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Anders Rønn–Nielsen and Eva B. Vedel Jensen

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Anders Rønn–Nielsen[1] and Eva B. Vedel Jensen[2]

[1]Department of Mathematical Sciences, University of Copenhagen
[2]Department of Mathematics, Aarhus University

Abstract

We consider a continuous, infinitely divisible random field in $\mathbb{R}^d$, $d = 1, 2, 3$, given as an integral of a kernel function with respect to a Lévy basis with convolution equivalent Lévy measure. For a large class of such random fields we compute the asymptotic probability that the excursion set at level x contains some rotation of an object with fixed radius as $x \to \infty$. Our main result is that the asymptotic probability is equivalent to the right tail of the underlying Lévy measure.

Keywords: convolution equivalence; excursion set; infinite divisibility; Lévy-based modelling

1 Introduction

In the present paper we investigate the extremal behaviour of excursion sets for a field $(X_t)_{t\in B}$ defined by

$$X_t = \int_{\mathbb{R}^d} f(|t-s|)\, M(\mathrm{d}s)\,, \tag{1.1}$$

where M is an infinitely divisible, independently scattered random measure on $\mathbb{R}^d$, f is some kernel function, and B is a compact index set. We will assume that the Lévy measure of the random measure M has a convolution equivalent right tail ([5, 6, 10]). In [13] it was shown under some regularity conditions that the distribution of $\sup_{t\in B} X_t$ has a similar convolution equivalent tail. In the present paper we will be interested in the excursion set

$$A_x = \{t \;:\; X_t > x\}\,.$$

Under the additional assumption (2.10) below, we derive the result that the asymptotic probability of the excursion set at level x containing some rotation of an object with a fixed radius r has a tail that is equivalent to the tail of the underlying Lévy measure. A more precise definition of the event that is studied asymptotically is found in Section 2 below. Measures with a convolution equivalent tail cover the important cases of an inverse Gaussian and a normal inverse Gaussian (NIG) basis, respectively, see [13].

Lévy models as defined in (1.1) provide a flexible and tractable modelling framework that recently has been used for a variety of modelling purposes, including modelling of turbulent flows ([4]), growth processes ([8]), Cox point processes ([7]), and brain imaging data ([9]). In [9], a model (1.1) with M following a NIG distribution was suitable for modelling the neuroscience data under consideration. For such data it is typically of interest to detect for which $t \in B$ a given field obtains values that are significantly large. The results in the present paper will make it possible to discuss whether a cluster of $t \in B$ with large observations jointly form an extreme observation.

For Gaussian random fields it is known that the distribution of the supremum of the field can be approximated by the expected Euler characteristic of an excursion set (see [3] and references therein). The supremum and excursion sets of a non–Gaussian field given by integrals with respect to an infinitely divisible random measure has already been studied, when the random measure has regularly varying tails. Results for the asymptotic distribution of the supremum are found in [11], and these results are refined in [1] and [2], where results are obtained on the asymptotic joint distribution of the number of critical points of the excursion sets. The arguments are – as in the present paper – based on finding the Lévy measure of a dense countable subset of the field. However, the remaining proofs rely heavily on the assumption of regularly varying tails and can therefore not be translated into the convolution equivalent framework.

Note that convolution equivalent distributions have heavier tails than Gaussian distributions and lighter tails than those of regularly varying distributions. The latter statement follows from the fact that convolution equivalent distributions have exponential tails while regularly varying distributions have power function tails.

The present paper is organised as follows. In Section 2 we define the random (1.1) and introduce the necessary assumptions. In Section 3 we show three technical lemmas concerning the asymptotic behaviour of deterministic fields. These results will be used in Section 4. In Section 4 we show the main result of the paper. The proof will be in several steps, utilising that X can be decomposed as X^1+X^2, where X^1 is a compound Poisson sum and X^2 has lighter tails than X^1. The proofs in this section will apply techniques that are similar to the proofs in [13].

2 Preliminaries

We shall make the same general assumptions as in [13] except for the additional assumption (2.10) below. For completeness, we will present all assumptions in the following. Consider an independently scattered random measure M on $\mathbb{R}^d$, $d = 1, 2, 3$. Then for a sequence of disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ in $\mathcal{B}(\mathbb{R}^d)$ the random variables $(M(A_n))_{n\in\mathbb{N}}$ are independent and satisfy $M(\cup A_n) = \sum M(A_n)$. Assume furthermore that $M(A)$ is infinitely divisible for all $A \in \mathcal{B}(\mathbb{R}^d)$. Then M is called a Lévy basis, see [4] and references therein.

For a random variable X let $C(\lambda \ddagger X)$ denote its cumulant function $\log E(e^{i\lambda X})$. We shall assume that the Lévy basis is stationary and isotropic such that for $A \in \mathcal{B}(\mathbb{R}^d)$ the variable $M(A)$ has a Lévy–Khintchine representation given by

$$C(\lambda \ddagger M(A)) = i\lambda a m_d(A) + \tfrac{1}{2}\lambda^2\theta m_d(A) + \int_{A\times\mathbb{R}} \left(e^{i\lambda u} - 1 - i\lambda u 1_{[-1,1]}(u)\right) F(\mathrm{d}s, \mathrm{d}u)\,, \tag{2.1}$$

where m_d is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $a \in \mathbb{R}$, $\theta \geq 0$ and F is a measure on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ on the form

$$F(A \times B) = m_d(A)\rho(B)\,. \tag{2.2}$$

We assume that ρ has an exponential tail with index $\beta > 0$, i.e. for all $y \in \mathbb{R}$

$$\frac{\rho((x-y,\infty))}{\rho((x,\infty))} \to e^{\beta y} \qquad \text{as } x \to \infty. \tag{2.3}$$

Furthermore, letting ρ_1 be a normalization of the restriction of ρ to $(1,\infty)$, we assume that ρ_1 has a convolution equivalent right tail, i.e.

$$\frac{(\rho_1 * \rho_1)((x,\infty))}{\rho_1((x,\infty))} \to 2M \qquad \text{as } x \to \infty, \tag{2.4}$$

where $M < \infty$. Here $\rho_1 * \rho_1$ denotes the convolution. In fact, $M = \int e^{\beta y}\,\rho_1(\mathrm{d}y)$, cf. [10, Corollary 2.1, (ii)]. Writing $\rho((x,\infty)) = L(x)e^{-\beta x}$, it is seen from (2.3) that for all $y \in \mathbb{R}$

$$\frac{L(x-y)}{L(x)} \to 1 \qquad \text{as } x \to \infty. \tag{2.5}$$

For each $a, b \in \mathbb{R}$, the limit (2.5) holds uniformly in $y \in [a,b]$, cf. [10, p. 408]. We furthermore assume

$$\int z^2\,\rho(\mathrm{d}z) < \infty\,. \tag{2.6}$$

Now assume that $f : [0,\infty) \to [0,\infty)$ is a strictly decreasing kernel function satisfying

$$\int_{\mathbb{R}^d} f(|s|)\mathrm{d}s < \infty\,, \tag{2.7}$$

and

$$f(x) \leq \frac{K_1}{(x+1)^d} \qquad \text{for all } x \geq 0 \tag{2.8}$$

for a finite, positive constant K_1. Assume furthermore that f is differentiable with f' satisfying

$$|f'(x)| \leq \frac{K_2}{(x+1)^d} \qquad \text{for all } x \geq 0 \tag{2.9}$$

for a finite, positive constant K_2. Finally, let $r > 0$ be fixed and assume that there exists g such that $f(x) \leq g(x)$ for all $x \geq 0$ and such that

$$g(x) = f'(r)(x-r) + f(r) \qquad \text{for all } x \in [0, 2r]\,. \tag{2.10}$$

Note that this in particular satisfied, if f is concave on $[0, 2r]$. We will furthermore choose g on $[2r, \infty)$ such that it satisfies (2.7)–(2.9).

Let B be a compact, convex subset of $\mathbb{R}^d$ with $m_d(B) > 0$ and define the set $B \oplus C_r = \{x + y \ : \ x \in B, y \in C_r(0)\}$, where $C_r(0)$ is the ball with center in 0 and radius r. We consider the family of random variables $(X_t)_{t \in B \oplus C_r}$ defined by

$$X_t = \int_{\mathbb{R}^d} f(|t - s|)\, M(\mathrm{d}s)\,.$$

See [13] for existence of the integrals.

Example 2.1 (Gaussian kernel function)**.** Suppose that $f(x) = e^{-\sigma x^2}$, $\sigma > 0$, then the assumptions (2.7)–(2.9) are satisfied, and f is concave on the interval $[0, \frac{1}{\sqrt{2\sigma}}]$. In particular, the assumption (2.10) is satisfied for $r \leq \frac{1}{2\sqrt{2\sigma}}$.

Example 2.2 (Matérn kernel function)**.** Suppose that

$$f(x) = \frac{1}{2^{\eta-1}\Gamma(\eta)} |\lambda x|^{\eta} K_{\eta}(\lambda |x|)\,,$$

where K_η is the modified Bessel function of the second kind, index $\eta \geq \frac{1}{2}$, and $\lambda > 0$. It can be shown that the Matérn kernel satisfies the assumptions (2.7)–(2.9). See [12, Example 2.5] and references therein for details. Furthermore [12, Example 2.5] provides identities for the derivatives of f from which it can be shown that f is concave in an interval $(0, \delta)$ close to 0, when $\eta > \frac{1}{2}$. In particular, the assumption (2.10) will be satisfied.

