot,\cdot)$ on $L(\lambda)$, such that for any $\mu, \mu' \in X$ with $\mu \neq \mu'$, the weight spaces $L(\lambda)_\mu$ and $L(\lambda)_{\mu'}$ are orthogonal. If $\mathbf{H}$ is a subgroup of $\mathbf{G}$ and V is an $\mathbf{H}$-submodule of $L(\lambda)$, then so is the orthogonal complement of V.*

The way we have constructed the $L(\lambda)$ means they are submodules of $k[\mathbf{G}]$, and since the big cell $\mathbf{UB}^+$ is dense, we see that the restriction map from $k[\mathbf{G}]$ to $k[\mathbf{U}^+]$ is injective on any $L(\lambda)$ as the action on these is uniquely determined by the action of the big cell.

This means that we can identify $L(\lambda)$ with a submodule of $k[\mathbf{U}^+]$, but the decomposition of the weight spaces will be different, as we on $k[\mathbf{G}]$ have $\mathbf{T}$ acting by translation and on $k[\mathbf{U}^+]$ it acts by conjugation. Thus, the λ weight space of $L(\lambda)$ obtains weight 0 when viewed as a submodule of $k[\mathbf{U}^+]$.

Lemma 5.6. *Let M be a $\mathbf{B}^+$-submodule of $k[\mathbf{U}^+]$ and assume that $M \cap k[\mathbf{U}^+]^{\mathbf{U}_r^+} = k$. Then $M \cap k[\mathbf{U}^+]^{U^+} = k$.*

Proof. By Lemma 5.3 what we need to show is that under the assumptions, if we have an $f \in k[\mathbf{U}^+]$ such that $f \circ L \in M$ then f has weight 0, so we need to show that $f \circ F \in M$.

Write $f = \sum_\mu f_\mu$ where f_μ has weight μ, and similarly write $f \circ L = \sum_\mu g_\mu$. By assumption, $f \circ L \in M$, which is a $\mathbf{T}$-submodule, so all g_μ are in M. Let ν be maximal with $f_\nu \neq 0$. We now have that $f_\nu \circ F = g_{p^r\nu}$ because of Lemma 5.4, since that tells us that $f_\nu \circ L$ has $f_\nu \circ F$ as the unique summand of weight $p^r\nu$. This means that $f_\nu \circ F$ lies in both M and $F^*(k[\mathbf{U}^+])$, which by Lemma 5.3 means that $f_\nu \in M \cap k[\mathbf{U}^+]^{\mathbf{U}_r^+}$, which by assumption is k, so $g_{p^r\nu}$ has weight 0 which means that $p^r\nu = 0$ so $\nu = 0$. But since ν was chosen to be maximal, this means that $f = f_0$, so f has weight 0, which was what we needed to show.

□

Lemma 5.7. *For any $\lambda \in X_r$ we have $L(\lambda)^{U^+} = L(\lambda)_\lambda$.*

Proof. This follows directly from Lemma 5.6, Proposition 4.21 and Proposition 4.22, by letting the M in 5.6 be the copy of $L(\lambda)$ inside $k[\mathbf{U}^+]$ as described previously (recall that the 0 weight space in $k[\mathbf{U}^+]$ corresponds to the λ weight space in $k[\mathbf{G}]$ when we make this identification).

□

We are now ready to prove that restricting simple $\mathbf{G}$-modules to G will again give us a simple module, as long as we put some restrictions on the highest weight.

Proposition 5.8. *Let $\lambda \in X_r$. Then $L(\lambda)$ remains simple when viewed as a G-module.*

Proof. Since $L(\lambda)^{U^+} = L(\lambda)_\lambda$ is 1-dimensional, we can let v^+ span this subspace. Since U^+ is a p-group, any non-zero U^+-submodule of $L(\lambda)$ will have a non-zero fixed point. So if M is a G-submodule of $L(\lambda)$ then M must intersect non-trivially with $L(\lambda)^{U^+}$, so it must contain v^+. This means that the G-submodule generated by v^+ is the unique simple G-submodule of $L(\lambda)$.

We now want to show that in fact, v^+ generates all of $L(\lambda)$ as a G-module, so let M be this module. For this, we use the non-degenerate bilinear form on $L(\lambda)$. Since weight spaces with distinct weights are orthogonal with respect to this form and the form is non-degenerate, we must have $(v^+, v^+) \neq 0$ since the λ weight space is 1-dimensional.

But now the orthogonal complement of M will again be a G-submodule, and since it does not contain v^+, this means that it is 0. Thus, we must have $M = L(\lambda)$ as desired.

□

We will also be able to prove that these $L(\lambda)$ are pair-wise non-isomorphic and form a complete set of irreducible G-modules, as long as we assume $\mathbf{G}$ to be semisimple and simply connected. For this, we will need to know the number of such modules, which is provided by the following results.

Theorem 5.9 ([Isa76, Corollary 15.11]). *If H is a finite group then the number of isomorphism classes of irreducible representations of H over k is equal to the number of p-regular conjugacy classes of H.*

Theorem 5.10 ([Car93, Theorem 3.7.6]). *If the derived subgroup $\mathbf{G}'$ is simply connected then the number of semisimple conjugacy classes in G is $|(Z(\mathbf{G})^0)^F|p^{rl}$ where l is the semisimple rank of $\mathbf{G}$.*

Proposition 5.11. *If $\mathbf{G}$ is semisimple and simply connected, then the number of isomorphism classes of simple G-modules is p^{rl}, where l is the semisimple rank of $\mathbf{G}$.*

Proof. Since $\mathbf{G}$ is semisimple, $Z(\mathbf{G})$ is finite and thus the connected component $Z(\mathbf{G})^0$ is trivial so $|(Z(\mathbf{G})^0)^F| = 1$. We also get that $\mathbf{G}$ is equal to its derived subgroup, so we can use 5.10 and get that the number of semisimple conjugacy classes of G is p^{rl}.

So by 5.9 we just need to show that an element of G is semisimple if and only if it is p-regular.

Since k is algebraically closed, we just need to show that under any embedding of G into some $GL_n(k)$, the elements corresponding to diagonalizable matrices are exactly the p-regular ones. But this just means that we need to show that a matrix in $GL_n(k)$ of finite order is diagonalizable if and only if it has order not divisible by p.

First, assume $A \in GL_n(k)$ is diagonalizable, so $D = XAX^{-1}$ is diagonal for some X. Now the order of D is not divisible by p, since there are no elements in k of order divisible by p (if x was such an element then we would have $(x^m)^p = 1$ so $(x^m)^p - 1 = (x^m - 1)^p = 0$ so $x^m = 1$).

Conversely, assume that the order of $A \in GL_n(k)$ is not divisible by p. By the multiplicative Jordan-decomposition, we can write $A = BC = CB$ with B diagonalizable and C unipotent. Thus, we are done if we can show that the order of any unipotent matrix has order divisible by p.

So assume that B is a unipotent matrix, so $(B - I)^m = 0$ for some m. Now let m' be given such that $p^{m'} \geq m$. Then $0 = (B - I)^{p^{m'}} = B^{p^{m'}} - I$ so the order of B is a power of p, and we are done. $\square$

We will also need a further result in order to distinguish how G acts on the various $L(\lambda)$. It will turn out that knowing how T acts gives us most of the information needed, but we will in some cases also need to know something about the action of $U_{-\alpha}$ for a simple root α.

Lemma 5.12. *Let $\lambda \in X_+$ and $\alpha \in S$. Then $U_{-\alpha}$ stabilizes the weight space $L(\lambda)_\lambda$ if and only if $\langle \lambda, \alpha^\vee \rangle = 0$.*

Proof. Since $\mathbf{G}$ is split, it is enough to show that the statement holds for $\mathbf{U}_{-\alpha}$, as the stabilizer of some subspace of $L(\lambda)$ when viewed as a G-module is just the fixed points under the Frobenius morphism of the stabilizer when viewed as a $\mathbf{G}$-module.

Let v span $L(\lambda)_\lambda$. Assume first that $\langle \lambda, \alpha^\vee \rangle = 0$. We wish to show that for any $a \in \mathbb{G}_a$ we have $x_{-\alpha}(a).v = cv$ for some $c \in k$.

We know that since $x_{-\alpha}$ is a homomorphism from $\mathbb{G}_a$, the action of $x_{-\alpha}(a)$ is given by a polynomial in a, so we can write $x_{-\alpha}.v = \sum_\mu f_\mu(a)v_\mu$ where $v_\mu \in L(\lambda)_\mu$ and $f_\mu \in k[a]$.

Now let $t \in \mathbf{T}$. We can compute $t.x_{-\alpha}(a).t^{-1}.v$ in two ways. We have

$$t.x_{-\alpha}(a).t^{-1}.v = (tx_{-\alpha}(a)t^{-1}).v = x_{-\alpha}(-\alpha(t)a).v = \sum_\mu f_\mu(-\alpha(t)a)v_\mu$$

but also since $v \in L(\lambda)_\lambda$ that

$$t.x_{-\alpha}(a).t^{-1}.v = t.x_{-\alpha}(a).(\lambda(t)^{-1}v) = t.\sum_\mu \lambda(t)^{-1}f_\mu(a)v_\mu = \sum_\mu \lambda(t)^{-1}f_\mu(a)\mu(t)v_\mu$$

Since the v_μ are linearly independent, this means that we have $f_\mu(-\alpha(a)) = \mu(t)\lambda(t)^{-1}f_\mu(a)$ for each μ, so if $f_\mu(a) = \sum_{i=0}^{m} c_i a^i$ we get that for each i we have $\mu(t)\lambda(t)^{-1}c_i = c_i(-\alpha(t))^i$ so all but one c_i must be 0, and we get that $f_\mu(a) = c_i a^i$ for some i. Since this was for an arbitrary $t \in \mathbf{T}$, we then get that $\mu - \lambda = i(-\alpha)$ so $\mu = \lambda + i(-\alpha)$.

So we see that the possible weights that can occur when we act by $x_{-\alpha}(a)$ are those of the form $\lambda - i\alpha$ for non-negative integers i. Now, if $\lambda - i\alpha$ is a root, then so is $s_\alpha(\lambda - i\alpha)$ (where s_α is the reflection corresponding to the simple root α). But we have

$$s_\alpha(\lambda - i\alpha) = s_\alpha(\lambda) - s_\alpha(i\alpha) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha + i\alpha$$

and since we know that λ is the highest weight of $L(\lambda)$, this forces $i \leq \langle \lambda, \alpha^\vee \rangle$, which then means that $i = 0$, and that $x_{-\alpha}(a).v = cv$ for a suitable constant $c \in k$ as desired.

Now we look at the other direction. So assume that $x_{-\alpha}(a).v \in kv$ for all $a \in \mathbb{G}_a$.

We will look at the image of φ_α in $\mathbf{G}$, which is isomorphic to SL_2. Call this subgroup L. Let M be the L-module generated by v. Now, we see that M is a highest weight module for L, so we know that it is generated by the vectors $u.v$ for $u \in \mathbf{U}_{-\alpha}$. If $\mathbf{U}_{-\alpha}$ stabilizes the subspace spanned by v, then this will imply that M is 1-dimensional. We know that this must mean that the highest weight of M (with respect to the action of the maximal torus in L) is 0.

On the other hand, we know that the maximal torus of L consists of those elements in $\mathbf{T}$ of the form $\alpha^\vee(\mathbb{G}_m)$, so for any t in this maximal torus, we know that the action on v is given by multiplication by $t^{\langle \lambda, \alpha^\vee \rangle}$, so the weight is $\langle \lambda, \alpha^\vee \rangle$, which must then be 0, as was the claim.

$\square$

Proposition 5.13. *Let $\mathbf{G}$ be semisimple and simply connected and let L be a finite-dimensional simple G-module. Then L is isomorphic to exactly one $L(\lambda)$ with $\lambda \in X_r$.*

Proof. Since $\mathbf{G}$ is semisimple and simply connected, we can write any $\lambda \in X$ as a $\mathbb{Z}$-linear combination of the fundamental weights $\omega_1, \dots, \omega_l$ where l is the rank of $\mathbf{G}$.

We then see that we have $\lambda \in X_r$ exactly if λ can be written as $\sum_{i=1}^{l} a_i\omega_i$ with $0 \leq a_i < p^r$ for $1 \leq i \leq l$.

Now it is clear by 5.11 that if we can show that distinct elements of X_r give rise to non-isomorphic simple modules even when restricted to G, then we are done, as there are exactly p^{rl} elements in X_r and this is also the number of simple G-modules.

Now, if $L(\lambda)$ and $L(\mu)$ (with $\lambda, \mu \in X_r$) are isomorphic as G-modules, then such an isomorphism will send U^+-invariant vectors to U^+-invariant vectors, and since the U^+-invariant vectors are exactly those of weight λ (for $L(\lambda)$) or μ (for $L(\mu)$), this means that for all $t \in T$, we have $\lambda(t) = \mu(t)$. Now write $\lambda = \sum_{i=1}^{l} a_i\omega_i$ and $\mu = \sum_{i=1}^{l} b_i\omega_i$. The preceding requirement is then that for all $x \in \mathbb{F}_{p^r}$ and $1 \leq i \leq l$, we have $x^{a_i} = x^{b_i}$, so we have $a_i \equiv b_i \pmod{p^r - 1}$.

Since all the a_i and b_i are between 0 and $p^r - 1$, this means that if $\lambda \neq \mu$ then for some i we have $a_i = 0$ and $b_i = p^r - 1$ (or vice versa, but we can assume this by symmetry). If this is the case, we cannot distinguish the modules by the T-action alone, but we can see that if α is the root corresponding to the i'th fundamental weight, then the root subgroup $U_{-\alpha}$ will stabilize the U^+-invariant vectors in $L(\lambda)$, but not the ones in $L(\mu)$ by Lemma 5.12, so the modules are not isomorphic.

$\square$

Note that the above proposition also holds without the assumption of $\mathbf{G}$ being simply connected (see [Hum06] Theorem 2.11), but the proof requires some other considerations, as we no longer have as good a description of the number of finite-dimensional simple modules, nor can we use the fundamental weights as was done in the above proof.

The last part of the above proposition fails for arbitrary reductive groups, as can be seen by looking at GL_n where there are infinitely many distinct weights in X_r, but clearly there are only finitely many isomorphism classes of finite-dimensional simple modules.

For $\mathbf{G} = GL_n$ we can still determine all the simple modules for G, by using that the derived subgroup of $\mathbf{G}$ is SL_n, which is semisimple and simply connected, so we can use 5.10 along with 5.9 and the arguments from the proof of 5.11 to find the number of isomorphism classes of irreducible G-modules. We see that this means that we need to find $|(Z(\mathbf{G})^0)^F|$, but since the center of $\mathbf{G}$ is isomorphic to $k^\times$, it is connected, and we know that the fixed points under F just correspond to $\mathbb{F}_{p^r}^\times$, so there are $p^r - 1$ elements.

We thus need to find $(p^r - 1)p^{r(n-1)}$ simple non-isomorphic G-modules. But we know that

$$X_r = \{(\lambda_1, \dots, \lambda_n) \,|\, 0 \le \lambda_i - \lambda_{i+1} < p^r \text{ for all } 1 \le i \le n-1\}$$

so if we restrict ourselves to just those $\lambda \in X_r$ with $\lambda_n = 0$, we get $p^{r(n-1)}$ such λ. Now we see that any module of the form $L(\lambda) \otimes \mathrm{Det}^m$ will be simple when restricted to SL_n, and we see that they are pairwise non-isomorphic if we require $0 \le m \le p^r - 2$, so we now have $(p^r - 1)p^{r(n-1)}$ pair-wise non-isomorphic simple G-modules as we wanted.

5.1 Composition Multiplicity of the Steinberg Module

If M is some $\mathbf{G}$-module with a good filtration, we would like to be able to calculate the composition multiplicity of the r'th Steinberg module St_r in M, seen as a G-module. We denote this multiplicity by $[M : \mathrm{St}_r]$ and note that since St_r is injective and projective (see Theorem 5.18), we have $[M : \mathrm{St}_r] = \dim(\mathrm{Hom}_G(\mathrm{St}_r, M))$.

In order to find a formula for this multiplicity, we will use some ideas from [BNP12], where they study the module $\mathcal{G}_r = \mathrm{ind}_G^{\mathbf{G}}(k)$.

The properties of this module we will need are the following (see also [BNP12]).

Proposition 5.14. *Let M and N be $\mathbf{G}$-modules. Then, for all $i \ge 0$,*

$$\mathrm{Ext}^i_G(M, N) \cong \mathrm{Ext}^i_{\mathbf{G}}(M, N \otimes \mathcal{G}_r)$$

Proof. Since $\mathbf{G}/G$ is isomorphic to $\mathbf{G}$ as an affine scheme, it is itself affine, and by Proposition 3.25 G is exact in $\mathbf{G}$. Now we can apply Proposition 3.26 together with Proposition 3.23 and get

$$\mathrm{Ext}^i_{\mathbf{G}}(M, N \otimes \mathcal{G}_r) = \mathrm{Ext}^i_{\mathbf{G}}(M, N \otimes \mathrm{ind}_G^{\mathbf{G}}(k)) \cong \mathrm{Ext}^i_{\mathbf{G}}(M, \mathrm{ind}_G^{\mathbf{G}}(k \otimes \mathrm{res}_G^{\mathbf{G}}(N))) \cong \mathrm{Ext}^i_G(M, N)$$

as was the claim. □

The above proposition allows us to change the calculation of $\dim(\mathrm{Hom}_G(\mathrm{St}_r, M))$ to a calculation of $\dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, M \otimes \mathcal{G}_r))$, and we already have a lot of tools to study Hom-spaces for $\mathbf{G}$-modules.

In order to better understand the $\mathbf{G}$-module $\mathcal{G}_r$, we will show that it is in fact isomorphic to $k[\mathbf{G}]$ but with a different action of $\mathbf{G}$ than the usual one. Let $k[\mathbf{G}]^\vee$ denote the $\mathbf{G}$-module that as a k-module is $k[\mathbf{G}]$ and where the action of $\mathbf{G}$ is given as follows: Embed $\mathbf{G}$ into $\mathbf{G} \times \mathbf{G}$ via the usual diagonal map Δ, and let $\varphi : \mathbf{G} \times \mathbf{G} \to \mathbf{G} \times \mathbf{G}$ be given by (F, id). Let $\mathbf{G} \times \mathbf{G}$ act on $k[\mathbf{G}]$ by letting the first factor act via the left regular representation and the second factor act via the right regular representation. Let $\mathbf{G}$ act on $k[\mathbf{G}]^\vee$ via the composition of the maps Δ and φ.

This means that we have $(g.f)(x) = f(F(g)^{-1}xg)$.

Proposition 5.15. *As a* $\mathbf{G}$*-module,* $\mathcal{G}_r$ *is isomorphic to* $k[\mathbf{G}]^\vee$*, and has a filtration with factors of the form* $\nabla(\lambda)\otimes\nabla(-w_0(\lambda))^{(r)}$*, where for each* $\lambda\in X_+$*, the corresponding factor has multiplicity 1.*

Proof. Let $L:\mathbf{G}\to\mathbf{G}$ be the Lang map as defined previously. By Theorem 5.2 and the following lemmas, the map L^* is a bijection from $k[\mathbf{G}]$ to $k[\mathbf{G}]^G$. But we have

$$\mathcal{G}_r = \operatorname{ind}_G^{\mathbf{G}}(k) = \{f\in k[\mathbf{G}]\,|\,f(gh)=f(g) \text{ for all } g\in\mathbf{G},\, h\in G\} = k[\mathbf{G}]^G$$

so we have a bijection from $k[\mathbf{G}]$ to $\mathcal{G}_r$. We then just need to check that this map is a morphism of $\mathbf{G}$-modules when we use the action of $\mathbf{G}$ on $k[\mathbf{G}]$ described above.

So we calculate

$$L^*(g.f)(x) = (g.f)(L(x)) = f(F(g)^{-1}L(x)g)$$

and

$$(g.(L^*(f)))(x) = L^*(f(g^{-1}x)) = f(L(g^{-1}x))$$
$$= f(F(g^{-1}x)(g^{-1}x)^{-1}) = f(F(g)^{-1}F(x)x^{-1}g) = f(F(g)^{-1}L(x)g)$$

and we see that it is indeed a morphism of $\mathbf{G}$-modules.

The existence of the claimed filtration now follows directly from Proposition 4.28. ☐

Before we go further, we will present a proof that St_r is projective as a G-module, using the above results.

We will first need two lemmas.

Define a new ordering on X by $\mu\leq_{\mathbb{Q}}\lambda$ if $\lambda-\mu$ is a non-negative linear combination of positive roots with *rational* coefficients.

Lemma 5.16 ([Hum06, Proposition 5.8(a)])**.** *If* $\lambda\in X_+$*,* $\mu\in X_r$ *and* $L(\mu)$ *is a composition factor of* $\nabla(\lambda)$ *as a* G*-module, then* $\mu\leq_{\mathbb{Q}}\lambda$*.*

Lemma 5.17 ([Hum06, Theorem 3.7(d)])**.** *If* $\lambda\in X_r$ *then* $\dim(\Delta(\lambda))\leq\dim(\mathrm{St}_r)$*.*

Theorem 5.18. *As a* G*-module,* St_r *is both projective and injective.*

Proof. We first prove that $\mathrm{Ext}^i_G(\Delta(\lambda),\mathrm{St}_r)=0$ for all $\lambda\in X_r$ and all $i\geq 1$.

By Proposition 5.14 we have $\mathrm{Ext}^i_G(\Delta(\lambda),\mathrm{St}_r)\cong\mathrm{Ext}^i_{\mathbf{G}}(\Delta(\lambda),\mathrm{St}_r\otimes\mathcal{G}_r)$, and by Proposition 5.15 we have a filtration of $\mathrm{St}_r\otimes\mathcal{G}_r$ with factors of the form $\mathrm{St}_r\otimes\nabla(\mu)\otimes\nabla(-w_0(\mu))^{(r)}$ which is isomorphic to $\nabla(\mu)\otimes\nabla((p^r-1)\rho-p^rw_0(\mu))$ by Theorem 4.29. By Theorem 4.25 this has a good filtration, and thus $\mathrm{St}_r\otimes\mathcal{G}_r$ has a good filtration, which yields the first claim by Theorem 4.24.

To prove the theorem, we need to show that $\mathrm{Ext}^i_G(L(\lambda),\mathrm{St}_r)=0$ for all $\lambda\in X_r$ and all $i\geq 1$. We do this by induction on λ (with respect to the ordering $\leq_{\mathbb{Q}}$). If we apply $\mathrm{Hom}_G(-,\mathrm{St}_r)$ to the short exact sequence $0\to N\to\Delta(\lambda)\to L(\lambda)\to 0$ we get a long exact sequence with terms

$$\mathrm{Ext}^i_G(N,\mathrm{St}_r)\to\mathrm{Ext}^{i+1}_G(L(\lambda),\mathrm{St}_r)\to\mathrm{Ext}^{i+1}_G(\Delta(\lambda),\mathrm{St}_r).$$

When $i\geq 1$ we get the claim by induction using Lemma 5.16, so we only need to consider the case $i=0$ where we have

$$\mathrm{Hom}_G(N,\mathrm{St}_r)\to\mathrm{Ext}^1_G(L(\lambda),\mathrm{St}_r)\to\mathrm{Ext}^1_G(\Delta(\lambda),\mathrm{St}_r)=0$$

where the last term is 0 by the previous claim. But $\mathrm{Hom}_G(N,\mathrm{St}_r)=0$ since St_r is simple as a G-module and since $\dim(N)<\dim(\Delta(\lambda))\leq\dim(\mathrm{St}_r)$, which completes the proof.

That also $\mathrm{Ext}^i_G(\mathrm{St}_r,L(\lambda))=0$ for all $\lambda\in X_r$ and all $i\geq 1$ can be shown in the same way. ☐

Some notes on the above proof: First it should be noted that the result can be proven much more easily as a special case of the theorem that if M is a $\mathbf{G}$-module which is injective as a $\mathbf{G}_r$-module, then M is injective as a G-module (see [Dru13, Theorem 2.3]).

Another note is that when $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_r$ (so for example when $p \geq 2h-2$ by Theorem 6.11), then the proof can be simplified, as then it is possible to use the precise same arguments as those for the vanishing of $\mathrm{Ext}^i_G(\Delta(\lambda), \mathrm{St}_r)$ to show the vanishing of $\mathrm{Ext}^i_G(L(\lambda), \mathrm{St}_r)$ by applying Theorem 6.10 and using that

$$\mathrm{Ext}^i_{\mathbf{G}}(L(\lambda), \mathrm{St}_r \otimes \nabla(\mu) \otimes \nabla(-w_0(\mu))^{(r)}) \cong \mathrm{Ext}^i_{\mathbf{G}}(\Delta(-w_0(\mu)), \mathrm{St}_r \otimes L(\lambda) \otimes \nabla(-w_0(\mu))^{(r)})$$

The formula for the multiplicity we can prove is then the following (the proof is due to Pillen, see [WW11, Proposition 2.5]).

