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**Abstract**

Let $X$ be a Lévy process and $V$ the reflection at boundaries 0 and $b > 0$. A number of properties of $V$ are studied, with particular emphasis on the behaviour at the upper boundary $b$. The process $V$ can be represented as solution of a Skorokhod problem $V(t) = V(0) + X(t) + L(t) - U(t)$ where $L, U$ are the local times (regulators) at the lower and upper barrier. Explicit forms of $V$ in terms of $X$ are surveyed as well more pragmatic approaches to the construction of $V$, and the stationary distribution $\pi$ is characterised in terms of a two-barrier first passage problem. A key quantity in applications is the loss rate $\ell^b$ at $b$, defined as $\mathbb{E}_\pi U(1)$. Various forms of $\ell^b$ and various derivations are presented, and the asymptotics as $b \to \infty$ is exhibited in both the light-tailed and the heavy-tailed regime. The drift zero case $\mathbb{E}X(1) = 0$ plays a particular role, with Brownian or stable functional limits being a key tool. Further topics include studies of the first hitting time of $b$, central limit theorems and large deviations results for $U$, and a number of explicit calculations for Lévy processes where the jump part is compound Poisson with phase-type jumps.

*Keywords:* Applied probability, Central limit theorem, Finite buffer problem, First passage problem, Functional limit theorem, Heavy tails, Integro-differential equation, Itô's formula, Linear equations, Local time, Loss rate, Martingale, Overflow, Phase-type distribution, Poisson's equation, Queueing theory, Siegmund duality, Skorokhod problem, Storage process

# Contents

# 1 Introduction

This article is concerned with a one-dimensional Lévy process $X = \{X(t)\}_{t\geq 0}$ reflected at two barriers $0, b$ and is a mixture of a literature survey and new results or proofs.

We denote the two-sided reflected process by $V = \{V(t)\}_{t\geq 0}$ (or $V^b$, when the dependence on $b$ needs to be stressed). The discrete time counterpart of $V$ is a two-sided reflected random walk $\big(V_n : n = 0, 1, 2, \ldots\big)$ defined by

$$V_n = \min\big(b, \max(0, V_{n-1} + Y_n)\big) \tag{1.1}$$

where $Y_1, Y_2, \ldots$ are i.i.d. (with common distribution say $F$) and initial condition $V_0 = v$ for some $v \in [0, b]$; the role of the Lévy process $X$ is then taken by the random walk $X_n = Y_1 + \cdots + Y_n$.

The study of such processes in discrete or continuous time has a long history and numerous applications. For a simple example, consider the case $X(t) = \sum_1^{N(t)} Z_i - ct$ of a compound Poisson process with drift, where $N$ is Poisson($\lambda$) and the $Z_i$ independent of $N$, i.i.d. and non-negative. Here one can think of $V$ as the amount of work in a system with a server working at rate $c$, jobs arriving at Poisson rate $\lambda$ and having sizes $Z_1, Z_2, \ldots$, and finite capacity $b$ of storage. If a job of size $y > b - x$ arrives when the content is $x$, only $b - x$ of the job is processed and $x + y - b$ is lost. One among many examples is a data buffer, where the unit is number of bits (discrete in nature, but since both $b$ and a typical job size $z$ are huge, a continuous approximation is motivated).

Studies of systems with such finite capacity are numerous, and we mention here waiting time processes in queues with finite capacity ([44], [45], [25], [47]), and a finite dam or fluid model ([11], [103], [122]). They are used in models of network traffic or telecommunications systems involving a finite buffer ([74], [130], [86]), and they also occur in finance, e.g. [59], [56]. In the queueing context, it should be noted that even if in the body of literature, there is no upper bound $b$ on the state space, the reason is mainly mathematical convenience: the analysis of infinite-buffer systems is in many respects substantially simpler than that of finite-buffer systems. In real life, infinite waiting rooms or infinite buffers do not occur, so that the infinite-buffer assumption is really just an approximation.

In continuous time, there is no obvious analogue of the defining equation (1.1). We follow here the tradition of representation as solution of a Skorokhod problem

$$V(t) = V(0) + X(t) + L(t) - U(t) \tag{1.2}$$

where $L, U$ are non-decreasing right-continuous processes such that

$$\int_0^\infty V(t)\,\mathrm{d}L(t) = 0\,, \quad \int_0^\infty \big(b - V(t)\big)\,\mathrm{d}U(t) = 0\,. \tag{1.3}$$

In other words, $L$ can only increase when $V$ is at the lower boundary 0, and $L$ only when $V$ is at the upper boundary $b$. Thus, $L$ represents the 'pushing up from 0' that is needed to keep $V(t) \geq 0$ for all $t$, and $U$ represents the 'pushing down from $b$' that is needed to keep $V(t) \leq b$ for all $t$. An illustration is given in Fig. 1, with the unreflected Lévy process in the upper panel, whereas the lower panel has the two-sided reflected process $V$ (blue) in the middle subpanel, $L$ (red) in the lower and $U$ (green) in the upper. Questions of existence and uniqueness are discussed in Sections $3, 4$.

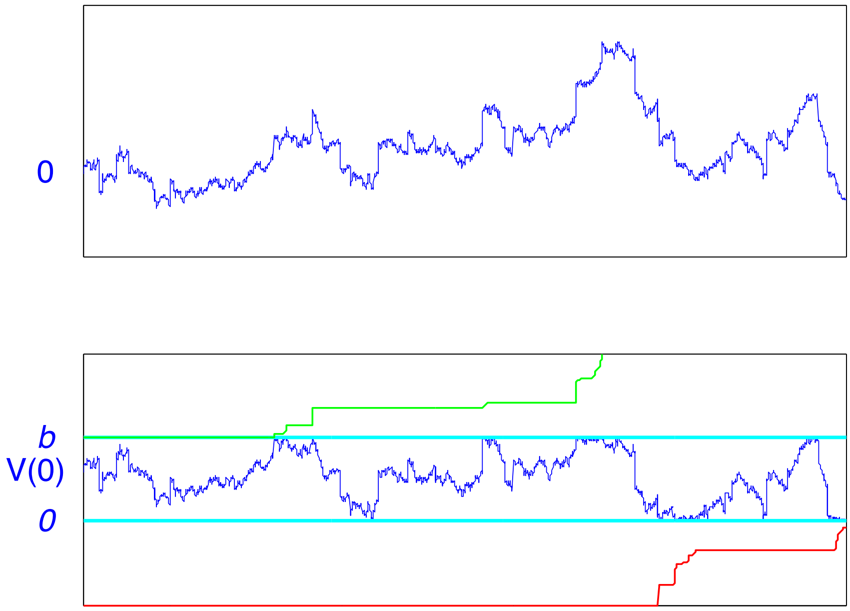


**Figure 1:** The processes $X, V$, $L$, $U$

As usual in applied probability, a first key question in the study of $V$ is the long-run behavior. A trivial case is monotone sample paths. For example if in continuous time the underlying Lévy process $X$ has non-decreasing and non-constant sample paths, then $V(t) = b$ for all large $t$. Excluding such degenerate cases, $V$ regenerates at suitable visits to 0 (see Section 5.1 for more detail), and a geometric trial argument easily gives that the mean regeneration time is finite. Thus by general theory ([11]), a stationary distribution $\pi = \pi^b$ exists and

$$\frac{1}{N+1}\sum_{n=0}^{N} f(V_n) \to \pi(f)\,, \quad \frac{1}{T}\int_0^T f(V_t)\,\mathrm{d}t \to \pi(f) \tag{1.4}$$

a.s. in discrete, resp. continuous time whenever $f$ is (say) bounded or non-negative. A further fundamental quantity is the overflow or loss at $b$ which is highly relevant for applications; in the dam context, it represents the amount of lost water and in the data buffer context the number of lost bits. The long-run behavior in discrete time is given by

$$\frac{1}{N}\sum_{n=1}^{N} f\big(Y_n + V_{n-1} - b\big)^+ \quad \to \quad \int_0^b \pi(\mathrm{d}x)\int_{b-x}^{\infty} f(y+x-b)\,F(\mathrm{d}y)\,, \tag{1.5}$$

as follows by conditioning on $Y_n = y$ and using (1.4). We denote by $\ell = \ell^b$ the limit on the r.h.s. of (1.5) and refer to it as the *loss rate*. For example, in data transmission models the loss rate can be interpreted as the bit loss rate in a finite data buffer.

The form of $\pi$ in the general continuous-time Lévy case is discussed in Section 5. In general, $\pi$ is not explicitly available (sometimes the Laplace transform is). The key result for us is a representation as a two-sided exit probability,

$$\pi[x,\infty) = \pi[x,b] = \mathbb{P}\big(V\big(\tau[x-b,x)\big) \geq x\big) \tag{1.6}$$

where $\tau[y,x) = \inf\big\{t \geq 0 :\ X(t) \notin [y,x)\big\}$, $y \leq 0 \leq x$.

In continuous time, the obvious definition of the loss rate is $\ell = \mathbb{E}_\pi U(1) = \mathbb{E}_\pi U(t)/t$. However, representations like (1.5) are not apriori obvious, except for special cases as $X$ being compound Poisson where

$$\ell = \int_0^b \pi(\mathrm{d}x) \int_{b-x}^\infty (x+y-b)\lambda\, G(\mathrm{d}y)\,,$$

where $\lambda$ is the Poisson rate and $G$ the jump size distribution. To state the main result, we need to introduce the basic Lévy setup:

$$X(t) = ct + \sigma B(t) + J(t)\,,$$

where $B$ is standard Brownian motion and $J$ an independent jump process with Lévy measure $\nu$ and jumps of absolute size $\leq 1$ compensated. That is, the Lévy exponent

$$\kappa(\alpha) = \log \mathbb{E}\mathrm{e}^{\alpha X(1)} = \frac{1}{t}\log \mathbb{E}\mathrm{e}^{\alpha X(t)}$$

is given by

$$\kappa(\alpha) = c\alpha + \sigma^2\alpha^2/2 + \int_{-\infty}^\infty \big(\mathrm{e}^{\alpha y} - 1 - y\mathbb{1}(|y| \leq 1)\,\nu(\mathrm{d}y)\,, \tag{1.7}$$

and one often refers to $(c, \sigma^2, \nu)$ as the *characteristic triplet* of $X$ (see the end of the section for further detail and references). We further write

$$m = \mathbb{E}X(1) = \mathbb{E}X(t)/t = \kappa'(0) = c + \int_{|y|>1} y\,\nu(\mathrm{d}y)$$

for the mean drift of $X$.

**Theorem 1.1.** *Assume that $m$ is well-defined and finite. Then*

$$\ell^b = \frac{1}{2b}\big\{2m\,\mathbb{E}V + \sigma^2 + \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^\infty \varphi(x,y)\nu(\mathrm{d}y)\big\}\,,$$

*where*

$$\varphi(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y \leq -x, \\ y^2 & \text{if } -x < y < b-x, \\ 2y(b-x)-(b-x)^2 & \text{if } y \geq b-x. \end{cases}$$

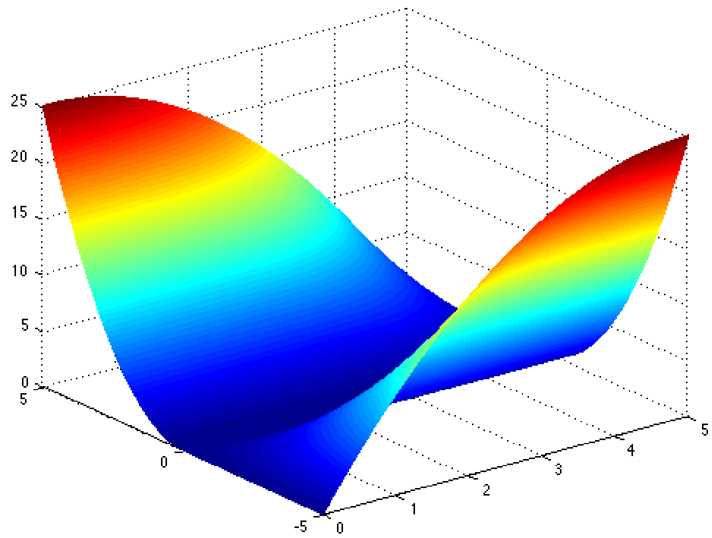


**Figure 2:** The function $\varphi(x, y)$ plotted for $b = 5$, $0 \le x \le b$, $-5 \le y \le 5$

Theorem 1.1 first appears in Asmussen & Pihlsgård [19], with a rather intricate and lengthy proof. Section 6 contains a more direct and shorter proof originating from Pihlsgård & Glynn [108]. In Section 8, we summarize the original approach of [19], and in Section 15, some new representations of $\ell^b$ are presented using yet another approach. Whereas the method in Section 8 uses asymptotic expansions of identities obtained by martingale optional stopping, the ones in Sections 6 and 15 contain stochastic calculus as a main ingredient.

Starting from Theorem 1.1, it is fairly straightforward to derive an alternative formula for $\ell^b$, which can be convenient (note that the form (1.6) for the tail probability $\pi[x, b]$ has a nicer form than the one for $\pi(\mathrm{d}x)$ that follows by differentiation).

**Corollary 1.2.** *The loss rate $\ell^b$ can be written*

$$\ell^b = \frac{1}{2b}\Big\{2m\mathbb{E}V + \sigma^2 + \int_0^b y^2\nu(\mathrm{d}y) + \int_b^\infty (2yb - b^2)\nu(\mathrm{d}y) \\ - 2\int_0^b \Big\{\int_{-\infty}^{-x} (x+y)\nu(\mathrm{d}y) + \int_{b-x}^\infty (x+y-b)\nu(\mathrm{d}y)\Big\}\pi[x, b]\mathrm{d}x\Big\}.$$

Similar discussion applies to the underflow of 0, but for obvious symmetry reasons, it suffices to consider the situation at the upper barrier. One should note, however, that with $\ell^0 = \mathbb{E}_\pi L(1)$ one has

$$0 = m + \ell^0 - \ell^b\,. \tag{1.8}$$

Thus, $\ell^0$ is explicit in terms of $\ell^b$. Relation (1.8) follows by a rate conservation principle, since in order for $X + L - U$ to preserve stationarity, the drift must be zero. One may note that no moment conditions on $X$ are needed for the existence of a stationary version of $V$. However, if $\mathbb{E}|X(1)| = \infty$, then one of $\ell^b$ or $\ell^0$ is infinite (or both are).

In many applications, the upper buffer size $b$ is large. This motivates that instead of going into the intricacies of exact computation of quantities like the loss rate $\ell^b$, one may look for approximate expression for $b \to \infty$. Early references in this direction iare Jelenković [74] who treated the random walk case with heavy tails, and Kim & Shroff [86], who considered the light-tailed case but only gave logarithmic asymptotics. Exact asymptotics for the light-tailed case is given in Asmussen & Pihlsgård [19] and surveyed in Section 10, whereas asymptotics for the heavy-tailed Lévy case first appears in Andersen [5] and is surveyed in Section 11. We assume negative drift, i.e. $m = \mathbb{E}X(1) < 0$, but by (1.8), the results can immediately be translated to positive drift. Note, however, that with negative drift one has $\ell^b \to 0$ as $b \to \infty$ (the results of Sections 10, 11 give the precise rates of decay), whereas with positive drift $\ell^b \to m$ (thus, (1.8) combined with Sections 10, 11 gives the convergence rate). The case of zero drift $m = 0$ has specific features as studied in Andersen & Asmussen [4], see Section 12; the key tool is here a functional limit theorem with either a Brownian or a stable process limit.

Going one step further in the study of the loss rate, one may ask for transient properties. One question is properties of the overflow time $\inf\{t > 0 : V^b(t) = b\}$ where one possible approach is regenerative process theory, Section 13 and another integro-differential equations, Section 14.2. Another question is properties of $U(t)$ for $t < \infty$. From the above, $U$ obeys the LLN $\mathbb{E}U(t)/t \to \ell$ as $t \to \infty$. Obvious questions are an associated CLT,

$$\sqrt{t}\big(U(t) - \ell\big) \to N(0, \sigma^2)$$

for some suitable $\sigma^2$, and large deviations properties like the asymptotics of

$$\mathbb{P}\big(U(t) > t(1+\epsilon)\ell\big) \quad \text{and} \quad \mathbb{P}\big(U(t) < t(1-\epsilon)\ell\big)\,.$$

These topics are treated (for the first time) in Sections 14.3 and 14.4.

Finally, the paper contains a number of explicit calculations for the special case where the jump part is compound Poisson with phase-type jumps. There is a considerable literature on this or closely related models. and we refer to Asmussen [12] for a survey and references. We have also included some material on one-sided reflection (Section 2) and two-sided reflection in discrete time (Section 3), which should serve both as a background and to give an understanding of the special problems that arise for the core topic of the paper, two-sided reflection in continuous time.

We conclude this introduction with some supplementary comments on the set-up on the Lévy model. Classical general references are Bertoin [28] and Sato [119], but see also Kyprianou [94] and Applebaum [7].

A simple case is jumps of bounded variation, which occurs if and only if

$$\int_{-\infty}^{\infty} |x|\, \nu(\mathrm{d}x) < \infty.$$

Then the expression (1.7) can be rewritten

$$\kappa(\alpha) = \widetilde{c}\alpha + \sigma^2\alpha/2 + \int_{-\infty}^{\infty} (\mathrm{e}^{\alpha y} - 1)\, \nu(\mathrm{d}y)\,, \tag{1.9}$$

where

$$\widetilde{c} = c - \int_{-1}^{1} y\,\nu(\mathrm{d}y)\,, \quad m = \widetilde{c} + \int_{-\infty}^{\infty} y\,\nu(\mathrm{d}y)\,. \tag{1.10}$$

With infinite variation, the integrals in (1.9) diverge, so that one needs the form (1.7).

To avoid trivialities, we assume throughout that the sample paths of $X$ are non-monotone; in terms of the parameters of $X$, this means that either

(a) $\sigma^2 > 0$,

(b) $\sigma^2 = 0$ and $X$ is of unbounded variation (i.e. $\int |y|\,\nu(\mathrm{d}y) = \infty$),

(c) $\sigma^2 = 0$, $X$ is of bounded variation, and the Lévy measure $\nu$ has support both in $(-\infty, 0)$ and $(0, \infty)$,

(d) $\sigma^2 = 0$, $X$ is of bounded variation, and either the Lévy measure $\nu$ has support in $(-\infty, 0)$ and $\widetilde{c} > 0$ in (1.9), or $\nu$ has support in $(0, \infty)$ and $\widetilde{c} < 0$ in (1.9).

# 2 One-sided reflection

Consider first the discrete time case and let $X_n = Y_1 + \cdots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. (with common distribution say $F$) so that $X$ is a random walk. The random walk one-sided reflected at 0 (i.e., corresponding to $b = \infty$) is then defined by the recursion

$$V_n^\infty = (V_{n-1}^\infty + Y_n)^+ = \max\bigl(0, V_{n-1}^\infty + Y_n\bigr) \tag{2.1}$$

starting from $V_0^\infty \geq 0$. The process $V^\infty$ also goes under the name a *Lindley process* (see [11, III.6] for a survey and many facts used in the following) and is a Markov chain with state space $[0, \infty)$.

For the following, it is important to note that the recursion (2.1) is explicitly solvable:

$$V_n^\infty = \max\bigl(V_0^\infty + X_n, X_n - X_1, \ldots, X_n - X_{n-1}, 0\bigr) \tag{2.2}$$

(for a proof, one may just note that the r.h.s. of (2.2) satisfies the recursion (2.1)). Reversing the order of $Y_1, \ldots, Y_n$ yields

$$V_n^\infty \stackrel{\mathscr{D}}{=} \max\bigl(V_0^\infty + X_n, X_{n-1}, \ldots, X_1\bigr)\,. \tag{2.3}$$

This shows in particular that $V_n^\infty$ is increasing in stochastic order so that a limit in distribution $V_\infty^\infty$ exists. By a standard random walk trichotomy ([11, VIII.2]), one of the following possibilities arises:

(a) $X_n \to \infty$ so that $V_\infty^\infty = \infty$ a.s.;

(b) $\limsup_{n\to\infty} X_n = \infty$, $\liminf_{n\to\infty} X_n = -\infty$ so that

$$\max\bigl(V_0 + X_n, X_{n-1}, \ldots, X_1\bigr) \to \infty$$

and $V_\infty^\infty = \infty$ a.s.;

(c) $X_n \to -\infty$ so that $V_\infty < \infty$ a.s.

For our purposes, it is sufficient to assume $\mathbb{E}|Y| < \infty$, and letting $m = \mathbb{E}Y$, the three cases then correspond to $m > 0$, $m = 0$, resp. $m < 0$, or, in Markov chain terms, roughly to the transient, null recurrent, resp. positive recurrent (ergodic) cases.

Consider from now on the ergodic case $m < 0$ (and, to avoid trivialities, assume that $\mathbb{P}(Y > 0) > 0$). Define $M = \max_{n\geq 0} X_n$. Since $X_n \to -\infty$, $V_0^\infty + X_n$ in (2.3) vanishes eventually, and letting $n \to \infty$ yields

$$V_\infty^\infty \stackrel{\mathscr{D}}{=} M\,. \tag{2.4}$$

It is often convenient to rewrite this is the form

$$\overline{\pi}^\infty(x) = \mathbb{P}(V_\infty^\infty > x) = \mathbb{P}\bigl(\tau(x) < \infty\bigr)\,, \tag{2.5}$$

where $\tau(x) = \inf\{n : X_n > x\}$ and $\pi^\infty$ is the distribution of $V_\infty^\infty$.

Explicit or algorithmically tractable forms of $\pi^\infty$ can only be found assuming some special structure, mainly skip-free properties or phase-type (or, more generally, matrix-exponential) forms, see [11, VIII.5]. Therefore asymptotics is a main part of the theory. The two main results are:

**Theorem 2.1** (light-tailed case)**.** *Assume $m < 0$, that $F$ is non-lattice and that there exists $\gamma > 0$ with $\mathbb{E}e^{\gamma Y} = 1$, $\mathbb{E}\bigl[Ye^{\gamma Y}\bigr] < \infty$. Then there exists $0 < C < \infty$ such that*

$$\overline{\pi}^\infty(x) = \mathbb{P}(V_\infty^\infty > x) \sim Ce^{-\gamma x}\,, \quad x \to \infty. \tag{2.6}$$

**Theorem 2.2** (heavy-tailed case)**.** *Assume $m < 0$, that*

$$\overline{F}_I(x) = \int_x^\infty \overline{F}(y)\,\mathrm{d}y$$

*is a subexponential tail*[1] *and that $F$ is long-tailed in the sense that $\overline{F}(x+x_0)/\overline{F}(x) \to 1$ for any $x_0$. Then*

$$\overline{\pi}^\infty(x) = \mathbb{P}(V_\infty^\infty > x) \sim \frac{1}{|m|}\overline{F}_I(x)\,, \quad x \to \infty. \tag{2.7}$$

*Sketch of proof of Theorem 2.1.* We use a standard exponential change of measure technique ([11, Ch. XIII]). Let $\widetilde{\mathbb{P}}$, $\widetilde{\mathbb{E}}$ refer to the case where $X$ has c.d.f.

$$\widetilde{F}(x) = \mathbb{E}\bigl[e^{\gamma X};\, X \leq x\bigr]$$

rather than $F(x)$. Using (2.5) and standard likelihood ratio identities gives

$$\overline{\pi}^\infty(x) = \mathbb{P}\bigl(\tau(x) < \infty\bigr) = \widetilde{\mathbb{E}}\bigl[e^{-\gamma X_{\tau(x)}};\, \tau(x) < \infty\bigr] = e^{-\gamma x}\widetilde{\mathbb{E}}e^{-\gamma\xi(x)}\,, \tag{2.8}$$

where $\xi(x) = X_{\tau(x)} - x$ is the overshoot. Thus, the result follows with $C = \mathbb{E}e^{-\gamma\xi(\infty)}$ once it is shown that $\xi(x)$ has a proper limit $\xi(\infty)$ in distribution. This is turn follows by renewal theory by noting that $\xi(x)$ has the same distribution as the time until the first renewal after $x$ in a renewal process with interarrivals distributed as $\xi(0)$ (the first ladder height). That $\xi(0)$ is non-lattice follows from $F$ being so, and $\widetilde{\mathbb{E}}\xi(0) < \infty$ follows from $\mathbb{E}[Xe^{\gamma X}] < \infty$. We omit the easy details. □

[1] By this we mean that there exists a subexponential distribution $G$ such that $\overline{F}_I(x) = \overline{G}(x)$ for all large $x$. For background on heavy-tailed distributions, see e.g.[13, X.1] , [57] and the start of Section 3.

The computation of $C = \mathbb{E}\mathrm{e}^{-\gamma\xi(\infty)}$ is basically of the same level of difficulty as the computation of $\pi^\infty$ itself and feasible in essentially the same situations. Result of type Theorem 2.1 commonly go under the name *Cramér-Lundberg asymptotics*, and the equation $\mathbb{E}\mathrm{e}^{\gamma Y} = 1$ is the *Lundberg equation.*

*Discussion of proofs of Theorem 2.2.* The form of the result can be understood from the 'one big jump' heuristics, stating that large values of sums and random walks arise as consequence of one big $Y_i$, while the remaining $Y_j$ are 'typical'; in particular, $X_{i-1} = \sum_{j=1}^{i-1} Y_j \approx im$ for large $i$. Splitting up after the value of $i$ and noting that the contribution from a finite segment $1, \ldots, i_0$ is insignificant, we therefore get

$$\begin{aligned}
\mathbb{P}(M > x) &= \sum_{i=1}^{\infty} \mathbb{P}\bigl(\tau(x) = i\bigr) \approx \sum_{i=1}^{\infty} \mathbb{E}\bigl[X_{i-1} \approx im, Y_i > x - im\bigr) \\
&\approx \sum_{i=1}^{\infty} \mathbb{P}(Y_i > x - im) = \sum_{i=1}^{\infty} \overline{F}(x - im) \\
&\approx \int_0^\infty \overline{F}(x - tm)\,\mathrm{d}t = \frac{1}{|m|}\int_x^\infty \overline{F}(u)\,\mathrm{d}u = \frac{1}{|m|}\overline{F}_I(x)\,.
\end{aligned}$$

The rigorous verification of (2.7) traditionally follows a somewhat different line where the essential tool is ladder height representations. The first step is to show that the first ladder height $X_{\tau(0)}$ has a tail asymptotically proportional to $\overline{F}_I$, and next one uses the representation of $M$ as a geometric sum of ladder heights to get the desired result. The details are not really difficult but too lengthy to be given here. See, e.g., [11, X.9] or [13, X.3]. A more recent proof by Zachary [129] (see also Foss, Korshunov & Zachary [57]) is, however, much more in line with the above heuristics. □

We next turn to continuous time where $X$ is a Lévy process. There is no recursion of equal simplicity as (2.1) here, so question on existence and uniqueness have to be treated by other means.

One approach simply adapts the representation (2.2) by rewriting the r.h.s. as

$$(V_0^\infty + X_n) \vee \max_{i=0,\ldots,n}(X_n - X_i) = X_n + \max\bigl(V_0^\infty, -\min_{i=0,\ldots,n} X_i\bigr)\,.$$

One then in complete analogue (which can be motivated for example by a discrete skeleton approximation) defines the continuous-time one-sided reflected process $V$ by

$$V^\infty(t) = X(t) + L(t) \tag{2.9}$$

where

$$L(t) = \max\bigl(V^\infty(0), -\min_{0\le s\le t} X(s)\bigr)\,. \tag{2.10}$$

Here $L$ is often denoted the *local time at* 0, though this terminology is somewhat unfortunate because 'local time' in used in many different meaning in the probability literature. Often also the term *regulator* is used.

The second approach uses the Skorokhod problem: in (2.10), take $L$ as a non-decreasing right-continuous processes such that

$$\int_0^\infty V^\infty(t)\,\mathrm{d}L(t) = 0\,. \tag{2.11}$$

In other words, $L$ can only increase when $V^\infty$ is at the boundary 0. Thus, $L$ represents the 'pushing up from 0' that is needed to keep $V^\infty(t) \geq 0$ for all $t$.

It is readily checked that the r.h.s. of (2.10) represents one possible choice of $L$. Thus, existence is clear. Uniqueness also holds:

**Proposition 2.3.** *Let $\{L^*(t)\}$ be any nondecreasing right-continuous process such that*

*(a) the process $\{V^*(t)\}$ given by $V^*(0) = V(0)$, $V^*(t) = X_t + L^*(t)$ satisfies $V^*(t) \geq 0$ for all $t$,*

*(b) $L^*$ can increase only when $V^* = 0$, i.e. $\int_0^T V^*(t)\,\mathrm{d}L^*(t) = 0$ for all $T$.*

*Then $L^*(t) = L(t)$, $V^*(t) = V^\infty(t)$.*

*Proof.* Let $D(t) = L(t) - L^*(t)$, $\Delta D(s) = D(s) - D(s-)$. The integration-by-parts formula for a right-continuous process of bounded variation gives

$$
\begin{aligned}
D^2(t) &= 2\int_0^t D(s)\mathrm{d}D(s) - \sum_{s\leq t}\bigl(\Delta D(s)\bigr)^2 \\
&= 2\int_0^t \bigl(L(s) - L^*(s)\bigr)\mathrm{d}L(s) - 2\int_0^t \bigl(L(s) - L^*(s)\bigr)\mathrm{d}L^*(s) - \sum_{s\leq t}\bigl(\Delta D(s)\bigr)^2 \\
&= 2\int_0^t \bigl(V^\infty(s) - V^*(s)\bigr)dL(s) - 2\int_0^t \bigl(V^\infty(s) - V^*(s)\bigr)\mathrm{d}L^*(s) - \sum_{s\leq t}\bigl(\Delta D(s)\bigr)^2 \\
&= -2\int_0^t V^*(s)\,\mathrm{d}L(s) - 2\int_0^t V^\infty(s)\mathrm{d}L^*(s) - \sum_{s\leq t}\bigl(\Delta D(s)\bigr)^2 .
\end{aligned}
$$

Here the two first integrals are nonnegative since $V^*(s), V^\infty(s)$ are so, and also the sum is clearly so. Thus $D(t)^2 \leq 0$, which is only possible if $L(t) \equiv L^*(t)$. □

Define $M = \max_{0\leq t<\infty} X(t)$ and assume $m = \mathbb{E}X(1) < 0$. The argument for (2.4) then immediately goes through to get the existence of a proper limit $V^\infty(\infty)$ of $V^\infty(t)$ and the representation

$$
V^\infty(\infty) \stackrel{\mathscr{D}}{=} M . \tag{2.12}
$$

Equivalently,

$$
\overline{\pi}^\infty(x) = \mathbb{P}(V^\infty(\infty) > x) = \mathbb{P}\bigl(\tau(x) < \infty\bigr) , \tag{2.13}
$$

where $\tau(x) = \inf\{n : X(t) > x\}$ and $\pi^\infty$ is the distribution of $V(\infty)$.

The loss rate $\ell = \ell^b$ is undefined in this setting since $b = \infty$. A closely related quantity is $\ell^0 = \mathbb{E}_{\pi^\infty} L(1)$ and one has

$$
\ell^0 = -m . \tag{2.14}
$$

This follows by a conservation law argument: in (2.9), take $t = 1$, consider the stationary situation and take expectations to get[2]

$$
\mathbb{E}_{\pi^\infty} V(1) = \mathbb{E}_{\pi^\infty} V(0) + \mathbb{E}_{\pi^\infty} X(1) + \mathbb{E}_{\pi^\infty} L(1) = \mathbb{E}_{\pi^\infty} V(1) + m + \ell^0 .
$$

[2]Strictly speaking, the argument requires $\mathbb{E}_{\pi^\infty} V(0) < \infty$ which amounts to a second moment assumption. For the general case, just use a truncation argument.

For an example of the relevance of $\ell^0$, consider the M/G/1 workload process. Here $\ell^0$ can be interpreted as the average unused capacity of the server or as the average idle time.

We next consider analogues of the asymptotic results in Theorems 2.1, 2.2. The main results are the following two theorems (for a more complete treatment, see [13, XI.2]):

**Theorem 2.4** (light-tailed case)**.** *Assume $m < 0$, that $X$ is not a compound Poisson process with lattice support of the jumps, and that there exists $\gamma > 0$ with $\kappa(\gamma) = 0$, $\kappa'(\gamma) < \infty$. Then there exists $0 < C < \infty$ such that*

$$\overline{\pi}^\infty(x) = \mathbb{P}\big(V^\infty(\infty) > x\big) \sim C\mathrm{e}^{-\gamma x}\,, \quad x \to \infty. \tag{2.15}$$

**Theorem 2.5** (heavy-tailed case)**.** *Assume $m < 0$, that*

$$\overline{\nu}(x) = \int_x^\infty \nu(\mathrm{d}y)$$

*is a subexponential tail and that $\nu$ is long-tailed in the sense that $\overline{\nu}(x+x_0)/\overline{\nu}(x) \to 1$ for any $x_0$. Then*

$$\overline{\pi}^\infty(x) = \mathbb{P}\big(V^\infty(\infty) > x\big) \sim \frac{1}{|m|}\overline{\nu}_I(x)\,, \quad x \to \infty, \tag{2.16}$$

*where* $\overline{\nu}_I(x) = \displaystyle\int_x^\infty \overline{\nu}(y)\,\mathrm{d}y$.

*Sketch of proof of Theorem 2.4.* The most substantial (but small) difference from the proof of Theorem 2.1 is the treatment of the overshoot process $\xi$ which has no longer the simple renewal process interpretation. However, the process $\xi$ is regenerative with regeneration points $\omega(1), \omega(2), \ldots$ where one can take

$$\omega(k) = \inf\{t > \omega(k-1) + U_k : \xi(t) = 0\}\,,$$

where $U_1, U_2, \ldots$ are independent uniform$(0,1)$ r.v.'s. One can then check that the non-compound Poisson property suffices for $\xi\big(\omega(1)\big)$ to be non-lattice and that $\kappa'(x) < \infty$ suffices for $\mathbb{E}\xi\big(\omega(1)\big) < \infty$. These two facts entail the convergence in distribution of $\xi(t)$ to a proper limit. $\square$

As in discrete time, $C$ can only be evaluated is special cases (general expressions are in Bertoin & Doney [29] but require the full Wiener-Hopf factorization, a problem of equal difficulty). However, if $X$ is upward skipfree (i.e., $\nu$ is concentrated on $(-\infty, 0)$), then $C = 1$ as is clear from $\xi(x) \equiv 0$. See also [13, XI.2] for the downward skipfree case as well as for related calculations, and [12] for the compound Poisson phase-type case.

For the proof of Theorem 2.5, we need a lemma:

**Lemma 2.6.** $\mathbb{P}\big(X(1) > x\big) \sim \overline{\nu}(x)\,.$

*Proof.* Write $X = X' + X'' + X'''$ where the characteristic triplets of $X', X'', X'''$ are $(c, \sigma^2, \nu')$, $(0, 0, \nu'')$ and $(0, 0, \nu''')$, resp., with $\nu', \nu'', \nu'''$ being the restrictions of $\nu$ to $[-1, 1]$, $(-\infty, -1)$ and $(1, \infty)$, respectively.

With $\beta''' = \overline{\nu}(1)$, the r.v. $X(1)'''$ is a compound Poisson sum of r.v.'s, with Poisson parameter $\beta'''$ and distribution $\nu'''/\beta'''$. Standard heavy-tailed estimates (e.g. [13, X.2]) then give

$$\mathbb{P}(X'''(1) > x) \sim \beta''' \frac{\overline{\nu}'''(x)}{\beta'''} = \overline{\nu}(x)\,, \qquad x > 1.$$

The independence of $X''(1)$ and $X'''(1) > 0$ therefore implies

$$\mathbb{P}\big(X''(1) + X'''(1) > x\big) \sim \overline{\nu}(x)\,,$$

cf. the proof of [13, X.3.2]. It is further immediate that $\kappa'(r) < \infty$ for all $r$. In particular, $X'(1)$ is light-tailed, and the desired estimate for $X(1) = X'(1) + X''(1) + X'''(1)$ then follows by [13, X.1.11]. □

*Proof of Theorem 2.5.* Define

$$M^d = \sup_{n=0,1,2,\ldots} X(n)\,.$$

Then

$$\mathbb{P}(M^d > u) \sim \frac{1}{|\mathbb{E}X(1)|} \int_u^\infty \overline{\nu}(y)\,\mathrm{d}y \tag{2.17}$$

by Theorem 2.4 and Lemma 2.6. Also clearly $\mathbb{P}(M^d > u) \le \mathbb{P}(M > u) = \psi(u)$. Given $\epsilon > 0$, choose $a > 0$ with $\mathbb{P}\big(\inf_{0\le t\le 1} X(t) > -a\big) \ge 1 - \epsilon$. Then

$$\mathbb{P}(M^d > u - a) \ge (1 - \epsilon)\mathbb{P}(M > u)\,.$$

But by subexponentiality, $\mathbb{P}(M^d > u - a) \sim \mathbb{P}(M^d > u)$. Putting these estimates together completes the proof. □

The proof of Theorem 2.5 is basically a special case of what is called *reduced load equivalence*. This principle states that if $X$ has negative drift and $X = X_1 + X_2$, where $X_1$ has heavy-tailed increments and $X_2$ has increments with lighter tails, then $M = \sup_t X(t)$ has the same tail behavior as $\sup_t\big(X_1(t) + \mathbb{E}X_2(t)\big)$. For precise versions of the principle, see e.g. Jelenković, Momcilović & Zwart [75].

# 3 Loss rate asymptotics for two-sided reflected random walks

We recall from Section 1 that a two-sided reflected random walk $\{V_n\}_{n=0,1,2,\ldots}$ is defined by the recursion

$$V_n = \min\big(b, \max(0, V_{n-1} + Y_n)\big) \tag{3.1}$$

where $Y_1, Y_2, \ldots$ are i.i.d. (with common distribution say $F$) and initial condition $V_0 = v$ for some $v \in [0, b]$. Let $X_n = Y_1 + \cdots + Y_n$ so that $X$ is a random walk.

Existence of $V$ is not an issue in discrete time because of the recursive nature of (3.1). Recall from (1.6) that the stationary distribution $\pi^b$ can be represented in terms of two-sided exit probabilities as

$$\mathbb{P}(V \geq x) = \pi^b[x, \infty) = \pi^b[x, b] = \mathbb{P}\big(X_{\tau[x-b,x)} \geq x\big) \tag{3.2}$$

where $V$ is a r.v. having the stationary distribution and $\tau[y, x) = \inf\big\{k \geq 0 : X_k \notin [y, x)\big\}$, $y \leq 0 \leq x$ (we defer the proof of this to Section 5).

The loss rate in discrete time as defined as the limit in (1.5) may be written as

$$\ell^b = \mathbb{E}(V + Y - b)^+ = \mathbb{E}\max(V + Y - b, 0)\,. \tag{3.3}$$

where $V$ is the stationary r.v. For later use we note the alternative form

$$\ell^b = \mathbb{E}(Y - b)^+ + \int_0^b \mathbb{P}(Y > b - y)\,\overline{\pi}(y)\,\mathrm{d}y, \tag{3.4}$$

which follows by partial integration in (3.3).

From now on we assume that $-\infty < m = \mathbb{E}Y < 0$. The following two results on the asymptotics of $\ell^b$ are close analogues of Theorems 2.1, 2.2:

**Theorem 3.1.** *Under the assumptions on $Y, \gamma$ in Theorem 2.1,*

$$\ell^b \sim D\mathrm{e}^{-\gamma b},\ b \to \infty,$$

*where $D$ is a constant given in (3.7) below.*

**Theorem 3.2.** *Let $Y_1, Y_2, \cdots$ be an i.i.d. sequence with mean $m < 0$ and let $\ell^b$ be the loss rate at $b$ of the associated random walk $X_n = Y_1 + \cdots + Y_n$, reflected in 0 and $b$. Assume $\overline{F}(x) \sim \overline{B}(x)$ for some distribution $B \in \mathcal{S}^*$. Then*

$$\ell^b \sim \overline{F}_I(b),\ b \to \infty, \quad \textit{where} \quad \overline{F}_I(b) = \int_b^\infty \overline{F}(y)\,\mathrm{d}y = \mathbb{E}(Y - b)^+.$$

We used here the standard notation for the classes $\mathcal{L}, \mathcal{S}$ and $\mathcal{S}^*$ of heavy-tailed distributions (see e.g. [89] or [13]): If $B$ is a distribution on $[0, \infty)$ we have $B \in \mathcal{L}$ ($B$ is *long-tailed*) iff

$$\lim_{x \to \infty} \frac{\overline{B}(x + y)}{\overline{B}(x)} = 1, \qquad \text{for all } y\,,$$

where $\overline{B}(x) = 1 - B(x)$. The class $\mathcal{S}$ of subexponential distributions is defined by the requirement

$$\lim_{x \to \infty} \frac{\overline{B^{*n}}(x)}{\overline{B}(x)} = n \qquad n = 2, 3, \ldots$$

where $B^{*n}$ denotes the $n$th convolution power of $B$. A subclass of $\mathcal{S}$ is $\mathcal{S}^*$, where we require that the mean $\mu_B$ of $B$ is finite and

$$\lim_{x \to \infty} \int_0^x \frac{\overline{B}(x - y)}{\overline{B}(x)} \overline{B}(y)\,\mathrm{d}y = 2\mu_B.$$

The classes are related by $\mathcal{S}^* \subseteq \mathcal{S} \subseteq \mathcal{L}$. More generally, we will say a measure $\nu$ belongs to, say, $\mathcal{S}$ if it is tail equivalent to a distribution in $\mathcal{S}$, that is $\nu([x,\infty)) \sim \overline{B}(x)$ for some $B$ in $\mathcal{S}$.

Theorem 3.1 is from Pihlsgård [107]. Theorem 3.2 was originally proved in Jelenković [74], but we provide a shorter proof by taking advantage of the representation of the stationary distribution provided by (1.6).