For $s \in B$ let $C_r(s)$ be the ball in $\mathbb{R}^d$ with radius r and center s and let $S^{d-1} = \{\alpha \in \mathbb{R}^d \ : \ |\alpha| = 1\}$ be the unit sphere. For $\alpha \in S^{d-1}$ define R_α to be the rotation map that rotates a fixed vector $\alpha_0 \in S^{d-1}$ into α. For convenience, we let $\alpha_0 = 1$, $\alpha_0 = (1, 0)$, $\alpha_0 = (1, 0, 0)$ for $d = 1, 2, 3$ respectively.

Let $D \subseteq C_r(0)$ be a set with radius r in the sense that there exists $\beta \in S^{d-1}$ such that $\{s + r\beta, s - r\beta\} \subseteq D$. Furthermore, define $D^\alpha(s) = R_\alpha D + s$. Recalling the definition of the excursion set, $A_x = \{t \in B \oplus C_r \ : \ X_t > x\}$, we will be interested in the event

$$\{\text{there exists } t \in B\,, \alpha \in S^{d-1} \ : \ D^\alpha(t) \subseteq A_x\}\,.$$

Alternatively, this can be expressed as

$$\{\sup_{t_0 \in B} \sup_{\alpha \in S^{d-1}} \inf_{t \in D^\alpha(t_0)} X_t > x\}\,.$$

Example 2.3. A possible choice of D is $C_r(0)$. Here the rotations of D are unnecessary. Another choice could be that $D = \{r\alpha_0, -r\alpha_0\}$. A third possibility is the line segment connecting the points $r\alpha_0$ and $-r\alpha_0$.

For the study of the extremal behaviour of $(X_t)_{t \in B \oplus C_r}$, it will crucial that the field $(X_t)_{t \in T}$ is itself infinitely divisible, with $T = (B \oplus C_r) \cap \mathbb{Q}^d$, where $\mathbb{Q}^d$ are the rational numbers in $\mathbb{R}^d$. For details, see [13] and references therein. The Lévy measure of $(X_t)_{t\in T}$ is the measure ν on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ defined by $\nu = F \circ V^{-1}$, where $V : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^T$ is given by

$$V(s, z) = (zf(|t-s|))_{t\in T} .$$

Because of the infinite divisibility of $(X_t)_{t\in T}$, we have the following decomposition, see e.g. [11],

$$X_t = X_t^1 + X_t^2 ,$$

where the fields $(X_t^1)_{t\in T}$ and $(X_t^2)_{t\in T}$ are independent. The first field, $(X_t^1)_{t\in T}$, is a compound Poisson sum

$$X_t^1 = \sum_{n=0}^{N} U_t^n ,$$

where N is Poisson distributed with parameter $\nu(A) < \infty$ and $A = \{x \in \mathbb{R}^T : \sup_{t\in T} x_t > 1\}$. The fields $(U_t^n)_{t\in T}$ are independent and identically distributed with common distribution $\nu_1 = \nu_A/\nu(A)$, where ν_A is the measure on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ obtained by restricting ν to A. Furthermore $(X_t^2)_{t\in T}$ is infinitely divisible with a Lévy measure ν_{A^c}, the restriction of ν to A^c.

As argued in [13], all the fields U^n, X^1, and X^2 have continuous extension to $B \oplus C_r$. It should furthermore be noted that each of the fields $(U_t^n)_{t\in B\oplus C_r}$ can be represented by $\big(Zf(|t-S|)\big)_{t\in B\oplus C_r}$, where $(Z,S) \in [0,\infty) \times \mathbb{R}^d$ has distribution F_1, that is the restriction of the measure F to the set

$$V^{-1}(A) = \{(s,z) \in \mathbb{R}^d \times \mathbb{R} \ : \ \sup_{t\in T} zf(|t-s|) > 1\} .$$

3 Asymptotic results for deterministic fields

An important property for the arguments in [13] is that for a continuous field $(y_t)_{t\in B\oplus C_r}$ it holds for all $s \in B$ that

$$\inf_{t\in B} \frac{x - y_t}{f(|t-s|)} - x + y_s \to 0$$

as $x \to \infty$. For the purpose of this paper we shall need a similar but more involved result concerning the asymptotic behaviour of

$$\inf_{t_0\in B} \inf_{\alpha\in S^{d-1}} \sup_{t\in D^\alpha(t_0)} \frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)} , \tag{3.1}$$

where S^{d-1} and $D^\alpha(t)$ are as defined in the introduction.

Lemma 3.1. *Let $(y_t)_{t\in B\oplus C_r}$ be a continuous field. Then there exists a function $\lambda_s((y_t)_{t\in B\oplus C_r})$ such that for each $s \in B$*

$$\inf_{t_0\in B} \inf_{\alpha\in S^{d-1}} \sup_{t\in D^\alpha(t_0)} \frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)} + \lambda_s((y_t)_{t\in B\oplus C_r}) \to 0$$

as $x \to \infty$. If $(y_t)_{t\in B\oplus C_r}$ is constantly equal to y, then $\lambda_s((y_t)_{t\in B\oplus C_r}) = y/f(r)$ for all s, and if y is a constant, $\lambda_s((y+y_t)_{t\in B\oplus C_r}) = y/f(r) + \lambda_s((y_t)_{t\in B\oplus C_r})$. Furthermore, $\lambda_s((y_t)_{t\in B\oplus C_r})$ only depends on $(y_t)_{t\in C_{r+\epsilon}(s)}$ for any $\epsilon > 0$.

Proof. Let $y^* = \sup_{t\in B\oplus C_r}$ and $y_* = \inf_{t\in B\oplus C_r}$. Then the expression in (3.1) is bounded from above by

$$\frac{x-y_*}{\sup_{t_0,\alpha}\inf_{t\in D^\alpha(t_0)} f(|t-s|)} - \frac{x}{f(r)} = \frac{x-y_*}{f(r)} - \frac{x}{f(r)} = \frac{-y_*}{f(r)}\,.$$

Similarly, the expression is bounded from below by $-y^*/f(r)$. The result for a constant field (y_t) is seen from this, and the result concerning adding a constant to (y_t) follows similarly, when the existence of the limit $\lambda_s((y_t)_{t\in B\oplus C_r})$ is established. For each $x>0$ we can choose $t_x \in B$ and $\alpha_x \in S^{d-1}$ such that

$$\inf_{t_0\in B}\inf_{\alpha\in S^{d-1}}\sup_{t\in D^\alpha(t_0)} \frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)} = \sup_{t\in D^{\alpha_x}(t_x)} \frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)}\,. \tag{3.2}$$

First, we show that $t_x \to s$. We find

$$\frac{x-\inf_{t\in C_r(s)} y_t}{f(r)} = \sup_{t\in C_r(s)} \frac{x-y_t}{f(r)} \geq \sup_{t\in D^{\alpha_x}(t_x)} \frac{x-y_t}{f(|t-s|)} \geq \frac{x-y^*}{\inf_{t\in D^{\alpha_x}(t_x)} f(|t-s|)}\,.$$

When using that $\inf_{t\in D^{\alpha_x}(t_x)} f(|t-s|) \leq f(r)$ this yields

$$\frac{x-y^*}{x-\inf_{t\in C_r(s)} y_t} \leq \frac{\inf_{t\in D^{\alpha_x}(t_x)} f(|t-s|)}{f(r)} \leq 1\,,$$

such that $\inf_{t\in D^{\alpha_x}(t_x)} f(|t-s|) \to f(r)$ as $x\to\infty$. Since it furthermore holds that $\inf_{t\in D^\alpha(t_0)} f(|t-s|) < f(r)$ for all $t_0 \neq s$ and $\alpha \in S^{d-1}$, we can conclude that $t_x \to s$. From this we can conclude that $\lambda_s((y_t)_{t\in B\oplus C_r})$ only depends on y_t for t close to $C_r(s)$.

In fact, we need a stronger version of this result. From differentiability of f in r we have for $u \geq 0$

$$\frac{1}{f(u)} - \frac{1}{f(r)} = b(u-r) + (u-r)\phi(u-r) \tag{3.3}$$

for $b>0$ and some continuous function ϕ with $\phi(0)=0$. Using that f is decreasing we find for each $K>0$ that

$$\begin{aligned}
x\Big(\frac{1}{f(u)} - \frac{1}{f(r)}\Big) &\leq -bK + \phi(-K/x) \qquad \text{for } 0<u<r-\frac{K}{x}\,,\\
x\Big(\frac{1}{f(u)} - \frac{1}{f(r)}\Big) &\geq bK + \phi(K/x) \qquad \text{for } u>r+\frac{K}{x}\,.
\end{aligned}$$

In particular, we can choose K and x_0 such that for all $x>x_0$

$$\begin{aligned}
\frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)} &< \frac{-y^*}{f(r)} \qquad \text{for } |t-s|<r-\frac{K}{x}\,,\\
\frac{x-y_t}{f(|t-s|)} - \frac{x}{f(r)} &> \frac{-y_*}{f(r)} \qquad \text{for } |t-s|>r+\frac{K}{x}\,.
\end{aligned}$$

With this choice of K we have for $x > x_0$ that

$$\begin{aligned}\inf_{t_0\in B}\inf_{\alpha\in S^{d-1}}\sup_{t\in D^\alpha(t_0)}&\frac{x-y_t}{f(|t-s|)}-\frac{x}{f(r)}\\&=\inf_{t_0\in B}\inf_{\alpha\in S^{d-1}}\sup_{t\in D^\alpha(t_0)\cap H_x}\frac{x-y_t}{f(|t-s|)}-\frac{x}{f(r)}\,,\end{aligned}\tag{3.4}$$

where $H_x = \{t\in\mathbb{R}^d \;:\; r-K/x \le |t-s| \le r+K/x\}$. Define

$$h(\ell)=\sup\{|\phi(u-r)| \;:\; r-\ell\le u\le r+\ell\}\,,$$

and note that $h(\ell)\to 0$ as $\ell\to 0$.