Theorem 5.19. *Let N be a finite-dimensional $\mathbf{G}$-module admitting a good filtration. Then*

$$\dim(\mathrm{Hom}_G(\mathrm{St}_r, N)) = \sum_{\lambda \in X_+} [N \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

Proof. First, we will compute $\mathrm{Ext}^i_{\mathbf{G}}(\mathrm{St}_r, N \otimes \nabla(\lambda) \otimes \nabla(-w_0(\lambda))^{(r)})$ for some $\lambda \in X_+$. We start by noting that by the definition of $\Delta(\lambda)$ it is isomorphic to $\mathrm{Ext}^i_{\mathbf{G}}(\mathrm{St}_r \otimes \Delta(\lambda)^{(r)}, N \otimes \nabla(\lambda))$ and by Theorem 4.29 this is isomorphic to $\mathrm{Ext}^i_{\mathbf{G}}(\Delta((p^r-1)\rho + p^r\lambda), N \otimes \nabla(\lambda))$. Finally, by Proposition 4.24, we know that if $i \geq 1$ this is 0 and for $i = 0$ we get

$$\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes \nabla(\lambda) \otimes \nabla(-w_0(\lambda))^{(r)}) \cong \mathrm{Hom}_{\mathbf{G}}(\Delta((p^r-1)\rho + p^r\lambda), N \otimes \nabla(\lambda))$$

If M is a $\mathbf{G}$-module admitting a filtration with factors of the form $\nabla(\lambda) \otimes \nabla(-w_0(\lambda)^{(r)}$ and S is a $\mathbf{G}$-submodule of M admitting a filtration of the same form, then so does the quotient $Q = M/S$. The short exact sequence $0 \to S \to M \to Q \to 0$ induces a long exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes S) \to \mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes M) \to \mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes Q) \to \mathrm{Ext}^1_{\mathbf{G}}(\mathrm{St}_r, N \otimes S)$$

and by the previous calculation, the last term vanishes, which shows that

$$\dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes M)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes S)) + \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes Q))$$

By repeating the above identity and using the filtration of $\mathcal{G}_r$ described in Proposition 5.15, we get that

$$\dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes \mathcal{G}_r)) = \sum_{\lambda \in X_+} \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, N \otimes \nabla(\lambda) \otimes \nabla(-w_0(\lambda))^{(r)}))$$

By Proposition 5.14 the left-hand side of the above is $\dim(\mathrm{Hom}_G(\mathrm{St}_r, N))$, and we can now use the calculations from the start of this proof to get that the right-hand side is

$$\sum_{\lambda \in X_+} \dim(\mathrm{Hom}_{\mathbf{G}}(\Delta((p^r-1)\rho + p^r\lambda), N \otimes \nabla(\lambda)))$$

and now the claim follows from applying Proposition 4.24 which says that

$$\dim(\mathrm{Hom}_{\mathbf{G}}(\Delta((p^r-1)\rho + p^r\lambda), N \otimes \nabla(\lambda))) = [N \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

□

6 A Conjecture of Donkin

This section is based on the joint paper with Daniel Nakano [KN14], where we extend previous work of Andersen ([And01]) on a conjecture of Donkin (Conjecture 6.1).

Assume for convenience in this section that $\mathbf{G}$ is semisimple and simply connected.

6.1 Good (p, r)-filtrations

For $\lambda \in X_+$ write $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$ (then $\lambda_1 \in X_+$ automatically).

Define $\nabla^{(p,r)}(\lambda) = L(\lambda_0) \otimes \nabla(\lambda_1)^{(r)}$.

A good (p, r)-filtration of a $\mathbf{G}$-module M is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

such that $\bigcup_i M_i = M$ and such that $M_i/M_{i-1} \cong \nabla^{(p,r)}(\lambda_i)$ for all $i \geq 1$ and for suitable $\lambda_i \in X_+$.

The following conjecture which was introduced by Donkin at an MSRI lecture in 1990 interrelates good filtrations with good (p, r)-filtrations via the Steinberg module.

Conjecture 6.1. *Let M be a finite-dimensional $\mathbf{G}$-module admitting a good (p, r)-filtration. Then $\mathrm{St}_r \otimes M$ has a good filtration.*

In [And01], Andersen showed that the above conjecture is true when $p \geq 2h - 2$. Donkin also conjectured that the reverse implication should hold, but this part of the conjecture will not be mentioned further in this dissertation.

In this section, we will expand on the results of Andersen by proving that when M has a good (p, r)-filtration then $\mathrm{St}_r \otimes M$ has a good filtration, provided a suitable inequality holds between p, r, h and the weights occurring in the good (p, r)-filtration of M. As a special case, we recover the results of Andersen, though our method of proof is markedly different. Our method of proof involves the use of Donkin's cohomological criterion for the existence of a good filtration (Theorem 4.24), and a careful analysis of the vanishing of extension groups with suitable conditions on weights of the form $w.0 + p\beta$ with $w \in W$ and $\beta \in \mathbb{Z}R$.

In order to prove Conjecture 6.1, it is clearly enough to prove that $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration for any $\lambda \in X_+$. However, due to a result of Andersen (included as Proposition 6.10), it turns out that it is enough to show that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for any $\lambda \in X_r$. The inequality we obtain allows us to prove that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration with smaller restrictions on p provided that the weight λ is also suitably smaller. This still leaves weights $\lambda \in X_+$ for which we do not know whether $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration when p is small. However, if $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$ and if λ_1 is large enough compared to λ_0 (made precise in Theorem 6.19), then we can still show that $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration, even if λ_0 is not small enough to satisfy the inequality we get with respect to p, r and h.

A natural question is for which $\lambda \in X_+$ does $\mathrm{St}_r \otimes L(\lambda)$ have a good filtration? When $p \geq 2h-2$ and if $\langle \lambda, \alpha_0^\vee \rangle \leq (p^r - 1)(h - 1)$ we have that $L(\lambda) \simeq \nabla^{(p,r)}(\lambda)$ (Proposition 6.13) so in these cases it does hold. However, we also prove that this is close to being the best bound of this type possible. Namely, we show that if $p = 2h-5$ and R is of type A, then there is a λ with $\langle \lambda, \alpha_0^\vee \rangle \leq (p-1)(h-1)$ and such that $\mathrm{St}_1 \otimes L(\lambda)$ does not have a good filtration. By examining the weights more closely, we prove that Conjecture 6.1 is true if R is of type A_2, A_3, B_2 or G_2, without restrictions on p, apart from $p \neq 7$ in the G_2 case.

6.2 Weights of the form $w \cdot 0$

Throughout this section we will make use of the observation that if $\lambda \leq \mu$ with $\lambda, \mu \in X$ then $\langle \lambda, \alpha_0^\vee \rangle \leq \langle \mu, \alpha_0^\vee \rangle$. This follows because $\mu = \lambda + \beta$ where β is a non-negative linear combination of the simple roots, and inner product of any simple root with $\alpha_0^\vee$ is greater or equal to zero.

Lemma 6.2. *Let $w \in W$ and set $R_w^{<0} = \{\alpha \in -R^+ \mid w^{-1}(\alpha) > 0\}$. Then*

$$w.0 = \sum_{\alpha \in R_w^{<0}} \alpha$$

In particular, $w_0.0 \leq w.0 \leq 0$.

Proof. Define $R_w^{>0} = \{\alpha \in R^+ \mid w^{-1}(\alpha) > 0\}$.

We first claim that $w(R^+) = R_w^{>0} \cup R_w^{<0}$. This follows by noting that in fact we also have $R_w^{>0} = \{w(\alpha) \mid \alpha \in R^+,\, w(\alpha) > 0\}$ and $R_w^{<0} = \{w(\alpha) \mid \alpha \in R^+,\, w(\alpha) < 0\}$.

We also have that $R^+ = R_w^{>0} \cup -R_w^{<0}$. Since $|R^+| = |w(R^+)|$, and clearly $R_w^{>0} \cup -R_w^{<0} \subseteq R^+$, it is enough to check that $R_w^{>0}$ and $-R_w^{<0}$ are disjoint. But if $\alpha \in R_w^{>0} \cap -R_w^{<0}$, then by the above note we have that $\alpha = w(\beta)$ for some $\beta \in R^+$ and also that $-\alpha = w(\gamma)$ for some $\gamma \in R^+$, and hence we have $w(\beta + \gamma) = \alpha - \alpha = 0$ so $\beta + \gamma = 0$, but this is not possible for positive roots β and γ.

We note that $w.0 = w(\rho) - \rho$ and since w permutes the roots and ρ is half the sum of all the positive roots, we get from the above that

$$w(\rho) = w\left(\frac{1}{2}\sum_{\alpha \in R^+} \alpha\right) = \frac{1}{2}\sum_{\alpha \in R^+} w(\alpha) = \frac{1}{2}\sum_{\beta \in w(R^+)} \beta = \frac{1}{2}\left(\sum_{\alpha \in R_w^{>0}} \alpha + \sum_{\alpha \in R_w^{<0}} \alpha\right)$$

so

$$w(\rho) - \rho = \sum_{\alpha \in R_w^{<0}} \alpha$$

and hence the first part of the claim. Since $w_0.0 = w_0(\rho) - \rho = -2\rho = \sum_{\alpha \in -R^+} \alpha$ this gives the second part of the claim. □

We can now prove bounds on the size of the inner products of $w \cdot 0$ with coroots.

Proposition 6.3. *Let $w \in W$. Then*

(a) *$\langle w \cdot 0, \alpha_0^\vee \rangle \geq -2(h-1)$;*

(b) *$\langle w \cdot 0, \alpha^\vee \rangle \leq h - 2$ for any $\alpha \in S$.*

Proof. (a) By Lemma 6.2 we have $w \cdot 0 \geq w_0 \cdot 0$, thus

$$\langle w \cdot 0, \alpha_0^\vee \rangle \geq \langle w_0 \cdot 0, \alpha_0^\vee \rangle = \langle -2\rho, \alpha_0^\vee \rangle = -2(h-1).$$

(b) Observe that $\langle w \cdot 0, \alpha^\vee \rangle = \langle w(\rho) - \rho, \alpha^\vee \rangle = \langle w(\rho), \alpha^\vee \rangle - \langle \rho, \alpha^\vee \rangle = \langle \rho, w^{-1}(\alpha^\vee) \rangle - 1$. But $\langle \rho, w^{-1}(\alpha^\vee) \rangle$ is at most $\langle \rho, \alpha_0^\vee \rangle$ because for any root β we have that $\langle \rho, \beta^\vee \rangle$ is the height of $\beta^\vee$. Part (b) now follows because $\langle \rho, \alpha_0^\vee \rangle = h - 1$. □

6.3 Dominant weights in the root lattice

We summarize the results on dominant weights which will be used in the subsequent sections in the following proposition.

Proposition 6.4. *Let $\lambda \in X_+$ and assume that $p \geq h-1$.*

(a) *If $\lambda = w \cdot 0 + p\beta$ for some $w \in W$ and $\beta \in \mathbb{Z}R$ and $\lambda \neq 0$, then $\langle \lambda, \alpha_0^\vee \rangle \geq 2(p-h+1)$.*

(b) *Suppose that $\mathrm{Ext}^i_{\mathbf{G}}(k, L(\lambda)) \neq 0$ for some $i \geq 1$. Then $\langle \lambda, \alpha_0^\vee \rangle \geq 2(p-h+1)$.*

Proof. (a) By Proposition 6.3(b), if $\alpha \in S$, we have $\langle w \cdot 0, \alpha^\vee \rangle \leq h-2$. Since λ is dominant

$$0 \leq \langle w \cdot 0 + p\beta, \alpha^\vee \rangle \leq (h-1) + p\langle \beta, \alpha^\vee \rangle.$$

Now by assumption $p \geq h-1$ which forces $\langle \beta, \alpha^\vee \rangle \geq 0$, so β must be dominant.

Next we observe that by Proposition A.1, we must have $\langle \beta, \alpha_0^\vee \rangle \geq 2$ since $\beta \neq 0$. Combining this fact with Proposition 6.3(a), we see that

$$\langle \lambda, \alpha_0^\vee \rangle = \langle w \cdot 0, \alpha_0^\vee \rangle + p\langle \beta, \alpha_0^\vee \rangle \geq -2(h-1) + 2p = 2(p-h+1)$$

as claimed.

(b) The linkage principle (Theorem 4.17) implies that $\lambda = w \cdot 0 + p\beta$ for some $w \in W$ and some $\beta \in \mathbb{Z}R$, so the result follows directly from part (a) since we must have $\lambda \neq 0$. $\square$

Note that when $p \geq h$ the above cannot be improved. This is because, for $\lambda = (p-h+1)\alpha_0$, there is a short exact sequence $0 \to L(\lambda) \to \nabla(\lambda) \to L(0) \to 0$, as can be seen by applying the Jantzen sum formula (Theorem 4.18).

6.4 A Cohomological Criterion

Theorem 4.24 will be the main tool used to prove the existence of good filtrations, but for our purposes it is convenient to provide a modified version of this cohomological criterion. We note that if Donkin's conjecture holds, parts (b) and (c) of the theorem below would give a cohomological criterion for the existence of good (p, r)-filtrations.

Theorem 6.5. *Let M be a $\mathbf{G}$-module. The following are equivalent*

(a) *$\mathrm{St}_r \otimes M$ has a good filtration.*

(b) *$\mathrm{Ext}^1_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\Delta(\mu), \mathrm{St}_r \otimes M)) = 0$ for all $\mu \in X_+$.*

(c) *$\mathrm{Ext}^i_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\Delta(\mu), \mathrm{St}_r \otimes M)) = 0$ for all $\mu \in X_+$, $i \geq 1$.*

Proof. Consider the Lyndon-Hochschild-Serre spectral sequence (Theorem 4.23)

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Ext}^q_{\mathbf{G}_r}(\Delta(\mu), \mathrm{St}_r \otimes M)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes M).$$

Since St_r is injective as a $\mathbf{G}_r$-module by Theorem 4.30, we also have that $\mathrm{St}_r \otimes M$ is injective as a $\mathbf{G}_r$-module. Therefore, this spectral sequence collapses and yields the isomorphism:

$$\mathrm{Ext}^i_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes M) \cong \mathrm{Ext}^i_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\Delta(\mu), \mathrm{St}_r \otimes M)) \tag{1}$$

for all $i \geq 0$. The theorem now follows from Theorem 4.24. $\square$

6.5 Good filtrations for $\mathrm{St}_r \otimes L(\lambda)$: bounds on λ

From the proof of Theorem 6.5, we have

$$\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes M) \cong \mathrm{Ext}^1_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\Delta(\mu), \mathrm{St}_r \otimes M)) \tag{2}$$

We will first show that we can restrict our attention to a finite set of weights $\mu \in X_+$ in order to verify that this extension group is zero.

Lemma 6.6. *Let $\lambda, \mu \in X_+$ and assume that $\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) \neq 0$. Then*

$$\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle + (p^r - 1)(h - 1).$$

Proof. Consider the short exact sequence $0 \to L(\lambda) \to \nabla(\lambda) \to Q \to 0$. One can tensor this sequence with St_r and apply $\mathrm{Hom}_{\mathbf{G}}(\Delta(\mu), -)$ to obtain the long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes Q) \to \mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) \to \mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes \nabla(\lambda)) \to \cdots$$

Since $\mathrm{St}_r \otimes \nabla(\lambda)$ has a good filtration by Theorem 4.25, it follows from Theorem 4.24 that we have $\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes \nabla(\lambda)) = 0$. From our hypothesis, $\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) \neq 0$ which implies that $\mathrm{Hom}_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes Q) \neq 0$.

The head of $\Delta(\mu)$ is $L(\mu)$, so μ must then be a weight of $\mathrm{St}_r \otimes Q$ and also a weight of $\mathrm{St}_r \otimes \nabla(\lambda)$. In particular, $\mu \leq (p^r - 1)\rho + \lambda$, and

$$\langle \mu, \alpha_0^\vee \rangle \leq \langle (p^r - 1)\rho + \lambda, \alpha_0^\vee \rangle = (p^r - 1)(h - 1) + \langle \lambda, \alpha_0^\vee \rangle.$$

□

Using the result in the preceding section we can then obtain another bound for μ which is needed in order to get $\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) \neq 0$, this time requiring $\langle \mu, \alpha_0^\vee \rangle$ to be large enough compared to λ, p and r.

Proposition 6.7. *Let $p \geq h - 1$ and assume that $\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) \neq 0$ for some $\lambda, \mu \in X_+$. Then*

$$\langle \mu, \alpha_0^\vee \rangle \geq (p^r - 1)(h - 1) + 2p^r(p - h + 1) - \langle \lambda, \alpha_0^\vee \rangle.$$

Proof. From (2) and using the fact that $\mathrm{St}_r \cong \mathrm{St}_r^*$ and $\Delta(\mu)^* \cong \nabla(-w_0(\mu))$ we have

$$\begin{aligned} \mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) &\cong \mathrm{Ext}^1_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda))) \\ &\cong \mathrm{Ext}^1_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, \nabla(-w_0(\mu)) \otimes L(\lambda))) \end{aligned}$$

so we can assume the last Ext-group is not 0. Set $\nu = -w_0(\mu)$.

If we take a composition series for $\nabla(\nu) \otimes L(\lambda)$ (as a $\mathbf{G}$-module), this gives us a filtration of $\mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, \nabla(\nu) \otimes L(\lambda))$ since St_r is projective as a $\mathbf{G}_r$-module by Theorem 4.30. Therefore, since we assume that $\mathrm{Ext}^1_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}(\mathrm{St}_r, \nabla(\nu) \otimes L(\lambda))) \neq 0$, there must be some $\sigma \in X_+$ such that $L(\sigma)$ is a composition factor of $\nabla(\nu) \otimes L(\lambda)$ and such that $\mathrm{Ext}^1_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, L(\sigma))) \neq 0$.

In particular, $\mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, L(\sigma)) \neq 0$. By Theorem 4.10 we have $L(\sigma) \cong L(\sigma_0) \otimes L(\sigma_1)^{(r)}$ where $\sigma = \sigma_0 + p^r\sigma_1$ with $\sigma_0 \in X_r$ and $\sigma_1 \in X_+$. Consequently,

$$0 \neq \mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, L(\sigma)) \cong \mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, L(\sigma_0) \otimes L(\sigma_1)^{(r)}) \cong \mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, L(\sigma_0)) \otimes L(\sigma_1)^{(r)}.$$

Since St_r is simple as a $\mathbf{G}_r$-module, $\sigma_0 = (p^r - 1)\rho$, and $\mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, L(\sigma)) \cong L(\sigma_1)^{(r)}$. We now have

$$0 \neq \mathrm{Ext}^1_{\mathbf{G}/\mathbf{G}_r}(k, \mathrm{Hom}_{\mathbf{G}_r}(\mathrm{St}_r, L(\sigma))) \cong \mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma_1)).$$

Now apply Proposition 6.4(b), which shows that $\langle \sigma_1, \alpha_0^\vee \rangle \geq 2(p - h + 1)$. This yields

$$\langle \sigma, \alpha_0^\vee \rangle \geq (p^r - 1)(h - 1) + 2p^r(p - h + 1).$$

Since $L(\sigma)$ was assumed to be a composition factor of $\nabla(\nu) \otimes L(\lambda)$ we get that $\sigma \leq \nu + \lambda$. Therefore,

$$(p^r - 1)(h - 1) + 2p^r(p - h + 1) \leq \langle \sigma, \alpha_0^\vee \rangle \leq \langle \nu + \lambda, \alpha_0^\vee \rangle.$$

It now follows that

$$\langle \mu, \alpha_0^\vee \rangle = \langle \mu, -w_0(\alpha_0^\vee) \rangle = \langle \nu, \alpha_0^\vee \rangle \geq (p^r - 1)(h - 1) + 2p^r(p - h + 1) - \langle \lambda, \alpha_0^\vee \rangle.$$

□

The preceding results allow us to provide sufficient conditions for $\mathrm{St}_r \otimes L(\lambda)$ to admit a good filtration.

Theorem 6.8. *Let $p \geq h - 1$ and assume that $\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) \neq 0$ for some $\lambda, \mu \in X_+$. Then*

$$(p^r - 1)(h - 1) + 2p^r(p - h + 1) - \langle \lambda, \alpha_0^\vee \rangle \leq \langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle + (p^r - 1)(h - 1)$$

Proof. This follows directly by combining Lemma 6.6 and Proposition 6.7. □

And we also obtain the following theorem, removing the mention of μ. We only state the result for $p \geq h$ as the conditions on λ are never satisfied for $p = h - 1$.

Theorem 6.9. *Assume that $p \geq h$ and let $\lambda \in X_+$ with $\langle \lambda, \alpha_0^\vee \rangle < p^r(p - h + 1)$. Then $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration.*

Proof. If $\mathrm{St}_r \otimes L(\lambda)$ does not have a good filtration, then by Theorem 4.24 there must be some $\mu \in X_+$ with $\mathrm{Ext}^1_{\mathbf{G}}(\Delta(\mu), \mathrm{St}_r \otimes L(\lambda)) \neq 0$. Hence, by Theorem 6.8

$$(p^r - 1)(h - 1) + 2p^r(p - h + 1) - \langle \lambda, \alpha_0^\vee \rangle \leq \langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle + (p^r - 1)(h - 1)$$

and in particular

$$(p^r - 1)(h - 1) + 2p^r(p - h + 1) - \langle \lambda, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle + (p^r - 1)(h - 1)$$

which gives $\langle \lambda, \alpha_0^\vee \rangle \geq p^r(p - h + 1)$, contradicting the choice of λ. □

6.6 Donkin's Conjecture

We first recall the result of Andersen (cf. [And01, Proposition 2.6]), which allows us to reduce the general question of whether $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration to just considering the case when $\lambda \in X_r$ (i.e., whether $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration). A proof is included below as we need a slightly more general version than the one originally given by Andersen.

Proposition 6.10. *Let V be a $\mathbf{G}$-module. The following are equivalent.*

(a) $\mathrm{St}_r \otimes V$ *has a good filtration.*

(b) $\mathrm{St}_r \otimes V \otimes \nabla(\lambda)^{(r)}$ *has a good filtration for all $\lambda \in X_+$.*

Proof. Since $\nabla(0)^{(r)} \cong k^{(r)} \cong k$, we clearly have that (b) implies (a). For the other direction (i.e. (a) implies (b)) assume that $\mathrm{St}_r \otimes V$ has a good filtration. Then by Theorem 4.25, $\mathrm{St}_r \otimes V \otimes \nabla(p^r\lambda)$ has a good filtration for all $\lambda \in X_+$. Since direct summands of modules with good filtrations themselves have good filtrations (by Theorem 4.24), it is sufficient to show that $\mathrm{St}_r \otimes \nabla(\lambda)^{(r)}$ is a direct summand of $\mathrm{St}_r \otimes \nabla(p^r\lambda)$, which would then imply that $\mathrm{St}_r \otimes V \otimes \nabla(\lambda)^{(r)}$ is a direct summand of $\mathrm{St}_r \otimes V \otimes \nabla(p^r\lambda)$.

For our purposes we need to show that there are maps $\varphi : \mathrm{St}_r \otimes \nabla(\lambda)^{(r)} \to \mathrm{St}_r \otimes \nabla(p^r\lambda)$ and $\psi : \mathrm{St}_r \otimes \nabla(p^r\lambda) \to \mathrm{St}_r \otimes \nabla(\lambda)^{(r)}$ such that $\psi \circ \varphi = \mathrm{id}$. Since $\mathrm{St}_r \otimes \nabla(\lambda)^{(r)} \cong \nabla((p^r-1)\rho + p^r\lambda)$ by Theorem 4.29, and this has a simple socle and 1-dimensional space of endomorphisms, it is sufficient to find such φ and ψ such that the weight space of weight $(p^r-1)\rho + p^r\lambda$ is not in the kernel of the composed map.

In order for maps φ and ψ as above to exist, we need $\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r \otimes \nabla(\lambda)^{(r)}, \mathrm{St}_r \otimes \nabla(p^r\lambda)) \neq 0$ and $\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r \otimes \nabla(p^r\lambda), \mathrm{St}_r \otimes \nabla(\lambda)^{(r)}) \neq 0$. We prove this below, and note that the arguments for this claim also show that choosing any non-zero maps φ and ψ will in fact give the desired property.

By Frobenius reciprocity (Proposition 3.22), we have

$$\mathrm{Hom}_{\mathbf{G}}(\nabla(\lambda)^{(r)}, \nabla(p^r\lambda)) \cong \mathrm{Hom}_{\mathbf{B}}(\nabla(\lambda)^{(r)}, p^r\lambda) \neq 0$$

since $p^r\lambda$ is the highest weight of $\nabla(\lambda)^{(r)}$. Hence, $\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r \otimes \nabla(\lambda)^{(r)}, \mathrm{St}_r \otimes \nabla(p^r\lambda)) \neq 0$.

On the other hand, we have $\mathrm{St}_r \otimes \nabla(\lambda)^{(r)} \cong \nabla((p^r-1)\rho + p^r\lambda)$, and by Frobenius reciprocity,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r \otimes \nabla(p^r\lambda), \mathrm{St}_r \otimes \nabla(\lambda)^{(r)}) &\cong \mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r \otimes \nabla(p^r\lambda), \nabla((p^r-1)\rho + p^r\lambda)) \\
&\cong \mathrm{Hom}_{\mathbf{B}}(\mathrm{St}_r \otimes \nabla(p^r\lambda), (p^r-1)\rho + p^r\lambda) \\
&\neq 0
\end{aligned}$$

since $(p^r-1)\rho + p^r\lambda$ is the highest weight of $\mathrm{St}_r \otimes \nabla(p^r\lambda)$. This shows that $\mathrm{St}_r \otimes \nabla(\lambda)^{(r)}$ is a direct summand of $\mathrm{St}_r \otimes \nabla(p^r\lambda)$ which completes the proof. $\square$

We can now present a proof of one direction of Donkin's Conjecture when $p \geq 2h-2$ which recovers Proposition 2.10 of [And01].

Theorem 6.11. *Let $p \geq 2h-2$. If M has a good (p,r)-filtration then $\mathrm{St}_r \otimes M$ has a good filtration.*

Proof. Since $p \geq 2h-2$ we have $p-h+1 \geq h-1$. Therefore, for any $\lambda \in X_r$,

$$\langle \lambda, \alpha_0^\vee \rangle \leq (p^r-1)(h-1) < p^r(h-1) \leq p^r(p-h+1).$$

Hence, according to Theorem 6.9, $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_r$. The statement of the theorem now follows from Proposition 6.10. $\square$

6.7 Other bounds

In this section, we will present another method to obtain sufficient conditions on p, r and λ to ensure that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration. This procedure starts with the $r = 1$ case and then uses an inductive argument similar to the one in [And01, Proposition 2.10], though the formulation below is more general. In some cases, it will be easier to deal with the $r = 1$ case.

