Minkowski Tensors

Stereological Estimation, Reconstruction and Stability Results

PhD Thesis

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Preface

This thesis presents research results obtained through my PhD studies at Department of Mathematics, Aarhus University, from 2012 to 2016. My studies were carried out under supervision of Markus Kiderlen and were partly financed by Centre of Stochastic Geometry and Advanced Bioimaging (CSGB) funded by a grant from the Villum Foundation.

The thesis consists of an introductory chapter followed by four self-contained papers:

Paper A A. Kousholt, M. Kiderlen and D. Hug, Surface tensor estimation from linear sections, *Math. Nachr.* 228(14-15), 1647–1672, 2015

Paper B A. Kousholt and M. Kiderlen, Reconstruction of convex bodies from surface tensors, *Adv. Appl. Math.*, 76, 1–33, 2016

Paper C A. Kousholt, Reconstruction of n-dimensional convex bodies from surface tensors, Accepted for publication in *Adv. Appl. Math.*, 2016

Paper D J. Hörrmann and A. Kousholt, Reconstruction of convex bodies from moments, Submitted, 2016.

The introductory chapter of the thesis provides background material and relates the four papers to each other and to existing literature within the field. In the introductory chapter, results from Papers A–D are referred to as follows: Theorem 3.1 in Paper A is referred to as Theorem A.3.1, Section 2 in Paper B is referred to as Section B.2, etc.

As main author, I contributed comprehensively to Papers A–C, both in the research and the writing phase. My work on Paper D started during a stay as visiting PhD student at Ruhr-Universität Bochum in May – June 2015. I had the pleasure of collaborating with Julia Hörrmann, who at the beginning of my visit had already established a lot of the results that constitute Paper D. My main contributions to Paper D are the results on reconstruction from noisy measurements.

The results of Papers A–D have been presented at the following conferences and workshops:

- 8th Internal CSGB Workshop, Tisvildeleje, Denmark, May 2014
- Workshop on Tensor Valuations in Stochastic Geometry and Imaging, Sandbjerg, Denmark, September 2014
- 18th Workshop on Stochastic Geometry, Stereology and Image Analysis, Lingen, Germany, March 2015

- Geometry and Physics of Spatial Random Systems, Bad Herrenalb, Germany, September 2015
- AU Workshop on Stochastic Geometry, Stereology and their Applications, Sandbjerg, Denmark, June 2016.

Halfway through my PhD studies, I obtained a master's degree in statistics. Paper A (except Sections A.4.2 and A.5) was included in my qualifying examination progress report. Early versions of the reconstruction algorithm and the stability results described in Paper B were also included.

My four years as a PhD student at Aarhus University have brought many exciting and challenging experiences. I owe huge thanks to a lot of people who have inspired and supported me on the way.

First and foremost, I would like to thank my supervisor Markus Kiderlen for his careful guidance and for many, many hours of educational and inspiring discussions. I also want to thank Eva B. Vedel Jensen for providing an excellent working environment at CSGB.

Moreover, I want to thank my co-authors Daniel Hug and Julia Hörrmann. Special thanks go to Julia Hörrmann for being an attentive host during my stay in Bochum. I also thank her colleagues at the Department of Mathematics, Ruhr-Universität Bochum for their hospitality that made my stay very pleasant.

I want to thank my friends and colleagues at Department of Mathematics, Aarhus University who have made my daily life very enjoyable. In particular, I thank my present and former office mates Ina Trolle Andersen, Sabrina Tang Christensen and Anne Marie Svane for lovely companionship during the last years.

Finally, I thank my family for their never-ending encouragement and support.

Astrid Kousholt
Aarhus, August 2016

Summary

Minkowski tensors are tensor-valued valuations that generalize notions like surface area and volume. Recently, Minkowski tensors have been established as robust and versatile descriptors of shape of spatial structures in applied sciences, see [5, 42, 43]. In this thesis, different aspects of Minkowski tensors of convex bodies are investigated.

From Crofton's formula for Minkowski tensors we derive stereological estimators of translation invariant surface tensors of convex bodies. The estimators are based on one-dimensional linear sections. In a design-based setting, we suggest three types of estimators. These are based on isotropic uniform random lines, non-isotropic random lines and vertical sections, respectively. In a model-based setting, we derive estimators of the specific surface tensors associated with a stationary process of convex particles.

We investigate how much information about a convex body a finite number of surface tensors contain. We show that the shape of a convex body is uniquely determined by a finite number of surface tensors if and only if the convex body is a polytope with nonempty interior. Further, stability results for surface tensors and harmonic intrinsic volumes are derived, and reconstruction algorithms that approximate convex bodies by polytopes are developed. The algorithms are based on a finite number of exact surface tensors or a finite number of possibly noisy measurements of harmonic intrinsic volumes. Using the derived stability results, consistency of the algorithms is established. In the case of noisy measurements, appropriate assumptions on the variance of the noise variables are required to obtain consistency. The algorithms are implemented and their feasibility is illustrated by examples.

As for surface tensors, we investigate how much information about a convex body can be retrieved from a finite number of its geometric moments, equivalently of its volume tensors. We give a sufficient condition for a convex body to be uniquely determined by a finite number of its geometric moments, and we show that among all convex bodies, those which are uniquely determined by a finite number of moments form a dense set. Further, we derive a stability result for convex bodies based on geometric moments. The stability result is improved considerably by using another set of moments, namely Legendre moments. We present a reconstruction algorithm that approximates a convex body from a finite number of possibly noisy Legendre moments. Consistency of the algorithm is established using the derived stability result for Legendre moments. Again, appropriate assumptions on the variance of the noise variables are required when the Legendre moments are disrupted by noise.

Resumé

Minkowski-tensorer er valuationer med værdier i rummet af symmetriske tensorer over det n-dimensionale euklidiske rum. De generaliserer begreber som overfladeareal og volumen og er med succes blevet brugt inden for det naturvidenskabelige område til at beskrive geometrien af rumlige strukturer, [5, 42, 43]. Formålet med denne afhandling er at belyse og undersøge forskellige aspekter af Minkowski-tensorer af konvekse legemer.

Ved at benytte Croftons formel for Minkowski-tensorer udledes stereologiske estimatorer for overfladetensorer. Vi beskriver tre typer designbaserede estimatorer, som bygger på henholdsvis isotropiske uniforme stokastiske linjer, stokastiske linjer med foretrukne retninger samt vertikale snit. Endvidere udledes en modelbaseret estimator for specifikke overfladetensorer af en stationær partikelproces med konvekse partikler.

Herefter undersøges det, hvor megen information endeligt mange overfladetensorer af et konvekst legeme indeholder om legemet. Specielt vises det, at et konvekst legemes form er entydigt bestemt af endeligt mange overfladetensorer, hvis og kun hvis legemet er en polytop med indre punkter. Endvidere udledes stabilitetsresultater for overfladetensorer og harmoniske indre volumener, og der udvikles algoritmer, der approksimerer et ukendt konvekst legeme med polytoper. Inputtet af algoritmerne er enten endeligt mange overfladetensorer af det ukendte legeme eller målinger (muligvis behæftet med støj) af endeligt mange harmoniske indre volumener af legemet. Ved at benytte de udledte stabilitetsresultater sikres det, at algoritmerne er konsistente. Når målingerne er behæftet med støj, kræves det, at variansen er kontrolleret på passende vis.

Tilsvarende undersøges det, hvor megen information endeligt mange geometriske momenter af et konvekst legeme indeholder om legemet. Dette er ækvivalent til at undersøge informationsmængden i endeligt mange volumentensorer. Vi giver en betingelse, der sikrer, at et konvekst legeme er entydigt bestemt af endeligt mange geometriske momenter og viser yderligere, at konvekse legemer, der er entydigt bestemt af endeligt mange momenter, udgør en tæt delmængde i rummet af konvekse legemer. Vi udleder stabilitetsresultater for geometriske momenter og forbedrer resultaterne betydeligt ved at erstatte geometriske momenter med Legendre-momenter. Endeligt udvikles en rekonstruktionsalgoritme baseret på endeligt mange Legendre-momenter. Algoritmen approksimerer et ukendt konvekst legeme med en polytop og tillader, at Legendre-momenterne er behæftet med støj. Det vises, at algoritmerne er konsistente ved at benytte stabilitetsresultatet for Legendre-momenter. Igen kræves det, at variansen på støjvariablene er passende kontrolleret for at sikre, at algoritmen er konsistent.

Introduction

The geometry of physical and biological spatial structures often reflects properties of the material under consideration. Therefore, understanding of the geometry can give insight into the physical and biological behaviour of the material. Essential properties of the geometry are summarized by scalar-valued size measurements such as volume, surface area and Euler characteristic. However, quantifying more complex geometric information such as shape, orientation, elongation and anisotropy requires descriptors of a different nature. A recently introduced set of tensor-valued geometric descriptors, namely Minkowski tensors, have received considerable attention. Minkowski tensors have with success been used to quantify shape information in materials science, see [6, 42, 43] and the references given there. Applications in biosciences have also appeared, see [5]. From a theoretical point of view, Minkowski tensors are also interesting. This is demonstrated by Alesker's characterization theorem stating that Minkowski tensors multiplied with powers of the metric tensor span the vector space of isometry covariant, continuous, tensor-valued valuations on the set of convex bodies, see [2].

The aim of the present thesis is to investigate different aspects of Minkowski tensors. In particular, we study the connection between a convex body and its Minkowski tensors. We develop estimation procedures for Minkowski tensors in a stereological setting based on information of the underlying convex body, and conversely, we explore what information about a convex body can be retrieved from a finite number of its Minkowski tensors.

This chapter serves as an introduction to Papers A–D that constitute the thesis. The research questions treated in the papers are presented together with the obtained results. Further, the results are related to each other and to the existing literature within the field. The first section provides background material and introduces the main notions of the thesis. As general references on convex, stochastic and integral geometry, we refer to the comprehensive monographs [38] and [41].

1 Background

We work in the n-dimensional Euclidean vector space $\mathbb{R}^n$ equipped with its usual inner product $\langle \cdot , \cdot \rangle$ and norm $\|\cdot\|$. The volume and the surface area of the unit ball B^n in $\mathbb{R}^n$ are denoted by κ_n and ω_n, respectively. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. Further, we let λ denote the Lebesgue measure on $\mathbb{R}^n$, and for $k = 0, \dots, n$, we let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure.

Slightly different variants of the definition of a **convex body** can be found in the literature. In [38], a convex body is a nonempty, compact, convex subset of $\mathbb{R}^n$. In [9], a convex body is further required to have nonempty interior. To ease notation, it is convenient to use a certain definition depending on the application in mind, and therefore, the definition of a convex body varies between Papers A–D. In Paper A, a convex body is a convex and compact subset of $\mathbb{R}^n$. In Papers B and C, we use the definition from [38], and in Paper D we follow [9]. In this introductory chapter, we use the definition from Paper A. However, when the results from Papers B–D are described, a convex body is defined as in the relevant paper.

The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$ and is closed under Minkowski addition,

$$K + L = \{x + y \mid x \in K, y \in L\}, \qquad K, L \in \mathcal{K}^n,$$

and scalar multiplication,

$$\alpha K = \{\alpha x \mid x \in K\}, \qquad \alpha \in \mathbb{R}, K \in \mathcal{K}^n.$$

A nonempty convex body K is uniquely determined by its **support function** h_K given by

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle$$

for $u \in S^{n-1}$. The set of nonempty convex bodies $\mathcal{K}^n \setminus \{\emptyset\}$ is equipped with the **Hausdorff metric** δ that can be expressed in terms of support functions,

$$\begin{aligned}\delta(K, L) &= \min\{\alpha \geq 0 \mid K \subseteq L + \alpha B^n, L \subseteq K + \alpha B^n\} \\ &= \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|\end{aligned}$$

for $K, L \in \mathcal{K}^n$. In addition to the Hausdorff metric, we use the L^2-metric on $\mathcal{K}^n \setminus \{\emptyset\}$. The L^2-distance between K and L is defined as the L^2-distance between their support functions,

$$\delta_2(K, L) = \|h_K - h_L\|_2.$$

The Hausdorff metric and the L^2-metric are equivalent.

For an abelian group G, a function $\phi\colon \mathcal{K}^n \to G$ is called a **valuation** (or additive) if $\phi(\emptyset) = 0$ and

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L) \tag{1}$$

whenever $K, L, K \cup L \in \mathcal{K}^n$.

1.1 Intrinsic volumes

Intrinsic volumes are important geometric characteristics of convex bodies. For a nonempty convex body K, the intrinsic volumes of K can, for instance, be defined using the Steiner formula that states that the volume of the parallel set $K + \epsilon B^n$ of K is a polynomial in $\epsilon > 0$,

$$\lambda(K + \epsilon B^n) = \sum_{j=0}^{n} \epsilon^{n-j} \kappa_{n-j} V_j(K). \tag{2}$$

The **intrinsic volumes** $V_0(K), \ldots, V_n(K)$ of K are defined by the coefficients in the polynomial on the right hand side of (2). The definition is extended letting $V_j(\emptyset) = 0$ for $j = 0, \ldots, n$. The Steiner formula (2) was first proved for polytopes and sufficiently smooth surfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ by Steiner in [44].

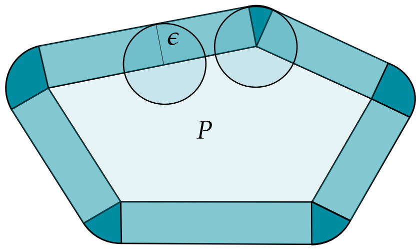

Figure 1: Parallel set $P + \epsilon B^2$ of polytope $P \subseteq \mathbb{R}^2$ with $\epsilon > 0$. By the Steiner formula, the area of the parallel set is $\pi V_0(P)\epsilon^2 + 2V_1(P)\epsilon + V_2(P)$.

The intrinsic volumes $V_0, \dots, V_n$ summarize geometric features: V_n is the volume, $2V_{n-1}$ is the surface area (on the set of convex bodies with nonempty interior), V_1 is proportional to the mean width, and V_0 is the Euler characteristic (the Euler characteristic is trivial on $\mathcal{K}^n$, but its extension to polyconvex sets (finite unions of convex sets) is an interesting geometric characteristic).

The term 'intrinsic volumes' was first used by McMullen in [29] and can be explained by the fact that the intrinsic volumes of a convex body K depend only on K and not on the dimension of the ambient space. In particular, the mth intrinsic volume $V_m(K)$ is the m-dimensional volume of K if K is contained in an affine subspace $E \subseteq \mathbb{R}^n$ of dimension m. In the literature, with different normalizations, the intrinsic volumes are also known as 'quermaßintegrals' and 'Minkowski functionals'.

Via the Steiner formula, the intrinsic volumes inherit several properties from the volume functional. The intrinsic volumes are continuous, translation and rotation invariant, real-valued valuations. According to Hadwiger's characterization theorem, intrinsic volumes are essentially the only functionals on $\mathcal{K}^n$ with these properties as $V_0, \dots, V_n$ constitute a basis for the vector space of continuous, translation and rotation invariant, real-valued valuations on $\mathcal{K}^n$, see, e.g., [41].

Due to their properties and to Hadwiger's characterization theorem, the intrinsic volumes are natural size descriptors that provide essential and complete information about invariant geometric features of convex bodies. However, the important functional properties of intrinsic volumes also set a limit to their use as descriptors. For instance, due to rotation invariance, intrinsic volumes do not capture the orientation of convex bodies. In order to describe more complex shape information of convex bodies, the scalar-valued intrinsic volumes are extended to a set of tensor-valued descriptors, namely Minkowski tensors.

1.2 Minkowski tensors

Let $\mathbb{T}^p$ be the vector space of symmetric tensors of rank p over $\mathbb{R}^n$, i.e. the space of symmetric multilinear functions of p variables in $\mathbb{R}^n$. Due to multilinearity, a tensor $T \in \mathbb{T}^p$ can be identified with the array

$$\{T(e_{i_1}, \dots, e_{i_p})\}_{i_1,\dots,i_p=1}^{n}$$

of components of T, where $(e_1,\dots,e_n)$ is the standard basis of $\mathbb{R}^n$. For tensors $T_1 \in \mathbb{T}^{p_1}$ and $T_2 \in \mathbb{T}^{p_2}$, the symmetric tensor product $T_1T_2 \in \mathbb{T}^{p_1+p_2}$ of T_1 and T_2 is defined by

$$\begin{aligned}&T_1T_2(x_1,\dots,x_{p_1+p_2})\\ &\quad= \frac{1}{(p_1+p_2)!}\sum_{\sigma\in S_{p_1+p_2}} T_1(x_{\sigma(1)},\dots,x_{\sigma(p_1)})\,T_2(x_{\sigma(p_1+1)},\dots,x_{\sigma(p_1+p_2)})\end{aligned}$$

for $x_1,\dots,x_{p_1+p_2} \in \mathbb{R}^n$, where $S_{p_1+p_2}$ is the symmetric group on $\{1,\dots,p_1+p_2\}$. Identifying $\mathbb{R}^n$ with its dual via the inner product, we write $x^p \in \mathbb{T}^p$ for the p-fold symmetric tensor product of $x \in \mathbb{R}^n$. The tensor x^p can then be identified with the array

$$\{x_{i_1}\cdots x_{i_p}\}_{i_1,\dots,i_p=1}^{n}.$$

A tensor-valued function $\phi\colon \mathcal{K}^n \to \mathbb{T}^p$ is called **isometry covariant** if the following two conditions hold:

1. $\phi(\rho(K))(x_1,\dots,x_p) = \phi(K)(\rho^{-1}x_1,\dots,\rho^{-1}x_p)$ for every ρ in the orthogonal group $\mathcal{O}(n)$, $K \in \mathcal{K}^n$ and $x_1,\dots,x_p \in \mathbb{R}^n$,
2. ϕ exhibits polynomial behaviour under translations, i.e.

$$\phi(K+t) = \sum_{j=0}^{p} \phi_{p-j}(K)\frac{t^j}{j!}$$

for $K \in \mathcal{K}^n$, $t \in \mathbb{R}^n$ and suitable functions $\phi_{p-j}\colon \mathcal{K}^n \to \mathbb{T}^{p-j}$.

The mathematical investigation of Minkowski tensors was initiated by McMullen in [30] and continued, among others, by Alesker in [1, 2] and Schneider in [36]. Minkowski tensors are defined using support measures (also called generalized curvature measures) that arise from a local version of the Steiner formula (2). Let $p(K,\cdot)\colon \mathbb{R}^n \to K$ be the metric projection on a nonempty convex body K, i.e. $p(K,x)$ is the unique nearest point of $x \in \mathbb{R}^n$ in K. Further, define

$$u(K,x) = \frac{x-p(K,x)}{\|x-p(K,x)\|}$$

for $x \notin K$. The unit vector $u(K,x)$ is an outer normal of K at the boundary point $p(K,x)$. For $\epsilon > 0$ and a Borel set $A \subseteq \mathbb{R}^n \times S^{n-1}$, the volume of the local parallel set

$$M_\epsilon(K,A) = \{x \in (K+\epsilon B^n)\setminus K \mid (p(K,x),u(K,x)) \in A\}$$

of K is a polynomial in ϵ of degree at most $n-1$, so

$$\lambda(M_\epsilon(K,A)) = \sum_{j=0}^{n-1} \epsilon^{n-j}\kappa_{n-j}\Lambda_j(K,A). \tag{3}$$

The coefficients of the polynomial on the right-hand side of this local version of the Steiner formula define the **support measures** $\Lambda_0(K,\cdot),\dots,\Lambda_{n-1}(K,\cdot)$ of K. Further, we define $\Lambda_j(\emptyset,\cdot) = 0$ for $j = 0,\dots,n-1$.

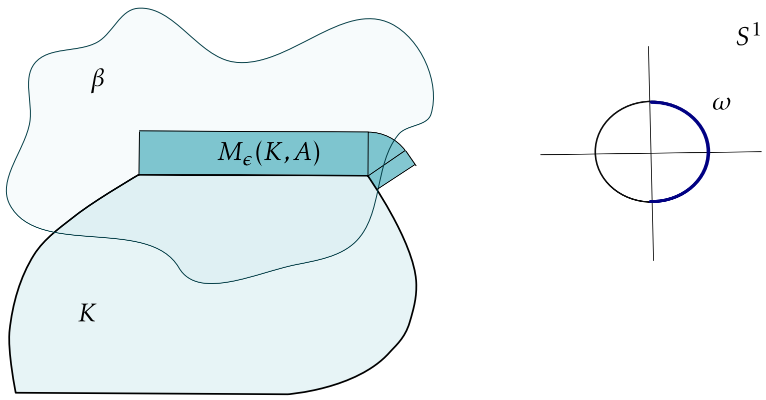

Figure 2: Local parallel set $M_\epsilon(K,A)$ of convex body K at $A = \beta \times \omega$ with Borel sets $\beta \subseteq \mathbb{R}^2$ and $\omega \subseteq S^1$.

Letting $A = \mathbb{R}^n \times S^{n-1}$ and comparing (2) and (3), we obtain the intrinsic volumes as the total mass of the support measures,

$$V_j(K) = \Lambda_j(K, \mathbb{R}^n \times S^{n-1}) \tag{4}$$

for $K \in \mathcal{K}^n$ and $j = 0, \dots, n-1$.

For a convex polytope $P \subseteq \mathbb{R}^n$, the support measures can be given an intuitive interpretation as they can be expressed in the form

$$\Lambda_j(P,A) = \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} \int_F \int_{N(P,F)\cap S^{n-1}} 1_A(x,u)\, \mathcal{H}^{n-j-1}(du)\, \mathcal{H}^j(dx) \tag{5}$$

for a Borel set $A \in \mathbb{R}^n \times S^{n-1}$, where $\mathcal{F}_j(P)$ is the set of j-faces of P, and $N(P,F)$ is the normal cone of K at P (consisting of all outer normals of K at P).