We will show the convergence result by contradiction. To obtain this, we assume that there is a sequence $x_1 < \tilde{x}_1 < x_2 < \tilde{x}_2 < \cdots$ and constants a and $\epsilon > 0$ such that

$$\sup_{t\in D^{\alpha_n}(t_n)}\frac{x_n-y_t}{f(|t-s|)}-\frac{x_n}{f(r)}\le a,\qquad \sup_{t\in D^{\tilde\alpha_n}(\tilde t_n)}\frac{\tilde x_n-y_t}{f(|t-s|)}-\frac{\tilde x_n}{f(r)}\ge a+\epsilon \tag{3.5}$$

for all n, where $\alpha_n=\alpha_{x_n}$, $\tilde\alpha_n=\alpha_{z_n}$ are the corresponding rotations, and $t_n=t_{x_n}$, $\tilde t_n = t_{\tilde x_n}$ corresponds two the relevant displacements, chosen according to *(3.2). By going to subsequences we can assume that $|t_n-s|$ is decreasing and that (α_n) is convergent. Let ℓ be chosen such that $h(\ell)<1/m$, where $m\in\mathbb{N}$ will be determined later. Let β_n be the rotation that is needed to rotate $D^{\alpha_{n+1}}(0)$ into $D^{\alpha_n}(0)$: $R_{\beta_n}D^{\alpha_{n+1}}(0) = D^{\alpha_n}(0)$. Choose $\delta>0$ according to the uniform continuity of $(z_t)_{t\in B\oplus C_r} = (y_t/f(|t-s|))_{t\in B\oplus C_r}$ such that $|z_{s_2}-z_{s_1}|<\epsilon/4$ if $|s_2-s_1|<\delta$. Furthermore, δ should be chosen so small that $\delta<\ell/2$. Choose $\tilde x > x_0$ such that $\delta + K/\tilde x < \ell$. Now choose n such that $|t_n-t_{n+1}|<\delta/2$, such that $|R_{\beta_n}u-u|<\delta/2$ for all $u\in B\oplus C_r$, and such that $K/x_n + |t_n-t_{n+1}| < K/\tilde x$.

Recall that $D^{\alpha_n}(t_n)$ can be parametrised by $\{R^{\alpha_n}t+t_n \;:\; t\in D\}$ and that similarly, $D^{\alpha_{n+1}}(t_{n+1})$ is parametrised by $\{R_{\alpha_{n+1}}t+t_{n+1} \;:\; t\in D\}$. Choose $D_{\tilde x}\subseteq D$ such that $D^{\alpha_n}(t_n)\cap H_{\tilde x} = \{R^{\alpha_n}t+t_n \;:\; t\in D_{\tilde x}\}$. By the definition of t_n we have that

$$\sup_{t\in\tilde D^{\alpha_n}\cap H_{\tilde x}}\frac{x_n-y_t}{f(|t-s|)}-\frac{x_n}{f(r)}\le a \tag{3.6}$$

Let furthermore $\tilde D_{\tilde x}^{\alpha_{n+1}}$ be the rotation centred in s of $D^{\alpha_n}(t_n)\cap H_{\tilde x}$ with an angle β_n: $\tilde D_{\tilde x}^{\alpha_{n+1}} = R_{\beta_n}(D^{\alpha_n}(t_n)\cap H_{\tilde x}-s)+s$. Now $\tilde D_{\tilde x}^{\alpha_{n+1}}$ has the form $\{R_{\alpha_{n+1}}t+\tilde t \;:\; t\in D_{\tilde x}\}$ for some $\tilde t$; in fact $\tilde t = R_{\beta_n}(t_n-s)+s$, but that will not be important in the following. Since for $t\in D_{\tilde x}$ each $R_{\alpha_{n+1}}t+\tilde t\in\tilde D_{\tilde x}^{\alpha_{n+1}}$ is the rotation around s of $R_{\alpha_n}t+t_n\in D^{\alpha_n}(t_n)\cap H_{\tilde x}$, we have that the distance to s is unchanged. Since furthermore, $|R_{\alpha_{n+1}}t+\tilde t-(R_{\alpha_n}t+t_n)|<\delta$ for $t\in D_{\tilde x}$ because of the choice of β_n, the inequality (3.6) now leads to

$$\sup_{t\in\tilde D_{\tilde x}^{\alpha_{n+1}}}\frac{x_n-y_t}{f(|t-s|)}-\frac{x_n}{f(r)}\le a+\epsilon/4\,,$$

which can be re–parametrised as

$$\sup_{t\in D_{\tilde x}} x_n\left(\frac{1}{f(|R_{\alpha_{n+1}}t+\tilde t-s|)}-\frac{1}{f(r)}\right)-z_{R_{\alpha_{n+1}}t+\tilde t}\le a+\epsilon/4\,. \tag{3.7}$$

Define in the same way $D_{\tilde{x}}^{\alpha_{n+1}}(t_{n+1}) = \{R_{\alpha_{n+1}}t + t_{n+1} \ : \ t \in D_{\tilde{x}}\}$ as a reduced version of $D^{\alpha_{n+1}}(t_{n+1})$. By the definition of t_{n+1} we have similarly

$$\sup_{t\in D_{\tilde{x}}} x_n \left(\frac{1}{f(|R_{\alpha_{n+1}}t + t_{n+1} - s|)} - \frac{1}{f(r)} \right) - z_{R_{\alpha_{n+1}}t+t_{n+1}} \leq a\,,$$

and by the uniform continuity of (z_t) and the small distance between t_{n+1} and $\tilde{t}$ we have

$$\sup_{t\in D_{\tilde{x}}} x_n \Big(\frac{1}{f(|R_{\alpha_{n+1}}t + t_{n+1} - s|)} - \frac{1}{f(r)} \Big) - z_{R_{\alpha_{n+1}}t+\tilde{t}} \leq a + \epsilon/4\,. \tag{3.8}$$

Note that $D_{\tilde{x}}^{\alpha_{n+1}}(t_{n+1})$ is a translation of $\tilde{D}_{\tilde{x}}^{\alpha_{n+1}}$. We shall parametrise all the intermediate translations by

$$D_{u,\tilde{x}} = \{R_{\alpha_{n+1}}t + \gamma(u) \ : \ t \in D_{\tilde{x}}\}$$

for $u \in [0,1]$. Here $\gamma(u) = \tilde{t} + u(t_{n+1} - \tilde{t})$ is a linear parametrisation of the line segment from $\tilde{t}$ to t_{n+1}. Note that $D_{0,\tilde{x}} = \tilde{D}_{\tilde{x}}^{\alpha_{n+1}}$ and $D_{1,\tilde{x}} = D_{\tilde{x}}^{\alpha_{n+1}}(t_{n+1})$. Now define $x(u) = \frac{K}{1-u+C}$ for $u \in [0,1]$, where $C, K > 0$ are chosen such that $x(0) = x_n$ and $x(1) = x_{n+1}$, see Lemma A.1 in the Appendix. Suppose we can show that

$$\sup_{t\in D_{\tilde{x}}} x(u) \left(\frac{1}{f(|R_{\alpha_{n+1}}t + \gamma(u)) - s|)} - \frac{1}{f(r)} \right) - z_{R_{\alpha_{n+1}}t+\tilde{t}} \leq a + \epsilon/2 \tag{3.9}$$

for all $u \in [0,1]$. Then choosing u such that $x(u) = \tilde{x}_n$ and defining $\tilde{t}_n = \gamma(u)$ gives the inequality

$$\sup_{t\in D_{\tilde{x}}} \tilde{x}_n \left(\frac{1}{f(|R_{\alpha_{n+1}}t + \tilde{t}_n) - s|)} - \frac{1}{f(r)} \right) - z_{R_{\alpha_{n+1}}t+\tilde{t}} \leq a + \epsilon/2\,.$$

Using the uniform continuity of (z_t) again together with a reparametrisation gives

$$\sup_{t\in D_{u,\tilde{x}}} \frac{\tilde{x}_n - y_t}{f(|t-s|)} - \frac{\tilde{x}_n}{f(r)} \leq a + 3\epsilon/4\,.$$

Note that $D^{\alpha_{n+1}}(\tilde{t}_n) \cap H_{\tilde{x}_n} \subseteq D_{u,\tilde{x}}$ due to the choices of $\tilde{x}$ and $x_n < \tilde{x}_n$. In combination with (3.4) this gives the desired contradiction to (3.5).