## Proof of Theorem 3.1 (light tails)

We introduce the following notation (standard in random walk theory):

- $M = \sup_{k\geq 0} X_k$.
- $\tau_+(u) = \inf\{k \geq 1 :\ X_k > u\},\ \tau_+^w(u) = \inf\{k \geq 1 :\ X_k \geq u\},\ u \geq 0$.
- $G_+(x) = \mathbb{P}(X_{\tau_+(u)} \leq x),\ G_+^w(x) = \mathbb{P}(X_{tau_+^w(u)}(u)) \leq x)$.
- $\tau_-^s(-u) = \inf\{k \geq 1 :\ X_k < -u\},\ u \geq 0$.
- The overshoot of level $u$, $B(u) = X_{\tau_+(u))-u},\ u \geq 0$.
- The weak overshoot of level $u$, $B^w(u) = X_{\tau_+^w(u)} - u,\ u \geq 0$.
- $B(\infty)$, a r.v. having the limiting distribution (if it exists) of $B(u)$ as $u \to \infty$.
- $B^w(\infty)$, a r.v. having the limiting distribution (if it exists) of $B^w(u)$ as $u \to \infty$.

Recall that $\kappa(\alpha) = \log \mathbb{E}\mathrm{e}^{\alpha Y_1}$ and that $\gamma > 0$ is the root of the Lundberg equation $\kappa(\alpha) = 0$ with $\kappa'(\gamma) < \infty$. We let $\mathbb{P}_L$ and $\mathbb{E}_L$ correspond to a measure which is exponentially tilted by $\gamma$, i.e.,

$$\mathbb{P}(G) = \mathbb{E}_L[\mathrm{e}^{-\gamma X_\tau}; G] \tag{3.5}$$

when $\tau$ is a stopping time w.r.t. $\{\mathscr{F}(n) = \sigma(Y_1, Y_2, \ldots, Y_n)\}$ and $G \in \mathscr{F}(\tau)$, $G \subseteq \{\tau < \infty\}$ where $\mathscr{F}(\tau)$ is the stopping time $\sigma$-field. Note that $\mathbb{E}_L Y = \kappa'(\gamma) > 0$ by convexity.

**Lemma 3.3.** *Assume that $Y$ is non-lattice. Then, for each $v \geq 0$,*

$$\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(u)\big) \sim \mathrm{e}^{-\gamma u}\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\mathbb{P}_L\big(\tau_-^s(-v) = \infty)\big),\ u \to \infty.$$

*Proof.* We first note that $\tau_+^w(u)$ is a stopping time w.r.t. $\mathscr{F}(n)$ and that $\{\tau_-^s(-v) > \tau_+^w(u)\} \in \mathscr{F}(\tau_+^w(u))$. Then (3.5) gives

$$\begin{aligned}\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(u)\big) &= \mathbb{E}_L\big[\mathrm{e}^{-\gamma Z(\tau_+^w(u))};\ \tau_-^s(-v) > \tau_+^w(u)\big] \\ &= \mathrm{e}^{-\gamma u}\mathbb{E}_L\big[\mathrm{e}^{-\gamma B(u)}; \tau_-^s(-v) > \tau_+(u)\big]\mathbb{P}_L\big(\tau_+^w(u) = \tau_+(u)\big) \\ &\quad + \mathrm{e}^{-\gamma u}\mathbb{P}_L\big[\tau_-^s(-v) > \tau_+^w(u)\,\big|\,\tau_+^w(u) \neq \tau_+(u)\big]\mathbb{P}_L\big(\tau_+^w(u) \neq \tau_+(u)\big).\end{aligned}$$

Since $Y$ is non-lattice, it follows that $G_+^w$ is so (see [11], Lemma 1.3, p. 222) and then the renewal theorem (see [11], Theorem 4.6, p. 155) applied to the renewal process governed by $G_+^w$, in which the forward recurrence time process coincides with the overshoot process $B^w = B^w(u)$, yields $B^w(u) \xrightarrow{\mathscr{D}} B^w(\infty)$ w.r.t. $\mathbb{P}_L$ where $B^w(\infty)$ has a density. Thus 0 is a point of continuity of $B^w(\infty)$ and we then get that

$\mathbb{P}_L(\tau_+^w(u) \neq \tau_+(u)) = \mathbb{P}_L(B^w(u) = 0) \to 0$ and $\mathbb{P}_L(\tau_+^w(u) = \tau_+(u)) \to 1,\ u \to \infty$. We now use that $B(u) \to B(\infty)$, $\{\tau_-^s(-v) > \tau_+^w(u)\} \uparrow \{\tau_-^s(-v) = \infty\}$ in $\mathbb{P}_L$-distribution and apply the argument used in the proof of Corollary 5.9, p. 368 in [11] saying that $B(u)$ and $\{\tau_-^s(-v) > \tau_+^w(u)\}$ are asymptotically independent. □

In the representation of $\ell^b$ in (3.4), It follows from the assumption $\kappa'(\gamma) < \infty$ that $\mathbb{E}(Y - b)^+ = \mathrm{o}(\mathrm{e}^{-\gamma b})$. In the second term we make the change of variables $v = b - y$ and get

$$\int_0^b \mathbb{P}(Y > b - y)\overline{\pi}(y)\mathrm{d}y = \int_0^\infty \mathbb{1}(v \le b)\mathbb{P}(Y > v)\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(b - v)\big)\mathrm{d}v$$

$$= \mathrm{e}^{-\gamma b}\int_0^\infty \mathrm{e}^{\gamma v}\mathbb{1}(v \le b)\mathbb{P}(Y > v)\mathrm{e}^{\gamma(b-v)}\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(b - v)\big)\,\mathrm{d}v. \quad (3.6)$$

Further, we have that $\mathbb{P}(\tau_-^s(-v) > \tau_+^w(b - v)) \le \mathbb{P}(M \ge b - v) \le \mathrm{e}^{-\gamma(b-v)}$ (the last inequality is just a variant of Lundberg's inequality), so

$$\mathrm{e}^{\gamma v}\mathbb{1}(v \le b)\mathbb{P}(Y > v)\mathrm{e}^{\gamma(b-v)}\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(b - v)\big) \le \mathrm{e}^{\gamma v}\mathbb{P}(Y > v)$$

and since $\int_0^\infty \mathrm{e}^{\gamma v}\mathbb{P}(Y > v)\mathrm{d}v < \infty$ the assertion follows with

$$D = \mathbb{E}_L\mathrm{e}^{-\gamma B(\infty)}\int_0^\infty \mathrm{e}^{\gamma v}\mathbb{P}(Y > v)\mathbb{P}_L(\tau_-^s(-v) = \infty)\,\mathrm{d}v \quad (3.7)$$

by (3.6), Lemma 3.3 and dominated convergence. □

**Remark 3.4.** The constants occurring in $D$ and above are standard in Wiener-Hopf theory for random walks. Note that alternative expressions for $D$ are in [107].

## Proof of Theorem 3.2 (heavy tails)

By (3.4), we need to prove that

$$\limsup_{b\to\infty} I(b) = 0 \quad \text{where} \quad I(b) = \int_0^b \frac{\mathbb{P}(Y > b - y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y\,. \quad (3.8)$$

For any $A > 0$

$$\limsup_{b\to\infty}\int_0^A \frac{\mathbb{P}(Y > b - y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y \le \limsup_{b\to\infty}\frac{\mathbb{P}(Y > b - A)}{\overline{F}_I(b)}\int_0^A \overline{\pi}^b(y)\,\mathrm{d}y = 0$$

so therefore

$$\limsup_{b\to\infty}\int_0^b I(b) = \limsup_{b\to\infty}\int_A^b \frac{\mathbb{P}(Y > b - y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y\,. \quad (3.9)$$

Define $m^+ = \int_0^\infty \mathbb{P}(Y > t)\,\mathrm{d}t$ and $F_e(y) = (1/m^+)\int_0^x \mathbb{P}(Y > t)\,\mathrm{d}t$. According to (2.7) we have $\overline{\pi}^\infty(y)|m| \sim \overline{F}_I(y)$ so that for large $A$ and $y > A$

$$\overline{\pi}^\infty(y) \le 2\overline{F}_I(y)/|m| = 2m^+\overline{F}_e(y)/|m|$$

From Proposition 11.5 in Section 11 (proved there for Lévy processes but valid also for random walks as it only relies on the representation (3.2) of $\pi$ as a two-barrier passage time probability),

$$0 \le \overline{\pi}^{\infty}(x) - \overline{\pi}^{b}(x) \le \overline{\pi}^{\infty}(b)\,. \tag{3.10}$$

Using this, we have:

$$\begin{aligned}
\limsup_{b\to\infty} \int_A^{b-A} \frac{\mathbb{P}(Y > b-y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y \ &\le 2\limsup_{b\to\infty} \int_A^{b-A} \frac{m^+\mathbb{P}(Y > b-y)\overline{F}_e(y)}{|m|\overline{F}_I(b)}\,\mathrm{d}y \\
= 2\limsup_{b\to\infty} \int_A^{b-A} \frac{\mathbb{P}(Y > b-y)\overline{F}_e(y)}{|m|\overline{F}_e(b)}\,\mathrm{d}y& \\
= 2\limsup_{b\to\infty} \frac{\overline{F}_e^{*2}(b)}{\overline{F}_e(b)} \int_A^{b-A} \frac{\mathbb{P}(Y > b-y)\overline{F}_e(y)}{|m|\overline{F}_e^{*2}(b)}\,\mathrm{d}y& \\
= 4\limsup_{b\to\infty} \int_A^{b-A} \frac{\mathbb{P}(Y > y)\overline{F}_e(b-y)}{|m|\overline{F}_e^{*2}(b)}\,\mathrm{d}y& \\
= 4\limsup_{b\to\infty} \frac{m^+}{|m|}\mathbb{P}(A < U \le b-A \mid U+V > b) = \frac{2m^+}{|m|}\overline{F}_e(A)\,.&
\end{aligned}$$

where $U$ and $V$ are independent with $U \stackrel{\mathscr{D}}{=} V \stackrel{\mathscr{D}}{=} F_e$ and we have used that for i.i.d. random variables in $\mathcal{S}$

$$\mathbb{P}(A < Y_1 < b-A \mid Y_1 + Y_2 > b) \to \tfrac{1}{2}\overline{F}(A), \quad b \to \infty$$

(cf. [13], pp. 294, 296, slightly adapted). By combining the result above with (3.9) we have

$$\limsup_{b\to\infty} I(b) \le \frac{2m^+}{|m|}\overline{F}_e(A) + \limsup_{b\to\infty} \int_{b-A}^{b} \frac{\mathbb{P}(Y > b-y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y\,. \tag{3.11}$$

Here the integral equals

$$\limsup_{b\to\infty} \int_0^{A} \frac{\mathbb{P}(Y > y)\overline{\pi}^b(b-y)}{\overline{F}_I(b)}\,\mathrm{d}y \le \limsup_{b\to\infty} \frac{\overline{\pi}^b(b-A)}{\overline{F}_I(b)} \int_0^A \mathbb{P}(X > y)\,\mathrm{d}y\,.$$

If we define $\sigma_A = \inf\{n \ge 0 \mid X_n < -A\}$, $M_n = \max_{k\le n} X_k$ and use the representation (3.2) of the stationary distribution we have:

$$\overline{\pi}^b(b-A) = \mathbb{P}(M_{\sigma_A} > b-A)\,.$$

By Theorem 1 of [58] we have $\mathbb{P}(M_{\sigma_A} > b-A) \sim \mathbb{E}\sigma_A\overline{F}(b)$ and therefore $\overline{\pi}^b(b-A)/\overline{F}_I(b) \to 0$ since the tail of $F$ is lighter than that of the integrated tail. Using (3.11) it thus follows that we can bound $\limsup I(b)$ by $2m^+\overline{F}_e(A)/|m|$. Letting $A \to \infty$ completes the proof.

# 4 Construction and the Skorokhod problem

We consider here the problem of how to rigorously define $V = V^b$ in the continuous time Lévy set-up.

First, we note that there is a simple pragmatic solution: Let $y \in [0, b]$ be the initial value. For $y < b$, take the segment up to the first hitting time $\tau(b)$ of $b$ as the initial segment of $V^\infty$ (the one-sided reflected process started from $y$) until $(b, \infty)$ is hit; we then let $V\bigl(\tau(b)\bigr) = b$. For $y = b$, we similarly take the segment up to the first hitting time $\tau^*(0)$ of 0 by using the one-sided reflection operator (with the sign reversed and change of origin) as constructed in Section 2; at time $\tau^*(0)$ where this one-sided reflected (at $b$) process hits $(-\infty, 0]$, we let $V\bigl(\tau^*(0)\bigr) = 0$. The whole process $V$ is then constructed by glueing segments together in an obvious way. Glueing also local times together, we obtain the desired solution of the Skorokhod problem. Uniqueness of this solution may be established using a proof nearly identical to that of Proposition 2.3.

Before we proceed to a more formal definition of $V$ we restate the Skorokhod problem: Given a cadlag process $\{X(t)\}$ we say a triplet $(\{V(t)\}, \{L(t)\}, \{U(t)\})$ of processes is the *solution to the Skorokhod problem on* $[0, b]$ if $V(t) = X(t) + L(t) - U(t) \in [0, b]$ for all $t$ and

$$\int_0^\infty V(t)\,\mathrm{d}L(t) = 0 \quad \text{and} \quad \int_0^\infty (b - V(t))\,\mathrm{d}U(t) = 0\,.$$

Note that the Skorokhod problem as introduced above is a purely deterministic problem. We refer to the mapping which associates a triplet $(\{V(t)\}, \{L(t)\}, \{U(t)\})$ to a cadlag process $X(t)$ as the *Skorokhod map.*

**Remark 4.1.** The Skorokhod problem on $[0, b]$ is a particular case of reflection of processes in convex regions of $\mathbb{R}^n$, which is treated in Tanaka [126] where a proof of existence and uniqueness is provided given that the involved processes are continuous or step functions. This is extended in [6] to include cadlag processes, which covers what is needed in this article. Apart from the generalizations to larger classes of functions, other papers have focused on more general domains than convex subsets of $\mathbb{R}^n$, e.g. Lions & Snitzman [100] and Saisho [116]. The case of Brownian motion in suitable regions has received much attention in recent decades, see e.g. Harrison & Reiman [70] and Chen & Yao [40]. In [69] Ch. 2 § 4, the Skorokhod problem on $[0, b]$ is introduced as the *two-sided regulator* and is used to treat Brownian motion with two-sided reflection; another early references on two-sided reflection problems is Chen & Mandelbaum [39]. A comprehensive treatment of the Skorokhod map and its continuity properties, as well as other reflection mappings and their properties, is given in Whitt [127].

Various formulas for the Skorokhod map have appeared in the literature, among them Cooper *et al.* [45]. See [91] for a survey of these formulas and the relation between them. An alternative approach to estimation of stationary quantities is to take advantage of the integral representation of the one-dimensional Skorokhod reflection, see Konstantopoulos & |ast [88], Anantharam & Konstantopoulos [2], and Buckingham *et al.* [36]. This is applicable when considering processes of finite variation, so that we can write $S(t) = A(t)+B(t)$ for non-decreasing cadlag processes $A$ and $B$. It is then possible to write $V(t)$ as an integral with respect to $A(\mathrm{d}t)$. This representation can for example be used to derive the Laplace transform of $V$ in terms of the Palm measure. □

As described in Section 2, specifically (2.9) and (2.10), an explicit expression for $V(t)$ is available when one is concerned with one-sided reflection. This is also the case when dealing with Skorokhod problem on $[0, b]$. Indeed, from Kruk *et al.* [91] we have:

$$V^b(t) = X(t) - \left((V(0)-b)^+ \wedge \inf_{u\in[0,t]} X(u)\right) \vee \sup_{s\in[0,t]} \left((V(0)-b) \wedge \inf_{u\in[s,t]} X(u)\right). \tag{4.1}$$

We shall assume $V(0) = 0$ a.s and in this case we have following simplification, which was originally proved in [5].

**Theorem 4.2.** *If $V(0) = 0$, then*

$$V^b(t) = \sup_{s\in[0,t]} \Big( \big(X(t) - X(s)\big) \wedge \inf_{u\in[s,t]} \big(b + X(t) - X(u)\big)\Big). \tag{4.2}$$

**Remark 4.3.** Before we provide a rigorous proof, we note the following intuitive explanation for the expression (4.2): For $v > 0$ consider the process $\{V_v(t)\}_{t>v}$ obtained by reflecting $X_v(t) = X(t) - X(v)$ at $b$ from below (in terms of recursions like (2.1) this is $V_n = b \vee (V_{n-1} + Y_n)$). Similarly to (2.9) and (2.10) we obtain $V_v(t) = X_v(t) \wedge \inf_{v<u<t}(b - X_v(t) - X_v(u))$. Then obviously $V_v(t) \le V(t)$ but since $V_v(t^*) = V(t^*)$ for $t^* = \sup_{0<u<t} V(y) = 0$, we have $V(t) = \sup_{0<v<t} V_v(t)$. □

The proof of (4.2) proceeds as follows: First we prove Proposition 4.4 and 4.5 which are the discrete time equivalents of (4.1) and (4.2). Then we prove Lemma 4.6, which states that the implied mapping of $X(t)$ in (4.2) is Lipschitz-continuous in the $J_1$ topology which is combined with an piecewise constant approximation to obtain the equivalence of (4.2) and (4.1). To emphasize the deterministic nature of the Skorokhod problem and for explicit treatment of the involved mappings, we switch notation and let $\boldsymbol{y} = \{y_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^\infty$ and consider the sequences $\boldsymbol{x}$ and $\boldsymbol{v}$ obtained by respectively taking cumulative sums of $\boldsymbol{y}$ and applying two-sided reflection, that is $x_n = y_1 + \cdots + y_n$ and $v_n = \min(b, \max(0, v_{n-1} + y_n)$ with $x_0 = v_0 = 0$. We let $\Gamma_{0,b}$ denote the two-sided reflection mapping, that is $\Gamma_{0,b}(\boldsymbol{x}) = \boldsymbol{v}$.

**Proposition 4.4.** *The solution of the two-sided reflection is given by*

$$\Gamma_{0,b}(\boldsymbol{x})(n) = \max_{k\in\{0,\cdots,n\}} \Big( \min_{j\in\{k,\cdots,n\}} (x_n - x_k, b + x_n - x_j)\Big). \tag{4.3}$$

*Proof.* We prove the claim by induction. The case $n = 1$ is trivial, so we assume (4.3) holds for some $n$, and consider the cases $y_{n+1} \le 0$ and $y_{n+1} > 0$ separately. For the former case we have

$$\begin{aligned}
\Gamma_{0,b}(\boldsymbol{x})(n+1) = v_{n+1} &= 0 \vee (v_n + y_{n+1}) \wedge b = 0 \vee (v_n + y_{n+1}) \\
&= 0 \vee \Big( \max_{k\in\{0,\cdots,n\}} \Big( \min_{j\in\{k,\cdots,n\}} (x_n - x_k, b + x_n - x_j)\Big) + y_{n+1}\Big) \\
&= 0 \vee \Big( \max_{k\in\{0,\cdots,n\}} \Big( \min_{j\in\{k,\cdots,n\}} (x_{n+1} - x_k, b + x_{n+1} - x_j)\Big)\Big).
\end{aligned} \tag{4.4}$$

Since $y_{n+1} \le 0$, we have

$$\min_{j\in\{k,\cdots,n+1\}} x_{n+1} - x_j = \min_{j\in\{k,\cdots,n\}} x_{n+1} - x_j,$$

so that (4.4) equals

$$\begin{aligned} 0 \vee \Big( &\max_{k\in\{0,\cdots,n\}} \big( \min_{j\in\{k,\cdots,n+1\}} (x_{n+1} - x_k, b + x_{n+1} - x_j) \big) \Big) \\ &= \max_{k\in\{0,\cdots,n+1\}} \big( \min_{j\in\{k,\cdots,n+1\}} (x_{n+1} - x_k, b + x_{n+1} - x_j) \big), \end{aligned} \tag{4.5}$$

as desired. The case $y_{n+1} > 0$ is similar:

$$\begin{aligned} v_{n+1} &= 0 \vee (v_n + y_{n+1}) \wedge b = (v_n + y_{n+1}) \wedge b \\ &= \Big( \max_{k\in\{0,\cdots,n\}} \big( \min_{j\in\{k,\cdots,n\}} (x_n - x_k, b + x_n - x_j) \big) + y_{n+1} \Big) \wedge b \\ &= \max_{k\in\{0,\cdots,n\}} \big( \min_{j\in\{k,\cdots,n\}} (x_{n+1} - x_k, b + X_{n+1} - x_j) \wedge b \big), \end{aligned}$$

which equals (4.5) as well. This completes the proof. □

Proposition 4.4 provides the discrete-time analogue of (4.2). Next, we provide the discrete-time analogue for (4.1), in the case $v_0 = 0$.

**Proposition 4.5.** *The solution of the two-sided reflection is given by*

$$\Gamma_{0,b}(\boldsymbol{x})(n) = \min_{k\in\{0,\ldots,n\}} \big[ \big( (x_n - x_k + b) \wedge \max_{i\in\{0,\ldots,n\}} (x_n - x_i) \big) \vee \max_{i\in\{k,\ldots,n\}} (x_n - x_i) \big] . \tag{4.6}$$

*Proof.* The proof is again by induction and again the case $n = 1$ is straightforward, so we assume the stated holds for some $n$. Then we have

$$\begin{aligned} \Gamma_{0,b}(\boldsymbol{x})(n+1) &= 0 \vee (v_n + y_{n+1}) \wedge b \\ &= 0 \vee \Big( \min_{k\in\{0,\ldots,n\}} \big[ \big( (x_n - x_k + b) \wedge \max_{i\in\{0,\ldots,n\}} (x_n - x_i) \big) \\ &\qquad\qquad \vee \max_{i\in\{k,\ldots,n\}} (x_n - x_i) \big] + y_{n+1} \Big) \wedge b \\ &= 0 \vee \min_{k\in\{0,\ldots,n\}} \big[ \big( (x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\ldots,n\}} (x_{n+1} - x_i) \big) \\ &\qquad\qquad \vee \max_{i\in\{k,\ldots,n\}} (x_{n+1} - x_i) \big] \wedge b \\ &= \min_{k\in\{0,\ldots,n\}} \big[ \big( (x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\ldots,n\}} ((x_{n+1} - x_i) \vee 0) \big) \\ &\qquad\qquad \vee \max_{i\in\{k,\ldots,n\}} ((x_{n+1} - x_i) \vee 0) \big] \wedge b \\ &= \min_{k\in\{0,\ldots,n\}} \big[ \big( (x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\ldots,n+1\}} (x_{n+1} - x_i) \big) \\ &\qquad\qquad \vee \max_{i\in\{k,\ldots,n+1\}} (x_{n+1} - x_i) \big] \wedge b. \end{aligned} \tag{4.7}$$

We notice that

$$
\begin{aligned}
&\big((x_{n+1}-x_k+b)\wedge \max_{i\in\{0,\dots,n+1\}}(x_{n+1}-x_i)\big)\vee \max_{i\in\{k,\dots,n+1\}}(x_{n+1}-x_i)\\
&\qquad=\begin{cases}\max_{i\in\{0,\dots,n+1\}}(x_{n+1}-x_i) & \text{if } k=0\\ \max_{i\in\{0,\dots,n+1\}}(x_{n+1}-x_i)\wedge b & \text{if } k=n+1,\end{cases}
\end{aligned}
$$

so that (4.7) equals

$$
\min_{k\in\{0,\dots,n+1\}}\big[\big((x_{n+1}-x_k+b)\wedge \max_{i\in\{0,\dots,n+1\}}(x_{n+1}-x_i)\big)\vee \max_{i\in\{k,\dots,n+1\}}(x_{n+1}-x_i)\big]\,.
$$

This proves the claim. □

We now proceed to the proof of (4.2). Let $\psi\in\mathcal{D}[0,\infty)$. From [91] we have:

$$
\Gamma_{0,b}(\boldsymbol{\psi})(t)=\psi(t)-\sup_{s\in[0,t]}\Big[\Big(\big(\psi(s)-b\big)\vee\inf_{u\in[0,t]}\psi(u)\Big)\wedge\inf_{u\in[s,t]}\psi(u)\Big], \tag{4.8}
$$

when the process is started at 0. In view of the two previous propositions it seems reasonable to conjecture that $\Gamma_{0,b}=\Xi$, where

$$
\Xi[\boldsymbol{\psi}](t)=\sup_{s\in[0,t]}\big[\big(\psi(t)-\psi(s)\big)\wedge\inf_{u\in[s,t]}\big(b+\psi(t)-\psi(u)\big)\big]\,. \tag{4.9}
$$

We prove this by first showing that $\Xi$ is Lipschitz-continuous in the $J_1$ topology.

**Lemma 4.6.** *The mapping $\Xi$ is Lipschitz-continuous in the uniform and $J_1$ metrics as a mapping from $D[0,T]$ for $T\in[0,\infty]$, with constant 2.*

*Proof.* We follow the proof of Corollary 1.5 in [90] closely. Fix $T<\infty$. We start by proving Lipschitz-continuity in the uniform metric. Define

$$
R_t[\boldsymbol{\psi}](s)=\big[\big(-\psi(s)\big)\wedge\inf_{u\in[s,t]}(b-\psi(u))\big];\quad S[\boldsymbol{\psi}](t)=\sup_{s\in[0,t]}R_t[\boldsymbol{\psi}](s). \tag{4.10}
$$

For $\psi_1,\psi_2\in D[0,T]$ we have

$$
\begin{aligned}
S[\boldsymbol{\psi_1}](t)-S[\boldsymbol{\psi_2}](t)&\le\sup_{s\in[0,t]}\big(R_t[\boldsymbol{\psi_1}](s)-R_t[\boldsymbol{\psi_2}](s)\big)\\
&\le\sup_{s\in[0,t]}\big[|-\psi_1(s)-(-\psi_2(s))|\vee\big|\inf_{u\in[s,t]}(b-\psi_1(u))-\inf_{u\in[s,t]}(b-\psi_2(u))\big|\big]\\
&\le\|\boldsymbol{\psi_1}-\boldsymbol{\psi_2}\|_T.
\end{aligned}
$$

The same inequality applies to $S[\boldsymbol{\psi_2}](t)-S[\boldsymbol{\psi_2}](t)$, so that taking the supremum leads to

$$
\|S[\boldsymbol{\psi_1}]-S[\boldsymbol{\psi_2}]\|_T\le\|\boldsymbol{\psi_1}-\boldsymbol{\psi_2}\|_T\,,
$$

and this proves Lipschitz-continuity, with constant 2:

$$
\|\Xi[\boldsymbol{\psi_1}]-\Xi[\boldsymbol{\psi_2}]\|_T\le\|\boldsymbol{\psi_1}-\boldsymbol{\psi_2}\|+\|S[\boldsymbol{\psi_1}]-S[\boldsymbol{\psi_t}]\|_T\le 2\|\boldsymbol{\psi_1}-\boldsymbol{\psi_2}\|_T.
$$

We now turn to the $J_1$-metric, and we let $\mathcal{M}$ denote the class of strictly increasing continuous functions from $[0,T]$ onto itself with continuous inverse. An elementary verification yields that for $\boldsymbol{\psi} \in D[0,T]$ and $\boldsymbol{\lambda} \in \mathcal{M}$ we have $\Xi[\boldsymbol{\psi} \circ \boldsymbol{\lambda}] = \Xi[\boldsymbol{\psi}] \circ \boldsymbol{\lambda}$. With $\boldsymbol{e}$ being the identity, this leads to

$$\begin{aligned} d_{J_1}(\Xi[\boldsymbol{\psi_1}], \Xi[\boldsymbol{\psi_2}]) &= \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{\|\Xi[\boldsymbol{\psi_1}] \circ \boldsymbol{\lambda} - \Xi[\boldsymbol{\psi_2}]\|_T \vee \|\boldsymbol{\lambda} - \boldsymbol{e}\|_T\} \\ &= \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{\|\Xi[\boldsymbol{\psi_1} \circ \boldsymbol{\lambda}] - \Xi[\boldsymbol{\psi_2}]\|_T \vee \|\boldsymbol{\lambda} - \boldsymbol{e}\|_T\} \\ &\leq \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{2\|\boldsymbol{\psi_1} \circ \boldsymbol{\lambda} - \boldsymbol{\psi_2}\|_T \vee \|\boldsymbol{\lambda} - \boldsymbol{e}\|_T\} \leq 2 d_{J_1}(\boldsymbol{\psi_1}, \boldsymbol{\psi_2}), \end{aligned}$$

where we used the Lipschitz-continuity in the uniform metric. This proves Lipschitz-continuity in the $J_1$ metric, again with constant 2; it is valid for every $T < \infty$ and hence also for $T = \infty$. ☐

We are now ready to prove that $\Gamma_{0,b} = \Xi$.

**Theorem 4.7.** *For $\psi \in D[0,\infty)$ we have $\Gamma[\boldsymbol{\psi}](t) = \Xi[\boldsymbol{\psi}](t)$.*

*Proof.* Let $\boldsymbol{\psi} \in D[0,\infty)$ be given, and define $\boldsymbol{\gamma}_n$ and $\boldsymbol{\psi}_n$ by $\gamma_n(t) = \lfloor nt \rfloor / n$, $\psi_n(t) = \psi(\gamma_n(t))$. Since $\boldsymbol{\gamma}_n \to \boldsymbol{e}$ in the uniform topology, we have $\boldsymbol{\gamma}_n \to_{d_{J_1}} \boldsymbol{e}$ and hence $(\boldsymbol{\psi}, \boldsymbol{\gamma}_n) \to (\boldsymbol{\psi}, \boldsymbol{e})$ in the strong version of the $J_1$ topology (see p. 83 in [127]). Since $\boldsymbol{e}$ is strictly increasing we may apply Theorem 13.2.2 in [127] to obtain $\boldsymbol{\psi}_n \to_{d_{J_1}} \boldsymbol{\psi}$. Fix $t < T$, and consider $\boldsymbol{\psi}$ as element of $D[0,T]$. Since the image $\psi_n([0,T])$ is finite, we may apply Props. 4.4 and 4.5, in conjunction with (4.8), to obtain $\Gamma_{0,b}[\boldsymbol{\psi}_n] = \Xi[\boldsymbol{\psi}_n]$. Finally, we let $n \to \infty$ and use the $J_1$-continuity of the $\Gamma_{0,b}$ mapping proved in [90], and the $J_1$-continuity of $\Xi$ proved in Lemma 4.6 to finish the proof. ☐

**Remark 4.8.** Letting $b \to \infty$ yields $\sup_{s \in [0,t]} [(\psi(t) - \psi(s))]$, which is indeed the standard one-sided reflection from (2.9) and (2.10). ☐

# 5 The stationary distribution

## 5.1 Ergodic properties

The following observation is easy but basic:

**Proposition 5.1.** *The two-sided reflected Lévy process $V = V^b$ admits a unique stationary distribution $\pi = \pi^b$. Furthermore, for any initial distribution $V$ converges in distribution and total variation to $\pi$.*

*Proof.* We appeal to the theory of regenerative processes ([11, Ch. VI]). The classical definition of a stochastic process to be regenerative means in intuitive terms that the process can be split into i.i.d. *cycles* (with the first cycle having a possibly different distribution). There is usually a multitude of ways to define a cycle. The naive approach in the case of $V^b$ is to take the instants of visits to state 0 (say) as regeneration points, but these will typically have accumulation points (cf. the theory of Brownian zeros!) and so a bit more of sophistication is needed. Instead we may, e.g., define the generic cycle length $T$ as starting at level 0 at time 0, waiting until level $b$ is hit and taking the cycle termination time $T$ as the next hitting time of 0 ('up to b from 0 and down again'). That is,

$$T = \inf\bigl\{t > \inf\{s > 0 : V^b(s) = b\} : V^b(t) = 0 \,\big|\, V^b(0) = 0\bigr\}.$$

The regenerative structure together with the easily verified fact $\mathbb{E}T < \infty$ then immediately gives the existence of $\pi^b$.

Tv. convergence just follows from coupling $V^b$ with the stationary version $\widehat{V}^b$ (cf. [11, VII.1]). Indeed, we may assume that $V^b$ and $\widehat{V}^b$ both have the same driving process $X$. Then $\widehat{V}^b(t) \geq V(t)$ for all $t$, and so $\tau = \inf\{t > 0 : V^b(t) = \widehat{V}^b(t)\}$ is bounded by $T_1$, hence a.s. finite. □

**Remark 5.2.** T.v. convergence in distribution is often alternatively established by verifying that the distribution of $T$ is spread-out ([11, VI.1]). In the present context, this is slightly more tedious but goes like this. $T$ decomposes as the independent sum $T_1 + T_2$ where $T_1$ is passage time from 0 to $b$ and $T_2$ the passage from $b$ to 0 so that it suffices to verify that one of $T_1, T_2$ is spread-out. This is obvious for Brownian motion since there $T_1, T_2$ are both absolutely continuous. In the case $\nu \neq 0$ of a non-vanishing jump component, suppose, e.g., that $\nu$ does not vanish on $(0, \infty)$. Then $b$ may be hit by a jump, i.e. $\mathbb{P}_0\big(\Delta X(T_1) > 0\big) > 0$. Then also $\mathbb{P}_0\big(\Delta X(T_1) > \epsilon\big) > 0$ for some $\epsilon > 0$ and the absolutely continuous part of $T_1$ may be taken as $\mathbb{P}_0(T_1 \in \cdot, \big(\Delta X(T_1) > \epsilon\big) > 0$.

Another approach is to take advantage of the fact that $V^b$ is a Markov process on a compact state space with a semi-group with easily verified smoothness properties, cf. [87] for some general theory, and yet another to invoke Harris recurrence in continuous time, cf. [23], [24]. We omit the details. □

**Remark 5.3.** The process $V^b$ is in fact geometrically ergodic, i.e.

$$\sup_A |\mathbb{P}_x(V^b(t) \in A) - \pi^b(A)| = \mathrm{O}(\mathrm{e}^{-\epsilon t}) \tag{5.1}$$

for some $\epsilon > 0$ where the O term is uniform in $x$. This follows again from the coupling argument by bounding the l.h.s. of (5.1) by $\mathbb{P}(\tau > t)$ and checking that $\tau$ has exponential moments (geometric trials argument!).

It is easy to derive rough bounds on the tail of $\tau$ and thereby lower bounds on $\epsilon$. To get the exact rate of decay in (5.1) seems more difficult, as is typically the case in Markov process theory (but see Linetsky [99] for the Brownian case).

## 5.2 First passage probability representation

The main result on the stationary distribution $\pi^b$ is as follows and states that $\pi^b$ can be computed via two-sided exit probabilities for the Lévy process.

**Theorem 5.4.** *The stationary distribution of the two-sided reflected Lévy process $V = V^b$ is given by*

$$\pi^b[x, b] = \mathbb{P}\big(V(\infty) \geq x\big) = \mathbb{P}\big(X\big(\tau[x - b, x)\big) \geq x\big) \tag{5.2}$$

*where $\tau[u, v) = \inf\{t \geq 0 : X(t) \notin [u, v)\}$, $u \leq 0 \leq v$.*

Note that in the definition of $\tau[u, v)$ we write $t \geq 0$, not $t > 0$.

We remark that the calculation of $\mathbb{P}\big(X\big(\tau[x-K,x)\big) \geq x\big)$ is a special case of scale function calculations for spectrally negative Lévy processes. For such a process, the scale function $W^q$ is usually defined as the function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-sx} W^q(x)\,\mathrm{d}x = \frac{1}{\kappa(-s)-q}\,.$$

However, it has a probabilistic interpretation related to (5.2) by means of

$$\mathbb{E}\big[\mathrm{e}^{-q\tau[a,b)}\mathbb{1}\big(X(\tau[a,b)) \geq b\big)\big] = \frac{W^q(|a|)}{W^q(|a|+b)} \tag{5.3}$$

The present state of the area of scale functions is surveyed in Kuznetsov *et al.* [92]. Classically, there has been very few explicit examples, but a handful more, most for quite special structures, have recently emerged (see, e.g., Hubalek & Kyprianou [71] and Kyprianou & Rivero [96]).

We shall present two approaches to the proof of Theorem 5.4. One is direct and specific for the model, the other uses general machinery for certain classes of stochastic processes with certain monotonicity properties.

## Direct verification

Write $V_0(t)$ for $V$ started from $V_0(0)=0$, let $T$ be fixed and for $0 \leq t \leq T$, let $\{R_x(t)\}$ be defined as $R_x(t) = x - X(T) + X(T-t)$ until $(-\infty,0]$ or $(b,\infty)$ is hit; the value is then frozen at 0, resp. $\infty$. We shall show that

$$V_0(T) \geq x \iff R_x(T) = 0; \tag{5.4}$$

this yields

$$\mathbb{P}\big(V_0(T) \geq x\big) = \mathbb{P}\big(\tau[x-b,x) \leq T,\, X\big(\tau[x-b,x)\big) \geq x\big)$$

and the proposition then follows by letting $T \to \infty$.

Let $\sigma = \sup\{t \in [0,T] : V_0(t) = 0\}$ (well–defined since $V_0(0)=0$). Then $V_0(T) = X(T) - X(\sigma) + U(\sigma) - U(T)$, so if $V_0(T) \geq x$ then $X(T) - X(\sigma) \geq x$, and similarly, for $t \geq \sigma$

$$x \leq V_0(T) = V_0(t) + X(T) - X(t) + U(t) - L(T) \leq b + X(T) - X(t),$$

implying $R_x(T-t) \leq b$. Thus absorbtion of $\{R_x(t)\}$ at $\infty$ is not possible before $T-\sigma$, and $X(T) - X(\sigma) \geq x$ then yields $R_x(T-\sigma) = 0$ and $R_x(T) = 0$.

Assume conversely $R_x(T) = 0$ and write the time of absorbtion in 0 as $T-\sigma$. Then $x - X(T) + X(\sigma) \leq 0$, and $R_x(t) \leq b$ for all $t \leq T-\sigma$ implies $x - X(T) + X(t) \leq b$ for all $t \geq \sigma$. If $V_0(t) < b$ for all $t \in [\sigma,T]$, then $U(T) - U(t) = 0$ for all such $t$ and hence

$$\begin{aligned} V_0(T) &= V_0(\sigma) + X(T) - X(\sigma) + L(T) - L(\sigma) \\ &\geq V_0(\sigma) + X(T) - X(\sigma) \geq 0 + x. \end{aligned}$$

If $V_0(t) = b$ for some $t \in [\sigma,T]$, denote by $\omega$ the last such $t$. Then $U(T) = U(\omega)$ and hence

$$\begin{aligned} V_0(T) &= V_0(\omega) + X(T) - X(\omega) + L(T) - L(\omega) \\ &\geq b + X(T) - X(\omega) + 0 \geq x. \end{aligned}$$

□

## Siegmund duality

Now consider the general approach. Let $\mathbb{T}=\mathbb{N}$ or $\mathbb{T}=[0,\infty)$, let $\{V(t)\}_{t\in\mathbb{T}}$ be a general Markov process with state space $E=[0,\infty)$ or $E=\mathbb{N}$, and let $V_x(t)$ be the version starting from $V_x(0)=x$. We write interchangeably $\mathbb{P}(V_x(t)\in A)$ and $\mathbb{P}_x(V(t)\in A)$. Then $\{V_x(t)\}$ is *stochastically monotone* if $x\le y$ implies $V_x(t)\le_{\mathrm{so}} V_x(t)$ (stochastical ordering) for all $t\in\mathbb{T}$, i.e. if $\mathbb{P}_x(V(t)\ge z)\le\mathbb{P}_y(V(t)\ge z)$ for all $t$ and $z$.

**Proposition 5.5.** *The existence of a Markov process $\{R(t)\}_{t\in\mathbb{T}}$ on $E\cup\{\infty\}$ such that*

$$\mathbb{P}_x\big(V(t)\ge y\big)=\mathbb{P}_y\big(R(t)\le x\big) \tag{5.5}$$

*is equivalent to (i) $\{V(t)\}$ is stochastically monotone and (ii) $\mathbb{P}_x\big(V(t)\ge y\big)$ is a right–continuous function of $x$ for all $t$ and $y$.*

*Proof.* If $\{R(t)\}$ exists, the l.h.s. of (5.5) is nondecreasing and right–continuous in $x$ and so necessity of (i), (ii) is clear. If conversely (i), (ii) hold, then the r.h.s. of (5.5) defines a probability measure for each $y$ that we can think of as the element $P^t(y,\cdot)$ of a transition kernel $P^t$ (thus $P^t(y,\{\infty\})=1-\lim_{x\to\infty}\mathbb{P}_x\big(V(t)\ge y\big)$), and we shall show that the Chapman-Kolmogorov equations $P^{t+s}=P^tP^s$ hold. This follows since

$$\begin{aligned}
P^{t+s}\big(y,[0,x]\big)&=\mathbb{P}_x\big(V(t+s)\ge y\big)=\int_E\mathbb{P}_x\big(V(t)\in\mathrm{d}z)\mathbb{P}_z\big(V(s)\ge y\big)\\
&=\int_E\mathbb{P}_x\big(V(t)\in\mathrm{d}z\big)\int_0^z P^s(y,\mathrm{d}u)=\int_0^z P^s(y,\mathrm{d}u)\mathbb{P}_x\big(V(t)\ge u\big)\\
&=\int_0^z P^s(y,\mathrm{d}u)P^t\big(u,[0,x]\big)=(P^tP^s)\big(y,[0,x]\big).
\end{aligned}$$

□

**Theorem 5.6.** *The state 0 is absorbing for $\{R(t)\}$. Furthermore, letting*

$$\tau=\inf\{t>0:\ R_x(t)\le 0\}=\inf\{t>0:\ R_x(t)=0\}\,,$$

*one has*

$$\mathbb{P}_0\big(V(T)\ge x\big)=\mathbb{P}_x(\tau\le T), \tag{5.6}$$

*and if $V(t)$ converges in total variation, say to $V=V(\infty)$, then*

$$\mathbb{P}_0(V\ge x)=\mathbb{P}_x(\tau<\infty), \tag{5.7}$$

*Proof. Proof.* Taking $x=y=0$ in (5.5) yields $\mathbb{P}_0\big(R(t)\le 0\big)=\mathbb{P}_0\big(V(t)\ge 0\big)=1$ so that indeed 0 is absorbing for $\{R(t)\}$. We then get

$$\mathbb{P}_x(\tau\le T)=\mathbb{P}_x\big(R(T)\le 0\big)=\mathbb{P}_0\big(V(T)\ge x\big).$$

Let $T\to\infty$. □

## Dual recursions

We turn to a second extension of (5.5), (5.7) which does not require the Markov property but, however, works more easily when $\mathbb{T} = \mathbb{N}$ than when $\mathbb{T} = [0, \infty)$. We there assume that $\{V_n\}_{n\in\mathbb{N}}$ is generated by a recursion of the form

$$V_{n+1} = f(V_n, U_n), \tag{5.8}$$

where $\{U_n\}$ (the *driving sequence*) is a stationary sequence of random elements taking values in some arbitrary space $F$ and $f : E \times F \to E$ is a function. The (time-homogeneous) Markov case arises when the $U_n$ are i.i.d. (w.l.o.g., uniform on $F = (0, 1)$), but also much more general examples are incorporated. We shall need the following lemma, which is essentially just summarizes the standard properties of generalized inverses as occuring in, e.g., quantile functions.