Proposition 6.12. *Let m be a positive integer and let $\Gamma_1 \subseteq X_m$ be a set of weights, such that $\mathrm{St}_m \otimes L(\lambda)$ has a good filtration for all $\lambda \in \Gamma_1$. Let $\Gamma_r = \sum_{i=0}^{r-1} p^{im}\Gamma_1$ be the set of weights λ of the form $\lambda = \lambda_0 + p^m\lambda_1 + \cdots + p^{(r-1)m}\lambda_{r-1}$ with all $\lambda_i \in \Gamma_1$.*

Then $\mathrm{St}_{rm} \otimes L(\lambda)$ has a good filtration for all $\lambda \in \Gamma_r$.

Proof. We proceed by induction on r. Using Theorem 4.10, we have

$$\mathrm{St}_{rm} = L((p^{rm} - 1)\rho) = L((p^m - 1)\rho + p^m(p^{(r-1)m} - 1)\rho) \cong \mathrm{St}_m \otimes \mathrm{St}_{(r-1)m}^{(m)}$$

and if $\lambda = \lambda_0 + p^m\lambda_1 + \cdots + p^{(r-1)m}\lambda_{r-1}$ with all $\lambda_i \in \Gamma_1$, then we can write $\lambda = \lambda_0 + p^m\mu$ where $\mu = \lambda_1 + p^m\lambda_2 + \cdots + p^{(r-2)m}\lambda_{r-1} \in \Gamma_{r-1}$, and $L(\lambda) \cong L(\lambda_0) \otimes L(\mu)^{(m)}$.

Now $\mathrm{St}_{rm} \otimes L(\lambda) \cong \mathrm{St}_m \otimes L(\lambda_0) \otimes (\mathrm{St}_{(r-1)m} \otimes L(\mu))^{(m)}$. By assumption $\mathrm{St}_m \otimes L(\lambda_0)$ has a good filtration since $\lambda_0 \in \Gamma_1$, and by induction we get that $\mathrm{St}_{(r-1)m} \otimes L(\mu)$ has a good filtration. The result now follows by Proposition 6.10. ☐

The case of the above that will be of most interest is when $m = 1$. As a special case one obtains the result: if $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_1$, then $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_r$.

One way to use Proposition 6.12 is to use Theorem 6.9 in the case $r = 1$ to get a set of weights to use, and then expand. The set of weights thus obtained for arbitrary r will generally contain weights not satisfying the inequality of Theorem 6.9 unless either $p \leq h$ or $p \geq 2h - 2$. In the case of $p \geq 2h - 2$ we can take $\Gamma_1 = X_1$ and obtain an alternative proof of Corollary 6.11. Furthermore, if $p < 2h - 2$ then there are weights in X_1 which do not satisfy the inequality in Theorem 6.9, so we cannot directly improve the bound on p this way.

6.8 Tensoring with other simple modules

The methods employed in the preceding sections use the condition that $\langle \lambda, \alpha_0^\vee \rangle$ is not too large to show that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration. In particular, our techniques do not need that $L(\lambda)$ remains simple when restricted to $\mathbf{G}_r$ (i.e., $\lambda \in X_r$).

Therefore, a natural question to ask is whether one can replace the conjecture that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_r$ (which is still only a conjecture when $p < 2h - 2$) with the stronger statement that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_+$ with $\langle \lambda, \alpha_0^\vee \rangle \leq (p^r - 1)(h - 1)$ (which also holds when $p \geq 2h - 2$ by Corollary 6.9).

However, we will show that this is not the case for smaller primes. For $p \geq 2h - 2$ with $\langle \lambda, \alpha_0^\vee \rangle \leq (p^r - 1)(h - 1)$, λ also satisfies $\langle \lambda, \alpha_0^\vee \rangle < p^r(p - h + 1)$, and we have the following result which does hold for smaller primes.

Proposition 6.13. *Let $\lambda \in X_+$ with $\langle \lambda, \alpha_0^\vee \rangle < p^r(p - h + 1)$. Then $L(\lambda) \cong \nabla^{(p,r)}(\lambda)$.*

Proof. Write $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$ and observe that

$$\langle \lambda_1, \alpha_0^\vee \rangle \leq \frac{1}{p^r}\langle \lambda, \alpha_0^\vee \rangle < p - h + 1$$

so $\langle \lambda_1 + \rho, \alpha_0^\vee \rangle < (p-h+1) + (h-1) = p$ and hence $L(\lambda_1) \cong \nabla(\lambda_1)$ by Proposition 4.15.

Consequently, by Theorem 4.10

$$\nabla^{(p,r)}(\lambda) = L(\lambda_0) \otimes \nabla(\lambda_1)^{(r)} \cong L(\lambda_0) \otimes L(\lambda_1)^{(r)} \cong L(\lambda_0 + p^r\lambda_1) = L(\lambda).$$

□

The following class of counterexamples shows that $\mathrm{St}_1 \otimes L(\lambda)$ need not have a good filtration when $\langle \lambda, \alpha_0^\vee \rangle \leq (p-1)(h-1)$.

Proposition 6.14. *Let R be of type A_n with $n \geq 3$ and assume that $p = 2h-5$ is a prime. Let $\lambda = p(\omega_1 + \omega_2 + \cdots + \omega_{n-1})$. Then $\langle \lambda, \alpha_0^\vee \rangle \leq (p-1)(h-1)$ but $\mathrm{St}_1 \otimes L(\lambda)$ does not have a good filtration.*

Proof. Since $h \geq 4$ (recall that $h = n+1$) we have

$$\begin{aligned} \langle \lambda, \alpha_0^\vee \rangle &= p(n-1) = (2h-5)(h-2) = 2h^2 - 9h + 10 = 2h^2 - 8h + 6 - (h-4) \\ &\leq 2h^2 - 8h + 6 = (2h-6)(h-1) = (p-1)(h-1) \end{aligned}$$

which was the first part of the claim.

Let $\mu = \omega_1 + \omega_2 + \cdots + \omega_{n-1}$ (so $\lambda = p\mu$). We claim that $\nabla(\mu) \not\cong L(\mu)$. First apply Theorem 4.14 with the positive root $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$. One has

$$\langle \mu + \rho, \alpha^\vee \rangle = (n-1) + (n-1) = 2h - 4 = 1 + p$$

so if we had $\nabla(\mu) \cong L(\mu)$ there would have to be a positive root β_0 with $\langle \mu + \rho, \beta_0^\vee \rangle = 1$ and $\alpha - \beta_0 \in R \cup \{0\}$.

However, the only positive root β_0 such that $\langle \mu + \rho, \beta_0^\vee \rangle = 1$ is $\beta_0 = \alpha_n$ since $\mu + \rho$ is dominant and all other simple roots γ have $\langle \mu + \rho, \gamma^\vee \rangle = 2$. Since $\alpha - \alpha_n$ is not a root (and $\alpha \neq \alpha_n$), this shows the claim.

Now one has $\mathrm{St}_1 \otimes L(\lambda) \cong \mathrm{St}_1 \otimes L(\mu)^{(1)} \cong L((p-1)\rho + p\mu)$ by Theorem 4.10. But since this is a simple module, the only way it can have a good filtration is if it is isomorphic to $\nabla((p-1)\rho + p\mu)$. But by Theorem 4.29 we have $\nabla((p-1)\rho + p\mu) \cong \mathrm{St}_1 \otimes \nabla(\mu)^{(1)}$. Since $L(\mu)^{(1)}$ is a submodule of $\nabla(\mu)^{(1)}$, it follows that if $\mathrm{St}_1 \otimes L(\mu)^{(1)} \cong \mathrm{St}_1 \otimes \nabla(\mu)^{(1)}$ then also $L(\mu)^{(1)} \cong \nabla(\mu)^{(1)}$, and thus $L(\mu) \cong \nabla(\mu)$, which is not the case. □

The existence of the aforementioned family of counterexamples means that if one wants to show that $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_1$ when p is small, then the methods needs to take into account more than just the size of $\langle \lambda, \alpha_0^\vee \rangle$.

6.9 Good filtrations on $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$

The goal of this section is to show that if $\lambda = \lambda_0 + p^r \lambda_1$ with $\lambda_0 \in X_r$ and $\lambda_1 \in X_+$ where λ_1 is "large enough" compared to λ_0 (and p and r), then $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration, even if λ_0 is not small enough compared to p and r to apply Theorem 6.9. We start with some preliminary lemmas. The first lemma involves weights of $\nabla(\lambda)$ and the second provides conditions on when $R^1 \operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda) = 0$.

Lemma 6.15. *Let $\lambda \in X_+$ and assume that μ is a weight of $\nabla(\lambda)$. Then $\langle \mu, \alpha^\vee \rangle \geq -\langle \lambda, \alpha_0^\vee \rangle$ for all $\alpha \in R^+$. In particular, the same inequality holds for any weight of $L(\lambda)$.*

Proof. Since the Weyl group acts transitively on the Weyl chambers, there is some $w \in W$ such that $\langle w(\mu), \alpha^\vee \rangle \leq 0$ for all $\alpha \in R^+$. Then $\langle w(\mu), \alpha^\vee \rangle \geq \langle w(\mu), \alpha_0^\vee \rangle$ for all $\alpha \in R$ for the same reason that the reverse inequality would hold if $w(\mu)$ was dominant.

For any $\alpha \in R^+$, $\langle \mu, \alpha^\vee \rangle = \langle w(\mu), w(\alpha^\vee) \rangle \geq \langle w(\mu), \alpha_0^\vee \rangle$. Since μ is a weight of $\nabla(\lambda)$, so is $w(\mu)$. Hence, $w(\mu) \geq w_0(\lambda)$ and $\langle w(\mu), \alpha_0^\vee \rangle \geq \langle w_0(\lambda), \alpha_0^\vee \rangle$. Furthermore, $\langle w_0(\lambda), \alpha_0^\vee \rangle = -\langle \lambda, \alpha_0^\vee \rangle$ which gives the first claim. If μ is a weight of $L(\lambda)$ then μ is also a weight of $\nabla(\lambda)$, thus the second claim follows. $\square$

Lemma 6.16. *Let $\lambda \in X$. If $\langle \lambda, \alpha^\vee \rangle \geq -1$ for all $\alpha \in S$ then $R^1 \operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda) = 0$.*

Proof. This follows by combining Theorem 4.12 with Proposition 4.13, since if $\lambda \notin X_+$, the conditions mean that there is some $\alpha \in S$ with $\langle \lambda, \alpha^\vee \rangle = -1$. $\square$

The preceding lemmas can be used to determine sufficient conditions for $\nabla(\nu) \otimes L(\lambda)$ to have a good filtration.

Proposition 6.17. *Let $\lambda, \nu \in X_+$ with $\langle \lambda, \alpha_0^\vee \rangle \leq \langle \nu, \alpha^\vee \rangle + 1$ for all $\alpha \in S$. Then $\nabla(\nu) \otimes L(\lambda)$ has a good filtration.*

Proof. First note that by Lemma 6.15, for any weight μ of $L(\lambda)$ and any $\alpha \in S$,

$$\langle \nu + \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle + \langle \mu, \alpha^\vee \rangle \geq \langle \nu, \alpha^\vee \rangle - \langle \lambda, \alpha_0^\vee \rangle \geq -1.$$

Now apply Proposition 3.23 which gives

$$\nabla(\nu) \otimes L(\lambda) = \operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\nu) \otimes L(\lambda) \cong \operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\nu \otimes L(\lambda)).$$

The weights of $L(\lambda)$ gives a filtration of $L(\lambda)$ as a $\mathbf{B}$-module, so we obtain a filtration of $\nu \otimes L(\lambda)$ with factors of the form $\nu + \mu$ where μ is a weight of $L(\lambda)$.

We wish to show that this filtration gives a filtration of $\operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\nu \otimes L(\lambda))$ with terms of the form $\operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\nu + \mu)$. In order to do this, it is sufficient to show that $R^1 \operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\nu + \mu) = 0$ for all weights μ of $L(\lambda)$. But, for any such μ and $\alpha \in S$ one has $\langle \nu + \mu, \alpha^\vee \rangle \geq -1$, so this follows by Lemma 6.16. Therefore, we have demonstrated that $\nabla(\nu) \otimes L(\lambda)$ has a filtration with factors of the form $\operatorname{ind}_{\mathbf{B}}^{\mathbf{G}}(\gamma)$ for suitable γ which finishes the proof. $\square$

As a direct consequence of the above, we get a sufficient condition on λ which guarantees that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration, with no requirement on p. For $p = h$ this condition is better than the one obtained from Theorem 6.9.

Theorem 6.18. *If $\lambda \in X_+$ with $\langle \lambda, \alpha_0^\vee \rangle \leq p^r$ then $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration.*

Proof. This follows directly from Proposition 6.17 since $\langle (p^r-1)\rho, \alpha^\vee\rangle = p^r - 1$ for all $\alpha \in S$. $\square$

We now present sufficient conditions to insure that $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration.

Theorem 6.19. *Let λ be a dominant weight and write $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$. Moreover, assume that $\langle \lambda_0, \alpha_0^\vee\rangle \le p^r(\langle \lambda_1, \alpha^\vee\rangle + 1)$ for all $\alpha \in S$. Then $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration.*

Proof. By Theorem 4.29

$$\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda) = \mathrm{St}_r \otimes \nabla(\lambda_1)^{(r)} \otimes L(\lambda_0) \cong \nabla((p^r-1)\rho + p^r\lambda_1) \otimes L(\lambda_0)$$

so the claim follows from Proposition 6.17 since $\langle (p^r-1)\rho + p^r\lambda_1, \alpha^\vee\rangle = p^r\langle \lambda_1, \alpha^\vee\rangle + p^r - 1$ for any $\alpha \in S$. $\square$

As a special case of above theorem, we see that if $p^r(\langle \lambda_1, \alpha^\vee\rangle + 1) \ge (p^r-1)(h-1)$ for all $\alpha \in S$, then for any $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$, $\mathrm{St}_r \otimes \nabla^{(p,r)}(\lambda)$ has a good filtration.

6.10 Root systems of small rank

For the root systems of type A_2, A_3, B_2 and G_2 we can show that $\mathrm{St}_r \otimes M$ has a good filtration for any $\mathbf{G}$-module M with a good (p,r)-filtration, without any restrictions on p, except for the case $p = 7$ in type G_2.

We do this by proving that $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_1$, since then the statement follows from Proposition 6.12 and Proposition 6.10.

In the following, we will call a weight $\lambda \in X_+$ *simple* if $L(\lambda) = \nabla(\lambda)$.

We start with a result similar to Lemma 6.6 and Proposition 6.7. We will not give a proof here, as the arguments are completely identical to those of the mentioned results.

Proposition 6.20. *Let $\lambda \in X_1$ and assume that $\mathrm{St}_1 \otimes L(\lambda)$ does not have a good filtration.*

Then there are weights $\mu, \sigma \in X_+$ with $\mu \ne \lambda$ such that $\mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma)) \ne 0$, $[\nabla(\lambda) : L(\mu)] \ne 0$ and $p\sigma \le \lambda + \mu$.

In particular, λ is not simple, $p\sigma \le 2\lambda$, $p\langle \sigma, \alpha_0^\vee\rangle \le 2\langle \lambda, \alpha_0^\vee\rangle$, and $p\langle \sigma, \alpha_0^\vee\rangle \le \langle \lambda + \mu, \alpha_0^\vee\rangle$.

In order to apply the above, we start by obtaining a version of Proposition 6.4 (b) when $p \le h-1$ (for $p \ge h$ we can use the proposition itself, and as remarked there, we cannot improve this). We do this by using that if $\mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma)) \ne 0$ then $[\Delta(\sigma) : k] \ne 0$ (Proposition 4.16), and then apply the Jantzen sum formula (Theorem 4.18) to see which σ satisfies this. Note that in type G_2, we instead use the tables in [Hag83] to do this.

Once we have obtained the above, we apply Proposition 6.20 in several steps.

The first step is to use it to reduce the set of weights we need to consider. In some cases, we will instead apply Theorem 6.18 for this.

In some cases, we will also need to apply the Jantzen sum formula to $\Delta(\lambda)$ for some of those weights λ we are left with, (or use the tables in [Hag83]), in order to know which simple modules can occur as composition factors of $\nabla(\lambda)$ (since the composition factors of $\nabla(\lambda)$ are the same as those of $\Delta(\lambda)$). This will also give a further reduction in the weights we need to consider, as we do not need to consider any simple weights.

In the following we will write all weights in the basis consisting of the fundamental weights.

Type A_2

Since we have $2h-2=4$, we need to consider the cases $p=2$ and $p=3$.

$p=2$

In this case we are done as soon as we apply Theorem 6.18 as there are no weights left to consider.

$p=3$

The only weight left to consider after applying Theorem 6.18 is $(2,2)=(p-1)\rho$ which is simple, so we are done.

Type A_3

Since $2h-2=6$, the cases we need to consider are $p=2$, $p=3$ and $p=5$.

$p=2$

The only weight left to consider after applying Theorem 6.18 is $(1,1,1)=(p-1)\rho$, which is simple so we are done.

$p=3$

We see that if $\mathrm{Ext}^1_{\mathbf{G}}(k,L(\sigma))\neq 0$ then $\langle\sigma,\alpha_0^\vee\rangle\geq 2$ since all the fundamental weights are simple. The weight $(0,2,0)$ shows that we cannot do any better, but this is the only such weight where equality holds (as can be checked using the Jantzen sum formula).

By Theorem 6.18, we need to consider the weights $(0,2,2)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(2,0,2)$, $(2,1,1)$, $(2,1,2)$, $(2,2,0)$, $(2,2,1)$ and $(2,2,2)$. But applying Theorem 4.14 we see that the only ones of these we need to consider are $(1,1,2)$, $(1,2,1)$, $(2,0,2)$, $(2,1,1)$ and $(2,1,2)$ as the rest are simple.

By Proposition 6.20 we can then further restrict to those weights λ such that we either have $3(0,2,0)\leq 2\lambda$ or $\langle\lambda,\alpha_0^\vee\rangle\geq 5$. This rules out the weights $(1,1,2)$, $(2,0,2)$ and $(2,1,1)$, so we are left with just $(1,2,1)$ and $(2,1,2)$.

Applying the Jantzen sum formula to $\Delta(1,2,1)$ we see that we have a short exact sequence $0\to L(1,2,1)\to\nabla(1,2,1)\to L(0,2,0)\to 0$, we can use Proposition 6.20 to rule out this weight, as we do not have $3(0,2,0)\leq(1,2,1)+(0,2,0)$ since $(1,2,1)+(0,2,0)-3(0,2,0)=(1,-2,1)=-\alpha_2$.

For the weight $(2,1,2)$ we again apply the Jantzen sum formula and get a short exact sequence $0\to L(2,1,2)\to\nabla(2,1,2)\to L(0,1,0)\to 0$. Like before, we can rule out this weight as we do not have $3(0,2,0)\leq(0,1,0)+(2,1,2)$ since $(0,1,0)+(2,1,2)-3(0,2,0)=(2,-4,2)=-2\alpha_2$.

$p=5$

After applying Theorem 6.9, we are left with the weights $(2,4,4)$, $(3,3,4)$, $(3,4,3)$, $(3,4,4)$, $(4,2,4)$, $(4,3,3)$, $(4,3,4)$, $(4,4,2)$, $(4,4,3)$ and $(4,4,4)$. But applying Theorem 4.14 we reduce this to just the weights $(3,3,4)$, $(3,4,3)$, $(4,2,4)$, $(4,3,3)$ and $(4,3,4)$. And the result for $(3,3,4)$ follows from the result for $(4,3,3)=-w_0(3,3,4)$.

By Proposition 6.4, if $\mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma)) \neq 0$ then $\langle \sigma, \alpha_0^\vee \rangle \geq 4$. Thus, by Proposition 6.20 we see that it will be sufficient, for each weight λ in the above list, to show that if $L(\mu)$ is a composition factor of $\nabla(\lambda)$ with $\mu \neq \lambda$ then $\langle \mu + \lambda, \alpha_0^\vee \rangle < 5 \cdot 4 = 32$, i.e. that $\langle \mu, \alpha_0^\vee \rangle \leq 19 - \langle \lambda, \alpha_0^\vee \rangle$.

Applying the Jantzen sum formula, we get the following (we do not need to compute the characters completely, as we only need bounds on the weights occurring):

For $\lambda = (4, 3, 3)$, all weights μ that occur have $\langle \mu, \alpha_0^\vee \rangle \leq 7 < 19 - \langle \lambda, \alpha_0^\vee \rangle = 9$.

For $\lambda = (4, 3, 4)$, all weights μ that occur have $\langle \mu, \alpha_0^\vee \rangle \leq 3 < 19 - \langle \lambda, \alpha_0^\vee \rangle = 8$.

For $\lambda = (4, 2, 4)$, all weights μ that occur have $\langle \mu, \alpha_0^\vee \rangle \leq 4 < 19 - \langle \lambda, \alpha_0^\vee \rangle = 9$.

For $\lambda = (3, 4, 3)$, all weights μ that occur have $\langle \mu, \alpha_0^\vee \rangle \leq 6 < 19 - \langle \lambda, \alpha_0^\vee \rangle = 9$.

Thus we have dealt with all the weights in this case.

Type B_2

Here $2h - 2 = 6$ so we need to deal with the cases $p = 2$, $p = 3$ and $p = 5$.

$p = 2$

After applying Theorem 6.18, the only weight left is $(1, 1) = (p - 1)\rho$ which is simple, so this case is done.

$p = 3$

In this case we see that if $\mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma)) \neq 0$ then $\langle \sigma, \alpha_0^\vee \rangle \geq 3$ (by using the Jantzen sum formula to check that all the weights $(1, 0)$, $(0, 1)$ and $(0, 2)$ are simple).

Thus by Proposition 6.20 we only need to consider weights λ with $2\langle \lambda, \alpha_0^\vee \rangle \geq 3 \cdot 3 = 9$, i.e. with $\langle \lambda, \alpha_0^\vee \rangle \geq 5$. This means we just need to consider the weights $(2, 1)$ and $(2, 2) = (p-1)\rho$. The latter is simple, as is $(2, 1)$ (by applying the Jantzen sum formula), so this case is done.

$p = 5$

After applying Theorem 6.9 we are left with the weights $(3, 4)$, $(4, 2)$, $(4, 3)$ and $(4, 4) = (p - 1)\rho$. The last of these is simple, and so are $(4, 2)$ and $(4, 3)$ (seen by applying the Jantzen sum formula). This leaves us with just the weight $(3, 4)$.

By Proposition 6.4, if $\mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma)) \neq 0$ then $\langle \sigma, \alpha_0^\vee \rangle \geq 4$.

Applying the Jantzen sum formula to the weight $(3, 4)$, we see that there is a short exact sequence $0 \to L(3, 4) \to \nabla(3, 4) \to L(0, 4) \to 0$, so we are done by Proposition 6.20 since it is not the case that $14 = \langle (3, 4) + (0, 4), \alpha_0^\vee \rangle \geq 5 \cdot 4 = 20$.

Type G_2

Here $2h - 2 = 10$ so we need to deal with the cases $p = 2$, $p = 3$, $p = 5$ and $p = 7$. However, we will not be able to deal with the case $p = 7$.

$p = 2$

Here the weights we need to consider after applying Theorem 6.18 are $(0, 1)$ and $(1, 1) = (p - 1)\rho$. The last of these is simple, and so is $(0, 1)$ (as can be seen from the table on p. 90 of [Hag83]). So we are done in this case.

$p = 3$

From the table on p. 85 of [Hag83] we see that if $\mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma)) \neq 0$ then $\langle \sigma, \alpha_0^\vee \rangle \geq 5$.

By Proposition 6.20 we thus only need to consider weights λ with $2\langle \lambda, \alpha_0^\vee \rangle \geq 3 \cdot 5 = 15$, i.e. with $\langle \lambda, \alpha_0^\vee \rangle \geq 8$. But the only restricted weight satisfying this are $(1, 2)$ and $(2, 2) = (p-1)\rho$. Since the latter of these is simple, we are left with $(1, 2)$.

Looking at the same table again, we see that if $L(\mu)$ is a composition factor of $\nabla(1, 2)$ then $\langle \mu, \alpha_0^\vee \rangle \leq 6$, so by Proposition 6.20we are done, since $\langle (1, 2), \alpha_0^\vee \rangle + 6 = 14 < 3 \cdot 5 = 15$.

$p = 5$

From the table on p. 83 of [Hag83] we see that if $\mathrm{Ext}^1_{\mathbf{G}}(k, L(\sigma)) \neq 0$ then $\langle \sigma, \alpha_0^\vee \rangle \geq 15$.

By Proposition 6.20 we see that we only need to consider weights λ with $2\langle \lambda, \alpha_0^\vee \rangle \geq 5 \cdot 15 = 75$. But there are no restricted weights satisfying this, so we are done.

7 The Steinberg Square

In this section we will study the $\mathbf{G}$-module $\mathrm{St}_r \otimes \mathrm{St}_r$.

By Theorem 4.25 we see that $\mathrm{St}_r \otimes \mathrm{St}_r$ is tilting (since $\mathrm{St}_r^* \cong \mathrm{St}_r$), so by Proposition 4.31 we have

$$\mathrm{St}_r \otimes \mathrm{St}_r \cong \bigoplus_{\nu \in X_+} t_r(\nu) T(\nu)$$

for suitable natural numbers $t_r(\nu)$.