The support measures were introduced by Schneider in [35] and are generalizations of curvature and area measures. The **area measures** $S_0(K,\cdot), \dots, S_{n-1}(K,\cdot)$ of a convex body K are Borel measures on the unit sphere defined as rescaled versions of the projection of the support measures on their second component. More precisely,

$$S_j(K,\omega) = n\kappa_{n-j} \binom{n}{j}^{-1} \Lambda_j(K, \mathbb{R}^n \times \omega) \tag{6}$$

for a Borel set $\omega \subseteq S^{n-1}$ and $K \in \mathcal{K}^n$. Top order area measures $S_{n-1}(K,\cdot)$, $K \in \mathcal{K}^n$ are called **surface area measures** and play a prominent role in this thesis. For a convex polytope $P \subseteq \mathbb{R}^n$ with $m \geq n+1$ facets, we immediately obtain from (5) and (6) that

$$S_{n-1}(P,\cdot) = \sum_{j=1}^m \alpha_j \delta_{u_j}, \tag{7}$$

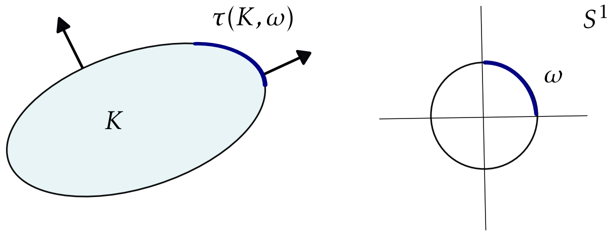

Figure 3: The reverse spherical image $\tau(K,\omega)$ of the convex body K at $\omega \subseteq S^1$.

where δ_u denotes the Dirac measure at $u \in S^{n-1}$, $u_1,\dots,u_m \in S^{n-1}$ are facet normals of P, and $\alpha_1,\dots\alpha_m > 0$ are the corresponding $(n-1)$-dimensional volumes of the facets. The simple structure of surface area measures of polytopes is very important when deriving uniqueness results and reconstruction algorithms in Papers B and C. For an arbitrary convex body K with nonempty interior, we have

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}(\tau(K,\omega))$$

where $\tau(K,\omega)$ is the reverse spherical image of K at ω (the set of boundary points of K with an outer normal in ω), see Figure 3.

Using the support measures, we can now introduce the main notion of this thesis. For a convex body $K \in \mathcal{K}^n$ and $r,s \in \mathbb{N}_0$, the **Minkowski tensors** of K are defined as

$$\Phi_j^{r,s}(K) = \frac{\omega_{n-j}}{r!\,s!\,\omega_{n-j+s}} \int_{\mathbb{R}^n\times S^{n-1}} x^r u^s \,\Lambda_j(K,d(x,u)) \tag{8}$$

for $j = 0,\dots,n-1$, and

$$\Phi_n^{r,0}(K) = \frac{1}{r!}\int_K x^r \lambda(dx). \tag{9}$$

For other choices of r,s and j, we define $\Phi_j^{r,s} = 0$. In Paper A, we use the notation $\Phi_{j,r,s} = \Phi_j^{r,s}$. For $j = n-1$ and $r = 0$, the tensors (8) are called **surface tensors**. The tensors (9) are called **volume tensors**. From (4) we obtain that $\Phi_j^{0,0} = V_j$ for $j = 0,\dots,n$, so the Minkowski tensors are extensions of the intrinsic volumes.

Example 1. In this example, we describe surface tensors of polytopes and calculate the surface tensors up to rank 2 of a planar rectangle. Surface tensors of polytopes have a simple structure due to the simple structure of surface area measures of polytopes. Let $P \subseteq \mathbb{R}^2$ be a convex polytope with $m \geq n+1$ facets. Let $u_1,\dots,u_m \in S^{n-1}$ be the facet normals of P, and let $\alpha_1,\dots,\alpha_m > 0$ be the corresponding $(n-1)$-dimensional volumes. Then we obtain from (6), (7) and (8) that

$$\Phi_{n-1}^{0,s}(P) = \frac{1}{s!\,\omega_{s+1}}\int_{S^{n-1}} u^s S_{n-1}(P,du) = \frac{1}{s!\,\omega_{s+1}}\sum_{j=1}^m \alpha_j u_j^s \tag{10}$$

for $s \in \mathbb{N}_0$. Now, let P be a rectangle in $\mathbb{R}^2$ with facet normals $\pm e_1, \pm e_2$ and corresponding facet lengths $\alpha,\beta > 0$. Then we obtain from (10) that

$$\Phi_1^{0,0}(P) = \alpha+\beta, \quad \Phi_1^{0,1}(P) = 0 \quad \text{and} \quad \Phi_1^{0,2}(P) = \frac{1}{4\pi}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},$$

where we have identified the surface tensors with their arrays of components.

The tensor-valued functional $\Phi_j^{r,s}$ is an isometry covariant, continuous valuation on $\mathcal{K}^n$. As an analogue to Hadwiger's characterization theorem, Alesker's characterization theorem states that every isometry covariant, continuous, tensor-valued valuation on $\mathcal{K}^n$ is a linear combination of so-called basic tensor valuations $Q^m\Phi_j^{r,s}$ with $m, r, s \in \mathbb{N}_0$ and either $j \in \{0, \dots, n-1\}$ or $(j, s) = (n, 0)$, see [2], where $Q \in \mathbb{T}^2$ is the metric tensor defined as $Q(x, y) = \langle x, y \rangle$ for $x, y \in \mathbb{R}^n$. In contrast to intrinsic volumes, the basic tensor valuations are not linearly independent. They are related by the McMullen relations,

$$Q \sum_{s \in \mathbb{N}_0} \Phi_{j-r+s}^{r-s,s-2} = 2\pi \sum_{s \in \mathbb{N}_0} s\Phi_{j-r+s}^{r-s,s} \tag{11}$$

for $r \in \mathbb{N}$ with $r \geq 2$ and $j \in \{0, \dots, n+r-2\}$, see, e.g., [38]. These relations are essentially the only linear dependencies among the valuations, see [21].

Minkowski tensors can be additively extended to polyconvex sets. However, in Papers A–D, we solely focus on Minkowski tensors of convex bodies. More precisely, we focus on two subfamilies of Minkowski tensors. Papers A-C deal with the translation invariant Minkowski tensors $\Phi_j^{0,s}$, $j \in \{0, 1, n-1\}$, $s \in \mathbb{N}_0$ derived from area measures. Particular attention is paid to the family of surface tensors. Paper D treats (a scaled version of) volume tensors.

1.3 Minkowski tensors as shape descriptors

In materials science and also to some extent in biosciences, Minkowski tensors have been established as a useful tool to characterize geometric features of spatial objects. In addition to Alesker's characterization theorem and the fact that Minkowski tensors are extensions of intrinsic volumes, the importance of Minkowski tensors as shape descriptors is hinted at by their close relation to prominent descriptors like the centre of mass,

$$\frac{\Phi_n^{1,0}(K)}{V_n(K)},$$

and the tensor of inertia,

$$2(\Phi_n^{2,0}(K) - \mathrm{Tr}(\Phi_n^{2,0}(K))Q),$$

where $\mathrm{Tr}(\Phi_n^{2,0}(K))$ is the trace of $\Phi_n^{2,0}(K)$ when considered as an $n \times n$ matrix, see [6].

In [24], the volume tensor of rank 2 of a convex body K is used to construct an ellipsoidal approximation of K. The orientation and elongation of K are summarized by the ellipsoid. If K is an ellipsoid centred at the origin, then the approximating ellipsoid is K itself. In a similar way, in [47], the mean particle volume tensor of rank 2 of a particle process is used to construct an ellipsoid, called the Miles ellipsoid, summarizing shape and elongation of the typical particle.

In the following, we mention a few examples from applied sciences, where Minkowski tensors successfully have been used as shape descriptors. In materials science, Minkowski tensors have been established as robust measures of anisotropy that can be applied to a broad range of spatial structures such as porous and granular material, see [42, 43]. In the listed papers, the ratios of the largest and the smallest eigenvalues of Minkowski tensors of rank 2 (when considered as matrices) are used to quantify anisotropy. In astrophysics, Minkowski tensors have been used to describe morphology of galaxies in [6]. An example from bioscience is [5], where

Minkowski tensors are used to describe and distinguish shapes of different types of two-dimensional neuronal cell networks.

The mentioned examples exclusively use lower rank Minkowski tensors ($r+s \leq 2$). The higher rank tensors have apparently not yet been explored in applied sciences.

2 Stereological estimation of surface tensors

In order to use Minkowski tensors as geometric descriptors, the tensors must be accessible, but in applications, an object under consideration may only be observable through linear sections, via a digital image or the like. Therefore, estimators of Minkowski tensors based on partial information of the underlying object have to be derived. Several estimation procedures based on digital images have already been developed, see [42, 43, 45, 17]. If a full digital image of the object is not available, these estimation procedures are not applicable. If the object, however, is observable through sections, then stereological estimators can be used instead.

Stereology is a subfield of stochastic geometry and spatial statistics. For general references, see, e.g., [4, 23, 25]. Stereology deals with estimation of characteristics of an object from a geometric sample of the object. A sample can, for instance, be the intersection of the object with a test set (a hyperplane, a full-dimensional set, a lattice, etc.) or the projection of the object onto a linear subspace.

Methods from stereology are widely used in microscopy. When a three-dimensional object is observed in a microscope, the microscope image is a two-dimensional sample of the object. Using stereological methods, characteristics like volume and surface area of the three-dimensional object can be estimated from measurements in the two-dimensional sample.

Statistical inference in stereology can be formulated in a model-based or design-based setting. In a model-based setting, the object of interest is considered random, while the test set is deterministic. The object of interest is assumed to be 'spatially homogeneous' such that the sample becomes representative for the entire structure. In a design-based setting, the object of interest is considered deterministic, and the randomness enters the design by randomized sampling.

In [24], Jensen et al. derive stereological estimators for certain Minkowski tensors ($j \in \{1,\dots,n-1\}, r,s \in \{0,1\}$ and $j = n, s = 0, r \in \mathbb{N}_0$) of convex bodies in $\mathbb{R}^n$. The estimators are formulated in a design-based setting and are based on random linear sections containing a fixed reference point. Using a combination of a design- and model-based approach, stereological methods are used in [47] and [34] to estimate mean volume tensors of a particle process of compact particles in $\mathbb{R}^3$, see, e.g., [41] for details on particle processes. The so-called optical rotator design is used, so the particles only need to be observable in a central thick slice. In the given applications in [47] and [34], it is assumed that the distribution of the typical particle of the process is invariant under rotations that fix a prescribed axis (called the vertical axis). Under this assumption, an estimator that only requires measurements in the central plane of a thick slice is developed in [26]. A simulation study indicates that the estimator in [26] is superior to the estimators described in [47] and [34], even though it is simpler and therefore easier to implement in microscopy.

2.1 From integral geometric formulae to stereological applications

In order to establish unbiased stereological estimators, appropriate integral geometric formulae are required. In [4], it is explained how integral geometric formulae are used to derive estimators in various designs, see also [25]. To illustrate, we give a simple example where an unbiased estimator of V_{n-1} is obtained from a classical integral geometric formula, namely the Crofton formula for intrinsic volumes.

In the following, we let $\mathcal{E}_k^n$ denote the set of k-dimensional affine subspaces of $\mathbb{R}^n$, and let μ_k^n denote the corresponding motion invariant measure normalized as in [41]. The Crofton formula for intrinsic volumes states that

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, \mu_k^n(dE) = c_{nkj}\, V_{n-k+j}(K) \tag{12}$$

for $K \in \mathcal{K}^n$, $k = 0, \ldots, n-1$, $j \leq k$ and a known constant $c_{nkj} \in \mathbb{R}$. When considered as a function of K, the Crofton integral on the left-hand side of (12) is a continuous, rotation and translation invariant, real-valued valuation. Therefore, it follows from Hadwiger's characterization theorem that the integral is a linear combination of intrinsic volumes, and since the integral is homogeneous of degree $n-k+j$, it is then proportional to $V_{n-k+j}(K)$. The constant c_{nkj} is determined by inserting $K = B^n$.

Now, let $K \in \mathcal{K}^n$. We assume that K is contained in a known compact reference set $A \subseteq \mathbb{R}^n$, but otherwise K is considered unknown. Let E be an **isotropic uniform random line** hitting the reference set A, i.e. the distribution of the random line E is given by

$$\mathbb{P}(E \in \mathcal{A}) = c(A) \int_{\mathcal{A}} \mathbf{1}(A \cap E' \neq \emptyset)\, \mu_1^n(dE')$$

for a Borel set $\mathcal{A} \subseteq \mathcal{E}_1^n$, where $c(A)$ is a normalizing constant. Since $K \subseteq A$, we obtain from (12) with $k = 1$ and $j = 0$ that

$$\mathbb{E}\, V_0(K \cap E) = c(A) \int_{\mathcal{E}_1^n} V_0(K \cap E')\, \mu_1^n(dE') = c(A)\, c_{n10}\, V_{n-1}(K),$$

so $(c(A)\, c_{n10})^{-1}\, V_0(K \cap E)$ is an unbiased estimator of $V_{n-1}(K)$. Using this estimator, the intrinsic volume $V_{n-1}(K)$ of K (and then the surface area of K) can be unbiasedly estimated when K is observable through one-dimensional linear sections.

2.2 Integral geometric formulae for Minkowski tensors

The work on establishing integral geometric formulae for Minkowski tensors was initiated by Schneider in [36], where translative kinematic integrals of the form

$$\int_{\mathbb{R}^n} \Phi_j^{r,0}(K \cap (L + x))\, \lambda(dx)$$

were calculated for $K, L \in \mathcal{K}^n$ and $j \in \{n-1, n\}$.

In [39], a complete set of Crofton formulae for Minkowski tensors giving explicit expressions of integrals of the form

$$\int_{\mathcal{E}_k^n} \Phi_j^{r,s}(K \cap E)\, \mu_k^n(dE) \tag{13}$$

were derived in dimension two and three. An application of Alesker's characterization theorem implies that the integral (13) is a linear combination of basic tensor valuations. However, the constants in the combination are very difficult to determine and due to the McMullen relations (11) not unique. The Crofton formulae in [39] are therefore derived by direct calculations.

Finally, in [20], a complete set of Crofton formulae for Minkowski tensors were derived in arbitrary dimension. Further, *intrinsic* versions of the Crofton formulae were derived. These formulae give expressions of integrals of the form (13), where the Minkowski tensor $\Phi_j^{r,s}(K\cap E)$ is replaced by the relative Minkowski tensor $\Phi_{j,r,s}^{(E)}(K\cap E)$ that is calculated with the affine subspace E as ambient space, see [20] or Paper A for details on relative tensors. In contrast to intrinsic volumes, Minkowski tensors are not intrinsic as $\Phi_j^{r,s}$ and $\Phi_{j,r,s}^{(E)}$ differ substantially.

Kinematic and Crofton formulae of translation invariant Minkowski tensors are also studied by Bernig and Hug in [7] via methods from algebraic integral geometry. They simplify the exterior (non-intrinsic) Crofton formulae from [20] and also establish Crofton formulae for the trace-free part of certain Minkowski tensors.

The stereological estimators in [24] are derived from rotational Crofton formulae that express Minkowski tensors as rotational averages. These formulae are established in [3].

2.3 Paper A

In Paper A, we derive stereological estimators of surface tensors based on linear sections. In contrast to [24], we adopt the classical stereological setting where the sectioning space is affine, and further, we restrict to the simplest case where it is one-dimensional (as in the example in Section 2.1). The relevant integral geometric formula is therefore a Crofton formula with lines.

A Crofton formula with lines is established in [20] as a special case of a formula that expresses intrinsic Crofton integrals

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}^{(E)}(K\cap E)\,\mu_k^n(dE)$$

for $K\in\mathcal{K}^n$ and $k,j,r,s\in\mathbb{N}_0$ with $0\le j\le k\le n-1$ as linear combinations of Minkowski tensors of K. Due to the generality treated in [20], the constants in the derived linear combinations are very lengthy and hard to evaluate, even in the special case with a one-dimensional sectioning space. For estimation purposes we need explicit constants, so we give an independent and elementary proof of the Crofton formula in the special case $k=1$ and $j=r=0$ yielding simple expressions for the constants. For $K\in\mathcal{K}^n$ and even rank $s\in\mathbb{N}_0$, the integral

$$\int_{\mathcal{E}_1^n} \Phi_{0,0,s}^{(E)}(K\cap E)\,\mu_1^n(dE) \tag{14}$$

is expressed as a linear combination of surface tensors of even rank at most s of K. For odd rank s, the integral (14) trivially vanishes as $\Phi_{0,0,s}^{(E)}(K\cap E)=0$ for $E\in\mathcal{E}_1^n$, see Theorem A.3.1.

Proceeding as in Section 2.1, the derived Crofton formula can, in principle, be used to construct an unbiased estimator for the linear combination of surface tensors of different ranks. This is, however, not our aim. Instead, for even $s \in \mathbb{N}_0$, we invert the linear system

$$C \begin{pmatrix} \Phi_{n-1}^{0,0}(K) \\ \Phi_{n-1}^{0,2}(K) \\ \vdots \\ \Phi_{n-1}^{0,s}(K) \end{pmatrix} = \begin{pmatrix} \int_{\mathcal{E}_1^n} \Phi_{0,0,0}^{(E)}(K \cap E)\, \mu_1^n(dE) \\ \int_{\mathcal{E}_1^n} \Phi_{0,0,2}^{(E)}(K \cap E)\, \mu_1^n(dE) \\ \vdots \\ \int_{\mathcal{E}_1^n} \Phi_{0,0,s}^{(E)}(K \cap E)\, \mu_1^n(dE) \end{pmatrix},$$

where the matrix C is determined by the Crofton formula. In this way, the surface tensors of even rank s are expressed as Crofton integrals

$$\Phi_{n-1}^{0,s}(K) = \int_{\mathcal{E}_1^n} \alpha(E, K \cap E)\, \mu_1^n(dE), \tag{15}$$

where the measurement function $\alpha : \mathcal{E}_1^n \times \mathcal{K}^n \to \mathbb{R}$ is a linear combination of relative Minkowski tensors of rank at most s of K. In fact, the measurement function only depends on K through $V_0(K \cap E)$, see Theorem A.3.4.

Our results do not express odd rank surface tensors as Crofton integrals with lines. This drawback is not caused by our method of proof. In fact, apart from the trivial cases $n = 1$ and $s = 1$, there does not exist neither bounded nor translation invariant measurement functions that express odd rank surface tensors as Crofton integrals, see Theorem A.3.6. Therefore, a method not based on Crofton integrals with lines is required to estimate odd rank surface tensors.

In a design-based setting, three different types of estimators based on random lines are derived from the integral formula (15). For all three types of estimators, it is sufficient to observe whether a test line hits or misses the convex body K in order to estimate the surface tensors of K. This follows as the measurement function in (15) only depends on K through $V_0(K \cap E)$, and it makes the estimators very easy to use in applications. In the following, the three different types of estimators are described.

1. Like the estimator in Secion 2.1, the first type of estimator derived from (15) is based on isotropic uniform random (IUR) lines. The estimator possesses some unfortunate statistical properties. This is unsurprising since we are estimating surface tensors with many unknown components based on the rather weak information obtained from a measurement in one single line. Several natural ways to improve the estimator are discussed and compared in Paper A.

2. As described in [25], when analyzing skin tissue in biology, it might be convenient to use test sections orthogonal to the skin surface to make the different layers of tissue distinguishable in the sample. In this case, a design based on vertical sections can be applied, see [25] or Paper A for a definition of vertical sections. The second type of estimator is based on vertical sections and is derived by combining (15) with a Blaschke-Petkantschin formula. As for the first type of estimator, possible ways to improve the vertical section estimator are described.

3. The third type of estimator discussed in Paper A is based on non-isotropic random lines. Such random lines prefer directions given by a directional density, and we investigate how this density should be chosen in order to minimize

variance of the estimator. In this design, our investigation mainly focuses on the estimation of the rank 2 surface tensor of planar convex bodies. For $K \in \mathcal{K}^2$, this tensor can be identified with the symmetric 2×2 matrix

$$\left\{ \frac{1}{8\pi} \int_{S^1} u_i u_j \, S_1(K, du) \right\}_{i,j=1}^{2}. \tag{16}$$

For each component and each fixed convex body $K \in \mathcal{K}^2$ there exists a directional density that minimizes the variance of the estimator, see Lemma A.4.9. This density is, unfortunately, not accessible as it depends on K, which is typically unknown in applications. However, for each component there does exist a density independent of K yielding an estimator with smaller variance than the IUR estimator, see Theorem A.4.10. For the three different components of (16), the suggested densities differ, so it requires three random lines with different distributions to estimate the entire tensor. Taking this into account, the IUR design (where each line can be used to estimate all three components) is superior to the non-isotropic design when the entire tensor is sought for, see Theorem A.4.11.

Turning to a model-based setting, we briefly discuss estimators of the specific surface tensors of a stationary process of convex particles in $\mathbb{R}^n$, see Section A.5 for a definition. We derive a rotational Crofton formula for relative specific surface tensors, and the 'inverse' version of this formula expresses specific surface tensors as rotational Crofton integrals, see Theorem A.5.2, and suggests an estimation procedure for specific surface tensors of even rank. Like the estimation procedures presented in the design-based setup, this procedure is based on one-dimensional linear sections. This work is related to [40], where the mean area moment tensor (specific surface tensor of rank 2) of a particle process in $\mathbb{R}^n$ is estimated based on $(n-1)$-dimensional test sections.