**Lemma 5.7.** *Assume that $f(x, u)$ is continuous and nondecreasing in $x$ for each fixed $u \in F$ and define $g(x, u) = \inf\{y : f(y, u) \geq x\}$. Then for fixed $u$ $g(x, u)$ is left–continuous in $x$, nondecreasing in $x$ and strictly increasing on the interval $\{x : 0 < g(x, u) < \infty\}$. Further, $f(y, u) = \sup\{x : g(x, u) \leq y\}$ and*

$$g(x, u) \leq y \iff f(y, u) \geq x. \tag{5.9}$$

W.l.o.g., we can take $\{U_n\}$ with doubly infinite time, $n \in \mathbb{Z}$, and define the dual process $\{R_n\}_{n\in\mathbb{N}}$ by

$$R_{n+1} = g(R_n, U_{-n}), \qquad n \in \mathbb{N}; \tag{5.10}$$

when the initial value $x = R_0$ is important, we write $R_n(x)$.

**Theorem 5.8.** *Equations* (5.5) *and* (5.7) *also hold in the set–up of* (5.8) *and* (5.10).

*Proof. Proof.* For $T \in \mathbb{N}$, define $V_0^{(T)}(y) = y$,

$$V_1^{(T)}(y) = f\big(V_0^{(T)}(y), U_{-(T-1)}\big), \ \ldots \ , V_T^{(T)}(y) = f\big(V_{T-1}^{(T)}(y), U_0\big).$$

We shall show by induction that

$$V_T^{(T)}(y) \geq x \iff R_T(x) \leq y \tag{5.11}$$

(from this (5.5) follows by taking expectations and using the stationarity; since $g(0, u) = 0$, (5.6) then follows as above). The case $T = 0$ of (5.11) is the tautology $y \geq x \iff x \leq y$. Assume (5.11) shown for $T$. Replacing $y$ by $f(y, U_{-T})$ then yields

$$V_T^{(T)}\big(f(y, U_{-T})\big) \geq x \iff R_T(x) \leq f(y, U_{-T}).$$

But $V_T^{(T)}\big(f(y, U_{-T})\big) = V_{T+1}^{(T+1)}(y)$ and by (5.9),

$$R_T(x) \leq f(y, U_{-T}) \quad \iff \quad R_{T+1}(x) = g(R_T(x), U_{-T}) \leq y.$$

Hence (5.11) holds for $T + 1$. □

**Example 5.9.** Consider the a discrete time random walk reflected at 0, $V_{n+1} = (V_n + \xi_n)^+$ with increments $\xi_0, \xi_1, \ldots$ which are i.i.d. or, more generally, stationary.

In the set-up of Proposition 5.5 and Theorem 5.6, we need (for the Markov property) to assume that $\xi_0, \xi_1, \ldots$ are i.i.d. We take $E = [0, \infty)$ and for $y > 0$, we then get

$$\mathbb{P}_y(R_1 \le x) = \mathbb{P}_x(V_1 \ge y) = \mathbb{P}(x + \xi_0 \ge y) = \mathbb{P}(y - \xi_0 \le x).$$

For $y = 0$, we have $\mathbb{P}_0(R_1 = 0) = 1$. These two formulas show that $\{R_n\}$ evolves as a random walk $\breve{X}_n = -\xi_0 - \xi_{-1} - \cdots - \xi_{-n+1}$ with increments $-\xi_0, -\xi_1, \ldots$ as long as $R_n > 0$, i.e. $R_n(x) = x - \breve{X}_n$, $n < \tau$, $R_n(x) = 0$, $n \ge \tau$; when $(-\infty, 0]$ is hit, the value is instantaneously reset to 0 and $\{R_n\}$ then stays in 0 forever. We see further that we can identify $\tau$ and $\tau(x)$, and thus (5.7) is the same as the maximum representation (2.5) of the stationary distribution of $V$.

Consider instead the approach via Theorem 5.8 (which allows for increments that are just stationary). We let again $E = [0, \infty)$, take $U_k = \xi_k$ and $f(x, u) = (x + u)^+$. It is easily seen that $g(y, u) = (y - u)^+$ and so $\{R_n\}$ evolves as $\breve{X}$ as long as $R_n > 0$, while 0 is absorbing. With , $X_n^* = -\breve{X}_n$ it follows that $\tau = \inf\{n : x + \breve{X}_n \le 0\}$ $= \inf\{n : X_n^* \ge x\}$. This last expression shows that (5.6) is the same as a classical result in queueing theory known as *Loynes' lemma*, [11, IX.2c]. □

**Example 5.10.** Consider two-sided reflection in discrete time,

$$V_{n+1} = \min\bigl[b, (V_n + \xi_n)^+\bigr]. \tag{5.12}$$

For Theorem 5.6, we take $\xi_0, \xi_1, \ldots$ i.i.d. and $E = [0, \infty)$ (not $[0, b]$!). For $y > B$, we then get

$$\mathbb{P}_y(R_1 \le x) = \mathbb{P}_x(V_1 \ge y) \le \mathbb{P}_x(V_1 > b) = 0$$

for all $x$, i.e. $\mathbb{P}_y(R_1 = \infty) = 1$. For $0 \le y \le b$, $\mathbb{P}_y(R_1 \le x) = \mathbb{P}_x(V_1 \ge y)$ becomes

$$\mathbb{P}\bigl((x + \xi_0)^+ \ge y\bigr) = \begin{cases} 1 & y = 0, \\ \mathbb{P}(y - X_0 \le x) & 0 < y \le b. \end{cases}$$

Combining these facts show that $\{R_n\}$ evolves as $\breve{X}$ as long as $R_n \in (0, b]$. States 0 and $\infty$ are absorbing, and from $y > b$ $\{R_n\}$ is in the next step absorbed at $\infty$. Thus for $R_0 = x \in (0, b]$, absorbtion at 0 before $N$, i.e. $\tau \le N$, cannot occur if $(b, \infty)$ is entered and with $X_n = \xi_0 + \cdots + \xi_{n-1}$, $\tau[u, v) = \inf\{n \ge 0 : X_n \notin [u, v)\}$, $u \le 0 < v$, we get

$$\begin{aligned} \mathbb{P}_0(V_N \ge x) &= \mathbb{P}_x(\tau \le N) \\ &= \mathbb{P}\bigl(\tau[x - b, x) \le N,\, X_{\tau[x-b,x)} \ge x\bigr), \end{aligned} \tag{5.13}$$

$$\mathbb{P}(V \ge x) = \mathbb{P}(X_{\tau[x-b,x)} \ge x) \tag{5.14}$$

(note that $\tau[x - b, x)$ is always finite). □

The Markov process approach of Theorem 5.6 is from Siegmund (1976a), and the theory is often referred to as *Siegmund duality*, whereas the recursive approach of

Theorem 5.8 is from Asmussen & Sigman (1996). None of the approaches generalizes readily to higher dimension, as illustrated by Blaszczyszyn & Sigman (1999) in their study of many–server queues. For stochastic recursions in general, see Brandt *et al.* (1990) and Borovkov & Foss (1992).

The two–barrier formula (5.14) is implicit in Lindley (1959) and explicit in Siegmund (1976a), but has often been overlooked so that there are a number of alternative treatments of stationarity in two-barrier models around.

When applying Siegmund duality when $\mathbb{T} = [0, \infty)$, it is often more difficult to rigorously identify $\{R_t\}$ than when $\mathbb{T} = \mathbb{N}$. Asmussen (1995) gives a Markov-modulated generalization for $\mathbb{T} = [0, \infty)$, and there is some general theory for the recursive setting in Ryan & Sigman (2000).

**Example 5.11.** An early closely related and historically important example is Moran's model for the dam ([103]), which is discrete-time with the analogue of $Y_k$ having the form $Y_k = A_k - c$.The inflow sequence $\{A_n\}$ is assumed i.i.d. and the release is constant, say $c$ per time unit (if the content just before the release is $x < c$, only the amount $x$ is released), and we let $b$ denote the capacity of the dam. We will consider a slightly more general model where also the release at time $n$ is random, say $B_n$ rather than $c$ (the sequence $\{B_n\}$ is assumed i.i.d. and independent of $\{A_n\}$).

We let $Q_n^A$ denote the content just before the $n$th input (just after the $(n-1)$th release) and $Q_n^B$ the content just after that (just before the $(n+1)$th release). Then

$$Q_n^A = \left[Q_{n-1}^B - B_{n-1}\right]^+, \tag{5.15}$$
$$Q_n^B = \left(Q_n^A + A_n\right) \wedge K, \tag{5.16}$$
$$Q_n^A = \left[(Q_{n-1}^A + A_{n-1}) \wedge K - B_{n-1}\right]^+, \tag{5.17}$$
$$Q_n^B = \left([Q_{n-1}^B - B_{n-1}]^+ + A_n\right) \wedge K. \tag{5.18}$$

The recursions (5.17), (5.18) are obviously closely related to (3.1), but not a special case.

The stationary distributions of the recursions (5.17), (5.18) can be studied by much the same methods as used for (3.1). Consider e.g. (5.18) which can be written as $Q_n^B = f(Q_{n-1}^B, \boldsymbol{U}_{n-1})$ where $\boldsymbol{u} = (a, b)$, $f(x, \boldsymbol{u}) = ([x-b]^+ + a) \wedge K$ and $\boldsymbol{U}_{n-1} = (A_n, B_{n-1})$. The inverse function $g$ of $f$ in the sense of Proposition 5.7 is then given by

$$g(x, a, b) = \begin{cases} 0 & x = 0 \text{ or } x \in (0, b],\ a \geq b, \\ x - (a - b) & x \in (0, b],\ a < b, \\ \infty & x > b. \end{cases} .$$

It follows that the dual process $\{R_n\}$ started from $x$ evolves as the unrestricted random walk $\{(x - A_0)^+ - S_n\}$, starting from $(x - A_0)^+$ and having random walk increments $Z_n = A_n - B_{n-1}$, and that $\mathbb{P}_e(Q_n^B \geq x)$ is the probability that this process will exit $(0, K]$ to the right. □

## 5.3 Further properties of $\pi^b$

We first ask when $\pi^b$ has an atom at $b$, i.e., when $\pi^b\{b\} > 0$ so that there is positive probability of finding the buffer full. The dual question is whether $\pi^b\{0\} > 0$. For the answers, we need the fact that in the finite variation case, the underlying Lévy process $X$ has the form

$$X(t) = \theta t + S_1(t) - S_2(t) \tag{5.19}$$

where $S_1, S_2$ are independent subordinators.

**Theorem 5.12.** (i) *In the infinite variation case, $\pi^b\{b\} = \pi^b\{0\} = 0$. In the finite variation case* (5.19)*:*

(ii) *$\pi^b\{b\} > 0$ and $\pi^b\{0\} = 0$ when $\theta > 0$;*

(iii) *$\pi^b\{b\} = 0$ and $\pi^b\{0\} > 0$ when $\theta < 0$;*

*Proof.* We have $\pi^b\{b\} = \mathbb{P}\big(X\big(\tau[0,b)\big) \geq b\big)$. In the unbounded variation case, $(-\infty, 0)$ is regular for $X$, meaning that $(-\infty, 0)$ is immediately entered when starting from $X(0) = 0$ ([94, p. x]), so that in this case $X\big(\tau[0,b)\big) = 0$ and $\pi^b\{b\} = 0$. Thus $\pi^b\{b\} > 0$ can only occur in the bounded variation case which is precisely (5.19). Similarly for $\pi^b\{0\}$.

One has

$$S_1(t)/t \xrightarrow{\text{a.s.}} 0\,, \quad S_2(t)/t \xrightarrow{\text{a.s.}} 0, \quad t \to 0 \tag{5.20}$$

(cf. [11, p. 254]). Thus if $\theta < 0$, $X$ takes on negative values close to arbitrarily close to $t = 0$ so that $\tau[0,b) = 0$, $X\big(\tau[0,b)\big) = 0$ and $\pi^b\{b\} = 0$.

If $\theta > 0$, we get $X(t) > 0$ for $0 < t < \varepsilon$ for some $\varepsilon$. This implies that $X$ has a chance to escape to $[b,\infty)$ before hitting $(-\infty, 0)$ which entails $\mathbb{P}\big(X\big(\tau[0,b)\big) \geq b\big) > 0$ and $\pi^b\{b\} > 0$.

Combining these facts with a sign reversion argument yields (ii), (iii). □

**Corollary 5.13.** *In the spectrally positive (downward skipfree) case, $\pi^b\{b\} = 0$.*

*Proof.* The conclusion follows immediately from Theorem 5.12(i) in the infinite variation case. In the finite variation case where $S_2 \equiv 0$, our basic assumption that the paths of $X$ are non-monotonic implies $\theta < 0$, and we can appeal to Theorem 5.12(iii). □

The next result relates one- and two-sided reflection.

**Theorem 5.14.** *Assume $m = \mathbb{E}X(1) < 0$ so that $\pi^\infty$ exists, and that $X$ is spectrally positive with $\nu\{b\} = 0$. Then $\pi^b$ is $\pi^\infty$ conditioned to $[0,b]$, i.e. $\pi^b(A) = \pi^\infty(A)/\pi^\infty[0,b]$ for $A \subseteq [0,b]$. Equivalently, $\pi^b$ is the distribution of $M = \sup_{t\geq 0} X(t)$ conditioned on $M \leq b$.*

*Proof.* For $x \in (0,b)$, define $p_1(x) = \mathbb{P}\big(X\big(\tau[x-b,x)\big) \geq x\big)$, $p_2(x) = \mathbb{P}\big(X\big(\tau[x-b,x)\big) \geq b\big)$. Then spectral positivity implies that $X$ is downward skipfree so that

$$p_2(x) = p_1(x) + \big(1 - p_1(x)\big)p_2(b)\,, \quad p_1(x) = \frac{p_2(x)}{1 - p_2(b)}\,.$$

In terms of stationary distributions, this means

$$\pi^b[x,b] = \frac{\pi^\infty[x,\infty)}{\pi^\infty[0,b)} = \frac{\pi^\infty[x,\infty)}{\pi^\infty[0,b]},$$

where the last equality follows from Corollary 5.13. □

**Corollary 5.15.** *Assume $m = \mathbb{E}X(1) > 0$ and that $X$ is spectrally negative with $\nu\{-b\} = 0$. Then $\pi^b$ is the distribution of $b - \underline{M}$ conditioned on $\underline{M} \geq -b$ where $\underline{M} = \inf_{t\geq 0} X(t)$.*

# 6 The loss rate via Itô's formula

The identification of the loss rate $\ell^b$ of a Lévy process $X$ first appeared in Asmussen & Pihlsgård [19]. The derivation is based on optional stopping of the Kella-Whitt martingale followed by lots of tedious algebra, see Section 8. In this section we will follow an alternative more natural approach presented in Pihlsgård & Glynn [108]. One important point of that paper is that the dynamics of the two-sided reflection are governed by stochastic integrals involving the feeding process. Thus, all that is required is that stochastic integration makes sense. Hence, the natural framework is to take the input $X$ to be a semimartingale. What we will do in the current section is to solve the more general problem of explicitly identifying the local times $L$ and $U$ (in terms of $X$ and $V$) when the feeding process $X$ is a semimartingale. The main result in [19] follows easily from what will be presented below.

We start with a brief discussion about semimartingales. A stochastic process $X$ is a semimartingale if it is adapted, cadlag and admits a decomposition

$$X(t) = X(0) + N(t) + B(t)$$

where $N$ is a local martingale, $B$ a process of a.s. finite variation on compacts with $N(0) = B(0) = 0$. Alternatively, a semimartingale is a stochastic process for which the stochastic integral

$$\int H(s)\mathrm{d}X(s) \tag{6.1}$$

is well defined for $H$ belonging to a satisfactory rich class of processes (more precisely, the predictable processes). In (6.1), we will in this exposition take $H$ to be an adapted process with left continuous paths with right limits. The class of semimartingales forms a vector space and contains e.g. all adapted processes with cadlag paths of finite variation on compacts and Lévy processes. For a thorough introduction to semimartingales we refer to Protter [113].

Let $X$ and $Y$ be semimartingales. $[X,X]$ denotes the quadratic variation process of $X$ and $[X,X]^c$ is the continuous part of $[X,X]$. $[X,Y]$ is the quadratic covariation process (by some authors referred to as the bracket process) of $X$ and $Y$.

Section 4 contains a discussion concerning the existence and uniqueness of the solution $(V,L,U)$ to the underlying Skorokhod problem in which no assumptions about the structure of $X$ are made, so it applies to the case where $X$ is a general semimartingale. We will start by presenting two preliminary results.

**Lemma 6.1.** *$V$, $L$ and $U$ are semimartingales.*

*Proof.* Since $L$ and $U$ are cadlag, increasing and finite (thus of bounded variation) it follows that they are semimartingales. Semimartingales form a vector space, so we are done. $\square$

**Lemma 6.2.** *It holds that $[V,V]^c = [X,X]^c$.*

*Proof.* $L-U$ is cadlag of bounded variation and it follows by Theorem 26, p. 71, in [113] that $[L-U,L-U]^c = 0$ which is well known to imply $[X,L-U]^c = 0$, see, e.g., Theorem 28, p. 75 in [113]. The claim now follows from

$$\begin{aligned}[V,V]^c &= [X+L-U,X+L-U]^c \\ &= [X,X]^c + [L-U,L-U]^c + 2[X,L-U]^c = [X,X]^c. \end{aligned} \qquad \square$$

We now establish the link between $(L,U)$ and $X$. We choose to mainly focus on the local time $U$, by partly eliminating $L$, but it should be obvious how to obtain the corresponding results for $L$.

**Theorem 6.3.** *Let $X$ be a semimartingale which is reflected at $0$ and $b$. Then the following relationship holds.*

$$2bU(t) = V(0)^2 - V(t)^2 + 2\int_{0+}^{t} V(s-)\,\mathrm{d}X(s) + [X,X]^c(t) + J_R(t) \tag{6.2}$$

*where $J_R$ is pure jump, increasing and finite with*

$$J_R(t) = \sum_{0<s\le t} \varphi\big(V(s-),\Delta X(s)\big), \tag{6.3}$$

*where*

$$\varphi(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y\le -x, \\ y^2 & \text{if } -x<y<b-x, \\ 2y(b-x)-(b-x)^2 & \text{if } y\ge b-x. \end{cases}$$

*Proof.* By the definition of the quadratic variation process $[V,V]$ and Lemma 6.2,

$$V(t)^2 - V(0)^2 - 2\int_{0+}^{t} V(s-)\mathrm{d}V(s) = [V,V](t) = [X,X]^c(t) + \sum_{0<s\le t}(\Delta V(s))^2. \tag{6.4}$$

Furthermore,

$$\mathrm{d}V(t) = \mathrm{d}X(t) + \mathrm{d}L(t) - \mathrm{d}U(t) \quad\text{and}\quad V(s-) = V(s) - \Delta V(s),$$

so it follows by the formulation of the Skorokhod problem that

$$\begin{aligned}
&\int_{0+}^{t} V(s-)\mathrm{d}V(s) \\
&\quad= \int_{0+}^{t} V(s-)\mathrm{d}X(s) + \int_{0+}^{t}(V(s)-\Delta V(s))\mathrm{d}L(s) - \int_{0+}^{t}(V(s)-\Delta V(s))\mathrm{d}U(s) \\
&\quad= \int_{0+}^{t} V(s-)\mathrm{d}X(s) - \int_{0+}^{t}\Delta V(s)\mathrm{d}L(s) - bU(t) + \int_{0+}^{t}\Delta V(s)\mathrm{d}U(s) \\
&\quad= \int_{0+}^{t} V(s-)\mathrm{d}X(s) - \sum_{0<s\le t}\Delta V(s)\Delta L(s) - bU(t) + \sum_{0<s\le t}\Delta V(s)\Delta U(s).
\end{aligned} \tag{6.5}$$

(6.2) and (6.3) follow by combining (6.4) and (6.5) with the fact that

$$\begin{aligned}\Delta V(s) &= \max(\min(\Delta X(s), b - V(s-)), 0) + \min(\max(\Delta X(s), -V(s-)), 0),\\ \Delta V(s)\Delta L(s) &= -V(s-)(-\min(\Delta X(s) + V(s-), 0)),\\ \Delta V(s)\Delta U(s) &= (b - V(s-))\max(\Delta X(s) + V(s-) - b, 0).\end{aligned}$$

Since $0 \leq \varphi(x, y) \leq y^2$ it follows that $J_R(t)$ is increasing and that

$$J_R(t) \leq \sum_{0<s\leq t} (\Delta X(s))^2 \leq [X, X](t) < \infty. \qquad \square$$

We will need the next result in order to go from the path-by-path representation in Theorem 6.3 to the loss rate $\ell^b$.

**Lemma 6.4.** *Suppose that $X$ is a Lévy process with characteristic triplet $(\mu, \sigma, \nu)$ and $\mathbb{E}|X(1)| < \infty$. Let*

$$I(t) = \int_{0+}^{t} V(s-)\,\mathrm{d}X(s).$$

*If $V(0) \overset{\mathscr{D}}{=} \pi$ it holds that $\mathbb{E}I(t) = tm\mathbb{E}V(0)$.*

*Proof.* Let $\widetilde{X}(t) = X(t) - \sum_{0<s\leq t} \Delta X(s)\mathbb{1}(|\Delta X(s)| \geq 1)$, so that

$$I(t) = \int_{0+}^{t} V(s-)\mathrm{d}\widetilde{X}(s) + \sum_{0<s\leq t} V(s-)\Delta X(s)\mathbb{1}(|\Delta X(s)| \geq 1).$$

We let $\widetilde{Y}(t) = \widetilde{X}(t) - t\mathbb{E}\widetilde{X}(1)$. Then $\widetilde{Y}$ is a martingale (and thus a local martingale) and it follows by Theorem 29, p. 128, in [113] that

$$J(t) = \int_{0+}^{t} V(s-)\,\mathrm{d}\widetilde{Y}(s)$$

is also a local martingale. Theorem 29, p. 75, in [113] tells us that

$$\begin{aligned}[][J, J](t) = \int_{0+}^{t} V(s-)^2\mathrm{d}[\widetilde{Y}, \widetilde{Y}](s) &= \int_{0+}^{t} V(s-)^2\mathrm{d}[\widetilde{X}, \widetilde{X}](s) \leq b^2[\widetilde{X}, \widetilde{X}](t)\\ &= b^2\Big(\sigma^2 t + \sum_{0<s\leq t} (\Delta X(s))^2\mathbb{1}(|\Delta X(s)| < 1)\Big)\end{aligned}$$

and it follows that $\mathbb{E}[J, J](t) < \infty$ for all $t \geq 0$, which implies that $J$ is a martingale, see Corollary 3, p. 73 in [113]. Then $\mathbb{E}J(t) = \mathbb{E}J(0) = 0$, and thus

$$\mathbb{E}\int_{0+}^{t} V(s-)\,\mathrm{d}\widetilde{X}(s) = \mathbb{E}\int_{0+}^{t} V(s-)\mathbb{E}\widetilde{X}(1)\,\mathrm{d}s = t\mathbb{E}V(0)\mathbb{E}\widetilde{X}(1).$$

Furthermore, since $\sum_{0<s\leq t} \Delta X(s)\mathbb{1}(|\Delta X(s)| \geq 1)$ is a compound Poisson process and $V(s-)$ is independent of $\Delta X(s)$, we get that

$$\mathbb{E}\sum_{0<s\leq t} V(s-)\Delta X(s)\mathbb{1}\big(|\Delta X(s)| \geq 1\big) = t\mathbb{E}V(0)\Big(\int_{1}^{\infty} x\,\nu(\mathrm{d}x) + \int_{-\infty}^{-1} x\,\nu(\mathrm{d}x)\Big)$$

and it follows that

$$\mathbb{E}I(t) = t\mathbb{E}V(0)\mathbb{E}\widetilde{X}(1) + t\mathbb{E}V(0)\left(\int_1^\infty x\,\nu(\mathrm{d}x) + \int_{-\infty}^{-1} x\,\nu(\mathrm{d}x)\right) = tm\mathbb{E}V(0). \qquad \square$$

The next corollary is an easy consequence of Theorem 6.3 and Lemma 6.4 and is precisely the main result in the paper [19]. Note that as we keep $b$ fixed it is no restriction to assume that the support of $\nu$ is $[-a,\infty)\setminus\{0\}$ for some $a \geq b$. Otherwise we just truncate $\nu$ at $-a$ (we then get a point mass of size $\nu((-\infty,-a])$ at $-a$). The truncation does not affect $V$ and hence not $\ell^b$.

**Corollary 6.5.** *Let $X$ be a Lévy process with characteristic triplet $(\mu,\sigma,\nu)$. If $\int_1^\infty y\nu(\mathrm{d}y) = \infty$, then $\ell^b = \infty$ and otherwise*

$$\ell^b = \frac{1}{2b}\big\{2m\mathbb{E}V + \sigma^2 + \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^\infty \varphi(x,y)\nu(\mathrm{d}y)\big\}. \tag{6.6}$$

*Proof.* The first part is obvious. The second part follows immediately from (6.2) and (6.3) and Lemma 6.4 if we note that for a Lévy process $[X,X]^c(t) = \sigma^2 t$. $\square$

# 7 Two martingales

We will need nothing more sophisticated here than taking the property of $\{M(t)\}_{t\geq 0}$ to be a martingale as

$$\mathbb{E}\big[M(t+s)\,\big|\,\mathscr{F}(t)\big] = M(t), \qquad t \geq 0,\ s > 0\,, \tag{7.1}$$

where $\big\{\mathscr{F}(t)\big\}_{t\geq 0}$ is the natural filtration generated by the Lévy process, i.e. $\mathscr{F}(t) = \sigma\big(X(v):\, 0 \leq v \leq t\big)$.

The applications of martingales in the present context are typically optional stopping, i.e. the identity $\mathbb{E}M(\tau) = M(0)$ for a stopping time $\tau$ when $M(0)$ is deterministic or $\mathbb{E}M(\tau) = \mathbb{E}M(0)$ in the general case. This is not universally true, but conditions need to be verified, for example

$$\mathbb{E}\sup_{t\leq\tau}|M(t)| < \infty\,. \tag{7.2}$$

## The Wald martingale

A classical example in the area of Lévy processes is the Wald martingale given by

$$M(t) = \mathrm{e}^{\alpha X(t) - t\kappa(\alpha)}\,; \tag{7.3}$$

The proof that this is a martingale is elementary using the property of independent stationary increments and the definition of the Lévy exponent $\kappa$.

**Remark 7.1.** For the Wald martingale $\mathrm{e}^{\theta X(t)-t\kappa(\theta)}$, there is an usually easier approach to justify stopping than (7.2): consider the exponentially tilted Lévy process with $\kappa_\theta(\alpha) = \kappa(\alpha+\theta) - \kappa(\theta)$. Then optional stopping is permissible if and only if $\mathbb{P}_\theta(\tau < \infty) = 1$. See [11, p. 362]. $\square$

**Example 7.2.** Consider Brownian motion with drift $\mu$ and variance constant $\sigma^2$, and the problem of computing the two-sided exit probability

$$\mathbb{P}\big(X\big(\tau[x-b,x)\big) \geq x\big) = \pi^b[x,b]$$

occurring in the calculation of the stationary distribution $\pi^b$.

We have $\kappa(\alpha) = \alpha\mu + \alpha^2\sigma^2/2$, and take $\alpha = \gamma = -2\mu/\sigma^2$ as the root of the Lundberg equation $\kappa(\alpha) = 0$. Then the martingale is $\mathrm{e}^{\gamma X(t)}$. Condition (7.2) holds for $\tau = \tau[x-b,x)$ since $x-b \leq X(t) \leq x$ for $t \leq \tau[x-b,x)$. Letting

$$\begin{aligned} p^+(x) &= \mathbb{P}\big(X\big(\tau[x-b,x)\big) \geq x\big) = \mathbb{P}\big(X\big(\tau[x-b,x)\big) = x\big)\,, \\ p^-(x) &= \mathbb{P}\big(X\big(\tau[x-b,x)\big) < x-b\big) = \mathbb{P}\big(X\big(\tau[x-b,x)\big) = x-b\big) \end{aligned}$$

(note the path properties of Brownian motion for the second expression!), optional stopping thus gives

$$1 = M(0) = \mathbb{E}M\big(\tau[x-b,x)\big) = p^+(x)\mathrm{e}^{\gamma x} + p^-(x)\mathrm{e}^{\gamma(x-b)}\,.$$

Together with $1 = p^+(x) + p^-(x)$ this gives

$$p^+(x) = \frac{1-\mathrm{e}^{\gamma(x-b)}}{\mathrm{e}^{\gamma x} - \mathrm{e}^{\gamma(x-b)}} = \frac{\mathrm{e}^{-\gamma x} - \mathrm{e}^{-\gamma b}}{1-\mathrm{e}^{-\gamma b}}\,. \tag{7.4}$$

The last expression identifies $\pi^b$ as the distribution of an exponential r.v. $W$ conditioned to $[0,b]$ when $\gamma > 0$, (i.e. $\mu < 0$) and of $-W$ when $\gamma < 0$ (i.e. $\mu > 0$). □

**Example 7.3.** Consider again the Brownian setting, but now with the problem of computing quantities like

$$r^+ = \mathbb{E}[\mathrm{e}^{-q\tau};\, X(\tau) = v]\,, \quad r^- = \mathbb{E}[\mathrm{e}^{-q\tau};\, X(\tau) = u]\,, \quad r = r^+ + r^- = \mathbb{E}\mathrm{e}^{-q\tau}$$

where $\tau = \inf\{t : X(t) \notin [u,v]\}$ with $u < 0 < b$ and $q > 0$, as occurring in the calculation of the scale function.

We take $\alpha$ as root of

$$q = \kappa(\alpha) = \alpha\mu + \alpha^2\sigma^2/2$$

(rather than the Lundberg equation $\kappa(\alpha) = 0$). Since $q > 0$, there are two roots, one positive and one negative,

$$\theta^+ = \theta^+(q) = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 q}}{2}\,, \quad \theta^- = \theta^-(q) = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 q}}{2}\,.$$

We therefore have two Wald martingales at disposal, $\mathrm{e}^{\theta^+ X(t) - qt}$ and $\mathrm{e}^{\theta^- X(t) - qt}$.

Instead of verifying condition (7.2) (trivial for $\theta^+$ but not $\theta^-$!), it is easier to note that in the present context, we have $\tau < \infty$ for all $\mu$, and this implies the conditions of Remark 7.1. Optional stopping thus gives

$$1 = r^+\mathrm{e}^{\theta^+ v} + r^+ r^- \mathrm{e}^{\theta^+ v}\,, \quad 1 = r^+\mathrm{e}^{\theta^- v} + r^+ r^- \mathrm{e}^{\theta^- v}\,.$$

These two linear equations can immediately be solved for $r^+, r^-$, and then also $r = r^+ + r^-$ is available. □

**Example 7.4.** Consider again the two-sided exit problem, but now with exponential ($\delta$) jumps at rate $\lambda$ in the positive directions added to the Brownian motion.

Inspired by Examples 7.2, 7.3 we look for solutions of the Lundberg equation

$$0 = \kappa(\alpha) = \alpha\mu + \alpha^2\sigma^2/2 + \lambda\frac{\delta}{\delta - \alpha}\,.$$

This is a cubic, which looks promising since we have three unknowns, the probability of exit below at $u$, the probability of continuous exit above at $v$, and the probability of exit above by a jump. However, only two of the three roots $\theta$ leads to a permissible Wald martingale; since $\log \mathbb{E}\mathrm{e}^{\alpha X(1)}$ is convex, there is at most two roots in the interval where this function is finite. □

## The Kella-Whitt martingale

Consider a modification $Z(t) = X(t) + B(t)$ of the Lévy process, where $\{B(t)\}_{t\geq 0}$ is adapted with $D$-paths, locally bounded variation, continuous part $\{B^c(t)\}$, and jumps $\Delta B(s) = B(s) - B(s-)$. The Kella-Whitt martingale

$$\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha Z(s)}\,\mathrm{d}s + \mathrm{e}^{\alpha Z(0)} - \mathrm{e}^{\alpha Z(t)} \\ + \alpha\int_0^t \mathrm{e}^{\alpha Z(s)}\,\mathrm{d}B^c(s) + \sum_{0\leq s\leq t} \mathrm{e}^{\alpha Z(s)}\big(1 - \mathrm{e}^{-\alpha\Delta B(s)}\big)\,. \quad (7.5)$$

Since the Kella-Whitt martingale (ii) is less standard than the Wald martingale (i), we add some discussion and references. The first occurence is in Kella & Whitt [84] where it was identified as a rewriting of the stochastic integral

$$\int_0^t \exp\big\{\alpha\big(X(s-) + B(s-)\big) + s\kappa(\alpha)\big\}\,\mathrm{d}W(s)$$

where $W$ is the Wald martingale. The stochastic integral representation immediately gives the *local* martingale property. To proceed from this, much subsequent work next shows the *global* martingale property by direct calculations specific for the particular application. However, recently Kella & Boxma [80] showed that this is automatic under minor conditions.

A simple but still useful case is the Kella-Whitt martingale with $B(t) \equiv 0$,

$$\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha X(s)}\,\mathrm{d}s + \mathrm{e}^{\alpha x} - \mathrm{e}^{\alpha X(t)} \quad (7.6)$$

A survey of applications of the Kella-Whitt martingale is in Asmussen [11, IX.3]; see also Kyprianou [94], [95].

# 8 The loss rate via the Kella-Whitt martingale

In this section we summarize the original derivation of the loss rate $\ell = \ell^b$ which is presented in Asmussen & Pihlsgård [19]. It is essentially based on optional stopping of the Kella-Whitt martingale for $V$. As stated in the Introduction and in Section 6, this is less straightforward than the direct Itô integration method used in Section 6. It is not difficult to see why the latter approach leads more directly to the result: the Kella-Whitt martingale, see Kella & Whitt [84], is itself obtained as a stochastic integral with respect to the Wald martingale (indexed by, say, $\alpha$) for $V$, so this method implicitly relies on Itô's formula and, more importantly, there is introduced an arbitrariness via $\alpha$ which is removed by letting $\alpha \to 0$. This requires a delicate analysis, which is to a large extent based on Taylor expansions and tedious algebra, and hence of limited probabilistic interest. This is perhaps the most serious drawback of the original approach. However, the martingale technique also has advantages. E.g., if the process $X$ is such that the equation $\kappa(\alpha) = 0$ has a non-zero root $\gamma$, we obtain an alternative formula for $\ell$, see Theorem 8.6 below, which turns out to be very useful when we derive asymptotics for $\ell^b$ as $b \to \infty$ when $X$ is light tailed. We see no immediate way of deriving this result directly via Itô's formula.

To follow the exposition in [19], we need to introduce some further notation. First, we split $L$ and $U$ into their continuous and jump parts, i.e.,

$$L(t) = L_c(t) + L_j(t) \quad \text{and} \quad U(t) = U_c(t) + U_j(t) \tag{8.1}$$

where $L_c(t)$ is the continuous part of $L$, $L_j(t)$ the jump part etc., i.e., $L_j(t) = \sum_{0 \le s \le t} \Delta L(s)$ and $L_c(t) = L(t) - L_j(t)$. Further, we treat the contributions to $L$ and $U$ coming from small and large jumps of $X$ separately: let

$$\begin{aligned} \underline{\Delta} L(s) &= \Delta L(s) \mathbb{1}\big(-k \le \Delta X(s) \le 0\big)\,, \qquad & \overline{\Delta} L(s) &= \Delta L(s) \mathbb{1}\big(\Delta X(s) < -k\big)\,, \\ \underline{\Delta} U(s) &= \Delta U(s) \mathbb{1}\big(0 \le \Delta X(s) \le k\big)\,, \qquad & \overline{\Delta} U(s) &= \Delta U(s) \mathbb{1}\big(\Delta X(s) > k\big) \end{aligned}$$

where $k$ is a constant such that $k > \max(1, b)$. Further, we let

$$\ell_j^b = \mathbb{E} U_j(1)\,, \quad \ell_c^b = \mathbb{E} U_c(1)\,, \quad \underline{\ell}_j^b = \mathbb{E} \sum_{0 \le s \le 1} \underline{\Delta} U(s)\,, \quad \overline{\ell}_j^b = \mathbb{E} \sum_{0 \le s \le 1} \overline{\Delta} U(s)\,,$$

and similarly at 0. Clearly $\ell_j^b = \underline{\ell}_j^b + \overline{\ell}_j^b$ and $\ell_j^0 = \underline{\ell}_j^0 + \overline{\ell}_j^0$. The Lévy exponent $\kappa(\alpha)$ can be rewritten as

$$c_k \alpha + \sigma^2 \alpha^2 / 2 + \int_{-\infty}^{\infty} \left[ \mathrm{e}^{\alpha x} - 1 - \alpha x \mathbb{1}\big(|x| \le k\big) \right] \nu(\mathrm{d}x), \tag{8.2}$$

where $c_k = c + \int_{|y| > k} y \, \nu(\mathrm{d}y)$.

The paper [19] relies on the original reference on the Kella-Whitt local martingale associated with Lévy processes, [84], and on Asmussen & Kella [18] for the generalisation to a multidimensional local martingale associated with Markov additive processes with finite state space Markov modulation. However, it was recently discovered in Kella & Boxma [80] that without any further assumptions, these local martingales are in fact martingales. This very useful result makes it possible to keep the treatment below slightly shorter than what was presented in [19].

The first step in the analysis is to show that $\ell^0$ and $\ell^b$ are well defined if the process $X$ is sufficiently well behaved.

**Lemma 8.1.** *If $\mathbb{E}|X(1)| < \infty$ then $\mathbb{E} L(t) < \infty$ and $\mathbb{E} U(t) < \infty$ (for all $t$).*

*Proof.* Assume (without loss of generality) that $V(0) = 0$. Let $\tau(0) = 0$ and, for $j \geq 1$, $\upsilon(j) = \inf\{t > \tau(j-1) : V(t) = b\}$, $\tau(j) = \inf\{t > \upsilon(j) : V(t) = 0\}$. We view $V$ as regenerative with $i$th cycle equal to $[\tau(i-1), \tau(i))$. Let $n(t)$ denote the number of cycles completed in $[0, t]$. Then

$$L(t) = \sum_{i=1}^{n(t)} C_i + R(t)$$

where $C_i$ is the contribution to $L(t)$ from the $i$th cycle and $R(t)$ is what comes from $[\tau(n(t)), t]$. Let $m(t)$ be the local time corresponding to $X$ reflected at 0. Then $C_i \stackrel{\mathscr{D}}{=} m(\upsilon(1)) + J_1$ where $J_1$ comes from a jump of $X$ ending the cycle. It is known, see Lemma 3.3 in [11], that $\mathbb{E}m(t) < \infty$ and this together with $\mathbb{E}|X(1)| < \infty$ yields $\mathbb{E}C_i < \infty$ and $\mathbb{E}R(t) < \infty$. Furthermore, $\mathbb{E}(\tau(i) - \tau(i-1)) > 0$ and this implies that $\mathbb{E}n(t) < \infty$, see Proposition 1.4 in [11]. It now follows from Wald's identity that

$$\mathbb{E}L(t) = \mathbb{E}n(t)\mathbb{E}C_1 + \mathbb{E}R(t) < \infty.$$

$\mathbb{E}U(t) < \infty$ is now immediate from the formulation of the Skorokhod problem. $\square$

The next step is the construction of the Kella-Whitt martingale for the reflected process $V$.

**Proposition 8.2.** *Assume that $\mathbb{E}|X(1)| < \infty$. For each $t$, let $M(t)$ be the random variable*

$$\begin{aligned}
&\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V(s)}\mathrm{d}s + \mathrm{e}^{\alpha V(0)} - \mathrm{e}^{\alpha V(t)} \\
&+\alpha\int_0^t \mathrm{e}^{\alpha V(s)}\mathrm{d}L_c(s) + \sum_{0\leq s\leq t} \mathrm{e}^{\alpha V(s)}(1 - \mathrm{e}^{-\alpha\Delta L(s)}) \\
&-\alpha\int_0^t \mathrm{e}^{\alpha V(s)}\mathrm{d}U_c(s) + \sum_{0\leq s\leq t} \mathrm{e}^{\alpha V(s)}(1 - \mathrm{e}^{\alpha\Delta U(s)}).
\end{aligned}$$

*Then*

$$\begin{aligned}
M(t) = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V(s)}\mathrm{d}s + \mathrm{e}^{\alpha V(0)} - \mathrm{e}^{\alpha V(t)} + \alpha L_c(t) + \sum_{0\leq s\leq t}(1 - \mathrm{e}^{-\alpha\Delta L(s)}) \\
- \alpha\mathrm{e}^{\alpha b}U_c(t) + \mathrm{e}^{\alpha b}\sum_{0\leq s\leq t}(1 - \mathrm{e}^{\alpha\Delta U(s)}) \qquad (8.3)
\end{aligned}$$

*and $M$ is a zero mean martingale.*

*Proof.* $L$ and $U$ solve the Skorokhod problem stated in the Introduction, so the first claim is clearly true. $L - U$ is of bounded variation and it follows by what was proven in [80] that $M$ is a martingale. $\square$

We proceeed by stating two lemmas.