Also write

$$\mathrm{soc}_{\mathbf{G}}(\mathrm{St}_r \otimes \mathrm{St}_r) = \bigoplus_{\nu \in X_+} s_r(\nu) L(\nu)$$

We will be interested in studying the numbers $t_r(\nu)$ and $s_r(\nu)$. In particular, some questions we would like to be able to answer are:

1. Is there a relation between those ν with $t_r(\nu) \neq 0$ and those ν with $s_r(\nu) \neq 0$?
2. For which $\nu \in X_+$ do we have $t_r(\nu) \neq 0$, and for which do we have $s_r(\nu) \neq 0$?
3. Given $t_1(\nu)$ and $s_1(\nu)$ for all $\nu \in X_+$, can we determine $t_r(\nu)$ and $s_r(\nu)$ for all r and all $\nu \in X_+$?
4. Can we determine $t_r(\nu)$ or $s_r(\nu)$ in terms of ν?

To summarize the answers:

1. The answer to this is "yes", at least when $p \geq 2h - 2$ (see Corollary 7.5).
2. We can only give a complete answer to this in some special cases. In general, we have to make do with some necessary conditions (see Proposition 7.7).
3. The answer to this is "yes", in some special cases (see Proposition 7.13).
4. For this question, we can only give an answer in some special cases.

In order to be able to study these, we will need to know a bit more about the socles of certain tilting modules.

An easy proposition, which will simplify a lot of things is the following.

Proposition 7.1. *For all $\nu \in X_+$ we have $s_r(\nu) = s_r(-w_0(\nu))$ and $t_r(\nu) = t_r(-w_0(\nu))$.*

Proof. Since $(\mathrm{St}_r \otimes \mathrm{St}_r)^* \cong \mathrm{St}_r \otimes \mathrm{St}_r$ we get from Proposition 4.32 that

$$s_r(\nu) = \dim(\mathrm{Hom}_{\mathbf{G}}(L(\nu), \mathrm{St}_r \otimes \mathrm{St}_r)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r \otimes \mathrm{St}_r, L(\nu)))$$

$$= \dim(\mathrm{Hom}_{\mathbf{G}}(L(-w_0(\nu)), \mathrm{St}_r \otimes \mathrm{St}_r)) = s_r(-w_0(\nu))$$

which was the first claim.

The second claim also follows from Proposition 4.32 since $\mathrm{St}_r \otimes \mathrm{St}_r$ being self-dual means that any $T(\nu)$ must appear precisely the same number of times as $T(\nu)^* \cong T(-w_0(\nu))$ in the decomposition. $\square$

For convenience, we will from now on use the following notation: For $\lambda = \lambda_0 + p^r\lambda_1 \in X_+$ with $\lambda_0 \in X_r$ we write $w_r(\lambda) = (p^r - 1)\rho + w_0(\lambda_0) + p^r\lambda_1$ (note that w_r is not an element in W but we do have $w_r(w_r(\lambda)) = \lambda$).

We also have the following property of w_r.

Lemma 7.2. *If $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^{r-1}\lambda_{r-1}$ with all $\lambda_i \in X_1$ then*

$$w_r(\lambda) = w_1(\lambda_0) + pw_1(\lambda_1) + \cdots + p^{r-1}w_1(\lambda_{r-1})$$

Proof. Note that $\lambda \in X_r$ so

$$w_r(\lambda) = (p^r - 1)\rho + w_0(\lambda) = (p-1)\rho + p(p^{r-1} - 1)\rho + w_0(\lambda_0) + pw_0(\lambda_1 + p\lambda_2 + \cdots + p^{r-2}\lambda_{r-1})$$
$$= w_1(\lambda_0) + pw_{r-1}(\lambda_1 + p\lambda_2 + \cdots + p^{r-2}\lambda_{r-1})$$

from which the claim easily follows by induction on r. $\square$

Lemma 7.3. *Assume that $p \geq 2h - 2$ and let $\lambda \in X_+$ with $\langle \lambda, \alpha_0^\vee \rangle \leq (p^r - 1)(h-1)$. Then $\operatorname{soc}_{\mathbf{G}}(T((p^r-1)\rho + \lambda)) = L(w_r(\lambda))$.*

Proof. Write $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$. Then by Theorem 4.34 we have

$$T((p^r-1)\rho + \lambda) \cong T((p^r-1)\rho + \lambda_0) \otimes T(\lambda_1)^{(r)}$$

Now we get by Theorem 4.35 that $\operatorname{soc}_{\mathbf{G}_r}(T((p^r-1)\rho + \lambda_0)) = L((p^r-1)\rho + w_0(\lambda_0))$, so for $\mu = \mu_0 + p^r\mu_1 \in X_+$ with $\mu_0 \in X_r$ we get, by Theorem 4.10

$$\operatorname{Hom}_{\mathbf{G}}(L(\mu), T((p^r-1)\rho + \lambda))$$
$$\cong \operatorname{Hom}_{\mathbf{G}/\mathbf{G}_r}(L(\mu_1)^{(r)}, \operatorname{Hom}_{\mathbf{G}_r}(L(\mu_0), T((p^r-1)\rho + \lambda_0)) \otimes T(\lambda_1)^{(r)})$$
$$\cong \begin{cases} \operatorname{Hom}_{\mathbf{G}}(L(\mu_1), T(\lambda_1)) & \text{if } \mu_0 = (p^r-1)\rho + w_0(\lambda_0) \\ 0 & \text{else} \end{cases}$$

so we just need to show that $\operatorname{soc}_{\mathbf{G}}(T(\lambda_1)) = L(\lambda_1)$. In fact, we claim that $T(\lambda_1)$ is simple, which clearly implies the claim.

To see this, we note that we have $\langle p^r\lambda_1, \alpha_0^\vee \rangle \leq (p^r-1)(h-1)$ so

$$\langle \lambda_1 + \rho, \alpha_0^\vee \rangle \leq (h-1) + (h-1) = 2h - 2 \leq p$$

which shows that $\nabla(\lambda_1)$ is simple by Proposition 4.15, and hence also that $L(\lambda_1)$ is tilting, or in other words that $T(\lambda_1)$ is simple (being the unique indecomposable tilting module of highest weight λ_1).

Thus we have shown that $\operatorname{soc}_{\mathbf{G}}(T((p^r-1)\rho + \lambda)) = L((p^r-1)\rho + w_0(\lambda_0) + p^r\lambda_1) = L(w_r(\lambda))$ as was the claim. $\square$

The way to apply the above lemma is given in the following.

Proposition 7.4. *If $t_r(\nu) \neq 0$ for some $\nu \in X_+$ then $\nu = (p^r-1)\rho + \lambda$ for some $\lambda \in X_+$ with $\langle \lambda, \alpha_0^\vee \rangle \leq (p^r-1)(h-1)$.*

Proof. Since $\mathrm{St}_r \otimes \mathrm{St}_r$ is injective as a $\mathbf{G}_r$-module by Theorem 4.30, the same must be true for $T(\nu)$ since this is a direct summand of $\mathrm{St}_r \otimes \mathrm{St}_r$ by assumption. Now it follows by Proposition 4.33 that $\nu = (p^r - 1)\rho + \lambda$ for some $\lambda \in X_+$.

Since ν must be a weight of $\mathrm{St}_r \otimes \mathrm{St}_r$ we must have $\langle \nu, \alpha_0^\vee \rangle \leq 2(p^r - 1)(h-1)$, and hence we get $\langle \lambda, \alpha_0^\vee \rangle = \langle \nu, \alpha_0^\vee \rangle - (p^r - 1)(h-1) \leq (p^r - 1)(h-1)$ as was the claim. □

Combining the above results, we get the following.

Corollary 7.5. *Assume that $p \geq 2h-2$. Then for any $\nu \in X_+$ we have $s_r(\nu) = t_r((p^r-1)\rho + w_r(\nu))$*

Proof. This follows directly from combining Lemma 7.3 and Proposition 7.4 (recall that we have $w_r(w_r(\nu)) = \nu$). □

A useful tool is the following.

Theorem 7.6. *Assume that $p \geq 2h-2$ and let $\nu \in X_+$ with $s_r(\nu) \neq 0$. Then $\mathrm{St}_r \otimes L(\nu)$ has a good filtration.*

Proof. By Corollary 7.5 and Proposition 7.4 we see that $\nu = w_r(\mu)$ for some dominant weight μ with $\langle \mu, \alpha_0^\vee \rangle \leq (p^r-1)(h-1)$. So if we write $\mu = \mu_0 + p^r \mu_1$ with $\mu_0 \in X_r$ we have $\nu = \nu_0 + p^r \nu_1$ with $\nu_0 = (p^r - 1)\rho + w_0(\mu_0) \in X_r$ and $\nu_1 = \mu_1$. In particular, we have $\langle \nu_1, \alpha_0^\vee \rangle \leq \frac{(p^r-1)(h-1)}{p^r} \leq h-1$ so $\langle \nu_1 + \rho, \alpha_0^\vee \rangle \leq (h-1) + (h-1) \leq p$ and hence $L(\nu_1) \cong \nabla(\nu_1)$ by Proposition 4.15.

By Theorem 4.10 we now have

$$L(\nu) \cong L(\nu_0) \otimes L(\nu_1)^{(r)} \cong L(\nu_0) \otimes \nabla(\nu_1)^{(r)}$$

so the claim follows by combining Theorem 6.11 and Theorem 6.10. □

In order to determine necessary conditions on $\nu \in X_+$ to have $s_r(\nu) \neq 0$ we can make use of.

Proposition 7.7. *Let $\nu = \nu_0 + p^r \nu_1 \in X_+$ with $\nu_0 \in X_r$. We have*

1. $s_r(\nu) \leq [\mathrm{St}_r \otimes \nabla(-w_0(\nu)) : \mathrm{St}_r]_\nabla$.
2. $s_r(\nu) \leq [\mathrm{St}_r \otimes \nabla(\nu) : \mathrm{St}_r]_\nabla$.
3. $s_r(\nu) \leq [\nabla((p^r - 1)\rho + p^r \nu_1) \otimes \nabla(\nu_0) : \mathrm{St}_r]_\nabla$.
4. $s_r(\nu) \leq [\mathrm{St}_r \otimes \nabla(\nu_0) : \nabla((p^r - 1)\rho - p^r w_0(\nu_1))]_\nabla$.

Proof. We have

$$s_r(\nu) = \dim(\mathrm{Hom}_{\mathbf{G}}(L(\nu), \mathrm{St}_r \otimes \mathrm{St}_r)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes L(-w_0(\nu))))$$

and the inclusion $L(-w_0(\nu)) \hookrightarrow \nabla(-w_0(\nu))$ gives the inequality

$$\dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes L(-w_0(\nu)))) \leq \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes \nabla(-w_0(\nu))))$$

and by Theorem 4.24 we have $\dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes \nabla(-w_0(\nu)))) = [\mathrm{St}_r \otimes \nabla(-w_0(\nu)) : \mathrm{St}_r]_\nabla$ which proves the first claim (since $\mathrm{St}_r \otimes \nabla(-w_0(\nu))$ has a good filtration by Theorem 4.25).

The second claim now follows since $s_r(\nu) = s_r(-w_0(\nu))$ by Proposition 7.1.

For the third claim we use that by Theorem 4.10 we have $L(\nu) \cong L(\nu_0) \otimes L(\nu_1)^{(r)}$, and the inclusions $L(\nu_0) \hookrightarrow \nabla(\nu_0)$ and $L(\nu_1)^{(r)} \hookrightarrow \nabla(\nu_1)^{(r)}$ gives like above an inequality (again using Proposition 7.1)

$$s_r(\nu) = s_r(-w_0(\nu)) \leq \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes \nabla(\nu_1)^{(r)} \otimes \nabla(\nu_0)))$$

and applying Theorem 4.29 we get

$$s_r(\nu) \leq \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \nabla((p^r-1)\rho + p^r\nu_1) \otimes \nabla(\nu_0))) = [\nabla((p^r-1)\rho + p^r\nu_1) \otimes \nabla(\nu_0) : \mathrm{St}_r]_{\nabla}$$

by Theorem 4.24 as claimed.

The final claim follows similarly by once again applying Theorem 4.29 to get

$$\begin{aligned}\dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes \nabla(\nu_0)\nabla(\nu_1)^{(r)})) &= \dim(\mathrm{Hom}_{\mathbf{G}}(\Delta((p^r-1)\rho - p^r w_0(\nu_1)), \mathrm{St}_r \otimes \nabla(\nu_0))) \\ &= [\mathrm{St}_r \otimes \nabla(\nu_0) : \nabla((p^r-1)\rho - p^r w_0(\nu_1))]_{\nabla}\end{aligned}$$

by Theorem 4.24. □

Theorem 7.8. *Assume that $p \geq 2h-2$ and let $\nu = \nu_0 + p^r\nu_1 \in X_+$ with $\nu_0 \in X_r$ and $s_r(\nu) \neq 0$. Then we have*

1. $s_r(\nu) = [\mathrm{St}_r \otimes L(-w_0(\nu)) : \mathrm{St}_r]_{\nabla}$.
2. $s_r(\nu) = [\mathrm{St}_r \otimes L(\nu) : \mathrm{St}_r]_{\nabla}$.
3. $s_r(\nu) = [\nabla((p^r-1)\rho + p^r\nu_1) \otimes L(\nu_0) : \mathrm{St}_r]_{\nabla}$.
4. $s_r(\nu) = [\mathrm{St}_r \otimes L(\nu_0) : \nabla((p^r-1)\rho - p^r w_0(\nu_1))]_{\nabla}$.

Proof. If $p \geq 2h-2$ and $s_r(\nu) \neq 0$ then also $s_r(-w_0(\nu)) \neq 0$ by Proposition 7.1, so $\mathrm{St}_r \otimes L(-w_0(\nu))$ has a good filtration by Theorem 7.6, and we have

$$[\mathrm{St}_r \otimes L(-w_0(\nu)) : \mathrm{St}_r]_{\nabla} = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes L(-w_0(\nu))))$$

by Theorem 4.24, so the first claim follows.

The second claim follows since $s_r(\nu) = s_r(-w_0(\nu))$ by Proposition 7.1.

For the third and fourth claims, we use that by Theorem 4.10 we have $L(\nu) \cong L(\nu_0) \otimes L(\nu_1)^{(r)}$, and from Proposition 7.4 together with Corollary 7.5 we see that $\langle p^r\nu_1, \alpha_0^\vee \rangle \leq (p^r-1)(h-1)$ (since $w_r(p^r\nu_1) = p^r\nu_1$). This gives $\langle \nu_1 + \rho, \alpha_0^\vee \rangle \leq (h-1)+(h-1) = 2h-2 \leq p$ and since $-w_0(\nu_1)$ satisfies the same inequality, we get $L(-w_0(\nu_1)) \cong \nabla(-w_0(\nu_1))$ by Proposition 4.15. Applying Theorem 4.29 we thus have $\mathrm{St}_r \otimes L(-w_0(\nu_1))^{(r)} \cong \Delta((p^r-1)\rho - p^r w_0(\nu_1))$ and $\mathrm{St}_r \otimes L(\nu_1)^{(r)} \cong \nabla((p^r-1) + p^r\nu_1)$. We can therefore use the previous part and rewrite

$$\begin{aligned} s_r(\nu) &= [\mathrm{St}_r \otimes L(\nu) : \mathrm{St}_r]_{\nabla} \\ &= \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes L(\nu))) \\ &= \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes L(\nu_1)^{(r)} \otimes L(\nu_0))) \\ &= \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \nabla((p^r-1)\rho + p^r\nu_1) \otimes L(\nu_0))) \\ &= [\nabla((p^r-1)\rho + p^r\nu_1) \otimes L(\ nu_0) : \mathrm{St}_r]_{\nabla} \end{aligned}$$

and

$$\begin{aligned}
s_r(\nu) &= [\mathrm{St}_r \otimes L(\nu) : \mathrm{St}_r]_\nabla \\
&= \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes L(\nu))) \\
&= \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r \otimes L(-w_0(\nu_1))^{(r)}, \mathrm{St}_r \otimes L(\nu_0))) \\
&= \dim(\mathrm{Hom}_{\mathbf{G}}(\Delta((p^r-1)\rho - p^r w_0(\nu_1)), \mathrm{St}_r \otimes L(\nu_0))) \\
&= [\mathrm{St}_r \otimes L(nu_0) : \nabla((p^r-1)\rho - p^r w_0(\nu_1))]_\nabla
\end{aligned}$$

□

As a corollary we get.

Corollary 7.9. *Let $\nu = \nu_0 + p^r \nu_1 \in X_+$ with $\nu_0 \in X_r$. Assume that $s_r(\nu) \neq 0$. Then*

1. $\nu \geq 0$.
2. $\nu_0 + p^r w_0(\nu_1) \geq 0$.
3. $\langle \nu_1, \alpha_0^\vee \rangle \leq \frac{\langle \nu_0, \alpha_0^\vee \rangle}{p^r}$.
4. $\langle \nu_1, \alpha_0^\vee \rangle \leq \frac{(p^r-1)(h-1)}{p^r}$.

Proof. If $s_r(\nu) \neq 0$ then by Proposition 7.7 we have $[\mathrm{St}_r \otimes \nabla(\nu) : \mathrm{St}_r]_\nabla \neq 0$ and thus $(p^r-1)\rho$ must be a weight of $\mathrm{St}_r \otimes \nabla(\nu)$, which means that $(p^r-1)\rho \leq (p^r-1)\rho + \nu$, so $\nu \geq 0$, as was the first claim.

Proposition 7.7 also shows that we have $[\mathrm{St}_r \otimes \nabla(\nu_0) : \nabla((p^r-1)\rho - p^r w_0(\nu_1))]_\nabla \neq 0$. Thus, $(p^r-1)\rho - p^r w_0(\nu_1)$ is a weight of $\mathrm{St}_r \otimes L(\nu_0)$, so we have $(p^r-1)\rho - p^r w_0(\nu_1) \leq (p^r-1)\rho + \nu_0$ which gives the second claim.

For the third claim, we have the inequality $\langle \nu_0 + p^r w_0(\nu_1), \alpha_0^\vee \rangle \geq 0$ from the previous claim.

We can rearrange this inequality by using that

$$\langle \nu_0 + p^r w_0(\nu_1), \alpha_0^\vee \rangle = \langle \nu_0, \alpha_0^\vee \rangle + p^r \langle w_0(\nu_1), \alpha_0^\vee \rangle = \langle \nu_0, \alpha_0^\vee \rangle - p^r \langle \nu_1, \alpha_0^\vee \rangle$$

which turns it into

$$\langle \nu_1, \alpha_0^\vee \rangle \leq \frac{\langle \nu_0, \alpha_0^\vee \rangle}{p^r}$$

which was the claim.

The final claim follows by noting that $\langle \nu_0, \alpha_0^\vee \rangle \leq (p^r-1)(h-1)$ since $\nu_0 \in X_r$.

□

A further corollary is the following, which can be applied inductively given some character data.

Corollary 7.10. *Assume that $p \geq 2h-2$ and let $\nu \in X_+$ with $s_r(\nu) \neq 0$. Write*

$$\operatorname{ch} \nabla(\nu) = \operatorname{ch} L(\nu) + \sum_{\mu \in X_+,\, \mu \neq \nu} a_\mu \operatorname{ch} L(\mu)$$

and assume that $\mathrm{St}_r \otimes L(\mu)$ has a good filtration whenever $a_\mu \neq 0$.

Then

$$s_r(\nu) = [\mathrm{St}_r \otimes \nabla(\nu) : \mathrm{St}_r]_\nabla - \sum_{\mu \in X_+} a_\mu s_r(\mu)$$

Proof. By Theorem 7.8 we see that $s_r(\nu) = [\mathrm{St}_r \otimes L(\nu) : \mathrm{St}_r]_\nabla$, and this is completely determined by the character of $\mathrm{St}_r \otimes L(\nu)$. Thus we have

$$s_r(\nu) = [\mathrm{St}_r \otimes \nabla(\nu) : \mathrm{St}_r]_\nabla - \sum_{\mu \in X_+} a_\mu [\mathrm{St}_r \otimes L(\mu) : \mathrm{St}_r]_\nabla$$

Now apply Theorem 7.8 again to see that whenever $a_\mu \neq 0$ we have $s_r(\mu) = [\mathrm{St}_r \otimes L(\mu) : \mathrm{St}_r]_\nabla$, which finishes the proof (the requirement in that proposition that $s_r(\mu) \neq 0$ is only needed to ensure that $\mathrm{St}_r \otimes L(\mu)$ has a good filtration, which we have assumed to hold here). $\square$

A similar result is the following, which has slightly different assumptions.

Corollary 7.11. *Assume that $p \geq 2h - 2$ and let $\nu = \nu_0 + p^r \nu_1 \in X_+$ with $\nu_0 \in X_r$ and $s_r(\nu) \neq 0$. Write*

$$\mathrm{ch}\,\nabla(\nu_0) = \mathrm{ch}\,L(\nu_0) + \sum_{\mu \in X_+,\, \mu \neq \nu} a_\mu\, \mathrm{ch}\,L(\mu)$$

and assume that $a_\mu \neq 0 \implies \mu \in X_r$. Then

$$s_r(\nu) = [\mathrm{St}_r \otimes \nabla(\nu_0) : \nabla((p^r - 1)\rho - p^r w_0(\nu_1))]_\nabla - \sum_{\mu \in X_+} a_\mu s_r(\mu + p^r \nu_1)$$

Proof. By Theorem 7.8 we have $s_r(\nu) = [\mathrm{St}_r \otimes L(\nu_0) : \nabla((p^r - 1)\rho - p^r w_0(\nu_1))]_\nabla$ which only depends on the character of $\mathrm{St}_r \otimes L(\nu_0)$. Note that by Theorem 6.11 the assumptions imply that $\mathrm{St}_r \otimes L(\mu)$ has a good filtration whenever $a_\mu \neq 0$, so we get

$$s_r(\nu) = [\mathrm{St}_r \otimes L(\nu_0) : \nabla((p^r - 1)\rho - p^r w_0(\nu_1))]_\nabla$$

$$= [\mathrm{St}_r \otimes \nabla(\nu_0) : \nabla((p^r - 1)\rho - p^r w_0(\nu_1))]_\nabla - \sum_{\mu \in X_+} a_\mu [\mathrm{St}_r \otimes L(\mu) : \nabla((p^r - 1)\rho - p^r w_0(\nu_1))]_\nabla$$

and by Theorem 7.8 we have $[\mathrm{St}_r \otimes L(\mu) : \nabla((p^r - 1)\rho - p^r w_0(\nu_1))]_\nabla = s_r(\mu + p^r \nu_1)$ since $\mu \in X_r$, which completes the proof. $\square$

Note that if $\nu \in X_r$ we cannot be sure that we can apply Corollary 7.11 since not all the $L(\mu)$ that occur need have $\mu \in X_r$. However, if $\nu \in X_r$ we can always apply Corollary 7.10 (by Theorem 6.9).

As a special case of this we get.

Corollary 7.12. *Assume that $p \geq 2h - 2$ and let $\nu \in X_+$ with $s_r(\nu) \neq 0$. Assume further that $\mathrm{St}_r \otimes L(\mu)$ as a good filtration for all μ such that $L(\mu)$ is a composition factor of $\nabla(\nu)$ and that for all such μ we have $s_r(\mu) = 0$. Then $s_r(\nu) = [\mathrm{St}_r \otimes \nabla(\nu) : \mathrm{St}_r]_\nabla$.*

Proof. This follows directly from Corollary 7.10 $\square$

In order to determine $s_r(\nu)$ and $t_r(\nu)$ from $s_1(\nu)$ and $t_1(\nu)$ we will use that by Theorem 4.29 we have $\mathrm{St}_r \otimes \mathrm{St}_r \cong (\mathrm{St}_1 \otimes \mathrm{St}_1) \otimes (\mathrm{St}_{r-1} \otimes \mathrm{St}_{r-1})^{(1)}$. But to use this properly, we will need a stronger assumption that just $p \geq 2h - 2$, namely that $s_1(\nu)$ is only non-zero for $\nu \in X_1$. This turns out to hold when $\mathbf{G}$ is SL_2 or when $\mathbf{G}$ is SL_3 and $p = 2$ (see Proposition 7.18 and Proposition 7.31), but it seems like this might not hold in any further generality (see Example 7.14 and Theorem 7.29).

Note that the following theorem also gives us the $t_r(\nu)$ by Corollary 7.5.

Proposition 7.13. *Assume that $p \geq 2h-2$ and that $s_1(\nu) = 0$ for all $\nu \notin X_1$.*

If $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^{r-1}\lambda_{r-1}$ with all $\lambda_i \in X_1$, then

$$s_r(\lambda) = \prod_{i=0}^{r-1} s_1(\lambda_i)$$

In particular, $s_r(\lambda)$ is non-zero precisely when λ has a p-adic expansion in terms of weights occurring as the highest weights in the socle of $\mathrm{St}_1 \otimes \mathrm{St}_1$.

Proof. By Theorem 4.29 we have $\mathrm{St}_r \otimes \mathrm{St}_r \cong (\mathrm{St}_1 \otimes \mathrm{St}_1) \otimes (\mathrm{St}_{r-1} \otimes \mathrm{St}_{r-1})^{(1)}$, so we will proceed by induction on r, the base case of $r = 1$ being clear.

Now we have

$$(\mathrm{St}_1 \otimes \mathrm{St}_1) \otimes (\mathrm{St}_{r-1} \otimes \mathrm{St}_{r-1})^{(1)} \cong \bigoplus_{\nu \in X_+} t_1(\nu) T(\nu) \otimes \left(\bigoplus_{\mu \in X_+} t_{r-1}(\mu) T(\mu) \right)^{(1)}$$

The assumptions, together with Lemma 7.3 imply that the only ν with $t_1(\nu) \neq 0$ are those with $\nu = (p-1)\rho + \nu_0$ where $\nu_0 \in X_1$. Similarly, by induction, the only μ with $t_{r-1}(\mu) \neq 0$ are those of the form $(p^{r-1}-1)\rho + \mu_0$ with $\mu_0 \in X_{r-1}$.