Thus the proof will be complete, if we can show (3.9). First, we observe that the cases $u = 0$ and $u = 1$ follows from (3.7) and (3.8). The result for a general $u \in (0,1)$ will follow, if we for any given $t \in D_{\tilde{x}}$ and all $u \in [0,1]$ can show that

$$x(u)F(u) \leq a + \tilde{z} + \epsilon/2\,, \tag{3.10}$$

where

$$F(u) = \frac{1}{f(|\tilde{\gamma}(u) - s|)} - \frac{1}{f(r)}\,,$$

$\tilde{z} = z_{R_{\alpha_{n+1}}t+\tilde{t}}$ and $\tilde{\gamma}(u) = R_{\alpha_{n+1}}t + \gamma(u)$. For ease of notation, t is suppressed. To obtain this, we will use that for all t such that $r \leq |t-s| \leq r + \ell$ it holds that

$$(b - 1/m)(|t-s| - r) \leq \left(\frac{1}{f(|t-s|)} - \frac{1}{f(r)} \right) \leq (b + 1/m)(|t-s| - r)\,, \tag{3.11}$$

and for $r - \ell \leq |t-s| \leq r$ it holds that

$$(b+1/m)(|t-s|-r) \leq \left(\frac{1}{f(|t-s|)} - \frac{1}{f(r)}\right) \leq (b-1/m)(|t-s|-r)\,, \qquad (3.12)$$

where we have applied (3.3) and that $h(\ell) < 1/m$. Note that the assumptions above give that $\big||\tilde{\gamma}(u)-s|-r\big| < \ell$ for all $u \in [0,1]$. Furthermore, note that $F(u) > 0$ if and only if $|\tilde{\gamma}(u)-s|-r > 0$. We shall consider the cases (i): $F(0), F(1) > 0$, (ii): $F(0), F(1) < 0$, (iii): $F(0) < 0, F(1) > 0$, (iv): $F(0) > 0, F(1) < 0$ separately.

In the case (i) we find using (3.11) that

$$(b-1/m)x(u)(|\tilde{\gamma}(u)-s|-r) \leq a + \tilde{z} + \epsilon/4 \qquad (3.13)$$

for $u = 0, 1$. Now let $G(u)$ be the linear interpolation such that $G(0) = (|\tilde{\gamma}(0)-s|-r)$ and $G(1) = (|\tilde{\gamma}(1)-s|-r)$. Then, since $(b-1/m)x(u)G(u) \leq a+\tilde{z}+\epsilon/2$ for $u = 0, 1$, and since $u \mapsto x(u)G(u)$ is monotone, Lemma A.1 in the Appendix gives that the above inequality is satisfied for all $u \in [0,1]$. Since furthermore, $u \mapsto |\tilde{\gamma}(u)-s|$ is seen to be convex, we have that (3.13) is satisfied for all $u \in [0,1]$. Thus also

$$(b+1/m)x(u)(|\tilde{\gamma}(u)-s|-r) \leq (a+\tilde{z}+\epsilon/4)\frac{b+1/m}{b-1/m}$$

holds for all u. Another reference to (3.11) then gives that

$$x(u)F(u) \leq (a+\tilde{z}+\epsilon/4)\frac{b+1/m}{b-1/m}\,. \qquad (3.14)$$

Now consider the case (ii). Since $F(u) < 0$ if both $F(0) < 0$ and $F(1) < 0$, the property (3.10) is trivially satisfied, if $a+\tilde{z}+\epsilon/2 \geq 0$. So assume that $a+\tilde{z}+\epsilon/2 < 0$. Then we find similarly using (3.12) that

$$x(u)F(u) \leq (a+\tilde{z}+\epsilon/4)\frac{b-1/m}{b+1/m} \qquad (3.15)$$

The case case (iii) is trivially satisfied, since $u \mapsto F(u)$ is increasing. For the case (vi), it is only of interest to show that $x(u)F(u) \leq a + \tilde{z} + \epsilon/4$ for all $u \in [0, u_0]$, where $F(u_0) = 0$. To obtain this, the technique from (i) can be repeated, since here $x(u)F(u) \leq a + \tilde{z} + \epsilon/4$ for $u = 0, u_0$.

Now the desired inequality (3.10) can be obtained from (3.14) and (3.15) by letting $m \to \infty$. Note that this can be done uniformly in t, since the field (z_t) is bounded. □

The following lemma describes λ_s for a particularly simple set D^α:

Lemma 3.2. *If $D^\alpha(t) = \{t+\alpha r, t-\alpha r\}$ for all $t \in B$ and $\alpha \in S^{d-1}$, then*

$$\lambda_s((y_t)_{t\in B\oplus C_r}) = \sup_{\alpha\in S^{d-1}} \frac{1}{2f(r)}(y_{s+\alpha r} + y_{s-\alpha r})$$

Proof. Define $u_{s,\alpha} = s + r\alpha$ for $\alpha \in S^{d-1}$ and $u_{s,t,\gamma,\alpha} = s + t\gamma + r\alpha$ for $t \geq 0$ and $\gamma \in S^{d-1}$. The latter parametrises points on the boundary of a ball with radius r and center in $s + t\gamma$. Note that $u_{s,0,\gamma,\alpha} = u_{s,\alpha}$ and that $\lim_{t\to 0} u_{s,t,\gamma,\alpha} = u_{s,\gamma,\alpha}$. Furthermore, $|u_{t\,\gamma,\alpha} - s| = |t\gamma + r\alpha| = \sqrt{t^2 + r^2 + 2tr\cos\angle(\alpha,\gamma)}$, where $\angle(\alpha,\gamma)$ denotes the angle between α and γ. In the one dimensional case, where $d = 1$, we e.g. have $\angle(1,-1) = \pi$. From differentiability of f in r we can write

$$\begin{aligned}
&\left|\frac{1}{t}\left(\frac{1}{f(|u_{s,t,\gamma,\alpha} - s|)} - \frac{1}{f(r)}\right) - \frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma)\right| \\
&\quad = \left|\frac{1}{t}\left(\frac{-f'(r)}{f(r)^2}\big(|t\gamma + r\alpha| - r\big)\right.\right. \\
&\qquad\qquad \left.\left. + \phi(|t\gamma + r\alpha| - r)\big(|t\gamma + r\alpha| - r\big)\right) - \frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma)\right|,
\end{aligned}$$

where ϕ is continuous with $\phi(0) = 0$. Using a second order Taylor approximation around 0 of $t \mapsto \sqrt{t^2 + r^2 + 2tr\cos\angle(\alpha,\gamma)}$ it is seen that $\big(|t\gamma + r\alpha| - r\big)/t$ converges to $\cos\angle(\alpha,\gamma)$ uniformly in α,γ as $t \to 0$. Thus for all $s \in B$

$$\sup_{\gamma,\alpha}\left|\frac{1}{t}\left(\frac{1}{f(|u_{s,t,\gamma,\alpha} - s|)} - \frac{1}{f(r)}\right) - \frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma)\right| \to 0$$

as $t \to 0$. Since $y_{u_{s,t,\gamma,\alpha}} \to y_{u_{s,\alpha}}$ uniformly in $\alpha,\gamma \in S^{d-1}$ due to uniform continuity of the (y_t)–field, we find that if (t_x) is a sequence decreasing to 0 such that $xt_x \to C$ as $x \to \infty$, then

$$\sup_{\gamma,\alpha}\left|\frac{x - y_{u_{s,t_x,\gamma,\alpha}}}{f(|u_{s,t_x,\gamma,\alpha} - s|)} - \frac{x}{f(r)} - \left(C\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma) - \frac{y_{u_{s,\alpha}}}{f(r)}\right)\right| \to 0$$

as $x \to \infty$. From this we find

$$\begin{aligned}
&\sup_{\gamma,\alpha}\left|\max_{t\in D^\alpha(s+\gamma t)}\left(\frac{x - y_t}{f(|t - s|)} - \frac{x}{f(r)}\right)\right. \\
&\qquad \left. - \max_{t\in D^\alpha(s)}\left(C\frac{-f'(r)}{f(r)^2}\cos\angle(t - s,\gamma) - \frac{y_t}{f(r)}\right)\right| \to 0
\end{aligned}$$

as $x \to \infty$. Next we claim that for all $\alpha,\gamma \in S^{d-1}$ and $C \geq 0$

$$\begin{aligned}
&\max_{t\in D^\alpha(s)}\left(C\frac{-f'(r)}{f(r)^2}\cos\angle(t - s,\gamma) - \frac{y_t}{f(r)}\right) \\
&\quad = \max\left\{C\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma) - \frac{y_{s+r\alpha}}{f(r)}, -C\frac{-f'(r)}{f(r)^2}\cos\angle(\alpha,\gamma) - \frac{y_{s-r\alpha}}{f(r)}\right\} \\
&\quad \geq \sup_{\alpha\in S^{d-1}}\frac{1}{2f(r)}(y_{s+r\alpha} + y_{s-r\alpha}),
\end{aligned}$$

with equality if

$$\alpha_0 = \operatorname*{argmax}_{\alpha\in S^{d-1}}\left\{\frac{1}{2f(r)}(y_{s+r\alpha} + y_{s-r\alpha}) \;:\; y_{s+r\alpha} \geq y_{s-r\alpha}\right\},$$

and furthermore $\gamma_0 = \alpha_0$ and $C_0 = f(r)/(-2f'(r))(y_{s+r\alpha} - y_{s-r\alpha})$. For the proposed choice of α_0, γ_0, C_0 it is easily seen that

$$C_0 \frac{-f'(r)}{f(r)^2} \cos\angle(\alpha_0, \gamma_0) - \frac{y_{s+r\alpha_0}}{f(r)} = -C_0 \frac{-f'(r)}{f(r)^2} \cos\angle(\alpha_0, \gamma_0) - \frac{y_{s-r\alpha_0}}{f(r)},$$

and that the common value equals the desired lower bound. It is furthermore seen that any other choice of α, γ, C can only increase one of the two terms above.