In Paper A, we work under the assumption of convexity of the underlying object. This might seem a rather restrictive assumption, but it appears to be reasonable in a large number of applications in biology. Although not explicitly stated, many of the results in Paper A carry over to the larger class of polyconvex sets. In particular, the estimators (A.4.2), (A.4.6), (A.4.15) and (A.4.21) can be applied to polyconvex sets. This follows by additivity of the listed estimators and of surface tensors.

In Paper A, we have derived different types of stereological estimators based on measurements in linear sections. We have treated the simplest case where the test sections are one-dimensional. This allowed us to give an elementary proof of the required Crofton formula that yielded accessible constants. Further, this approach ensured that the measurement functions depend on the underlying body in a simple way making the estimators easy to use in applications. A natural continuation of the research presented in Paper A would be to establish estimators based on planar (or even k-dimensional) test sections. Very recently, in [22], the complicated constants appearing in the intrinsic Crofton formulae in [20] have been substantially simplified making the formulae better fitted for estimation purposes. In this way, a first step towards the development of stereological estimators based on k-dimensional test sections has already been taken. The next step is to translate the Crofton formulae into stereological estimators. As in Paper A, it is required to solve a linear system in order to estimate the individual surface tensors.

3 Shape from surface tensors

In Paper A, the surface tensors of a convex body are estimated from certain information about the convex body. In Papers B and C, a reversed problem is treated. We investigate how much information about a convex body can be retrieved from its surface tensors. Results of this research can help to quantify the shape information contained in one or several tensors. Further, they can also show the limitations of tensor-valued shape descriptors by exhibiting sets with (a finite number of) identical tensors but rather different shapes according to some natural criterion.

A convex body K with nonempty interior is uniquely determined up to translation by its surface area measure, see e.g., [38, Thm. 8.1.1]. Using this and an application of Stone-Weierstrass's theorem, it is argued in Paper B that K is likewise determined up to translation by its surface tensors. When the **shape** of a convex body is defined as the equivalence class of all translations of the body, the uniqueness statement can equivalently be expressed by saying that the shape of K is determined by the surface tensors of K.

In general, all surface tensors are required to determine the shape of a given convex body. However, in connection with applications, most likely, only a finite number of surface tensors will be available, so from a practical point of view, it is useful to quantify the information that can be retrieved from surface tensors up to a certain finite rank. This is the starting point of Papers B and C. The main aim of our research in this area is to establish uniqueness, stability and reconstruction results. More precisely, we aim to:

1. describe the set of convex bodies whose shapes are uniquely determined by a finite number of surface tensors,
2. derive an upper bound on the distance between the shapes of two convex bodies with a finite number of identical surface tensors,
3. develop reconstruction algorithms that approximate the shape of a convex body from a finite number of surface tensors.

As a measure of distance in shape between two nonempty convex bodies K and L, we use a translation invariant version of the Hausdorff distance called the **translative Hausdorff distance**,

$$\delta^t(K,L) = \inf_{x\in\mathbb{R}^n} \delta(K, L+x).$$

We also make use of the translation invariant version of the L^2-metric given by

$$\delta_2^t(K,L) = \inf_{x\in\mathbb{R}^n} \delta_2(K, L+x).$$

In Papers B and C, it turns out that a recently introduced set of geometric functionals on $\mathcal{K}^n$ called **harmonic intrinsic volumes** are very useful when deriving stability results and reconstruction algorithms. Harmonic intrinsic volumes of a convex body $K \in \mathcal{K}^n$ are moments of the area measures of K with respect to an orthonormal sequence of spherical harmonics. Let $\mathcal{H}_m^n$ denote the vector space of spherical harmonics of degree m and let $N(n,m)$ denote the dimension of $\mathcal{H}_m^n$. Then the harmonic intrinsic volumes of K are given by

$$\psi_{jmk}(K) = \int_{S^{n-1}} H_{mk}(u)\, S_j(K, du)$$

for $j = 0,\dots,n-1$, $m \in \mathbb{N}_0$, $k = 1,\dots,N(n,m)$, where $H_{m1},\dots,H_{mN(n,m)}$ form an orthonormal basis of $\mathcal{H}_m^n$. Each polynomial on S^{n-1} of degree at most $d \in \mathbb{N}_0$ is a linear combination of spherical harmonics of degree at most d, see, e.g., [13, Cor. 3.2.6], so for $s \in \mathbb{N}_0$, the tensors $\Phi_j^{0,0}(K),\dots,\Phi_j^{0,s}(K)$ are determined by $\psi_{jmk}(K)$ for $m = 0,\dots,s$ and $k = 1,\dots,N(n,m)$. Obviously, the converse also holds. In Paper B, it is shown that for $n = 2$ harmonic intrinsic volumes can be obtained as values of surface tensors meaning that for $j = 0,1$, $m \in \mathbb{N}_0$ and $k = 1,\dots,N(2,m)$ there exist vectors $v_{jmk}^1,\dots,v_{jmk}^m \in \mathbb{R}^n$ such that

$$\psi_{jmk}(K) = \Phi_j^{0,m}(K)(v_{jmk}^1,\dots,v_{jmk}^m).$$

Harmonic intrinsic volumes were first introduced in [16] (with a different normalization), where they among other things were used to obtain inversion formulas for the intensity of a non-isotropic Boolean model. Further, they appear in [7] as components of the previously mentioned trace-free tensors. For a detailed introduction to harmonic intrinsic volumes, see [16], Papers B and Paper C.

Investigations of problems similar to those treated in Papers B and C can be found in the literature. In [32], a planar convex body is reconstructed from a finite number of measurements of its support function, and in [11], an origin-symmetric convex body $K \in \mathcal{K}^n$ is reconstructed from a finite number of values of its brightness function $b_K : S^{n-1} \to \mathbb{R}$ defined as

$$b_K(u) = V_{n-1}(K \mid u^\perp)$$

for $u \in S^{n-1}$, where $K \mid u^\perp$ is the projection of K onto the orthogonal complement $u^\perp$ of u. The reconstruction problem in [11] is approached in the following way: From a finite number of given values of the brightness function of an unknown convex body, the surface area measure of a convex polytope with brightness function values identical to the specified values are determined by solving a constrained least squares problem. The polytope is then reconstructed from its area measure using Algorithm MinkData proposed in [28]. Algorithm MinkData reconstructs a polytope from its surface area measure by solving a nonlinear optimization problem. In [8], uniqueness results for lightness functions (generalizations of the brightness function) are derived and stability versions of the obtained results are discussed.

The mentioned examples [32, 11, 8] are from the mathematical field of *geometric tomography*. In this field, information about a geometric object is retrieved from section or projection data, see [9]. Geometric tomography bears a resemblance to stereology, but the aim of geometric tomography is to reconstruct the entire object under consideration, whereas the aim of stereology is to estimate certain characteristics of the object.

The components of surface tensors are moments of the surface area measure of the underlying convex body, so the problem of reconstructing the shape of a convex body from its surface tensors is a '*shape from moments*'-problem. In the classical setting, 'moments' refers to the moments of the Lebesgue measure restricted to underlying object, see (17). The literature on this type of problem is vast, and the problem has applications in, for instance, X-ray tomography, see [31]. We will return to 'shape from moments'-problems in Section 4.

In the following subsections, we give a description of the results obtained in Papers B and C and discuss ideas for future work.

3.1 Paper B

In Paper B, the described research questions are partly answered. First, we show that for a convex body $K \subseteq \mathbb{R}^n$ and a natural number s there exists a convex polytope $P \subseteq \mathbb{R}^n$ such that K and P have identical surface tensors up to rank s, see Theorem B.4.1. This result is important in connection with the development of reconstruction algorithms. In the context of uniqueness, it is likewise interesting as the result implies that the shape of a convex body is uniquely determined by a finite number of surface tensors only if the convex body is a polytope. We further show that the shape of a convex polytope P with nonempty interior and at most $m \geq n+1$ facets is uniquely determined by the surface tensors of P up to rank $2m$, see Theorem B.4.3. When combined, the two results state that the shape of a convex body K is uniquely determined by a finite number of the surface tensors of K if and only if K is a polytope with nonempty interior.

In addition to the uniqueness results, we derive stability results for the tensors $(\Phi_1^{0,s})_s$, which are surface tensors for $n = 2$. The first order area measure $S_1(K, \cdot)$ of a convex body K with sufficiently smooth boundary has a density with respect to the spherical Lebesgue measure. This density involves the support function h_K of K and therefore establishes a connection between h_K and the harmonic intrinsic volumes $(\psi_{1mk}(K))_{mk}$ derived from $S_1(K, \cdot)$. This connection is utilized to derive a stability result for the harmonic intrinsic volumes $(\psi_{1mk}(K))_{mk}$. Let $s_o \in \mathbb{N}_0$, $\rho \geq 0$ and $K, L \in \mathcal{K}^n$ with $K, L \subseteq RB^n$ for some $R > 0$. Now, assume that

$$\sum_{k=1}^{N(n,m)} \Big(\psi_{1mk}(K) - \psi_{1mk}(L)\Big)^2 \leq \rho$$

for $m = 0, \dots s_o$. For $0 < \alpha < \frac{3}{2}$, the stability result states that

$$\delta_2^t(K,L)^2 \leq c_1(n,\alpha,R)\Big((s_o+1)(n+s_o-1)\Big)^{-\alpha} + \rho M(n)$$

where $c > 0$ is a constant depending on n, α and R, and $M > 0$ is a constant depending on n, see Theorem B.4.8. The proof of the stability result uses a generalized version of Wirtinger's inequality (Corollary B.4.7) involving a higher order spherical expansion. Due to the relation between harmonic intrinsic volumes and Minkowski tensors, the stated stability result can be converted into a stability result for the tensors stating that

$$\delta(K,L) \leq c_2(n,\alpha,R) s_o^{-\frac{2\alpha}{n+1}}$$

for $0 < \alpha < \frac{3}{2}$, when $\Phi_1^{0,s}(K) = \Phi_1^{0,s}(L)$ for $s = 0, \dots, s_o$, see Theorem B.4.9. Spherical harmonics and their properties are important for the proofs leading to the stability results. This partly explains the introduction of harmonic intrinsic volumes in Paper B.

We develop two reconstruction algorithms in Paper B:

1. the first algorithm is based on exact surface tensors up to a certain rank of an unknown convex body in $\mathbb{R}^2$,

2. the second algorithm is based on measurements disrupted by noise of surface tensors up to a certain rank of an unknown convex body in $\mathbb{R}^2$.

The structures of the reconstruction algorithms are similar to the structure of the algorithm in [11], and the presented algorithms also make use of Algorithm MinkData. The output of the first algorithm is a polygon with surface tensors identical to the given surface tensors of the unknown convex body. The existence of such a polygon follows from the uniqueness result previously described, and its facet normals and facet lengths are determined by solving a least squares problem. From the facet normals and lengths, the polygon is constructed using Algorithm MinkData (which is very simple in the case $n = 2$).

The stability result for surface tensors ensures that the reconstruction algorithm is consistent. If K_s is an output of the algorithm based on surface tensors up to rank $s \in \mathbb{N}$ of an unknown convex body $K_0 \subseteq \mathbb{R}^2$, then the shape of K_s converges to the shape of K_0, when s increases. To apply the stability result, the existence of a ball that contains all reconstructions $(K_s)_{s\in\mathbb{N}}$ is required. This condition is easily verified as the surface area of each reconstruction K_s is identical to the surface area of K_0 and hence bounded.

We recommend using harmonic intrinsic volumes instead of surface tensors evaluated in the standard basis when only noisy measurements of surface tensors are available for reconstruction. Therefore, the input of the second algorithm is noisy measurements of harmonic intrinsic volumes up to a certain degree of an unknown convex body in $\mathbb{R}^2$. The output is a polygon that fits the given measurements in a least squares sense. As for the first algorithm, the facet normals and facet lengths of the polygon are found as a solution to a least squares problem and the polygon is constructed using Algorithm MinkData. The consistency of the algorithm is obtained from the stability result for harmonic intrinsic volumes under certain assumptions on the variance of the noise on the measurements. We use harmonic intrinsic volumes instead of surface tensors evaluated at the standard basis in order to obtain stronger consistency results, see the discussion in Section B.6.4 for details.

To illustrate the feasibility of the reconstruction algorithms, they have been implemented in MatLab and several examples of reconstructions are presented in the paper, see, e.g., Figures B.3 and B.4.

It is natural to ask if the reconstruction algorithms presented in Paper B can be extended to an n-dimensional setting. Paper C shows that the least squares approach can, in fact, be used to find an approximating polytope given finitely many surface tensors of an unknown convex body in $\mathbb{R}^n$. However, the stability results in Paper B involve the tensors $(\Phi_1^{0,s})_s$ and results for surface tensors $(\Phi_{n-1}^{0,s})_s$ require fundamentally different method of proof. In the two-dimensional setting, the tensors $(\Phi_1^{0,s})_s$ are surface tensors, therefore, we restrict to the two-dimensional case in Paper B such that the stability results ensure consistency of the algorithms. Stability results for surface tensors will be one of the main contributions of Paper C.

3.2 Paper C

Paper C is a sequel to Paper B, where we treat some of the questions left open by Paper B. In particular, we establish stability results for surface tensors $(\Phi_{n-1}^{0,s})_s$ and harmonic intrinsic volumes $(\psi_{n-1mk})_{mk}$ derived from surface area measures, see Theorems C.4.3 and C.4.5. First an upper bound on the Dudley distance between surface area measures of convex bodies is derived. The bound is expressed in terms of the distance between the harmonic intrinsic volumes up to a certain degree of the convex bodies. Then, using the connection between surface tensors and harmonic intrinsic volumes in combination with existing stability results for surface area measures, the bound on the Dudley distance is converted into a bound of the translative Hausdorff distance between two convex bodies with identical surface tensors up to a certain rank. More precisely, let $s_o \in \mathbb{N}$, $0 < \varepsilon < 1$ and let $K, L \in \mathcal{K}^n$ contain a ball of radius r and be contained in a ball of radius R for some $r, R > 0$. Assume that $\Phi_{n-1}^{0,s}(K) = \Phi_{n-1}^{0,s}(L)$ for $s = 0, \ldots, s_o$. Then

$$\delta^t(K,L) \leq c_3(n,r,R,\varepsilon) s_o^{-\frac{1-\varepsilon}{4n}}$$

where $c_3 > 0$ is a constant depending on n, r, R and ε.

The reconstruction algorithms developed in Paper B are generalized to an n-dimensional setting. The generalizations follow the lines of the algorithms in Paper B. There are, however, certain difficulties to be tackled. For instance, for the reconstruction algorithm based on noisy measurements the least squares problem might not have a solution. We therefore extend the domain of the objective function from the set of surface area measures to the set finite measures on S^{n-1} with centroid at the origin. This ensures the existence of a measure μ that solves the least squares problem. In order to decide if μ is a surface area measure, we show and use that the first and second order moments of a Borel measure ν on S^{n-1} determine if ν is a surface area measure, see Lemma C.5.2. If μ is not a surface area measure there does not exist a polytope that fits the given input measurements in a least squares sense, and the generalized reconstruction algorithm does not have an output polytope. However, this situation only occurs when the measurements are too noisy, see Lemma C.6.3.

For each $s \in \mathbb{N}_0$, let K_s denote an output polytope of the reconstruction algorithm based on surface tensors up to rank s (or measurements of harmonic intrinsic volumes up to of degree s). In order to apply the stability results to ensure consistency of the reconstruction algorithms, it is necessary to ensure the existence of radii $r, R > 0$ (independent of s) such that $rB^n \subseteq K_s + x_s \subseteq RB^n$ for some $x_s \in \mathbb{R}^n$. In contrast to the similar problem in Paper B, this is far from trivial when $n \geq 3$. It is no longer sufficient that K_s and K_0 have identical surface area. Instead, the existence of the radii r and R is shown using that K_s and K_0 have identical (or sufficiently close) rank 2 surface tensor when $s \geq 2$, see Lemma C.5.4. Now, using the stability results, we obtain consistency of the reconstruction algorithms. The consistency of the algorithm based on noisy measurements requires that the variances of the noise terms decrease appropriately with s.

As in Paper B, we illustrate the feasibility of the algorithms by examples. Using the algorithm based on exact surface tensors, we reconstruct a prolate spheroid and a pyramid in $\mathbb{R}^3$, see Figures C.1 and C.2. Using the algorithm based on noisy measurements of harmonic intrinsic volumes, we reconstruct an oblate spheroid in $\mathbb{R}^3$ under different levels of noise, see Figure C.3.

We strengthen and complete the uniqueness results presented in Paper B by showing that the shape of a convex polytope $P \subseteq \mathbb{R}^n$ with at most $m \geq n+1$ facets is uniquely determined by the surface tensors of P up to rank $m-n+2$, see Theorem C.3.2. Further, for each $m \geq n+1$, we construct a convex polytope that is *not* uniquely determined up to translation by its surface tensors up to rank $m-n+1$. This implies that the rank $m-n+2$ cannot be reduced, and in this sense, the uniqueness result is optimal.

We conclude this section by giving some forward looking remarks. The optimality of the exponents in the stability results in Papers B and C are open questions. It would be interesting to settle these questions either by showing that the derived upper bounds are optimal or by improving them. It is generally believed that the stability result for surface area measures [18, Thm. 3.1] used to derive the stability result presented in Paper C is not optimal. In this case, the bound in Paper C is, likewise, not optimal.

The stability results in Papers B and C for surface tensors have been used to ensure consistency of reconstruction algorithms based on surface tensors. Since we have established stability results for tensors $(\Phi_1^{0,s})_s$ it is natural to seek a reconstruction algorithm based on this type of tensor. An idea would be to use the connection between first order area measures and support functions, see (B.13), combined with an existing reconstruction algorithm based on support functions, [32, 10].

In future work, it would likewise be natural to attempt establishing stability results for tensors $\Phi_j^{0,s}$ with $1 < j < n-1$ (the case of $j = 0$ is not interesting, as $S_0(K, \cdot)$ is the spherical Lebesgue independent of K). Stability results [38, Thm. 8.5.4] involving area measures $S_j(K, \cdot)$, $j = 2, \ldots, n-2$ might be a helpful tool.

4 Shape from volume tensors

Paper D deviates from Papers A–C as it deals with volume tensors instead of surface tensors. The paper is, however, closely related to Papers B and C as the research questions treated in Paper D are similar to the questions investigated in those papers. The aim of Paper D is to establish uniqueness, stability and reconstruction results for volume tensors. The obtained results and the used methods are different from the results and methods in Papers B and C due to the structural differences between surface and volume tensors. An example hereof is the difference between the uniqueness results. Polytopes are uniquely determined by a finite number of surface tensors, whereas convex superlevel sets of polynomials are uniquely determined by a finite number of volume tensors, see Theorem B.4.3 and Corollary D.3.2.

In the literature, the (geometric) **moments** of a compact set $K \subseteq \mathbb{R}^n$ are defined as

$$\mu_\alpha(K) = \int_K x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, \lambda(dx) \tag{17}$$

for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. We refer to $|\alpha| = \sum_{j=1}^n \alpha_j$ as the order of $\mu_\alpha(K)$. The moments are rescaled versions of the components of volume tensors. Following the existing literature in this area, we adopt the notation of moments in Paper D. In addition to the moments (17), we consider another type of moments, namely Legendre moments, that are described in the the next section. To distinguish between the two series of moments, we refer to (17) as geometric moments.

The problem of reconstructing a geometric object from its moments has received considerable attention in the last decades. We mention a few examples from the rich literature on moment problems. Using Prony's method (see, e.g., [15]), Milanfar et al. in [31] show that the vertices of a simply connected planar m-gon are uniquely determined by the moments of the m-gon up to order $2m-3$. Further, they present an algorithm that reconstructs planar convex polygons from noisy measurements of their moments. In [12], Gravin et al. present an algorithm that reconstructs an n-dimensional convex polytope from a finite number of its moments. In addition to the results for polytopes, there are uniqueness results based on moments for sublevel sets of homogeneous polynomials and so-called quadrature domains, see [27, 14].

4.1 Paper D

In Paper D, we first establish the following uniqueness result. Let $C \subseteq \mathbb{R}^n$ be compact and let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d \in \mathbb{N}_0$. Then a convex body K of the form $K = C \cap \{p \geq 0\}$ is uniquely determined in $\{L \in \mathcal{K}^n \mid L \subseteq C\}$ by its moments up to order d, see Corollary D.3.2. The proof of this result uses existing uniqueness results [33] for functionals applied to indicator functions for convex bodies. We then show that the moments up to second order of a convex body determine an upper bound of the circumradius of the body. From this fact and the established uniqueness result, we obtain that the set

$$\bigcup_{m \in \mathbb{N}_0} \{K \in \mathcal{K}^n \mid K \text{ is uniquely determined by } \mu_\alpha(K), |\alpha| \leq m\}$$

is dense in $\mathcal{K}^n$, see Remark D.3.7 and Theorem D.3.8.

For stability and reconstruction, we restrict to planar convex bodies in the unit square $[0,1]^2$. Inspired by stability results [46] for functions on the unit interval, we derive stability results for sufficiently smooth functions on the unit square. Via an approximation argument, this result is applied to differences of indicator functions for convex bodies contained in the unit square. This yields an upper bound of the Nikodym distance between two convex bodies in terms of the difference of a finite number of their geometric moments. The **Nikodym distance** δ_N between two convex bodies $K, L \subseteq [0,1]^2$ is defined as the area of their symmetric difference, i.e.

$$\delta_N(K,L) = V_2((K \cup L) \setminus (K \cap L)).$$

It follows from the definition that δ_N can be expressed as the L^2-norm of the difference $\mathbf{1}_K - \mathbf{1}_L$ of the indicator functions of K and L. This is very convenient in the set-up of Paper D. The Nikodym distance and the Hausdorff distance are equivalent on the set of convex bodies contained in the unit square.