Now we can apply Theorem 4.34 to get

$$\mathrm{St}_r \otimes \mathrm{St}_r \cong \bigoplus_{\nu, \mu \in X_+} t_1(\nu) t_{r-1}(\mu) T(\nu + p\mu)$$

$$\cong \bigoplus_{\nu, \mu \in X_+} t_1(\nu) t_{r-1}(\mu) T((p-1)\rho + \nu_0 + p((p^{r-1}-1)\rho + \mu_0))$$

and note that $(p-1)\rho + p(p^{r-1}-1)\rho = (p^r - 1)\rho$ which means that we get

$$\mathrm{St}_r \otimes \mathrm{St}_r \cong \bigoplus_{\nu, \mu \in X_+} t_1(\nu) t_{r-1}(\mu) T((p^r-1)\rho + (\nu_0 + p\mu_0))$$

Now we get that

$$t_r((p^r-1)\rho + (\nu_0 + p\mu_0)) = t_1((p-1) + \nu_0) t_{r-1}((p^{r-1}-1)\rho + \mu_0)$$

which by Corollary 7.5 gives

$$s_r(w_r(\nu_0 + p\mu_0)) = s_1(w_1(\nu_0)) s_{r-1}(w_{r-1}(\mu_0))$$

Write $\mu_0 = \sigma_0 + p\sigma_1 + \cdots + p^{r-2}\sigma_{r-2}$ with all $\sigma_i \in X_1$, so by Lemma 7.2 we have

$$w_{r-1}(\mu_0) = w_1(\sigma_0) + p w_1(\sigma_1) + \cdots + p^{r-2} w_1(\sigma_{r-2})$$

and by induction we have

$$s_{r-1}(w_r(\mu_0)) = \prod_{i=0}^{r-2} s_1(w_1(\sigma_i))$$

Now the result follows by writing

$$\lambda = w_r(w_r(\lambda)) = w_r(w_1(\lambda_0) + p w_1(\lambda_1) + \cdots + p^{r-1} w_1(\lambda_{r-1}))$$

by Lemma 7.2. $\square$

As mentioned, the assumption that $s_1(\nu)$ is only non-zero for $\nu \in X_1$ does not seem to hold in much generality, as demonstrated by the following example.

Example 7.14. Let $\mathbf{G} = SL_4$ and $p = 7$ (then $p \geq 2h - 2 = 6$). Let $\nu = (6, 1, 1) + 7(0, 0, 1)$ (we adopt the notation from 4.6). We claim that then $s_1(\nu) \neq 0$ even though $\nu \notin X_1$.

First we note that by Theorem 4.14 we have $L(6, 1, 1) \cong \nabla(6, 1, 1) \cong \Delta(6, 1, 1)$ and clearly $L(0, 0, 1) \cong \nabla(0, 0, 1) \cong \Delta(0, 0, 1)$, so we have

$$\dim(\mathrm{Hom}_{\mathbf{G}}(L(\nu), \mathrm{St}_1 \otimes \mathrm{St}_1)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_1, \nabla(6\rho + 7(1, 0, 0)) \otimes \nabla(1, 1, 6)))$$

$$= [\nabla(6\rho + 7(1, 0, 0)) \otimes \nabla(1, 1, 6) : \mathrm{St}_1]_\nabla$$

(by applying Theorem 4.10, Theorem 4.29 and Theorem 4.24).

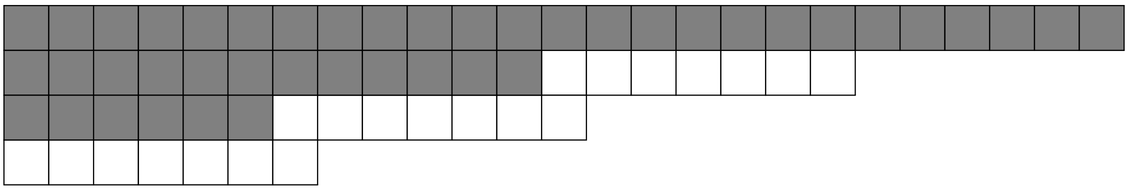
Let $\lambda = 6\rho + 7(1, 0, 0)$, $\mu = (1, 1, 6)$ and $\sigma = 6\rho$. Let

$$m = \frac{|\tilde{\lambda}| + |\tilde{\mu}| - |\tilde{\sigma}|}{4} = \frac{(6 \cdot 6 + 7) + 3 \cdot 7 - 6 \cdot 6}{4} = \frac{4 \cdot 7}{4} = 7$$

so by Proposition 4.36 we have

$$[\nabla(6\rho + 7(1, 0, 0)) \otimes \nabla(1, 1, 6) : \mathrm{St}_1]_\nabla = c^{\tilde{\sigma}+7\tilde{\omega}}_{\tilde{\lambda},\tilde{\mu}}$$

where $\tilde{\omega} = (1, 1, 1, 1)$. So we wish to show that there exists an SSYT of shape $(\tilde{\sigma} + 7\tilde{\omega})/\tilde{\lambda}$ and with type $\tilde{\mu}$. So the shape will be the white part of



and the claim is that is it possible to insert 8 1's, 7 2's and 6 3's in that shape and get an SSYT whose reverse reading word is a lattice permutation. One possible such way is

												1	1	1	1	1	1	1
						2	2	2	2	2	2	2						
1	3	3	3	3	3	3												

where the reverse reading word is $(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 1)$ which is indeed a lattice permutation.

Note that the above example actually works for arbitrary primes if one replaces the weight ν by $(p - 1, 1, 1) + p(0, 0, 1)$. We chose to only put the details for the specific case as it made for nicer illustrations.

Even though we cannot be sure that $s_r(\nu)$ is only non-zero for $\nu \in X_r$, we at least have the following, which can sometimes be helpful.

Proposition 7.15. *If $\nu = p^r\lambda$ for some $\lambda \in X_+$ with $\lambda \neq 0$ then $s_r(\nu) = 0$.*

Proof. By Theorem 4.10 we have $L(\nu) \cong L(\lambda)^{(r)}$ so we get

$$s_r(\nu) = \dim(\mathrm{Hom}_{\mathbf{G}}(L(\nu), \mathrm{St}_r \otimes \mathrm{St}_r)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, \mathrm{St}_r \otimes L(-w_0(\lambda))^{(r)}))$$
$$= \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_r, L((p^r-1)\rho - p^r w_0(\lambda)))) = 0$$

since we assumed that $\lambda \neq 0$. $\square$

Removing the requirement that $s_1(\nu) \neq 0 \implies \nu \in X_1$ is not very straightforward. As can be seen from the proof of Proposition 7.13 we will also need to consider some terms of the form $T(\sigma) \otimes T((p^{r-1}-1)\rho + \lambda)$ (where we can not even be sure that $\lambda \in X_{r-1}$). We do know that σ will be small enough that $T(\sigma) \cong \nabla(\sigma)$, so there is some hope that it can be done, but it will not be easy.

However, we can get the following weaker statement.

Proposition 7.16. *Assume that $p \geq 2h-2$ and let $\lambda \in X_r$ with $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^{r-1}\lambda_{r-1}$ where all $\lambda_i \in X_1$.*

Then

$$s_r(\lambda) \geq \prod_{i=0}^{r-1} s_1(\lambda_i)$$

Proof. By Theorem 4.29 we have $\mathrm{St}_r \otimes \mathrm{St}_r \cong (\mathrm{St}_1 \otimes \mathrm{St}_1) \otimes (\mathrm{St}_{r-1} \otimes \mathrm{St}_{r-1})^{(1)}$, so we will proceed by induction on r, the base case of $r = 1$ being clear.

Now we have

$$(\mathrm{St}_1 \otimes \mathrm{St}_1) \otimes (\mathrm{St}_{r-1} \otimes \mathrm{St}_{r-1})^{(1)} \cong \bigoplus_{\nu \in X_+} t_1(\nu) T(\nu) \otimes \left(\bigoplus_{\mu \in X_+} t_{r-1}(\mu) T(\mu) \right)^{(1)}$$

By Proposition 7.4 the only ν that occur are those of the form $(p-1)\rho + \nu_0$ and those μ that occur have the form $(p^{r-1}-1)\rho + \mu_0$, so we split up the first sum further to get

$$\bigoplus_{\nu_0 \in X_1} t_1((p-1)\rho + \nu_0) T((p-1)\rho + \nu_0) \otimes \left(\bigoplus_{\mu_0 \in X_+} t_{r-1}((p^{r-1}-1)\rho + \mu_0) T((p^{r-1}-1)\rho + \mu_0) \right)^{(1)}$$

$$\oplus \bigoplus_{\nu_0 \notin X_1} t_1((p-1)\rho + \nu_0) T((p-1)\rho + \nu_0) \otimes \left(\bigoplus_{\mu_0 \in X_+} t_{r-1}((p^{r-1}-1)\rho + \mu_0) T((p^{r-1}-1)\rho + \mu_0) \right)^{(1)}$$

By Theorem 4.34 the first summand is isomorphic to

$$\bigoplus_{\nu_0 \in X_1, \mu \in X_+} t_1((p-1)\rho + \nu_0) t_{r-1}((p^{r-1}-1)\rho + \mu_0) T((p-1)\rho + \nu_0 + p((p^{r-1}-1)\rho + \mu_0))$$
$$\cong \bigoplus_{\nu_0 \in X_1, \mu \in X_+} t_1((p-1)\rho + \nu_0) t_{r-1}((p^{r-1}-1)\rho + \mu_0) T((p^r-1)\rho + \nu_0 + p\mu_0)$$

In particular, for $\lambda \in X_r$ we have by Proposition 7.5 that $s_r(\lambda)$ is the number of times the module $T((p^r-1)\rho + w_r(\lambda))$ occurs in the decomposition of $\mathrm{St}_r \otimes \mathrm{St}_r$, so if we write $\lambda = \lambda_0 + p\lambda'$ then by Proposition 7.2 we have $w_r(\lambda) = w_1(\lambda_0) + pw_{r-1}(\lambda')$ so this number of times will be at least $t_1((p-1)\rho + w_1(\lambda_0)) t_{r-1}((p^{r-1}-1)\rho + w_{r-1}(\lambda'))$ which by Proposition 7.5 equals $s_1(\lambda_0)s_{r-1}(\lambda')$, and now the claim follows by induction. $\square$

We also get as a corollary.

Corollary 7.17. *Assume that* $p \geq 2h-2$ *and let* $\nu = \nu_0 + p\nu_1 + \cdots + p^{r-1}\nu_{r-1}$ *with all* $\nu_i \in X_1$*. If* $s_1(\nu_i) \neq 0$ *for all* i *then* $s_r(\nu) \neq 0$*.*

Proof. This follows directly from Proposition 7.16. □

7.1 SL_2

In this section we will let $\mathbf{G} = SL_2$. In this case, we can completely determine $s_r(\nu)$ and $t_r(\nu)$ for all $\nu \in X_+$.

First, we note the following.

Proposition 7.18. *If* $\mathbf{G} = SL_2$ *and* $s_r(\nu) \neq 0$ *for some* $\nu \in X_+$*, then* $\nu \in X_r$*.*

Proof. Since we have $h = 2$ this follows directly from Corollary 7.9, since we have

$$\frac{(p^r - 1)(h-1)}{p^r} = \frac{p^r - 1}{p^r} < 1$$

□

Note that the above means that by Proposition 7.13 and Corollary 7.5 it is in fact enough to find $s_1(\nu)$ for all $\nu \in X_1$ in order to find all $s_r(\nu)$ and $t_r(\nu)$ (since we automatically have $p \geq 2h-2$).

Note also that X_1 is identified with the set of integers ν with $0 \leq \nu \leq p-1$.

Something that makes everything much easier in the case $\mathbf{G} = SL_2$ is the following.

Proposition 7.19. *If* $\mathbf{G} = SL_2$ *and* $\nu \in X_1$ *then* $L(\nu) \cong \nabla(\nu)$*.*

Proof. We have $\langle \nu + \rho, \alpha_0^\vee \rangle = \nu + 1 \leq p$ so the statement follows directly from Proposition 4.15. □

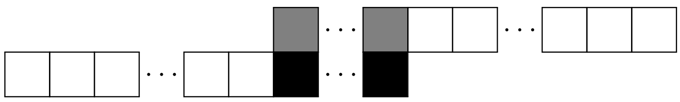
To simplify the calculations, we will use the following lemma.

Lemma 7.20. *Let* $\tilde{\lambda}$*,* $\tilde{\mu}$ *and* $\tilde{\nu}$ *be partitions with* $l(\tilde{\lambda}) = l(\tilde{\mu}) = 1$ *and* $l(\tilde{\nu}) \leq 2$*. Then*

$$c^{\tilde{\nu}}_{\tilde{\lambda},\tilde{\mu}} = \begin{cases} 1 & \text{if } \tilde{\nu}_2 \leq \tilde{\lambda}_1 \leq \tilde{\nu}_1 \text{ and } |\tilde{\lambda}| + |\tilde{\mu}| = |\tilde{\nu}| \\ 0 & \text{else} \end{cases}$$

Proof. If either $\tilde{\lambda}_1 \not\leq \tilde{\nu}_1$ or $|\tilde{\lambda}| + |\tilde{\mu}| \neq |\tilde{\nu}|$ then clearly $c^{\tilde{\nu}}_{\tilde{\lambda},\tilde{\mu}} = 0$.

If $\tilde{\nu}_2 > \tilde{\lambda}_1$ then the shape of $\tilde{\nu}/\tilde{\lambda}$ will look like



and since the gray boxes will need to contain 1's, the black boxes would need to contain numbers strictly grater than 1, but since $l(\tilde{\mu}) = 1$ we are only allowed to place 1's, so there are no SSYTs of this shape and type $\tilde{\mu}$.

If $\tilde{\nu}_2 \leq \tilde{\lambda}_1 \leq \tilde{\nu}_1$ and $|\tilde{\lambda}| + |\tilde{\mu}| = |\tilde{\nu}|$ then we see that we can indeed construct an SSYT of shape $\tilde{\nu}/\tilde{\lambda}$ and type $\tilde{\mu}$ since there will be no overlap between the two rows, so putting 1's in all boxes give a valid SSYT. Since the reverse reading word will consist of only 1's, it will trivially be a lattice permutation, so we do indeed have $c^{\tilde{\nu}}_{\tilde{\lambda},\tilde{\mu}} = 1$ in this case (there can never be more than one way to make an SSYT of a given shape when the type has length 1). □

Theorem 7.21. *If $\mathbf{G} = SL_2$ and $0 \le \nu \le p-1$ then*

$$s_1(\nu) = \begin{cases} 1 & \text{if } \nu \text{ is even} \\ 0 & \text{if } \nu \text{ is odd} \end{cases}$$

Proof. By Theorem 7.8 we have $s_1(\nu) = [\mathrm{St}_1 \otimes L(\nu) : \mathrm{St}_1]_\nabla$ which by Proposition 7.19 equals $[\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla$.

Let

$$m = \frac{(p-1) + \nu - (p-1)}{2} = \frac{\nu}{2}$$

so by Proposition 4.36 we have $[\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla = 0$ unless m is an integer, i.e. unless ν is even.

If ν is even then Proposition 4.36 says that $[\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla = c^{(p-1)\tilde{\rho}+m\tilde{\omega}}_{(p-1)\tilde{\rho},\tilde{\nu}}$ where $\tilde{\omega} = (1,1)$. We can thus apply Lemma 7.20 which says that what we need to show is that $(p-1) \le (p-1)+m$, that $m \le (p-1)$ and that $(p-1)+\nu = (p-1)+2m$, and all these are clear since $0 \le \nu \le p-1$ so also $0 \le m \le p-1$ (and the last one is by definition of m). $\square$

We can now describe $s_r(\nu)$ completely. The following is a special case of [DH05, Theorem 2.1].

Corollary 7.22. *If $\mathbf{G} = SL_2$ and $\nu \in X_r$ then*

$$s_r(\nu) = \begin{cases} 1 & \text{if } \nu \text{ has a } p\text{-adic expansion where all terms are even} \\ 0 & \text{else} \end{cases}$$

Proof. This follows directly by combining Theorem 7.21 with Proposition 7.13. $\square$

A special case of the above worth noting is the following.

Corollary 7.23. *If $\mathbf{G} = SL_2$ and $p = 2$ then $\mathrm{St}_r \otimes \mathrm{St}_r$ is indecomposable and isomorphic to $T(2(p^r-1)\rho) = T(2^{r+1}-2)$.*

Proof. By Corollary 7.22 we have $\mathrm{soc}_{\mathbf{G}}(\mathrm{St}_r \otimes \mathrm{St}_r) = L(0)$, so the module is indecomposable and tilting. Since the highest weight is $(p^r-1)\rho + (p^r-1)\rho = 2(p^r-1)\rho = 2^{r+1}-2$ the statement now follows. $\square$

7.2 SL_3

In this section, we consider the case $\mathbf{G} = SL_3$. In this case we can determine $s_1(\nu)$ and $t_1(\nu)$ for all $\nu \in X_+$, and for $p = 2$ we will determine $s_r(\nu)$ and $t_r(\nu)$ for all $\nu \in X_+$ and all r.

We will adopt the notation from 4.6, so all dominant weights will be written in terms of the fundamental weights.

Unfortunately, we will be able to have $s_1(\nu) \neq 0$ with $\nu \notin X_1$, but we do have the following to limit the possibilities.

Proposition 7.24. *Let $\mathbf{G} = SL_3$. If $\nu = \nu_0 + p^r\nu_1 \in X_+$ with $\nu_0 \in X_r$ and $s_r(\nu) \neq 0$ then $\langle \nu_1, \alpha_0^\vee \rangle \le 1$ and if $\nu_1 \neq 0$ then $\langle \nu_0, \alpha_0^\vee \rangle \ge p^r$.*

Proof. Since $h = 3$, we can apply Corollary 7.9 to see that

$$\langle \nu_1, \alpha_0^\vee \rangle \leq \frac{(p^r - 1)(h-1)}{p^r} = \frac{2(p^r-1)}{p^r} < 2$$

which gives the first claim.

The second claim follows similarly since Corollary 7.9 gives

$$1 \leq \langle \nu_1, \alpha_0^\vee \rangle \leq \frac{\langle \nu_0, \alpha_0^\vee \rangle}{p^r}$$

and hence $\langle \nu_0, \alpha^\vee \rangle \geq p^r$ as claimed. $\square$

A big part of why we are able to completely determine $s_1(\nu)$ here is the following result, which allows us to apply Corollary 7.10 or Corollary 7.11.

Proposition 7.25. *Let $\mathbf{G} = SL_3$ and $\nu = (\nu_1, \nu_2) \in X_1$.*

1. *If either $\nu_1 + \nu_2 \leq p - 2$ or $\max(\nu_1, \nu_2) = p - 1$ then $L(\nu) \cong \nabla(\nu)$.*
2. *If neither of the above holds, there is a short exact sequence $0 \to L(\nu) \to \nabla(\nu) \to \nabla(\mu) \to 0$ where $\mu = (p - \nu_2 - 2, p - \nu_1 - 2)$.*

Proof. The first claim follows directly from Theorem 4.14.

The second claim follows by applying the Jantzen sum formula (Theorem 4.18), since we note that the only root that adds anything to the sum is α_0, where we get $\langle \nu + \rho, \alpha_0^\vee \rangle = \nu_1 + \nu_2 + 2$, so since $\alpha_0 = (1,1)$, this contributes the term $\chi(\nu - (\nu_1 + \nu_2 + 2 - p)(1,1)) = \chi(p - \nu_2 - 2, p - \nu_1 - 2)$.

The result now follows by noting that when we set $\mu = (p - \nu_2 - 2, p - \nu_1 - 2)$ then we get $\langle \mu + \rho, \alpha_0^\vee \rangle = 2p - (\nu_1 + \nu_2) - 2 \leq p$ since we assumed that $\nu_1 + \nu_2 > p - 2$. Thus $L(\mu) \cong \nabla(\mu)$ by Proposition 4.15, which yields the claim. $\square$

Like for SL_2 we will need a lemma that tells us how to compute those Littlewood-Richardson coefficients that can show up.

Lemma 7.26. *Let $\tilde{\lambda}$, $\tilde{\nu}$ and $\tilde{\mu}$ be partitions satisfying*

- *$l(\tilde{\lambda}) \leq 2$, $l(\tilde{\nu}) \leq 2$ and $l(\tilde{\mu}) \leq 3$.*
- *$\tilde{\lambda}_2 \geq \tilde{\mu}_3$.*
- *$\tilde{\lambda}_1 \geq \tilde{\mu}_2$.*
- *$\tilde{\mu}_1 - \tilde{\lambda}_1 \geq \tilde{\nu}_2$.*
- *$\tilde{\mu}_3 \geq \tilde{\nu}_2$.*
- *$\tilde{\mu}_2 - \tilde{\lambda}_2 \geq \tilde{\nu}_2$.*
- *$|\tilde{\lambda}| + |\tilde{\nu}| = |\tilde{\mu}|$.*

Then $c^{\tilde{\mu}}_{\tilde{\lambda},\tilde{\nu}} = \tilde{\nu}_2 + 1$.

Proof. First we note that all columns of $\tilde{\mu}/\tilde{\lambda}$ have at most one box, since we have $\tilde{\mu}_3 \leq \tilde{\lambda}_2$ and $\tilde{\mu}_2 \leq \tilde{\lambda}_1$.

We also note that in order to make an SSYT of shape $\tilde{\mu}/\tilde{\lambda}$ whose reverse reading word is a lattice permutation, we must only put 1's in the top row. But since the top row then contains $\tilde{\mu}_1 - \tilde{\lambda}_1 \geq \tilde{\nu}_2$ 1's, we are guaranteed that the resulting SSYT will have a lattice permutation as its reverse reading word, no matter how we place the remaining 1's and 2's, since the number of 2's we need to place is $\tilde{\nu}_2$, and we have already placed at least that many 1's in the top row.

Since there are no columns with more than one box, to make an SSYT we just need to place numbers in non-decreasing order in the two remaining rows. Thus, any such SSYT is uniquely determined by the number of 2's placed in the bottom row, and since the number of boxes in the bottom row is $\tilde{\mu}_3 \geq \tilde{\nu}_2$ and the number of boxes in the second row is $\tilde{\mu}_2 - \tilde{\lambda}_2 \geq \tilde{\nu}_2$, we see that we can pick any number of 2's between 0 and $\tilde{\nu}_2$ to place there.

This gives precisely $\tilde{\nu}_2 + 1$ possible choices, as was the claim (these choices actually work since we have assumed that $|\tilde{\lambda}| + |\tilde{\nu}| = |\tilde{\mu}|$). □

Using the above lemma we then get the following.

Lemma 7.27. *Let $\nu = (\nu_1, \nu_2) \in X_1$ with $\nu_2 \leq \nu_1$. Then*

$$[\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla = \begin{cases} \nu_2 + 1 & \textit{if } 3 \mid \nu_1 + 2\nu_2 \\ 0 & \textit{else} \end{cases}$$

Proof. Let

$$m = \frac{(p-1)|\tilde{\rho}| + |\tilde{\nu}| - (p-1)|\tilde{\rho}|}{3} = \frac{|\tilde{\nu}|}{3} = \frac{\nu_1 + 2\nu_2}{3}$$

so by Proposition 4.36 we see that if 3 does not divide $\nu_1 + 2\nu_2$ we have $[\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla = 0$.

If 3 does divide $\nu_1 + 2\nu_2$ then by Proposition 4.36 we have

$$[\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla = s^{\tilde{\rho}_1 + m\tilde{\omega}}_{\tilde{\rho}_1, \tilde{\nu}}$$

where $\rho_1 = (p-1)\rho$ and $\tilde{\omega} = (1, 1, 1)$.

We claim that if we set $\tilde{\lambda} = \tilde{\rho}_1$ and $\tilde{\mu} = \tilde{\rho}_1 + m\tilde{\omega}$ then the partitions $\tilde{\lambda}$, $\tilde{\nu}$ and $\tilde{\mu}$ satisfy the conditions of Lemma 7.26, which implies that $[\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla = \nu_2 + 1$ as claimed.

The conditions we need to show are:

- $l(\tilde{\lambda}) \leq 2$, $l(\tilde{\nu}) \leq 2$ and $l(\tilde{\mu}) \leq 3$: This is clear from the definitions.
- $\tilde{\lambda}_2 \geq \tilde{\mu}_3$: We have

$$\tilde{\lambda}_2 = (p-1) = \frac{3(p-1)}{3} \geq \frac{3\nu_1}{3} \geq \frac{\nu_1 + 2\nu_2}{3} = m = \tilde{\mu}_3$$

- $\tilde{\lambda}_1 \geq \tilde{\mu}_2$: We have

$$\tilde{\lambda}_1 = 2(p-1) = (p-1) + \frac{3(p-1)}{3} \geq (p-1) + \frac{3\nu_1}{3} \geq (p-1) + \frac{\nu_1 + 2\nu_2}{3} = (p-1) + m = \tilde{\mu}_2$$

- $\tilde{\mu}_1 - \tilde{\lambda}_1 \geq \tilde{\nu}_2$: We have

$$\tilde{\mu}_1 - \tilde{\lambda}_1 = 2(p-1) + m - 2(p-1) = m = \frac{\nu_1 + 2\nu_2}{3} \geq \frac{3\nu_2}{3} = \nu_2$$

- $\tilde{\mu}_3 \geq \tilde{\nu}_2$: We have
$$\tilde{\mu}_3 = m = \frac{\nu_1 + 2\nu_2}{3} \geq \frac{3\nu_2}{3} = \nu_2$$
- $\tilde{\mu}_2 - \tilde{\lambda}_2 \geq \tilde{\nu}_2$: We have
$$\tilde{\mu}_2 - \tilde{\lambda}_2 = (p-1) + m - (p-1) = m = \frac{\nu_1 + 2\nu_2}{3} \geq \frac{3\nu_2}{3} = \nu_2$$
- $|\tilde{\lambda}| + |\tilde{\nu}| = |\tilde{\mu}|$: We have
$$|\tilde{\lambda}| + |\tilde{\nu}| = 3(p-1) + \nu_1 + 2\nu_2 = 3(p-1) + 3m = 3(p-1) + m|\tilde{\omega}| = |\tilde{\mu}|$$

□

We can now determine $s_1(\nu)$ for all $\nu \in X_1$ and when $p \geq 5$. The assumption of $p \geq 5$ is not really needed, but we will later go through the cases $p = 2$ and $p = 3$ explicitly anyway.