Now let (α_n) and (γ_n) be sequences in S^{d-1}, let (t_n) be a sequence of positive numbers, and let (x_n) be a sequence increasing to infinity. Then the results above show that

$$\liminf_{n\to\infty} \max_{t\in D_n^\alpha(s+\gamma_n t_n)} \left(\frac{x - y_t}{f(|t-s|)} - \frac{x}{f(r)} \right) \geq \sup_{\alpha\in S^{d-1}} \frac{1}{2f(r)} (y_{s+r\alpha} + y_{s-r\alpha})$$

and that there is equality if $\alpha_n = \alpha_0$, $\gamma_n = \gamma_0$ and $x_n t_n \to C_0$ with α_0, γ_0, C_0 as proposed above. Combined with Lemma 3.1 this gives the desired result. $\square$

Lemma 3.3. *Let $n \in \mathbb{N}$ and assume for each $i = 1, \ldots, n$ that $(y_t^i)_{t\in B\oplus C_r}$ has the form*

$$y_t^i = z^i f(|t - s^i|) \qquad \text{for all } t \in B \oplus C_r,$$

where all $z^i \geq 0$ and $s^i \in \mathbb{R}^d$. Let g be as defined in (2.10). *Define for $s \in \mathbb{R}^d$*

$$\varphi(s) = f(r) 1_{B\oplus C_r}(s) + 1_{(B\oplus C_r)^c}(s) \sup_{t\in B} g(|t-s|). \tag{3.16}$$

Then it holds that

$$\sup_{s\in B} \lambda_s \Big(\Big(\sum_{i=1}^n y_t^i \Big)_{t\in B\oplus C_r} \Big) \leq \frac{1}{f(r)} \sum_{i=1}^n z^i \varphi(s^i)$$

and

$$\sup_{t_0\in B} \sup_{\alpha\in S^{d-1}} \inf_{t\in D^\alpha(t_0)} \sum_{i=1}^n y_t^i \leq \sum_{i=1}^n z^i \varphi(s^i).$$

Proof. Assume $s^i \in B \oplus C_r$. For each $\alpha \in S^{d-1}$ and $s \in B$ we find that if $\min\{|s + r\alpha - s^i|, |s - r\alpha - s^i|\} = r - \delta$ for some $\delta > 0$, then $\max\{|s + r\alpha - s^i|, |s - r\alpha - s^i|\} \geq r + \delta$. Using the assumption (2.10) then gives

$$\frac{1}{2}(y_{s+r\alpha}^i + y_{s-r\alpha}^i) \leq \frac{z^i}{2}(g(r-\delta) + g(r+\delta)) = z^i f(r) = z^i \varphi(s^i).$$

This inequality is clearly also satisfied, if both $|s + r\alpha - s^i| \geq r$ and $|s - r\alpha - s^i| \geq r$. If $s^i \in (B \oplus C_r)^c$ then for all choices of $s \in B$ and $\alpha \in S^{d-1}$ it holds that

$$\frac{1}{2}(y_{s+r\alpha}^i + y_{s-r\alpha}^i) \leq \frac{z^i}{2}\big(g(|s + r\alpha - s^i|) + g(|s - r\alpha - s^i|)\big) \leq z^i \varphi(s^i).$$

Recalling that $\{s+r\alpha, s-r\alpha\} \subseteq D^\alpha(s)$ combined with Lemma 3.2, it is now seen that for each $s \in B$

$$\begin{aligned}\lambda_s\Big(\Big(\sum_{i=1}^n y_t^i\Big)_{t\in B\oplus C_r}\Big) &\le \sup_{\alpha\in S^{d-1}} \frac{1}{2f(r)}\Big(\sum_{i=1}^n y^i_{s+\alpha r} + \sum_{i=1}^n y^i_{s-\alpha r}\Big)\\ &\le \sum_{i=1}^n \frac{1}{2f(r)} \sup_{\alpha\in S^{d-1}} \big(y^i_{s+\alpha r} + y^i_{s-\alpha r}\big) \le \frac{1}{f(r)}\sum_{i=1}^n z^i\varphi(s^i)\,.\end{aligned}$$

Taking supremum over $s \in B$ gives the first statement. For the second statement, we similarly find for each $t_0 \in B$ and $\alpha \in S^{d-1}$ that

$$\begin{aligned}\inf_{t\in D^\alpha(t_0)} \sum_{i=1}^n y_t^i &\le \min\Big\{\sum_{i=1}^n y^i_{t_0+r\alpha}, \sum_{i=1}^n y^i_{t_0-r\alpha}\Big\}\\ &\le \frac{1}{2}\Big(\sum_{i=1}^n y^i_{t_0+\alpha r} + \sum_{i=1}^n y^i_{t_0-\alpha r}\Big) \le \sum_{i=1}^n z^i\varphi(s^i)\,.\end{aligned}$$

The result follows by taking supremum over $t_0 \in B$ and $\alpha \in S^{d-1}$. □

4 The main theorem

In this section, we will derive the main result that is Theorem 4.7 below. For $x > 0$ we define the following set

$$\Lambda(x) = \{(y_t)_{t\in B\oplus C_r} \ :\ \sup_{t_0\in B}\sup_{\alpha\in S^{d-1}}\inf_{t\in D^\alpha(t_0)} y_t > x\}\,.$$

Note that for a random field $(Y_t)_{t\in B\oplus C_r}$ with excursion set $A_x = \{t \in B\oplus C_r \ :\ Y_t > x\}$ we have

$$P((Y_t)_{t\in B\oplus C_r} \in \Lambda(x)) = P(\text{there exists } t \in B, \alpha \in S^{d-1} \ :\ D^\alpha(t) \subseteq A_x)\,.$$

The first step will be determining the asymptotic behaviour of excursion sets for a field U with distribution ν_1.

Theorem 4.1. *Assume that $(U_t)_{t\in B\oplus C_r}$ has distribution ν_1 and let $(y_t)_{t\in B\oplus C_r}$ be continuous. Then*

$$\frac{P\big((U_t+y_t)_{t\in B\oplus C_r} \in \Lambda(x)\big)}{L(x/f(r))\exp(-\beta x/f(r))} \to \frac{1}{\nu(A)}\int_B \exp(\beta\lambda_s((y_t)_{t\in B\oplus C_r}))\,\mathrm{d}s \qquad \text{as } x\to\infty\,. \tag{4.1}$$

Furthermore,

$$\frac{P\big((U_t)_{t\in B\oplus C_r} \in \Lambda(x)\big)}{L(x/f(r))\exp(-\beta x/f(r))} \to \frac{1}{\nu(A)} m_d(B) \qquad \text{as } x\to\infty\,, \tag{4.2}$$

and

$$\frac{P\big((U_t+y_t)_{t\in B\oplus C_r} \in \Lambda(x)\big)}{P\big((U_t)_{t\in B\oplus C_r} \in \Lambda(x)\big)} \to \frac{\int_B \exp(\beta\lambda_s((y_t)_{t\in B\oplus C_r}))\,\mathrm{d}s}{m_d(B)} \qquad \text{as } x\to\infty\,. \tag{4.3}$$

Proof. The results (4.2) and (4.3) are direct consequences of (4.1), so we focus on the proof of (4.1). We can assume that $(y_t)_{t\in B\oplus C_r}$ is non–negative: Simply write $x = x' - x_0$ for a suitable x_0 such that $(x_0 + y_t)_{t\in B\oplus C_r}$ is non–negative, and find the limit of

$$\frac{P\big((U_t + x_0 + y_t)_{t\in B\oplus C_r} \in \Lambda(x')\big)}{L(x'/f(r))\exp(-\beta x'/f(r))}$$

as $x' \to \infty$. We find

$$\begin{aligned}
&P\big((U_t + y_t)_{t\in B\oplus C_r} \in \Lambda(x)\big)\\
&\quad= \frac{1}{\nu(A)} F(\{(s,z)\in\mathbb{R}^d\times\mathbb{R} \,:\, \sup_{t_0\in B}\sup_{\alpha\in S^{d-1}}\inf_{t\in D^\alpha(t_0)} zf(|t-s|) + y_t > x\})\\
&\quad= \frac{1}{\nu(A)} F\Big(\Big\{(s,z)\in\mathbb{R}^d\times\mathbb{R} \,:\, z > \inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)} \frac{x-y_t}{f(|t-s|)}\Big\}\Big)\\
&\quad= \frac{1}{\nu(A)} \int_B L\Big(\inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)} \frac{x-y_t}{f(|t-s|)}\Big) \exp\Big(-\beta \inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)} \frac{x-y_t}{f(|t-s|)}\Big)\,\mathrm{d}s\\
&\qquad+ \frac{1}{\nu(A)} \int_{\mathbb{R}^d\setminus B} L\Big(\inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)} \frac{x-y_t}{f(|t-s|)}\Big) \exp\Big(-\beta \inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)} \frac{x-y_t}{f(|t-s|)}\Big)\,\mathrm{d}s\,.
\end{aligned} \tag{4.4}$$