For two convex bodies $K, L \subseteq [0,1]^2$ with identical geometric moments up to order $m \in \mathbb{N}$, the derived stability result states that

$$\delta_N(K,L) \leq \frac{c_4}{m}$$

with some constant $c_4 > 0$, see Theorem D.4.2 and Remark D.4.4. When assuming that the geometric moments up to order m are close (but not identical), the stability result is, unfortunately, very poor as the upper bound in this case increases exponentially in m. However, by introducing another set of moments, namely Legendre moments, the stability result can be improved considerably. For a convex body $K \subseteq [0,1]^2$, the **Legendre moments** of K are defined as

$$\lambda_{ij}(K) = \int_K L_i(x_1) L_j(x_2)\, \lambda(dx)$$

for $i, j \in \mathbb{N}_0$, where $L_i\colon [0,1] \to \mathbb{R}$ for $i \in \mathbb{N}_0$ are shifted and normalized Legendre polynomials, see Section D.2.

To obtain reasonable stability results for surface tensors and geometric moments, these quantities are required to be identical for the two convex bodies in question. By considering harmonic intrinsic volumes and Legendre moments instead, we obtain stability results that allow the difference between these quantities to be non-zero. In this way, Legendre moments form a counterpart to harmonic intrinsic volumes.

We develop an algorithm that constructs an approximating polygon of a convex body $K \subseteq [0,1]^2$ from a finite number of exact moments of K. The moments of the output polygon fit the given moments in a least squares sense, but are typically not identical to them. Recall that in the case of exact surface tensors, the output polytope of the reconstruction algorithm has surface tensors identical to the given ones. This follows from Theorem B.4.1 that states that for any convex body $K \in \mathcal{K}^n$ and rank $s \in \mathbb{N}_0$, there is a polytope P such that K and P have identical surface tensors up to rank s. In the case of moments, there does not exist a similar result. In fact, any ellipsoid constitutes a counterexample as ellipsoids are uniquely determined by their moments up to order 2, see Example D.1. Due to the error on the moments of the output polygon, only the stability result based on Legendre moments is applicable. Therefore, we concentrate on reconstruction based on Legendre moments.

For an unknown convex body with a finite number of Legendre moments available, an approximating polygon that fits the given moments are found as a solution to a least squares problem. In contrast to surface tensors, Legendre moments of polygons are not easily parametrized by facet normals and facets lengths, so another parametrization is used to solve the least squares problem. A convex polygon $P \subseteq \mathbb{R}^2$ with $m \geq 3$ fixed facet normals $u_1, \ldots, u_m \in S^1$ can be described as

$$P = \bigcap_{j=1}^{m} \{x \in \mathbb{R}^2 \mid \langle x, u_j \rangle \leq h_j\}$$

for some $h_1, \ldots, h_m \in (-\infty, \infty)$, and the Legendre moments of P can be expressed as a polynomial in $(h_1, \ldots, h_m)$, see Lemma D.6.1. For a fixed set of normals $u_1, \ldots, u_m \in S^1$, the set of vectors $(h_1, \ldots, h_m)$ describing a convex polygon contained in $[0,1]^2$ and with facet normals $u_1, \ldots, u_m$ is determined by $5m$ linear equations, so restricting to polygons with a prescribed set of facet normals and parametrizing by $(h_1, \ldots, h_m)$, the least squares problem for Legendre moments can be solved as a polynomial optimization problem. The consistency of the algorithm is ensured by the stability result based on Legendre moments when the number of used Legendre moments and the number of prescribed facets of the output polygon increase. We also present a reconstruction algorithm that allows for measurements disrupted by noise of Legendre moments.

The reconstruction algorithms in Paper D have not yet been implemented, so in order to explore the performance of the algorithms in practice, this would be the natural next step.

5 Concluding remarks

Minkowski tensors are important from various points of view and appear in several different disciplines such as physics, microscopy, stochastic and integral geometry, convex geometry and valuation theory. The four papers constituting this thesis add new results to the existing literature on Minkowski tensors. Stereological estimation procedures for surface tensors have been established, uniqueness and stability results have been derived for surface and volume tensors, and corresponding reconstruction algorithms have been developed. Ideas for future work in direct continuation of the work presented in Papers A–D have already been suggested in the previous sections.

As described in Section 1.3, Minkowski tensors of lower rank ($r+s \leq 2$) have successfully been established as shape descriptors in applied sciences. The results of this thesis suggest that higher rank tensors are likewise useful as shape descriptors as they contain important shape information. This is for instance indicated by the uniqueness and stability results of Papers B–D and by the reconstruction examples.

The theory and applications of Minkowski tensors are continuously evolving, and as a final remark, we mention that a natural local version of Minkowski tensors that generalizes intrinsic volumes, support measures and Minkowski tensors has recently been introduced in [37]. Local Minkowski tensors take values in the space of tensor-valued measures and for a convex body $K \in \mathcal{K}^n$, they are defined as

$$\phi_j^{r,s}(K,\eta) = \frac{\omega_{n-j}}{r!\,s!\,\omega_{n-j+s}} \int_\eta x^r u^s \Lambda_j(K, d(x,u))$$

for a Borel set $\eta \in \mathbb{R}^n \times S^{n-1}$, $r,s \in \mathbb{N}_0$ and $j \in \{0,\dots,n-1\}$. Classification results for local Minkowski tensors analogous to Hadwiger's and Alesker's characterization theorems have already been given in [37] for polytopes and in [19] for the general case. Local Minkowski tensors might constitute a refined tool for applications.

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Paper A

Surface tensor estimation from linear sections

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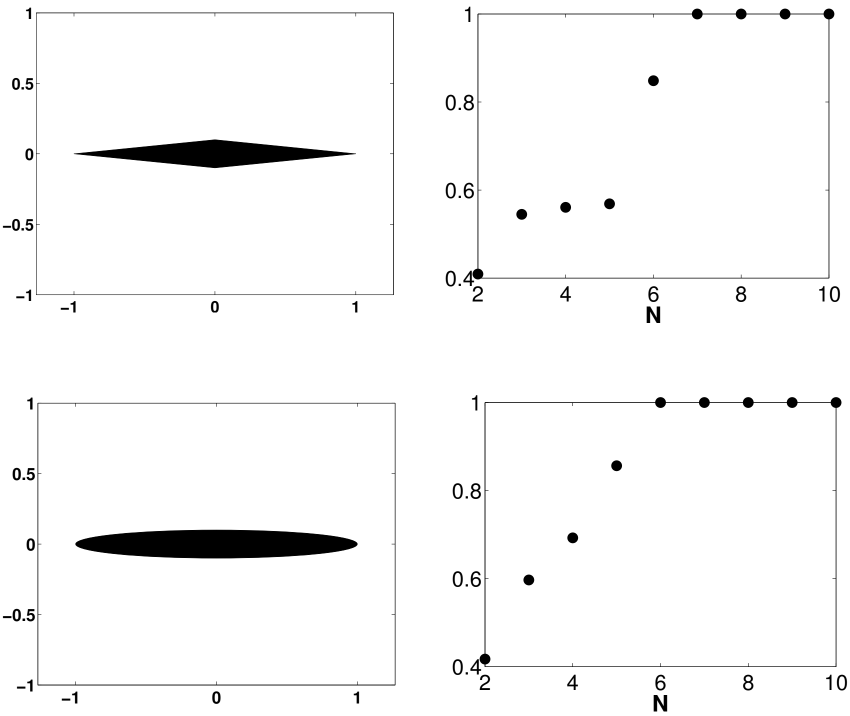

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Surface tensor estimation from linear sections

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From Crofton's formula for Minkowski tensors we derive stereological estimators of translation invariant surface tensors of convex bodies in the n-dimensional Euclidean space. The estimators are based on one-dimensional linear sections. In a design based setting we suggest three types of estimators. These are based on isotropic uniform random lines, vertical sections, and non-isotropic random lines, respectively. Further, we derive estimators of the specific surface tensors associated with a stationary process of convex particles in the model based setting.

1 Introduction

In recent years, there has been an increasing interest in Minkowski tensors as descriptors of morphology and shape of spatial structures of physical systems. For instance, they have been established as robust and versatile measures of anisotropy in [7], [25], [26]. In addition to the applications in materials science, [6] indicates that the Minkowski tensors lead to a putative taxonomy of neuronal cells. From a pure theoretical point of view, Minkowski tensors are, likewise, interesting. This is illustrated by Alesker's characterization theorem [1], stating that the basic tensor valuations (products of the Minkowski tensors and powers of the metric tensor) span the space of tensor-valued valuations satisfying some natural conditions.

This paper presents estimators of certain Minkowski tensors from measurements in one-dimensional flat sections of the underlying geometric structure. We restrict attention to translation invariant Minkowski tensors of convex bodies, more precisely, to those that are derived from the top order surface area measure; see Section 2 for a definition. As usual, the estimators are derived from an integral geometric formula. Specifically, we use a Crofton formula for Minkowski tensors. We adopt the classical setting where the sectioning space is affine and the integration is with respect to the suitably normalized motion invariant measure. Rotational Crofton formulae where the sectioning space is a linear subspace and the rotation invariant probability measure on the corresponding Grassmannian is used are established in [3]. The latter formulae were the basis for local stereological estimators of certain Minkowski tensors in [12] (for $j \in \{1, \dots, n-1\}$, $s, r \in \{0, 1\}$ and $j = n$, $s = 0$, $r \in \mathbb{N}$ in the notation of (2.1) and (2.2) , below).

Kanatani [14], [15] was apparently the first to use tensorial quantities to detect and analyse structural anisotropy via basic stereological principles. He expresses the expected number $N(m)$ of intersections per unit length of a probe with a test line of given direction m as the cosine transform of the spherical distribution density f of the surface of the given probe in $\mathbb{R}^n$ for $n = 2, 3$. The relation between N and f is studied by expanding f into spherical harmonics and by using the fact that these are eigenfunctions of the cosine transform. In order to express his results independently of a particular coordinate system, Kanatani uses tensors. For a fixed s, he considers the vector space V_s of all symmetric tensors spanned by the elementary tensor products $u^{\otimes s}$ of vectors u from the unit

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sphere S^{n-1}. Let $\hat{T}$ denote the deviator part (or trace-free part) of some symmetric tensor T. The tensors $\widehat{(u^{\otimes k})}$, for $k \leq s$ and $u \in S^{n-1}$, then span V_s and the components of $\widehat{(u^{\otimes k})}$ with respect to an orthonormal basis of $\mathbb{R}^n$ are spherical harmonics of degree k when considered as functions of u. Hence, $u \mapsto \widehat{(u^{\otimes k})}$ is an eigenfunction of the cosine transform (Kanatani calls it "Buffon transform"), which in fact is the underlying integral transform when considering Crofton integrals with lines, as we shall see below in (3.7). In [13], [16], he suggests to use these "fabric tensors" to detect surface motions and the anisotropy of the crack distribution in rock.

General Crofton formulas in $\mathbb{R}^n$ with flats of arbitrary dimension and for general Minkowski tensors (defined in (2.1)) of arbitrary rank are given in [10]. Theorem 3.1 is a special case of one of these results for translation invariant surface tensors and one-dimensional sections, that is, sections with lines. In comparison to [10], we get simplified constants in the case considered and obtain this result by an elementary independent proof. In contrast to Kanatani's approach, our proof does not rely on spherical harmonics. Here we focus on relative Crofton formulas in which the Minkowski tensors of the sections with lines are calculated relative to the section lines and not in the ambient space (Crofton formulas of the second type may be called extrinsic Crofton formulas). A quite general investigation of integral geometric formulas for translation invariant Minkowski tensors, including extrinsic Crofton formulas, is provided in [8].

In Theorem 3.1 we prove that the relative Crofton integral for tensors of arbitrary even rank s of sections with lines is equal to a linear combination of surface tensors of rank at most s. From this we deduce by the inversion of a linear system that any translation invariant surface tensor of even rank s can be expressed as a Crofton integral. The involved measurement functions then are linear combinations of relative tensors of rank at most s. This implies that the measurement functions only depend on the convex body through the Euler characteristic of the intersection of the convex body and the test line.

Our results do not allow to write surface tensors of odd rank as Crofton integrals based on sections with lines. This drawback is not a result of our method of proof. Indeed, apart from the trivial case of tensors of rank one, there does not exist a translation invariant or a bounded measurement function that expresses a surface tensor of odd rank as a Crofton integral. A precise and slightly more general statement is provided in Theorem 3.6.

In Section 4 the integral formula for surface tensors of even rank is transferred to stereological formulae in a design based setting. Three types of unbiased estimators are discussed. Section 4.1 describes an estimator based on isotropic uniform random lines. Due to the structure of the measurement function, it suffices to observe whether the test line hits or misses the convex body in order to estimate the surface tensors. However, the resulting estimators possess some unfortunate statistical properties. In contrast to the surface tensors of full dimensional convex bodies, the estimators are not positive definite. For convex bodies, which are not too eccentric (see (4.8)), this problem is solved by using n orthogonal test lines in combination with a measurement of the projection function of order $n-1$ of the convex body.

In applications it might be inconvenient or even impossible to construct the isotropic uniform random lines, which are necessary for the use of the estimator described above. Instead, it might be a possibility to use vertical sections; see Definition 4.5. A combination of Crofton's formula and a result of Blaschke-Petkantschin type allows us to formulate a vertical section estimator. The estimator, which is discussed in Section 4.2, is based on two-dimensional vertical flats.

The third type of estimator presented in the design based setting is based on non-isotropic linear sections; see Section 4.3. For a fixed convex body in $\mathbb{R}^2$ there exists a density for the distribution of test line directions in an importance-sampling approach that leads to minimal variance of the non-isotropic estimator, when we consider one component of a rank 2 tensor, interpreted as a matrix. In practical applications, this density is not accessible, as it depends on the convex body, which is typically unknown. However, there does exist a density independent of the underlying convex body yielding an estimator with smaller variance than the estimator based on isotropic uniform random lines. If all components of the tensor are sought for, the non-isotropic approach requires three test lines, as two of the four components of a rank 2 Minkowski tensor coincide due to symmetry. It should be avoided to use a density suited for estimating one particular component of the tensor to estimate any other component, as this would increase the variance of the estimator. In this situation, however, a smaller variance can be obtained by applying an estimator based on three isotropic random lines (each of which can be used for the estimation of all components of the tensor).

In Section 5 we turn to a model-based setting. We discuss the estimation of the *specific (translation invariant) surface tensors* associated with a stationary process of convex particles; see (5.1) for a definition. In [23] the problem of estimating the area moment tensor (rank 2) associated with a stationary process of convex particles via planar sections is discussed. We consider estimators of the specific surface tensors of arbitrary even rank based on one-dimensional linear sections. Using the Crofton formula for surface tensors, we derive a rotational Crofton formula for the specific surface tensors. Further, the specific surface tensor of rank s of a stationary process of convex particles is expressed as a rotational average of a linear combination of specific tensors of rank at most s of the sectioned process.

2 Preliminaries

We work in the n-dimensional Euclidean vector space $\mathbb{R}^n$ with inner product $\langle .\,, . \rangle$ and induced norm $\| \cdot \|$. Let $B^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the unit ball and $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ the unit sphere in $\mathbb{R}^n$. By κ_n and ω_n we denote the volume and the surface area of B^n, respectively. The Borel σ-algebra of a topological space X is denoted by $\mathcal{B}(X)$. Further, let λ denote the n-dimensional Lebesgue measure on $\mathbb{R}^n$, and for an affine subspace E of $\mathbb{R}^n$, let λ_E denote the Lebesgue measure defined on E. The k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. For $A \subseteq \mathbb{R}^n$, let $\dim A$ be the dimension of the affine hull of A, and let $\mathrm{conv}(A)$ be the convex hull of A.

Let $\mathbb{T}^p$ be the vector space of symmetric tensors of rank p over $\mathbb{R}^n$, that is, the space of symmetric multilinear functions of p variables in $\mathbb{R}^n$. Due to linearity, a tensor $T \in \mathbb{T}^p$ can be identified with the array $\{T(e_{i_1}, \ldots, e_{i_p})\}_{i_1,\ldots,i_p=1}^n$, where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$. Via this identification $\mathbb{T}^p$ is embedded in $\mathbb{R}^{n^p}$, and the euclidean norm on $\mathbb{R}^{n^p}$ induces a norm $\| \cdot \|_{\mathbb{T}^p}$ on $\mathbb{T}^p$. For symmetric tensors $a \in \mathbb{T}^{p_1}$ and $b \in \mathbb{T}^{p_2}$, let $ab \in \mathbb{T}^{p_1+p_2}$ denote the symmetric tensor product of a and b. We identify $x \in \mathbb{R}^n$ with the rank 1 tensor $z \mapsto \langle z, x\rangle$ and write $x^p \in \mathbb{T}^p$ for the p-fold symmetric tensor product of x. The metric tensor $Q \in \mathbb{T}^2$ is defined by $Q(x, y) = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$, and for a linear subspace L of $\mathbb{R}^n$, we define $Q(L) \in \mathbb{T}^2$ by $Q(L)(x, y) = \langle x|L, y|L\rangle$, where x/L is the orthogonal projection of x onto L.

As general references on convex geometry and Minkowski tensors, we use [21] and [10]. Let $\mathcal{K}^n$ denote the set of convex bodies (that is, compact, convex sets) in $\mathbb{R}^n$. We will write $h(K, \cdot)$ for the support function of a non-empty convex body K. In order to define the Minkowski tensors, we introduce the support measures $\Lambda_0(K, \cdot), \ldots, \Lambda_{n-1}(K, \cdot)$ of a convex body $K \in \mathcal{K}^n$. Let $p(K, x)$ be the metric projection of $x \in \mathbb{R}^n$ on a non-empty convex body K, and define $u(K, x) := \frac{x-p(K,x)}{\|x-p(K,x)\|}$ for $x \notin K$. For $\epsilon > 0$ and a Borel set $A \in \mathcal{B}\big(\mathbb{R}^n \times S^{n-1}\big)$, the Lebesgue measure of the local parallel set

$$M_\epsilon(K, A) := \{x \in (K + \epsilon B^n) \setminus K \mid (p(K, x), u(K, x)) \in A\}$$

of K is a polynomial in ϵ, hence

$$\lambda(M_\epsilon(K, A)) = \sum_{k=0}^{n-1} \epsilon^{n-k} \kappa_{n-k} \Lambda_k(K, A).$$

This local version of the Steiner formula defines the support measures $\Lambda_0(K, \cdot), \ldots, \Lambda_{n-1}(K, \cdot)$ of a non-empty convex body $K \in \mathcal{K}^n$. If $K = \emptyset$, we define the support measures to be the zero measures. The intrinsic volumes $V_0(K), \ldots, V_{n-1}(K)$ of K appear as total masses of the support measures, $V_j(K) = \Lambda_j\big(K, \mathbb{R}^n \times S^{n-1}\big)$ for $j = 0, \ldots, n-1$. Furthermore, the area measures $S_0(K, \cdot), \ldots, S_{n-1}(K, \cdot)$ of K are rescaled projections of the corresponding support measures on the second component. More explicitly, they are given by

$$\binom{n}{j} S_j(K, \omega) = n\kappa_{n-j} \Lambda_j(K, \mathbb{R}^n \times \omega)$$

for $\omega \in \mathcal{B}\big(S^{n-1}\big)$ and $j = 0, \ldots, n-1$.

For a convex body $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$, and $j \in \{0, 1, \ldots, n-1\}$, we define the *Minkowski tensors* as

$$\Phi_{j,r,s}(K) := \frac{\omega_{n-j}}{r!s!\omega_{n-j+s}} \int_{\mathbb{R}^n \times S^{n-1}} x^r u^s \,\Lambda_j(K, d(x, u)) \tag{2.1}$$

and

$$\Phi_{n,r,0}(K) := \frac{1}{r!} \int_K x^r \,\lambda(dx). \tag{2.2}$$

The definition of the Minkowski tensors is extended by letting $\Phi_{j,r,s}(K) = 0$, if $j \notin \{0, 1, \dots, n\}$, or if r or s is not in $\mathbb{N}_0$, or if $j = n$ and $s \neq 0$. For $j = n-1$, the tensors (2.1) are called surface tensors. In the present work, we only consider translation invariant surface tensors which are obtained for $r = 0$. In [10] the functions $Q^m \Phi_{j,r,s}$ with $m, r, s \in \mathbb{N}_0$ and either $j \in \{0, \dots, n-1\}$ or $(j, s) = (n, 0)$ are called the basic tensor valuations.

For $k \in \{1, \dots, n\}$, let $\mathcal{L}_k^n$ be the set of k-dimensional linear subspaces of $\mathbb{R}^n$, and let $\mathcal{E}_k^n$ be the set of k-dimensional affine subspaces of $\mathbb{R}^n$. For $L \in \mathcal{L}_k^n$, we write $L^\perp$ for the orthogonal complement of L. For $E \in \mathcal{E}_k^n$, we let $\pi(E)$ denote the linear subspace in $\mathcal{L}_k^n$ which is parallel to E, and we define $E^\perp := \pi(E)^\perp$. The sets $\mathcal{L}_k^n$ and $\mathcal{E}_k^n$ are endowed with their usual topologies and Borel σ-algebras. Let ν_k^n denote the unique rotation invariant probability measure on $\mathcal{L}_k^n$, and let μ_k^n denote the unique motion invariant measure on $\mathcal{E}_k^n$ normalized so that $\mu_k^n(\{E \in \mathcal{E}_k^n | E \cap B^n \neq \emptyset\}) = \kappa_{n-k}$ (see, e.g., [24]).