**Lemma 8.3.** *$\ell^b$ satisfies the following equation:*

$$\begin{aligned}\alpha(1-\mathrm{e}^{\alpha b})\ell^b = &-\kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V(0)} + \alpha\mathbb{E}X(1) - \alpha\mathrm{e}^{\alpha b}\overline{\ell}_j^b + \alpha\overline{\ell}_j^0 \\ &+\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2 + \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}L(s))^2 \\ &-\mathrm{e}^{\alpha b}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}U(s)}) - \mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L(s)}) + \mathrm{o}(\alpha^2),\end{aligned} \tag{8.4}$$

*where $\mathrm{o}(\alpha^2)/\alpha^2 \to 0$ if $\alpha \to 0$.*

*Proof.* If we take $t = 1$ in Proposition 8.2 and use the stationarity of $V$, we get

$$0 = \kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V(0)} + \alpha\ell_c^0 + \mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\Delta L(s)}) - \alpha\mathrm{e}^{\alpha b}\ell_c^b + \mathrm{e}^{\alpha b}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\Delta U(s)}). \tag{8.5}$$

We write

$$\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\Delta U(s)}) = \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\underline{\Delta}U(s)}) + \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}U(s)}), \tag{8.6}$$

$$\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\Delta L(s)}) = \sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\underline{\Delta}L(s)}) + \sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L(s)}) \tag{8.7}$$

and apply the expansion

$$\mathrm{e}^{\alpha x} = 1 + \alpha x + \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^3}{6}\mathrm{e}^{\theta\alpha x}, \qquad \theta \in (0,1) \tag{8.8}$$

to the first parts of the r.h.s. of (8.6) and (8.7) and get for the part in (8.6):

$$\begin{aligned}\mathrm{e}^{\alpha b}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\underline{\Delta}U(s)}) &= \mathrm{e}^{\alpha b}\mathbb{E}\big(-\alpha\sum_{0\le s\le 1}\underline{\Delta}U(s) - \frac{\alpha^2}{2}\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2\big) + \mathrm{o}(\alpha^2) \\ &= -\alpha\mathrm{e}^{\alpha b}\underline{\ell}_j^b - \mathrm{e}^{\alpha b}\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2 + \mathrm{o}(\alpha^2) \\ &= -\alpha\mathrm{e}^{\alpha b}(\ell_j^b - \overline{\ell}_j^b) - \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2 + \mathrm{o}(\alpha^2),\end{aligned} \tag{8.9}$$

because $\mathbb{E}\sum_{0\le s\le 1}\alpha^3(\underline{\Delta}U(s))^3\mathrm{e}^{\theta\alpha\underline{\Delta}U(s)}/6 = \mathrm{o}(\alpha^2)$, $\ell_j^b = \underline{\ell}_j^b + \overline{\ell}_j^b$ and $\mathrm{e}^{\alpha b}\alpha^2/2 = \alpha^2/2 + \mathrm{o}(\alpha^2)$. We proceed similarly for the part in (8.7) and get

$$\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\underline{\Delta}L(s)}) = \alpha(\ell_j^0 - \overline{\ell}_j^0) - \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}L(s))^2 + \mathrm{o}(\alpha^2). \tag{8.10}$$

If we combine (8.5), (8.6), (8.7), (8.9) and (8.10) we get

$$\begin{aligned}0 = \kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V(0)} &+ \alpha\ell^0 - \alpha\mathrm{e}^{\alpha b}\ell^b - \alpha\overline{\ell}_j^0 + \alpha\mathrm{e}^{\alpha b}\overline{\ell}_j^b - \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2 \\ -\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}L(s))^2 &+ \mathrm{e}^{\alpha b}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}U(s)}) + \mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L(s)}) + \mathrm{o}(\alpha^2).\end{aligned}$$

The claim now follows if we make the substitution $\ell^0 = \ell^b - \mathbb{E}X(1)$ and rearrange terms. □

**Lemma 8.4.** *Let, for $x > 0$, $\overline{\nu}(x) = \nu(x,\infty)$ and, for $x < 0$, $\underline{\nu}(x) = \nu(-\infty, x)$. In stationarity it then holds that as $\alpha \to 0$,*

$$
\begin{aligned}
&\kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V(t)} \\
&\quad = \mathrm{o}(\alpha^2) + \int_0^b \mathrm{e}^{\alpha x}\pi(\mathrm{d}x)\int_{-\infty}^{\infty} \mathrm{e}^{\alpha y}\mathbb{1}(|y|\geq k)\nu(\mathrm{d}y) \\
&\qquad - \int_{-\infty}^{\infty} I(|y|\geq k)\nu(\mathrm{d}y) + \alpha\Big(c_k - \int_0^b x\pi(\mathrm{d}x)\int_{-\infty}^{\infty}\mathbb{1}(|y|\geq k)\nu(\mathrm{d}y)\Big) \\
&\qquad + \alpha^2\Big(c_k\int_0^b x\pi(\mathrm{d}x) + \sigma^2/2 + \int_{-k}^{k} y^2/2\nu(\mathrm{d}y) \quad , \\
&\qquad\qquad - \int_0^b x^2/2\pi(\mathrm{d}x)\int_{-\infty}^{\infty}\mathbb{1}(|y|\geq k)\nu(\mathrm{d}y)\Big)
\end{aligned}
\tag{8.11}
$$

$$
\begin{aligned}
\mathrm{e}^{\alpha b}\mathbb{E}\sum_{0\leq s\leq 1}(1-\mathrm{e}^{\alpha\overline{\Delta}U(s)}) &= \mathrm{e}^{\alpha b}\int_0^b \pi(\mathrm{d}x)\int_k^{\infty}(1-\mathrm{e}^{\alpha(y-b+x)})\nu(\mathrm{d}y) \\
&= (1+\alpha b+\alpha^2 b^2/2)\overline{\nu}(k) - \int_0^b \mathrm{e}^{\alpha x}\pi(\mathrm{d}x)\int_k^{\infty}\mathrm{e}^{\alpha y}\nu(\mathrm{d}y) + \mathrm{o}(\alpha^2),
\end{aligned}
\tag{8.12}
$$

$$
\begin{aligned}
\mathbb{E}\sum_{0\leq s\leq 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L(s)}) &= \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^{-k}(1-\mathrm{e}^{\alpha(x+y)})\nu(\mathrm{d}y) \\
&= \underline{\nu}(-k) - \int_0^b \mathrm{e}^{\alpha x}\pi(\mathrm{d}x)\int_{-\infty}^{-k}\mathrm{e}^{\alpha y}\nu(\mathrm{d}y) + \mathrm{o}(\alpha^2),
\end{aligned}
\tag{8.13}
$$

$$
\begin{aligned}
\alpha\mathrm{e}^{\alpha b}\overline{\ell}_j^b &= \alpha\mathrm{e}^{\alpha b}\int_0^b \pi(\mathrm{d}x)\int_k^{\infty}(y-b+x)\nu(\mathrm{d}y) \\
&= (\alpha+\alpha^2 b)\int_0^b \pi(\mathrm{d}x)\int_k^{\infty}(y-b+x)\nu(\mathrm{d}y) + \mathrm{o}(\alpha^2),
\end{aligned}
\tag{8.14}
$$

$$
\alpha\overline{\ell}_j^0 = -\alpha\int_0^b \pi(\mathrm{d}x)\int_{-\infty}^{-k}(x+y)\nu(\mathrm{d}y), \tag{8.15}
$$

$$
\alpha m = \alpha c_k + \alpha\int_{-\infty}^{\infty} y\mathbb{1}(|y|\geq k)\nu(\mathrm{d}y), \tag{8.16}
$$

$$
\frac{\alpha^2}{2}\mathbb{E}\sum_{0\leq s\leq 1}(\underline{\Delta}U(s))^2 = \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{b-x}^{k}(y-b+x)\nu(\mathrm{d}y), \tag{8.17}
$$

$$
\frac{\alpha^2}{2}\mathbb{E}\sum_{0\leq s\leq 1}(\underline{\Delta}L(s))^2 = \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{-k}^{-x}(x+y)^2\nu(\mathrm{d}y). \tag{8.18}
$$

*Proof.* Clearly $\mathbb{E}\mathrm{e}^{\alpha V(s)} = \int_0^b \mathrm{e}^{\alpha x}\pi(\mathrm{d}x)$. (8.11) follows if we use the representation for $\kappa(\alpha)$ in (8.2) and expand the integrands corresponding to the compact sets $[-k,k]$ and $[0,b]$ according to (8.8). The remaining statements all follow by conditioning on $V(s-)$ and applying (8.8) where appropriate. $\square$

We are now ready to identify $\ell^b$ in terms of $\pi$ and $(c, \sigma^2, \nu)$. We remind of the remark just before Corollary 6.5.

**Theorem 8.5.** *If $\int_1^\infty y\nu(\mathrm{d}y) = \infty$, then $\ell^b = \infty$ and otherwise*

$$\ell^b = \frac{1}{2b}\left\{2m\mathbb{E}V + \sigma^2 + \int_0^b \pi(\mathrm{d}x) \int_{-\infty}^\infty \varphi(x,y)\nu(\mathrm{d}y)\right\}, \tag{8.19}$$

*where*

$$\varphi(x,y) = \begin{cases} -(x^2 + 2xy) & \text{if } y \le -x, \\ y^2 & \text{if } -x < y < b - x, \\ 2y(b-x) - (b-x)^2 & \text{if } y \ge b - x. \end{cases}$$

*Proof.* The first claim is obvious. We use (8.4) and identify the terms in the right hand side via Lemma 8.4 and get,

$$\begin{aligned}
\alpha(1 - \mathrm{e}^{\alpha b})\ell^b &= -c_k\alpha^2 \int_0^b x\pi(\mathrm{d}x) - \frac{\sigma^2\alpha^2}{2} - \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_0^{b-x} y^2\nu(\mathrm{d}y) \\
&- \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{-x}^0 y^2\nu(\mathrm{d}y) + (\overline{\nu}(k) + \underline{\nu}(-k))\frac{\alpha^2}{2}\int_0^b x^2\pi(\mathrm{d}x) \\
&+ \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{b-x}^k ((x-b)^2 + 2y(x-b))\nu(\mathrm{d}y) - \alpha^2 b\int_k^\infty y\nu(\mathrm{d}y) \\
&+ \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{-k}^{-x} (x^2 + 2xy)\nu(\mathrm{d}y) + \alpha^2 b\int_0^b (b-x)\overline{\nu}(k) + \mathrm{o}(\alpha^2).
\end{aligned} \tag{8.20}$$

We divide both sides of (8.20) by $\alpha(1 - \mathrm{e}^{\alpha b})$ and let first $\alpha \to 0$ and then $k \to \infty$ and get the limit (8.19) (note that $c_k \to \mathbb{E}X(1)$ as $k \to \infty$). □

The next result, which follows almost directly from the proof of Lemma 8.3, gives an alternative expression for $\ell^b$ whenever we can find a non-zero root $\gamma$ of $\kappa(\alpha) = 0$ (a genuine root in the sense that the real part as well as the imaginary part of $\mathbb{E}\mathrm{e}^{\gamma X(1)}$ are finite, cf. Lemma 10.7 where the meaning of $\kappa(\alpha) = 0$ is different). Note that in the original version (Theorem 3.2 in [19]) it is required that $\gamma$ is real but this is not necessary.

**Theorem 8.6.** *Assume that there exists a non-zero root $\gamma$ of the equation $\kappa(\alpha) = 0$. Then*

$$\ell^b = \frac{1}{\mathrm{e}^{\gamma b} - 1}\{\mathrm{e}^{\gamma b} I_1 + I_2 - \mathbb{E}X(1)\} \tag{8.21}$$

*where*

$$\begin{aligned}
I_1 &= \int_0^b \pi(\mathrm{d}x)\int_{b-x}^\infty ((y - b + x) + \gamma^{-1}(1 - \mathrm{e}^{\gamma(y-b+x)}))\nu(\mathrm{d}y) \\
I_2 &= \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^{-x} ((x+y) + \gamma^{-1}(1 - \mathrm{e}^{\gamma(x+y)}))\nu(\mathrm{d}y)
\end{aligned}$$

*Proof.* Let $\epsilon > 0$. We truncate the Lévy measure at $\epsilon$ and $-\epsilon$. By arguing precisely as when we derived (8.4) and taking $\alpha = \gamma$, we get

$$\gamma(\mathrm{e}^{\gamma b} - 1)\ell^b = -\gamma \mathbb{E}X(1) + \mathrm{e}^{\gamma b} I_1^\epsilon + I_2^\epsilon + \mathrm{O}(\epsilon), \tag{8.22}$$

where

$$I_1^\epsilon = \int_0^b \pi(\mathrm{d}x) \int_{(b-x)\vee\epsilon}^\infty (\gamma(y-b+x) + (1-\mathrm{e}^{\gamma(y-b+x)}))\nu(\mathrm{d}y),$$
$$I_2^\epsilon = \int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{-(x\vee\epsilon)} (\gamma(x+y) + (1-\mathrm{e}^{\gamma(x+y)}))\nu(\mathrm{d}y),$$

and the claim follows if we divide both sides of (8.22) by $\gamma(\mathrm{e}^{\gamma b} - 1)$, let $\epsilon \downarrow 0$ and apply monotone convergence. $\square$

The identification of $\ell^b$ is almost trivial in the discrete time case, see Section 3. However, the continuous time case is much more involved and less intuitive, no matter the choice of method for deriving the expression(s) for $\ell^b$ (the direct Itô approach presented in Section 6 or the methods used in the current section). In order to provide the presentation with some intuition, we present an alternative heuristic derivation of the formula for $\ell^b$ as given in (8.21). Recall the definitions of $\ell_c^b$, $\ell_j^b$ etc. given above. We will derive four equations involving $\ell_c^b$, $\ell_j^b$, $\ell_c^0$ and $\ell_j^0$ and solve for the unknowns. The first equation follows directly from the Skorokhod problem formulation and the stationarity of $V$:

$$\ell_c^0 + \ell_j^0 - \ell_c^b - \ell_j^b = -m. \tag{8.23}$$

The second equation is

$$\ell_j^b = \int_0^b \pi(\mathrm{d}x) \int_{b-x}^\infty (y-b+x)\nu(\mathrm{d}y) \tag{8.24}$$

and the third is

$$\ell_j^0 = -\int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{-x} (x+y)\nu(\mathrm{d}y). \tag{8.25}$$

In order to obtain the fourth equation, we take $\alpha = \gamma$ in (8.3), which yields

$$\gamma\ell_c^0 - \gamma\mathrm{e}^{\gamma b}\ell_c^b = -\mathrm{e}^{\gamma b}\int_0^b \pi(\mathrm{d}x) \int_{b-x}^\infty (1-\mathrm{e}^{\gamma(y-b+x)})\nu(\mathrm{d}y)$$
$$- \int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{-x} (1-\mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y). \tag{8.26}$$

(8.24), (8.25) and (8.26) apply at least if the jump part of $X$ is of bounded variation, i.e., if $\int_{-1}^1 |x|\nu(\mathrm{d}x) < \infty$. In this case they follow by straightforward conditioning on the value of $V$ immediately prior to a jump of $X$. By combining (8.23), (8.24), (8.26) and (8.26), we may identify the unknowns and the expression for $\ell^b$ given in (8.21) follows from $\ell^b = \ell_c^b + \ell_j^b$.

**Example 8.7.** Assume that $X$ is Brownian motion with drift $\mu$ and variance $\sigma^2$, i.e., $\kappa(\alpha) = \mu\alpha + \sigma^2\alpha^2/2$. Then $\gamma = -2\mu/\sigma^2$ and Theorem 8.6 gives us $\ell^b = -\mu/(\mathrm{e}^{-2b\mu/\sigma^2} - 1)$.

**Example 8.8.** Suppose that $X$ is a strictly stable Lévy process with index $\alpha \in (0,2) \setminus \{1\}$ (note that if $\alpha = 1$, then $\ell^b = \infty$ and if $\alpha = 2$, then $\ell^b = \sigma^2/2b$), i.e.,

$$\nu(\mathrm{d}x) = \begin{cases} c_+ x^{-(\alpha+1)}\mathrm{d}x & \text{if } x > 0, \\ c_-|x|^{-(\alpha+1)}\mathrm{d}x & \text{if } x < 0, \end{cases}$$

where $c_+, c_- \geq 0$ are such that $c_+ + c_- > 0$; see, e.g., Bertoin [28], pp. 216–218. Let $\beta = (c_+ - c_-)/(c_+ + c_-)$ and $\rho = 1/2 + (\pi\alpha)^{-1}\arctan(\beta\tan(\pi\alpha/2))$.

If $\alpha \in (0,1)$ then $\ell^b$ is 0 if $\beta = -1$ (then $X$ is the negative of a subordinator) and $\infty$ otherwise. We now consider the case $\alpha \in (1,2)$, which implies that $\mathbb{E}X(1) = \sigma = 0$. It follows from Theorem 1 in Kyprianou [93] and some rescaling manipulations that if $X$ is not spectrally one-sided, i.e., if $\rho \in (1 - 1/\alpha, 1/\alpha)$, then

$$\pi(\mathrm{d}x) = (bB(\alpha\rho, \alpha(1-\rho)))^{-1}(1 - x/b)^{\alpha\rho-1}(x/b)^{\alpha(1-\rho)-1}\mathrm{d}x,$$

where $B(\cdot,\cdot)$ is the beta function. Further, it turns out that in this example,

$$\int_{-\infty}^{\infty} \varphi(x,y)\nu(\mathrm{d}y) = 2(\alpha(\alpha-1)(2-\alpha))^{-1}(c_- x^{2-\alpha} + c_+(b-x)^{2-\alpha}),$$

and it follows from Theorem 8.5 (Theorem 3.1 in [19]) that

$$\ell^b = \frac{c_- B(2 - \alpha\rho, \alpha\rho) + c_+ B(2 - \alpha(1-\rho), \alpha(1-\rho))}{B(\alpha\rho, \alpha(1-\rho))\alpha(\alpha-1)(2-\alpha)b^{\alpha-1}}.$$

# 9 Phase-type jumps

A key step in the analysis of two-sided reflection is the computation of the stationary distribution or equivalently two-sided exit probabilities. This is not possible in general, but requires additional structure. One example is the spectrally negative case with the scale function available. Another one, that we concentrate on here, is phase-type jumps in both directions and an added Brownian component. This class of Lévy models has the major advantage of being dense (in the sense of $D$-convergence) in the class of all Lévy processes. Further, not only are explicit computations available for two-sided exit probabilities but also in a number of other problems standard in fluctuation theory for Lévy processes, see the survey in Asmussen [12] and the extensive list of references there.

## 9.1 Phase-type distributions

Phase-type distributions are absorption time distributions in finite continuous-time Markov processes (equivalently, lifelength distributions in terminating finite Markov processes). Let $\{J(t)\}_{t\geq 0}$ be Markov with a finite state space $E \cup \{\Delta\}$ such that $\Delta$ is absorbing and the rest transient. I.e., the process ends eventually up in $\Delta$ so that the absorption time (lifetime) $\zeta = \inf\{t : J(t) = \Delta\}$ is finite a.s. For $i, j \in E$, $i \neq j$, write $t_{ij}$ for the transition rate $i \to j$ and $t_i$ for the transition rate $i \to \Delta$. Define $t_{ii} = t_i + \sum_{j \in E}$ and let $\boldsymbol{T}$ be the $E \times E$ matrix with $ij$th element $t_{ij}$. If $\boldsymbol{\alpha}$ is an $E$-row vector with elements $\alpha_i$ summing to 1, we then define a phase-type (PH) distribution $F$ with representation $(E, \boldsymbol{\alpha}, \boldsymbol{T})$ (or just $(\boldsymbol{\alpha}, \boldsymbol{T})$) as the distribution of $\zeta$ corresponding to the initial distribution $\mathbb{P}_{\boldsymbol{\alpha}}$ of $\{J(t)\}$ given by $\mathbb{P}_{\boldsymbol{\alpha}}\big(J(0) = i\big) = 1$.

The situation is illustrated in the following figure, where we have represented the states by colored bullets, such that $\Delta$ corresponds to black. The process can be illustrated by a traditional graph, as above the horizontal line, or, as below, as a line of length $\zeta$ with segments colored according to the sample path. This last representation is the one to be used in subsequent figures.

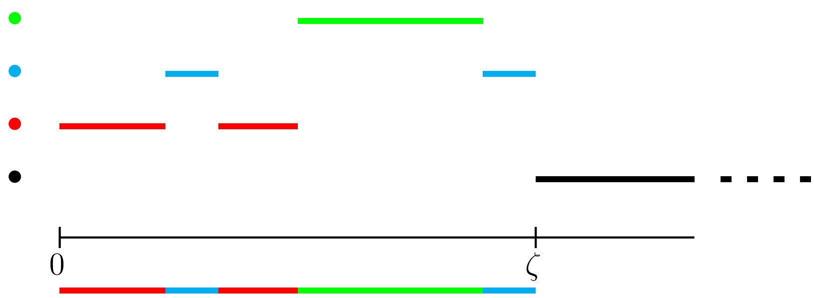


**Figure 3**

Analytic expressions for PH distributions, say for the p.d.f., c.d.f., etc., typically have matrix form. All that will matter to us is the form of the m.g.f. of PH$(\boldsymbol{\alpha}, \boldsymbol{T})$,

$$\mathbb{E}e^{s\zeta} = \boldsymbol{\alpha}(-s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t}\,, \tag{9.1}$$

where $\boldsymbol{t}$ is the column vector with elements $t_i$ (the exit rate vector).

The exponential distribution corresponds to $E$ having only one state, a mixture of exponentials to $t_{ij} = 0$ for $i \neq j$, and an Erlang$(p, \delta)$ distribution (a gamma$(p, \delta)$ distribution) to $E = \{1, \ldots, p\}$, $t_{i(i+1)} = \delta$ for $i < p$, all other off-diagonal elements 0, $\boldsymbol{\alpha} = (1\ 0\ \ldots\ 0)$.

## 9.2 The PH Lévy model

Any one-point distribution, say at $z > 0$, is the limit as $p \to \infty$ of the Erlang$(p, p/z)$ distribution. The PH class is closed under mixtures, and so its closure contains all distributions on $(0, \infty)$ with finite support. Hence the PH class is dense.

The class of compound Poisson processes is dense in $D$ in the class of Lévy processes. Hence the denseness properties of PH imply that the class of differences of two compound Poisson processes with PH jumps are dense. In our key examples, we will work in this class with an added Brownian component,

$$X(t) = \mu t + \sigma B(t) + \sum_{i=1}^{N^+(t)} Y_i^+ - \sum_{j=1}^{N^-(t)} Y_j^- \tag{9.2}$$

where $N^\pm$ is Poisson($\lambda^\pm$) and the $Y^\pm$ PH($E^\pm, \boldsymbol{\alpha}^\pm, \boldsymbol{T}^\pm$), with $n^\pm$ states. Then by (9.1), we have (in obvious notation) that

$$\begin{aligned}\kappa(s) &= \mu s + \sigma^2 s^2/2 \\ &\qquad + \lambda^+\big[\boldsymbol{\alpha}^+(-s\boldsymbol{I}^+ - \boldsymbol{T}^+)^{-1}\boldsymbol{t}^+ - 1\big] + \lambda^-\big[\boldsymbol{\alpha}^-(s\boldsymbol{I}^- - \boldsymbol{T}^-)^{-1}\boldsymbol{t}^- - 1\big]\end{aligned}$$

where $\boldsymbol{t}^+, \boldsymbol{t}^-$ are the exit vectors (minus the row sums of the phase generators).

Expanding the inverses as ratios between minors and the determinant, it follows that $\kappa(s) = r_1(s)/r_2(s)$ where $r_1, r_2$ are polynomials, with degree $n^+ + n^-$ of

$$r_2(s) = \det(-s\boldsymbol{I}^+ - \boldsymbol{T}^+)\det(s\boldsymbol{I}^- - \boldsymbol{T}^-)$$

and degree $n^+ + n^- + 2$ of $r_1$ if $\sigma^2 > 0$, $n^+ + n^- + 1$ if $\sigma^2 = 0$, $\mu \neq 0$, and $n^+ + n^-$ if $\sigma^2 = 0$, $\mu = 0$. Obviously, $\kappa(s)$ therefore has an analytic continuation to the whole of the complex plane with the zeros of $r_2$ removed.This representation is fundamental for the paper. We further let

$$\Theta = \{s \in \mathbb{C} : \mathbb{E}e^{\Re(s)X(1)} < \infty\};$$

then $\Theta$ is a strip of the form $\Theta = \big\{s \in \mathbb{C} : \underline{\theta} < \Re(s) < \overline{\theta}\big\}$ for suitable $\underline{\theta} < 0 < \overline{\theta}$ ($-\overline{\theta}$ is the eigenvalue of largest real part of $\boldsymbol{T}^+$ and $\underline{\theta}$ the eigenvalue of largest real part of $\boldsymbol{T}^-$).

The situation is illustrated in Fig. 4. The green-shaded area is the strip $\Theta \subset \mathbb{C}$ where the m.g.f. converges. The red squares are the singularities, i.e. the roots of $r_2$ or, equivalently, the union of the sets of roots of $\det\big(-s\boldsymbol{I}^+ - \boldsymbol{T}^+\big)$ and $\det\big(s\boldsymbol{I}^- - \boldsymbol{T}^-\big)$. The blue circles are the roots of $r_1$ or, equivalently, of $\kappa$ which will show up in numerous computational schemes of the paper.

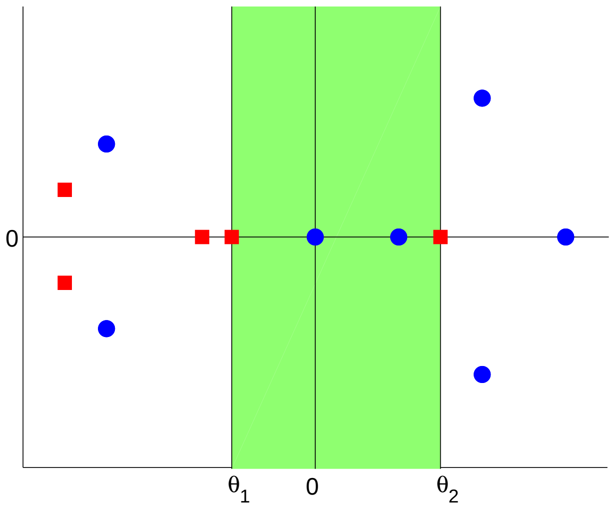


**Figure 4:** Features of $\kappa$

To avoid tedious distinctions between the various cases arising according to whether $\sigma^2$, $\mu$ are non-zero or not, we will assume that $\sigma^2 > 0$. This assumption has a further motivation from a common procedure (e.g. Asmussen and Rosinski [20]) of replacing small jumps by a Brownian motion with the same mean and variance.

## 9.3 Two-sided exit

Recall that $\tau[a,b) = \inf\{t \geq 0 : X(t) \notin [a,b)\}$ with $a \leq 0 < b$; we want to compute $\mathbb{P}\big(X\big(\tau[a,b)\big) \geq b\big)$.

Write

$$
\begin{aligned}
p_c^+ &= \mathbb{P}\Big(X\big(\tau[a,b)\big) = b\Big)\,, \quad p_c^- = \mathbb{P}\Big(X\big(\tau[a,b)\big) = a\Big)\,,\\
p_i^+ &= \mathbb{P}\Big(X\big(\tau[a,b)\big) > b,\ \text{upcrossing occurs in phase } i\Big)\,,\ i = 1,2,\ldots,n^+,\\
p_j^- &= \mathbb{P}\Big(X\big(\tau[a,b)\big) < a,\ \text{downcrossing occurs in phase } j\Big)\,,\ j = 1,2,\ldots,n^-\,.
\end{aligned}
$$

In more detail, we can imagine each upward jump of the process to be governed by a terminating Markov process $J$ with generator $\boldsymbol{T}^+$, and if the first exit time from $[a,b)$ is $t$, 'upcrossing in phase $i$' then means $J\big(b - X(\tau[a,b)-)\big) = i$ (similarly for the downward jumps). See Fig. 5 where $F^+$ has two phases, `red` and `green`, and $F^-$ just one, `blue` (we denote by $F^\pm$ the distributions of $Y^\pm$); thus on the figure, there is upcrossing in the `green` phase.

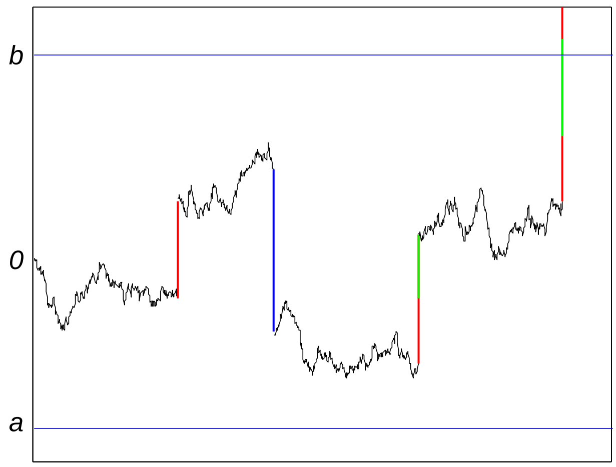


**Figure 5:** The two-sided exit problem

We have $\mathbb{P}\big(X\big(\tau[a,b)\big) \geq b\big) = p_c^+ + p_1^+ + \cdots + p_{n^+}^+$ and need $n^+ + n^- + 2$ equations to be able to solve for the unknowns. The first equation is the obvious

$$
p_c^+ + \sum_{i=1}^{n^+} p_i^+ + p_c^- + \sum_{j=1}^{n^-} p_j^- = 1.
$$

The following notation will be used. Let $\boldsymbol{e}_i^+, \boldsymbol{e}_i^-$ denote the $i$th unit row vectors and let $\widehat{F}_i^\pm[s] = \boldsymbol{e}_i^\pm(-s\boldsymbol{I}^\pm - \boldsymbol{T}^\pm)^{-1}\boldsymbol{t}^\pm$ denote the m.g.f. of the phase-type distributions $F_i^\pm$ with initial vector $\boldsymbol{e}_i^\pm$ and phase generator $\boldsymbol{T}^\pm$. Let further $0 = \rho_1, \rho_2, \ldots, \rho_{n^+ + n^+ + 2}$ denote the roots of $\kappa(\rho) = 0$, i.e., of the polynomial equation $r_1(\rho) = 0$.

### Heuristics via the Wald martingale

If the drift $\kappa'(0)$ is non-zero, a $\gamma \neq 0$ with $\mathbb{E}\mathrm{e}^{\gamma X(1)} < \infty$, $\kappa(\gamma) = 0$ exists and we can take $\rho_1 = 0$, $\rho_2 = \gamma$. Thus $\mathrm{e}^{\rho_2 X(t)}$ is an (integrable) martingale. Optional stopping at $\tau[a,b)$ then yields $1 = \mathbb{E}\mathrm{e}^{\rho_2 X(\tau[a,b))}$, which, taking over- and undershoots into account, means

$$1 = \mathrm{e}^{\rho_k b}\Big(p_c^+ + \sum_{i=1}^{n^+} p_i^+ \widehat{F}_i^+[\rho_k]\Big) + \mathrm{e}^{\rho_k a}\Big(p_c^- + \sum_{j=1}^{n^-} p_j^- \widehat{F}_j^-[-\rho_k]\Big) \tag{9.3}$$

for $k = 2$. This is one equation more, but only one. If $\rho_k$, $k > 2$, is one of the remaining $n^+ + n^-$ roots and $\mathbb{E}|\mathrm{e}^{\rho_k X(1)}| < \infty$, we can then proceed as for $\rho_k$ to conclude that (9.3) holds also for this $k$, and get in this way potentially the needed $n^+ + n^-$ remaining equations. But the problem is that typically $\mathbb{E}|\mathrm{e}^{\rho_k X(1)}| < \infty$ fails. Now both sides of (9.3) are analytic functions. But the validity for two $k$ is not enough to apply analytic continuation.

**Computation via the Kella-Whitt martingale**

We will use the simple form (7.6) of the Kella-Whitt martingale. This gives that $K$ defined according to

$$K(t) = \kappa(\alpha)\int_0^{t\wedge\tau[a,b)} \mathrm{e}^{\alpha X(s)}\mathrm{d}s + 1 - \mathrm{e}^{\alpha X(t\wedge\tau[a,b))}, \qquad \alpha \in \Theta,$$

is a local martingale. In fact, $K$ is a martingale as follows from Kella & Boxma [80]. Further, we have the bound

$$|K(t)| \le |\kappa(\alpha)|t\mathrm{e}^{|\alpha|\max(|a|,b)} + 1 + \mathrm{e}^{|\alpha|(x+V^+)} + \mathrm{e}^{|\alpha|(b-x+V^-)}$$

where $V^+$ and $V^-$ (the overshoot and undershoot of $b$ and $a$, respectively, at $\tau[a,b)$) are phase-type distributed. From $\mathbb{E}\tau[a,b) < \infty$ we then get $\mathbb{E}\sup_{t\le\tau[a,b)}|K(t)| < \infty$, so optional stopping at $\tau[a,b)$ is permissible. Letting $\phi(\alpha) = \mathbb{E}\int_0^{\tau[a,b)} \mathrm{e}^{\alpha X(s)}\mathrm{d}s$, this gives

$$\begin{aligned} 0 &= \kappa(\alpha)\phi(\alpha) + 1 \\ &\quad - \mathrm{e}^{\alpha b}\Big(p_c^+ + \sum_{i=1}^{n^+} p_i^+ \widehat{F}_i^+[\alpha]\Big) - \mathrm{e}^{\alpha a}\Big(p_c^- + \sum_{j=1}^{n^-} p_j^- \widehat{F}_j^-[-\alpha]\Big), \end{aligned} \tag{9.4}$$

It is easily seen that the function $\phi(\alpha)$ is well defined for all $\alpha \in \mathbb{C}$, not just for $\alpha \in \Theta$, and analytic when the common singularities of $\kappa$ and the $\widehat{F}_i^+, \widehat{F}_i^-$ are removed. Therefore by analytic continuation (9.4) is valid for all $\alpha$ in this domain. In particular we may take $\alpha$ as any of the $\rho_k$ to obtain (9.3) for $k = 1, \ldots, n^+ + n^- + 2$.

**Example 9.1.** Take as a simple example all jumps to be exponentially distributed (with parameters $\mu^+$, $\mu^-$) and $\mu = -1$. Then

$$\kappa(\alpha) = \frac{\lambda^+\alpha}{\mu^+ - \alpha} - \frac{\lambda^-\alpha}{\mu^- + \alpha} + \frac{\sigma^2\alpha^2}{2} - \alpha.$$

The method described above allows us to explicitly compute the c.d.f. $\pi^b[0,x]$ (in terms of the parameters of the model and $b$). Even for this simple case, the resulting expressions are quite complicated and rather than presenting them, we display numerical results in Fig. 6 in the form of plots of the c.d.f. of $\pi^b$, taking $\lambda^+ = \lambda^- = \mu^+ = \mu^- = 1$, $b = 2$, and letting $\sigma^2$ vary. □

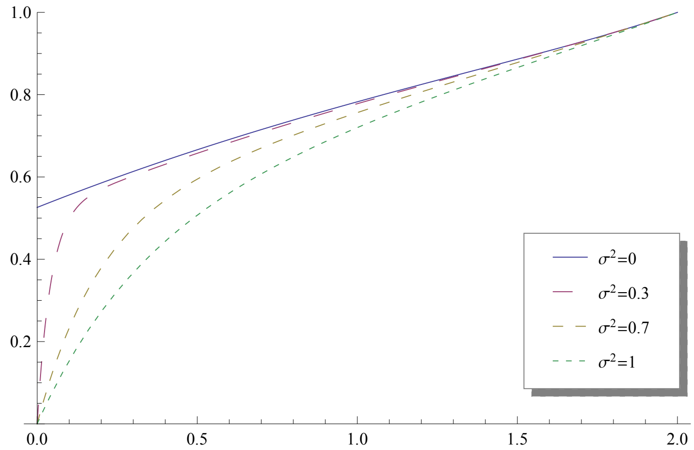


**Figure 6**

## 9.4 The scale function

Though the scale function does not appear in the rest of the paper, we give for the sake of completeness a sketch of its computation in the PH model. In view of (5.3), we need to evaluate

$$\mathbb{E}\left[\mathrm{e}^{-q\tau[a,b)}\mathbb{1}(X(\tau[a,b)) \geq b)\right]. \tag{9.5}$$

To this end, we use the Kella-Whitt martingale with $B(t) = -qt/\alpha$ which takes the form

$$\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha X(s)-qs}\,\mathrm{d}s + 1 - \mathrm{e}^{\alpha X(t)-qt} - q\int_0^t \mathrm{e}^{\alpha X(s)-qs}\,\mathrm{d}s$$

Optional stopping at $\tau[a,b)$ gives

$$\begin{aligned} 1 = &-\big(\kappa(\alpha)-q\big)\mathbb{E}\int_0^{\tau[a,b)} \mathrm{e}^{\alpha X(s)-qs}\,\mathrm{d}s \\ &+ \mathrm{e}^{\alpha b}\Big(p_c^+ y_c^+ + \sum_{i=1}^{n^+} p_i^+ \widehat{F}_i^+[\alpha] y_i^+\Big) + \mathrm{e}^{\alpha a}\Big(p_c^- y_c^- + \sum_{j=1}^{n^-} p_j^- \widehat{F}_j^-[-\alpha] y_j^-\Big) \end{aligned}$$

where $y_c^+$ is the expectation of $\mathrm{e}^{-q\tau[a,b)}$ given continuous exit above, $y_i^+$ the expectation of $\mathrm{e}^{-q\tau[a,b)}$ given exit above in phase $i$, and similarly for the $y_c^-, y_j^-$. As in Section 9.3, we may now choose $\rho_1^q, \ldots, \rho_{n^++n^-+2}^q$ as the roots of $\kappa(s) = q$ to conclude that

$$1 = \mathrm{e}^{\rho_k^q b}\Big(p_c^+ y_c^+ + \sum_{i=1}^{n^+} p_i^+ \widehat{F}_i^+[\rho_k^q] y_i^+\Big) + \mathrm{e}^{\rho_k^q a}\Big(p_c^- y_c^- + \sum_{j=1}^{n^-} p_j^- \widehat{F}_j^-[-\rho_k^q] y_j^-\Big)$$

for $k = 1, \ldots, n^+ + n^- + 2$. These linear equations may be solved for the $p_c^+ y_c^+, p_i^+ y_i^+$, $p_c^- y_c^-, p_j^- y_j^-$, and (9.5) can then be computed as $p_c^+ y_c^+ + \sum p_i^+ y_i^+$.

## 9.5 The loss rate

As before, we take $N^\pm$ to be Poisson($\lambda^\pm$) and $Y^\pm$ to be PH($E^\pm, \boldsymbol{\alpha}^\pm, \boldsymbol{T}^\pm$) with $n^\pm$ phases, respectively. If we let $x \to b$ and $x - b \to a$ in (9.3), we obtain, for

$k = 1, \dots, n^+ + n^- + 2$,

$$\mathrm{e}^{-\rho_k x} = p_c^+ + \sum_{i=1}^{n^+} p_i^+ \widehat{F}_i^+[\rho_k] + \mathrm{e}^{-\rho_k b}\Big(p_c^- + \sum_{j=1}^{n^-} p_i^- \widehat{F}_j^-[-\rho_k]\Big) \tag{9.6}$$

(where, as above, we let $\rho_1 = 0$). We let $\boldsymbol{a}_k$ be the vector

$$\boldsymbol{a}_k = \big(1 \ \widehat{F}_1^+(\rho_k) \ \dots \ \widehat{F}_{n^+}^+(\rho_k) \ \mathrm{e}^{-\rho_k b} \ \mathrm{e}^{-\rho_k b}\widehat{F}_1^-(-\rho_k) \ \dots \ \mathrm{e}^{-\rho_k b}\widehat{F}_{n^-}^-(-\rho_k)\big)$$

and construct the matrix $\boldsymbol{A}$ according to

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{n^+ + n^- + 2} \end{pmatrix}.$$

If we let

$$\boldsymbol{p} = \big(p_c^+ \ p_1^+ \ \dots \ p_{n^+}^+ \ p_c^- \ p_1^- \ \dots \ p_{n^-}^-\big)^\mathsf{T}$$

and take $\boldsymbol{e}_+$ to be a row vector with the first $n^+ + 1$ elements equal to one and zero otherwise, we may compute $\overline{\pi}(x) = p_c^+ + \sum_{i=1}^{n^+} p_i^+$ as $\boldsymbol{e}_+\boldsymbol{p}$ where $\boldsymbol{p}$ solves the set of linear equations $\boldsymbol{A}\boldsymbol{p} = \boldsymbol{h}(x)$, where

$$\boldsymbol{h}(x) = (\mathrm{e}^{-\rho_1 x} \ \dots \ \mathrm{e}^{-\rho_{n^+ + n^- + 2} x})^\mathsf{T},$$

i.e., formally

$$\overline{\pi}(x) = \boldsymbol{g}\boldsymbol{h}(x) = \boldsymbol{g}\exp\{\boldsymbol{H}x\}\boldsymbol{e}, \tag{9.7}$$

where $\boldsymbol{g} = \boldsymbol{e}_+\boldsymbol{A}^{-1}$ and $\boldsymbol{H} = \operatorname{diag}(-\rho_1, -\rho_2, \dots, -\rho_{n^+ + n^- + 2})$ (the rightmost part in (9.7) will prove itself useful below).