First, we show that the second term in (4.4) is $o\big(L(x/f(r))\exp(-\beta x/f(r))\big)$. Let $y^* = \sup_{s\in B\oplus C_r} y_s$. Utilising the fact that $L(x)\exp(-\beta x)$ is decreasing the second term is for $x > y^*$

$$\le \frac{1}{\nu(A)} \int_{\mathbb{R}^d\setminus B} L\Big(\frac{x-y^*}{f_0(s)}\Big) \exp\Big(-\beta\frac{x-y^*}{f_0(s)}\Big)\,\mathrm{d}s\,, \tag{4.5}$$

where we have introduced the notation $f_0(s) = \sup_{t_0,\alpha}\inf_{t\in D^\alpha(t_0)} f(|t-s|)$. From the arguments similar to the proof of [13, Theorem 3.1] it can be seen that for all $\gamma > 0$ there exists $x_0 > 0$ and $C > 0$ such that

$$\frac{L(ax)}{L(x)} \le Ce^{(a-1)\gamma x} \qquad \text{for all } x \ge x_0, a \ge 1\,. \tag{4.6}$$

Note that $f_0(s) < f(r)$ for all $s \in \mathbb{R}^d \setminus B$ due to convexity of B. Combining this with (2.5), (4.6) and the fact that $L(x)\exp(-\gamma x) \to 0$ for all $\gamma > 0$, gives that the integrand in (4.5) is $o\big(L(x/f(r))\exp(-\beta x/f(r))\big)$. If we denote the integrand of (4.5) by $h(s;x)$, it follows by the dominated convergence theorem that (4.5) is $o(L(x/f(r))\exp(-\beta x/f(r)))$ if we can find an integrable function g such that

$$\frac{h(s;x)}{L(x/f(r))\exp(-\beta x/f(r))} \le g(s)\,, \qquad s\in\mathbb{R}^d\,.$$

Let $0 < \gamma < \beta$. Then, using (4.6) and the boundedness of $L((x-y^*)/f(r))/L(x/f(r))$, we can find a constant $\tilde{C}$ and $x_0 > y^*$ such that for $x \ge x_0$

$$\begin{aligned}
&\frac{h(s;x)}{L(x/f(r))\exp(-\beta x/f(r))}\\
&\qquad\le \tilde{C}\exp(\beta y^*/f(r))\exp\left(-(\beta-\gamma)\left(\frac{1}{f_0(s)} - \frac{1}{f(r)}\right)(x_0 - y^*)\right).
\end{aligned} \tag{4.7}$$

Now, choose $R > 0$ such that $B \oplus C_r \subseteq C_R(0)$ and $\sup_{t\in B} f(|t-s|) < f(r)$ for all $s \notin C_R(0)$. Then, using (2.8), we get for $s \notin C_R(0)$

$$f_0(s) \le \sup_{t\in B} f(|t-s|) \le \sup_{t\in C_R(0)} f(|t-s|) \le \sup_{t\in C_R(0)} \frac{1}{(|t-s|+1)^d} = \frac{1}{(|s|-R+1)^d}\,.$$

It follows that the function (4.7) is integrable.

The theorem now follows by applying dominated convergence to the first term of (4.4). From Lemma 3.1 we have for $s \in B$

$$\frac{L\Big(\inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)}\frac{x-y_t}{f(|t-s|)}\Big)\exp\Big(-\beta\inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)}\frac{x-y_t}{f(|t-s|)}\Big)}{L(x/f(r))\exp(-\beta x/f(r))} \to e^{\beta\lambda_s((y_t)_t)}.$$

Using again that $L(x)\exp(-\beta x)$ is decreasing we find for x large

$$\begin{aligned}&\left|\frac{L\Big(\inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)}\frac{x-y_t}{f(|t-s|)}\Big)\exp\Big(-\beta\inf_{t_0,\alpha}\sup_{t\in D^\alpha(t_0)}\frac{x-y_t}{f(|t-s|)}\Big)}{L(x/f(r))\exp(-\beta x/f(r))} - e^{\beta\lambda_s((y_t)_t)}\right|\\ &\qquad\le \frac{L((x-y^*)/f(r))\exp(-\beta(x-y^*)/f(r))}{L(x/f(r))\exp(-\beta x/f(r))} + e^{\beta\lambda_s((y_t)_t)}\\ &\qquad\le (C+1)e^{\beta y^*}\,,\end{aligned}$$

where C is chosen such that $L((x-y^*)/f(r))/L(x/f(r)) \le C$. The result is integrable over B. □

The next step will be to extend the result of Theorem 4.1 to the case $P((U^1+\cdots+U^n+y_t)_t \in \Lambda(x))$, where U^i, $i=1,\ldots,n$, are independent with common distribution ν_1. Recall that each $(U^i_t)_{t\in B\oplus C_r}$ can be represented by $\big(Z^i f(|t-S^i|)\big)_{t\in B\oplus C_r}$, where (Z^i,S^i) has distribution F_1. For this purpose we will need the following lemma and corollary.

Lemma 4.2. *Let (Z,S) be distributed according to F_1. Then,*

$$\frac{P(Z\varphi(S) > x)}{L(x/f(r))\exp(-\beta x/f(r))} \to \frac{m_d(B\oplus C_r)}{\nu(A)}\,.$$

In particular, we have

$$E\exp(\beta/f(r)Z\phi(S)) < \infty\,.$$

Proof. Similar to the proof of Theorem 4.1 we can write

$$\begin{aligned}P(Z\phi(S) > x) &= \frac{1}{\nu(A)}F(\{(s,z)\in\mathbb{R}^d\times\mathbb{R} \,:\, z\varphi(s) > x\})\\ &= \frac{1}{\nu(A)}\int_{B\oplus C_r} L(x/f(r))\exp(-\beta x/f(r))\,\mathrm{d}s\\ &\quad + \frac{1}{\nu(A)}\int_{B\oplus C_r} L\Big(\frac{x}{\sup_{t\in B} g(|t-s|)}\Big)\exp\Big(-\beta\frac{x}{\sup_{t\in B} g(|t-s|)}\Big)\,\mathrm{d}s\,.\end{aligned}$$

The first term equals $L(x/f(r))\exp(-\beta x/f(r))$ times the desired limit. The second term is $o(L(x/f(r))\exp(-\beta x/f(r))$ by a dominated convergence argument, since $\sup_{t\in B} g(|t-s|) < f(r)$ for all $s \in (B\oplus C_r)^c$. The second result follows from [10, Corollary 2.1 (ii)]. □

Corollary 4.3. Let $U^1, U^2, \dots$ be independent and identically distributed with distribution ν_1. For all $n \in \mathbb{N}$ it holds that

$$E \exp\Big(\beta \sup_{s\in B} \lambda_s\big((U_t^1 + \cdots + U_t^n)_{t\in B\oplus C_r}\big)\Big) < \infty$$

Proof. Since each U^i has the form $\big(Z^i f(|t - S^i|)\big)_{t\in B\oplus C_r}$, the result follows from Lemma 3.3 and Lemma 4.2. $\square$

Theorem 4.4. *Let $U^1, U^2, \dots$ be independent and identically distributed with distribution ν_1 and assume that $(y_t)_{t\in B\oplus C_r}$ is continuous. For all $n \in \mathbb{N}$ it holds that*

$$\frac{P\big((U_t^1 + \cdots + U_t^n + y_t)_t \in \Lambda(x)\big)}{P\big((U_t^1)_t \in \Lambda(x)\big)} \to \frac{n}{m_d(B)} \int_B E e^{\beta\lambda_s((U_t^1+\cdots+U_t^{n-1}+y_t)_t)}\,\mathrm{d}s$$

as $x \to \infty$.

Proof. As in the proof of Theorem 4.1 we can assume that $(y_t)_{t\in B\oplus C_r}$ is non–negative. The result is shown by induction over n. For $n = 1$, the result is shown in Theorem 4.1. Assume now that the theorem is correct for some $n \in \mathbb{N}$. Let for convenience $V = U^1 + \cdots + U^n$ and recall the representation $U_t^i = Z^i f(|t - S^i|)$. Then we have

$$\begin{aligned} &P\big((V_t + U_t^{n+1} + y_t)_t \in \Lambda(x)\big) \\ &\quad = P\Big(\sum_{i=1}^n Z^i\varphi(S^i) > x/2, Z^{n+1}\varphi(S^{n+1}) > x/2, (V_t + U_t^{n+1} + y_t)_t \in \Lambda(x)\Big) \\ &\qquad + P\Big(\sum_{i=1}^n Z^i\varphi(S^i) \le x/2\,,\ (V_t + U_t^{n+1} + y_t)_t \in \Lambda(x)\Big) \\ &\qquad + P(Z^{n+1}\varphi(S^{n+1}) \le x/2\,,\ (V_t + U_t^{n+1} + y_t)_t \in \Lambda(x))\,. \end{aligned} \tag{4.8}$$

The first term is bounded from above by

$$P\Big(\sum_{i=1}^n Z^i\varphi(S^i) > x/2\Big) P(Z^{n+1}\varphi(S^{n+1}) > x/2)\,.$$

In Lemma 4.2 it was shown that the distribution of each $Z^i\varphi(S^i)$ is convolution equivalent. Thus both factors are asymptotically equivalent with $\rho_1((x/(2f(r)), \infty))$, and then it follows from the proof of [5, Lemma 2] that the product is $o((\rho_1 * \rho_1)((x/f(r), \infty)))$. In particular, the product above is $o(\rho_1((x/f(r), \infty)))$ due to the convolution equivalence.