If $K \in \mathcal{K}^n$ is contained in an affine subspace $E \in \mathcal{E}_k^n$ for some $k \in \{1, \dots, n\}$, then the Minkowski tensors can be evaluated in this subspace. For a linear subspace $L \in \mathcal{L}_k^n$, let $\pi_L : S^{n-1} \setminus L^\perp \to L \cap S^{n-1}$ be given by

$$\pi_L(u) := \frac{u|L}{\|u|L\|}.$$

Then we define the jth support measure $\Lambda_j^{(E)}(K, \cdot)$ of K relative to E as the image measure of the restriction of $\Lambda_j(K, \cdot)$ to $\mathbb{R}^n \times \left(S^{n-1} \setminus E^\perp\right)$ under the mapping $\mathbb{R}^n \times \left(S^{n-1} \setminus E^\perp\right) \to \mathbb{R}^n \times \left(S^{n-1} \cap \pi(E)\right)$ given by $(x, u) \mapsto (x, \pi_{\pi(E)}(u))$. The measure $\Lambda_j^{(E)}(K, \cdot)$ is intrinsic in the sense that its restriction to $E \times \left(S^{n-1} \cap \pi(E)\right)$ is the jth support measure of K calculated with respect to E as the ambient space. The Minkowski tensors of K relative to E are then defined as

$$\Phi_{j,r,s}^{(E)}(K) := \frac{\omega_{k-j}}{r!s!\omega_{k-j+s}} \int_{E \times (S^{n-1} \cap \pi(E))} x^r u^s \, \Lambda_j^{(E)}(K, d(x, u))$$

for $r, s \in \mathbb{N}_0$ and $j \in \{0, \dots, k-1\}$, and

$$\Phi_{k,r,0}^{(E)}(K) := \frac{1}{r!} \int_K x^r \, \lambda_E(dx).$$

As before, the definition is extended by letting $\Phi_{j,r,s}^{(E)}(K) = 0$ for all other choices of j, r and s.

In [10], Crofton integrals of the form

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}^{(E)}(K \cap E) \, \mu_k^n(dE),$$

where $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$ and $0 \leq j \leq k \leq n-1$, are expressed as linear combinations of the basic tensor valuations. When $j = k$ the integral formula becomes

$$\int_{\mathcal{E}_k^n} \Phi_{k,r,s}^{(E)}(K \cap E) \, \mu_k^n(dE) = \begin{cases} \Phi_{n,r,0}(K) & \text{if } s = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

see [10, Theorem 2.4]. In the case where $j < k$, the formulas become lengthy with coefficients in the linear combinations that are difficult to evaluate, see [10, Theorem 2.5 and 2.6]. In the following, we are interested in using the integral formulas for the estimation of the surface tensors, and therefore we need more explicit integral formulas. We only treat the special case where $k = 1$, that is, we consider integrals of the form

$$\int_{\mathcal{E}_1^n} \Phi_{j,r,s}^{(E)}(K \cap E) \, \mu_1^n(dE).$$

Since $\dim(E) = 1$, the tensor $\Phi_{j,r,s}^{(E)}(K)$ is by definition the zero function when $j > 1$, so the only non-trivial cases are $j = 0$ and $j = 1$. When $j = 1$ formula (2.3) gives a simple expression for the integral. In the case where $j = 0$ and $r = 0$, we provide an independent and elementary proof of the integral formula, which also leads to explicit and fairly simple constants.

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3 Linear Crofton formulae for tensors

We start with the main result of this section, which provides a linear Crofton formula relating an average of tensor valuations defined relative to varying section lines to a linear combination of surface tensors.

Theorem 3.1 *Let $K \in \mathcal{K}^n$. If $s \in \mathbb{N}_0$ is even, then*

$$\int_{\mathcal{E}_1^n} \Phi_{0,0,s}^{(E)}(K \cap E)\, \mu_1^n(dE) = \frac{2\omega_{n+s+1}}{\pi\, s!\,\omega_{s+1}^2 \omega_n} \sum_{k=0}^{\frac{s}{2}} c_k^{(\frac{s}{2})}\, Q^{\frac{s}{2}-k}\, \Phi_{n-1,0,2k}(K), \tag{3.1}$$

with constants

$$c_k^{(m)} = (-1)^k \binom{m}{k} \frac{(2k)!\,\omega_{2k+1}}{1-2k} \tag{3.2}$$

for $m \in \mathbb{N}_0$ and $k = 0, \dots, m$.

For odd $s \in \mathbb{N}_0$ the Crofton integral on the left-hand side is zero.

Before we give a proof of Theorem 3.1, let us consider the measurement function $\Phi_{0,0,s}^{(E)}(K \cap E)$ on the left-hand side of (3.1). Let $k \in \{1, \dots, n\}$. Slightly more general than in (3.1), we choose $s \in \mathbb{N}_0$ and $E \in \mathcal{E}_k^n$. Then

$$\Phi_{0,0,s}^{(E)}(K \cap E) = \frac{1}{s!\,\omega_{k+s}} \int_{S^{n-1} \cap \pi(E)} u^s\, \mathcal{H}^{k-1}(du)\, V_0(K \cap E),$$

since the surface area measure of order 0 of a non-empty set is up to a constant the invariant measure on the sphere. From calculations equivalent to [22, (24)–(26)] (or from a special case of Lemma 4.3 in [10]) we get that

$$\int_{S^{n-1} \cap \pi(E)} u^s\, \mathcal{H}^{k-1}(du) = \begin{cases} 2\dfrac{\omega_{s+k}}{\omega_{s+1}} Q(\pi(E))^{\frac{s}{2}} & \text{if } s \text{ is even,} \\ 0 & \text{if } s \text{ is odd.} \end{cases} \tag{3.3}$$

Hence

$$\Phi_{0,0,s}^{(E)}(K \cap E) = \frac{2}{s!\,\omega_{s+1}} \cdot Q(\pi(E))^{\frac{s}{2}}\, V_0(K \cap E), \tag{3.4}$$

when s is even, and $\Phi_{0,0,s}^{(E)}(K \cap E) = 0$ when s is odd. This implies that the Crofton integral in (3.1) is zero for odd s, and the tensors $\Phi_{n-1,0,s}(K)$ are hereby not accessible in this situation. This is even true for more general measurement functions; see Theorem 3.6. To show Theorem 3.1 we can restrict to even s from now on.

In the proof of Theorem 3.1 we use the following identity for binomial sums.

Lemma 3.2 *Let $m, n \in \mathbb{N}_0$. Then*

$$\sum_{j=0}^{m} (-1)^j \frac{\binom{2n}{2j}\binom{n-j}{m-j}}{\binom{n-\frac{1}{2}}{j}} = \frac{\binom{n}{m}}{1-2m}.$$

Lemma 3.2 can be proven by using the identity

$$\sum_{j=0}^{k} (-1)^j \frac{\binom{2n}{2j}\binom{n-j}{m-j}}{\binom{n-\frac{1}{2}}{j}} = \frac{(-1)^k(2k+1)\binom{2n}{2(k+1)}\binom{n-k-1}{m-k-1}}{(2m-1)\binom{n-\frac{1}{2}}{k+1}} - \frac{\binom{n}{m}}{(2m-1)}, \tag{3.5}$$

where $n, k \in \mathbb{N}_0$, and $m \in \mathbb{N}$ such that $k < m$. Identity (3.5) follows by induction on k.

P r o o f o f T h e o r e m 3 . 1 Let $K \in \mathcal{K}^n$ and let $s \in \mathbb{N}_0$ be even. If $n = 1$, formula (3.1) follows from the identity

$$\sum_{j=0}^{m} (-1)^j \frac{\binom{m}{j}}{1-2j} = \frac{\sqrt{\pi}\,\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} \tag{3.6}$$

with $m = \frac{s}{2}$. The left-hand side of (3.6) is a sum of alternating terms of the same form as the right-hand side of the binomial sum in Lemma 3.2. Using Lemma 3.2 and then changing the order of summation yields (3.6).

Now assume that $n \geq 2$. Using (3.4) we can rewrite the integral as

$$\begin{aligned}\int_{\mathcal{E}_1^n} &\Phi_{0,0,s}^{(E)}(K \cap E)\, \mu_1^n(dE) \\ &= \frac{2}{s!\omega_{s+1}} \int_{\mathcal{L}_1^n} Q(L)^{\frac{s}{2}} \int_{L^\perp} V_0(K \cap (L+x))\, \lambda_{L^\perp}(dx)\, \nu_1^n(dL) \\ &= \frac{2}{s!\omega_{s+1}\omega_n} \int_{S^{n-1}} u^s V_{n-1}(K \mid u^\perp)\, \mathcal{H}^{n-1}(du)\end{aligned}$$

by the convexity of K and an invariance argument for the second equality. A projection formula (see [9, (A.45)] or [21, (5.80)]) and Fubini's theorem then imply that

$$\begin{aligned}\int_{\mathcal{E}_1^n} &\Phi_{0,0,s}^{(E)}(K \cap E)\, \mu_1^n(dE) \\ &= \frac{1}{s!\omega_{s+1}\omega_n} \int_{S^{n-1}} \int_{S^{n-1}} u^s |\langle u, v\rangle|\, \mathcal{H}^{n-1}(du)\, S_{n-1}(K, dv).\end{aligned} \tag{3.7}$$

We now fix $v \in S^{n-1}$ and simplify the inner integral by introducing spherical coordinates (see, e.g., [20]). Then

$$\begin{aligned}\int_{S^{n-1}} &u^s |\langle u, v\rangle|\, \mathcal{H}^{n-1}(du) \\ &= \int_{-1}^{1} \int_{S^{n-1}\cap v^\perp} (1-t^2)^{\frac{n-3}{2}} (tv + \sqrt{1-t^2}w)^s |t|\, \mathcal{H}^{n-2}(dw)\, dt \\ &= \sum_{j=0}^{s} \binom{s}{j} v^j \int_{-1}^{1} (1-t^2)^{\frac{n-3}{2}} t^j \sqrt{1-t^2}^{\,s-j} |t|\, dt \int_{S^{n-1}\cap v^\perp} w^{s-j}\, \mathcal{H}^{n-2}(dw).\end{aligned}$$

The integral with respect to t is zero if j is odd. If j is even, then it is equal to the beta integral

$$B\left(\frac{j+2}{2}, \frac{n+s-j-1}{2}\right) = \frac{2\omega_{n+s+1}}{\omega_{j+2}\, \omega_{n+s-j-1}}.$$

Hence, since s is even, we conclude from (3.3) that

$$\begin{aligned}\int_{S^{n-1}} u^s |\langle u, v\rangle|\, \mathcal{H}^{n-1}(du) &= 4\omega_{n+s+1} \sum_{j=0}^{\frac{s}{2}} \binom{s}{2j} v^{2j} \frac{1}{\omega_{2j+2}\, \omega_{s-2j+1}} Q(v^\perp)^{\frac{s-2j}{2}} \\ &= 4\omega_{n+s+1} \sum_{j=0}^{\frac{s}{2}} \sum_{i=0}^{\frac{s}{2}-j} (-1)^i \binom{s}{2j} \binom{\frac{s}{2}-j}{i} \frac{1}{\omega_{2j+2}\, \omega_{s-2j+1}} Q^{\frac{s}{2}-j-i} v^{2(i+j)},\end{aligned}$$

where we have used that $Q(v^\perp) = Q - v^2$. By substitution into (3.7) and the definition of $\Phi_{n-1,0,2(i+j)}(K)$, we obtain that

$$\int_{\mathcal{E}_1^n} \Phi_{0,0,s}^{(E)}(K \cap E)\, \mu_1^n(dE) = \frac{4\omega_{n+s+1}}{s!\omega_{s+1}\omega_n} S, \tag{3.8}$$

where

$$S = \sum_{j=0}^{\frac{s}{2}} \sum_{i=0}^{\frac{s}{2}-j} (-1)^i \binom{s}{2j} \binom{\frac{s}{2}-j}{i} \frac{(2(i+j))!\omega_{2(i+j)+1}}{\omega_{2j+2}\, \omega_{s-2j+1}} Q^{\frac{s}{2}-j-i} \Phi_{n-1,0,2(i+j)}(K).$$

Re-indexing and changing the order of summation, we arrive at

$$S = \frac{\Gamma(\frac{s}{2}+\frac{1}{2})}{4\pi^{\frac{s+3}{2}}} \sum_{k=0}^{\frac{s}{2}} (-1)^k (2k)! \omega_{2k+1} Q^{\frac{s}{2}-k} \Phi_{n-1,0,2k}(K) \sum_{j=0}^{k} (-1)^j \binom{s}{2j} \binom{\frac{s}{2}-j}{k-j} \binom{\frac{s-1}{2}}{j}^{-1}$$

$$= \frac{1}{2\pi\omega_{s+1}} \sum_{k=0}^{\frac{s}{2}} (-1)^k \binom{\frac{s}{2}}{k} \frac{(2k)!\,\omega_{2k+1}}{1-2k} Q^{\frac{s}{2}-k} \Phi_{n-1,0,2k}(K),$$

where we have used Lemma 3.2 with $n = \frac{s}{2}$ and $m = k$. □

Setting $s = 2$ we immediately get the following corollary.

Corollary 3.3 *Let $K \in \mathcal{K}^n$. Then*

$$\int_{\mathcal{E}_1^n} \Phi_{0,0,2}^{(E)}(K \cap E)\, \mu_1^n(dE) = a_n \left(\Phi_{n-1,0,2}(K) + \frac{1}{4\pi} Q V_{n-1}(K) \right),$$

where

$$a_n = \frac{\Gamma(\frac{n}{2})}{2\Gamma(\frac{n+3}{2})\sqrt{\pi}}.$$

The Crofton formula in Theorem 3.1 expresses the integral of the measurement function $\Phi_{0,0,s}^{(E)}(K \cap E)$ as a linear combination of certain surface tensors of $K \in \mathcal{K}^n$. This could, in principle, be used to obtain unbiased stereological estimators of the linear combinations. However, it is more natural to ask what measurement one should use in order to obtain $\Phi_{n-1,0,s}(K)$ as a Crofton-type integral. For even s the tensor $\Phi_{n-1,0,s}(K)$ appears in the last term of the sum on the right-hand side of (3.1). But surface tensors of lower rank appear in the remaining terms of the sum. Therefore, we need to express the lower rank tensors $\Phi_{n-1,0,2k}(K)$ for $k = 0, \dots, \frac{s}{2} - 1$ as integrals. This is done by replacing s in Theorem 3.1 with $2k$ for $k = 0, \dots, \frac{s}{2} - 1$. This way, we get $\frac{s}{2} + 1$ linear equations, which give rise to the linear system

$$\begin{pmatrix} C_0 \int_{\mathcal{E}_1^n} \Phi_{0,0,0}^{(E)}(K \cap E)\, \mu_1^n(dE) \\ C_2 \int_{\mathcal{E}_1^n} \Phi_{0,0,2}^{(E)}(K \cap E)\, \mu_1^n(dE) \\ \vdots \\ C_s \int_{\mathcal{E}_1^n} \Phi_{0,0,s}^{(E)}(K \cap E)\, \mu_1^n(dE) \end{pmatrix} = C \begin{pmatrix} \Phi_{n-1,0,0}(K) \\ \Phi_{n-1,0,2}(K) \\ \vdots \\ \Phi_{n-1,0,s}(K) \end{pmatrix}$$

where

$$C = \begin{pmatrix} c_0^{(0)} & 0 & 0 & \dots & 0 \\ c_0^{(1)} Q & c_1^{(1)} & 0 & & \vdots \\ \vdots & & \ddots & & 0 \\ c_0^{(\frac{s}{2})} Q^{\frac{s}{2}} & c_1^{(\frac{s}{2})} Q^{\frac{s}{2}-1} & \dots & c_{\frac{s}{2}-1}^{(\frac{s}{2})} Q & c_{\frac{s}{2}}^{(\frac{s}{2})} \end{pmatrix}$$

and $C_j = \frac{\pi j! \omega_{j+1}^2 \omega_n}{2\omega_{n+j+1}}$ for $j = 0, 2, 4, \dots, s$. Our aim is to express $\Phi_{n-1,0,s}(K)$ as an integral, hence we have to invert the system. Notice that the constants $c_i^{(i)}$ are non-zero, which ensures that the system actually is invertible. The system can be inverted by the matrix

$$D = \begin{pmatrix} d_{00} & 0 & 0 & \dots & 0 \\ d_{10} Q & d_{11} & 0 & & \vdots \\ d_{20} Q^2 & d_{21} Q & d_{22} & 0 & \\ \vdots & & & \ddots & 0 \\ d_{\frac{s}{2}0} Q^{\frac{s}{2}} & d_{\frac{s}{2}1} Q^{\frac{s}{2}-1} & \dots & & d_{\frac{s}{2}\frac{s}{2}} \end{pmatrix}, \tag{3.9}$$

where $d_{ii} = \frac{1}{c_i^{(i)}}$ for $i = 0, \ldots, \frac{s}{2}$, and $d_{ij} = -\frac{1}{c_i^{(i)}} \sum_{k=j}^{i-1} c_k^{(i)} d_{kj}$ for $i = 1, \ldots, \frac{s}{2}$ and $j = 0, \ldots, i-1$. In particular, we have

$$\Phi_{n-1,0,s}(K) = \sum_{j=0}^{\frac{s}{2}} d_{\frac{s}{2}j} Q^{\frac{s}{2}-j} C_{2j} \int_{\mathcal{E}_1^n} \Phi_{0,0,2j}^{(E)}(K \cap E)\, \mu_1^n(dE). \tag{3.10}$$

Notice that only the dimension of the matrix (3.9) depends on s, hence we get the same integral formulas for the lower rank tensors for different choices of s. Formula (3.4) and the above considerations give the following 'inverse' version of the Crofton formula stated in Theorem 3.1.

Theorem 3.4 *Let $K \in \mathcal{K}^n$ and let $s \in \mathbb{N}_0$ be even. Then*

$$\int_{\mathcal{E}_1^n} G_s(\pi(E)) V_0(K \cap E)\, \mu_1^n(dE) = \Phi_{n-1,0,s}(K), \tag{3.11}$$

where

$$G_{2m}(L) := \sum_{j=0}^{m} \frac{2d_{mj} C_{2j}}{(2j)!\, \omega_{2j+1}} Q^{m-j} Q(L)^j$$

for $L \in \mathcal{L}_1^n$ and $m \in \mathbb{N}_0$.

It should be remarked that the measurement function in (3.11) is just a linear combination of the relative tensors of even rank at most s, but we prefer the present form to indicate the dependence on K more explicitly.

Example 3.5 For $s = 4$ the matrices are

$$C = \begin{pmatrix} 2 & 0 & 0 \\ 2Q & 8\pi & 0 \\ 2Q^2 & 16\pi Q & -\frac{64\pi^2}{3} \end{pmatrix}$$

and

$$D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{8\pi} Q & \frac{1}{8\pi} & 0 \\ -\frac{3}{64\pi^2} Q^2 & \frac{3}{32\pi^2} Q & -\frac{3}{64\pi^2} \end{pmatrix}. \tag{3.12}$$

Since $C_0 = \frac{2\pi\omega_n}{\omega_{n+1}}$, $C_2 = \frac{16\pi^3\omega_n}{\omega_{n+3}}$ and $C_4 = \frac{256\pi^5\omega_n}{3\omega_{n+5}}$, we have

$$G_4(L) = -\frac{\omega_n}{32\pi\omega_{n+1}} \big(3Q^2 - 6(n+1) Q Q(L) + (n+1)(n+3) Q(L)^2\big).$$

Due to the remark after Equation (3.10), the matrices C and D can be used to calculate G_2. We obtain

$$G_2(L) = \frac{\omega_n}{4\omega_{n+1}} \Big((n+1) Q(L) - Q\Big)$$

for $L \in \mathcal{L}_1^n$.

In Theorem 3.4 we only considered the situation, where s is even. It is natural to ask whether $\Phi_{n-1,0,s}(K)$ can also be written as a linear Crofton integral when s is odd. The case $s = 1$ is trivial, as the tensor $\Phi_{n-1,0,1}(K) = 0$ for all $K \in \mathcal{K}^n$. If $n = 1$, then $\Phi_{n-1,0,s}(K) = 0$ for all odd s, since the area measure of order 0 is the Hausdorff measure on the sphere. Apart from these trivial examples, $\Phi_{n-1,0,s}$ cannot be written as a linear Crofton-type integral, when s is odd and the measurement function satisfies some rather weak assumptions. This is shown in Theorem 3.6. The theorem involves a measurement function $\alpha : \mathcal{M} \to \mathbb{T}^s$, where $\mathcal{M} = \{(E, K) \in \mathcal{E}_1^n \times \mathcal{K}^n \mid K \subseteq E\}$. We say that α is integrable if $E \mapsto \alpha(E, K \cap E)$ is measurable and $E \mapsto \|\alpha(E, K \cap E)\|_{\mathbb{T}^s}$ is integrable with respect to μ_1^n for all $K \in \mathcal{K}^n$. In particular, this condition ensures that

$$\int_{\mathcal{E}_1^n} \|\alpha(E, \emptyset)\|_{\mathbb{T}^s}\, \mu_1^n(dE) < \infty. \tag{3.13}$$

Let $\mathcal{E}^* := \{E \in \mathcal{E}_1^n \mid E \cap B^n \neq \emptyset\}$. In addition to the integrability of α, we assume that for any $K \in \mathcal{K}^n$ with $K \subseteq B^n$ there is a finite constant $C_K = C$ such that

$$\int_{\mathcal{E}^*} \|\alpha(E + x, (K \cap E) + x)\|_{\mathbb{T}^s}\, \mu_1^n(dE) \le C \tag{3.14}$$

for all $x \in \mathbb{R}^n$.