With this formula for $\overline{\pi}(x)$ at hand we may proceed to the computation of $\ell^b$. We will take as a starting point the alternative formula for $\ell^b$ which is presented in the Introduction, i.e.,

$$\ell^b = \frac{1}{2b}\big(2m\,\mathbb{E}V + \sigma^2 + J_1 + J_2 - 2J_3 - 2J_4\big) \tag{9.8}$$

where

$$\begin{aligned}
J_1 &= J_1(b) = \int_0^b y^2 \nu(\mathrm{d}y)\,, \\
J_2 &= J_2(b) = \int_b^\infty (2yb - b^2)\nu(\mathrm{d}y)\,, \\
J_3 &= J_3(b) = \int_0^b \int_{-\infty}^{-x} (x+y)\nu(\mathrm{d}y)\,\overline{\pi}(x)\,\mathrm{d}x\,, \\
J_4 &= J_4(b) = \int_0^b \int_{b-x}^{\infty} (x+y-b)\nu(\mathrm{d}y)\,\overline{\pi}(x)\,\mathrm{d}x\,.
\end{aligned}$$

It thus remains to identify $m, \mathbb{E}V$ and $J_1, J_2, J_3, J_4$. If we note that

$$\nu(\mathrm{d}x) = \begin{cases} \lambda^+ \boldsymbol{\alpha}^+ \exp\{\boldsymbol{T}^+ x\} \boldsymbol{t}^+ \mathrm{d}x & \text{if } x > 0, \\ \lambda^- \boldsymbol{\alpha}^- \exp\{-\boldsymbol{T}^- x\} \boldsymbol{t}^- \mathrm{d}x & \text{if } x < 0, \end{cases}$$

we see that the computation of $\ell^b$ is more or less a matter of routine (though tedious!). However, for the sake of completeness and clarity we will perform the calculations in some detail anyway. Clearly,

$$m = \mu - \lambda^+ \boldsymbol{\alpha}^+ (\boldsymbol{T}^+)^{-1} \boldsymbol{e} + \lambda^- \boldsymbol{\alpha}^- (\boldsymbol{T}^-)^{-1} \boldsymbol{e},$$
$$\mathbb{E}V = \int_0^b \overline{\pi}(x) \mathrm{d}x = \int_0^b \boldsymbol{g} \boldsymbol{h}(x) \mathrm{d}x = \boldsymbol{g} \boldsymbol{k},$$

where

$$\boldsymbol{k} = \begin{pmatrix} b \\ \rho_2^{-1}(1 - \mathrm{e}^{-\rho_2 b}) \\ \vdots \\ \rho_{n^+ + n^- + 2}^{-1}(1 - \mathrm{e}^{-\rho_{n^+ + n^- + 2} b}) \end{pmatrix}.$$

Let $\otimes$ and $\oplus$ denote Kronecker matrix multiplication and addition, respectively, where $\oplus$ is defined for square matrices by $\boldsymbol{A}_1 \oplus \boldsymbol{A}_2 = \boldsymbol{A}_1 \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{A}_2$. It is not difficult to show that

$$\int \nu(\mathrm{d}y) = \begin{cases} \lambda^+ \boldsymbol{a}^+ (\boldsymbol{T}^+)^{-1} \exp\{\boldsymbol{T}^+ y\} \boldsymbol{t}^+ & \text{if } y > 0, \\ -\lambda^- \boldsymbol{a}^- (\boldsymbol{T}^-)^{-1} \exp\{-\boldsymbol{T}^- y\} \boldsymbol{t}^- & \text{if } y < 0, \end{cases} \tag{9.9}$$

$$\int y \nu(\mathrm{d}y) = \begin{cases} \lambda^+ \boldsymbol{a}^+ (\boldsymbol{T}^+)^{-1} (y \boldsymbol{I} - (\boldsymbol{T}^+)^{-1}) \exp\{\boldsymbol{T}^+ y\} \boldsymbol{t}^+ & \text{if } y > 0, \\ -\lambda^- \boldsymbol{a}^- (\boldsymbol{T}^-)^{-1} (y \boldsymbol{I} - (\boldsymbol{T}^-)^{-1}) \exp\{-\boldsymbol{T}^- y\} \boldsymbol{t}^- & \text{if } y < 0, \end{cases} \tag{9.10}$$

$$\int y^2 \nu(\mathrm{d}y) = \lambda^+ \boldsymbol{a}^+ (\boldsymbol{T}^+)^{-1} (y^2 \boldsymbol{I} - 2y (\boldsymbol{T}^+)^{-1} + 2 (\boldsymbol{T}^+)^{-2}) \exp\{\boldsymbol{T}^+ y\} \boldsymbol{t}^+, \text{ if } y > 0. \tag{9.11}$$

[Note that when we write $\int f(y)\mathrm{d}y$ (without integration limits) for some function $f$ we mean the primitive (indefinite integral), i.e. $\int f(y)\mathrm{d}y$ is a function such that its derivative with respect to $y$ equals $f(y)$.] It follows from (9.7), (9.9), (9.10) and the fact that all eigenvalues of $\boldsymbol{T}^+$ and $\boldsymbol{T}^-$ have negative real part, see e.g. [11], p. 83, that

$$\begin{aligned} J_3 &= \int_0^b \lambda^- \boldsymbol{a}^- (\boldsymbol{T}^-)^{-2} \exp\{\boldsymbol{T}^- x\} \boldsymbol{t}^- \, \overline{\pi}(x) \, \mathrm{d}x \\ &= \int_0^b \lambda^- \boldsymbol{a}^- (\boldsymbol{T}^-)^{-2} \exp\{\boldsymbol{T}^- x\} \boldsymbol{t}^- \, \boldsymbol{g} \exp\{\boldsymbol{H} x\} \boldsymbol{e} \, \mathrm{d}x \\ &= \lambda^- \big[(\boldsymbol{a}^- (\boldsymbol{T}^-)^{-2}) \otimes \boldsymbol{g}\big] \Big[\int_0^b \exp\{(\boldsymbol{T}^- \oplus \boldsymbol{H}) x\} \mathrm{d}x\Big] [\boldsymbol{t}^- \otimes \boldsymbol{e}] \\ &= \lambda^- \big[(\boldsymbol{a}^- (\boldsymbol{T}^-)^{-2}) \otimes \boldsymbol{g}\big] \big[(\boldsymbol{T}^- \oplus \boldsymbol{H})^{-1} \big(\exp\{(\boldsymbol{T}^- \oplus \boldsymbol{H}) b\} - \boldsymbol{I}\big)\big] [\boldsymbol{t}^- \otimes \boldsymbol{e}] \end{aligned}$$

where we used the standard identities

$$(\boldsymbol{A}_1\boldsymbol{B}_1\boldsymbol{C}_1)(\boldsymbol{A}_2\boldsymbol{B}_2\boldsymbol{C}_2) = (\boldsymbol{A}_1\otimes\boldsymbol{A}_2)(\boldsymbol{B}_1\otimes\boldsymbol{B}_2)(\boldsymbol{C}_1\otimes\boldsymbol{C}_2)$$
$$\exp\{\boldsymbol{R}x\}\otimes\exp\{\boldsymbol{S}x\} = \exp\{(\boldsymbol{R}\oplus\boldsymbol{S})x\}.$$

Similarly, $J_4$ becomes

$$-\lambda^+\big[(\boldsymbol{a}^+(\boldsymbol{T}^+)^{-2}\exp\{\boldsymbol{T}^+b\})\otimes\boldsymbol{g}\big]\big[(-\boldsymbol{T}^+\oplus\boldsymbol{H})^{-1}(\exp\{(-\boldsymbol{T}^+\oplus\boldsymbol{H})b\}-\boldsymbol{I})\big][\boldsymbol{t}^+\otimes\boldsymbol{e}].$$

Finally, it follows easily from (9.9), (9.10) and (9.11), that

$$J_1 = \lambda^+\boldsymbol{a}^+(\boldsymbol{T}^+)^{-1}\big[(b^2\boldsymbol{I}-2b(\boldsymbol{T}^+)^{-1}+2(\boldsymbol{T}^+)^{-2})\exp\{\boldsymbol{T}^+b\}-2(\boldsymbol{T}^+)^{-2}\big]\boldsymbol{t}^+$$
$$J_2 = -\lambda^+\boldsymbol{a}^+(\boldsymbol{T}^+)^{-1}\big[b^2\boldsymbol{I}-2b(\boldsymbol{T}^+)^{-1}\big]\exp\{\boldsymbol{T}^+b\}\boldsymbol{t}^+,$$

and thereby all terms in (9.8) have been evaluated.

# 10 Loss rate asymptotics: light tails

In this section we derive asymptotics of $\ell^b$ as $b\to\infty$ when $X$ is assumed to be light-tailed with $-\infty<\mathbb{E}X(1)<0$. By light-tailed, we simply mean that the set $\Theta=\{\alpha\in\mathbb{R}:\mathbb{E}\mathrm{e}^{\alpha X(1)}<\infty\}$ has a non-empty intersection with $(0,\infty)$.

We start by introducing the following notation.

- $M(t)=\sup_{0\le s\le t}X(s)$, $M(\infty)=\sup_{0\le t<\infty}X(t)$.
- $\tau_+(u)=\inf\{t>0:X(t)>u\}$, $\tau_+^w(u)=\inf\{t>0:X(t)\ge u\}$, $u\ge 0$.
- $\tau_-(-v)=\inf\{t>0:X(t)<-v\}$, $v\ge 0$.
- The overshoot of level $u$, $B(u)=X(\tau_+(u))-u$, $u\ge 0$.
- The weak overshoot of level $u$, $B^w(u)=X(\tau_+^w(u))-u$, $u\ge 0$.
- $B(\infty)$, a r.v. having the limiting distribution (if it exists) of $B(u)$ as $u\to\infty$.

Furthermore, we will assume that the Lundberg equation $\kappa(\alpha)=0$ has a solution $\gamma>0$ with $\kappa'(\gamma)<\infty$. We let $\mathbb{P}_L$ and $\mathbb{E}_L$ correspond to a measure which is exponentially tilted by $\gamma$, i.e.,

$$\mathbb{P}(G)=\mathbb{E}_L(\mathrm{e}^{-\gamma X(\tau)};G) \tag{10.1}$$

when $\tau$ is a stopping time and $G\in\mathscr{F}(\tau)$, $G\subseteq\{\tau<\infty\}$. Note that $\mathbb{E}_LX(1)=\kappa'(\gamma)>0$ by convexity of $\kappa$.

We need the following two lemmas. The first is just a reformulation of Theorem 8.6 and the second describes the asymptotic probability, as $u\to\infty$, of the event that $X$'s first exit of the set $[-v,u)$ occurs at the upper barrier.

**Lemma 10.1.** *For the integrals $I_1$ and $I_2$ in Theorem 8.6 we have the following alternative formulas.*

$$I_1=\int_b^\infty((y-b)+\gamma^{-1}(1-\mathrm{e}^{\gamma(y-b)}))\nu(\mathrm{d}y)+\int_0^b\overline{\pi}(x)\mathrm{d}x\int_{b-x}^\infty(1-\mathrm{e}^{\gamma(y-b+x)})\nu(\mathrm{d}y),$$
$$I_2=\int_{-\infty}^0(y+\gamma^{-1}(1-\mathrm{e}^{\gamma y}))\nu(\mathrm{d}y)+\int_0^b\overline{\pi}(x)\mathrm{d}x\int_{-\infty}^{-x}(1-\mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y).$$

*Proof.* Just change order of integration and perform partial integration. Then switch back to the original order of integration. □

**Lemma 10.2.** *Assume that $X$ is not compound Poisson with lattice jump distribution. Then, for each $v \geq 0$,*

$$\mathbb{P}\big(\tau_-(-v) > \tau_+^w(u)\big) \sim \mathrm{e}^{-\gamma u}\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\mathbb{P}_L\big(\tau_-(-v) = \infty\big), \;\; u \to \infty.$$

*Proof.* It is easily seen that $\tau_+^w(u)$ is a stopping time and that $\{\tau_-(-v) > \tau_+^w(u)\} \in \mathscr{F}(\tau_+^w(u))$. By the Blumenthal zero-one law (e.g. [28, p. 19]), it follows that $\mathbb{P}(\tau_+(0) = 0)$ is either 0 or 1. In the first case the sample paths of $M$ are step functions a.s. and it follows in the same way as in the proof of Lemma 3.3 (Lemma 2.3 in [107]) that $\mathbb{P}(\tau_+^w(u) \neq \tau_+(u)) \to 0, \; u \to \infty$. In the second case it follows by the strong Markov property applied at $\tau_+^w(u)$ that $\mathbb{P}(\tau_+^w(u) \neq \tau_+(u)) = 0$. From (10.1) we then get,

$$\begin{aligned}
\mathbb{P}\big(\tau_-(-v) > \tau_+^w(u)\big) &= \mathbb{E}_L[\mathrm{e}^{-\gamma X(\tau_+^w(u))}; \tau_-(-v) > \tau_+^w(u)] \\
&= \mathrm{e}^{-\gamma u}\mathbb{E}_L[\mathrm{e}^{-\gamma B(u)}; \tau_-(-v) > \tau_+(u)]\mathbb{P}\big(\tau_+^w(u) = \tau_+(u)\big) \\
&\quad + \mathrm{e}^{-\gamma u}\mathbb{P}_L\big(\tau_-(-v) > \tau_+^w(u) \,\big|\, \tau_+^w(u) \neq \tau_+(u)\big)\mathbb{P}_L\big(\tau_+^w(u) \neq \tau_+(u)\big) \\
&\sim \mathrm{e}^{-\gamma u}\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\mathbb{P}_L\big(\tau_-(-v) = \infty\big).
\end{aligned}$$

In the last step we used $B(u) \to B(\infty)$, see [29] and [107], $\big\{\tau_-(-v) > \tau_+(u)\big\} \uparrow \big\{\tau_-(-v) = \infty\big\}$ (both in $\mathbb{P}_L$-distribution) and asymptotic independence between $B(u)$ and $\big\{\tau_-(-v) > \tau_+(u)\big\}$, see the proof of Corollary 5.9, p. 368, in [11]. □

**Remark 10.3.** In the proof of Lemma 10.2 above we had to treat the cases $\mathbb{P}(\tau_+(0) = 0) = 1$ and $\mathbb{P}(\tau_+(0) = 0) = 0$ (corresponding to completely different short time behaviors of $X$) in slightly different ways. In traditional terminology, these cases correspond to whether the point 0 is *regular*, or *irregular*, for the set $(0, \infty)$, see [28], p. 104 or [119], p. 313 and p. 353. As a small digression, we shall briefly discuss this issue. It turns out that 0 is regular for $(0, \infty)$ if and only if

$$\int_0^1 t^{-1}\mathbb{P}(X(t) > 0)\,\mathrm{d}t = \infty,$$

see Theorem 47.2 and the remark at the bottom of p. 353 in [119]. Perhaps more interestingly, we can characterize the short time behavior of $X$ via its Lévy triplet. We will not give a complete account for all types of Lévy processes (this is done in Theorem 47.5 on p. 355 in [119]), but note that whenever the paths of $X$ are of infinite variation then 0 is regular for $(0, \infty)$ and if $X$ is the sum of a compound Poisson process and a non-positive drift then 0 is irregular for $(0, \infty)$. □

Next, we state the main result about the asymptotics for $\ell^b$.

**Theorem 10.4.** *Suppose that $X$ fulfills the conditions in Lemma 10.2. Then, as $b \to \infty$, $\ell^b \sim C\mathrm{e}^{-\gamma b}$ where*

$$\begin{aligned}
C = &-m + \mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)} \int_0^\infty \mathrm{e}^{\gamma x}\mathbb{P}_L\big(\tau_-(-x) = \infty\big)\int_x^\infty (1 - \mathrm{e}^{\gamma(y-x)})\,\nu(\mathrm{d}y)\,\mathrm{d}x \\
&+ \int_{-\infty}^0 \big[y + \gamma^{-1}(1 - \mathrm{e}^{\gamma y})\big]\,\nu(\mathrm{d}y) + \int_0^\infty \mathbb{P}\big(\tau_+^w(x) < \infty\big)\int_{-\infty}^{-x}(1 - \mathrm{e}^{\gamma(x+y)})\,\nu(\mathrm{d}y)\,\mathrm{d}x.
\end{aligned} \tag{10.2}$$

*Proof.* It follows from Lemma 10.1 and $\kappa'(\gamma) < \infty$ that

$$\begin{aligned}
\mathrm{e}^{\gamma b} I_1 &= \mathrm{o}(1) + \mathrm{e}^{\gamma b} \int_0^b \mathbb{P}\big(\tau_-(x-b) > \tau_+^w(x)\big)\,\mathrm{d}x \int_{b-x}^\infty \big(1 - \mathrm{e}^{\gamma(y-b+x)}\big)\,\nu(\mathrm{d}y) \\
&= \mathrm{o}(1) + \int_0^b \mathrm{e}^{\gamma z}\mathrm{e}^{\gamma(b-z)}\mathbb{P}\big(\tau_-(-z) > \tau_+^w(b-z)\big)\,\mathrm{d}z \int_z^\infty \big(1 - \mathrm{e}^{\gamma(y-z)}\big)\,\nu(\mathrm{d}y) \\
&\to \mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)} \int_0^\infty \mathrm{e}^{\gamma x}\mathbb{P}_L\big(\tau_-(-x) = \infty\big) \int_x^\infty \big(1 - \mathrm{e}^{\gamma(y-x)}\big)\,\nu(\mathrm{d}y)\,\mathrm{d}x, \quad b \to \infty.
\end{aligned}$$

The convergence follows from the pointwise convergence in Lemma 10.2 and dominated convergence, which is applicable because

$$\mathrm{e}^{\gamma b}\overline{\pi}(b-x)\mathbb{1}(x \le b) \le \mathrm{e}^{\gamma b}\mathbb{P}\big(M(\infty) > b-x\big)\mathbb{1}(x \le b) \le \mathrm{e}^{\gamma x}$$

and

$$\int_0^\infty \mathrm{e}^{\gamma x}\,\mathrm{d}x \int_x^\infty \big(1 - \mathrm{e}^{\gamma(y-x)}\big)\,\nu(\mathrm{d}y) = \int_0^\infty \big(\gamma^{-1}\mathrm{e}^{\gamma y} - y\mathrm{e}^{\gamma y} - \gamma^{-1}\big)\,\nu(\mathrm{d}y) > -\infty.$$

In $I_2$ we bound $\overline{\pi}(x)\mathbb{1}(x \le b)$ by 1, note that

$$\int_0^\infty \mathrm{d}x \int_{-\infty}^{-x} (1 - \mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y) = \int_{-\infty}^0 (-y - \gamma^{-1} + \gamma^{-1}\mathrm{e}^{\gamma y})\nu(\mathrm{d}y) < \infty$$

and apply dominated convergence which together with $\overline{\pi}(x) \to \mathbb{P}\big(\tau_+^w(x) < \infty\big)$ gives

$$I_2 \to \int_{-\infty}^0 \big[y + \gamma^{-1}(1 - \mathrm{e}^{\gamma y})\big]\,\nu(\mathrm{d}y) + \int_0^\infty \mathbb{P}\big(\tau_+^w(x) < \infty\big) \int_{-\infty}^{-x} \big[1 - \mathrm{e}^{\gamma(x+y)}\big]\,\nu(\mathrm{d}y)\,\mathrm{d}x.$$

The assertion now follows from Theorem 8.6. □

If $X$ is spectrally one-sided the constant in Theorem 10.4 simplifies significantly.

**Corollary 10.5.** *Let $X$ satisfy the conditions in Lemma 10.2. If $\nu(-\infty, 0) = 0$, then*

$$C = -m\Big\{1 + \frac{1}{\mathbb{E}_L X(1)} \int_0^\infty (\mathrm{e}^{\gamma x} - 1) \int_x^\infty (1 - \mathrm{e}^{\gamma(y-x)})\nu(\mathrm{d}y)\mathrm{d}x\Big\}.$$

*If $\nu(0, \infty) = 0$, then*

$$C = -m + \int_{-\infty}^0 \big[y + \gamma^{-1}(1 - \mathrm{e}^{\gamma y})\big]\,\nu(\mathrm{d}y) + \int_0^\infty \mathrm{e}^{-\gamma x} \int_{-\infty}^{-x} \big[1 - \mathrm{e}^{\gamma(x+y)}\big]\,\nu(\mathrm{d}y)\,\mathrm{d}x.$$

*Proof.* In the spectrally positive case we have that $\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)} = -m/\mathbb{E}_L X(1)$, see, e.g., Bertoin & Doney [29], and that

$$\mathbb{P}_L\big(\tau_-(-x) = \infty\big) = 1 - \mathbb{P}_L\big(\tau_-(-x) < \infty\big) = 1 - \mathbb{E}\mathrm{e}^{\gamma X(\tau_-(-x))} = 1 - \mathrm{e}^{-\gamma x}.$$

In the spectrally negative case,

$$\mathbb{P}\big(\tau_+^w(x) < \infty\big) = \mathbb{E}_L \mathrm{e}^{-\gamma X(\tau_+^w(x))} = \mathrm{e}^{-\gamma x}.$$

The claim now follows from Theorem 10.4. □

We next turn our attention towards asymptotics for $\ell^b$ as $b \to \infty$ in the PH example. In principle, we should be able to describe the asymptotics by carefully analyzing what comes out of (9.8), but we prefer to apply Theorem 10.4. Recall that we assume negative drift of the feeding process $X$, i.e. $\mathbb{E}X(1) < 0$. This means that $X(t) \xrightarrow{\text{a.s.}} -\infty,\ t \to \infty$, and that there exists a real positive root $\gamma$ of the equation $\kappa(\alpha) = 0$ such that $\mathbb{E}\mathrm{e}^{\gamma X(1)} = 1$, i.e. $\gamma$ is a genuine root of the Lundberg equation corresponding to $X$.

**Theorem 10.6.** *In the PH Lévy model,*

$$\begin{aligned} C = {} & \boldsymbol{e}^{\mathsf{T}}\boldsymbol{B}^{-1}\boldsymbol{e}_1^{\mathsf{T}}\gamma\lambda^+ \\ & \times [t]\big\{ -\boldsymbol{\alpha}^+(\gamma\boldsymbol{I}+\boldsymbol{T}^+)^{-2}\boldsymbol{e} \\ & \qquad + \big((\boldsymbol{e}^{\mathsf{T}}\tilde{\boldsymbol{B}}^{-1}) \otimes (\boldsymbol{\alpha}^+(\gamma\boldsymbol{I}+\boldsymbol{T}^+)^{-1})\big)\{\tilde{\boldsymbol{J}} \oplus (\gamma\boldsymbol{I}+\boldsymbol{T}^+)\}^{-1}\boldsymbol{e}\big\} \\ & + \lambda^-\boldsymbol{\alpha}^-\big\{(\boldsymbol{T}^-)^{-1} - (-\gamma\boldsymbol{I}+\boldsymbol{T}^-)^{-1}\big\}\boldsymbol{e} + (-\mu - \lambda^+\boldsymbol{\alpha}^+(\boldsymbol{T}^+)^{-1}\boldsymbol{e} \\ & + \lambda^-\boldsymbol{\alpha}^-(\boldsymbol{T}^-)^{-1}\boldsymbol{e}) + \gamma\lambda^-((\boldsymbol{e}^{\mathsf{T}}\boldsymbol{B}^{-1}) \otimes (\boldsymbol{\alpha}^-(-\gamma\boldsymbol{I}+\boldsymbol{T}^-)^{-1}))(\boldsymbol{J} \oplus \boldsymbol{T}^-)^{-1}\boldsymbol{e}\,. \end{aligned} \tag{10.3}$$

For the proof, we need two lemmas. The first is classical and relates to the locations in the complex plane of the roots of $\kappa(\alpha) = q,\ q \geq 0$ (see [12], [46] and references there).

**Lemma 10.7.** *Let $X$ be defined according to* (9.2)*.*

*(i) Consider the equation $\kappa(\alpha) = 0$. If $m \leq 0$ then 0 is the only root with zero real part. There are $n^-$ roots with negative real part and $n^+ + 1$ roots with positive real part.*

*(ii) Consider the equation $\kappa(\alpha) = q$ with $q > 0$. Then (regardless of the value of $m$) there are no roots with zero real part, $n^- + 1$ with negative real part and $n^+ + 1$ with positive real part.*

**Lemma 10.8.** *Assume $m < 0$. Then $\gamma > 0$ is a simple root, i.e. of algebraic multiplicity 1, and if $\rho$ is any other root with $\Re(\rho) > 0$, then $\Re(\rho) > \gamma$.*

*Proof.* Part (i) in Lemma 10.7 tells us that there are $n^+ + 1$ roots of $\kappa(\alpha) = 0$ with positive real part. Clearly, $\gamma$ is one of these. Let $\rho = \Re(\rho) + \mathrm{i}\Im(\rho)$ be one of the remaining roots (with positive real part) and suppose that $0 < \Re(\rho) \leq \gamma$. Now,

$$\begin{aligned} 1 = \mathbb{E}\mathrm{e}^{\rho X(1)} &= \mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\left(\cos(\Im(\rho)X(1)) + \mathrm{i}\sin(\Im(\rho)X(1))\right) \\ &= \mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\cos(\Im(\rho)X(1)) + \mathrm{i}\mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\sin(\Im(\rho)X(1)). \end{aligned} \tag{10.4}$$

From (10.4), the elementary inequality $|\cos(\Im(\rho)X(1))| \leq 1$ and the convexity of $\kappa(\cdot)$ in $(0, \gamma]$, it follows that $\Re(\rho) < \gamma$ is impossible (no matter the distribution of $X(1)$). Note that in the case under consideration, $\kappa(\alpha)$ is a rational function (i.e. $\kappa(\alpha) = p(\alpha)/q(\alpha)$ where $p$ and $q$ are polynomials) and from the fact that $0 < \kappa'(\gamma) < \infty$, see Section 10, we may conclude that the algebraic multiplicity of the root $\gamma$ equals one, i.e. $p(\alpha) = (\alpha - \gamma)r(\alpha)$ where $r(\alpha)$ does not contain the factor $(\alpha - \gamma)$. If $\Re(\rho) = \gamma$ and $\Im(\rho) \neq 0$ then it is easily seen that $1 = \mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\cos(\Im(\rho)X(1))$ is possible provided that $X(1)$ is lattice with span $2\pi/|\Im(\rho)|$, a case which is clearly ruled out by the structure of $X$. $\square$

*Proof of Theorem 10.6.* We have to compute $\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}$, $\mathbb{P}_L(\tau_-(-x) = \infty)$ and $\mathbb{P}(\tau_+^w(x) < \infty)$ for $x > 0$, see Theorem 10.4. Because of (thanks to!) the Brownian component in $X$ we need not distinguish between $\tau_+^w(x)$ and $\tau_+(x)$, cf. Remark 10.3. Define

$$
\begin{aligned}
p_c^+(t) &= \mathbb{P}(X(\tau_+(x)) = x, \tau_+(x) \leq t), \\
p_i^+(t) &= \mathbb{P}(X(\tau_+(x)) > x, \tau_+(x) \leq t, \text{ upcrossing occurs in state } i)\,,
\end{aligned}
$$

$i = 1, 2, \ldots, n^+$. If we let $\phi(\alpha, t) = \mathbb{E}\int_0^{\tau_+(x)\wedge t} \mathrm{e}^{\alpha X(s)}\mathrm{d}s$ it follows by optional stopping of the Kella-Whitt martingale at $\tau_+(x)$ that

$$
\begin{aligned}
0 = \kappa(\alpha)\phi(\alpha, t) + 1 - \mathrm{e}^{\alpha x}\Big(p_c^+(t) + \sum_{i=1}^{n^+} p_i^+(t)\widehat{F}_i^+(\alpha)\Big) \\
- \mathbb{E}[\mathrm{e}^{\alpha X(t)}; t < \tau_+(x)], \qquad \alpha \in \Theta. \qquad (10.5)
\end{aligned}
$$

Let $\rho_2, \rho_3, \ldots, \rho_{n^++2}$ denote the roots with positive real part (we tacitly assume that these are distinct and ordered so that $\rho_2 = \gamma$). If we mimic the derivation of (9.3), and take $\alpha = \rho_k$, we get

$$
0 = 1 - \mathrm{e}^{\rho_k x}\Big(p_c^+(t) + \sum_{i=1}^{n^+} p_i^+(t)\widehat{F}_i^+(\rho_k)\Big) - \mathbb{E}[\mathrm{e}^{\rho_k X(t)}; t < \tau_+(x)]. \qquad (10.6)
$$

If we let $t \to \infty$ in (10.6), it follows by $X(t) \xrightarrow{\text{a.s.}} -\infty$ and dominated convergence that

$$
\mathrm{e}^{-\rho_k x} = p_c^+ + \sum_{i=1}^{n^+} p_i^+ \widehat{F}_i^+(\rho_k), \qquad k = 2, 3, \ldots, n^+ + 2. \qquad (10.7)
$$

Let $\boldsymbol{B}$ be the matrix with $k$th row equal to $(1\ \widehat{F}_1^+(\rho_k)\ \ldots\ \widehat{F}_{n^+}^+(\rho_k))$. Then it is easily seen that

$$
\mathbb{P}(\tau_+(x) < \infty) = \boldsymbol{e}^\mathsf{T}\boldsymbol{B}^{-1}\exp\{\boldsymbol{J}x\}\boldsymbol{e}, \qquad (10.8)
$$

where $\boldsymbol{J} = \mathrm{diag}(-\rho_2, \ldots, -\rho_{n^++2})$. Since $\mathbb{P}(\tau_+(x) < \infty) = \mathbb{P}(M(\infty) > x)$ and we know that $\mathbb{P}(M(\infty) > x) \sim \mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\mathrm{e}^{-\gamma x}$, $x \to \infty$, we can use (10.8) and what we know about the elements of $\boldsymbol{J}$ to identify $\mathbb{E}_L\mathrm{e}^{-\gamma B(\infty)}$ as the sum of the elements in the first column of $\boldsymbol{B}^{-1}$. Now, it is well known, see e.g. [14], that w.r.t. $\mathbb{P}_L$, $X$ is still the sum of a Brownian motion with drift and a compound Poisson process with phase-type distributed jumps, with Lévy exponent $\kappa_L(\alpha) = \kappa(\alpha + \gamma)$. Furthermore, if we define $\boldsymbol{d} = (\gamma\boldsymbol{I} - \boldsymbol{T}^-)^{-1}\boldsymbol{t}^-$ and let $\boldsymbol{D}$ be the diagonal matrix with the $d_i$ on the diagonal then (w.r.t. $\mathbb{P}_L$) the intensity matrix corresponding to negative jumps is $\boldsymbol{T}_\gamma^- = \boldsymbol{D}^{-1}\boldsymbol{T}^-\boldsymbol{D} - \gamma\boldsymbol{I}$. It is clear that the equation $\kappa_L(\alpha) = 0$ has $n^- + 1$ roots with negative real part $\tilde{\rho}_k$, $k = 1, 2, \ldots, n^- + 1$ (all of the form $\tilde{\rho}_k = \rho - \gamma$ where $\kappa(\rho) = 0$ and $\Re(\rho) \leq 0$). Define $\widehat{\widetilde{F}}_i$ in the same way as $\widehat{F}_i^-$ with $\boldsymbol{T}^-$ replaced by $\boldsymbol{T}_\gamma^-$. In a fashion similar to the derivation of (9.3), we obtain (in the obvious notation)

$$
0 = 1 - \mathrm{e}^{-\tilde{\rho}_k x}\Big(\tilde{p}_c^-(t) + \sum_{i=1}^{n^-} \tilde{p}_i^-(t)\widehat{\widetilde{F}}_i^-(-\tilde{\rho}_k)\Big) - \mathbb{E}_L[\mathrm{e}^{\tilde{\rho}_k X(t)}; t < \tau_-(-x)]. \qquad (10.9)
$$

From (10.9) it follows that

$$\mathrm{e}^{\tilde{\rho}_k x} = \tilde{p}_c^- + \sum_{i=1}^{n^-} \tilde{p}_i^- \widehat{\tilde{F}}_i^-(-\tilde{\rho}_k), \qquad k = 1, 2, \ldots, n^- + 1, \tag{10.10}$$

and if we define $\tilde{\boldsymbol{B}}$ as the matrix with $k$th row equal to $\big(1\ \widehat{\tilde{F}}_1^-(-\tilde{\rho}_k)\ \ldots\ \widehat{\tilde{F}}_{n^-}^-(-\tilde{\rho}_k)\big)$, it is clear that

$$\mathbb{P}_L\big(\tau_-(-x) = \infty\big) = 1 - \boldsymbol{e}^\mathsf{T} \tilde{\boldsymbol{B}}^{-1} \exp\{\tilde{\boldsymbol{J}} x\} \boldsymbol{e}, \tag{10.11}$$

where $\tilde{\boldsymbol{J}} = \mathrm{diag}(\tilde{\rho}_1, \ldots, \tilde{\rho}_{n^-+1})$. All that now remains in order to describe the asymptotics of $\ell^b$ is to evaluate the integrals in (10.2); we omit the details. □

An important lesson to learn from this example is that the case where $X$ is spectrally one-sided is much easier than the general case. In fact, if we e.g. take $X$ to be spectrally positive then according to Corollary 10.5, $\ell^b \sim C\mathrm{e}^{-\gamma b}$, $b \to \infty$, where

$$C = -m\big\{1 - \gamma\lambda^+ \boldsymbol{\alpha}^+ (\gamma \boldsymbol{I} + \boldsymbol{T}^+)^{-1} \left\{(\gamma \boldsymbol{I} + \boldsymbol{T}^+)^{-1} - (\boldsymbol{T}^+)^{-1}\right\} \boldsymbol{e}/\kappa'(\gamma)\big\},$$

i.e. we need only know $\gamma$ to compute $C$ (the same thing holds when $X$ is spectrally negative, but $C$ comes out in a slightly different way, again see Corollary 10.5), whereas in the general case all roots of $\kappa(\alpha) = 0$ are required in order to completely describe the asymptotic behavior of $\ell^b$.

# 11 Loss rate asymptotics: heavy tails

The main result of this section states that under some heavy-tailed conditions, $\ell^b \sim \int_b^\infty \overline{\nu}(y)\mathrm{d}y$ which in view of Lemma 2.6, can be interpreted as stating that Theorem 3.2 still holds when the random walk is replaced by a Lévy process. More precisely:

**Theorem 11.1.** *Let $X$ be a Lévy process with Lévy measure $\nu \in \mathcal{S}$ and finite negative mean $m = \mathbb{E}X(1) < 0$. Consider the conditions*

- ***A**: $\mathbb{E}X(1)^2 < \infty$ and $\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y / \overline{\nu}_I(b) = \mathrm{O}(b)$ .*
- ***B**: $\overline{\nu}(b) \sim L(b)b^{-\alpha}$ where $L$ is a locally bounded slowly varying function and $1 < \alpha < 2$.*

*If either **A** or **B** holds, then*

$$\ell^b \sim \int_b^\infty \overline{\nu}(y)\mathrm{d}y. \tag{11.1}$$

It is worth noting, that the requirement on the tail of $\nu$ in **A** is very weak. Indeed, suppose $\overline{\nu}_I(x) \sim \overline{B}(x)$ where $B$ is either lognormal, Benktander or heavy-tailed Weibull. Then we recognize $a(x) = \int_x^\infty \overline{B}(y)\,\mathrm{d}y/\overline{B}(x)$ as the mean-excess function and it is known (see [65]), that $a(x) = \mathrm{o}(x)$. Furthermore, it is easily checked that the condition is satisfied when $B$ is a Pareto or Burr distribution, provided that the second moment is finite. Another remark is that we may use the results of [52] to express sufficient conditions for Theorem 11.1 in terms of the distribution of $X(1)$.

We will also derive Theorem 11.3 below, which gives an expression for the m.g.f. of the stationary distribution in the case of one-sided reflection. This result is interesting in its own right as well as useful in the proof of Theorem 11.1. Recall the decomposition of the one-sided reflected process, $V^\infty(t) = V^\infty(0) + X(t) + L(t)$, let $L^c(t)$ and $L^j(t)$ denote the continuous and jump parts of the local time, respectively, and recall that $\Theta = \{\alpha \in \mathbb{C} : \mathbb{E}e^{\Re(\alpha)X(1)} < \infty\}$.

**Lemma 11.2.** *Consider a Lévy process $X$, let $V^\infty$ be the process one-sided reflected at 0 and let $L^c$ and $L^j$ be the continuous and jump part of the corresponding local time $L$, respectively. Then, for $\alpha \in \Theta$ and $V^\infty(0) = x \geq 0$,*

$$M(t) = \kappa(\alpha) \int_0^t e^{\alpha V^\infty(s)}\,ds + e^{\alpha x} - e^{\alpha V^\infty(t)} + \alpha L^c(t) + \sum_{0 \leq s \leq t} (1 - e^{-\alpha \Delta L(s)}) \quad (11.2)$$

*is a martingale.*

*Proof.* The proof is similar to (but slightly easier than) the proof of Proposition 8.2, once we note that $L$ can increase only when $V$ is zero. □

**Theorem 11.3.** *Suppose $-\infty < m = \mathbb{E}X(1) < 0$, so that $V^\infty(\infty) = \lim_{t\to\infty} V^\infty(t)$ exists in distribution. For $\alpha \in \Theta$ we have*

$$\mathbb{E}e^{\alpha V^\infty(\infty)} = -\frac{1}{\kappa(\alpha)}\Big(\alpha \mathbb{E}_{\pi^\infty} L^c(1) + \mathbb{E}_{\pi^\infty} \sum_{0 \leq s \leq 1} (1 - e^{-\alpha \Delta L(s)})\Big). \quad (11.3)$$

*Proof.* Replacing $x$ by a r.v. distributed as $V^\infty(\infty)$ in (11.2) and taking expectations at $t = 1$ gives

$$0 = \kappa(\alpha)\,\mathbb{E}_{\pi^\infty} \int_0^1 e^{\alpha V^\infty(s)}\,ds + \alpha \mathbb{E}_{\pi^\infty} L^c(1) + \mathbb{E}_{\pi^\infty} \sum_{0 \leq s \leq 1} \left(1 - e^{-\alpha \Delta L(s)}\right).$$

Now just note that the expectation of the integral equals $\mathbb{E}e^{\alpha V^\infty(\infty)}$. □

If $X$ has no negative jumps, the term $\mathbb{E}_{\pi^\infty} \sum_{0\leq s\leq 1}(1 - e^{-\alpha\Delta L(s)})$ disappears, and $\mathbb{E}_{\pi^\infty} L^c(1) = \mathbb{E}_{\pi^\infty} L(1) = -m$, and we see that Theorem 11.3 indeed is a generalization of Corollary 3.4 in Chap. IX [11] which is itself a generalization of the Pollaczeck-Khinchine formula.

Next, we use the results above to obtain an expression for the mean of the stationary distribution in the case of one-sided reflection.

**Corollary 11.4.** *If $X$ is square integrable then $V^\infty$ is integrable and we have*

$$\mathbb{E}V^\infty = \frac{1}{2m}\Big(\mathbb{E}_{\pi^\infty} \sum_{0 \leq s \leq 1} \Delta L(s)^2 - \mathbb{V}ar(X(1))\Big) \quad (11.4)$$

$$= \frac{1}{2m}\Big(\int_{-\infty}^{\infty} y^2 \nu(dy) + \sigma^2 - \int_0^\infty \int_{-\infty}^{-x} (x+y)^2 \nu(dy) \pi^\infty(dx)\Big). \quad (11.5)$$

*Proof.* Since $X(1)$ is non-degenerate, we have by Lemma 4 chapter XV.1 in [55] that there exists $\epsilon > 0$ such that $\kappa(\mathrm{i}t) \neq 0$ for $t \in (-\epsilon, \epsilon) \setminus \{0\}$, and we may use (11.3) to obtain the characteristic function $\psi$ of $V$. We wish to show that $\psi$ is differentiable at 0. Define

$$g(t) = \mathbb{E}_{\pi^\infty} \sum_{0 \le s \le 1} (1 - \mathrm{e}^{-\mathrm{i}t\Delta L(s)}) \,, \quad \ell_1 = \mathbb{E}_{\pi^\infty} L^c(1) \,.$$

By Doob's inequality, we have that $\mathbb{E}X(1)^2 < \infty$ implies $\mathbb{E}L^2(1) < \infty$ and therefore $\mathbb{E}_{\pi^\infty} L^2(1) < \infty$, which in turn implies that $g$ is twice differentiable at 0. We see that

$$g'(0) = \mathrm{i}\, \mathbb{E}_{\pi^\infty} \sum_{0 \le s \le 1} \Delta L(s) = \mathrm{i}\, \mathbb{E}_{\pi^\infty} L^j(1) \,, \quad g''(0) = \mathbb{E}_{\pi^\infty} \sum_{0 \le s \le 1} \Delta L(s)^2 \,,$$

$\mathrm{i}\ell_1 + g'(0) = \mathrm{i}\mathbb{E}_{\pi^\infty} L(1) = -\mathrm{i}m$. Since $X$ is square integrable, we may use formula (2.4.1) p. 27 in [101] to get $\kappa(\mathrm{i}t) = \kappa'(0)\mathrm{i}t + \mathrm{o}(t)$. By combining this with equation (11.3), we conclude that

$$\lim_{t \to 0} \frac{\mathbb{E}e^{\mathrm{i}tV} - 1}{t} = \lim_{t \to 0} \frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{t\kappa(\mathrm{i}t)} = \lim_{t \to 0} \frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{\kappa'(0)\mathrm{i}t^2} \,,$$

provided that the limit exists. We may confirm that this is true, by applying l'Hospital's rule twice to the real and imaginary part separately

$$\begin{aligned}\lim_{t \to 0} \frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{\kappa'(0)\mathrm{i}t^2} &= \lim_{t \to 0} \frac{-\mathrm{i}\ell_1 - g'(t) - \mathrm{i}\kappa'(\mathrm{i}t)}{2\mathrm{i}\kappa'(0)t} \\ &= \lim_{t \to 0} \frac{-g''(t) + \kappa''(\mathrm{i}t)}{2\mathrm{i}\kappa'(0)} = \frac{-g''(0) + \kappa''(0)}{2\mathrm{i}\kappa'(0)}.\end{aligned}$$

We see that $\psi$ is differentiable. In itself, this does not entail integrability of $V$, but a short argument using the Law of Large Numbers and the fact that $V$ is non-negative, yields that $V$ is integrable. The first moment is

$$\mathbb{E}V = \frac{-g''(0) + \kappa''(0)}{2(-1)\kappa'(0)}$$

which is (11.4). We obtain (11.5) by conditioning on the value of the process prior to a jump. □

We proceed to the proof of Theorem 11.1. In order to establish (11.1), we need to prove that 1 is a lower bound for $\liminf_b \ell^b / \overline{\nu}_I(b)$ and an upper bound for $\limsup_b \ell^b / \overline{\nu}_I(b)$. The former is established in Proposition 11.6 and is seen to hold without the conditions assumed in Theorem 11.1. In the proof of the latter, we use Proposition 11.5 to establish the inequality

$$\frac{m}{b} \int_0^b \overline{\pi}^b(x) \,\mathrm{d}x \le \frac{m}{b} \int_0^b \overline{\pi}^\infty(x) \,\mathrm{d}x - m\overline{\pi}^\infty(b) \tag{11.6}$$

and the proof then follows two distinct routes depending on which of the conditions **A** or **B** is assumed. Under assumption **A**, we are allowed to rewrite the integral on the right-hand side of (11.6) as $\int_0^\infty \overline{\pi}^\infty(x) \,\mathrm{d}x - \int_b^\infty \overline{\pi}^\infty(x) \,\mathrm{d}x$. The first of these integrals is the mean of the stationary distribution in the case of one-sided reflection. This observation and Corollary 11.4 are important keys to the proof in this case. Under assumption **B** the proof essentially consists of combining the inequality (11.6) with repeated applications of Karamata's theorem.