The two remaining terms in (4.8) divided by $P((U_t^1)_t \in \Lambda(x))$ can be rewritten as follows

$$\begin{aligned} &\int_{C_x} \frac{P((U_t^{n+1} + \sum_{i=1}^n z^i f(|t - s^i|) + y_t)_t \in \Lambda(x))}{P((U_t^1)_t \in \Lambda(x))}\, F_1^{*\otimes n}(\mathrm{d}(z^1, s^1; \dots; z^n, s^n)) \\ &\qquad + \int_{\tilde{C}_x} \frac{P((V_t + zf(|t - s|) + y_t)_t \in \Lambda(x))}{P((U_t^1)_t \in \Lambda(x))}\, F_1(\mathrm{d}(z, s))\,. \end{aligned} \tag{4.9}$$

Here $F_1^{*\otimes n}$ is the $n-$fold product measure of F_1, and it has been used that $(V_t)_t$ can be represented by $\big(\sum_{i=1}^n Z^i f(|t-S^i|)\big)_t$. Furthermore,

$$\begin{aligned}
C_x &= \big\{(z^1,s^1;\dots;z^n,s^n) \,:\, \sum_{i=1}^n z^i\varphi(s^i) \le x/2\big\} \\
\tilde{C}_x &= \big\{(z,s) \,:\, z\varphi(s) \le x/2\big\}
\end{aligned}$$

Using Theorem 4.1 and the induction assumption, the two integrands of (4.9) times 1_{C_x} and $1_{\tilde{C}_x}$ respectively, converge to, as $x \to \infty$,

$$f_1(z^1,s^1;\dots;z^n,s^n) = \frac{1}{m_d(B)}\int_B e^{\beta\lambda_s((y_t+\sum_{i=1}^n z^i f(|t-s^i|))_{t\in B\oplus C_r})}\,\mathrm{d}s$$

and

$$f_2(z,s) = \frac{n}{m_d(B)}\int_B Ee^{\beta\lambda_s((U_t^1+\dots+U_t^{n-1}+zf(|t-s|)+y_t)_{t\in B\oplus C_r})}\,\mathrm{d}s\,,$$

respectively. We want to show that (4.9) converges to

$$\begin{aligned}
&\int f_1(z^1,s^1;\dots;z^n,s^n)\,F_1^{*\otimes n}(\mathrm{d}(z^1,s^1;\dots;z^n,s^n)) + \int f_2(z,s)\,F_1(\mathrm{d}(z,s)) \\
&\quad = \frac{n+1}{m_d(B)}\int_B Ee^{\beta\lambda_s((U_t^1+\dots+U_t^n+y_t)_t)}\,\mathrm{d}s.
\end{aligned}$$

Using Fatou's lemma, it is enough to find integrable functions $g_1(z^1,s^1;\dots;z^n,s^n;x)$ and $g_2(z,s;x)$ that are upper bounds of the two integrands of (4.9) such that the two limits $g_1(z^1,s^1;\dots;z^n,s^n) = \lim_{x\to\infty} g_1(z^1,s^1;\dots;z^n,s^n;x)$ and $g_2(z,s) = \lim_{x\to\infty} g_2(z,s;x)$ exist with

$$\int_{C_x} g_1(z^1,s^1;\dots;z^n,s^n;x)\,F_1^{*\otimes n}(\mathrm{d}(z^1,s^1;\dots;z^n,s^n)) + \int_{\tilde{C}_x} g_2(z,s;x)\,F_1(\mathrm{d}(z,s)) \tag{4.10}$$

converging to the similar integrals with $g_1(z^1,s^1;\dots;z^n,s^n)$ and $g_2(z,s)$. Using Lemma 3.3 we find that as functions $g_1(z;x)$ and $g_2(z^1,\dots,z^n;x)$ we can use

$$g_1(z^1,s^1;\dots;z^n,s^n;x) = \frac{P(Z^1\varphi(S^1) > x - y^* - \sum_{i=1}^n z^i\varphi(s^i))}{P((U_t)_t \in \Gamma(x))}\,,$$

where as previously $y^* = \sup_{t\in B\oplus C_r} y_t$, and

$$g_2(z,s;x) = \frac{P(\sum_{i=1}^n Z^i\varphi(S^i) > x - y^* - z\varphi(s))}{P((U_t)_t \in \Gamma(x))}\,.$$

Noting that $P((U_t)_t \in \Gamma(x)) \sim m_d(B)/m_d(B\oplus C_r)P(Z^1\varphi(S^1) > x)$ due to Theorem 4.1 and Lemma 4.2, we find that

$$\begin{aligned}
g_1(z^1,s^1;\dots;z^n,s^n;x) &\to g_1(z^1,s^1;\dots;z^n,s^n) \\
&= \frac{m_d(B\oplus C_r)}{m_d(B)}e^{\beta/f(r)(y^*+\sum_{k=1}^n z^i\varphi(s^i))}\,,
\end{aligned}$$

and since the distribution of $\sum_{i=1}^n Z^i\varphi(S^i)$ is convolution equivalent, [6, Corollary 2.11] gives

$$
\begin{aligned}
g_2(z,s;x) &\to g_2(z,s) \\
&= \frac{m_d(B\oplus C_r)}{m_d(B)} n\cdot e^{\beta/f(r)(y^*+z\varphi(s))}\big(Ee^{\beta/f(r)Z^1\varphi(S^1)}\big)^{n-1}.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\int g_1(z^1,s^1;\dots;z^n,s^n)\,F_1^{*\otimes n}(\mathrm{d}(z^1,s^1;\dots;z^n,s^n)) + \int g_2(z,s)\,F_1(\mathrm{d}(z,s)) \\
&\quad = \frac{m_d(B\oplus C_r)}{m_d(B)}(n+1)\cdot e^{\beta/f(r)y^*}\big(Ee^{\beta/f(r)Z^1\varphi(S^1)}\big)^n. \qquad (4.11)
\end{aligned}
$$

Since the tails of $\sum_{i=1}^n Z^i\varphi(S^i)$ and $Z^1\varphi(S^1)$ in particular are exponential with index $\beta/f(r)$, we have according to [5, Lemma 2] that (4.10) is asymptotically equal to

$$
e^{\beta/f(r)y^*}\frac{P\big(\sum_{i=1}^{n+1} Z^i\varphi(S^i)>x\big)}{P(Z^1\varphi(S^1)>x)}
$$

which, by another reference to [6, Corollary 2.11], is seen to converge to (4.11). □

For a dominated convergence argument, we need the lemma below.

Lemma 4.5. *Let $U^1, U^2, \dots$ be independent and identically distributed with distribution ν_1, and assume that (Z,S) has distribution F_1. There exists a constant K such that for all $n\in\mathbb{N}$ and all $x\geq 0$*

$$
P((U_t^1+\cdots+U_t^n)_t\in\Lambda(x)) \leq K^n P(Z\varphi(S)>x).
$$

Proof. Since $Z\varphi(S)$ has a convolution equivalent tail according to Corollary 4.1 it follows from [6, Lemma 2.8] that there exists K such that

$$
P\Big(\sum_{i=1}^n Z^i\varphi(S^i)>x\Big) \leq K^n P(Z\varphi(S)>x).
$$

The result now follows directly from Lemma 3.3. □

Recall that we can write the field $(X_t)_{t\in T}$ as

$$
X_t = X_t^1 + X_t^2,
$$

where the field X^1 is obtained from the fields $U^1, U^2, \dots$ and an independent Poisson distributed variable N with parameter $\nu(A)$ by

$$
X_t^1 = \sum_{n=1}^N U_t^n.
$$

Theorem 4.6. *For each $s\in B$ we have $E\exp\big(\beta\lambda_s((X_t^1)_{t\in B\oplus C_r})\big)<\infty$ and for a continuous field, $(y_t)_{t\in B\oplus C_r}$*

$$
\lim_{x\to\infty}\frac{P((X_t^1+y_t)_t\in\Lambda(x))}{L(x/f(r))\exp(-\beta x/f(r))} = \int_B E\big(e^{\beta\lambda_s((X_t^1+y_t)_{t\in B\oplus C_r})}\big)\,\mathrm{d}s.
$$