Theorem 7.28. *Assume that $p \geq 5$ and let $\mathbf{G} = SL_3$. Let $\nu = (\nu_1, \nu_2) \in X_1$. Then*
$$s_1(\nu) = \begin{cases} 0 & \textit{unless } 3 \mid \nu_1 + 2\nu_2 \\ \min(\nu_1, \nu_2) + 1 & \textit{if either } \nu_1 + \nu_2 \leq p - 2 \textit{ or } \max(\nu_1, \nu_2) = p - 1 \\ \nu_1 + \nu_2 + 2 - p & \textit{else} \end{cases}$$

Proof. First we note that by combining Proposition 7.7 and Lemma 7.27 we see that if 3 does not divide $\nu_1 + 2\nu_2$ then $s_1(\nu) = 0$.

So from now on we assume that $3 \mid \nu_1 + 2\nu_2$. Further, by using Proposition 7.1 we can assume that $\nu_2 \leq \nu_1$ (since we can otherwise just replace ν by $-w_0(\nu)$). If either $\nu_1 + \nu_2 \leq p - 2$ or $\max(\nu_1, \nu_2) = p - 1$ then by Proposition 7.25 we have $L(\nu) \cong \nabla(\nu)$ and by Theorem 7.8 we get that $s_1(\nu) = [\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla$ which equals $\nu_2 + 1$ by Lemma 7.27. Since we assumed that $\nu_2 \leq \nu_1$ this is precisely $\min(\nu_1, \nu_2) + 1$ as was the claim.

So now we assume that $\nu_1 + \nu_2 > p - 2$ and that neither ν_1 nor ν_2 equals $p - 1$. We still assume that $\nu_2 \leq \nu_1$.

By combining Corollary 7.10 with Proposition 7.25 we see that
$$s_1(\nu) = [\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla - s_1(p - \nu_2 - 2, p - \nu_1 - 2)$$
and by the above we see that
$$s_1(p - \nu_2 - 2, p - \nu_1 - 2) = \min(p - \nu_2 - 2, p - \nu_1 - 2) + 1 = p - \nu_1 - 2 + 1$$
since we assumed that $\nu_1 \geq \nu_2$ and since the weight $(p - \nu_2 - 2, p - \nu_1 - 2)$ satisfies the above criteria.

From Lemma 7.27 we now have
$$s_1(\nu) = [\mathrm{St}_1 \otimes \nabla(\nu) : \mathrm{St}_1]_\nabla - (p - \nu_1 - 2 + 1) = \nu_2 + 1 - (p - \nu_1 - 2 + 1) = \nu_1 + \nu_2 + 2 - p$$
as was the claim. □

We now need to also describe $s_1(\nu)$ when $\nu \notin X_1$.

Lemma 7.29. *Let $\nu = (\nu_1, \nu_2) \in X_1$. Let $m = \frac{\nu_1+2\nu_2-p}{3}$.*
Then

$$[\mathrm{St}_1 \otimes \nabla(\nu) : \nabla((p-1)\rho + p(1,0))]_\nabla = \begin{cases} 2m+1-\nu_2 & \text{if } m \in \mathbb{Z}_{\geq 0} \text{ and } 2\nu_1+\nu_2 \geq 2p \\ 0 & \text{else} \end{cases}$$

Proof. Let $\rho_1 = (p-1)\rho$ and $\mu = \rho_1 + p(1,0)$.

By Proposition 4.36 we just need to show that $c^{\tilde{\mu}+m(1,1,1)}_{\tilde{\rho}_1,\tilde{\nu}} = 2m+1-\nu_2$ whenever m is a non-negative integer and $2\nu_1+\nu_2 \geq 2p$, and 0 whenever $2\nu_1+\nu_2 < 2p$.

First we note that in the skew-shape $(\tilde{\mu}+m(1,1,1))/\tilde{\rho}_1$ there are no columns with more than one box, since in the third row of $\tilde{\mu}+m(1,1,1)$ we have m boxes,

$$m = \frac{\nu_1+2\nu_2-p}{3} \leq \frac{3p-3-p}{3} = \frac{2p-3}{3} \leq p-1$$

and $p-1$ is the number of boxes in the second row of $\tilde{\rho}_1$. The same argument works for the second row, as there we have $(p-1)+m$ boxes, and the number of boxes in the first row of $\tilde{\rho}_1$ is $2(p-1)$.

The above arguments mean that in order to make an SSYT of skew-shape $(\tilde{\mu}+m(1,1,1))/\tilde{\rho}_1$, we just need to have the rows be non-decreasing. We further see that to make the reverse reading word be a lattice permutation, we can only add 1's in the top row, but once we have added these, we have $p+m$ 1's in the top row, and since $\nu_2 \leq p-1 \leq p+m$, this means that we can place the 2's any way we like, as long as the rows are non-decreasing.

By the above we see that there is such an SSYT exactly when we have enough 1's to put in the top row, i.e. when

$$\nu_1+\nu_2 \geq p+m = p + \frac{\nu_1+2\nu_2-p}{3} = \frac{2p+\nu_1+2\nu_2}{3}$$

which happens precisely when $2\nu_1+\nu_2 \geq 2p$.

Thus, each such SSYT is uniquely determined by the number of 2's in the last row, so we need to see what the possible numbers are for these. First, we claim that we have enough 2's to fill up the bottom row, so we need to show that $\nu_2 \geq m$ which is true since the assumption $\nu_1 \leq p-1 \leq p$ gives

$$m = \frac{\nu_1+2\nu_2-p}{3} = \frac{2\nu_2}{3} + \frac{\nu_1-p}{3} \leq \frac{2\nu_2}{3} \leq \nu_2$$

Next, we note that we need to add enough 2's in the bottom row that there is room for the remaining 2's in the second row. So if we are to add a 2's in the bottom row, we need $\nu_2 - a \leq m$ which means that we need $a \geq \nu_2 - m$ (note that we have $\nu_2 - m \geq m$ since $2m = \frac{2\nu_1+4\nu_2-2p}{3}$ so $2m \geq \nu_2$ is equivalent to $2\nu_1+\nu_2 \geq 2p$ which was part of the assumption). This means that all a with $\nu_2 - m \leq a \leq m$ are possible, which gives $m-(\nu_2-m)+1 = 2m+1-\nu_2$ possibilities, as was the claim. $\square$

We can now also determine $s_1(\nu)$ for those $\nu \notin X_1$. Again the assumption of $p \geq 5$ is not actually needed.

Theorem 7.30. *Let $\mathbf{G} = SL_3$ and assume that $p \geq 5$. Let $\nu = (\nu_1, \nu_2) \in X_1$ and $m = \frac{\nu_1+2\nu_2-p}{3}$. Then*

$$s_1(\nu + p(0,1)) = \begin{cases} 2m+1-\nu_2 & \text{if } m \text{ is a non-negative integer and } 2\nu_1+\nu_2 \geq 2p \\ 0 & \text{else} \end{cases}$$

Proof. Since $p \geq 5 \geq 2h - 2 = 4$ we can apply Theorem 7.8 to see that

$$s_1(\nu + p(1,0)) = [\mathrm{St}_1 \otimes L(\nu) : \nabla((p-1)\rho + p(0,1))]_\nabla$$

so if $L(\nu) \cong \nabla(\nu)$ the claim follows directly from Lemma 7.29.

If this is not the case, then we can apply Proposition 7.25 to see that

$$s_1(\nu + p(0,1)) = [\mathrm{St}_1 \otimes \nabla(\nu) : \nabla((p-1)\rho + p(1,0))]_\nabla - [\mathrm{St}_1 \otimes \nabla(\mu) : \nabla((p-1)\rho + p(1,0))]_\nabla$$

where $\mu = (p - \nu_2 - 2, p - \nu_1 - 2)$.

The claim now follows like above if we can show that $[\mathrm{St}_1 \otimes \nabla(\mu) : \nabla((p-1)\rho + p(1,0))]_\nabla = 0$. But by Lemma 7.29 if this is non-zero, then

$$(p - \nu_2 - 2) + 2(p - \nu_1 - 2) - p \geq 0 \qquad \text{and} \qquad 2(p - \nu_2 - 2) + (p - \nu_1 - 2) \geq 2p$$

The first inequality gives $2\nu_1 + \nu_2 + 6 \leq 2p$ and the second one gives $\nu_1 + 2\nu_2 + 6 \leq p$. Adding these gives $3\nu_1 + 3\nu_2 + 12 \leq 3p$ and thus $\nu_1 + \nu_2 \leq p - 4 \leq p - 2$ which by Proposition 7.25 would imply that $L(\nu) \cong \nabla(\nu)$ which contradicts our assumption. $\square$

We have now completely described $s_1(\nu)$ for all $\nu \in X_+$ for SL_3 when $p \geq 5$, since by Proposition 7.24, if $s_1(\nu) \neq 0$ and $\nu \notin X_1$ then $\nu = \nu_0 + p(1,0)$ or $\nu = \nu_0 + p(0,1)$ with $\nu_0 \in X_1$, and by Proposition 7.1 we can then get $s_1(\nu)$ from Theorem 7.30 either directly or by replacing ν by $-w_0(\nu)$.

We thus also get all $t_1(\nu)$ when $p \geq 5$ by Proposition 7.5.

We will now look at the two remaining cases for SL_3, namely $p = 2$ and $p = 3$.

For $p = 2$ the result is a special case of [BDM11a, Proposition 3.2(a)].

Proposition 7.31. *If* $\mathbf{G} = SL_3$ *and* $p = 2$ *then* $\mathrm{soc}_{\mathbf{G}}(\mathrm{St}_1 \otimes \mathrm{St}_1) = L(0,0) \oplus 2\,\mathrm{St}_1$.

Proof. We wish to show that $s_1(0,0) = 1$, $s_1(1,1) = 2$ and that $s_1(\nu) = 0$ for all other $\nu \in X_+$.

By Proposition 7.7 together with Lemma 7.27 we see that we only need to consider $\nu = (\nu_1, \nu_2)$ such that $\nu_1 + 2\nu_2$ is divisible by 3, and since ν must be a weight of $\mathrm{St}_1 \otimes \mathrm{St}_1$ we also see that we can assume that $\nu_1 + \nu_2 \leq 4$.

This leaves the weights $(0,0)$, $(1,1)$, $(3,0)$, $(0,3)$ and $(2,2)$. By Proposition 7.1, once we show that $s_1(3,0) = 0$ we also see that $s_1(0,3) = 0$. That $s_1(2,2) = 0$ follows from Proposition 7.15.

- $(0,0)$: We have $s_1(0,0) = \dim(\mathrm{Hom}_{\mathbf{G}}(L(0,0), \mathrm{St}_1 \otimes \mathrm{St}_1)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_1, \mathrm{St}_1)) = 1$.

- $(1,1)$: Since $L(1,1) = \mathrm{St}_1$ we have

$$s_1(1,1) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_1, \mathrm{St}_1 \otimes \mathrm{St}_1)) = [\mathrm{St}_1 \otimes \mathrm{St}_1 : \mathrm{St}_1]_\nabla = 2$$

by Lemma 7.27.

- $(3,0)$: Here we first apply the Jantzen sum formula (Theorem 4.18) so see that we have a short exact sequence $0 \to L(3,0) \to \nabla(3,0) \to L(0) \to 0$. We would now like to apply Corollary 7.10, but that requires $p \geq 2h - 2$. However, if we can show that $\mathrm{St}_1 \otimes L(3,0)$ has a good filtration, the proof still works. Since we have $(3,0) = (1,0) + 2(1,0)$ we see that $L(3,0) \cong L(1,0) \otimes L(1,0)^{(1)}$ by Theorem 4.10. Now we easily see from Theorem 4.14 that $L(1,0) \cong \nabla(1,0)$ and hence $\mathrm{St}_1 \otimes L(3,0)$ has a good filtration by Theorem 6.10 and Theorem 4.25.

Thus, we can apply Corollary 7.10 to see that

$$s_1(3,0) = [\mathrm{St}_1 \otimes \nabla(3,0) : \mathrm{St}_1]_\nabla - s_1(0,0) = [\mathrm{St}_1 \otimes \nabla(3,0) : \mathrm{St}_1]_\nabla - 1$$

so we just need to show that $[\mathrm{St}_1 \otimes \nabla(3,0) : \mathrm{St}_1]_\nabla \leq 1$, which is clear by Proposition 4.36 since whenever $l(\tilde{\nu}) \leq 1$ we have $c^{\tilde{\mu}}_{\tilde{\lambda},\tilde{\nu}} \leq 1$.

□

We see above that when $p = 2$ we have $s_1(\nu) \neq 0 \implies \nu \in X_1$, so we would like to be able to apply Proposition 7.13. For this we need $p \geq 2h - 2$, but the only place in the proof it is needed is to be able to apply the tensor decomposition from Proposition 4.34, and for SL_3 this is actually known to hold for all p (see the remarks to [Jan03, II.11.16]). Hence, we do get all $s_r(\nu)$ and all $t_r(\nu)$ when $p = 2$ (the same argument also shows that we can apply Proposition 7.5).

For $p = 3$ we get the following, which is a special case of [BDM11a, Proposition 3.2(a)].

Proposition 7.32. *If* $\mathbf{G} = SL_3$ *and* $p = 3$ *then*

$$\mathrm{soc}_{\mathbf{G}}(\mathrm{St}_1 \otimes \mathrm{St}_1) = L(0,0) \oplus L(1,1) \oplus 3\,\mathrm{St}_1 \oplus L(5,2) \oplus L(2,5)$$

Proof. We wish to show that $s_1(0,0) = 1$, $s_1(1,1) = 1$, $s_1(2,2) = 3$, $s_1(5,2) = 1$, $s_1(2,5) = 1$ and $s_1(\nu) = 0$ for all other $\nu \in X_+$.

By Proposition 7.7 and Lemma 7.27 we only need to consider weights $\nu = (\nu_1, \nu_2)$ such that $3 \mid \nu_1 + 2\nu_2$, and if $s_1(\nu) \neq 0$ then ν must be a weight of $\mathrm{St}_1 \otimes \mathrm{St}_1$ so in particular, we must have $\langle \nu, \alpha_0^\vee \rangle \leq 2(p-1)(h-1) = 8$.

This means that we need to consider the weights $(0,0)$, $(1,1)$, $(3,0)$, $(0,3)$, $(2,2)$, $(4,1)$, $(1,4)$, $(3,3)$, $(6,0)$, $(0,6)$, $(5,2)$, $(2,5)$, $(7,1)$, $(1,7)$ and $(4,4)$. By Proposition 7.15 we immediately see that we do not need to consider the weights $(3,0)$, $(0,3)$, $(3,3)$, $(6,0)$ and $(0,6)$. Also, by Proposition 7.1, we can restrict ourselves to those $\nu = (\nu_1, \nu_2)$ with $\nu_2 \leq \nu_1$.

- $(0,0)$: We have $s_1(0,0) = \dim(\mathrm{Hom}_{\mathbf{G}}(L(0,0), \mathrm{St}_1 \otimes \mathrm{St}_1)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_1, \mathrm{St}_1)) = 1$.

- $(1,1)$: Applying Proposition 7.25 and using the results from Section 6.10 we see that we can apply Corollary 7.10 and Lemma 7.27 to get

$$s_1(1,1) = [\mathrm{St}_1 \otimes \nabla(1,1) : \mathrm{St}_1]_\nabla - s_1(0,0) = 2 - 1 = 1$$

- $(2,2)$: Since we have $L(2,2) = \mathrm{St}_1$ we get

$$s_1(2,2) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_1, \mathrm{St}_1 \otimes \mathrm{St}_1)) = [\mathrm{St}_1 \otimes \nabla(2,2) : \mathrm{St}_1]_\nabla = 3$$

by Lemma 7.27.

- $(4,1)$: We have $(4,1) = (1,1) + 3(1,0)$ so by Theorem 4.10 we have $L(4,1) \cong L(1,1) \otimes L(1,0)^{(1)}$ and since $L(1,0) \cong \nabla(1,0)$ by Proposition 7.25 we see that (applying Theorem 4.10)

$$s_1(4,1) = \dim(\mathrm{Hom}_{\mathbf{G}}(L(4,1), \mathrm{St}_1 \otimes \mathrm{St}_1)) = \dim(\mathrm{Hom}_{\mathbf{G}}(\mathrm{St}_1 \otimes L(1,0)^{(1)}, \mathrm{St}_1 \otimes L(1,1)))$$

$$\leq \dim(\mathrm{Hom}_{\mathbf{G}}(\nabla(2\rho + 3(1,0)), \mathrm{St}_1 \otimes \nabla(1,1)) = [\mathrm{St}_r \otimes \nabla(1,1) : \nabla(2\rho + 3(1,0))]_\nabla$$

and the claim now follows by Lemma 7.29 since if $\nu = (\nu_1, \nu_2) = (1,1)$ we have $2\nu_1 + \nu_2 = 3 < 2p = 6$.

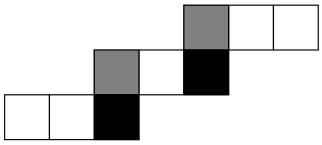
• $(5,2)$: We have $(5,2) = (2,2) + 3(1,0)$ and since we have $L(2,2) = \mathrm{St}_1 = \nabla(2,2)$ and by Proposition 7.25 we have $L(1,0) \cong \nabla(1,0)$, Theorem 4.29 shows that $L(5,2) \cong \nabla(5,2)$, so we get

$$s_1(5,2) = [\mathrm{St}_1 \otimes \mathrm{St}_1 : \nabla(2\rho + 3(1,0))]_\nabla$$

which is $2\frac{2+2\cdot 2-3}{3}+1-2 = 1$ by Lemma 7.29 since $0 \leq 2+2\cdot 2-3$ is divisible by 3 and $2\cdot 2+2 \geq 2\cdot 3 = 6$.

• $(7,1)$: By Proposition 7.7 we have $s_1(7,1) \leq [\mathrm{St}_1 \otimes \nabla(7,1) : \mathrm{St}_1]_\nabla$ so it is sufficient to show that the latter is 0.

By Proposition 4.36 it equals $c^{(7,5,3)}_{(4,2),(8,1)}$ so we need to show that there is no SSYT of skew-shape $(7,5,3)/(4,2)$ and type $(8,1)$. But the shape is



and since the gray boxes must contain a 1 or greater, both of the black boxes must contain a 2 or greater. But since the type needs to be $(8,1)$ we can only place one 2, so there is no SSYT of this shape and type.

• $(4,4)$: We can write $(4,4) = (1,1) + 3(1,1)$ so by Proposition 7.7 we get the inequality $s_r(4,4) \leq [\nabla(2\rho + 3(1,1)) \otimes \nabla(1,1) : \mathrm{St}_1]_\nabla$ and by Proposition 4.36 the latter equals $c^{(4,2)+m(1,1)}_{(10,5),(2,1)}$ where $m = \frac{15+3-6}{3} = 4$. But since $10 > 4 + m = 8$ we see that $(10,5) \not\subseteq (4,2) + m(1,1)$ so this is 0.

$\square$

By the above we see that if $\mathbf{G} = SL_3$ and $p = 3$ then we can still apply the arguments from Lemma 7.3 and Proposition 7.5 to also get all $t_1(\nu)$.

We could have simplified the calculations in the cases $\nu = (7,1)$ and $\nu = (4,4)$ by appealing to Proposition 7.24, but we chose to include the full details to illustrate how the combinatorics behave.

8 The Steinberg Square for Finite Groups of Lie type

In this section we will fix a natural number r and let $G = \mathbf{G}^{F^r}$.

We will study the restriction of $\mathrm{St}_r \otimes \mathrm{St}_r$ to G. Since St_r is projective as a G-module (by Theorem 5.18) the same is true for $\mathrm{St}_r \otimes St_r$, so if we write

$$\mathrm{soc}_G(\mathrm{St}_r \otimes \mathrm{St}_r) = \bigoplus_{\nu \in X_r} f_r(\nu) L(\nu)$$

then we get

$$\mathrm{St}_r \otimes \mathrm{St}_r \cong \bigoplus_{\nu \in X_r} f_r(\nu) P(\nu)$$

where $P(\nu)$ denotes the projective cover of $L(\nu)$ as a G-module.

So in order to better understand $\mathrm{St}_r \otimes \mathrm{St}_r$ as a G-module, we wish to understand the numbers $f_r(\nu)$ for $\nu \in X_r$.

We will be able to get similar results about $f_r(\nu)$ to the ones for $s_r(\nu)$, except that in this case the formulas will be considerably more complicated.

A nice thing about the Steinberg square as a G-module is that, except in some special cases, it has all simple G-modules as composition factors (see [HSTZ13, Theorem 1.2]).

First we will need a result analogous to Proposition 7.1.

Proposition 8.1. *For all $\nu \in X_r$ we have $f_r(\nu) = f_r(-w_0(\nu))$.*

Proof. Since we have $(\mathrm{St}_r \otimes \mathrm{St}_r)^* \cong \mathrm{St}_r \otimes \mathrm{St}_r$ we see that the number of times any $P(\nu)$ occurs in the decomposition must be the same as the number of times $P(\nu)^* \cong P(-w_0(\nu))$ occurs, which gives the claim. $\square$

We can then get some conditions analogous to those in Proposition 7.7. These are all very similar but which one is the most practical to use can vary.

Proposition 8.2. *Let $\nu \in X_r$. We have the following.*

1. $$f_r(\nu) \leq \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes \nabla(-w_0(\nu)) \otimes \nabla(\lambda) : \nabla((p^r - 1)\rho + p^r \lambda)]_\nabla$$

2. $$f_r(\nu) \leq \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla((p^r - 1)\rho + p^r \lambda)]_\nabla$$

3. $$f_r(\nu) \leq \sum_{\lambda \in X_+} [\nabla((p^r - 1)\rho - p^r w_0(\nu)) \otimes \nabla(\lambda) : \nabla((p^r - 1)\rho + p^r \lambda)]_\nabla$$

4. $$f_r(\nu) \leq \sum_{\lambda \in X_+} [\nabla((p^r - 1)\rho + p^r \nu) \otimes \nabla(\lambda) : \nabla((p^r - 1)\rho + p^r \lambda)]_\nabla$$

Proof. By definition we have

$$f_r(\nu) = \dim(\mathrm{Hom}_G(L(\nu), \mathrm{St}_r \otimes \mathrm{St}_r)) = \dim(\mathrm{Hom}_G(\mathrm{St}_r, \mathrm{St}_r \otimes L(-w_0(\nu))))$$

and the injection $L(-w_0(\nu)) \hookrightarrow \nabla(-w_0(\nu))$ then gives the inequality

$$f_r(\nu) \leq \dim(\mathrm{Hom}_G(\mathrm{St}_r \otimes \nabla(-w_0(\nu))))$$

Applying Theorem 5.19 we then get

$$f_r(\nu) \leq \dim(\mathrm{Hom}_G(\mathrm{St}_r, \mathrm{St}_r \otimes \nabla(-w_0(\nu)))) = \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes \nabla(-w_0(\nu)) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

as was the first claim.

The second claim follows from the first by applying Proposition 8.1.

The third claim follows by noting that for any G-module M we have $M \cong M^{(r)}$, so arguing as above, we get

$$f_r(\nu) \leq \dim(\mathrm{Hom}_G(\mathrm{St}_r, \mathrm{St}_r \otimes \nabla(-w_0(\nu))^{(r)}))$$

and applying Theorem 4.29 yields the claim.

The final claim then follows from the third claim by once again applying Proposition 8.1. □

Theorem 8.3. *Assume $p \geq 2h-2$ and let $\nu \in X_r$. Then we have*

1.

$$f_r(\nu) = \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes L(-w_0(\nu)) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

2.

$$f_r(\nu) = \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes L(\nu) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

Proof. By definition we have

$$f_r(\nu) = \dim(\mathrm{Hom}_G(L(\nu), \mathrm{St}_r \otimes \mathrm{St}_r)) = \dim(\mathrm{Hom}_G(\mathrm{St}_r, \mathrm{St}_r \otimes L(-w_0(\nu))))$$

By Theorem 6.11 we see that $\mathrm{St}_r \otimes L(\nu)$ has a good filtration, so we can apply Theorem 5.19 to get

$$f_r(\nu) = \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes L(-w_0(\nu)) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

as was the first claim.

The second claim follows from the above by applying Proposition 8.1. □

Note that in the above proof, the assumption that $p \geq 2h-2$ can be replaced by the assumption that $\mathrm{St}_r \otimes L(\nu)$ has a good filtration, as is conjectured to hold for all $\nu \in X_r$ without restrictions on p, and is known to hold in some small cases (by the results in Section 6.10).

In particular, we can also apply the above whenever we have $L(\nu) \cong \nabla(\nu)$.

By Proposition 8.2 we see that it will be necessary to know something about when we have $[\mathrm{St}_r \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla \neq 0$.

Proposition 8.4. *If* $[\mathrm{St}_r \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla \neq 0$ *for some* $\nu, \lambda \in X_+$ *then* $\nu \geq (p^r-1)\lambda$.

In particular, $\langle \nu, \alpha_0^\vee \rangle \geq (p^r-1)\langle \lambda, \alpha_0^\vee \rangle$.

Proof. If $[\mathrm{St}_r \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla \neq 0$ then $(p^r-1)\rho + p^r\lambda$ is a weight of $\mathrm{St}_r \otimes \nabla(\nu)\nabla(\lambda)$ so we have $(p^r-1)\rho + p^r\lambda \leq (p^r-1)\rho + \nu + \lambda$ and hence $\nu \geq (p^r-1)\lambda$ as was the first claim.