**Proposition 11.5.** *Let $X$ be a Lévy process, and let $\overline{\pi}^{\infty}(y), \overline{\pi}^{b}(y)$ be the tails of the reflected (one/two-sided) distributions. Then we have the following inequalities for $x > 0, b > 0$*

$$0 \le \overline{\pi}^{\infty}(x) - \overline{\pi}^{b}(x) \le \overline{\pi}^{\infty}(b)\,. \tag{11.7}$$

*Proof.* The inequalities in (11.7) are trivial for $x > b$. Let $0 \le x \le b$. The inequality $\overline{\pi}^{b}(x) \le \overline{\pi}^{\infty}(x)$ follows from the representations (1.6) and (2.13). The inequality $\overline{\pi}^{\infty}(x) - \overline{\pi}^{b}(x) \le \overline{\pi}^{\infty}(b)$, follows by dividing the sample paths of $X$ which cross above $x$ into those which do so by first passing below $x-b$, and those which stay above $x-b$. To be precise, define $\tau(y) = \inf\{t > 0 : X(t) \ge y\}$ and $\sigma(y) = \inf\{t > 0 : X(t) < y\}$ to be the first passage times above and below $y$ respectively. Then we can consider the event that a path crosses below $x - b$ before eventually passing above $x$, and since such a path must pass an interval of length at least $b$, we find that

$$\begin{aligned}\mathbb{P}\big(\sigma(x-b) < \tau(x) < \infty\big) &\le \mathbb{P}\Big(\sup_{t>0} X\big(\sigma(x-b)+t\big) - X\big(\sigma(x-b)\big) > b\Big)\\ &= \mathbb{P}\big(\tau(b) < \infty\big)\,.\end{aligned}$$

where we used the strong Markov property in the last equality. Next, we apply (2.13) to find

$$\begin{aligned}\overline{\pi}^{\infty}(x) &= \mathbb{P}\big(\tau(x) < \infty\big) = \mathbb{P}\big(\tau(x) < \sigma(x-b)\big) + \mathbb{P}\big(\sigma(x-b) < \tau(x) < \infty\big)\\ &\le \overline{\pi}^{b}(x) + \mathbb{P}\big(\tau(b) < \infty\big) = \overline{\pi}^{b}(x) + \overline{\pi}^{\infty}(b)\,,\end{aligned}$$

where we have used the equality $\mathbb{P}\big(\tau(x) < \sigma(x-b) \le \infty\big) = \overline{\pi}^{b}(x)$, which is a restatement of (1.6). □

**Proposition 11.6.** *For any Lévy process we have* $1 \le \liminf_{b\to\infty} \dfrac{\ell^b}{\overline{\nu}_I(b)}$ .

*Proof.* We have

$$\int_0^b \pi^b(\mathrm{d}x) \int_b^\infty (y - b + x)\,\nu(\mathrm{d}y) \le \ell^b$$

since the left-hand side is the contribution to $\ell^b$ by the jumps larger than $b$. Now just note that

$$\overline{\nu}_I(b) \le \int_b^\infty (y-b)\nu(\mathrm{d}y) + \int_0^b x\pi^b(\mathrm{d}x)\overline{\nu}(b) = \int_0^b \pi^b(\mathrm{d}x)\int_b^\infty (y-b+x)\nu(\mathrm{d}y)\,.$$

□

We are now ready for the proof of Theorem 11.1.

*Proof.* Thanks to Proposition 11.6, we only need to prove

$$\limsup_b \ell^b/\overline{\nu}_I(b) \leq 1\,. \tag{11.8}$$

Define

$$\mathcal{I}_1 = \frac{m}{b}\int_0^b x\pi^b(\mathrm{d}x)\,,\ \mathcal{I}_2 = \frac{\sigma^2}{2b}\,,\ \mathcal{I}_3 = \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^{\infty}\varphi_b(x,y)\nu(\mathrm{d}y)\,.$$

where the function $\varphi_b(\cdot,\cdot)$ is that of Theorem 1.1, with the dependence on $b$ made explicit. From Proposition 11.5 we have $\overline{\pi}^\infty(x)-\overline{\pi}^\infty(b) \leq \overline{\pi}^b(x)$ and since $m$ is assumed to be negative, we have $m\overline{\pi}^b(x) \leq m(\overline{\pi}^\infty(x) - \overline{\pi}^\infty(b))$. Applying this inequality to expression for the loss rate in Theorem 1.1 we obtain the following inequality:

$$\ell^b \leq \frac{m}{b}\int_0^b \overline{\pi}^\infty(x)\,\mathrm{d}x - m\overline{\pi}^\infty(b) + \mathcal{I}_2 + \mathcal{I}_3\ . \tag{11.9}$$

First, we assume **A** holds. By (2.16) we have

$$\lim_b \frac{-m\overline{\pi}^\infty(b)}{\overline{\nu}_I(b)} = 1\,, \tag{11.10}$$

so we will be done if we can show

$$\limsup_b \frac{1}{\overline{\nu}_I(b)}\left[\frac{m}{b}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y + \mathcal{I}_2 + \mathcal{I}_3\right] = 0\,. \tag{11.11}$$

We start by rewriting the term in the brackets above. Using Corollary 11.4 and the assumption that $\mathbb{E}X(1)^2 < \infty$ we have that $\int_0^\infty \overline{\pi}^\infty(y)\mathrm{d}y < \infty$ and using (11.4)

$$\begin{aligned}\frac{m}{b}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y &= \frac{m}{b}\int_0^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{m}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y\\ &= \frac{\mathbb{E}_{\pi^\infty}[\sum_{0\leq s\leq 1}\Delta L(s)^2] - \mathbb{V}ar(X(1))}{2b} + \frac{|m|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y\,.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\mathcal{I}_2 + \mathcal{I}_3\\ &\quad= \frac{\sigma^2}{2b} + \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\Big(\int_{-\infty}^{-x} -(x^2+2xy)\nu(\mathrm{d}y) + \int_{-x}^{b-x} y^2\nu(\mathrm{d}y)\\ &\qquad+ \int_{b-x}^\infty \big[2y(b-x)-(b-x)^2\big]\nu(\mathrm{d}y)\Big)\\ &\qquad= \frac{\sigma^2}{2b} + \frac{1}{2b}\int_{-\infty}^\infty y^2\nu(\mathrm{d}y) + \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^{-x}\big[-(x^2+2xy)-y^2\big]\nu(\mathrm{d}y)\\ &\qquad+ \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{b-x}^\infty\big[2y(b-x)-(b-x)^2-y^2\big]\nu(\mathrm{d}y)\\ &\quad= \frac{\sigma^2}{2b} + \frac{1}{2b}\int_{-\infty}^\infty y^2\nu(\mathrm{d}y) - \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y)\\ &\qquad- \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{b-x}^\infty (y-(b-x))^2\nu(\mathrm{d}y)\\ &\quad= \frac{\mathbb{V}ar(X(1)) - \mathbb{E}_{\pi^b}\sum_{0\leq s\leq 1}\Delta L(s)^2}{2b} - \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{b-x}^\infty (y-(b-x))^2\nu(\mathrm{d}y).\end{aligned}$$

The last equation follows from Example 25.12 p. 163 in [119], as does the following:

$$
\begin{aligned}
\mathbb{E}_{\pi^b} \sum_{0\le s\le 1} \Delta L(s)^2 &= \mathbb{E}_{\pi^b} \sum_{0\le s\le 1} (V(s-)+\Delta X(s))^2 \mathbb{1}(V(s-)+\Delta X(s)<0) \\
&= \int_0^b \Big( \mathbb{E} \sum_{0\le s\le 1} (x+\Delta X(s))^2 \mathbb{1}(x+\Delta X(s)<0) \Big) \pi^b(\mathrm{d}x) \\
&= \int_0^b \pi^b(\mathrm{d}x) \int_{-\infty}^{-x} (x+y)^2 \nu(\mathrm{d}y)\,,
\end{aligned}
$$

where we use Theorem 2.7 p. 41 in [94] in the last equation. Next, we note the fact that

$$
\mathbb{E}_{\pi^\infty} \sum_{0\le s\le 1} \Delta L(s)^2 \le \mathbb{E}_{\pi^b} \sum_{0\le s\le 1} \Delta L(s)^2,
$$

which can be verified using partial integration and (11.7). Using this in the last equation above, we may continue our calculation and obtain

$$
\begin{aligned}
\mathcal{I}_2 + \mathcal{I}_3 \le{}& \frac{\mathbb{V}ar(X(1)) - \mathbb{E}_{\pi^\infty} \sum_{0\le s\le 1} \Delta L(s)^2}{2b} \\
&- \frac{1}{2b} \int_0^b \pi^b(\mathrm{d}x) \int_{b-x}^{\infty} (y-(b-x))^2 \nu(\mathrm{d}y)\,.
\end{aligned}
$$

Comparing the expressions above we see that fractions cancel, and the expression in the brackets in (11.11) is less than

$$
\frac{|m|}{b} \int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b} \int_0^b \int_{b-x}^\infty (y-(b-x))^2 \nu(\mathrm{d}y) \pi^b(\mathrm{d}x)\,.
$$

Applying partial integration

$$
\begin{aligned}
&\frac{|m|}{b} \int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b} \int_0^b \int_{b-x}^\infty (y-(b-x))^2 \nu(\mathrm{d}y) \pi^b(\mathrm{d}x) \\
&\quad = \frac{|\mathbb{E}X(1)|}{b} \int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b} \int_b^\infty (y-b)^2 \nu(\mathrm{d}y) - \frac{1}{b} \int_0^b \overline{\pi}^b(x) \overline{\nu}_I(b-x)\mathrm{d}x \\
&\quad \le \frac{|m|}{b} \int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b} \int_b^\infty (y-b)^2 \nu(\mathrm{d}y) \\
&\quad = \frac{|m|}{b} \int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{b} \int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y.
\end{aligned}
$$

Returning to (11.11) and applying the results above we get

$$
\begin{aligned}
&\limsup_b \frac{1}{\overline{\nu}_I(b)} \left[ \frac{m}{b} \int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y + \mathcal{I}_2 + \mathcal{I}_3 \right] \\
&\qquad \le \limsup_b \frac{1}{\overline{\nu}_I(b)} \left[ \frac{|m|}{b} \int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{b} \int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y \right] \\
&\qquad = \limsup_b \frac{\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y}{b\overline{\nu}_I(b)} \left[ \frac{\int_b^\infty |m| \overline{\pi}^\infty(y)\,\mathrm{d}y}{\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y} - 1 \right] = 0,
\end{aligned}
$$

where the last equality follows since the term in the brackets tends to 0, and the fraction outside it is bounded by assumption. This proves that (11.1) holds under condition **A**.

We now assume condition **B** and start by noticing the following consequences of the assumptions

$$\int_b^\infty \overline{\nu}(y)\,\mathrm{d}y \sim \int_b^\infty \frac{L(y)}{y^\alpha}\,\mathrm{d}y \sim \frac{b^{-\alpha+1}L(b)}{\alpha-1}\,, \qquad b\to\infty\,, \tag{11.12}$$

where the last equivalence follows by Proposition 1.5.10 of [31] and the fact that $\alpha>1$. Since by Proposition 1.3.6 of [31], we have $b^{-\alpha+2}L(b)\to\infty$, (11.12) implies $b\overline{\nu}_I(b)\to\infty$.

The inequality (11.9) still holds, as does the limit in (11.10), so we proceed to analyze $m\int_0^b \overline{\pi}^\infty(y)\mathrm{d}y/(\overline{\nu}_I(b)b)$. Since $b\overline{\nu}_I(b)\to\infty$ as $b\to\infty$ we see that for any $A$

$$\lim_{b\to\infty}\frac{m}{b\overline{\nu}_I(b)}\int_0^A \overline{\pi}^\infty(y)\,\mathrm{d}y = 0\,. \tag{11.13}$$

Because of the result above we have for any $A$

$$\lim_{b\to\infty}\frac{m}{b\overline{\nu}_I(b)}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y = \lim_{b\to\infty}\frac{m}{b\overline{\nu}_I(b)}\int_A^b \overline{\pi}^\infty(y)\,\mathrm{d}y$$

and using $|m|\overline{\pi}^\infty(b)\sim\overline{\nu}_I(b)\sim b^{-\alpha+1}L(b)/(\alpha-1)$ we have

$$\begin{aligned}\lim_{b\to\infty}\frac{m}{b\overline{\nu}_I(b)}\int_A^b \overline{\pi}^\infty(y)\,\mathrm{d}y &= \lim_{b\to\infty}-\frac{1}{b\overline{\nu}_I(b)}\int_A^b \overline{\nu}_I(y)\,\mathrm{d}y\\ &= -\lim_{b\to\infty}\frac{1}{b\overline{\nu}_I(b)}\int_A^b \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y\end{aligned}$$

in the sense that if either limit exits so does the other and they are equal. Furthermore, since $-\alpha+1>-1$ and $L$ is locally bounded, we may apply Proposition 1.5.8 in [31] to obtain

$$-\lim_{b\to\infty}\frac{1}{b\overline{\nu}_I(b)}\int_A^b \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y = -\lim_{b\to\infty}\frac{1}{b\overline{\nu}_I(b)}\frac{b^{-\alpha+2}L(b)}{(-\alpha+2)(\alpha-1)} = -\frac{1}{-\alpha+2}.$$

That is, we obtain

$$\lim_{b\to\infty}\frac{m}{b\overline{\nu}_I(b)}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y = -\frac{1}{-\alpha+2}. \tag{11.14}$$

Returning to (11.9) we have

$$\begin{aligned}\limsup_b \frac{\ell^b}{\overline{\nu}_I(b)} &= \limsup_b\left[\frac{m}{b\overline{\nu}_I(b)}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{m\overline{\pi}^\infty(b)}{\overline{\nu}_I(b)} + \frac{\mathcal{I}_2}{\overline{\nu}_I(b)} + \frac{\mathcal{I}_3}{\overline{\nu}_I(b)}\right]\\ &= -\frac{1}{-\alpha+2}+1+\limsup_b\left[\frac{\mathcal{I}_2}{\overline{\nu}_I(b)}+\frac{\mathcal{I}_3}{\overline{\nu}_I(b)}\right].\end{aligned} \tag{11.15}$$

Since $b\overline{\nu}_I(b) \to \infty$ we have

$$\limsup_b \mathcal{I}_2/\overline{\nu}_I(b) = \limsup_b \frac{\sigma^2}{2b\overline{\nu}_I(b)} = 0\,,$$

and we may continue our calculation from (11.15)

$$-\frac{1}{-\alpha+2}+1+\limsup_b\Big[\frac{\mathcal{I}_2}{\overline{\nu}_I(b)}+\frac{\mathcal{I}_3}{\overline{\nu}_I(b)}\Big]=-\frac{1}{-\alpha+2}+1+\limsup_b\Big[\frac{\mathcal{I}_3}{\overline{\nu}_I(b)}\Big] \tag{11.16}$$

So we turn our attention to $\mathcal{I}_3$. First we divide the integral into two:

$$2b\mathcal{I}_3 = \underbrace{\int_0^b \pi^b(\mathrm{d}x)\Big(\int_{-\infty}^{-x} -(x^2+2xy)\nu(\mathrm{d}y)+\int_{-x}^{0} y^2\nu(\mathrm{d}y)\Big)}_{A(b)} \tag{11.17}$$

$$+\underbrace{\int_0^b \pi^b(\mathrm{d}x)\Big(\int_0^{b-x} y^2\nu(\mathrm{d}y)+\int_{b-x}^{\infty} 2(b-x)y-(b-x)^2\nu(\mathrm{d}y)\Big)}_{B(b)}\,. \tag{11.18}$$

We may assume $\nu$ is bounded from below; otherwise truncate $\nu$ at $-L$ for some $L>0$ chosen large enough to ensure that the mean of $X(1)$ remains negative. This truncation may increase the loss rate, which is not a problem, since we are proving an upper bound. Thus, we may assume that $A(b)$ is bounded:

$$A(b) \le \int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^0 y^2\nu(\mathrm{d}y) \le \int_{-\infty}^0 y^2\nu(\mathrm{d}y) < \infty\,.$$

And therefore, since $b\overline{\nu}_I(b)\to\infty$, we have

$$\frac{A(b)}{2b\overline{\nu}_I(b)} \to 0\,. \tag{11.19}$$

Turning to $B(b)$, we first perform partial integration

$$\begin{aligned}
B(b) &= \int_0^b y^2\nu(\mathrm{d}y)+\int_b^\infty 2by-b^2\nu(\mathrm{d}y)-\int_0^b \overline{\nu}_I(b-x)\overline{\pi}^b(x)\,\mathrm{d}x\\
&\le \int_0^b y^2\nu(\mathrm{d}y)+\int_b^\infty 2by-b^2\nu(\mathrm{d}y)\\
&= \int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y-b^2\overline{\nu}(b)+\int_b^\infty 2by-b^2\nu(\mathrm{d}y)\\
&= \int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y+2b\int_b^\infty \overline{\nu}(y)\,\mathrm{d}y\,.
\end{aligned}$$

Since $y\overline{\nu}(y)\sim y^{-\alpha+1}L(y)$ way may apply Proposition 1.5.8 from [31] to get

$$\int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y \sim 2\frac{L(b)b^{-\alpha+2}}{2-\alpha}\,,$$

and therefore

$$\lim_{b} \frac{1}{2b\overline{\nu}_I(b)} \int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y = \frac{\alpha-1}{2-\alpha}\,.$$

Combining this with our inequality for $B(b)$ above, we have

$$\limsup_{b\to\infty} \frac{B(b)}{2b\overline{\nu}_I(b)} \le \frac{\alpha-1}{2-\alpha} + 1 = \frac{1}{2-\alpha}\,.$$

Finally, by combining this with (11.15), (11.19) and (11.16), we obtain (11.8). □

# 12 Loss rate symptotics: no drift

In both sections 10 and 11 it was assumed that the underlying stochastic process had negative mean, and as discussed in Section 1 this also gives the asymptotic behavior in the case of positive drift. Thus, it remains to give an asymptotic expression as $b \to \infty$ for the loss rate in the zero-mean case. The result is as follows:

**Theorem 12.1.** *(a) Let $\{X(t)\}$ be a Lévy process with $m = \mathbb{E}X(1) = 0$ and*

$$\psi^2 = \mathbb{V}ar\big(X(1)\big) = \kappa''(0) = \sigma^2 + \int_{-\infty}^{\infty} y^2\,\nu(\mathrm{d}y) < \infty\,.$$

*Then*

$$\ell^b \sim \frac{1}{2b}\mathbb{V}ar(X(1))\,, \qquad b \to \infty\,. \tag{12.1}$$

*(b) Let $\{X(t)\}$ be a Lévy process with Lévy measure $\nu$. Assume $\mathbb{E}X(1) = 0$ and that for some $1 < \alpha < 2$, there exist slowly varying functions $L_1(x)$ and $L_2(x)$ such that for $L(x) = L_1(x) + L_2(x)$, we have*

$$\overline{\nu}(x) = x^{-\alpha}L_1(x) \qquad \nu(-x) = x^{-\alpha}L_2(x) \qquad \lim_{x\to\infty}\frac{L_1(x)}{L(x)} = \frac{\beta+1}{2} \tag{12.2}$$

*where $\nu(x) = \nu(-\infty, x])$ and $\overline{\nu}(x) = \nu[x,\infty)$. Then, setting*

$$\rho = 1/2 + (\pi\alpha)^{-1}\arctan(\beta\tan(\pi\alpha/2))\,,$$
$$c_+ = (\beta+1)/2\,, \quad c_- = (1-\beta)/2\,,$$

*we have $\ell^b \sim \gamma L(b)/b^{\alpha-1}$ where*

$$\gamma = \frac{c_- B(2-\alpha\rho, \alpha\rho) + c_+ B(2-\alpha(1-\rho), \alpha(1-\rho))}{B(\alpha\rho, \alpha(1-\rho))(\alpha-1)(2-\alpha)}$$

*and $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function.*

By comparing to Example 8.8, we see that the loss rate behaves asymptotically like that of a stable Lévy process.

To prove Theorem 12.1, we will use the fact that by properly scaling our Lévy process we may construct a sequence of Lévy processes which converges weakly to either a Brownian Motion or a stable process. Since $\ell^b$ has been calculated for both Brownian Motion and stable processes in Examples 8.7 and 8.8, we may use this convergence to obtain loss rate asymptotics in the case of zero drift, provided that the loss rate is continuous in the sense, that weak convergence (in the sense of Proposition 12.4 below) of the involved processes implies convergence of the associated loss rates. The required continuity results are established in Theorem 12.2 and Theorem 12.3.

**Theorem 12.2.** *Let $\{X^n\}_{n=0,1,\ldots}$ be a sequence of Lévy processes with associated loss rates $\ell^{b,n}$. Suppose $X^n \xrightarrow{\mathscr{D}} X^0$ in $D[0,\infty)$ and that the family $(X(1)^n)_{n=1}^{\infty}$ is uniformly integrable. Then $\ell^{b,n} \to \ell^{b,0}$ as $n \to \infty$.*

We shall also need:

**Theorem 12.3.** *Let $\{X_n\}_{n=1,2,\ldots}$ be a sequence of weakly convergent infinitely divisible random variables, with characteristic triplets $(c_n, \sigma_n, \nu_n)$. Then for $\alpha > 0$:*

$$\lim_{a\to\infty} \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0 \iff \left(\left(\left|X_n\right|^\alpha\right)_{|n\geq 1} \text{ is uniformly integrable.}\right.$$

The result is certainly not unexpected, but does not appear to be in the literature; the closest we could find is Theorem 25.3 in [119].

## Weak convergence of Lévy processes

We prove here Theorems 12.2 and 12.3. We will need the following weak convergence properties, where $D[0,\infty)$ is the metric space of cadlag functions on $[0,\infty)$ endowed with the Skorokhod topology (see Chap. 3, Sec. 16 in [32] or Chap. 3 in [127]).

**Proposition 12.4.** *Let $X^0, X^1, X^2, \ldots$ be Lévy processes with characteristic triplet $(c_n, \sigma_n, \nu_n)$ for $X^n$. Then the following properties are equivalent:*

*(i)* $X(t)^n \xrightarrow{\mathscr{D}} X(t)^0$ *for some* $t > 0$;

*(ii)* $X(t)^n \xrightarrow{\mathscr{D}} X(t)^0$ *for all* $t$;

*(iii)* $\{X(t)^n\} \xrightarrow{\mathscr{D}} \{X(t)^0\}$ *in* $D[0,\infty)$;

*(iv)* $\tilde{\nu}_n \to \tilde{\nu}_0$ *weakly, where* $\tilde{\nu}_n$ *is the bounded measure*

$$\tilde{\nu}_n(\mathrm{d}y) = \sigma_n^2 \delta_0(\mathrm{d}y) + \frac{y^2}{1+y^2}\nu_n(\mathrm{d}y) \tag{12.3}$$

*and* $\tilde{c}_n \to \tilde{c}_0$ *where*

$$\tilde{c}_n = c_n + \int \left(\frac{y}{1+y^2} - y\mathbb{1}|y| \leq 1\right) \nu_n(\mathrm{d}y)$$

See e.g. [76] pp. 244–248, in particular Lemma 13.15 and 13.17. If one of (i)–(iv) hold, we write simply $X^n \xrightarrow{\mathscr{D}} X^0$.

The following proposition is standard:

**Proposition 12.5.** *Let $p > 0$ and let $X_n \in L^p$, $n = 0, 1, \ldots$, such that $X_n \xrightarrow{\mathscr{D}} X_0$. Then $\mathbb{E}|X_n|^p \to \mathbb{E}|X_0|^p$ if and only if the family $\big(|X_n|^p\big)_{n\geq 1}$ is uniformly integrable.*

First, we prove Theorem 12.3. This is achieved through several preliminary results, of which the first is Lemma 12.6 which essentially states we may disregard the behavior of the Lévy measures on the interval $[-1, 1]$ in questions regarding uniform integrability. It is therefore sufficient to prove Theorem 12.3 for compound Poisson distributions, which is done in Proposition 12.8.

We start by examining the case where the Lévy measures have uniformly bounded support, i.e., there exists $A > 0$ such that $\nu_n([-A, A]^c) = 0$ for all $n$. We know from Lemma 25.6 and Lemma 25.7 in [119] that this implies the existence of finite exponential moments of $X_n$ and therefore the $m$th moment of $X_n$ exists and is finite for all $n, m \in \mathbb{N}$.

**Lemma 12.6.** *Suppose $X_n \xrightarrow{\mathscr{D}} X_0$ and the Lévy measures have uniformly bounded support. Then $\mathbb{E}[(X_n)^m] \to \mathbb{E}[(X_0)^m]$ for $m = 1, 2, \cdots$. In particular* (cf. Proposition 12.5) *the family $\big(|X_n|^\alpha\big)_{n\geq 1}$ is uniformly integrable for all $\alpha > 0$.*

*Proof.* Since the Lévy measures are uniformly bounded, the characteristic exponent from (1.7) is

$$\kappa_n(t) = c_n t + \sigma_n^2 t^2/2 + \int_{-A}^{A} \big(\mathrm{e}^{ty} - 1 - ty\mathbb{1}|y| \leq 1\big)\nu_n(\mathrm{d}y)\,. \tag{12.4}$$

With the aim of applying Proposition 12.4 we rewrite (12.4) as

$$\kappa_n(t) = \tilde{c}_n t + \int_{-A}^{A} \left(\mathrm{e}^{ty} - 1 - \frac{ty}{1+y^2}\right) \frac{1+y^2}{y^2}\tilde{\nu}_n(\mathrm{d}y)\,. \tag{12.5}$$

(the integrand is defined to be 0 at $y = 0$) where $\tilde{\nu}_n$ is given by (12.3) and

$$\tilde{c}_n = c_n + \int_{-A}^{A} \left(\frac{y}{1+y^2} - y\mathbb{1}|y| \leq 1\right) \nu_n(\mathrm{d}y)\,.$$

According to Proposition 12.4 the weak convergence of $\{X_n\}_{n\geq 1}$ implies $\tilde{c}_n \to \tilde{c}_0$ and $\tilde{\nu}_n \xrightarrow{\mathscr{D}} \tilde{\nu}_0$. Since the integrand in (12.5) is bounded and continuous, this implies that $\kappa_n(t) \to \kappa_0(t)$, which in turn implies that all exponential moments converge. In particular, the family $\big(\mathrm{e}^{X_n} + \mathrm{e}^{-X_n}\big)_{n\geq 1}$ is uniformly integrable, which implies that $\big(|X_n|^\alpha\big)_{n\geq 1}$ is so. $\square$

Next, we express the condition of uniform integrability using the tail of the involved distributions. We will need the following lemma on weakly convergent compound Poisson distributions.

**Lemma 12.7.** *Let $U_0, U_1, \ldots$ be a sequence of positive independent random variables such that $U_n > 1$, and let $N_0, N_1, \ldots$ be independent Poisson random variables with rates $\lambda_0, \lambda_1, \ldots$ Set $X_n = \sum_1^{N_n} U_{i,n}$ (empty sum $= 0$) with the $U_{i,n}$ being i.i.d for fixed $n$ with $U_{i,n} \stackrel{\mathscr{D}}{=} U_n$. Then $X_n \xrightarrow{\mathscr{D}} X_0$ if and only if $U_n \xrightarrow{\mathscr{D}} U_0$ and $\lambda_n \to \lambda_0$.*

*Proof.* The 'if' part follows from the continuity theorem for characteristic functions. For the converse, we observe that $e^{-\lambda_n} \to e^{-\lambda_0} = \mathbb{P}(X_0 \le 1/2)$ since $1/2$ is a continuity point of $X_0$ (note that $\mathbb{P}(X_0 \le x) = \mathbb{P}(X_0 = 0)$ for all $x < 1$). Taking logs yields $\lambda_n \to \lambda_0$ and the necessity of $U_n \xrightarrow{\mathscr{D}} U_0$ then is obvious from the continuity theorem for characteristic functions. □

Using the previous result, we are ready to prove part of Theorem 12.3 for a class of compound Poisson distributions:

**Proposition 12.8.** *Let $U_0, U_1, \ldots, N_0, N_1, \ldots,$ and $X_0, X_1, \ldots$ be as in Lemma 12.7. Assume $X_n \xrightarrow{\mathscr{D}} X_0$. Then for $\alpha > 0$.*

$$\lim_{a\to\infty} \sup_n \mathbb{E}\big[X_n^\alpha \mathbb{1} X_n > a\big] = 0 \iff \lim_{a\to\infty} \sup_n \mathbb{E}\big[U_n^\alpha \mathbb{1} U_n > a\big] = 0\,.$$

*Proof.* To prove that the l.h.s. implies the r.h.s., we let $G_n(x) = \mathbb{P}(X_n \le x)$, $F_n(x) = \mathbb{P}(U_n \le x)$, $\overline{F}_n(x) = 1 - F_n(x)$, $\overline{G}_n(x) = 1 - G_n(x)$, and let $F_n^{*m}(x), G_n^{*m}(x)$ denote the $m$-fold convolutions. Then

$$\overline{G}_n(x) = \sum_{m=1}^{\infty} \frac{\lambda_n^m}{m!} e^{-\lambda_n} \overline{F_n^{*m}}(x)\,, \qquad x > 0$$

which implies $\overline{G}_n(x) \ge \lambda_n e^{-\lambda_n} \overline{F}_n(x)$. Letting $\beta = \sup_n e^{\lambda_n}/\lambda_n$, which is finite by Lemma 12.7, we get $\overline{F}_n(x) \le \beta \overline{G}_n(x)$. Therefore:

$$\begin{aligned}
\mathbb{E}[U_n^\alpha \mathbb{1} U_n > a] &= \int_0^\infty \alpha t^{\alpha-1} \mathbb{P}(U_n > a \vee t)\mathrm{d}t = a^\alpha \overline{F}_n(a) + \alpha \int_a^\infty t^{\alpha-1} \overline{F}_n(t)\mathrm{d}t \\
&\le \beta a^\alpha \overline{G}_n(a) + \beta\alpha \int_a^\infty t^{\alpha-1} \overline{G}_n(t)\mathrm{d}t = \beta \mathbb{E}[X_n^\alpha \mathbb{1} X_n > a]\,.
\end{aligned}$$

Taking supremum and limits completes the first part of the proof.

For the converse we note that by Lemma 12.7 we have $F_n^{*1} \xrightarrow{\mathscr{D}} F_0^{*1}$ and it follows from the continuity theorem for characteristic functions that $F_n^{*m} \xrightarrow{\mathscr{D}} F_0^{*m}$. Fix $m \in \mathbb{N}$. Since $\left(\sum_{i=1}^m U_{i,n}\right)^\alpha \le m^\alpha \sum_{i=1}^m U_{i,n}^\alpha$ and the family $(m^\alpha \sum_{i=1}^m U_{i,n}^\alpha)_{n\ge1}$ is uniformly integrable, we have that also the family $(\sum_{i=1}^m U_{i,n})_{n\ge1}^\alpha$ is uniformly integrable. As noted above we have $\sum_{i=1}^m U_{i,n} \xrightarrow{\mathscr{D}} \sum_{i=1}^m U_{i,0}$, so Proposition 12.5 implies $\mathbb{E}(\sum_{i=1}^m U_{i,n})^\alpha \to \mathbb{E}(\sum_{i=1}^m U_{i,0})^\alpha$.

We next show $\mathbb{E}X_n^\alpha \to \mathbb{E}X_0^\alpha$ and thereby the assertion of the proposition. We have:

$$\begin{aligned}
\lim_n \mathbb{E}X_n^\alpha &= \lim_n \sum_{m=0}^\infty \mathbb{E}\Big(\sum_{i=1}^m U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!} e^{-\lambda_n} = \sum_{m=0}^\infty \lim_n \mathbb{E}\Big(\sum_{i=1}^m U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!} e^{-\lambda_n} \\
&= \sum_{m=0}^\infty \mathbb{E}\Big(\sum_{i=1}^m U_{i,0}\Big)^\alpha \frac{\lambda_0^m}{m!} e^{-\lambda_0} = \mathbb{E}X_0^\alpha\,,
\end{aligned}$$

where we used dominated convergence with the bound

$$\mathbb{E}\Big(\sum_{i=1}^m U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!} e^{-\lambda_n} \le \gamma m^{\alpha+1} \beta^m / m!\,,$$

where $\gamma = \sup_n \mathbb{E}U_n^\alpha$ and $\beta = \sup_n \lambda_n$. □

*Proof of Theorem 12.3.* Using the Lévy-Khinchine representation, we may write

$$X_n = X_n^{(1)} + X_n^{(2)} + X_n^{(3)}\,, \tag{12.6}$$

where the $(X_n^{(i)})_{n\geq 1}$ are sequences of infinitely divisible distributions having characteristic triplets $(0, 0, [\nu_n]_{\{y<-1\}})$ , $(c_n, \sigma_n, [\nu_n]_{\{|y|\leq 1\}})$ and $(0, 0, [\nu_n]_{\{y>1\}})$, respectively, which are independent for each $n$. Assume the family $(|X_n|^\alpha)_{n\geq 1}$ is uniformly integrable. We wish to apply Proposition 12.8 to the family $((X_n^{(3)})^\alpha)_{n\geq 1}$, and therefore we need to show that this family is uniformly integrable. First, we rewrite (12.6) as $X_n - X_n^{(2)} = X_n^{(1)} + X_n^{(3)}$ and use Lemma 12.6 together with the inequality $|x-y|^\alpha \leq 2^\alpha(|x|^\alpha + |y|^\alpha)$ to conclude that the family $(|X_n - X_n^{(2)}|^\alpha)_{n\geq 1}$ is uniformly integrable, which in turn implies that the family $(|X_n^{(1)} + X_n^{(3)}|^\alpha)_{n\geq 1}$ is uniformly integrable.

Assuming w.l.o.g. that 1 is a continuity point of $\nu_0$, we have that $X_n^{(1)}$ is weakly convergent and therefore tight. This implies that there exists $r > 0$ such that $\mathbb{P}(|X_n^{(1)}| \leq r) \geq 1/2$ for all $n$, which implies that for all $n$ and for all $t$ so large that $(t^{1/\alpha} - r)^\alpha > t/2$, we have:

$$\begin{aligned}(1/2)\mathbb{P}\big((X_n^{(3)})^\alpha > t\big) &\leq \mathbb{P}\big(|X_n^{(1)}| \leq r\big)\mathbb{P}\big(X_n^{(3)} > t^{1/\alpha}\big)\\ &= \mathbb{P}\big(|X_n^{(1)}| \leq r, X_n^{(3)} > t^{1/\alpha}\big) \leq \mathbb{P}\big(X_n^{(1)} + X_n^{(3)} > t^{1/\alpha} - r\big)\\ &\leq \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > (t^{1/\alpha} - r)^\alpha\big) \leq \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > t/2\big)\,.\end{aligned}$$

This implies that $((X_n^{(3)})^\alpha)$ is uniformly integrable, since $(|X_n^{(1)} + X_n^{(3)}|^\alpha)$ is so. Applying Proposition 12.8 yields

$$\lim_a \sup_n \int_a^\infty y^\alpha \nu_n(\mathrm{d}y) = 0\,. \tag{12.7}$$

Together with a similar relation for $\int_{-\infty}^{-a}$ this gives

$$\lim_{a\to\infty} \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0\,.$$

For the converse, we assume $\lim_a \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0$, and return to our decomposition (12.6). As before, we apply Lemma 12.6 to obtain that the family $(X_n^{(2)})$ is uniformly integrable. Furthermore, applying Proposition 12.8, we obtain that the families $(|X_n^{(1)}|^\alpha)$ and $(|X_n^{(3)}|^\alpha)$ are uniformly integrable, and since $|X_n|^\alpha \leq 3^\alpha(|X_n^{(1)}|^\alpha + |X_n^{(2)}|^\alpha + |X_n^{(3)}|^\alpha)$, the proof is complete. □

Next, we prove Theorem 12.2.
We consider a sequence of Lévy processes $\{X^n\}$ such that $X^n \xrightarrow{\mathscr{D}} X^0$ and use obvious notation like $\ell^{b,n}, \pi^{b,n}$ etc. Furthermore, we let $\tau^n(A)$ denote the first exit time of $X^n$ from $A$. Here $A$ will always be an interval.

We first show that weak convergence of $X^n$ implies weak convergence of the stationary distributions.

**Proposition 12.9.** $X^n \xrightarrow{\mathscr{D}} X^0 \Rightarrow \pi^{b,n} \xrightarrow{\mathscr{D}} \pi^{b,0}$.

*Proof.* According to Theorem 13.17 in [76] we may assume $\Delta_{n,t} = \sup_{v \le t} |X^n(v) - X^0(v)| \xrightarrow{\mathbb{P}} 0$. Then

$$\begin{aligned}
&\mathbb{P}\bigl(X^0_{\tau^0[y+\epsilon-b,y+\epsilon)} \ge y+\epsilon,\, \tau^0[y+\epsilon-b, y+\epsilon) \le t\bigr) \\
&\quad \le \mathbb{P}\bigl(X^n_{\tau^n[y-b,y)} \ge y,\, \tau^n[y-b,y) \le t\bigr) + \mathbb{P}(\Delta_{n,t} > \epsilon) \\
&\quad \le \mathbb{P}\bigl(X^n_{\tau^n[y-b,y)} \ge y\bigr) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,.
\end{aligned}$$

Letting first $n \to \infty$ gives

$$\liminf_{n\to\infty} \overline{\pi}^{b,n}(y) \ge \mathbb{P}\bigl(X^0_{\tau^0[y+\epsilon-b,y+\epsilon)} \ge y+\epsilon,\, \tau^0[y+\epsilon-b,y+\epsilon) \le t\bigr)\,,$$

and letting next $t \to \infty$, we obtain

$$\liminf_{n\to\infty} \overline{\pi}^{b,n}(y) \ge \overline{\pi}^{b,0}(y+\epsilon)\,. \tag{12.8}$$

Similarly,

$$\begin{aligned}
\mathbb{P}\bigl(X^n_{\tau^n[y-b,y)} \ge y,\, \tau^n[y-b,y) \le t\bigr) &\le \mathbb{P}\bigl(X^0_{\tau^0[y-\epsilon-b,y-\epsilon)} \ge y-\epsilon\bigr) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,, \\
\limsup_{n\to\infty} \mathbb{P}\bigl(X^n_{\tau^n[y-b,y)} \ge y,\, \tau^n[y-b,y) \le t\bigr) &\le \overline{\pi}^{b,0}(y-\epsilon)\,.
\end{aligned} \tag{12.9}$$

However,

$$\mathbb{P}\bigl(\tau^n[y-b,y) > t\bigr) \le \mathbb{P}\bigl(\tau^0[y-\epsilon-b, y+\epsilon) > t\bigr) + \mathbb{P}(\Delta_{n,t} > \epsilon)\,,$$

so that

$$\limsup_{n\to\infty} \mathbb{P}\bigl(\tau^n[y-b,y) > t\bigr) \le \mathbb{P}\bigl(\tau^0[y-\epsilon-b,y+\epsilon) > t\bigr)\,.$$

Since the r.h.s. can be chosen arbitrarily small, it follows by combining with (12.9) that

$$\limsup_{n\to\infty} \overline{\pi}^{b,n}(y) = \limsup_{n\to\infty} \mathbb{P}\bigl(X^n_{\tau^n[y-b,y)} \ge y\bigr) \le \overline{\pi}^{b,0}(y-\epsilon)\,.$$

Combining with (12.8) shows that $\overline{\pi}^{b,n}(y) \to \overline{\pi}^{b,0}(y)$ at each continuity point $y$ of $\overline{\pi}^{b,0}$, which implies convergence in distribution. □

The following elementary lemma gives two properties of the function $\varphi = \varphi_b$ from Theorem 1.1. The proof is omitted.

**Lemma 12.10.** *The function $\varphi_b(x,y)$ is continuous in the region $(x,y) \in [0,b] \times \mathbb{R}$ and satisfies $0 \le \varphi_b(x,y) \le 2y^2 \wedge 2b|y|$.*

We are now ready to prove Theorem 12.2.