Proof. The first result follows from $\lambda_s((X_t^1)_{t\in B\oplus C_r}) \le \frac{1}{f(r)}\sum_{n=0}^N Z^i\varphi(S^i)$ and from $E\exp(\beta/f(r)Z^1\varphi(S^1))$ being finite. For the proof of the limit result, we use that

$$P((X_t^1+y_t)_t\in\Lambda(x)) = e^{-\nu(A)}\sum_{n=1}^\infty \frac{\nu(A)^n}{n!}P((U_t^1+\cdots+U_t^n+y_t)_t\in\Lambda(x))\,.$$

Utilising Lemma 4.5 and the notation $y^*=\sup_{t\in B\oplus C_r} y_t$, we find

$$\begin{aligned}
&\sum_{n=1}^\infty \frac{\nu(A)^n}{n!}\frac{P((U_t^1+\cdots+U_t^n+y_t)_t\in\Lambda(x))}{P(Z\varphi(S)>x-y^*)}\\
&\quad\le \sum_{n=1}^\infty \frac{\nu(A)^n}{n!}\frac{P((U_t^1+\cdots+U_t^n)_t\in\Lambda(x-y^*))}{P(Z\varphi(S)>x-y^*)}\\
&\quad\le \sum_{n=1}^\infty \frac{K^n\nu(A)^n}{n!}\frac{P(Z\varphi(S)>x-y^*)}{P(Z\varphi(S)>x-y^*)} = \sum_{n=1}^\infty \frac{K^n\nu(A)^n}{n!}<\infty\,,
\end{aligned}$$

and furthermore, we obtain from Lemma 4.2 and Theorem 4.4 that

$$\begin{aligned}
&\lim_{x\to\infty}\frac{P((U_t^1+\cdots+U_t^n+y_t)_t\in\Lambda(x))}{P(Z\varphi(S)>x-y^*)}\\
&\quad= \frac{n}{e^{\beta/f(r)y^*}m_d(B\oplus C_r)}\int_B Ee^{\beta\lambda_s((U_t^1+\cdots+U_t^{n-1}+y_t)_t)}\,\mathrm{d}s\,.
\end{aligned}$$

with the convention that $U_t^1+\cdots+U_t^{n-1}=0$ if $n=1$. Then, dominated convergence gives

$$\begin{aligned}
&\lim_{x\to\infty}\frac{P((X_t^1+y_t)_t\in\Lambda(x))}{P(Z\varphi(S)>x-y^*)}\\
&\quad= \frac{e^{-\nu(A)}}{e^{\beta/f(r)y^*}m_d(B\oplus C_r)}\sum_{n=1}^\infty\frac{\nu(A)^n}{n!}n\int_B Ee^{\beta\lambda_s((U_t^1+\cdots+U_t^{n-1}+y_t)_t)}\,\mathrm{d}s\\
&\quad= \frac{\nu(A)}{e^{\beta/f(r)y^*}m_d(B\oplus C_r)}\sum_{n=0}^\infty e^{-\nu(A)}\frac{\nu(A)^n}{n!}\int_B Ee^{\beta\lambda_s((U_t^1+\cdots+U_t^{n}+y_t)_t)}\,\mathrm{d}s\\
&\quad= \frac{\nu(A)}{e^{\beta/f(r)y^*}m_d(B\oplus C_r)}\int_B E\big(e^{\beta\lambda_s((U_t^1+\cdots+U_t^N+y_t)_t)}\big)\\
&\quad= \frac{\nu(A)}{e^{\beta/f(r)y^*}m_d(B\oplus C_r)}\int_B E\big(e^{\beta\lambda_s((X_t^1+y_t)_t)}\big)
\end{aligned}$$

which with a final reference to Theorem 4.1 and Lemma 4.2 concludes the proof. □

The theorem below is the main result of our paper. In the formulation of the theorem, we explicitly state the assumptions under which the limit holds.

Theorem 4.7. *Under the assumptions* (2.1)–(2.6) *on* M *and* (2.7)–(2.10) *on* f, *then it holds that* $E\exp\big(\beta\lambda_{t_0}((X_t)_{t\in B\oplus C_r}\big)<\infty$ *and*

$$\lim_{x\to\infty}\frac{P(\sup_{t_0\in B}\sup_{\alpha\in S^{d-1}}\inf_{t\in D^\alpha(t_0)}X_t>x)}{L(x/f(r))\exp(-\beta x/f(r))} = E\exp\big(\beta\lambda_{t_0}((X_t)_{t\in B\oplus C_r}\big)m_d(B)$$

as $x\to\infty$ *with* $t_0\in B$ *arbitrarily chosen, and where* λ_{t_0} *is as defined in Lemma 3.1.*

Proof. First we note that $E \exp(\gamma \sup_{t \in B \oplus C_r} X_t^2) < \infty$ for all $\gamma > 0$ according to [13, Lemma 4.1]. Since furthermore

$$\lambda_{t_0}((X_t)_t) \leq \lambda_{t_0}((X_t^1 + \sup_t X_t^2)_t) = \lambda_{t_0}((X_t^1)_t) + \sup_t X_t^2 / f(r)$$

due to Lemma 3.1, the first statement follows from the first statement in Theorem 4.6. Let π be the distribution of $(X_t^2)_{t \in B \oplus C_r}$. We find that

$$\frac{P((X_t)_t \in \Lambda(x))}{P((X_t^1)_t \in \Lambda(x))} = \int \frac{P((X_t^1 + y_t)_t \in \Lambda(x))}{P((X_t^1)_t \in \Lambda(x))} \, \pi(\mathrm{d}y) = \int f(y;x) \, \pi(\mathrm{d}y) \, ,$$

with

$$f(y;x) = \frac{P((X_t^1 + y_t)_t \in \Lambda(x))}{P((X_t^1)_t \in \Lambda(x))} \, .$$

From Theorem 4.6 it is seen that

$$f(y;x) \to f(y) = \frac{\int_B E\big(e^{\beta \lambda_s((X_t^1 + y_t)_{t \in B \oplus C_r})}\big) \, \mathrm{d}s}{\int_B E\big(e^{\beta \lambda_s((X_t^1)_{t \in B \oplus C_r})}\big) \, \mathrm{d}s} \qquad \text{as } x \to \infty \, .$$

If we can show that

$$\int f(y;x) \, \pi(\mathrm{d}y) \to \int f(y) \, \pi(\mathrm{d}y) \tag{4.12}$$

as $x \to \infty$, then the theorem follows with another reference to Theorem 4.6 and by recalling that $(X_t)_{t \in B \oplus C_r}$ is stationary. According to Fatou's lemma, (4.12) follows if we can find integrable non-negative functions $g(y;x)$ and $g(y)$ such that

$$f(y;x) \leq g(y;x) \, , \tag{4.13}$$

$$g(y;x) \to g(y) \, , \tag{4.14}$$

$$\int g(y;x) \, \pi(\mathrm{d}y) \to \int g(y) \, \pi(\mathrm{d}y) \, . \tag{4.15}$$

For this purpose, let

$$g(y;x) = \frac{P((X_t^1 + \sup_t y_t)_t \in \Lambda(x))}{P((X_t^1)_t \in \Lambda(x))} \, .$$

Then, (4.13) is satisfied. Furthermore, using Theorem 4.6 and Lemma 3.1, we find that (4.14) is fulfilled with $g(y) = e^{\beta / f(r) \sup_t y_t}$. To prove (4.15), we have that

$$\int g(y;x) \, \pi_2(\mathrm{d}y) = \frac{P(\sup_{t_0,\alpha} \inf_{t \in D^\alpha(t_0)} X_t^1 + \sup_t X_t^2 > x)}{P(\sup_{t_0,\alpha} \inf_{t \in D^\alpha(t_0)} X_t^1 > x)} \, .$$

Note that $\sup_{t_0,\alpha} \inf_{t \in D^\alpha(t_0)} X_t^1$ has a convolution equivalent tail according to Theorem 4.6 and [10, Lemma 2.4 (i)]. Since $E \exp(\gamma \sup_t X_t^2) < \infty$ for all $\gamma > 0$ we have from [10, Lemma 2.1] and [10, Lemma 2.4 (ii)] that

$$\begin{aligned} &\lim_{x \to \infty} \frac{P(\sup_{t_0,\alpha} \inf_{t \in D^\alpha(t_0)} X_t^1 + \sup_t X_t^2 > x)}{P(\sup_{t_0,\alpha} \inf_{t \in D^\alpha(t_0)} X_t^1 > x)} \\ &\quad = E\big(\exp(\beta / f(r) \sup_t X_t^2)\big) = \int g(y) \, \pi(\mathrm{d}y) \, . \end{aligned}$$

It follows that (4.15) is fulfilled. $\square$

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A Appendix

The following simple lemma will be used in Lemma 3.1:

Lemma A.1. *Let $0 < x_n < x_{n+1}$ be given. Then there exists constants $C, D > 0$ such that $x : [0,1] \to [0,\infty)$ defined by*

$$x(u) = \frac{C}{1 - u + D} \tag{A.1}$$

is strictly increasing with $x(0) = x_n$ and $x(1) = x_{n+1}$. Furthermore, if $g(u) = au + b$, then $u \mapsto x(u)g(u)$ is monotone on $[0,1]$.

Proof. Any function on the form (A.1) is clearly strictly increasing on $[0,1]$. The constants C, D are found by straightforward manipulations. The last result is obtained by differentiating $u \mapsto x(u)g(u)$. $\square$

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