The second claim is now clear. $\square$

Corollary 8.5. *If* $[\mathrm{St}_r \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla \neq 0$ *for some* $\lambda \in X_+$ *and some* $\nu \in X_r$ *then* $\langle \lambda, \alpha_0^\vee \rangle \leq h-1$.

Proof. We have $\langle \nu, \alpha_0^\vee \rangle \leq (p^r-1)(h-1)$, so the claim follows directly from Proposition 8.4. $\square$

The above also immediately allows us to calculate $f_r(\nu)$ when ν is "small".

Proposition 8.6. *Assume that* $p \geq 2h-2$ *and let* $\nu \in X_r$ *with* $\langle \nu, \alpha_0^\vee \rangle < p^r - 1$. *Then* $f_r(\nu) = s_r(\nu)$.

Proof. By Theorem 8.3 we have

$$f_r(\nu) = \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes L(\nu) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

but Proposition 8.4 shows that the assumptions imply that all summands except possibly the one for $\lambda = 0$ are 0. Thus we get

$$f_r(\nu) = [\mathrm{St}_r \otimes L(\nu) : \mathrm{St}_r]_\nabla = s_r(\nu)$$

by Theorem 7.8. $\square$

8.1 $SL_2(p^r)$

When $\mathbf{G} = SL_2$ (so $G = SL_2(p^r)$) it turns out that the calculations of $f_r(\nu)$ are almost identical to those for $s_r(\nu)$, so we will be able to completely determine these for all r and all $\nu \in X_r$.

As previously for SL_2 we identify X_r with the set of integers ν satisfying $0 \leq \nu \leq p^r - 1$.

Theorem 8.7. *Let* $\mathbf{G} = SL_2$, $p \geq 3$ *and* $\nu \in X_r$. *Then*

$$f_r(\nu) = \begin{cases} 1 & \textit{if } \nu \leq p^r - 2 \textit{ and } \nu \textit{ has a } p\textit{-adic expansion with all even terms} \\ 2 & \textit{if } \nu = p^r - 1 \\ 0 & \textit{else} \end{cases}$$

Proof. If $\nu \leq p^r - 2$ then by Proposition 8.6 we have $f_r(\nu) = s_r(\nu)$, and thus the claim follows by Corollary 7.22.

If $\nu = p^r - 1$ we have $L(\nu) = \mathrm{St}_r$ and we see from Proposition 8.4 and Theorem 8.3 that

$$f_r(\nu) = [\mathrm{St}_r \otimes \mathrm{St}_r : \mathrm{St}_r]_\nabla + [\mathrm{St}_r \otimes \mathrm{St}_r \otimes L(1) : \nabla(2p^r - 1)]_\nabla$$

The first summand is precisely $s_r(\nu) = 1$ by Theorem 7.8 and Corollary 7.22, so we just need to show that the second summand is also 1.

But this follows by Proposition 4.36 since we have $(p^r-1) + (p^r-1) + 1 = 2p^r - 1$, and for any partitions $\tilde{\lambda}$ and $\tilde{\mu}$ we have $c_{\tilde{\lambda},\tilde{\mu}}^{\tilde{\lambda}+\tilde{\mu}} = 1$. $\square$

Theorem 8.8. *If* $\mathbf{G} = SL_2$ *and* $p = 2$ *then* $\mathrm{St}_r \otimes \mathrm{St}_r \cong P(0) \oplus \mathrm{St}_r$.

Proof. The claim is that $f_r(0) = 1$, $f_r(2^r - 1) = 1$ and $f_r(\nu) = 0$ for all other $\nu \in X_r$.

When $\nu \leq 2^r - 2$ we see by Proposition 8.6 that $f_r(\nu) = s_r(\nu)$, so in this case the claim follows by Corollary 7.22.

For $\nu = 2^r - 1$ we have $L(\nu) = \mathrm{St}_r$ and we see from Proposition 8.4 and Theorem 8.3 that

$$f_r(\nu) = [\mathrm{St}_r \otimes \mathrm{St}_r : \mathrm{St}_r]_\nabla + [\mathrm{St}_r \otimes \mathrm{St}_r \otimes L(1) : \nabla(2^{r+1} - 1)]_\nabla$$

and by Theorem 7.8 the first summand is $s_r(\nu) = 0$ by Corollary 7.22.

Thus we just need to show that $[\mathrm{St}_r \otimes \mathrm{St}_r \otimes L(1) : \nabla(2^{r+1} - 1)]_\nabla = 1$, and this follows from Proposition 4.36 since we have $(2^r - 1) + (2^r - 1) + 1 = 2^{r+1} - 1$. □

Note that the above is a special case of [Tsu90, Theorem 3].

Also worth noting is the following, which is a special case of [AJL83, Lemma 4.1].

Corollary 8.9. *Let* $\mathbf{G} = SL_2$ *and* $p \geq 3$. *Let* $\nu \in X_r$ *with* $\nu \neq 0$ *and such that* ν *has a* p*-adic expansion where all terms are even. Then we have isomorphisms of* G*-modules*

1. $T(2(p^r - 1)) \cong P(0) \oplus \mathrm{St}_r$
2. $T(2(p^r - 1) - \nu) \cong P(\nu)$

Proof. By comparing the values of $s_r(\nu)$ and $f_r(\nu)$ (using Corollary 7.22 and Theorem 8.7) we see that it is sufficient to show that $\dim(\mathrm{Hom}_G(\mathrm{St}_r, T(2(p^r - 1)))) \neq 0$. But since $T(2(p^r - 1))$ has a good filtration, we can use Theorem 5.19. One of the factors in this good filtration is $\nabla(2(p^r - 1))$, and we claim that in fact $\dim(\mathrm{Hom}_G(\mathrm{St}_r, \nabla(2(p^r - 1)))) \neq 0$.

By looking at the formula from Theorem 5.19 we see that it will in fact be sufficient to find a $\lambda \in X_+$ such that $(p^r - 1) + 2(p^r - 1) + \lambda = (p^r - 1) + p^r\lambda$, and it is easily seen that this holds for $\lambda = 2$. □

8.2 $SL_3(p^r)$

When $\mathbf{G} = SL_3$ (so $G = SL_3(p^r)$) it is a lot more difficult to determine the $f_r(\nu)$. We will only be able to determine these completely when $r = 1$ and $p = 2$ or $p = 3$.

We start with a result similar to Proposition 8.4, but which gives more precise information for SL_3.

Proposition 8.10. *Let* $\nu = (\nu_1, \nu_2) \in X_+$ *and* $\lambda = (\lambda_1, \lambda_2) \in X_+$. *If*

$$[\mathrm{St}_r \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla((p^r - 1)\rho + p^r\lambda)]_\nabla \neq 0$$

then $\nu_1 + 2\nu_2 - (p^r - 1)(\lambda_1 + 2\lambda_2)$ *is a non-negative integer divisible by* 3.

Proof. This follows directly from Proposition 4.37. □

We can then compute $f_1(\nu)$ for all $\nu \in X_1$ and $p = 2$.

Proposition 8.11. *Let* $\mathbf{G} = SL_3$ *and* $p = 2$. *Then*

$$\mathrm{soc}_G(\mathrm{St}_1 \otimes \mathrm{St}_1) = L(0, 0) \oplus L(1, 0) \oplus L(0, 1) \oplus 3\,\mathrm{St}_1$$

Proof. By Proposition 7.25 we have $L(\nu) \cong \nabla(\nu)$ for all $\nu \in X_1$ when $p = 2$, so the inequalities in Proposition 8.2 are in fact equalities in this case.

For $\nu \in X_1$ we need to consider a sum with terms of the form $[\mathrm{St}_1 \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla(\rho + 2\lambda)]_\nabla$, and we will use Proposition 8.10 to restrict which λ we need to consider for each $\nu \in X_1$. We can also assume that $\nu_1 \geq \nu_2$ by Proposition 8.1.

We can then calculate $f_1(\nu)$ as follows:

- $\nu = (0,0)$: By Proposition 8.6 and Proposition 7.31 we have $f_1(0,0) = s_1(0,0) = 1$.
- $\nu = (1,0)$: Here we only need to consider $\lambda = (1,0)$ where we get

$$[\mathrm{St}_1 \otimes \nabla(1,0) \otimes \nabla(1,0) : \nabla(\rho + 2(1,0))]_\nabla = 1$$

since we have $\rho + 2(1,0) = \rho + (1,0) + (1,0)$.

- $\nu = (1,1)$: Here we have $L(1,1) = \mathrm{St}_1$ and we need to consider $\lambda = (0,0)$ and $\lambda = (1,1)$.

  For $\lambda = (0,0)$ we get a contribution of $[\mathrm{St}_1 \otimes \mathrm{St}_1 : \mathrm{St}_1]_\nabla = s_1(1,1) = 2$ by Proposition 7.31.

  For $\lambda = (1,1)$ we get a contribution of $[\mathrm{St}_1 \otimes \mathrm{St}_1 \otimes \mathrm{St}_1 : \nabla(\rho + 2\rho)]_\nabla = 1$ since $\rho + 2\rho = \rho + \rho + \rho$.

  Adding these we get $f_1(1,1) = 3$ as claimed.

□

And we can also compute $f_1(\nu)$ when $p = 3$.

Proposition 8.12. *Let* $\mathbf{G} = SL_3$ *and* $p = 3$. *Then*

$$\mathrm{soc}_G(\mathrm{St}_1 \otimes \mathrm{St}_1) = L(0,0) \oplus L(1,1) \oplus L(2,0) \oplus L(0,2) \oplus L(2,1) \oplus L(1,2) \oplus 4\,\mathrm{St}_1$$

Proof. By Proposition 7.25 we see that the only weights $\nu \in X_1$ where we do not have $L(\nu) \cong \nabla(\nu)$ are $\nu = (1,0)$ and $\nu = (0,1)$. But for these weights we have $f_1(\nu) = s_1(\nu)$ by Proposition 8.6 and by Proposition 7.32 we have $s_1(\nu) = 0$ for these.

For the remaining weights we thus have equalities in Proposition 7.7, so we get that $f_1(\nu)$ is a sum with terms of the form $[\mathrm{St}_1 \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla(2\rho + 3\lambda)]_\nabla$ and we will use Proposition 8.10 to restrict which λ we need to consider for each $\nu \in X_1$. We will also by Proposition 8.1 only consider those $\nu = (\nu_1, \nu_2)$ with $\nu_1 \geq \nu_2$.

We can now compute $f_1(\nu)$ as follows.

- $\nu = (0,0)$: By Proposition 8.6 and Proposition 7.32 we have $f_1(0,0) = s_1(0,0) = 1$.
- $\nu = (1,1)$: Here we only need to consider $\lambda = (0,0)$, where the contribution is precisely $s_1(1,1) = 1$ by Proposition 7.32.
- $\nu = (2,0)$: Here we need to consider $\lambda = (1,0)$ where the contribution is

$$[\mathrm{St}_1 \otimes \nabla(2,0) \otimes \nabla(1,0) : \nabla(2\rho + 3(1,0))]_\nabla = 1$$

since $2\rho + 3(1,0) = 2\rho + (2,0) + (1,0)$.

- $\nu = (2,1)$: Here we need to consider $\lambda = (2,0)$ and $\lambda = (0,1)$.

  For $\lambda = (2,0)$ we get the contribution

$$[\mathrm{St}_1 \otimes \nabla(2,1) \otimes \nabla(2,0) : \nabla(2\rho + 3(2,0))]_\nabla$$

and we claim that this is 0. The reason for this is that if it was not 0 then there would be a $\mu \in X_+$ such that

$$[\mathrm{St}_1 \otimes \nabla(2,1) : \nabla(\mu)]_\nabla \neq 0 \qquad \text{and} \qquad [\nabla(\mu) \otimes \nabla(2,0) : \nabla(2\rho + 3(2,0))]_\nabla \neq 0$$

The latter of these would imply that if $\mu = (\mu_1, \mu_2)$ then $\mu_2 \leq 2$ since we would need to have $\tilde{\mu} \subseteq 2\tilde{\rho} + 3(2,0)$. But then the first one implies that $|\tilde{\mu}| < |2\tilde{\rho}| + |(3,1)|$ which is not possible since we have $|2\tilde{\rho} + 3(2,0)| = |2\tilde{\rho}| + |(3,1)| + |(2,0)|$.

For $\lambda = (0,1)$ we get the contribution

$$[\mathrm{St}_1 \otimes \nabla(2,1) \otimes \nabla(0,1) : \nabla(2\rho + 3(0,1))]_\nabla$$

and the claim is that this is 1. So we need to see for which μ it is possible to have both

$$[\mathrm{St}_1 \otimes \nabla(2,1) : \nabla(\mu)]_\nabla \neq 0 \qquad \text{and} \qquad [\nabla(\mu) \otimes \nabla(0,1) : \nabla(2\rho + 3(0,1))]_\nabla \neq 0$$

We note that since $|2\tilde{\rho}+3(1,1)| = |2\tilde{\rho}|+|(3,1)|+|(1,1)|$ we must have $|\tilde{\mu}| = |2\tilde{\rho}|+|(3,1)| = 6+4 = 10$. We now see that $\tilde{\mu}$ must be contained in

$$\ydiagram{7,5}$$

so $\tilde{\mu}$ is obtained by removing two boxes from this diagram such that the result is a partition and such that it is possible to put a 1 in one of the boxes and a 2 in the other in such a way that we get an SSYT where the reverse reading word is a lattice permutation. This rules out removing two boxes in the same row, as we would then need to place the 1 to the left of the 2, and the reverse reading word would become $(2,1)$ which is not a lattice permutation.

Thus, we see that the only possibility is $\tilde{\mu} = (6,4)$ (so $\mu = (2,4)$). Hence, the claim boils down to showing that $[\mathrm{St}_1 \otimes \nabla(2,1) : \nabla(2,4)]_\nabla = 1$. Applying Proposition 4.36 we see that it equals the number of SSYTs of type $(3,1)$ and shape

$$\ydiagram{2+2,2}$$

whose reverse reading word is a lattice permutation. But we see that the only such SSYT is

$$\begin{ytableau} \none & \none & 1 & 1 \\ 1 & 2 \end{ytableau}$$

which yields the claim.

- $\nu = (2,2)$: Here we need to consider $\lambda = (0,0)$ and $\lambda = (1,1)$.

  For $\lambda = (0,0)$ we get a contribution of $s_1(\nu) = 3$ by Proposition 7.32.

  For $\lambda = (1,1)$ we get a contribution of

$$[\mathrm{St}_1 \otimes \mathrm{St}_1 \otimes \nabla(1,1) : \nabla(2\rho + 3(1,1))]_\nabla$$

and the claim is that this is 1. But this is clear since $2\rho + 3(1,1) = 2\rho + 2\rho + (1,1)$.

□

As can be seen from the above calculations, everything gets increasingly complicated when we increase p. It turns out that a lot of the behavior of the $f_1(\nu)$ and more generally the $f_r(\nu)$ depends on whether 3 divides $p^r - 1$.

The following relates the multiplicity of St_r in $\mathrm{soc}_G(\mathrm{St}_r \otimes \mathrm{St}_r)$ and in $\mathrm{soc}_{\mathbf{G}}(\mathrm{St}_r \otimes \mathrm{St}_r)$.

Proposition 8.13. *Let $\mathbf{G} = SL_3$ and assume that 3 does not divide $p^r - 1$. Then*

$$f_r((p^r-1)\rho) = s_r((p^r-1)\rho) + 1$$

Proof. Since $L((p^r-1)\rho) = \mathrm{St}_r \cong \nabla((p^r-1)\rho)$, the inequalities in Proposition 8.2 are equalities, and we get

$$f_r((p^r-1)\rho) = \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes \mathrm{St}_r \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

For $\lambda = (0,0)$ the contribution to the sum is precisely $s_r((p^r-1)\rho)$, so we need to show that the remaining sum equals 1.

In fact, by Proposition 8.10 we see that the only $\lambda = (\lambda_1, \lambda_2)$ that can contribute to the sum are those where 3 divides $\lambda_1 + 2\lambda_2$, since 3 does not divide $p^r - 1$. But by Corollary 8.5 we must also have $\lambda_1 + \lambda_2 \leq h - 1 = 2$, so the only non-zero possible λ is $\lambda = (1,1) = \rho$, and the claim is thus that

$$[\mathrm{St}_1 \otimes \mathrm{St}_1 \otimes \nabla(\rho) : \nabla((p^r-1)\rho + p^r\rho)]_\nabla = 1$$

which is clear since

$$(p^r-1)\rho + p^r\rho = (p^r-1)\rho + (p^r-1)\rho + \rho$$

□

The above shows that when we decompose $\mathrm{St}_r \otimes \mathrm{St}_r$, we get an extra copy of St_r when we view it as a G-module instead of as a $\mathbf{G}$-module. The following corollary then shows that this extra copy always comes from a specific part of the decomposition, namely from the restriction of $T(2(p^r-1)\rho)$ to G (note that this summand will always be in $\mathrm{St}_r \otimes \mathrm{St}_r$ with multiplicity 1 since it corresponds to the number of times $L(0)$ occurs in the $\mathbf{G}$-socle).

Corollary 8.14. *Let $\mathbf{G} = SL_3$ and assume that 3 does not divide $p^r - 1$. Assume that $\nu \in X_+$ with $\nu \neq 0$ is given such that $s_r(\nu) \neq 0$ and such that St_r is a composition factor of $T((p^r-1)\rho + \nu)$ as a G-module.*

Then $\nu = (p^r-1)\rho$.

Proof. By Proposition 8.13 it is sufficient to show that St_r is a composition factor of $T(2(p^r-1)\rho)$ as a G-module, since whenever St_r is a composition factor, it will in fact be a direct summand, as it is both injective and projective (by Theorem 5.18).

So we need to show that $\dim(\mathrm{Hom}_G(\mathrm{St}_r, T(2(p^r-1)\rho))) \geq 1$, and since $T(2(p^r-1)\rho)$ has a good filtration where one of the factors is $\nabla(2(p^r-1)\rho)$, it is sufficient to show that

$$\dim(\mathrm{Hom}_G(\mathrm{St}_r, \nabla(2(p^r-1)\rho))) \geq 1$$

By Theorem 5.19 this equals

$$\sum_{\lambda \in X_+} [\nabla(2(p^r-1)\rho) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

so it is sufficient to find a λ such that $(p^r-1)\rho + p^r\lambda = 2(p^r-1)\rho + \lambda$, and clearly $\lambda = \rho$ satisfies this. □

Continuing with the same assumptions, we then show that when $L(\nu)$ occurs in the $\mathbf{G}$-socle of $\mathrm{St}_r \otimes \mathrm{St}_r$ (for some $\nu \in X_r$), then it does not occur any more times in the G-socle than in the $\mathbf{G}$-socle, unless we have $L(\nu) = \mathrm{St}_r$.

Proposition 8.15. *Let* $\mathbf{G} = SL_3$ *and assume that* 3 *does not divide* $p^r - 1$*. If* $\nu \in X_r$ *with* $\nu \neq (p^r-1)\rho$ *and* $s_r(\nu) \neq 0$ *then* $f_r(\nu) = s_r(\nu)$*.*

Proof. Write $\nu = (\nu_1, \nu_2)$. If $s_r(\nu) \neq 0$ then by Proposition 7.7 together with Proposition 4.36 we see that 3 divides $\nu_1 + 2\nu_2$.

By Proposition 8.2 we see that

$$f_r(\nu) \leq \sum_{\lambda \in X_+} [\mathrm{St}_r \otimes \nabla(\nu) \otimes \nabla(\lambda) : \nabla((p^r-1)\rho + p^r\lambda)]_\nabla$$

and by Proposition 8.10 we see that if $\lambda = (\lambda_1, \lambda_2) \in X_+$ is given such that the corresponding summand is not 0, then $\nu_1 + 2\nu_2 - (p^r-1)(\lambda_1 + 2\lambda_2)$ is a nonnegative integer divisible by 3.

But since $\nu_1 + 2\nu_2$ is divisible by 3 and $p^r - 1$ is not divisible by 3, this means that $\lambda_1 + 2\lambda_2$ is divisible by 3, and since we have $\lambda_1 + \lambda_2 \leq h - 1 = 2$ by Corollary 8.5, this means that the only possibility is $\lambda = (1,1)$. But since $\nu \neq (p^r-1)\rho$ we have $\nu_1 + 2\nu_2 < 3(p^r-1) = (p^r-1)(\lambda_1 + 2\lambda_2)$, so we also do not get any contribution from this λ.

We note that we can apply both Theorem 7.8 and Theorem 8.3 since the requirements there of $p \geq 2h-2$ is only needed to ensure that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_r$ and for SL_3 we know this to hold without restrictions on p by the results in Section 6.10. Thus, by the above we have

$$f_r(\nu) = [\mathrm{St}_r \otimes L(\nu) : \mathrm{St}_r]_\nabla = s_r(\nu)$$

as claimed. □

We have now had several results about what happens when $p^r - 1$ is not divisible by 3. If we instead assume that 3 divides $p - 1$, then we see that the those $L(\nu)$ that occur in the G-socle of $\mathrm{St}_1 \otimes \mathrm{St}_1$ are precisely the same as those that occur in the $\mathbf{G}$-socle (when $\nu \in X_1$).

Proposition 8.16. *Let* $\mathbf{G} = SL_3$ *and assume that* $p \equiv 1 \pmod 3$*. Let* $\nu \in X_1$*. Then*

$$s_1(\nu) \neq 0 \Leftrightarrow f_1(\nu) \neq 0$$

Proof. Since we have $s_1(\nu) \leq f_1(\nu)$ for all $\nu \in X_1$, it is clear that if $s_1(\nu) \neq 0$ then $f_1(\nu) \neq 0$.

So assume that $f_1(\nu) \neq 0$. If $\nu = (\nu_1, \nu_2)$ then by Proposition 8.10 we must have that 3 divides $\nu_1 + 2\nu_2$, since by assumption 3 divides $p - 1$. Thus, we see that $s_1(\nu) \neq 0$ by Theorem 7.28. $\square$

A Appendix - Dominant Weights in the Root Lattice

In this appendix, we will prove that if $\beta \in \mathbb{Z}R$ is a non-zero dominant weight, then $\langle \beta, \alpha_0^\vee \rangle \geq 2$.

Since we assume β to be non-zero, this simply means that we need to rule out the possibility that $\langle \beta, \alpha_0^\vee \rangle = 1$. Since any dominant weight can be written as a non-negative integral linear combination of the fundamental weights, we see that this simply means ruling out certain of the fundamental weights being in the root lattice.

We will need the following description of $\alpha_0^\vee$. These have been calculated by taking the coroot of each of the highest short roots listed in [Hum78], p. 66 Table 2.

Type A_n: $\alpha_0^\vee = \alpha_1^\vee + \alpha_2^\vee + \cdots + \alpha_n^\vee$.

Type B_n: $\alpha_0^\vee = 2\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$.

Type C_n: $\alpha_0^\vee = \alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_n^\vee$.

Type D_n: $\alpha_0^\vee = \alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee$.

Type E_6: $\alpha_0^\vee = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 3\alpha_4^\vee + 2\alpha_5^\vee + \alpha_6^\vee$.

Type E_7: $\alpha_0^\vee = 2\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 3\alpha_5^\vee + 2\alpha_6^\vee + \alpha_7^\vee$.

Type E_8: $\alpha_0^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 5\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee + 2\alpha_8^\vee$.

Type F_4: $\alpha_0^\vee = 2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee$.

Type G_2: $\alpha_0^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee$.

In particular, looking at the expressions for $\alpha_0^\vee$ in each of the types of root system, we see that the fundamental weights we need to rule out are the ones corresponding to the simple coroots with a coefficient of 1 in $\alpha_0^\vee$:

Type A_n: We need to rule out all the fundamental weights.

Type B_n: We just need to rule out the fundamental weight ω_n.

Type C_n: We just need to rule out the fundamental weight ω_1.

Type D_n: We need to rule out the fundamental weights ω_1, ω_{n-1} and ω_n.

Type E_6: We need to rule out the fundamental weights ω_1 and ω_6.

Type E_7: We just need to rule out the fundamental weight ω_7.

Type E_8: We don't need to rule out anything here.

Type F_4: We don't need to rule out anything here.

Type G_2: We don't need to rule out anything here.

A.1 Type A_n

We need to show that no fundamental weight can be written as an integral linear combination of the simple roots. Since any weight can be written in a unique way as a linear combination of the simple roots, this means we just need to write $\omega_i = \sum_{j=1}^{n} a_{i,j}\alpha_j$ and show that at least one $a_{i,j}$ is not an integer. Let $a_{i,0} = a_{i,n+1} = 0$ for convenience. We will determine the $a_{i,j}$ by regarding the inner products $\langle \omega_i, \alpha_l^\vee \rangle$ one at a time, starting from $l = 1$ and using them to express $a_{i,j+1}$ in terms of $a_{i,j}$, which will give an expression for $a_{i,j+1}$ in terms of $a_{i,1}$. When we then do this for all l, we end up with some equations that determine $a_{i,1}$.

We see that for $l < i$ we have $0 = \langle \omega_i, \alpha_l^\vee \rangle = 2a_{i,l} - a_{i,l-1} - a_{i,l+1}$ so $a_{i,l+1} = 2a_{i,l} - a_{i,l-1}$ and this gives $a_{i,l+1} = (l+1)a_{i,1}$. We also have $1 = \langle \omega_i, \alpha_i^\vee \rangle = 2a_{i,i} - a_{i,i-1} - a_{i,i+1}$ which means that $a_{i,i+1} = 2a_{i,i} - a_{i,i-1} - 1 = (i+1)a_{i,1} - 1$. For $l > i$ we similarly get $a_{i,l+1} = 2a_{i,l} - a_{i,l-1}$, but this time, this gives $a_{i,l+1} = (l+1)a_{i,1} - (l-i+1)$.