*Proof.* Recall the definition (12.3) of the bounded measure $\tilde{\nu}$ and let $\tilde{\varphi}_b(x,y) = \varphi(x,y)(1+y^2)/y^2$ for $y \ne 0$, $\tilde{\varphi}_b(x,0) = 1$. Note that $\tilde{\varphi}_b(x,y)$ is continuous on $(0,b) \times \mathbb{R}$, but discontinuous at $y=0$ if $x=0$ or $x=b$. We also get

$$\int_{-\infty}^{\infty} \tilde{\varphi}_b(x,y)\tilde{\nu}_n(\mathrm{d}y) = \sigma_n^2 + \int_{-\infty}^{\infty} \varphi_b(x,y)\nu_n(\mathrm{d}y)\,,$$

so that

$$a_n = \sigma_n^2 + \int_0^b \pi^{b,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \varphi_b(x,y)\nu_n(\mathrm{d}y) = \int_0^b \pi^{b,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \tilde{\varphi}_b(x,y)\tilde{\nu}_n(\mathrm{d}y)\,.$$

Let $\tilde{\nu}_n^1, \tilde{\nu}_n^2$ denote the restrictions of $\tilde{\nu}_n$ to the sets $|y| \le a$, resp. $|y| > a$. Using $0 \le \varphi_b(x,y) \le 2b|y|$, and uniform integrability (Theorem 12.3) we can choose $a$ such that

$$0 \le \int_{[-a,a]^c} \tilde{\varphi}_b(x,y)\tilde{\nu}_n^2(\mathrm{d}y) < \epsilon$$

for all $x$ and $n$ (note that $\tilde{\nu}_n \le \nu_n$ on $\mathbb{R}\backslash\{0\}$). We may also further assume that $a$ and $-a$ are continuity points of $\nu_0$ which implies $\tilde{\nu}_n^1 \to \tilde{\nu}_0^1$ weakly. In particular,

$$\sup_n \tilde{\nu}_n^1([-a,a]) < \infty. \tag{12.10}$$

Define

$$f_n(x) = \int_{-a}^{a} \varphi_b(x,y)\nu_n(\mathrm{d}y) + \sigma_n^2 = \int_{-a}^{a} \tilde{\varphi}_b(x,y)\tilde{\nu}_n^1\mathrm{d}y$$

we wish to prove that $\int f_n \,\mathrm{d}\pi^{b,n} \to \int f_0 \,\mathrm{d}\pi^{b,0}$ which, by using the generalized continuous-mapping theorem (e.g. [127]), will follow if

$$\pi^{b,0}(F) = 0 \tag{12.11}$$

where

$$F = \left\{x \mid \exists (x_n)_{n\ge1} : x_n \to x,\ f_n(x_n) \nrightarrow f_0(x)\right\}. \tag{12.12}$$

The proof of this follows different routes depending on whether or not $\sigma_0^2$ is zero. First, we assume $\sigma_0^2 = 0$ and consider the functions

$$\begin{aligned} f_n^-(x) &= \sigma_n^2 + \int_{(-\infty,0]} \varphi_b(x,y)\nu_n^1(\mathrm{d}y) = \int_{(-\infty,0]} \tilde{\varphi}_b(x,y)\tilde{\nu}_n^1(\mathrm{d}y)\,,\\ f_n^+(x) &= \int_{(0,\infty)} \varphi_b(x,y)\nu_n^1(\mathrm{d}y) = \int_{(0,\infty)} \tilde{\varphi}_b(x,y)\tilde{\nu}_n^1(\mathrm{d}y)\,. \end{aligned}$$

It follows from the definition of $\tilde{\nu}_n^1$, that the assumption $\sigma_0^2 = 0$ implies that $\tilde{\nu}_n^1$ has no mass at 0, and since this is the only possible discontinuity point of the integrands, we have $f_n^-(x) \to f_0^-(x)$ and $f_n^+(x) \to f_0^+(x)$ for $x \in [0,b]$. Furthermore, it can be checked that $x \mapsto f_n^-(x)$ is increasing, $x \mapsto f_n^+(x)$ is decreasing, and, using the bound $\varphi_b(x,y) \le 2y^2$, that both functions are uniformly bounded. That is, the functions $f_n^-$ and $f_n^+$ form two uniformly bounded sequences of continuous, monotone functions which converge to a continuous limit and as such, they converge uniformly. From this we get

$$\begin{aligned} \sup_{0\le y\le b} |f_n(y) - f_0(y)| &= \sup_{0\le y\le b} |f_n^-(y) - f_0^-(y) + f_n^+(y) - f_0^+(y)|\\ &\le \sup_{0\le y\le b} |f_n^-(y) - f_0^-(y)| + \sup_{0\le y\le b} |f_n^+(y) - f_0^+(y)| \to 0\,. \end{aligned}$$

Using the calculation above, we see that if we consider any $x \in [0, b]$ and sequence $(x_n)_{n\geq 1}$ converging to $x$, we have

$$\begin{aligned}|f_n(x_n) - f_0(x)| &\leq |f_n(x_n) - f_0(x_n)| + |f_0(x_n) - f_0(x)| \\ &\leq \sup_{0\leq y\leq b} |f_n(y) - f_0(y)| + |f_0(x_n) - f_0(x)| \to 0\,,\end{aligned} \tag{12.13}$$

where we use continuity of $f_0$ in the last part of the statement. This gives us that $F$ in (12.12) is the empty set, and hence we obtain (12.11) in the case $\sigma_0^2 = 0$.

Next, we consider the case where $\sigma_0^2 > 0$. We note that $\sigma_0^2 > 0$ implies that $\{X^0\}$ is a process of unbounded variation and using Theorem 6.5 in [94], this implies that 0 is regular for $(0, \infty)$. By comparing this to the representation (1.6) of the stationary distribution, we see that this implies $\pi^{b,0}(\{0, b\}) = 0$. Consider $x \in (0, b)$ and a sequence $(x_n)_{n\geq 1}$ converging to $x$. Assume w.l.o.g. that $x_n \in [\epsilon, b-\epsilon]$ for some $\epsilon > 0$. Since $\tilde{\varphi}_b(x, y)$ is continuous on the compact set $[\epsilon, b-\epsilon] \times [-a, a]$, we can use (12.11) to see that given $\epsilon_1$, there exists $\epsilon_2$ such that $|f_n(x') - f_n(x'')| < \epsilon_1$ for all $n$ whenever $|x' - x''| < \epsilon_2$ and $x', x'' \in [\epsilon, b - \epsilon]$. Since $x_n \to x$ this means, that given any $\epsilon_1 > 0$, we may use an inequality similar to (12.13) to conclude that for $n$ large enough

$$|f_n(x_n) - f_0(x)| \leq \epsilon_1 + |f_n(x) - f_0(x)|$$

and by taking $\limsup_n$ we see that the convergence $f_n(x_n) \to f_0(x)$ holds when $x \in (0, b)$, and can only fail $x = 0$ or $x = b$. Using that $\{0, b\}$ has $\pi^{b,0}$-measure 0, we have $\int f_n \,\mathrm{d}\pi^{b,n} \to \int f_0 \,\mathrm{d}\pi^{b,0}$ in this case as well. By combining this with the uniform integrability estimate above, we get that for any $\epsilon > 0$: $|a_n - a_0| \leq |f_n - f_0| + 2\epsilon$. (note that $f_i$ depends on $\epsilon$) and hence $\limsup_n |a_n - a_0| \leq 2\epsilon$, which implies $a_n \to a_0$.

By uniform integrability $\mathbb{E}X^n(1) \to \mathbb{E}X^0(1)$, and further $\pi^{b,n} \xrightarrow{\mathscr{D}} \pi^{b,0}$ implies $\int_0^b \overline{\pi}^{b,n}(y)\mathrm{d}y \to \int_0^b \overline{\pi}^{b,0}(y)\mathrm{d}y$. Remembering $a_n \to a_0$ and inspecting the expression (6.6) for the loss rate shows that indeed $\ell^{b,n} \to \ell^{b,0}$. □

## Proof of Theorem 12.1

First we note the effect that scaling and time-changing a Lévy process has on the loss rate:

**Proposition 12.11.** *Let $\beta, \delta > 0$ and define $X^{\beta,\delta}(t) = X(\delta t)/\beta$. Then the loss rate $\ell^{b/\beta}(X^{\beta,\delta})$ for $X^{\beta,\delta}$ equals $\delta/\beta$ times the loss rate $\ell^b(X) = \ell^b$ for $X$.*

*Proof.* It is clear that scaling by $\beta$ results in the same scaling of the loss rate. For the effect of $\delta$, note that the loss rate is the expected local time in stationarity per unit time and that one unit of time for $X^{\beta,\delta}$ corresponds to $\delta$ units of time for $X$. □

*Proof of Theorem 12.1 a).* Define $X^b(t) = X(tb^2)/b$. Then by Proposition 12.11 we have

$$b\ell^b(X) = \ell^1(X^b)$$

By the central limit theorem we have $X^b(1) \xrightarrow{\mathscr{D}} N(0, \psi^2)$ as $b \to \infty$. By Proposition 12.4, this is equivalent to $X^b \xrightarrow{\mathscr{D}} \psi B$ where $B$ is standard Brownian motion. We may apply Theorem 12.2, since

$$\mathbb{E}\big[(X^b(1))^2\big] = \mathbb{V}ar(X^1(1))\,,$$

that is, $\{X^b(1)\}_{b=1}^{\infty}$ is bounded in $L^2$ and therefore uniformly integrable. Thus

$$\lim_b b\ell^b(X) = \lim_b \ell^1\big(X^b\big) = \ell^1(\psi B) = \psi^2/2\,,$$

where the last equality follows directly from the expression for the loss rate in Theorem 1.1. □

*Proof of Theorem 12.1 b).* First we note that the stated conditions implies that the tails of $\nu$ are regularly varying, and therefore they are subexponential. Then by [52] we have that the tails of $\mathbb{P}(X(1) < x)$ are equivalent to those of $\nu$ and hence we may write

$$\mathbb{P}(X(1) > x) = x^{-\alpha} L_1(x) g_1(x)\,, \quad \mathbb{P}(X(1) < -x) = x^{-\alpha} L_2(x) g_2(x)$$

where $\lim_{x\to\infty} g_i(x) = 1$, $i = 1, 2$. The next step is to show that the fact that the tails of the distribution function are regularly varying allows us to apply the stable central limit theorem. Specifically, we show that the assumptions of Theorem 1.8.1 in [118] are fulfilled.

We notice that if we define $M(x) = L_1(x)g_1(x) + L_2(x)g_2(x)$ then $M(x)$ is slowly varying and

$$x^{\alpha}\big[\mathbb{P}(X(1) < -x) + \mathbb{P}(X(1) > x)\big] = M(x)\,. \tag{12.14}$$

Furthermore:

$$\frac{P(X(1) > x)}{\mathbb{P}(X(1) < -x) + \mathbb{P}(X(1) > x)} = L_1(x)g_1(x)/M(x) \sim L_1(x)/L(x) \to \frac{\beta+1}{2}\,, \tag{12.15}$$

as $x \to \infty$ since $L(x) \sim M(x)$. Define $L_0(x) = L(x)^{(-1/\alpha)}$ and let $L_0^{\#}(x)$ denote the de Bruin conjugate of $L_0$ (cf. [31] p. 29) and set $f(n) = n^{(1/\alpha)} L_0^{\#}(n^{(1/\alpha)})$. Let $f^{\leftarrow}$ be the generalized inverse of $f$. By asymptotic inversion of regularly varying functions ([31], p. 28-29) we have $f^{\leftarrow}(n) \sim (nL_0(n))^{\alpha}$ which implies

$$\frac{f^{\leftarrow}(n)L(n)}{n^{\alpha}} \sim \frac{(nL_0(n))^{\alpha}L(n)}{n^{\alpha}} = 1$$

and since $f^{\leftarrow}(f(n)) \sim n$ we have

$$\frac{nM(f(n))}{f(n)^{\alpha}} \sim \frac{nL(f(n))}{f(n)^{\alpha}} \sim \frac{f^{\leftarrow}(f(n))L(f(n))}{f(n)^{\alpha}} \to 1 \tag{12.16}$$

and therefore, if we define $\sigma = (\Gamma(1-\alpha)\cos(\alpha\pi/2))^{1/\alpha}$ we have

$$\frac{nM(\sigma^{-1}f(n))}{(\sigma^{-1}f(n))^{\alpha}} \sim \frac{nM(f(n))}{(\sigma^{-1}f(n))^{\alpha}} \to \sigma^{\alpha} \tag{12.17}$$

using slow variation of $M$. By combining (12.14), (12.15) and (12.17) we may apply the stable CLT Theorem 1.8.1 [118] [3] to obtain $X^b/f(b) \xrightarrow{\mathscr{D}} Z$ where $Z$ is a r.v. with characteristic function $\psi$, where

$$\psi(u) = \exp(-|\sigma u|^{\alpha}(1 - \mathrm{i}\beta\,\mathrm{sgn}(u)\tan(\alpha\pi/2))\,. \qquad u \in \mathbb{R}$$

[3] Note that the constants there should be replaced by their inverses.

Recalling that $\kappa$ is the characteristic exponent of $X$, this is equivalent to

$$\mathrm{e}^{b\kappa(\mathrm{i}u/f(b))} \to \psi(u)$$

and therefore

$$\mathrm{e}^{(bL_0(b))^\alpha \kappa(\mathrm{i}u/f(f^{\leftarrow}(b)))} \sim \mathrm{e}^{f^{\leftarrow}(b)\kappa(\mathrm{i}u/f(f^{\leftarrow}(b)))} \to \psi(u)$$

that is, for $\tilde{X}^b(t) = X(t(bL_0(b))^\alpha)/f(f^{\leftarrow}(b))$ we have $\tilde{X}^b(1) \xrightarrow{\mathscr{D}} Z$, and using $f(f^{\leftarrow}(b)) \sim b$ as well as the definition of $L_0(b)$, we see that the same applies to $X^b(1) = X(t(b^\alpha/L(b)))/b$. Setting $d = (\beta+1)/2$ and $c = (1-\beta)/2$ we calculate (cf. [118])

$$\begin{aligned} &-|\sigma t|^\alpha(1 - \mathrm{i}\beta\,\mathrm{sgn}(t)\tan(\alpha\pi/2) = -|\sigma t|^\alpha(1 + \mathrm{i}(d-c)\,\mathrm{sgn}(t)\tan(\alpha\pi/2) \\ &\quad d\alpha \int_{-\infty}^{0} (\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)(-t)^{-\alpha-1}\,\mathrm{d}t + c\alpha \int_0^\infty (\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)t^{-\alpha-1}\,\mathrm{d}t\,. \end{aligned}$$

That is, the characteristic triplet of $Z$ is $(\tau, 0, \nu)$, where

$$\nu(\mathrm{d}u) = \begin{cases} \dfrac{\alpha c}{(-u)^{\alpha+1}}\,\mathrm{d}u & u < 0 \\ \dfrac{\alpha d}{u^{\alpha+1}}\,\mathrm{d}u & u > 0 \end{cases} \tag{12.18}$$

and $\tau$ is a centering constant. We wish to use Theorem 12.2 and have to prove uniform integrability. Note that by combining Proposition 11.10 and Corollary 8.3 in [119], we have that the Lévy measure of $X^b$ is $\nu_b$, where

$$\nu_b(B) = b^\alpha L(b)^{-1}\nu(\{x : b^{-1}x \in B\})\,.$$

Using the assumptions in (12.2), this implies

$$\overline{\nu}_b(a) = b^\alpha L(b)^{-1}\overline{\nu}(ab) = L(b)^{-1}a^{-\alpha}L_1(ab)\,, \quad \nu_b(-a) = L(b)^{-1}a^{-\alpha}L_2(ab)\,.$$

Using partial integration and the remarks above, we find:

$$\begin{aligned} \int_{[-a,a]^c} |y|\nu_b(\mathrm{d}y) &= a\overline{\nu}_b(a) + \int_a^\infty \overline{\nu}_b(t)\mathrm{d}t + a\nu_b(-a) + \int_{-\infty}^{-a} \nu_b(t)\mathrm{d}t \\ &= a^{-\alpha+1}L(b)^{-1}\alpha L(ab) + \int_a^\infty t^{-\alpha}L(b)^{-1}L(tb)\mathrm{d}t\,. \end{aligned}$$

Furthermore, using Potter's Theorem (Theorem 1.5.6 in [31]) we have that for $\delta > 0$ such that $1 + \delta < \alpha$ there exists $\xi > 0$ such that

$$\frac{L(ab)}{L(b)} \le 2\max(a^\delta, a^{-\delta}), \qquad ab > \xi, b > \xi\,.$$

Using this, we get that

$$\lim_a \sup_{b>\xi} a^{-\alpha+1}\frac{L(ab)}{L(b)} \le 2\lim_a a^{-\alpha+1}\max(a^\delta, a^{-\delta}) = 0 \tag{12.19}$$

and similarly for the integral:

$$\sup_{b>\xi} \lim_{a} \int_{a}^{\infty} t^{-\alpha} \frac{L(tb)}{L(b)} \mathrm{d}t \le 2 \lim_{a} \int_{a}^{\infty} t^{-\alpha} \max(t^{\delta}, t^{-\delta}) \mathrm{d}t = 0\,. \tag{12.20}$$

By combining (12.19) and (12.20) we get

$$\lim_{a\to\infty} \sup_{b>\xi} \int_{[-a,a]^c} |y| \nu_b(\mathrm{d}y) = 0\,.$$

By Proposition 12.11 we have $b^{\alpha-1}L(b)^{-1}\ell^b(X) = \ell^1\big(X^b\big)$, and since we have proved uniform integrability, we may apply Theorem 12.2. Letting $b \to \infty$ and using Example 8.8 which states that the loss rate for our stable distribution is $\gamma$ (see also [93]), yields the desired result. □

# 13 The overflow time

We define the overflow time as

$$\omega(b,x) = \inf\big\{t > 0 : \, V^b(t) = b \,\big|\, V^b(0) = x\big\}\,, \qquad 0 \le x < b.$$

It can also be interpreted in terms of the one-sided reflected process as

$$\omega(b,x) = \inf\big\{t > 0 : \, V^\infty(t) \ge b \,\big|\, V^\infty(0) = x\big\}\,, \qquad 0 \le x \le b\,.$$

It has received considerable attention in the applied literature. We consider here evaulation of characteristics of $\omega(b,x)$, in particular expected values and distributions, both exact and asymptotically as $b \to \infty$. When no ambiguity exists, we write $\omega$ instead of $\omega(b,x)$.

As may be guessed, the Brownian case is by far the easiest:

**Example 13.1.** Let $X$ be $\mathrm{BM}(\mu, \sigma^2)$ with $\mu \ne 0$ [the case $\mu = 0$ requires a separate treatment which we omit]. Consider the Kella-Whitt martingale with $B(t) = x + L(t) - qt/\alpha$ where $L$ is the local time at 0 for the one-sided reflected process,

$$\kappa(\alpha) \int_0^t \mathrm{e}^{\alpha V^\infty(s) - qs}\,\mathrm{d}s + \mathrm{e}^{\alpha x} - \mathrm{e}^{\alpha V^\infty(t) - qt} + \alpha \int_0^t \mathrm{e}^{-qs}\,\mathrm{d}L(s) - q \int_0^t \mathrm{e}^{\alpha V^\infty(s) - qs}\,\mathrm{d}s$$

where we used that $L$ can only increase when $V^\infty$ is at 0 and so

$$\int_0^t \mathrm{e}^{\alpha V^\infty(s) - qs}\,\mathrm{d}L(s) = \int_0^t \mathrm{e}^{-qs}\,\mathrm{d}L(s)\,.$$

Take first $\gamma = -2\mu/\sigma$ as the root of $0 = \kappa(s) = s\mu + s^2\sigma^2/2$ and $q = 0$. Optional stopping at $\omega$ then gives $0 = \mathrm{e}^{\gamma x} - \mathrm{e}^{\gamma b} + \gamma \mathbb{E} L(\omega)$. Using $V^\infty = x + B + L$ and $\mathbb{E}B(\omega) = \mu \mathbb{E}\omega$ then gives

$$\mathbb{E}\omega(b,x) = \frac{b - x - (\mathrm{e}^{\gamma b} - \mathrm{e}^{\gamma x})/\gamma}{\mu} \tag{13.1}$$

Take next $q > 0$ and $\theta^{\pm}$ as the two roots of $\kappa(\alpha) = q$, cf. Example 7.3. We then get

$$0 = e^{\theta^+ x} - e^{\theta^+ x}\mathbb{E}e^{-q\omega} - q\,\mathbb{E}\int_0^{\omega} e^{\alpha V^{\infty}(s) - qs}\,ds\,.$$

Together with the similar equation with $\theta^-$ this can then be solved to obtain $\mathbb{E}e^{-q\omega}$ (the other unknown is $\mathbb{E}\int_0^{\omega} e^{\alpha V^{\infty}(s)-qs}\,ds$).

Early calculations of these and some related quantities are in Glynn & Iglehart [62] who also discuss the probabilistically obvious fact that $\omega(b, 0)$ is exponentially distributed in the Brownian case (as in the spectrally positive Lévy case), cf. Athreya & Werasinghee [22]. □

## 13.1 Exact results in the PH model

Recall from Section 3 that the process $V^{\infty}$ with one-sided reflection at 0 can be constructed as $V(t) = V(0) + X(t) + L(t)$, where

$$L(t) = -\min_{0 \le s \le t} (V(0) + X(s))$$

is the local time. For our phase-type model with a Brownian component, $L(t)$ decomposes as $L^c(t) + L^d(t)$, where $L_c$ is the continuous part (the contribution to $L$ from the segments between jumps where $V$ behaves as a reflected Brownian motion) and $L^d(t)$ the compensation of jumps of $X$ that would have taken $V$ below 0.

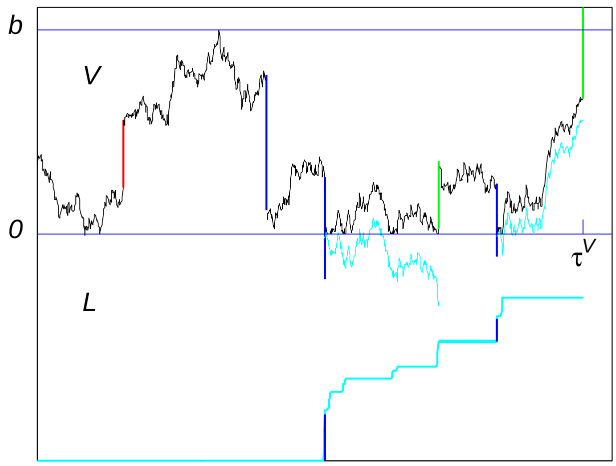


**Figure 7:** One-sided reflected process $V = V^{\infty}$ and local time $L$

The situation is illustrated on Fig. 7. We have again phases `red, green` for $F^+$ and `blue` for $F^-$. The `cyan` Brownian segments are how Brownian motion would have evolved without reflection. In the lower panel, the `cyan` segments of $L$ correspond to compensation when the Brownian motion would otherwise have taken $V^{\infty}$ below 0, and the `blue` jumps are the compensation from jumps of $X$ that would otherwise have taken $V^{\infty}$ below 0.

To compute the Laplace transform of $\omega$, we use the Kella-Whitt martingale with $B(t) = V(0) + L(t) - qt/\alpha$. Thus $\alpha Z(t) = \alpha V(t) - qt$, and the martingale takes the form

$$
\begin{aligned}
\kappa(\alpha)&\int_0^t \mathrm{e}^{\alpha V(s)-qs}\,\mathrm{d}s + \mathrm{e}^{\alpha V(0)} - \mathrm{e}^{\alpha V(t)-qt}\\
&\quad + \alpha\int_0^t \mathrm{e}^{\alpha V(s)-qs}\big(\mathrm{d}L^c(s) - q\,\mathrm{d}s/\alpha\big) + \sum_{0\le s\le t}\mathrm{e}^{\alpha V(s)-qs}(1-\mathrm{e}^{-\alpha\Delta L^d(s)})\\
&= \big(\kappa(\alpha)-q\big)\int_0^t \mathrm{e}^{\alpha V(s)-qs}\,\mathrm{d}s + \mathrm{e}^{\alpha V(0)} - \mathrm{e}^{\alpha V(t)-qt}\\
&\quad + \alpha\int_0^t \mathrm{e}^{-qs}\,\mathrm{d}L^c(s) + \sum_{0\le s\le t}\mathrm{e}^{-qs}(1-\mathrm{e}^{-\alpha\Delta L^d(s)})\,,
\end{aligned}
$$

where in the last step we used that $L$ can only increase when $V$ is at 0. Now introduce the following unknowns: $z_c^+$, the expectation of $\mathrm{e}^{-q\omega}$ evaluated on the event of continuous upcrossing of level $b$ only; $z_i^+$, the expectation of $\mathrm{e}^{-q\omega}$ evaluated on the event of upcrossing in phase $i=1,\dots,n^+$ only; $\ell_c = \mathbb{E}\int_0^\omega \mathrm{e}^{-qs}\,\mathrm{d}L^c(s)$; and $m_j$, the expected value of the sum of the $\mathrm{e}^{-qs}$ with $s\le\omega$ such that at time $s$ there is a downcrossing of level 0 in phase $j=1,\dots,n^-$. Optional stopping then gives

$$
\begin{aligned}
0 = \big(\kappa(\alpha)-q\big)&\mathbb{E}\int_0^\omega \mathrm{e}^{\alpha V(s)-qs}\,\mathrm{d}s + \mathrm{e}^{\alpha V(0)}\\
&- \mathrm{e}^{\alpha b}\Big(z_c^+ + \sum_{i=1}^{n^+}\widehat{F}_i^+[\alpha]z_i^+\Big) + \alpha\ell_c + \sum_{j=1}^{n^-} m_j(1-\widehat{F}_j^-[-\alpha])\,.
\end{aligned}
$$

Taking $\alpha$ as one of the same roots as in Section 9.4, we get

$$
0 = \mathrm{e}^{\rho_k^q V(0)} - \mathrm{e}^{\rho_k^q b}\Big(z_c^+ + \sum_{i=1}^{n^+}\widehat{F}_i^+[\rho_k^q]z_i^+\Big) + \rho_k^q\ell_c + \sum_{j=1}^{n^-} m_j(1-\widehat{F}_j^-[-\rho_k^q])\,,
$$

a set of linear equations from which the unknowns and hence $\mathbb{E}\mathrm{e}^{-q\omega} = z_c^+ + z_1^+ + \cdots + z_{n^+}^+$ can be computed.

## 13.2 Asymptotics via regeneration

The asymptotic study of $\omega(b,x)$ is basically a problem in extreme value theory since

$$
\mathbb{P}(\omega(b,x)\le t) = \mathbb{P}_x\big(\max_{0\le s\le t} V^\infty(s)\ge b\big)\,. \tag{13.2}
$$

This is fairly easy if $m=\mathbb{E}X(1)>0$ since then the max in (13.2) is of the same order as $X(t)$ which is in turn of order $mt$. Hence we assume $m<0$ in the following.

For processes with dependent increments such as $V^\infty$, the asymptotic study of the quantitites in (13.2) is most often (with Gaussian processes as one of the exceptions) done via regeneration, cf. [11, VI.4]

For $V^\infty$, we define (inspired by the discussion in Section 5.1) a cycle by starting at level 0, waiting until level 1 (say) has been passed and taking the cycle termination time $T$ as the next hitting time of 0 ('up to 1 from 0 and down again'). That is,

$$
T = \inf\big\{t > \inf\{s>0:\, V^\infty(s)\ge 1\}:\, V^\infty(t)=0 \,\big|\, V^\infty(0)=0\big\}\,.
$$

The key feature of the regenerative setting is that the asymptotic discussion can be reduced to the study of the behavior within a regenerative cycle. The quantities needed are

$$m_T = \mathbb{E}_0 T\,, \quad a(z) = \mathbb{P}_0\bigl(\max_{0\le s\le T} V^\infty(s) \ge z\bigr)\,.$$

Indeed one has by [11, VI.4] that:

**Theorem 13.2.** *As $b \to \infty$, it holds for any fixed $x$ that $a(b)\mathbb{E}\omega(b,x) \to m_T$ and that $a(b)\omega(b,x)/m_T$ has a limiting standard exponential distribution.*

For the more detailed implementation, we note:

**Proposition 13.3.** *(a) Assume that the Lévy measure $\nu$ is heavy-tailed, more precisely that $\overline{\nu}(z) = \int_z^\infty \nu(\mathrm{d}y)$ is a subexponential tail. Then $a(z) \sim m_T\overline{\nu}(x)$ as $z \to \infty$;*

*(b) Assume that the Lévy measure $\nu$ is light-tailed, more precisely that the Lundberg equation $\kappa(\gamma) = 0$ has a solution $\gamma > 0$ with $\kappa'(\gamma) < \infty$. Then $a(z) \sim C_T \mathrm{e}^{-\gamma z}$ for some constant $C_T$ as $z \to \infty$*

*Sketch of proof.* For (a), involve 'the principle of one big jump' saying that exceedance of $z$ occurs as a single jump of order $z$ (which occur at rate $\overline{\nu}(z)$. The rigorous proof, using the regenerative representation $\overline{\pi}^\infty(x) = a(x)/m_T$ of the stationary distribution of $V^\infty$ and known results on $\pi^\infty$, can be found in [10], [17].

For (b), let $\mathbb{P}_\gamma$, $\mathbb{E}_\gamma$ refer to the exponentially tilted case $\kappa_\gamma(\alpha) = \kappa(\alpha+\gamma) - \kappa(\alpha)$ with $V^\infty(0) = 0$. By standard likelihood ratio identities,

$$a(z) = \mathbb{P}_0\bigl(\omega(z,0) < T\bigr) = \mathbb{E}_\gamma\bigl[\exp\bigl\{-\gamma X\bigl(\omega(z,0)\bigr)\bigr\};\, \omega(z,0) < T\bigr] \tag{13.3}$$

Now $m_\gamma = \kappa'(\gamma) > 0$ so that $\mathbb{P}_\gamma\bigl(V^\infty(t) \to \infty\bigr) = 1$. Hence $\bigl\{\omega(z,0) < T\bigr\} \uparrow \{T = \infty\}$ where $\mathbb{P}_\gamma(T = \infty) > 0$, and

$$X\bigl(\omega(z,0)\bigr) + L\bigl(\omega(z,0)\bigr) = V^\infty\bigl(\omega(z,0)\bigr) = z + \xi(z)$$

where $\xi(z)$, the overshoot, converges in $\mathbb{P}_\gamma$-distribution to a limit $\xi(\infty)$ (in fact, the same as when overshoot distribution are taken w.r.t. $X$, not $V^\infty$), and $L\bigl(\omega(z,0)\bigr)$ converges in $\mathbb{P}_\gamma$-distribution to the finite r.v. $L(\infty)$. Combining with (13.3) and suitable independence estimates along the lines of Stam's lemma ([11, pp. 368–369]), the result follows with

$$C_T = \mathbb{E}_\gamma \mathrm{e}^{-\gamma\xi(\infty)} \cdot \mathbb{E}_\gamma\bigl[\mathrm{e}^{\gamma L(\infty)};\, T = \infty\bigr]\,.$$

That $C_T < \infty$ is seen by a comparison with Theorem 2.1 since clearly

$$a(z) \le \mathbb{P}_0\bigl(\max_{0\le s\le T} V^\infty(s) \ge z\bigr) = \overline{\pi}^\infty(z)\,.$$

□

In the heavy-tailed case, Theorem 13.2 and Proposition 13.3 determine the order of $\omega(b,x)$ as $\overline{\nu}(b)$. In the light-tailed case, we are left with the computation of the constant $C_T$. In general, one can hardly hope for an explicit expression beyond special cases. Note, however, that for the spectrally negative case one can find the Laplace transform $\mathbb{E}_x \mathrm{e}^{-q\omega(z,x)}$ as $Z^{(q)}(x)/Z^{(q)}(b)$, where

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y$$

is the 'second scale function'. See Pistorius [109] and Kyprianou [94, p. 228], with extensions in Ivanovs & Palmowski [73]. We return to $\mathbb{E}\omega$ in Section 14.2.

# 14 Studying $V$ as a Markov process

## 14.1 Preliminaries

An alternative approach to computing probabilities and expectations associated with $V$ and its loss process $U$ is to take advantage of the fact that $V$ is a Markov jump-diffusion process. As a consequence, a great number of probabilities / expectations can be computed by solving linear integro-differential equations, subject to suitable boundary conditions related to the boundary behaviour of $V$ and the specific functional under consideration.

Our exposition is somewhat simpler if we require that the jump component of $X$ be of bounded variation (BV). So, we will henceforth assume that

$$\int_{|y|\le 1} |y|\,\nu(\mathrm{d}y) < \infty\,. \tag{14.1}$$

In this setting,

$$X(t) - X(0) = \mu t + \sigma\,B(t) + \sum_{0<s\le t} \Delta X(s)\,,$$

where $\Delta X(s) = X(s) - X(s-)$ and the sum converges absolutely for each $t < \infty$ because of (14.1). Cf. the discussion at the end of Section 1. Without (14.1), the jump part would a.s. have unbounded variation. In order to deal with Lévy processes having non-BV jumps, one needs to modify the equations and arguments of this section slightly. We discuss this non-BV extension briefly at the end in Section 14.6.

The key to establishing suitable integro-differential equations in this context is the systematic use of Itô's formula in the form

$$\begin{aligned} f\big(V(s)\big) - f\big(V(s)\big) &- \sum_{0<s\le t}\big[f\big(V(s)\big) - f\big(V(s-)\big)\big] \\ &= \int_0^t \Big[\mu f'\big(V(s)\big) + \frac{\sigma^2}{2} f''\big(V(s)\big)\Big]\,\mathrm{d}s + \int_0^t f'\big(V(s)\big)\,\mathrm{d}B(s) \\ &\quad + \int_0^t f'\big(V(s)\big)\,\mathrm{d}L_c(s) - \int_0^t f'\big(V(s)\big)\,\mathrm{d}U_c(s) \end{aligned} \tag{14.2}$$

[Note that in our context, $V(s)$ could be replaced by $b$ in the last integral and by 0 i the next-to-last.] This follows for compound Poisson jumps by using Itô's formula in a form involving boundary modifications (see [43]) on intervals between jumps of $X$ (where $V$ is continuous), and in the general case by approximation by such a process. Equation (14.1) is the basic form of Itô's formula that we will systematically apply in what follows.

With the aid of Itô's formula, we will illustrate the use of Markov process arguments in deriving various integro-differential equations associated with $V$ and its loss process $U$. In fact, we will generalize from consideration of $U$ to additive functionals ([33]) of the form

$$\Lambda(t) = \int_0^t f\big(V(s)\big)\,\mathrm{d}s + \sum_{0<s\le t} \widetilde{f}\big(V(s-), \Delta X(s)\big) + r_1 L_c(t) + r_2 U_c(t)\,. \tag{14.3}$$

Note that we recover $U$ if we set $f \equiv 0$, $\widetilde{f}(x,y) = [x+y-b]^+$, $r_1 = 0$ and $r_2 = 1$. We assume throughout that $f$ is bounded, that $\widetilde{f}(x,0) = 0$, and that

$$\sup_{0\le x\le b} \int |\widetilde{f}(x,y)|\,\nu(\mathrm{d}y) < \infty\,.$$

An additional notational simplification will be useful: we set

$$r(x,y) = \begin{cases} 0 & x+y \le 0 \\ x+y & 0 \le x+y \le b \\ b & x+y \ge b \end{cases}$$

and observe that $V(s) = r\big(V(s-), \Delta X(s)\big)$ whenever $\Delta X(s) \neq 0$.

The integro-differential equations to follow are typically expressed in terms of the operator $\mathcal{L}$ defined on twice differentiable functions $\varphi :\ [0,b] \to \mathbb{R}$ and given by

$$(\mathcal{L}\varphi)(x) = \mu\varphi'(x) + \frac{\sigma^2}{2}\varphi''(x) + \int_{\mathbb{R}} \big[\varphi\big(r(x,y)\big) - \varphi(x)\big]\,\nu(\mathrm{d}y)\,,$$

The expression on the r.h.s. is familiar from the theory of generators of Markov processes, but given the multitude of formulations of this theory, we will not pursue this aspect. The generator view, though, may sometimes be helpful to heuristically understand the form of the results. For example, we will study the expectation $s(x) = \mathbb{E}_x T_z$ where $T_z = \inf\{t > 0 :\ V(t) \ge z\}$ is a level crossing time of $V$ (and as usual $\mathbb{P}_x, \mathbb{E}_x$ refer to the case $V(0) = x$). Intuitively, one should have

$$s(x) \approx h + \mathbb{E}_x s\big(x + V(h)\big) \approx h + s(x) + h\,(\mathcal{L}s)(x)$$

for small $h$, so that the equation to solve for computing $s$ should be $(\mathcal{L}s)(x) = -1$ (subject to suitable boundary conditions). Our detailed analysis aims at making this rigorous.

We will encounter functions $h(\theta, x)$ depending on two arguments and write then as usual $h_\theta(\theta,x)$, $h_x(\theta,x)$ for the partial derivatives. When working with $h$ as a function of $x$ for a fixed $\theta$, we write $h(\theta)$ rather than $h(\theta,\cdot)$.

## 14.2 Level crossing times

Consider the level crossing time

$$T_z = \inf\{t > 0 : V(t) \geq z\}$$

defined for $V$ (with $0 < z \leq b$) and

$$\tau(w) = \inf\{t > 0 : \Lambda(t) \geq w\}$$

defined (with $w > 0$) for the additive functional $\Lambda$ in (14.3). In this section, we formulate the integro-differential equations appropriate for computing characteristics of these quantities.

**Theorem 14.1.** *Fix $\theta \leq 0$. Suppose that there exists a function $h(\theta) = h(\theta, \cdot) : [0, b] \to \mathbb{R}$ that is twice continuously differentiable in $[0, z]$ satisfying the integro-differential equation*

$$\big(\mathcal{L}h(\theta)\big)(x) + \theta h(\theta, x) = 0 \tag{14.4}$$

*for $0 \leq x \leq z$, subject to the boundary conditions*

$$h(\theta, x) = 1 \quad \text{for } x \geq z, \quad h_x(\theta, 0) = 0\,.$$

*Then $h(\theta, x) = \mathbb{E}_x \mathrm{e}^{\theta T_z}$.*

*Proof.* Observe that $U_c(T_z) = 0$. Hence, Itô's formula yields

$$\mathbb{E}\mathrm{e}^{\theta(T_z \wedge t)} h\big(\theta, V(T_z \wedge t)\big) - h\big(\theta, V(0)\big) = \sum_{j=1}^{4} T_j,$$

where

$$\begin{aligned}
T_1 &= \int_0^{T_z \wedge t} \mathrm{e}^{bs} \big[\big(\mathcal{L}h(\theta)\big)\big(V(s)\big) + \theta h\big(\theta, V(s)\big)\big]\,\mathrm{d}s\ , \\
T_2 &= \int_0^{T_z \wedge t} \mathrm{e}^{bs} h_x\big(\theta, V(s)\big)\,\mathrm{d}s\,, \\
T_3 &= \int_0^{T_z \wedge t} \mathrm{e}^{bs} h_x\big(\theta, V(s)\big)\,\sigma\,\mathrm{d}B(s)\,\mathrm{d}s\,, \\
T_4 &= \sum_{0 < s \leq T_z \wedge t} \mathrm{e}^{bs} \big[h\big(\theta, V(s)\big) - h\big(\theta, V(s-)\big)\big]\,, \\
&\quad - \int_0^{T_z \wedge t} \mathrm{e}^{bs} \int_{\mathbb{R}} \big[h\big(\theta, r(V(s), y)\big) - h\big(\theta, V(s)\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s
\end{aligned}$$

Here $T_1 + T_2 = 0$ because of the integro-differential equation (14.4) and the boundary condition at $x = 0$, whereas $T_3 + T_4$ form a martingale. Hence

$$\mathbb{E}\mathrm{e}^{\theta(T_z \wedge t)} h\big(\theta, V(T_z \wedge t)\big) = h(\theta, x)$$

for $t \geq 0$. By monotone convergence and the boundary condition for $x \geq z$,

$$\mathbb{E}[\mathrm{e}^{\theta T_z} h(\theta, V(T_z));\, T_z \leq t] \uparrow \mathbb{E}[\mathrm{e}^{\theta T_z};\, T_z < \infty]$$

as $t \to \infty$. On the other hand, the boundedness of $h(\theta)$ and the fact that $\theta \le 0$ ensure that

$$\mathbb{E}\big[\mathrm{e}^{\theta t} h\big(\theta, V(t)\big);\, T_z > t\big] \to 0\,.$$

The conclusion follows by noting that $T_z < \infty$ a.s. because of the reflection at 0. □

We can now formally obtain our integro-differential equation for $\mathbb{E}_x T_z$ by differentiating (14.4) and the boundary conditions. In particular note that $\mathbb{E}_x T_z = h_\theta(0, x)$. Formally differentiating (14.4) w.r.t. $\theta$ yields

$$(\mathcal{L}h_\theta)(x) + h(\theta, x) + \theta h_\theta(\theta, x) = 0$$

for $0 \le x \le z$, subject to

$$h_\theta(\theta, x) = 0 \quad \text{for } x \ge z, \quad h_{x\theta}(\theta, 0) = 0\,.$$

Letting $\widetilde{h}(x) = \mathbb{E}T_z$, we conclude from $h(0, x) = 1$ for $0 \le x \le z$ that $\widetilde{h}$ should satisfy

$$(\mathcal{L}\widetilde{h})(x) = -1$$

for $0 \le x \le z$, subject to

$$\widetilde{h}(x) = 0 \quad \text{for } x \ge z, \quad \widetilde{h}'(0) = 0\,.$$

By working with the martingale

$$\widetilde{h}\big(V(T_z \wedge t)\big) + T_z \wedge t\,,$$

this can be rigorously verified by sending $t \to \infty$, using the boundedness of $\widetilde{h}$, and exploiting the fact that $\widetilde{h}\big(V(T_z)\big) = 0$.

Bounds on $\mathbb{E}_x T_z$ can be obtained similarly, in the presence of a non-negative twice continuousy differentiable function $\gamma$ for which

$$(\mathcal{L}\gamma)(x) \le -1$$

for $0 \le x \le z$. In this case,

$$\gamma\big(V(T_z \wedge t)\big) + T_z \wedge t$$

is a non-negative supermartingale. Because $\gamma$ is non-negative, $\mathbb{E}_x T_z \wedge t \le \gamma(x)$ for $t \ge 0$, yielding the bound

$$\mathbb{E}_x T_z \le \gamma(z)$$

for $0 \le x \le z$ upon application of the monotone convergence theorem.

We next turn to the computation of $\mathbb{E}_x \mathrm{e}^{\theta\tau(w)}$ with $\Lambda(t)$ as in (14.3) (note for the following result the quantities $f, \widetilde{f}, r_1, r_2$ occurring in the definition).