We call a measurement function α translation invariant, if $\alpha(E + z, (K \cap E) + z) = \alpha(E, K \cap E)$ for all $K \in \mathcal{K}^n$, $E \in \mathcal{E}_1^n$ and $z \in \mathbb{R}^n$. Moreover, α is said to be bounded, if $\sup\{\|\alpha(E, K \cap E)\|_{\mathbb{T}^s} \mid (E, K) \in \mathcal{E}_1^n \times \mathcal{K}^n\} < \infty$. Assuming that α is integrable, (3.14) is satisfied, for instance, if α is translation invariant or bounded.

Theorem 3.6 *Let $n \ge 2$ and let $s > 1$ be odd. There is no integrable map $\alpha : \mathcal{M} \to \mathbb{T}^s$ such that* (3.14) *and*

$$\int_{\mathcal{E}_1^n} \alpha(E, K \cap E)\, \mu_1^n(dE) = \Phi_{n-1,0,s}(K) \tag{3.15}$$

for all $K \in \mathcal{K}^n$ are satisfied.

P r o o f . Assume that there exists an integrable map α such that (3.14) and (3.15) are satisfied. Let $K \in \mathcal{K}^n$ with $K \subseteq B^n$ and let $r > 0$. Since $(x, E) \mapsto \alpha(E + x, (K \cap E) + x)$ is measurable and $K \cap E = \emptyset$ for $E \in \mathcal{E}_1^n \setminus \mathcal{E}^*$,

$$\begin{aligned}
&\int_{rB^n} \int_{\mathcal{E}_1^n} \|\alpha(E + x, (K \cap E) + x)\|_{\mathbb{T}^s}\, \mu_1^n(dE)\, \lambda(dx) \\
&\quad \le \int_{rB^n} \int_{\mathcal{E}^*} \|\alpha(E + x, (K \cap E) + x)\|_{\mathbb{T}^s}\, \mu_1^n(dE)\, \lambda(dx) \\
&\qquad\qquad + \int_{rB^n} \int_{\mathcal{E}_1^n} \|\alpha(E + x, \emptyset)\|_{\mathbb{T}^s}\, \mu_1^n(dE)\, \lambda(dx) \\
&\quad \le C V_n(rB^n) + V_n(rB^n) \int_{\mathcal{E}_1^n} \|\alpha(E, \emptyset)\|_{\mathbb{T}^s}\, \mu_1^n(dE) < \infty,
\end{aligned}$$

where we used that μ_1^n is translation invariant, (3.13) and (3.14). Hence, the map $(x, E) \mapsto \alpha(E + x, (K \cap E) + x)$ is integrable on $rB^n \times \mathcal{E}_1^n$, which justifies that Fubini's theorem can be applied in the following.

By Fubini's theorem, the translation invariance of μ_1^n, (3.15) and the fact that $\Phi_{n-1,0,s}$ is translation invariant, we then obtain that

$$\begin{aligned}
&\frac{1}{V_n(rB^n)} \int_{\mathcal{E}_1^n} \int_{rB^n} \alpha(E + x, (K \cap E) + x)\, \lambda(dx)\, \mu_1^n(dE) \\
&\quad = \frac{1}{V_n(rB^n)} \int_{rB^n} \int_{\mathcal{E}_1^n} \alpha(E + x, (K \cap E) + x)\, \mu_1^n(dE)\, \lambda(dx) \\
&\quad = \frac{1}{V_n(rB^n)} \int_{rB^n} \int_{\mathcal{E}_1^n} \alpha(E, (K + x) \cap E)\, \mu_1^n(dE)\, \lambda(dx) \\
&\quad = \frac{1}{V_n(rB^n)} \int_{rB^n} \Phi_{n-1,0,s}(K + x)\, \lambda(dx) \\
&\quad = \Phi_{n-1,0,s}(K).
\end{aligned}$$

Then

$$\begin{aligned}
&\Phi_{n-1,0,s}(K) - \Phi_{n-1,0,s}(-K) \\
&\quad = \frac{1}{V_n(rB^n)} \left(\int_{\mathcal{E}_1^n} \int_{rB^n} \alpha(E + x, (K \cap E) + x)\, \lambda(dx)\, \mu_1^n(dE) \right. \\
&\qquad\qquad \left. - \int_{\mathcal{E}_1^n} \int_{rB^n} \alpha(E + x, ((-K) \cap E) + x)\, \lambda(dx)\, \mu_1^n(dE) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{V_n(rB^n)} \left(\int_{\mathcal{E}^*} \int_{rB^n} \alpha(E + x, (K \cap E) + x)\, \lambda(dx)\, \mu_1^n(dE) \right. \\
&\qquad \left. - \int_{\mathcal{E}^*} \int_{rB^n} \alpha(E + x, ((-K) \cap E) + x)\, \lambda(dx)\, \mu_1^n(dE) \right) \\
&= \frac{1}{V_n(rB^n)} \left(\int_{\mathcal{E}^*} \int_{rB^n} \alpha(E + x, (K \cap E) + x)\, \lambda(dx)\, \mu_1^n(dE) \right. \\
&\qquad \left. - \int_{\mathcal{E}^*} \int_{rB^n} \alpha(-E + x, -(K \cap E) + x)\, \lambda(dx)\, \mu_1^n(dE) \right),
\end{aligned}
$$

where we used that $E \in \mathcal{E}^*$ if and only if $-E \in \mathcal{E}^*$ and the fact that μ_1^n is reflection invariant.

For $K \in \mathcal{K}^n$ with $K \subseteq B^n$ and $E \in \mathcal{E}_1^n$, the intersection $K \cap E$ is either the empty set, a singleton or a line segment in B^n. Hence, there exists a vector $z_E \in \mathbb{R}^n$ with $\|z_E\| \leq 2$ such that $-(K \cap E) = (K \cap E) + z_E$ and $-E = E + z_E$. Let $B_1 \Delta B_2$ denote the symmetric difference of two sets $B_1, B_2 \subseteq \mathbb{R}^n$. For $r > 2$ we have $(rB^n + z_E)\Delta(rB^n) \subseteq (r+2)B^n \setminus (r-2)B^n$, and therefore

$$
\begin{aligned}
&\left\| \Phi_{n-1,0,s}(K) - \Phi_{n-1,0,s}(-K) \right\|_{\mathbb{T}^s} \\
&\leq \frac{1}{V_n(rB^n)} \int_{\mathcal{E}^*} \int_{(rB^n + z_E)\Delta(rB^n)} \|\alpha(E + x, (K \cap E) + x)\|_{\mathbb{T}^s}\, \lambda(dx)\, \mu_1^n(dE) \\
&\leq \frac{1}{V_n(rB^n)} \int_{\mathcal{E}^*} \int_{(r+2)B^n \setminus (r-2)B^n} \|\alpha(E + x, (K \cap E) + x)\|_{\mathbb{T}^s}\, \lambda(dx)\, \mu_1^n(dE) \\
&\leq \frac{1}{V_n(rB^n)} \int_{(r+2)B^n \setminus (r-2)B^n} \int_{\mathcal{E}^*} \|\alpha(E + x, (K \cap E) + x)\|_{\mathbb{T}^s}\, \mu_1^n(dE)\, \lambda(dx) \\
&\leq C \frac{(r+2)^n - (r-2)^n}{r^n} \longrightarrow 0 \qquad \text{as } r \to \infty.
\end{aligned}
$$

Hence, we get $\Phi_{n-1,0,s}(K) = \Phi_{n-1,0,s}(-K)$. Since s is odd, we also have $\Phi_{n-1,0,s}(K) = -\Phi_{n-1,0,s}(-K)$. Therefore, $\Phi_{n-1,0,s}(K) = 0$ for all convex bodies $K \subseteq B^n$. By translation invariance and homogeneity this implies that $\Phi_{n-1,0,s}(K) = 0$ for all $K \in \mathcal{K}^n$. This is not the case, since $\Phi_{n-1,0,s}$ is a member of a basis in the vector space of translation invariant tensor valuations. See also the special case $v = 0$ of Lemma 5.3 in [11]. The contradiction proves the theorem. □

4 Design based estimation

In this section we use the integral formula (3.11) in Theorem 3.4 to derive unbiased estimators of the surface tensors $\Phi_{n-1,0,s}(K)$ of $K \in \mathcal{K}^n$, when s is even. We assume throughout this chapter that $n \geq 2$. Three different types of estimators based on 1-dimensional linear sections are presented. First, we establish estimators based on isotropic uniform random lines, then estimators based on random lines in vertical sections and finally estimators based on non-isotropic uniform random lines.

4.1 Estimation based on isotropic uniform random lines

In this section we construct estimators of $\Phi_{n-1,0,s}(K)$ based on isotropic uniform random lines. Let $K \in \mathcal{K}^n$. We assume that (the unknown set) K is contained in a compact reference set $A \subseteq \mathbb{R}^n$, the latter being known. Now let E be an *isotropic uniform random (IUR) line in* $\mathbb{R}^n$ *hitting* A, i.e., the distribution of E is given by

$$
\mathbb{P}(E \in \mathcal{A}) = c_1(A) \int_{\mathcal{A}} \mathbf{1}(E' \cap A \neq \emptyset)\, \mu_1^n(dE') \tag{4.1}
$$

for $\mathcal{A} \in \mathcal{B}(\mathcal{E}_1^n)$, where $c_1(A)$ is the normalizing constant

$$c_1(A) = \left(\int_{\mathcal{E}_1^n} \mathbf{1}(E' \cap A \neq \emptyset)\, \mu_1^n(dE') \right)^{-1}.$$

By (3.1) with $s = 0$ the normalizing constant becomes $c_1(A) = \frac{\omega_n}{2\kappa_{n-1}} V_{n-1}(A)^{-1}$, when A is a convex body. Then Theorem 3.4 implies that

$$c_1(A)^{-1} G_s(\pi(E)) V_0(K \cap E) \tag{4.2}$$

is an unbiased estimator of $\Phi_{n-1,0,s}(K)$, when s is even.

Example 4.1 Using the expressions of G_2 and G_4 in Example 3.5 we get that

$$-\frac{V_{n-1}(A)}{32\pi^2}\big(3Q^2 - 6(n+1)QQ(\pi(E)) + (n+1)(n+3)Q(\pi(E))^2\big)V_0(K \cap E)$$

is an unbiased estimator of $\Phi_{n-1,0,4}(K)$, and

$$\frac{V_{n-1}(A)}{4\pi}\Big((n+1)Q(\pi(E)) - Q\Big)V_0(K \cap E) \tag{4.3}$$

is an unbiased estimator of $\Phi_{n-1,0,2}(K)$, when A is a convex body. For $n = 3$, these estimators read

$$-\frac{3V_2(A)}{32\pi^2}\Big(Q^2 - 8QQ(\pi(E)) + 8Q(\pi(E))^2\Big)V_0(K \cap E)$$

and

$$\frac{V_2(A)}{\pi}\Big(Q(\pi(E)) - \frac{1}{4}Q\Big)V_0(K \cap E). \tag{4.4}$$

An investigation of the estimators in Example 4.1 shows that they possess some unfavourable statistical properties. With probability $1 - \frac{V_{n-1}(K)}{V_{n-1}(A)}$ the test line E does not hit K. In this situation the estimators are simply zero and contain, hereby, no information on the shape of K. Furthermore, if $K \cap E \neq \emptyset$, the matrix representation of the estimator (4.3) of $\Phi_{n-1,0,2}(K)$ is, in contrast to $\Phi_{n-1,0,2}(K)$, not positive semi-definite. In fact, the eigenvalues of the matrix representation of $(n+1)Q(\pi(E)) - Q$ are n (with multiplicity 1) and -1 (with multiplicity $n - 1$). It is not surprising that estimators based on the measurement of one single line, are not sufficient, when we are estimating tensors with many unknown parameters. To improve the estimators, they can be extended in a natural way to use information from N *IUR* lines for some $N \in \mathbb{N}$. In addition, the integral formula (3.11) can be rewritten in the form

$$\begin{aligned}\Phi_{n-1,0,s}(K) &= \int_{\mathcal{L}_1^n}\int_{L^\perp} G_s(L)V_0(K \cap (L+x))\,\lambda_{L^\perp}(dx)\,\nu_1^n(dL)\\ &= \int_{\mathcal{L}_1^n} G_s(L)V_{n-1}(K|L^\perp)\,\nu_1^n(dL),\end{aligned} \tag{4.5}$$

which implies that

$$\frac{1}{N}\sum_{i=1}^N G_s(L_i)V_{n-1}(K|L_i^\perp) \tag{4.6}$$

is an unbiased estimator of $\Phi_{n-1,0,s}(K)$, when $L_1, \ldots L_N \in \mathcal{L}_1^n$ are N isotropic lines (through the origin) for an $N \in \mathbb{N}$. When K is full-dimensional this estimator never vanishes. In the case where $s = 2$ the estimator becomes

$$\frac{1}{N}\frac{\omega_n}{4\omega_{n+1}}\sum_{i=1}^N \big((n+1)Q(L_i) - Q\big)V_{n-1}(K|L_i^\perp). \tag{4.7}$$

In stereology it is common practice to use orthogonal test lines. If we set $N = n$ and let $L_1, \dots, L_n$ be isotropic, pairwise orthogonal lines, then the estimator (4.7) becomes positive definite exactly when

$$(n+1)V_{n-1}\big(K|L_i^{\perp}\big) > \sum_{j=1}^{n} V_{n-1}\big(K \mid L_j^{\perp}\big) \tag{4.8}$$

for all $i = 1, \dots, n$. This is a condition on K requiring that K is not too eccentric. A sufficient condition for (4.8) to hold makes use of the radius $R(K)$ of the smallest ball containing K and the radius $r(K)$ of the largest ball contained in K. If

$$\frac{r(K)}{R(K)} > \left(1 - \frac{1}{n}\right)^{\frac{1}{n-1}}, \tag{4.9}$$

then (4.8) is satisfied, and hence the estimator (4.7) with n orthogonal, isotropic lines is positive definite. In $\mathbb{R}^2$ this means that $2r(K) > R(K)$ is sufficient for a positive definite estimator (4.7), and in particular for all ellipses for which the length of the longer main axis does not exceed twice the length of the smaller main axis, (4.7) yields positive definite estimators. For ellipses, this criterion is also necessary as the following example shows.

Example 4.2 Consider the situation where $n = 2$ and K is an ellipse, $K = \big\{x \in \mathbb{R}^2 \mid x^{\top}Bx \leq 1\big\}$, given by the matrix

$$B = \begin{pmatrix} \alpha^{-2} & 0 \\ 0 & (k\alpha)^{-2} \end{pmatrix},$$

where $\alpha > 0$ and $k \in (0, 1]$. The parameter k determines the eccentricity of K. If $k \in \big(\frac{1}{2}, 1\big]$, and L_1 and L_2 are orthogonal, isotropic random lines in $\mathbb{R}^2$, the estimator (4.7) becomes positive definite by the above considerations. Now let $k \in (0, 1/2]$. Since $n = 2$, each pair of orthogonal lines is determined by a constant $\phi \in \big[0, \frac{\pi}{2}\big)$ by letting $L_1 = u_{\phi}^{\perp}$ and $L_2 = u_{\phi+\frac{\pi}{2}}^{\perp}$, where $u_{\phi} = (\cos(\phi), \sin(\phi))^{\top}$. Then

$$V_{n-1}(K \mid L_1^{\perp}) = 2h(K, u_{\phi}) = 2\alpha\sqrt{\cos^2(\phi) + k^2\sin^2(\phi)}$$

and

$$V_{n-1}(K \mid L_2^{\perp}) = 2\alpha\sqrt{\sin^2(\phi) + k^2\cos^2(\phi)}.$$

Condition (4.8) is satisfied if and only if

$$\phi \in \left[\sin^{-1}\left(\sqrt{\frac{1-4k^2}{5(1-k^2)}}\right), \cos^{-1}\left(\sqrt{\frac{1-4k^2}{5(1-k^2)}}\right)\right],$$

and the probability that the estimator is positive definite, when L_1 and L_2 are orthogonal, isotropic lines (corresponding to ϕ being uniformly distributed on $[0, \frac{\pi}{2}]$) is

$$\frac{2}{\pi}\left(\cos^{-1}\left(\sqrt{\frac{1-4k^2}{5(1-k^2)}}\right) - \sin^{-1}\left(\sqrt{\frac{1-4k^2}{5(1-k^2)}}\right)\right),$$

which converges to $\frac{2}{\pi}\Big(\cos^{-1}\Big(\sqrt{\frac{1}{5}}\Big) - \sin^{-1}\Big(\sqrt{\frac{1}{5}}\Big)\Big) \approx 0.41$ as k converges to 0.

In $\mathbb{R}^2$ the estimator (4.7) can alternatively be combined with a systematic sampling approach with N isotropic random lines. Let $N \in \mathbb{N}$, and let ϕ_0 be uniformly distributed on $[0, \frac{\pi}{N}]$. Moreover, let $\phi_i = \phi_0 + i\frac{\pi}{N}$ for $i = 1, \dots, N-1$. Then $u_{\phi_0}, \dots, u_{\phi_{N-1}}$ are N systematic isotropic uniform random directions in the upper half of S^1, where $u_{\phi} = (\cos(\phi), \sin(\phi))^{\top}$. As the estimator (4.7) is a tensor of rank 2, it can be identified with the symmetric 2×2 matrix, where the (i, j)'th entry is the estimator evaluated at (e_i, e_j), where (e_1, e_2) is the standard basis of $\mathbb{R}^2$. The estimator becomes

$$S_N(K, \phi_0) = \frac{1}{N}\sum_{i=0}^{N-1}\begin{pmatrix} 3\cos^2(\phi_i) - 1 & 3\cos(\phi_i)\sin(\phi_i) \\ 3\cos(\phi_i)\sin(\phi_i) & 3\sin^2(\phi_i) - 1 \end{pmatrix} V_1\left(K \mid u_{\phi_i}^{\perp}\right). \tag{4.10}$$

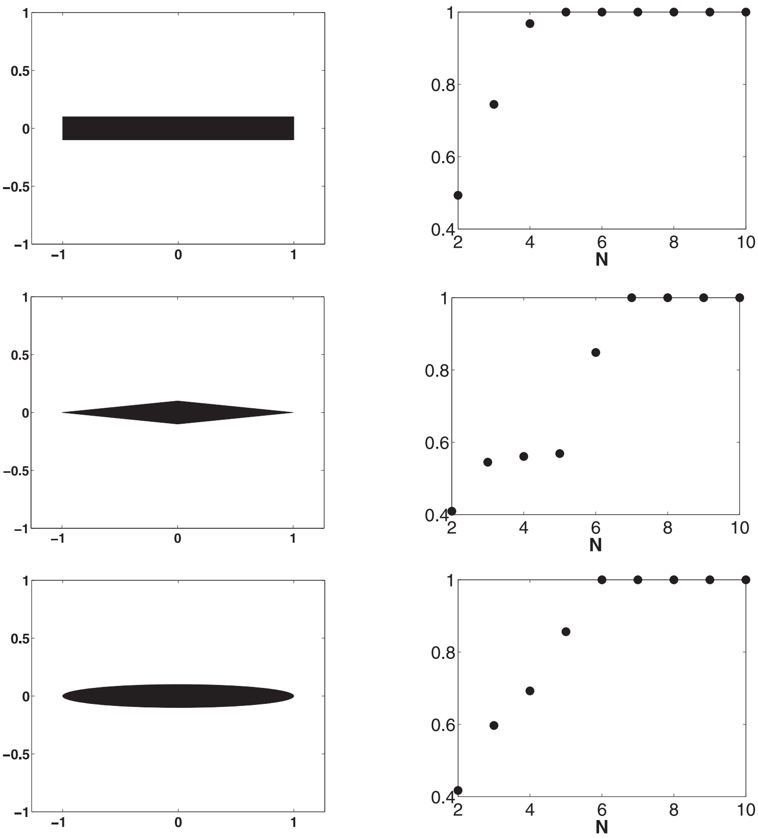

Fig. 1 The probability that $S_N(K_i, \phi_0)$ is positive definite for $i = 1, 2, 3$, when ϕ_0 is uniformly distributed on $[0, \frac{\pi}{N}]$ plotted against the number of equidistant lines N.

Example 4.3 To investigate how the estimator $S_N(K, \phi_0)$ performs we estimate the probability that the estimator is positive definite for three different origin-symmetric convex bodies in $\mathbb{R}^2$; a parallelogram, a rectangle, and an ellipse. Thus let

$$K_1 = \mathrm{conv}\{(1, \epsilon), (-1, \epsilon), (-1, -\epsilon), (1, -\epsilon)\},$$

$$K_2 = \mathrm{conv}\{(1, 0), (0, \epsilon), (-1, 0), (0, -\epsilon)\}$$

and

$$K_3 = \left\{ x \in \mathbb{R}^2 \mid x^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\epsilon}} \end{pmatrix} x \leq 1 \right\}$$

with $\epsilon = 0.1$. The support functions, and hence the intrinsic volumes $V_1\big(K_i|u_\phi^\perp\big)$, of K_1, K_2 and K_3 have simple analytic expressions, and the estimator $S_N(K_i, \phi_0)$ can be calculated for $\phi_0 \in [0, \frac{\pi}{N}]$ and $i = 1, 2, 3$. The eigenvalues of the estimators can be calculated numerically, and the probability that the estimators $S_N(K_i, \phi_0)$ are positive definite, when ϕ_0 is uniformly distributed on $[0, \frac{\pi}{N}]$, can hereby be estimated. For each choice of N, the estimate of the probability is based on 500 equally spread values of ϕ_0 in $[0, \frac{\pi}{N}]$. The estimate of the probability that $S_N(K_i, \phi_0)$ is positive definite is plotted against the number of equidistant lines N for $i = 1, 2, 3$ in Figure 1. The plots in Figure 1 show that even though we consider rather eccentric shapes, the number N of lines needed to get a positive definite estimator with probability 1 is in all cases less than 7.