Combining these, we see that we get $a_{i,n} = na_{i,1} - (n-i)$. We now need to look at the final inner product, $\langle \omega_i, \alpha_n^\vee \rangle$, which gives

$$0 = 2a_{i,n} - a_{i,n-1} = 2(na_{i,1} - (n-i)) - ((n-1)a_{i,1} - (n-1-i)) = (n+1)a_{i,1} - (n+1-i)$$

if $i \neq n$ and

$$1 = 2a_{n,n} - a_{n,n-1} = 2na_{n,1} - (n-1)a_{n,1} = (n+1)a_{n,1}$$

if $i = n$, which gives $a_{i,1} = \frac{n+1-i}{n+1}$ in both cases, and this is clearly not an integer for any i.

A.2 Type B_n

Keep the notation from the type A_n calculations. We are now only interested in the $i = n$ case, so write $a_j = a_{n,j}$ for simplicity. We can use the same calculations as before to see that we get $a_j = ja_1$ for all j, since this did not include any inner product of the form $\langle \cdot, \alpha_n^\vee \rangle$. Now we use that we have $1 = \langle \omega_n, \alpha_n^\vee \rangle = 2a_n - 2a_{n-1} = 2(a_n - a_{n-1}) = 2(na_1 - (n-1)a_1) = 2a_1$ and hence $a_1 = \frac{1}{2}$, which shows that ω_n is not in the root lattice in this case, as desired.

A.3 Type C_n

Keep the notation from type A_n but write $a_j = a_{1,j}$ as we are only interested in the $i = 1$ case. The same calculations as before show that $a_j = ja_1 - (j-1)$ for $j \leq n-1$, and we then continue by using $0 = \langle \omega_1, \alpha_{n-1}^\vee \rangle = 2a_{n-1} - a_{n-2} - 2a_n$ so we get

$$a_n = \frac{1}{2}(2a_{n-1} - a_{n-2}) = \frac{1}{2}(2(n-1)a_1 - 2(n-2) - ((n-2)a_1 - (n-3))) = \frac{1}{2}(na_1 - (n-1))$$

We then combine this with the final inner product to get

$$0 = \langle \omega_1, \alpha_n^\vee \rangle = 2a_n - a_{n-1} = na_1 - (n-1) - ((n-1)a_1 - (n-2)) = a_1 - 1$$

and hence $a_1 = 1$. Now we see that $a_n = \frac{1}{2}(n - (n-1)) = \frac{1}{2}$ and since this is not an integer, we are done.

A.4 Type D_n

Again keep the notation from type A_n. Here the cases of interest are those where $i \in \{1, n-1, n\}$. As before, we use the same calculations to get $a_{i,j} = ja_{i,1} - \delta_{i,1}(j-1)$ for $j \leq n-2$. We then use $0 = \langle \omega_i, \alpha_{n-2}^\vee \rangle = 2a_{i,n-2} - a_{i,n-3} - a_{i,n-1} - a_{i,n}$ which gives $a_{i,n-1} + a_{i,n} = (n-1)a_{i,1} - \delta_{i,1}(n-2)$. Using the final two inner products, we get $\delta_{i,n-1} = \langle \omega_i, \alpha_{n-1}^\vee \rangle = 2a_{i,n-1} - a_{i,n-2}$ so

$$a_{i,n-1} = \frac{1}{2}(a_{i,n-2} + \delta_{i,n-1}) = \frac{1}{2}((n-2)a_{i,1} - \delta_{i,1}(n-3) + \delta_{i,n-1})$$

and similarly, $a_{i,n} = \frac{1}{2}((n-2)a_{i,1} - \delta_{i,1}(n-3) + \delta_{i,n})$.

Adding the last two equalities and combining with the above gives

$$(n-1)a_{i,1} - \delta_{i,1}(n-2) = (n-2)a_{i,1} - \delta_{i,1}(n-3) + \frac{1}{2}(\delta_{i,n-1} + \delta_{i,n})$$

and hence $a_{i,1} = \delta_{i,1} + \frac{1}{2}(\delta_{i,n-1} + \delta_{i,n})$ and the only way this can be an integer is if $i = 1$ in which case we get $a_{i,1} = 1$. But then $a_{i,n-1} = \frac{1}{2}((n-2) - (n-3)) = \frac{1}{2}$ and since this is not an integer, we are done.

A.5 Type E_6

This time the calculations will be somewhat different than the previous ones, due to the ordering of the roots. We will therefore consider the inner products in a different order. The cases of interest are those where $i \in \{1, 6\}$.

So first we compute $\delta_{i,1} = \langle \omega_i, \alpha_1^\vee \rangle = 2a_{i,1} - a_{i,3}$ which gives $a_{i,3} = 2a_{i,1} - \delta_{i,1}$. Next, we consider $0 = \langle \omega_i, \alpha_3^\vee \rangle = 2a_{i,3} - a_{i,1} - a_{i,4}$ which gives $a_{i,4} = 2a_{i,3} - a_{i,1} = 3a_{i,1} - 2\delta_{i,1}$ and then we look at $0 = \langle \omega_i, \alpha_4^\vee \rangle = 2a_{i,4} - a_{i,2} - a_{i,3} - a_{i,5}$ giving $a_{i,2} + a_{i,5} = 4a_{i,1} - 3\delta_{i,1}$. Combining this with $0 = \langle \omega_i, \alpha_2^\vee \rangle = 2a_{i,2} - a_{i,4}$ which gives $a_{i,2} = \frac{1}{2}a_{i,4} = \frac{1}{2}(3a_{i,1} - 2\delta_{i,1})$, we get that we have $a_{i,5} = 4a_{i,1} - 3\delta_{i,1} - \frac{1}{2}(3a_{i,1} - 2\delta_{i,1}) = \frac{1}{2}(5a_{i,1} - 4\delta_{i,1})$. We then continue by using that we have $0 = \langle \omega_i, \alpha_5^\vee \rangle = 2a_{i,5} - a_{i,4} - a_{i,6}$ giving $a_{i,6} = 2a_{i,5} - a_{i,4} = 3a_{i,1} - 2\delta_{i,1}$ and finally $\delta_{i,6} = \langle \omega_i, \alpha_6^\vee \rangle = 2a_{i,6} - a_{i,5}$ giving $a_{i,6} = \frac{1}{2}(a_{i,5} + \delta_{i,6}) = \frac{1}{4}(5a_{i,1} - 4\delta_{i,1} + 2\delta_{i,6})$. Combining these, we get $5a_{i,1} - 4\delta_{i,1} + 2\delta_{i,6} = 12a_{i,1} - 8\delta_{i,1}$, so $a_{i,1} = \frac{1}{7}(4\delta_{i,1} + 2\delta_{i,6})$ which is not an integer for $i \in \{1, 6\}$.

A.6 Type E_7

Here we can reuse the calculations from type E_6, but the case of interest is $i = 7$, so write $a_j = a_{7,j}$.

This means we get $a_5 = \frac{3}{2}a_1$ and $a_6 = 3a_1$ (since $\delta_{i,1} = 0$), and we continue by considering $0 = \langle \omega_7, \alpha_6^\vee \rangle = 2a_6 - a_5 - a_7$ giving $a_7 = 2a_6 - a_5 = \frac{9}{2}a_1$. Finally, we have $1 = \langle \omega_7, \alpha_7^\vee \rangle = 2a_7 - a_6$ giving $a_7 = \frac{1}{2}(a_6 + 1) = \frac{1}{2}(3a_1 + 1)$. Combining these, we get $3a_1 + 1 = 9a_1$ so $a_1 = \frac{1}{6}$ which is not an integer.

A.7 Summary

We now summarize the calculations of this appendix in the following.

Proposition A.1. *If $\beta \in \mathbb{Z}R$ is a non-zero dominant weight, then $\langle \beta, \alpha_0^\vee \rangle \geq 2$.*

B Appendix - Tensor Products of Simple Modules for $GL_n(\mathbb{C})$

In this section the combinatorial concepts needed in order to compute the decomposition of tensor products of simple $GL_n(\mathbb{C})$-modules will be introduced.

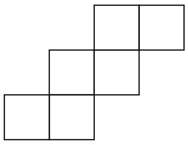
In this section, a partition will be a non-increasing sequence of non-negative integers containing 0. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition, we set $|\lambda| = \sum_i \lambda_i$ and write $\lambda \vdash |\lambda|$. Let $l(\lambda) = \max\{i \mid \lambda_i \neq 0\}$ and call this the length of λ.

If λ and μ are partitions, we say that μ is contained in λ and write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i.

Definition B.1. Let λ and μ be partitions with $\mu \subseteq \lambda$.A semistandard young tableau (from now on abbreviated SSYT) of skew-shape λ/μ is a 2-dimensional array of positive integers $T_{i,j}$ with $1 \leq i \leq l(\lambda)$ and $\mu_i + 1 \leq j \leq \lambda_i$ such that whenever the numbers are defined, $T_{i,j} \leq T_{i+1,j}$ and $T_{i,j} < T_{i,j+1}$.

We will write an SSYT graphically by putting the array T inside an $l(\lambda) \times \lambda_1$ box, where for each $T_{i,j}$ which exists, we put a box with $T_{i,j}$ in it.

Example B.2. If $\lambda = (4, 3, 2)$ and $\mu = (2, 1)$ then an SSYT of shape λ/μ will look like (before we put in the values)



Example B.3. A possible full SSYT of same shape as in Example B.2 could look like

		1	1
	1	2	
3	4		

Note that when looking at an SSYT like this, the requirements become that each row is non-decreasing, and each column is strictly increasing.

Definition B.4. If T is an SSYT, we define the type of T to be the sequence $\alpha(T) = (\alpha_1, \alpha_2, \dots)$ where $\alpha_m = |\{(i,j) \mid T_{i,j} = m\}|$

Example B.5. If T is the SSYT from Example B.3 then $\alpha(T) = (3, 1, 1, 1)$.

Definition B.6. If T is an SSYT of skew-shape λ/μ with $l(\lambda) = n$, we define the reverse reading word of T to be the sequence $(T_{1,\lambda_1}, T_{1,\lambda_1 - 1}, \dots, T_{1,\mu_1+1}, T_{2,\lambda_2}, \dots, T_{n,\lambda_n}, \dots, T_{n,\mu_n+1})$. That is, the sequence obtained by reading the entries in T one row at a time from *right to left* and top to bottom.

Example B.7. If T is the SSYT from Example B.3 then the reverse reading word of T is $(1, 1, 2, 1, 4, 3)$.

Definition B.8. A lattice permutation is a sequence of positive integers $(x_1, x_2, \dots, x_n)$ such that for any positive integer i and any subsequence of the form $x = (x_1, \dots, x_k)$, i appears in x at least as many times as $i + 1$ does.

Example B.9. If T is the SSYT from Example B.3 then the reverse reading word of T (Example B.7) is not a lattice permutation, since there is a 4 before the first 3.

If instead we take T to be the SSYT

		1	1
	1	2	
2	3		

then the reverse reading word becomes $(1, 1, 2, 1, 3, 2)$ which is a lattice permutation.

Definition B.10. Let λ, μ, ν be partitions. Define the Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ as follows:

If $\lambda \nsubseteq \nu$ then $c^{\nu}_{\lambda,\mu} = 0$. Otherwise, it is the number of SSYT T of skew-shape ν/λ and type μ such that the reverse reading word of T is a lattice permutation.

The above is not the usual definition of the Littlewood-Richardson coefficients, which are instead defined in terms of decomposition of certain symmetric functions, but due to Theorem A1.3.3 in [Sta99], the above definition is equivalent to the usual one.

One feature of these coefficients that is not quite obvious from this definition, however, it that $c^{\nu}_{\lambda,\mu} = c^{\nu}_{\mu,\lambda}$ which is clear from the usual definition.

The reason these coefficients will be interesting is that the simple modules for $GL_n(\mathbb{C})$ can be indexed by tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geq \lambda_{i+1}$ for all i (such a λ corresponds to a dominant weight for $GL_n(\mathbb{C})$ and the simple module is just $L(\lambda)$ which, since this is over $\mathbb{C}$, is also $\nabla(\lambda)$). If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a dominant weight and $\overline{\lambda} = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0)$ then it is easy to see that $L(\lambda) \cong L(\overline{\lambda}) \otimes_{\mathbb{C}} \mathrm{Det}^{\lambda_n}$ where Det^i is the 1-dimensional $GL_n(\mathbb{C})$-module where $g \in GL_n(\mathbb{C})$ acts by the scalar $\det(g)^i$.

Since it is also easy to check that the tensor product of two simple $GL_n(\mathbb{C})$-modules is semisimple (for example by reducing to $SL_n(\mathbb{C})$ and using that the center of $GL_n(\mathbb{C})$ acts semisimply), we would like to be able to decompose a tensor product $L(\lambda) \otimes_{\mathbb{C}} L(\mu)$ as a direct sum of $L(\nu)$ for suitable dominant weights ν. By the above remark, we can assume that λ and μ are partitions (by tensoring with suitable powers of the determinant module), and it turns out (see [Sta99] Appendix 2) that if λ and μ are partitions, then

$$L(\lambda) \otimes_{\mathbb{C}} L(\mu) \cong \bigoplus_{\nu \vdash (|\lambda|+|\mu|),\, l(\nu) \leq n} c^{\nu}_{\lambda,\mu} L(\nu)$$

so we can in fact determine this decomposition completely, as long as we can calculate all the Littlewood-Richardson coefficients.

Example B.11. Let us consider the example $L(\lambda)$ where $\lambda = (1, 0, \ldots, 0)$. It is easy to see that this is simply $\mathbb{C}^n$ with the usual matrix multiplication as the action of $GL_n(\mathbb{C})$. If we want to calculate $L(\lambda) \otimes_{\mathbb{C}} L(\lambda)$ then the above rules tell us we need to find partitions μ such that $\lambda \subseteq \mu$ (this holds for any non-zero partition), $|\mu| = 2|\lambda| = 2$ and such that there exists an SSYT of skew-shape μ/λ of type λ whose reverse reading word is a lattice permutation.

Since the reverse reading word will in this case only contain one element, the requirement that it be a lattice permutation is simply that that element is a 1, which is already automatic since the type had to be $\lambda = (1, 0, \ldots, 0)$.

So we see that any partition $\mu \vdash 2$ works, and also that for any such μ, $c^{\mu}_{\lambda,\lambda} = 1$.

We conclude that $L(\lambda) \otimes_{\mathbb{C}} L(\lambda) \cong L(\mu) \oplus L(\nu)$ where $\mu = (2, 0, \ldots, 0)$ and $\nu = (1, 1, 0, \ldots, 0)$ (as long as $n \geq 2$. If $n = 1$ then $L(\lambda)$ is the trivial module and so is the tensor product considered).

Now, if $n \geq 2$ then we have here a decomposition into two simple modules, and in Example 3.21 we also have a decomposition into two $GL_n(\mathbb{C})$-submodules, namely the modules $S^2(M)$ and $\Lambda^2(M)$. This must then mean that those modules are irreducible (when $K = \mathbb{C}$) and that one of them is isomorphic to $L(\mu)$ and the other is isomorphic to $L(\nu)$. By looking at how the diagonal matrices in $GL_n(\mathbb{C})$ act, we can see that $S^2(M) \cong L(\mu)$ and $\Lambda^2(M) \cong L(\nu)$.

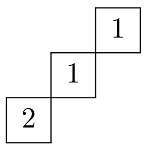
Example B.12. Let us consider an example, which is slightly more complicated. We consider $\lambda = (m, 0, \ldots, 0)$ and $\mu = (1, 1, \ldots, 1, 0, \ldots, 0)$ (l 1's). To decompose $L(\lambda) \otimes_{\mathbb{C}} L(\mu)$ we need to find those partitions ν such that $|\nu| = |\lambda| + |\mu| = m + l$, $\lambda \subseteq \nu$ and such that there exists an SSYT T of skew-shape ν/λ of type μ such that the reverse reading word of T is a lattice permutation. On the other hand, as noted after defining the Littlewood-Richardson coefficients, we can also find those ν such that $\mu \subseteq \nu$ and such that there is an SSYT T of skew-shape ν/μ and type λ, and such that the reverse reading word of T is a lattice permutation.

If we choose to look at it in the second way, we see that since the desired type of the SSYT is λ, it must consist of only 1's, so since it has to be an SSYT, this means that whenever $T_{i,j}$ is defined, $T_{i+1,j}$ is not, so this must mean that we have either $\nu = (m, 1, \ldots, 1, 0, \ldots, 0)$ (l 1's) or $\nu = (m + 1, 1, \ldots, 1, 0, \ldots, 0)$ ($l - 1$ 1's).

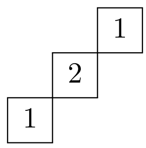
In both cases, we see that the coefficient is 1, as there can never be more than one SSYT of a given shape and type λ, so we get that $L(\lambda) \otimes_{\mathbb{C}} L(\mu) \cong L(\nu_1) \oplus L(\nu_2)$ where $\nu_1 = (m, 1, \ldots, 1, 0, \ldots, 0)$ and $\nu_2 = (m + 1, 1, \ldots, 1, 0, \ldots, 0)$ are the possibilities from above.

Once again, we need to be careful if $n = l$ (it does not make sense to look at the module $L(\mu)$ if $n < l$), as in that case, one of those modules does not occur. But in that case, the module $L(\mu)$ can easily be seen to be the determinant module, and the tensor product is indeed irreducible in that case.

In the above examples the Littlewood-Richardson coefficients appearing were all 0 or 1. This is not the case in general, as for example there are two $SSYT$ of skew-shape ν/λ and type μ if $\lambda = \mu = (2, 1)$ and $\nu = (3, 2, 1)$, namely



and



Let us now consider what happens for $SL_n(\mathbb{C})$ instead of $GL_n(\mathbb{C})$. Here the simple modules have the form $L(\lambda)$ where λ is a partition with $l(\lambda) \leq n - 1$. On the other hand, if we take the corresponding simple $GL_n(\mathbb{C})$-module $L(\lambda)$ and view it as an $SL_n(\mathbb{C})$-module, then it stays simple, so we can use the same tools as for $GL_n(\mathbb{C})$ to decompose tensor products of simple $SL_n(\mathbb{C})$-modules. We just need to know what happens to a simple $GL_n(\mathbb{C})$-module $L(\lambda)$ when reduced to $SL_n(\mathbb{C})$ when λ is not a partition of length at most $n - 1$. But as noted previously, we have $L(\lambda) \cong L(\overline{\lambda}) \otimes_{\mathbb{C}} \mathrm{Det}^{\lambda_n}$ where $\overline{\lambda} = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0)$ is a partition of length at most $n - 1$. Since the determinant module for $GL_n(\mathbb{C})$ reduces to the trivial module for $SL_n(\mathbb{C})$, this means that as $SL_n(\mathbb{C})$-modules, $L(\lambda) \cong L(\overline{\lambda})$.

Example B.13. Let us consider the $SL_n(\mathbb{C})$-module $\mathbb{C}^n$ with the usual matrix multiplication as the action. As in the previous example, it is easy to see that this module is isomorphic to $L(\lambda)$ where

$\lambda = (1, 0, \ldots, 0)$, so we can compute $L(\lambda) \otimes_{\mathbb{C}} L(\lambda)$ in the same way as before and get $L(\mu) \oplus L(\nu)$ with $\mu = (2, 0, \ldots, 0)$ and $\nu = (1, 1, 0, \ldots, 0)$. If $n \geq 3$ then this is also the decomposition as $SL_n(\mathbb{C})$-modules, but if $n = 2$ then the module $L(\nu)$ has to be identified with the module $L(0)$ (which is the trivial module) as described above, which fits the earlier remarks that this is in fact the determinant module for $GL_n(\mathbb{C})$.

Example B.14. Let us consider the simple $SL_n(\mathbb{C})$-modules $L(\lambda)$ and $L(\mu)$ with $\lambda = (m, 0, \ldots, 0)$ and $\mu = (1, 1, \ldots, 1, 0, \ldots, 0)$ (l 1's). As before, we get that $L(\lambda) \otimes_{\mathbb{C}} L(\mu) \cong L(\nu_1) \oplus L(\nu_2)$ where $\nu_1 = (m, 1, \ldots, 1, 0, \ldots,)$ (l 1's) and $\nu_2 = (m + 1, 1, \ldots, 1, 0, \ldots, 0)$ ($l - 1$ 1's).

If $n \geq l + 2$ this is again the decomposition as $SL_n(\mathbb{C})$-modules, but if $n = l + 1$ (if $n < l + 1$ it does not make sense to consider the module $L(\mu)$ in the first place), then $L(\nu_1) \cong L(\nu_3)$ with $\nu_3 = (m - 1, 0, \ldots, 0)$ as $SL_n(\mathbb{C})$-modules, so in this case we get that the decomposition becomes $L(\lambda) \otimes_{\mathbb{C}} L(\mu) \cong L(\nu_2) \oplus L(\nu_3)$.

References

[AJL83] Henning Haahr Andersen, Jens Jørgensen, and Peter Landrock. The projective indecomposable modules of SL(2, p^n). *Proc. London Math. Soc. (3)*, 46(1):38–52, 1983.

[And80a] Henning Haahr Andersen. The Frobenius morphism on the cohomology of homogeneous vector bundles on G/B. *Ann. of Math. (2)*, 112(1):113–121, 1980.

[And80b] Henning Haahr Andersen. The strong linkage principle. *J. Reine Angew. Math.*, 315:53–59, 1980.

[And83] Henning Haahr Andersen. Filtrations of cohomology modules for Chevalley groups. *Ann. Sci. École Norm. Sup. (4)*, 16(4):495–528 (1984), 1983.

[And01] Henning Haahr Andersen. p-filtrations and the Steinberg module. *J. Algebra*, 244(2):664–683, 2001.

[BDM11a] C. Bowman, S. R. Doty, and S. Martin. Decomposition of tensor products of modular irreducible representations for SL_3. *Int. Electron. J. Algebra*, 9:177–219, 2011. With an appendix by C. M. Ringel.

[BDM11b] C. Bowman, S. R. Doty, and S. Martin. Decomposition of tensor products of modular irreducible representations for SL_3: The $p \geq 5$ case. *arXiv:1111.5811 [math.RT]*, 2011.

[BNP12] Christopher P. Bendel, Daniel K. Nakano, and Cornelius Pillen. On the vanishing ranges for the cohomology of finite groups of Lie type. *Int. Math. Res. Not. IMRN*, (12):2817–2866, 2012.

[Car93] Roger W. Carter. *Finite groups of Lie type.* Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[Cha81] Leonard Chastkofsky. Projective characters for finite Chevalley groups. *J. Algebra*, 69(2):347–357, 1981.

[Cur60] Charles W. Curtis. Representations of Lie algebras of classical type with applications to linear groups. *J. Math. Mech.*, 9:307–326, 1960.

[DH05] Stephen Doty and Anne Henke. Decomposition of tensor products of modular irreducibles for SL_2. *Q. J. Math.*, 56(2):189–207, 2005.

[Don81] Stephen Donkin. A filtration for rational modules. *Math. Z.*, 177(1):1–8, 1981.

[Don88] Stephen Donkin. Skew modules for reductive groups. *J. Algebra*, 113(2):465–479, 1988.

[Don93] Stephen Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[Dru13] Christopher M. Drupieski. On projective modules for Frobenius kernels and finite Chevalley groups. *Bull. Lond. Math. Soc.*, 45(4):715–720, 2013.

[Hag83] Jan H. Hagelskjær. *Induced modules for algebraic groups.* Speciale. Aarhus University, 1983.

[HSTZ13] Gerhard Heide, Jan Saxl, Pham Huu Tiep, and Alexandre E. Zalesski. Conjugacy action, induced representations and the Steinberg square for simple groups of Lie type. *Proc. Lond. Math. Soc. (3)*, 106(4):908–930, 2013.

[Hum76] James E. Humphreys. *Ordinary and modular representations of Chevalley groups.* Lecture Notes in Mathematics, Vol. 528. Springer-Verlag, Berlin-New York, 1976.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978. Second printing, revised.

[Hum06] James E. Humphreys. *Modular representations of finite groups of Lie type*, volume 326 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[Isa76] I. Martin Isaacs. *Character theory of finite groups.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Pure and Applied Mathematics, No. 69.

[Jan73] Jens Carsten Jantzen. Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen. *Bonn. Math. Schr.*, (67):v+124, 1973.

[Jan80] Jens Carsten Jantzen. Weyl modules for groups of Lie type. In *Finite simple groups. II, Proceedings of the Symposium held at the University of Durham, Durham, July 31–August 10, 1978*, pages 291–300. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1980.

[Jan81] Jens Carsten Jantzen. Zur Reduktion modulo p der Charaktere von Deligne und Lusztig. *J. Algebra*, 70(2):452–474, 1981.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[Kem76] George R. Kempf. Linear systems on homogeneous spaces. *Ann. of Math. (2)*, 103(3):557–591, 1976.

[KN14] Tobias Kildetoft and Daniel K. Nakano. On good (p, r)-filtrations for rational G-modules. *arXiv:1403.7011 [math.GR]*, 2014.

[Mat90] Olivier Mathieu. Filtrations of G-modules. *Ann. Sci. École Norm. Sup. (4)*, 23(4):625–644, 1990.

[Rin91] Claus Michael Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991.

[Spr09] T. A. Springer. *Linear algebraic groups.* Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, second edition, 2009.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ste63] Robert Steinberg. Representations of algebraic groups. *Nagoya Math. J.*, 22:33–56, 1963.

[Tsu90] Yukio Tsushima. On certain projective modules for finite groups of Lie type. *Osaka J. Math.*, 27(4):947–962, 1990.

[WW11] Jinkui Wan and Weiqiang Wang. The $\mathrm{GL}_n(q)$-module structure of the symmetric algebra around the Steinberg module. *Adv. Math.*, 227(4):1562–1584, 2011.