**Theorem 14.2.** *Fix $\theta \le 0$. Suppose that there exists a function $k(\theta) : [0, b] \times \mathbb{R} \to \mathbb{R}$ of $x, \lambda$ that is twice continuously differentiable in $x$ and continuously differentiable in $\lambda$ on $[0, b] \times (-\infty, w]$, and satisfies*

$$0 = \mu k_x(\theta, x, \lambda) + \frac{\sigma^2}{2} k_{xx}(\theta, x, \lambda) + k_\lambda(\theta, x, \lambda) f(x) + \theta k(\theta, x, \lambda) \tag{14.5}$$

$$+ \int_{\mathbb{R}} \big[k\big(\theta, r(x, y), \lambda + \widetilde{f}(x, y)\big) - k(\theta, x, \lambda)\big]\, \nu(\mathrm{d}y) \tag{14.6}$$

*for $0 \le x \le b$, $\lambda \le w$, subject to the boundary conditions*

$$r_1 k_\lambda(\theta, x, \lambda) + k_x(\theta, x, \lambda) = 0 \quad \textit{and} \quad r_2 k_\lambda(\theta, x, \lambda) - k_x(\theta, x, \lambda) = 0$$

*for $\lambda \le w$. If $\mathbb{E}_\pi \Lambda(1) \ge 0$, then $k(\theta, x, \lambda) = \mathbb{E}_x \mathrm{e}^{\theta\tau(w)}$.*

*Proof.* An application of Itô's formula guarantees that

$$\mathrm{e}^{\theta(\tau(w)\wedge t)} k\big(\theta, V(\tau(w)\wedge t)), \Lambda(\tau(w)\wedge t)\big) - k\big(\theta, V(0), 0\big) = \sum_{j=1}^{6} T_j$$

where

$$\begin{aligned}
T_1 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs} \int_{\mathbb{R}} [t]\big[k\big(\theta, r(V(s), y), \Lambda(s) + \widetilde{f}(V(s), y)\big) \\
&\qquad\qquad - k\big(\theta, V(s), \Lambda(s)\big)\big]\, \nu(\mathrm{d}y)\, \mathrm{d}s\,, \\
T_2 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs} [t]\Big[\mu k_x\big(\theta, V(s), \Lambda(s)\big)\frac{\sigma^2}{2} k_{xx}\big(\theta, V(s), \Lambda(s)\big) \\
&\qquad + k_\lambda\big(\theta, V(s), \Lambda(s)\big) f\big(V(s)\big) + \theta k_\lambda\big(\theta, V(s), \Lambda(s)\big)\Big]\, \mathrm{d}s\,, \\
T_3 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs}\big[k_x\big(\theta, V(s), \Lambda(s)\big) + r_1 k_\lambda\big(\theta, V(s), \Lambda(s)\big)\big]\, \mathrm{d}L_c(s)\,, \\
T_4 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs}\big[-k_x\big(\theta, V(s), \Lambda(s)\big) + r_2 k_\lambda\big(\theta, V(s), \Lambda(s)\big)\big]\, \mathrm{d}U_c(s)\,, \\
T_5 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs} k_x\big(\theta, V(s), \Lambda(s)\big)\, \sigma\, \mathrm{d}B(s)\,, \\
T_6 &= \sum_{0<s\le t} \mathrm{e}^{bs}[t]\big[k\big(\theta, r(V(s-), \Delta X(s)), \Lambda(s-) + \widetilde{f}(V(s-), \Delta X(s))\big) \\
&\qquad - k\big(\theta, V(s), \Lambda(s-)\big)\big] \\
&\quad - \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs} \int_{\mathbb{R}} [t]\big[k\big(\theta, r(V(s-), y), \Lambda(s-) + \widetilde{f}(V(s-), y)\big) \\
&\qquad\qquad - k\big(\theta, V(s-), \Lambda(s-)\big)\big]\, \mathrm{d}s\,.
\end{aligned}$$

Here $T_1 + T_2 = 0$ because of the integro-differential equation (14.6), $T_3 = T_4 = 0$ because of the boundary conditions, and $T_5, T_6$ are martingales. Consequently,

$$k\big(\theta, V(0), 0\big) = \mathbb{E}_x\big[\mathrm{e}^{\theta\tau(w)} k\big(\theta, V(\tau(w)\wedge t)), \Lambda(\tau(w)\wedge t)\big)\big]$$

Now $\tau(w) < \infty$ a.s. because $\mathbb{E}_\pi \Lambda(1) < \infty$. Because $\theta \le 0$ and $k$ is bounded, the r.h.s. converges to

$$\mathbb{E}_x\big[\mathrm{e}^{\theta\tau(w)} k\big(\theta, V(\tau(w))), \Lambda(\tau(w))\big)\,.$$

The proof is completed upon recognizing that $k(\theta, x, \lambda) = 1$ for $\lambda \ge w$. □

## 14.3 Poisson's equation and the CLT

A natural complement to the computation of the loss rate $\ell^b$ is the development of a central limit theorem (CLT) for the cumulative loss. In particular, we wish to obtain a CLT of the form

$$\frac{U(t) - \ell^b t}{\sqrt{t}} \xrightarrow{\mathscr{D}} \eta N(0,1) \tag{14.7}$$

as $t \to \infty$. This CLT lends itself to the approximation

$$U(t) \overset{\mathcal{D}}{\approx} \ell^b t + \eta\sqrt{t}\, N(0,1) \tag{14.8}$$

when $t$ is large, where $\overset{\mathcal{D}}{\approx}$ means 'has approximately the same distribution as' (and carries no rigorous meaning, other than through (14.7)). The key new parameter to be computed in the approximation (14.8) is the *time-average variance constant* $\eta^2$. Computing $\eta^2$, in turn, involves representing $U(t)$ in terms of the solution to *Poisson's equation* which is well-known to play a fundamental role for Markov process CLTs (cf. e.g. Bhattacharyya [30], Glynn [61], Glynn & Meyn [63], [11, I.7, II.4d]). See also Williams [128] for the CLT for $U$ in the Brownian case.

We develop the theory in terms of a general additive functional $V$ of the form (14.3) and its associated boundary processes $L$ and $U$. Given a function $g:\ [0,b] \to \mathbb{R}$ and a scalar $c$, we say that the pair $(g,c)$ is a solution to Poisson's equation for the additive functional $\Lambda$ if

$$g\big(V(t)\big) + \Lambda(t) - ct$$

is a martingale. When $\Lambda = U$, $c$ must clearly equal $\ell^b$.

**Theorem 14.3.** *Assume that $f$ is bounded and that*

$$\sup_{0\le x\le b} \int_{\mathbb{R}} \big(|\widetilde{f}(x,y)| + \widetilde{f}(x,y)^2\big)\,\nu(\mathrm{d}y) < \infty\,. \tag{14.9}$$

*If there exists a twice continuously differentiable function $g:\ [0,b] \to \mathbb{R}$ satisfying*

$$\sup_{0\le x\le b} \int_{\mathbb{R}} \big|g\big(r(x,y)\big)\big|\,\nu(\mathrm{d}y) < \infty\,. \tag{14.10}$$

*and a scalar $c$ such that the pair $(g,c)$ satisfies the integro-differential equation*

$$(\mathcal{L}g)(x) = -\Big(f(x) + \int_{\mathbb{R}} \widetilde{f}(x,y)\,\nu(\mathrm{d}y) - c\Big) \tag{14.11}$$

*for $0 \le x \le b$, subject to the boundary conditions*

$$g'(0) = -r_1, \qquad g'(b) = r_2\,, \tag{14.12}$$

*then*

$$g\big(V(t)\big) + \Lambda(t) - ct$$

*is a martingale. Furthermore,*

$$\frac{\Lambda(t) - \ell^b t}{\sqrt{t}} \xrightarrow{\mathscr{D}} \eta N(0,1)$$

*as $t \to \infty$, where*

$$\eta^2 = \int_0^b \Big[\sigma^2 g'(x)^2 + \int_{\mathbb{R}} \big(\widetilde{f}(x,y) + g\big(r(x,y)\big) - g(x)\big)^2\big]\,\nu(\mathrm{d}y)\Big]\,\pi(\mathrm{d}x)\,.$$

*Proof.* We note that Itô's formula guarantees that

$$
\begin{aligned}
g\big(V(t)\big) &- g\big(V(0)\big) + \Lambda(t) - ct \\
&= \int_0^t (\mathcal{L}g)\big(V(s)\big)\,\mathrm{d}s + \int_0^t g'\big(V(s)\big)\sigma\,\mathrm{d}B(s) \\
&\quad + \sum_{0<s\le t} \big[g\big(V(s)\big) - g\big(V(s-)\big)\big] \\
&\quad - \int_0^t \int_{\mathbb{R}} \big[g\big(r(V(s-),y)\big) - g\big(V(s-)\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s \\
&\quad + \int_0^t f\big(X(s)\big)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}} \widetilde{f}\big(V(s-),y\big)\,\nu(\mathrm{d}y)\,\mathrm{d}s \\
&\quad + \sum_{0<s\le t} f\big(V(s-),\Delta X(s)\big) - \int_0^t \int_{\mathbb{R}} \widetilde{f}\big(V(s-),y\big)\,\nu(\mathrm{d}y)\,\mathrm{d}s \\
&\quad + r_1 L_c(t) + r_2 U_c(t) - ct + g'(0)L_c(t) - g'(b)U_c(t) \\
&= \int_0^t g'\big(V(s)\big)\sigma\,\mathrm{d}B(s) \\
&\quad + \sum_{0<s\le t} \big[g\big(V(s)\big) - g\big(V(s-)\big) + \widetilde{f}\big(V(s-),\Delta X(s)\big)\big] \\
&\quad - \int_0^t \int_{\mathbb{R}} \big[g\big(r(V(s-),y)\big) - g\big(V(s)\big) + \widetilde{f}\big(V(s-),y\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s \\
&= M(t) \qquad (\text{say})\,,
\end{aligned}
$$

where (14.9) and (14.10) were used to obtain the second equality. In the presence of (14.9), (14.10), and the boundedness of $g$ and $g'$, it follows that $M(t)$ is a martingale. Furthermore, the quadratic variation has the form

$$
\begin{aligned}
[M,M](t) &= \int_0^t g'\big(V(s)\big)^2 \sigma^2\,\mathrm{d}s \\
&\quad + \sum_{0<s\le t} \big[g\big(V(s)\big) - g\big(V(s-)\big)\big\} + \widetilde{f}\big(V(s-),\Delta X(s)\big)\big]^2 \\
&= \int_0^t \Big[\sigma^2 g\big(r(V(s-),y)\big)^2 \\
&\quad + \int_{\mathbb{R}} \big[g\big(V(s)\big) - g\big(V(s-)\big) + \widetilde{f}\big(V(s-),\Delta X(s)\big)\big]^2\Big]\,\nu(\mathrm{d}y)\,\mathrm{d}s \\
&\quad + M_1(t)
\end{aligned}
$$

where $M_1(t)$ is a martingale. It is easily seen that $[M,M](t)/t \to \eta^2$ a.s. as $t \to \infty$. Finally, to verify condition a) of the martingale CLT in [54, p. 340], we need to show that

$$
\frac{1}{\sqrt{t}}\mathbb{E}_\pi \sup_{0\le s\le t} \big|M(s) - M(s-)\big| \to 0 \tag{14.13}
$$

as $t \to \infty$ (this needs only to be verified for $V(0)$ distributed as $\pi$ because we can couple $V$ to the stationary version from any initial distribution). Of course, a sufficient condition for (14.13) is to establish that

$$\frac{1}{t}\mathbb{E}_\pi \sup_{0\le s\le t}\big|M(s)-M(s-)\big|^2 \to 0\,. \tag{14.14}$$

It is well known that (14.14) is immediate if

$$\mathbb{E}_\pi \sup_{0\le s\le 1}\big(M(s)-M(s-)\big)^2 < \infty\,. \tag{14.15}$$

But (14.15) is bounded by

$$\begin{aligned}
&\mathbb{E}_\pi \sum_{0\le s\le 1}\big[M(s)-M(s-)\big]^2\\
&\quad = \mathbb{E}_\pi \sum_{0\le s\le 1}\big[g\big(V(s)\big)-g\big(V(s-)\big)+\widetilde{f}(V(s-),\Delta X(s))\big]^2\\
&\quad = \mathbb{E}_\pi \int_{\mathbb{R}}\big[g\big(r(V(s-),y)\big)-g\big(V(s-)\big)+\widetilde{f}(V(s-),y)\big]^2\nu(\mathrm{d}y)\,\mathrm{d}s\\
&\quad = \int_0^b\int_{\mathbb{R}}\big[g\big(r(x,y)\big)-g\big(x,\big)+\widetilde{f}(x,y)\big]^2\nu(\mathrm{d}y)\,\pi(\mathrm{d}x)
\end{aligned}$$

due to the boundedness of $g$ and condition (14.9). The martingale CLT then yields the desired conclusion. □

Theorem 14.3 therefore provides the CLT for general additive functionals associated with $V$, provided that one can solve the integro-differential equation (14.11) subject to the boundary condition (14.12). Finally, we note that the fact that $g\big(V(t)\big)+\Lambda(t)-ct$ is, in great generality, a martingale, implies that

$$\mathbb{E}_x\Lambda(t) = ct + g(x) - \mathbb{E}_x g\big(V(t)\big)\,,$$

where, as usual, $\mathbb{E}_x$ refers to the case $V(0)=x$. Since

$$\mathbb{E}_x g\big(V(t)\big) \to \mathbb{E}_\pi g\big(V(t)\big)$$

as $t\to\infty$ (since $V$ is regenerative with absolutely continuous cycles), we conclude that

$$\mathbb{E}_x\Lambda(t) = ct + g(x) - \mathbb{E}_\pi g\big(V(t)\big) + \mathrm{o}(1)\,,$$

as $t\to\infty$. Hence, the solution $g$ to Poisson's equation also provides a 'correction' to the value of $\mathbb{E}_x\Lambda(t)$ that reflects the influence of the initial condition on the expected value of an additive functional.

## 14.4 Large deviations for the loss process

We turn next to obtaining a family of integro-differential equations from which the large deviations behaviour of the additive functional $\Lambda(\cdot)$ can be derived (for earlier work in this direction in the Brownian case, see Zhang & Glynn [131] and Forde *et al.* [56]). The key to the analysis is the following result:

**Theorem 14.4.** *Fix $\theta \in \mathbb{R}$. Suppose that*

$$\sup_{0\le x\le b} \int_{\mathbb{R}} \mathrm{e}^{\theta \widetilde{f}(x,y)}\, \nu(\mathrm{d}y) < \infty\,. \tag{14.16}$$

*If there exists a positive twice differentiable function $u(\theta) : [0, b] \to \mathbb{R}$ and a scalar $\psi(\theta)$ such that the pair $\big(u(\theta), \psi(\theta)\big)$ satisfies the integro-differential equation*

$$\begin{aligned} 0 &= \mu u_x(\theta,x) + \frac{\sigma^2}{2} u_{xx}(\theta,x) + \big(\theta f(x) - \psi(\theta)\big) u(\theta,x) \\ &\quad + \int_{\mathbb{R}} \big[\mathrm{e}^{\theta \widetilde{f}(x,y)} u\big(\theta, r(x,y)\big) - u(\theta,x)\big]\, \nu(\mathrm{d}y) \end{aligned} \tag{14.17}$$

*for $0 \le x \le b$, subject to the boundary conditions*

$$u_x(\theta,0) = -r_1\theta\,, \quad u_x(\theta,b) = r_2\theta\,, \tag{14.18}$$

*then $M(\theta,t) = \mathrm{e}^{\theta\Lambda(t)} u\big(\theta, V(t)\big)$ is a martingale.*

*Proof.* Define $A(s) = \exp\big\{\theta\Lambda(s) - \psi(\theta)s\big\}$ and

$$S(t) = \sum_{0<s\le t} A(s)\big[\exp\big\{\theta \widetilde{f}(V(s-), \Delta X(s))\big\} u\big(\theta, V(s)\big) - u\big(\theta, V(s-)\big)\big]\,.$$

Itô's formula shows that $M(\theta,t) - M(\theta,0)$ equals

$$\begin{aligned} &[t] S(t) + \int_0^t \big[\theta f\big(V(s)\big) - \psi(\theta)\big] A(s) u\big(\theta, V(s-)\big)\,\mathrm{d}s \\ &\qquad + \int_0^t r_1\theta A(s)\,\mathrm{d}L_c(s) + \int_0^t r_2\theta A(s)\,\mathrm{d}U_c(s) \\ &\qquad + \int_0^t \big[\mu u_x\big(\theta, V(s)\big) + \frac{\sigma^2}{2} u_{xx}\big(\theta, V(s)\big)\big] A(s)\mathrm{d}s \\ &= \int_0^t A(s) u_x\big(\theta, V(s)\big)\sigma\,\mathrm{d}B(s) + S(t) \\ &\qquad - \int_0^t A(s) \int_{\mathbb{R}} \big[\exp\big\{\theta \widetilde{f}\big(V(s), y\big)\big\} u\big(\theta, r\big(V(s), y\big)\big) - u\big(\theta, V(s)\big)\big]\, \nu(\mathrm{d}y)\,\mathrm{d}s\,, \end{aligned}$$

where the second equality uses the fact that $\big(u(\theta), \psi(\theta)\big)$ satisfy (14.17) and (14.18). Given the boundedness of $u(\theta)$ and (14.16), the fact that $M(\theta,t)$ is integrable and is a martingale is clear. □

As a consequence of the martingale property and the fact that $u(\theta)$ is bounded above and below by finite positive constants, it is straightforward to establish that

$$\frac{1}{t} \log \mathbb{E}_x \mathrm{e}^{\theta\Lambda(t)} \to \psi(\theta)$$

as $t \to \infty$. Suppose that there exists $\theta^* > 0$ for which $\psi(\cdot)$ exists in a neighbourhood of $\theta^*$ and is continuously differentiable there. If we let $a = \psi'(\theta^*)$, then

$$\frac{1}{t} \log \mathbb{P}_x\big(\Lambda(t) \ge at\big) \to \psi(\theta^*) - \theta^* a\,;$$

see, for example, the proof of the Gärtner-Ellis theorem in [49, 45–51]. Hence, the integro-differential equation (14.17) is intimately connected to the study of large deviations for $\Lambda$.

## 14.5 Discounted expectations for additive functionals

As our final illustration of how integro-differential equations naturally arise when computing expectations of additive functionals of reflected Lévy processes, we consider the calculation of an infinite horizon discounted expectation. Specifically, we let the discounting factor at $t$ be

$$\Gamma(t) = \int_0^t g\big(V(s)\big)\,\mathrm{d}s + \sum_{0<s\le t} \widetilde{g}\big(V(s-),\Delta X(s)\big) + u_1 L_c(s) + u_2 U_c(t)$$

for given functions $g, \widetilde{g}$ (where $\widetilde{g}$ is such that $\widetilde{g}(x,0)=0$ for $0 \le x \le b$), and set

$$D = \int_0^\infty \mathrm{e}^{-\Gamma(s)}\,\mathrm{d}\Lambda(s)\,.$$

As for $f, \widetilde{f}$, we assume that $g$ is bounded and that

$$\sup_{0\le x\le b} \int_{\mathbb{R}} \big|\widetilde{g}(x,y)\big|\,\nu(\mathrm{d}y) < \infty.$$

**Theorem 14.5.** *Assume that $f, \widetilde{f}$, $g, \widetilde{g}$, $u_1, u_2$ are non-negative with $g$ strictly positive. If there exists a twice continuously differentiable function $k : [0,b] \to [0,\infty)$ satisfying the integro-differential equation*

$$\begin{aligned} 0 = {} & \mu k'(x) + \frac{\sigma^2}{2}k''(x) - g(x)k(x) \\ & + \int_{\mathbb{R}} \big[\mathrm{e}^{-\widetilde{g}(x,y)}k\big(r(x,y)\big) - k(x)\big]\,\nu(\mathrm{d}y) \\ & + f(x) + \int_{\mathbb{R}} \widetilde{f}(x,y)\,\nu(\mathrm{d}y) \end{aligned}$$

*for $0 \le x \le b$, subject to the boundary conditions*

$$k'(0) - u_1 k(0) = -r_1\,, \quad k'(b) + u_2 k(b) = -r_2\,,$$

*then $\mathbb{E}_x D = k(x)$ for $0 \le x \le b$.*

*Proof.* Ito's formula ensures that

$$D(s) + \mathrm{e}^{-\Gamma(t)}k\big(V(s)\big) - k\big(V(0)\big) = \sum_{j=1}^{8} T_j$$

where

$$
\begin{aligned}
T_1 &= \int_0^t \mathrm{e}^{-\Gamma(s)}\big[\mu k'\big(V(s)\big) + \frac{\sigma^2}{2}\,k'\big(V(s)\big) - g(x)k\big(V(s)\big)\big]\,,\\
T_2 &= \int_0^t \mathrm{e}^{-\Gamma(s)}\int_{\mathbb{R}}\big[\mathrm{e}^{-\widetilde{g}\left(V(s),y\right)}k\big(r\big(V(s),y\big)\big) - k\big(V(s)\big)\big]\,\nu(\mathrm{d}y)\,,\\
T_3 &= \int_0^t \mathrm{e}^{-\Gamma(s)} f\big(V(s)\big)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\Gamma(s)}\int_{\mathbb{R}}\widetilde{f}\big(V(s),y\big)\,\nu(\mathrm{d}y)\,\mathrm{d}s\,,\\
T_4 &= \int_0^t \mathrm{e}^{-\Gamma(s)}\big[r_1 - u_1 k\big(V(s)\big) + k'\big(V(s)\big)\big]\,\mathrm{d}L_c(s)\,,\\
T_5 &= \int_0^t \mathrm{e}^{-\Gamma(s)}\big[r_2 - u_2 k\big(V(s)\big) - k'\big(V(s)\big)\big]\,\mathrm{d}U_c(s)\,,\\
T_6 &= \int_0^t k'\big(V(s)\big)\,\sigma\,\mathrm{d}B(s)\,,\\
T_7 &= \sum_{0<s\le t}\mathrm{e}^{-\Gamma(s)}\widetilde{f}\big(V(s),\Delta X(s)\big) - \int_0^t \mathrm{e}^{-\Gamma(s)}\widetilde{f}\big(V(s),y\big)\,\nu(\mathrm{d}y)\,\mathrm{d}s\,,\\
T_8 &= \sum_{0<s\le t}\mathrm{e}^{-\Gamma(s)}\big[\mathrm{e}^{-\widetilde{g}(V(s-),\Delta X(s))}k\big(r\big(V(s),\Delta X(s)\big)\big) - k\big(V(s-)\big)\big]\\
&\quad - \int_0^t \mathrm{e}^{-\Gamma(s)}\int_{\mathbb{R}}\big[\mathrm{e}^{-\widetilde{g}(V(s-),y)}k\big(r\big(V(s),y\big)\big) - k\big(V(s-)\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s\,.
\end{aligned}
$$

Here $T_1+T_2+T_3=0$ because of the integro-differential equation, $T_4=T_5=0$ because of the boundary conditions satisfied by $k$, and $T_6, T_7, T_8$ are all martingales. Consequently,

$$
k(x) = \mathbb{E}_x\int_0^t \mathrm{e}^{-\Gamma(s)}\,\mathrm{d}\Lambda(s) + \mathbb{E}_x\mathrm{e}^{-\Gamma(t)}k\big(V(t)\big)\,.
$$

Sending $t\to\infty$, the non-negativity assumption ensures that

$$
\int_0^t \mathrm{e}^{-\Gamma(s)}\,\mathrm{d}\Lambda(s)\ \uparrow\ \mathbb{E}_x D\,,
$$

while the non-negativity of $g, \widetilde{g}$, $u_1, u_2$, positivity of $g$ and boundedness of $k$ ensure that

$$
\int_0^t \mathrm{e}^{-\Gamma(s)}k\big(V(t)\big) \to 0\,,
$$

proving the theorem. □

## 14.6 Jumps of infinite variation

Lévy processes are permitted to have a jump part of infinite variation as long as the FV condition (14.1) is weakened to

$$
\int_{|y|<1} y^2\,\nu(\mathrm{d}y) < \infty\,. \tag{14.19}
$$

In this setting, one must compensate the small jumps, by considering the random measure

$$\int_{|y|<1} y\big[\mu(\mathrm{d}y,\mathrm{d}s) - \nu(\mathrm{d}y)\,\mathrm{d}s\big] < \infty \tag{14.20}$$

where $\mu$ is the Poisson random measure having intensity measure $\nu \otimes m$ (where $m$ is Lebesgue measure). The centered random measure is well-defined, and forms a square-integrable martingale when integrated over $s$ (due to (14.19)). Thus, in the non-BV jump setting we can write the Lévy process $X$ as

$$\begin{aligned} X(t) - X(0) &= at + \sigma B(t) \\ &\quad + \sum_{0<s\le t} \Delta X(s)\mathbb{1}\big(\big|\Delta X(s)\big| \ge 1\big) + \int_0^t \int_{|y|<1} y\big[\mu(\mathrm{d}y,\mathrm{d}s) - \nu(\mathrm{d}y)\,\mathrm{d}s\big] \end{aligned}$$

for some suitably defined constant $a$; observe that when the stronger FV condition (14.1) holds,

$$a = \mu + \int_{|y|<1} y\,\nu(\mathrm{d}y)\,.$$

In order to develop an Itô-type formla in this setting, we note than when $\nu[-\epsilon,\epsilon] = 0$ for some $\epsilon > 0$, we can write (for $f$ twice differentiable)

$$\begin{aligned} &f\big(V(t)\big) - f\big(V(0)\big) \\ &\quad = \int_0^t \int_{\mathbb{R}} \big[f\big(V(s-)+y\big) - f\big(V(s-)\big)\big]\mu(\mathrm{d}y,\mathrm{d}s) + \frac{\sigma^2}{2}\int_0^t f''\big(V(s)\big)\,\mathrm{d}s \\ &\qquad + \int_0^t f'\big(V(s)\big)\Big[a\,\mathrm{d}s + \sigma\,\mathrm{d}B(s) - \int_{|y|<1} y\,\nu(\mathrm{d}y)\,\mathrm{d}s + \mathrm{d}L_c(s) - \mathrm{d}U_c(s)\Big] \\ &\quad = \int_0^t \int_{\mathbb{R}} \big[f\big(V(s-)+y\big) - f\big(V(s-)\big)\big]\big(\mu(\mathrm{d}y,\mathrm{d}s) - \nu(\mathrm{d}y)\,\mathrm{d}s\big) \\ &\qquad + \int_0^t \int_{|y|\ge 1} \big[f\big(V(s-)+y\big) - f\big(V(s-)\big)\big]\nu(\mathrm{d}y)\,\mathrm{d}s \\ &\qquad + \int_0^t \int_{|y|<1} \big[f\big(V(s-)+y\big) - f\big(V(s-)\big) - yf'\big(V(s-)\big)\big]\nu(\mathrm{d}y)\,\mathrm{d}s \qquad (14.21) \\ &\qquad + \int_0^t f'\big(V(s)\big)\,\sigma\,\mathrm{d}B(s) + f'(0)\big(L_c(t) - L_c(0)\big) - f'(b)\big(U_c(t) - U_c(0)\big)\,. \end{aligned}$$

By sending $\epsilon \downarrow 0$ and utilising (14.19), we find that this formula extends to the general case in the general Lévy setting. We note that the smoothness of $f$ guarantees that

$$f\big(V(s-)+y\big) - f\big(V(s-)\big) - yf'\big(V(s-)\big)$$

is of order $y^2$ when $y$ is small, thereby guaranteeing that the term (14.21) on the r.h.s. is well-defined. As a consequence of the martingale property of the centered stochastic integral,

$$\mathbb{E}_x f\big(V(t)\big) - \mathbb{E}_x f\big(V(0)\big) = \int_0^t (\widetilde{\mathcal{L}}f)\big(V(s)\big)\,\mathrm{d}s\,,$$

where for some suitable $\widetilde{\mu}$

$$(\widetilde{\mathcal{L}}\varphi)(x) = \widetilde{\mu}\varphi'(x) + \frac{\sigma^2}{2}\varphi''(x)$$
$$+ \int_{|y|>1} \big[\varphi\big(r(x,y)\big) - \varphi(x)\big]\,\nu(\mathrm{d}y) + \int_{|y|\le 1} \big[\varphi\big(r(x,y)\big) - \varphi(x) - y\varphi'(x)\big]\,\nu(\mathrm{d}y)\,,$$

provided that $\varphi'(0 = \varphi'(b) = 0$. The integro-differential operator $\widetilde{\mathcal{L}}$ replaces the operator $\mathcal{L}$ that appeared earlier in the BV case (it can be easily verified that $\widetilde{\mathcal{L}} = \mathcal{L}$ in the BV case). For example, to compute $\mathbb{E}_x T_z$, the Itô argument above establishes that if $h$ satisfies $(\widetilde{\mathcal{L}}h)(x) = -1$ subject to $h'(0) = 0$ and $h(x) = 0$ for $x \ge z$, then $h(x) = \mathbb{E}_x T_z$. In a similar fashion all the other integro-differential equations derived earlier in this section can be generalised to Lévy processes having non-BV jumps.

# 15 Additional representations for the loss rate

In Sections 6 and 8, two representations for $\ell^b$ were provided, in which $\ell^b$ was represented in terms of an integral against the stationary distribution $\pi$ for the 'interior process' $V$. In this section, we return to the computation of $\ell^b$ and provide a simple argument establishing that there are infinitely many such representations of $\ell^b$ in terms of $\pi$.

The notation is the same as in Section 14; recall in particular the function $r(x,y)$ associated with two-sided reflection and the integro-differential operator $\mathcal{L}$.

We first write the local time $U(t)$ at $b$ in terms of the jump component and its continuous component, so that

$$U(t) - U(0) = \sum_{0<s\le t} \Delta U(s) + U_c(t)\,,$$

where, as usual, $\Delta U(s) = U(s) - U(s-)$ for $s > 0$. Clearly,

$$\ell^b = \ell^b_j + \ell^b_c\,,$$

where

$$\ell^b_j = \lim_{t\to\infty} \frac{1}{t} \sum_{0<s\le t} \Delta U(s)\,, \quad \ell^b_c = \lim_{t\to\infty} \frac{1}{t} U_c(t) \text{ a.s.}$$

We now show how $\ell^b_j$ and $\ell^b_c$ can be individually calculated in terms of $\pi$. Dealing with $\ell^b_j$ is easy. Note that

$$\widetilde{M}(t) = \sum_{0<s\le t} \Delta U(s) - \int_0^t \int_{\mathbb{R}} [V(s) + y - b]^+ \nu(\mathrm{d}y)\,\mathrm{d}s$$

is a martingale, and hence

$$\mathbb{E}\frac{1}{t} \sum_{0\le s\le t} \Delta U(s) = \mathbb{E}\frac{1}{t} \int_0^t \int_{\mathbb{R}} [V(s) + y - b]^+ \nu(\mathrm{d}y)\,\mathrm{d}s\,.$$

Consequently,

$$\ell_j^b = \int_0^b \int_{\mathbb{R}} [x+y-b]^+ \nu(\mathrm{d}y)\,\pi(\mathrm{d}x)\,.$$

It remains only to compute $\ell_c^b$. For a given twice differentiable function $h : [0,b] \to \mathbb{R}$, Itô's formula ensures that

$$\begin{aligned} h\big(V(t)\big) - h\big(V(0)\big) = &\sum_{0<s\le t} \big[h\big(V(s)\big) - h\big(V(s-)\big)\big] \\ &+ \int_0^t \Big[\mu h'\big(V(s)\big) + \frac{\sigma^2}{2} h''\big(V(s)\big)\Big]\,\mathrm{d}s \qquad (15.1) \\ &+ \sigma \int_0^t h'\big(V(s)\big)\big)\,\mathrm{d}B(s) + h'(0)L_c(t) - h'(b)U_c(t)\,, \end{aligned}$$

where $L_c(\cdot)$ is the continuous component of $L(\cdot)$. Letting

$$\begin{aligned} M(t) = &\sum_{0<s\le t} \big[h\big(V(s)\big) - h\big(V(s-)\big)\big] + \sigma \int_0^t h'\big(V(s)\big)\,\mathrm{d}B(s) \\ &- \int_0^t \int_{\mathbb{R}} \big[h\big(r\big(V(s-),y\big)\big) - h\big(V(s-)\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s\,, \end{aligned}$$

and rewriting (15.1) in terms of $\mathcal{L}$, we get

$$h\big(V(t)\big) - h\big(V(0)\big) = M(t) + \int_0^t (\mathcal{L}h)\big(V(s)\big)\,\mathrm{d}s + h'(0)L_c(t) - h'(b)U_c(t)$$

Further, $M(\cdot)$ is a square integrable martingale, and since $h$ and its derivatives are bounded, it follows by taking stationary expectations at $t=1$ that

$$0 = \int_0^b (\mathcal{L}h)(x)\,\pi(\mathrm{d}x) + h'(0)\ell_c^0 - h'(b)\ell_c^b\,, \qquad (15.2)$$

where $\ell_c^0 = \lim_{t\to\infty} \frac{1}{t} L_c(t)$ a.s.

As a consequence, we can now compute $\ell_c^0$ and $\ell_c^b$ by choosing two (twice differentiable) functions $h_1$ and $h_2$. According to (15.2),

$$\begin{pmatrix} h_1'(b) & -h_1'(0) \\ h_2'(b) & -h_2'(0) \end{pmatrix} \begin{pmatrix} \ell_c^b \\ \ell_c^0 \end{pmatrix} = \begin{pmatrix} \int_0^b (\mathcal{L}h_1)(x)\,\pi(\mathrm{d}x) \\ \int_0^b (\mathcal{L}h_2)(x)\,\pi(\mathrm{d}x) \end{pmatrix} \qquad (15.3)$$

Thus, if $h_1$ and $h_2$ are chosen so that the coefficient matrix on the l.h.s. of (15.3) is non-singular, this yields formulae for $\ell_c^0$ and $\ell_c^b$ in terms of $\pi$. Consequently, there are infinitely many representations of $\ell^b$ in terms of $\pi$ (two of which have been introduced in Sections 6 and 8).

Even in situations where $\pi$ is not easily computable, the above approach provides a mechanism for easily computing bounds on $\ell^b$. For example, by choosing $h_1$ so that $h_1'(0) = 0$ and $h_1'(b) = 1$ (and $h_2(\cdot)$ arbitrarily), we can compute bounds on $\ell_c^b$ in terms of the supremum of $(\mathcal{L}h_1)(x)$.

# 16 Markov-modulation

Models with the parameters varying according to the state of a finite Markov chain or -process have a long history and are popular in many areas: in statistics, they go unded the name of *hidden Markov models* (e.g. Cappé, Moulines & Ryden [38]), in finance the term *Markov regime switching* is used (e.g. Elliott, Chan & Siu [51]), and in queueing the first occurrence was with the *Markov-modulated Poisson process*. We consider here Lévy processes with the characteristic triplet $(c_i, \sigma_i^2, \nu_i)$ depending on the state $J(t) = i$ of an underlying finite ergodic Markov process $J$, with the extension that additional jumps may occur at state changes of $J$. This is important since then the model class becomes dense in the whole of $D[0, \infty)$, cf. [11, Ch. XI] where also the connection to Markov additive processes is explained.

In this section we generalize the results from Section 6 and Section 8 to hold for a Markov-modulated Lévy process $X$. We will use the same technique as in Section 6 (a direct application of Ito's formula for general semimartingales) to derive a formula for $\ell^b$. In [19] an approach based on optional stopping of a multi-dimensional version of the Kella-Whitt martingale is used to obtain $\ell^b$, but this will not be presented here, since it is very complicated and does not really shed any probabilistic light upon the underlying Skorokhod problem. Further, the direct Ito approach leads directly to an easier expression for $\ell^b$.

We start by constructing $X$. We assume that we are given an underlying probability space with filtration $\mathscr{F}$, which satisfies the usual conditions, i.e., it is augmented and right-continuous. Let $J$ (the modulating process) be a right-continuous irreducible Markov process with state space $\{1, \ldots, p\}$, intensity matrix $\boldsymbol{Q} = (q_{ij})$ and stationary row vector $\boldsymbol{\alpha} = (\alpha_i)$. Let $X^1, \ldots, X^p$ be Lévy processes (with respect to $\mathscr{F}$) with characteristic triplets $(c_i, \sigma_i^2, \nu_i)$, $i = 1, \ldots, p$, which are independent of $J$ and each other and satisfy $\mathbb{E}|X^i(1)| < \infty$, $i = 1, \ldots, p$. Further, let $\{U^{ij} : 1 \le i, j \le p\}$ and $\{U_n^{ij} : n \ge 1,\ 1 \le i, j \le p\}$ be independent random variables which are also independent of $X^1, \ldots, X^p$ and $J$, such that for each $i, j, n$, $U^{ij}$ and $U_n^{ij}$ are identically distributed with distribution $H^{ij}$ and $\mathbb{E}|U^{ij}| < \infty$. Let $T_0, T_1, \ldots$ be the jump epochs of $J$ (with $T_0 = 0$). It is assumed that for every $i, j, n$, $U_n^{ij}$ is measurable with respect to $\mathscr{F}(T_n)$ and that $U^{ij} \in \mathscr{F}(0)$. We then define the process $X$ according to

$$X(t) = \sum_{n \ge 1} \sum_{\substack{1 \le i,j \le p \\ i \ne j}} (X^i(T_n) - X^i(T_{n-1}) + U_n^{ij}) \mathbb{1}(J(T_{n-1}) = i, J(T_n) = j, T_n \le t)$$

$$+ \sum_{n \ge 1} \sum_{i=1}^{p} (X^i(t) - X^i(T_{n-1})) \mathbb{1}(J(T_{n-1}) = i, T_{n-1} \le t < T_n), \quad (16.1)$$

or, equivalently, $X(0) = 0$ and

$$\mathrm{d}X(s) = \sum_{i=1}^{p} \mathbb{1}(J(s) = i) \mathrm{d}X^i(s) + \sum_{n \ge 1} \sum_{\substack{1 \le i,j \le p \\ i \ne j}} U_n^{ij} \mathbb{1}(s = T_n, J(T_n-) = i, J(T_n) = j). \quad (16.2)$$

We denote the stationary measure of $(V,J)$ by $\pi(\cdot,\cdot)$ ($(V,J)$ is assumed to be stationary throughout this section). Let $\tilde{H}^{ij}=H^{ji}$ and $\tilde{J}$ be time-reversed version of $J$ (note that $\tilde{J}$ has intensity matrix $\tilde{\boldsymbol{Q}}=\boldsymbol{A}^{-1}\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{A}$ where $\boldsymbol{A}$ is the diagonal matrix with $\boldsymbol{\alpha}$ on the diagonal, and that $\boldsymbol{\alpha}$ is also stationary for $\tilde{J}$). $\tilde{X}$ is constructed by using (16.1) with $H^{ij}$ replaced by $\tilde{H}^{ij}$ and $J$ replaced by $\tilde{J}$. In the same way as in Proposition 2.11 in [11], p. 314, we obtain the following representation of $\pi$ in the Markov-modulated case.

$$\pi([y,b],i)=\alpha_i\mathbb{P}_i(\tilde{X}(\tau[y-b,y))\geq y), \tag{16.3}$$

where $\tau[u,v)=\inf\{t\geq 0:\tilde{X}(t)\notin[u,v)\},\ u\leq 0\leq v$, and $\mathbb{P}_i(\cdot)=\mathbb{P}(\cdot\mid\tilde{J}(0)=i)$.

Now, we turn our attention towards the identification of $\ell^b$. The only differences between the Markov-modulated case and the standard Lévy process case are that we now have to treat time segments corresponding to different states of $J$ separately and that state changes in $J$ generate jumps in $X$. E.g., we get the following equivalent to (6.4) (where $\mathrm{d}X(s)$ is given by (16.2))

$$\begin{aligned}
V(t)^2-V(0)^2&-\int_{0+}^t 2V(s-)\mathrm{d}X(s)\\
&=-2bU(t)+\int_{0+}^t \mathrm{d}[X,X]^c(s)\\
&\quad+\sum_{0<s\leq t}\{-2\Delta V(s)\Delta L(s)+2\Delta V(s)\Delta U(s)+(\Delta V(s))^2\}.
\end{aligned}$$

where (cf. Corollary 2.5 and Corollary 2.9 on p. 313 in [11])

$$m=\mathbb{E}_\pi X(1)=\sum_{i=1}^p\alpha_i\Big(m_i+\sum_{j\neq i}q_{ij}\mathbb{E}U^{ij}\Big) \tag{16.4}$$

with $m_i=\mathbb{E}X^i(1)$. Thus, we have

$$\begin{aligned}
-2m\mathbb{E}V=-2b\ell^b&+\mathbb{E}_\pi\int_{0+}^1\mathrm{d}[X,X]^c(s)\\
&+\mathbb{E}_\pi\sum_{0<s\leq 1}\{-2\Delta V(s)\Delta L(s)+2\Delta V(s)\Delta U(s)+(\Delta V(s))^2\}.
\end{aligned}$$

What remains is to identify terms which is fairly straightforward. It is easily seen that

$$\mathbb{E}\int_{0+}^1\mathrm{d}[X,X]^c(s)=\sum_{i=1}^p\alpha_i\sigma_i^2\,,\quad \mathbb{E}V=\sum_{i=1}^p\int_0^b x\pi(\mathrm{d}x,i).$$

For the sum of jumps we get (condition on $\big(V(s-),J(s-)\big)$),

$$\begin{aligned}
&\mathbb{E}\sum_{0<s\leq 1}\Delta V(s)\Delta L(s)\\
&=\sum_{i=1}^p\int_0^b\pi(\mathrm{d}x,i)\int_{-\infty}^\infty x(x+y)\mathbb{1}(y\leq -x)\Big(\nu_i(\mathrm{d}y)+\sum_{j\neq i}q_{ij}H^{ij}(\mathrm{d}y)\Big),
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\sum_{0<s\le 1}\Delta V(s)\Delta U(s)\\
&\quad=\sum_{i=1}^{p}\int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty}(b-x)(y-b+x)\mathbb{1}(y\ge b-x)\Big(\nu_i(\mathrm{d}y)+\sum_{j\ne i}q_{ij}H^{ij}(\mathrm{d}y)\Big),\\
&\mathbb{E}\sum_{0<s\le 1}(\Delta V(s))^2\\
&\quad=\sum_{i=1}^{p}\int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty}y^2\mathbb{1}(-x<y<b-x)\Big(\nu_i(\mathrm{d}y)+\sum_{j\ne i}q_{ij}H^{ij}(\mathrm{d}y)\Big)\\
&\qquad+\sum_{i=1}^{p}\int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty}(b-x)^2\mathbb{1}(y\ge b-x)\Big(\nu_i(\mathrm{d}y)+\sum_{j\ne i}q_{ij}H^{ij}(\mathrm{d}y)\Big)\\
&\qquad+\sum_{i=1}^{p}\int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty}x^2\mathbb{1}(y\le -x)\Big(\nu_i(\mathrm{d}y)+\sum_{j\ne i}q_{ij}H^{ij}(\mathrm{d}y)\Big).
\end{aligned}
$$

Putting the pieces together, we get the final expression for $\ell^b$ in the Markov-modulated case,

$$
\begin{aligned}
\ell^b=\frac{1}{2b}\bigg\{&2\Big(\sum_{i=1}^{p}\int_0^b x\pi(\mathrm{d}x,i)\Big)\Big(\sum_{i=1}^{p}\alpha_i\Big(m_i+\sum_{j\ne i}q_{ij}\mathbb{E}U^{ij}\Big)\Big)\\
&+\sum_{i=1}^{p}\alpha_i\sigma_i^2+\sum_{i=1}^{p}\int_0^b\pi(\mathrm{d}x,i)\int_{-\infty}^{\infty}\varphi(x,y)\Big(\nu_i(\mathrm{d}y)+\sum_{j\ne i}q_{ij}H^{ij}(\mathrm{d}y)\Big)\bigg\}.
\end{aligned}
\tag{16.5}
$$

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