To apply the estimator (4.2) it is only required to observe whether the test line hits or misses the convex body K. The estimator (4.6) requires more sophisticated information in terms of the projection function. In the following example the coefficient of variation of versions of the estimators (4.4) and (4.7) are estimated and compared in a three-dimensional set-up.

Example 4.4 Let K_l' be the prolate spheroid in $\mathbb{R}^3$ with main axis parallel to the standard basis vectors e_1, e_2 and e_3, and corresponding lengths of semi-axes $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = l$. For $l = 1, \dots, 5$, let K_l denote the ellipsoid obtained by rotating K_l' first around e_1 with an angle $\frac{3\pi}{16}$, and then around e_2 with an angle $\frac{5\pi}{16}$. Note, that the eccentricity of K_l increases with l. In this example, based on simulations, we estimate and compare the coefficient of variation (CV) of the developed estimators of $\Phi_{2,0,2}(K_l)$ for $l = 1, \dots, 5$.

Formula (4.4) provides an unbiased estimator of the tensor $\Phi_{2,0,2}(K_l)$ for $l = 1, \dots, 5$. The estimator is based on one *IUR* line hitting a reference set A, and can in a natural way be extended to an estimator based on three orthogonal *IUR* lines hitting A. We estimate the variance of both estimators. Let, for $l = 1, \dots, 5$, the reference set A_l be a ball of radius $R_l > 0$. The choice of the reference set influences the variance of the estimator. In order to minimize this effect in the comparison of the CV's, the radii of the reference sets are chosen such that the probability that a test line hits K_l is constant for $l = 1, \dots, 5$. By formula (4.1) the probability that an *IUR* line hitting A_l hits K_l is $\frac{V_2(K_l)}{V_2(A_l)}$. The radius is chosen, such that this probability is $\frac{1}{7}$. We further estimate the variance of the projection estimator (4.7) based on one isotropic line and on three orthogonal isotropic lines.

As $\Phi_{2,0,2}(K_l)$ is a tensor of rank 2, it can be identified with the symmetric matrix $\{\Phi_{2,0,2}(K_l)(e_i, e_j)\}_{i,j=1}^3$ of size 3×3. Thus, in order to estimate $\Phi_{2,0,2}(K_l)$, the matrix $\left\{\hat{\Phi}_{2,0,2}(K_l)(e_i, e_j)\right\}_{i,j=1}^3$ is calculated. Here, $\hat{\Phi}_{2,0,2}(K_l)$ refers to any of the four estimators described above. Due to symmetry, there are six different components of the matrices.

The estimates of the variances are based on 1500–10000 estimates of the tensor, depending on the choice of the estimator and the eccentricity of K_l. Using the estimates of the variances, we estimate the absolute value of the CV's by

$$\widehat{CV}_{ij} = \frac{\sqrt{\widehat{\mathrm{Var}}(\hat{\Phi}_{2,0,2}(K_l)(e_i, e_j))}}{|\Phi_{2,0,2}(K_l)(e_i, e_j)|},$$

for $i, j = 1, 2, 3$ and $l = 1, \dots, 5$. As K_l is an ellipsoid, the tensor $\Phi_{2,0,2}(K_l)$ can be calculated numerically. The CV's of the four estimators are plotted in Figure 2 for each of the six different components of the associated matrix. As K_1 is a ball, the off-diagonal elements of the matrix associated with $\Phi_{2,0,2}(K_1)$ are zero. Thus, the CV is in this case calculated only for the estimators of the diagonal-elements.

The projection estimators give, as expected, smaller CV's, than the estimators based on the Euler characteristic of the intersection between the test lines and the ellipsoid. For the estimators based on one test line the CV of the projection estimator is typically around 38% of the corresponding estimator (4.4). For the estimators based on three orthogonal test lines, the CV of the projection estimator is typically 9% of the estimator (4.4), when $l = 2, \dots, 5$. Due to the fact that K_1 is a ball, the variance of the projection estimator based on three orthogonal lines is 0, when $l = 1$.

It is interesting to compare the increase of efficiency when using the estimator based on three orthogonal test lines instead of three i.i.d. test lines. The CV of an estimator based on three i.i.d. test lines is $\frac{1}{\sqrt{3}}$ of the CV of the estimator (4.4), (the "+" signs in Figure 2). The CV, when using three orthogonal test lines, is typically around 92% of that CV. For $l = 2, \dots, 5$, the CV's of the projection estimator based on three orthogonal lines, are typically 20% of the CV, when using three i.i.d lines, indicating that spatial random systematic sampling increases precision without extra workload.

The CV's of the estimators of the diagonal-elements $\Phi_{2,0,2}(K_l)(e_i, e_i)$ are almost constant in l. Hence the eccentricity of K_l does not affect the CV's for these choices of l. There is a decreasing tendency of the CV's of the estimators of the off-diagonal elements. This might be explained by the fact that the true value of $\Phi_{2,0,2}(K_l)(e_i, e_j)$ is close to zero, when $i \neq j$ and l is small.

The above example shows that only the projection estimator based on three orthogonal test lines has a satisfactory precision. For $l = 2$ the CV's are approximately $\frac{1}{3}$ for the diagonal-elements and 1 for the off-diagonal elements. Further variance reduction of the projection estimator can be obtained by using a larger

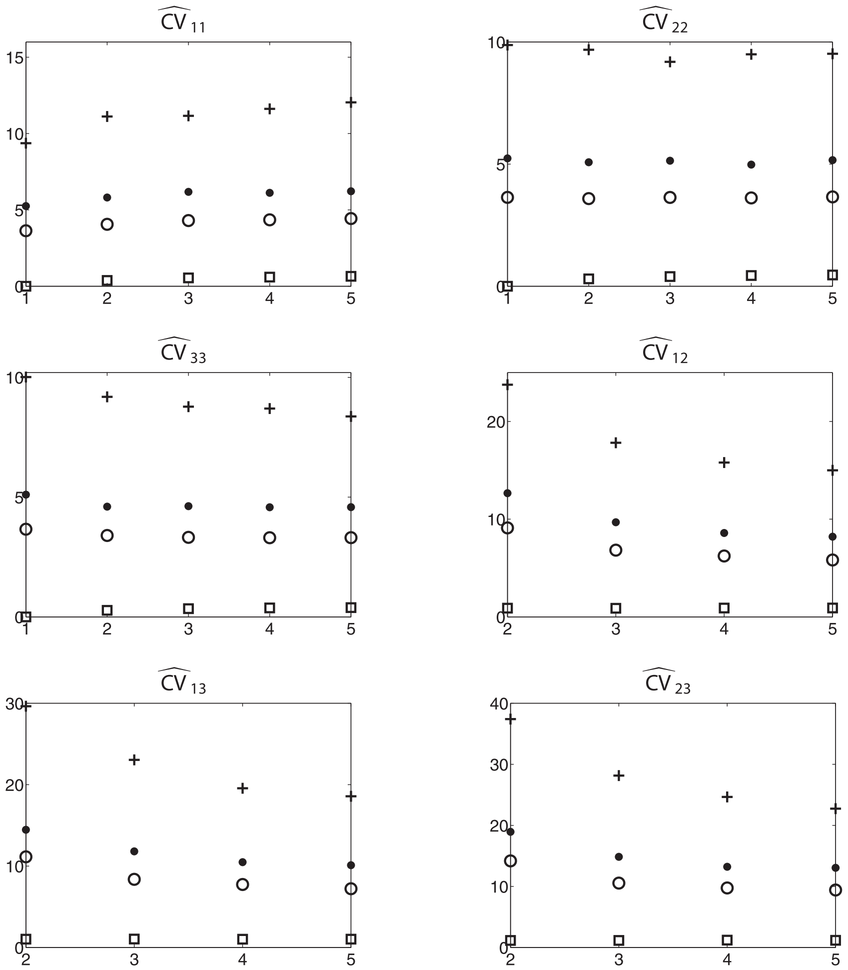

Fig. 2 The estimated coefficients of variation $\widehat{CV}_{ij}$ of the estimators of $\Phi_{2,0,2}(K_l)(e_i, e_j)$ plotted against l for $i, j \in \{1, 2, 3\}$. The CV of the estimator (4.4) based on *one* line is designated by "+", while the CV of the corresponding estimator based on *three* lines is designated by "•". The CV of the projection estimator is designated by "∘" and "□" for one and three lines, respectively.

number of systematic random test directions. For $n = 2$ this can be effectuated by choosing equidistant points on the upper half circle; see (4.10). For $n = 3$ the directions must be chosen evenly spread; see [19] for details.

If the projections are not available or too costly to obtain, systematic sampling in the position of the test lines with given orientations can be applied. In $\mathbb{R}^2$ this corresponds to a Steinhaus-type estimation procedure (see, e.g., [5]). In $\mathbb{R}^3$ the fakir method described in [18] can be applied.

4.2 Estimation based on vertical sections

In the previous section we constructed an estimator of $\Phi_{n-1,0,s}(K)$ based on isotropic uniform random lines. As described in [17], it is sometimes inconvenient or impossible to use the *IUR* design in applications. For instance, in biology when analysing skin tissue, it might be necessary to use sample sections, which are normal to the surface of the skin, so that the different layers become clearly distinguishable in the sample. Instead of using *IUR* lines it is then a possibility to use vertical sections introduced by Baddeley in [4]. The idea is to fix a direction (the normal of the skin surface), and only consider flats parallel to this direction. After randomly selecting a flat among these flats, we want to pick a line in the flat in such a way that this line is an isotropic uniform random line in $\mathbb{R}^n$. Like in the classical formulae for vertical sections, we select this line in a non-uniform way according to a Blaschke-Petkantschin formula (see (4.13)). This idea is used to deduce estimators of $\Phi_{n-1,0,s}(K)$ from the Crofton formula (3.11).

When introducing the concept of vertical sections we use the following notation. For $0 \leq k \leq n$ and $L \in \mathcal{L}_k^n$, let

$$\mathcal{L}_r^L = \begin{cases} \{M \in \mathcal{L}_r^n \mid M \subseteq L\} & \text{if } 0 \leq r \leq k, \\ \{M \in \mathcal{L}_r^n \mid L \subseteq M\} & \text{if } k < r \leq n, \end{cases}$$

and, similarly, let $\mathcal{E}_r^E = \{F \in \mathcal{E}_r^n \mid F \subseteq E\}$ for $E \in \mathcal{E}_k^n$ and $0 \leq r \leq k$. Let ν_r^L denote the unique rotation invariant probability measure on $\mathcal{L}_r^L$, and let μ_r^E denote the motion invariant measure on $\mathcal{E}_r^E$ normalized as in [24].

Let $L_0 \in \mathcal{L}_1^n$ be fixed. This is the *vertical axis* (the normal of the skin surface in the example above). Let the reference set $A \subseteq \mathbb{R}^n$ be a compact set.

Definition 4.5 Let $1 < k < n$. A random k-flat H in $\mathbb{R}^n$ is called a *vertical uniform random (VUR) k-flat hitting A* if the distribution of H is given by

$$P(H \in \mathcal{A}) = c_2(A) \int_{\mathcal{L}_k^{L_0}} \int_{A|L^\perp} \mathbf{1}(L + x \in \mathcal{A})\, \lambda_{L^\perp}(dx)\, \nu_k^{L_0}(dL)$$

for $\mathcal{A} \in \mathcal{B}(\mathcal{E}_k^n)$, where $c_2(A) > 0$ is a normalizing constant.

The distribution of H is concentrated on the set

$$\{E \in \mathcal{E}_k^n \mid E \cap A \neq \emptyset, L_0 \subseteq \pi(E)\}.$$

When the reference set A is a convex body, the normalizing constant becomes

$$c_2(A) = \binom{n-1}{k-1} \frac{\kappa_{n-1}}{\kappa_{k-1}\kappa_{n-k}} \frac{1}{V_{n-k}(A|L_0^\perp)}.$$

(Note that we do not indicate the dependence of $c_2(A)$ on k by our notation.) This can be shown, e.g., by using the definition of $\nu_k^{L_0}$ together with [24, (13.13)], Crofton's formula in the space $L_0^\perp$, and the equality

$$\mathbf{1}_{A|L^\perp}(x) = V_0\left(\left(A|L_0^\perp\right) \cap (x + L)\right) \tag{4.11}$$

for $A \in \mathcal{K}^n$, $L \in \mathcal{L}_k^{L_0}$ and $x \in L^\perp$. For later use note that when $k = 2$ the normalizing constant becomes

$$c_2(A) = \frac{\omega_{n-1}}{2\kappa_{n-2} V_{n-2}(A|L_0^\perp)}. \tag{4.12}$$

To construct an estimator, which is based on a vertical uniform random flat, we cannot use Theorem 3.4 immediately as in the *IUR*-case. It is necessary to use a Blaschke-Petkantschin formula first; see [17, (2.8)]. It states that for a fixed $L_0 \in \mathcal{L}_1^n$ and an integrable function $f : \mathcal{E}_1^n \to \mathbb{R}$, we have

$$\begin{aligned} \int_{\mathcal{E}_1^n} f(E)\, \mu_1^n(dE) = \frac{\pi\omega_{n-1}}{\omega_n} \int_{\mathcal{L}_2^{L_0}} \int_{M^\perp} \int_{\mathcal{E}_1^{M+x}} & f(E) \sin(\angle(E, L_0))^{n-2} \\ & \times \mu_1^{M+x}(dE)\, \lambda_{M^\perp}(dx)\, \nu_2^{L_0}(dM), \end{aligned} \tag{4.13}$$

where $\angle(E_1, E_2)$ is the (smaller) angle between $\pi(E_1)$ and $\pi(E_2)$ for two lines $E_1, E_2 \in \mathcal{E}_1^n$. For $K \in \mathcal{K}^n$ and even $s \in \mathbb{N}_0$, Equation (4.13) can be applied coordinate-wise to the mapping $E \mapsto \Phi_{0,0,s}^{(E)}(K \cap E)$ and combined with the Crofton formula in Theorem 3.1. The result is an integral formula for two-dimensional vertical sections.

Theorem 4.6 *Let $L_0 \in \mathcal{L}_1^n$ be fixed. If $K \in \mathcal{K}^n$ and $s \in \mathbb{N}_0$ is even, then*

$$\int_{\mathcal{L}_2^{L_0}} \int_{M^\perp} \int_{\mathcal{E}_1^{M+x}} \Phi_{0,0,s}^{(E)}(K \cap E) \sin(\angle(E, L_0))^{n-2}\, \mu_1^{M+x}(dE)\, \lambda_{M^\perp}(dx)\, \nu_2^{L_0}(dM)$$

$$= \frac{2\omega_{n+s+1}}{s!\pi^2\omega_{n-1}\omega_{s+1}^2} \sum_{k=0}^{\frac{s}{2}} c_k^{(\frac{s}{2})} Q^{\frac{s}{2}-k} \Phi_{n-1,0,2k}(K), \tag{4.14}$$

where the constants $c_k^{(m)}$ *are given in Theorem* 3.1. *For odd* $s \in \mathbb{N}_0$ *the integral on the left-hand side is zero.*

If Theorem 3.1 is replaced by Theorem 3.4 in the above line of arguments, we obtain an explicit measurement function for vertical sections leading to one single tensor.

Theorem 4.7 *Let* $L_0 \in \mathcal{L}_1^n$ *be fixed. If* $K \in \mathcal{K}^n$ *and* $s \in \mathbb{N}_0$ *is even, then*

$$\frac{\omega_n}{\pi\omega_{n-1}} \Phi_{n-1,0,s}(K) = \int_{\mathcal{L}_2^{L_0}} \int_{M^\perp} \int_{\mathcal{E}_1^{M+x}} G_s(\pi(E)) V_0(K \cap E) \times \sin(\angle(E, L_0))^{n-2} \mu_1^{M+x}(dE)\, \lambda_{M^\perp}(dx)\, \nu_2^{L_0}(dM),$$

where G_s *is given in Theorem* 3.4.

Let $s \in \mathbb{N}_0$ be even and assume that $K \in \mathcal{K}^n$ is contained in a reference set $A \in \mathcal{K}^n$. Using Theorem 4.7 we are able to construct unbiased estimators of the tensors $\Phi_{n-1,0,s}(K)$ of K based on a vertical uniform random 2-flat. If H is an *VUR* 2-flat hitting A with vertical direction $L_0 \in \mathcal{L}_1^n$, then it follows from Theorem 4.7 and (4.12) that

$$V_{n-2}(A|L_0^\perp) \int_{\mathcal{E}_1^H} G_s(\pi(E)) V_0(K \cap E) \sin(\angle(E, L_0))^{n-2} \mu_1^H(dE) \tag{4.15}$$

is an unbiased estimator of $\Phi_{n-1,0,s}(K)$. Hence the surface tensors can be estimated by a two-step procedure. First, let H be a *VUR* 2-flat hitting the convex body A with vertical direction L_0. Given H, the integral

$$\int_{\mathcal{E}_1^H} G_s(\pi(E)) V_0(K \cap E) \sin(\angle(E, L_0))^{n-2} \mu_1^H(dE) \tag{4.16}$$

is estimated in the following way. Let $E \in \mathcal{E}_1^H$ be an *IUR* line in H hitting A, i.e. the distribution of E is given by

$$P(E \in \mathcal{A}) = c_3(A) \int_{\mathcal{A}} \mathbf{1}(A \cap E \neq \emptyset)\, \mu_1^H(dE), \qquad \mathcal{A} \in \mathcal{B}(\mathcal{E}_1^H),$$

where

$$c_3(A) = \frac{\pi}{2} V_1(A \cap H)^{-1}$$

is the normalizing constant. The integral (4.16) is then estimated unbiasedly by

$$c_3(A)^{-1} G_s(\pi(E)) V_0(K \cap E) \sin(\angle(E, L_0))^{n-2}. \tag{4.17}$$

Example 4.8 Consider the case $s = 2$. Let H be a *VUR* 2-flat hitting $A \in \mathcal{K}^n$ with vertical direction L_0. Given H, let E be an *IUR* line in H hitting A. Then

$$\frac{\kappa_{n-2} V_{n-2}(A|L_0^\perp) V_1(A \cap H)}{\omega_{n+1}} \Big((n+1) Q(\pi(E)) - Q\Big) V_0(K \cap E) \sin(\angle(E, L_0))^{n-2}$$

is an unbiased estimator of $\Phi_{n-1,0,2}(K)$.

Using [24, (13.13)] and an invariance argument, the integral (4.16) can alternatively be expressed in the following way

$$\int_{\mathcal{E}_1^H} G_s(\pi(E)) V_0(K \cap E) \sin(\angle(E, L_0))^{n-2} \mu_1^H(dE)$$
$$= \frac{1}{\omega_2} \int_{S^{n-1} \cap \pi(H)} G_s(u^\perp \cap \pi(H)) \sin(\angle(u^\perp \cap \pi(H), L_0))^{n-2} \times \int_{[u]} V_0(K \cap H \cap (u^\perp + x))\, \lambda_{[u]}(dx)\, \mathcal{H}^1(du)$$

$$= \frac{1}{\omega_2} \int_{S^{n-1} \cap \pi(H)} G_s(u^\perp \cap \pi(H)) \cos(\angle(u, L_0))^{n-2} w(K \cap H, u)\, \mathcal{H}^1(du),$$

where $[u]$ denotes the linear hull of a unit vector u and

$$w(M, u) = h(M, u) + h(M, -u) = \max\{\langle x, u\rangle \mid x \in M\} - \min\{\langle x, u\rangle \mid x \in M\}$$

is the width of $M \in \mathcal{K}^n$ in direction u if $M \neq \emptyset$ and zero otherwise. Hence, given H,

$$G_s(U^\perp \cap \pi(H)) \cos(\angle(U, L_0))^{n-2} w(K \cap H, U) \tag{4.18}$$

is an unbiased estimator of the integral (4.16) if U is uniform on $S^{n-1} \cap \pi(H)$. As in the *IUR* set-up in Section 4.1 we have two estimators: an estimator (4.17), where it is only necessary to observe whether the random line E hits or misses K, and the alternative estimator (4.18), which requires more information. The latter estimator has a better precision at least when the reference set A is large. Variance reduction can be obtained by combining the estimators with a systematic sampling approach.

4.3 Estimation based on non-isotropic random lines

In this section we consider estimators based on non-isotropic random lines. It is well-known from the theory of importance sampling that variance reduction of estimators can be obtained by modifying the sampling distribution in a suitable way (see, e.g., [2]). The estimators in this section are developed with inspiration from this theory. Let again $K \in \mathcal{K}^n$, and let $f : \mathcal{L}_1^n \to [0, \infty)$ be a probability density with respect to the invariant measure ν_1^n on $\mathcal{L}_1^n$ such that f is positive ν_1^n-almost surely. Then by Theorem 3.4 we have trivially

$$\int_{\mathcal{E}_1^n} \frac{G_s(\pi(E)) V_0(K \cap E)}{f(\pi(E))} f(\pi(E))\, \mu_1^n(dE) = \Phi_{n-1,0,s}(K). \tag{4.19}$$

Let $A \subseteq \mathbb{R}^n$ be a compact reference set containing K, and let E be an f-weighted random line in $\mathbb{R}^n$ hitting A, that is, the distribution of E is given by

$$P(E \in \mathcal{A}) = c_4(A) \int_{\mathcal{A}} \mathbf{1}(E \cap A \neq \emptyset) f(\pi(E))\, \mu_1^n(dE)$$

for $\mathcal{A} \in \mathcal{B}(\mathcal{E}_1^n)$, where

$$c_4(A) = \left(\int_{\mathcal{E}_1^n} \mathbf{1}(E \cap A \neq \emptyset) f(\pi(E))\, \mu_1^n(dE) \right)^{-1}$$

is a normalizing constant. Then

$$\frac{c_4(A)^{-1} G_s(\pi(E)) V_0(K \cap E)}{f(\pi(E))}$$

is an unbiased estimator of $\Phi_{n-1,0,s}(K)$. Notice that if we let the density f be constant, then this procedure coincides with the *IUR* design in Section 4.1.

Our aim is to decide which density f should be used in order to decrease the variance of the estimator of $\Phi_{n-1,0,s}(K)$. Furthermore, we want to compare this variance with the variance of the estimator based on an *IUR* line. From now on, we restrict the investigation to the situation where $n = 2$ and $s = 2$. Furthermore, we assume that the reference set A is a ball in $\mathbb{R}^2$ of radius R for some $R > 0$. Then $c_4(A) = (2R)^{-1}$ independently of f.

Since $\Phi_{1,0,2}(K)$ can be identified with a symmetric 2×2 matrix, we have to estimate three unknown components. We consider the variances of the three estimators separately. The components of the associated matrix of $G_2(L)$ for $L \in \mathcal{L}_1^n$ are defined by

$$g_{ij}(L) = G_2(L)(e_i, e_j), \tag{4.20}$$

for $i, j = 1, 2$, where (e_1, e_2) is the standard basis of $\mathbb{R}^2$. More explicitly, by Example 3.5, the associated matrix of $G_2(L)$ of the line $L = [u]$, for $u \in S^1$, is

$$\{g_{ij}([u])\}_{ij} = \frac{3}{8} \begin{pmatrix} u_1^2 - \frac{1}{3} & u_1 u_2 \\ u_1 u_2 & u_2^2 - \frac{1}{3} \end{pmatrix}.$$

Now let

$$\hat{\varphi}_{ij}(E, K \cap E) := 2R\, g_{ij}(\pi(E)) V_0(K \cap E).$$

Then

$$\frac{\hat{\varphi}_{ij}(E, K \cap E)}{f(\pi(E))} \tag{4.21}$$

is an unbiased estimator of $\Phi_{1,0,2}(K)(e_i, e_j)$, when E is an f-weighted random line in $\mathbb{R}^2$ hitting A.

For a given $K \in \mathcal{K}^2$ the weight function f minimizing the variance of the estimators of the form (4.21) can be determined.

Lemma 4.9 *For a fixed $K \in \mathcal{K}^2$ contained in the reference set A with* $\dim K \geq 1$ *and $i, j \in \{1, 2\}$, the estimator* (4.21) *has minimal variance if and only if $f = f_K^*$ holds $\nu_1^2 - a.s.$, where*

$$f_K^*(L) \propto \sqrt{2RV_1(K|L^\perp)}\, |g_{ij}(L)| \tag{4.22}$$

is a density with respect to ν_1^2 that depends on i, j and K.

P r o o f . As K is compact, f_K^* is a well-defined probability density, and since $\dim K \geq 1$, the density f_K^* is non-vanishing ν_1^2-almost surely. The second moment of the estimator (4.21) is

$$\mathbb{E}_f\left(\frac{\hat{\varphi}_{ij}(E, K \cap E)}{f(\pi(E))}\right)^2 = 2R \int_{\mathcal{L}_1^2} V_1(K|L^\perp) \frac{g_{ij}(L)^2}{f(L)}\, \nu_1^2(dL), \tag{4.23}$$

where $\mathbb{E}_f$ denotes expectation with respect to the distribution of an f-weighted random line in $\mathbb{R}^2$ hitting A. The right-hand side of (4.23) is the second moment of the random variable

$$\frac{\sqrt{2RV_1(K|L^\perp)}\, g_{ij}(L)}{f(L)},$$

where the distribution of the random line L has density f with respect to ν_1^2. By [2, Chapter 5, Theorem 1.2] the second moment of this variable is minimized, when f is proportional to $\sqrt{2RV_1(K|L^\perp)}\, |g_{ij}(L)|$. Since the proof of [2, Chapter 5, Theorem 1.2] follows simply by an application of Jensen's inequality to the function $t \mapsto t^2$, equality can be characterized due to the strict convexity of this function, (see, e.g., [9, (B.8)]). Equality holds if and only if $\sqrt{2RV_1(K|L^\perp)}\, |g_{ij}(L)|$ is a constant multiple of $f(L)$ (or equivalently $f = f_K^*$) almost surely. □

The proof of Lemma 4.9 generalizes directly to arbitrary dimension n. As a consequence of Lemma 4.9, we obtain that for any convex body $K \in \mathcal{K}^2$, optimal non-isotropic sampling provides a strictly smaller variance of the estimator (4.21) than isotropic sampling. Indeed, noting that (4.21) with a constant function f reduces to the usual estimator (4.3) (with $n = 2$, $A = RB^2$) based on *IUR* lines, this follows from the fact that f_K^* cannot be constant. If f_K^* was constant almost surely, then $V_1(K|u^\perp) \propto |g_{ij}([u])|^{-2}$ for almost all $u \in S^1$. The left-hand side is essentially bounded, whereas the right-hand side is not. This is a contradiction.

A further consequence of Lemma 4.9 is that there does not exist an estimator of the form (4.21) independent of K that has uniformly minimal variance for all $K \in \mathcal{K}^2$ with $\dim K \geq 1$. Unfortunately, f_K^* is not accessible, as it depends on K, which is typically unknown. Even though estimators of the form (4.21) cannot have uniformly minimal variance for all $K \in \mathcal{K}^2$ with $\dim K \geq 1$, we now show that there is a non-isotropic sampling design which always yields smaller variance than the isotropic sampling design. Let

$$f^*(L) \propto |g_{ij}(L)|$$

be a density with respect to ν_1^2. As $|g_{ij}(L)|$ is bounded and non-vanishing for ν_1^2-almost all L, f^* is well-defined and non-zero ν_1^2-almost everywhere. For convex bodies of constant width, the density f^* coincides with the optimal density f_K^*.

Theorem 4.10 *Let $K \in \mathcal{K}^2$, and let $A = RB^2$ for some $R > 0$ be such that $K \subseteq A$. Then*

$$Var_{f^*}\left(\frac{\hat{\varphi}_{ij}(E, K \cap E)}{f^*(\pi(E))}\right) < Var_{IUR}\big(\hat{\varphi}_{ij}(E, K \cap E)\big). \tag{4.24}$$

Proof. Using the fact that both estimators are unbiased, it is sufficient to show that there is a $0 < \lambda < 1$ with

$$\mathbb{E}_{f^*}\left(\frac{\hat{\varphi}_{ij}(E, K\cap E)}{f^*(\pi(E))}\right)^2 \leq \lambda\, \mathbb{E}_{IUR}\big(\hat{\varphi}_{ij}(E, K\cap E)\big)^2, \tag{4.25}$$

for all $K \in \mathcal{K}^2$. Using (4.23), the left-hand side of this inequality is

$$2R\int_{\mathcal{L}_1^2} |g_{ij}(L)|\, \nu_1^2(dL) \int_{\mathcal{L}_1^2} |g_{ij}(L)| V_1(K|L^\perp)\, \nu_1^2(dL)$$

and the right-hand side is

$$2R\int_{\mathcal{L}_1^2} g_{ij}(L)^2\, V_1(K|L^\perp)\, \nu_1^2(dL).$$

Since $u \mapsto V_1(K|u^\perp)$ is the support function of an origin-symmetric convex body in $\mathbb{R}^2$ (see [21, (5.80)]), the inequality (4.25) holds if

$$\begin{aligned}\int_0^{2\pi} |g_{ij}([u_\phi])|\,\frac{d\phi}{2\pi}\int_0^{2\pi} |g_{ij}([u_\phi])|\, h(Z,u_\phi)\,\frac{d\phi}{2\pi}\\ \leq \lambda\int_0^{2\pi} g_{ij}([u_\phi])^2 h(Z,u_\phi)\,\frac{d\phi}{2\pi}\end{aligned} \tag{4.26}$$

for any origin-symmetric convex body $Z \subset \mathbb{R}^2$. Here $u_\phi = (\cos(\phi), \sin(\phi))^\top$ for $\phi \in [0, 2\pi]$. Since origin-symmetric convex bodies in $\mathbb{R}^2$ can be approximated by sums of origin-symmetric line segments (see, e.g., [21, Cor. 3.5.7]) and the integrals in (4.26) depend linearly on the support function of Z, it is sufficient to show (4.26) for all origin-symmetric line segments Z of length two. Hence, we may assume that Z is an origin-symmetric line segment with endpoints $\pm(\cos(\gamma), \sin(\gamma))^\top$, where $\gamma \in [0, \pi)$. We now substitute the support function

$$h(Z, u_\phi) = |\cos(\phi - \gamma)|$$

for $\phi \in [0, 2\pi)$ into (4.26).

First, we consider the estimation of the first diagonal element of $\Phi_{1,0,2}(K)$, that is, $i, j = 1$ and $g_{ij}([u_\phi]) = \frac{3}{8}(\cos^2(\phi) - \frac{1}{3})$ for $\phi \in [0, 2\pi]$. The integrals in (4.26) then become

$$P_{f^*}(\gamma) := \frac{3}{8}\int_0^{2\pi}\left|\cos^2(\phi) - \frac{1}{3}\right|\frac{d\phi}{2\pi}\,\frac{3}{8}\int_0^{2\pi}\left|\cos^2(\phi) - \frac{1}{3}\right| |\cos(\phi-\gamma)|\,\frac{d\phi}{2\pi}$$

and

$$P_{IUR}(\gamma) := \frac{9}{64}\int_0^{2\pi}\left(\cos^2(\phi) - \frac{1}{3}\right)^2 |\cos(\phi-\gamma)|\,\frac{d\phi}{2\pi}.$$

Let $\kappa = \arccos(\frac{1}{\sqrt{3}})$. Then

$$M := \frac{3}{8}\int_0^{2\pi}\left|\cos^2(\phi) - \frac{1}{3}\right|\frac{d\phi}{2\pi} = \frac{\sqrt{2}+\kappa}{4\pi} - \frac{1}{16},$$

and elementary, but tedious calculations show that

$$\begin{aligned}P_{f^*}(\gamma) &= \frac{M}{\pi}\left(\frac{2\sqrt{2}}{3\sqrt{3}}\cos(\gamma) - \frac{1}{4}\cos^2(\gamma)\right)\mathbf{1}_{[0,\frac{\pi}{2}-\kappa]}(\gamma)\\ &\quad + \frac{M}{\pi}\left(\frac{1}{4}\cos^2(\gamma) + \frac{1}{3\sqrt{3}}\sin(\gamma)\right)\mathbf{1}_{(\frac{\pi}{2}-\kappa,\frac{\pi}{2}]}(\gamma)\end{aligned}$$

for $\gamma \in [0, \frac{\pi}{2}]$. Further, $P_{f^*}(\gamma) = P_{f^*}(\pi - \gamma)$ for $\gamma \in [\frac{\pi}{2}, \pi]$. For the *IUR* estimator we get that

$$P_{IUR}(\gamma) = \frac{1}{20\pi}\left(-\frac{3}{8}\cos^4(\gamma) + \cos^2(\gamma) + \frac{1}{2}\right)$$

for $\gamma \in [0, \frac{\pi}{2}]$, and $P_{IUR}(\gamma) = P_{IUR}(\pi - \gamma)$ for $\gamma \in [\frac{\pi}{2}, \pi]$. The functions P_{f^*} and P_{IUR} are plotted in Figure 3. Basic calculus for the comparison of these two functions shows that $P_{f^*} < P_{IUR}$. This implies that $P_{f^*} \le \lambda P_{IUR}$, where $\lambda = \max_{\gamma \in [0,\pi]} \frac{P_{f^*(\gamma)}}{P_{IUR}(\gamma)}$ is smaller than 1 as P_{f^*} and P_{IUR} are continuous on the compact interval $[0, \pi]$. Hereby (4.26) is satisfied for $i = j = 1$. Interchanging the roles of the coordinate axes in (4.26) yields the same result for $i = j = 2$.

We now consider estimation of the off-diagonal element, that is, $i = 1,\ j = 2$. Then the left-hand and the right-hand side of (4.26) become

$$Q_{f^*}(\gamma) := \frac{3}{8}\int_0^{2\pi} |\cos(\phi)\sin(\phi)|\,\frac{d\phi}{2\pi}\,\frac{3}{8}\int_0^{2\pi} |\cos(\phi)\sin(\phi)||\cos(\phi-\gamma)|\,\frac{d\phi}{2\pi} \tag{4.27}$$

and

$$Q_{IUR}(\gamma) := \frac{9}{64}\int_0^{2\pi} \cos^2(\phi)\sin^2(\phi)|\cos(\phi-\gamma)|\,\frac{d\phi}{2\pi} \tag{4.28}$$

for $\gamma \in [0, \pi]$. We have

$$\frac{3}{8}\int_0^{2\pi} |\cos(\phi)\sin(\phi)|\,\frac{d\phi}{2\pi} = \frac{3}{8\pi},$$

and then

$$Q_{f^*}(\gamma) = \frac{3}{32\pi^2}\Big(\sin(\gamma) + \cos(\gamma) - \sin(\gamma)\cos(\gamma)\Big)$$

for $\gamma \in [0, \frac{\pi}{2}]$, and $Q_{f^*}(\gamma) = Q_{f^*}(\gamma - \frac{\pi}{2})$ for $\gamma \in [\frac{\pi}{2}, \pi]$. For $\gamma \in [0, \pi]$ we further find that

$$Q_{IUR}(\gamma) = \frac{3}{320\pi}\Big(4 - \frac{1}{2}\sin^2(2\gamma)\Big).$$

The functions Q_{IUR} and Q_{f^*} are plotted in Figure 4. Basic calculus shows that

$$\min_{0\le\gamma\le\pi} Q_{f^*} = \frac{3}{32\pi^2}\left(\sqrt{2} - \frac{1}{2}\right), \qquad \max_{0\le\gamma\le\pi} Q_{f^*} = \frac{3}{32\pi^2}, \tag{4.29}$$

and

$$\min_{0\le\gamma\le\pi} Q_{IUR} = \frac{21}{640\pi}, \qquad \max_{0\le\gamma\le\pi} Q_{IUR} = \frac{3}{80\pi}. \tag{4.30}$$

Hence

$$Q_{f^*}(\gamma) \le \frac{3}{32\pi^2} \le \lambda\frac{21}{640\pi} \le \lambda\, Q_{IUR}(\gamma)$$

for $\gamma \in [0, \pi]$ with $\lambda = \frac{3}{\pi} < 1$. Hereby (4.26) holds for all origin-symmetric convex bodies $Z \subset \mathbb{R}^2$ and $i = 1,\ j = 2$, and the claim is shown. □

If E is an f^*-weighted random line suited for estimating one particular component of $\Phi_{1,0,2}(K)$, then E should not be used to estimate any of the other components, as this would increase the variance of these estimators considerably. Hence, if we estimate all of the components of the tensor using the estimator based on f^*-weighted lines, we need three lines; one for each component. If we want to compare this approach with an estimation procedure based on *IUR* lines, requiring the same workload, we will use *three IUR* lines. Note however, that all three *IUR* lines can be used to estimate *all* three components of the tensor. This implies that we should actually compare the variance of the estimator based on *one* f^*-weighted random line with the variance of an estimator based on *three IUR* lines. It turns out that the estimator based on *three* independent *IUR* lines has always smaller variance, than the estimator based on *one* f-weighted line, no matter how the density f is chosen.

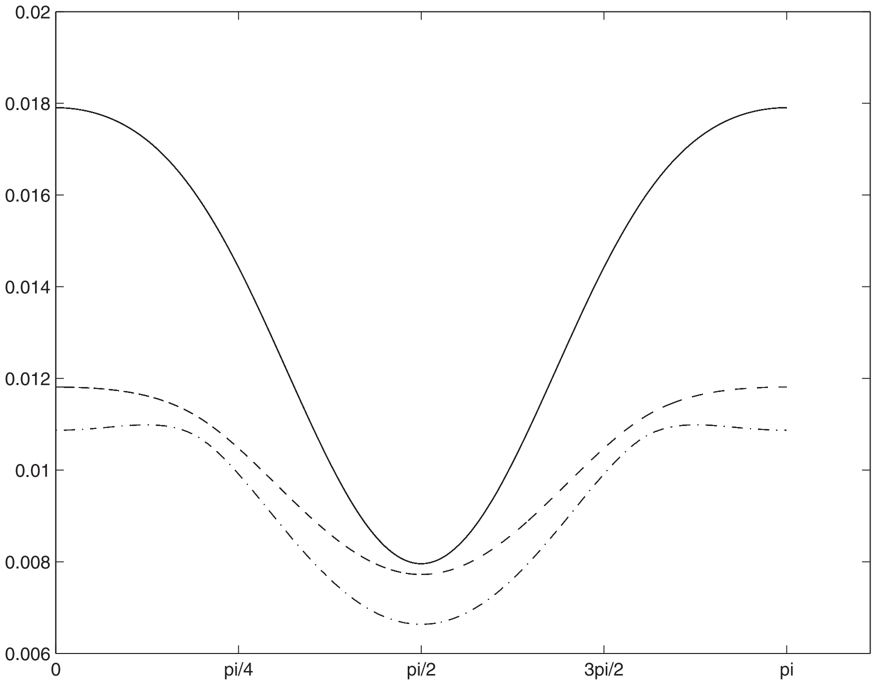

Fig. 3 The straight line is P_{IUR}, the dashed line is P_{f^*}, and the dash-dotted line is P_{opt}.

Theorem 4.11 *Let $K \in \mathcal{K}^2$ with* $\dim(K) \geq 1$. *Let $A = RB^2$ with some $R > 0$ be such that $K \subseteq A$. Let f be a density with respect to ν_1^2, which is non-zero ν_1^2-almost everywhere. Let E_1, E_2 and E_3 be independent IUR lines in $\mathbb{R}^2$ hitting A. Then*

$$Var\left(\frac{1}{3}\sum_{k=1}^{3}\hat{\varphi}_{ij}(E_k, K \cap E_k)\right) < Var_f\left(\frac{\hat{\varphi}_{ij}(E, K \cap E)}{f(\pi(E))}\right)$$

for $i, j \in \{1, 2\}$.

P r o o f . By Lemma 4.9, the variance of the estimator (4.21) is bounded from below by the variance of the same estimator with $f = f_K^*$. Hence, it is sufficient to compare the second moments of

$$\frac{1}{3}\sum_{k=1}^{3}\hat{\varphi}_{ij}(E_k, K \cap E_k)$$

and (4.21) with $f = f_K^*$. The latter is

$$2R\left(\int_{\mathcal{L}_1^2}|g_{ij}(L)|\sqrt{V_1(K|L^\perp)}\,\nu_1^2(dL)\right)^2,$$

so let

$$P_{opt}(\gamma) := \left(\frac{3}{8}\int_0^{2\pi}\left|\cos^2(\phi) - \frac{1}{3}\right|\sqrt{|\cos(\phi-\gamma)|}\,\frac{d\phi}{2\pi}\right)^2$$

and

$$Q_{opt}(\gamma) := \left(\frac{3}{8}\int_0^{2\pi}|\cos(\phi)\sin(\phi)|\sqrt{|\cos(\phi-\gamma)|}\,\frac{d\phi}{2\pi}\right)^2$$

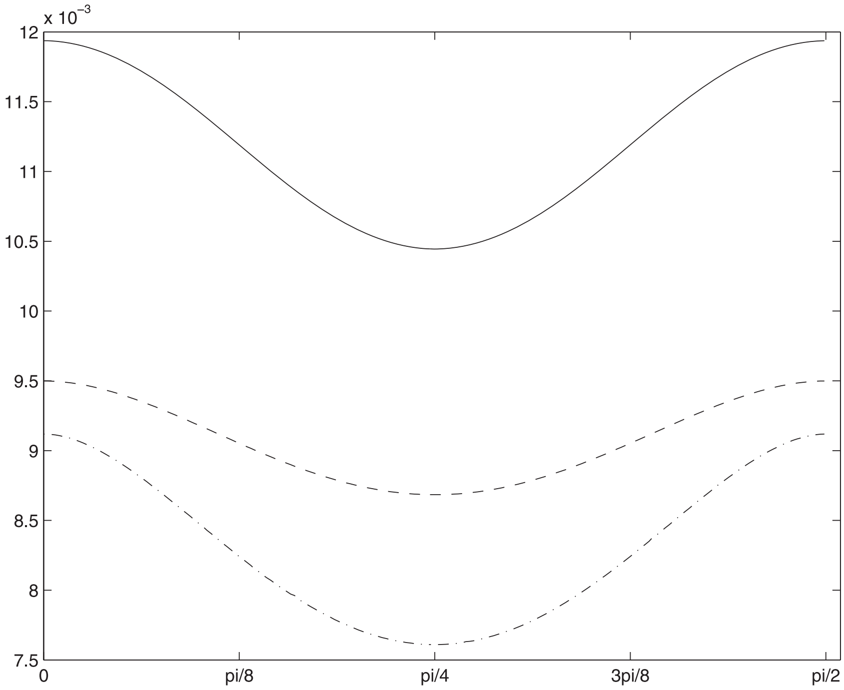

Fig. 4 The straight line is Q_{IUR}, the dashed line is Q_{f^*}, and the dash-dotted line is Q_{opt}.

for $\gamma \in [0, \pi]$. Using the notation of the previous proof, by (4.27), (4.28), (4.29) and (4.30) we have

$$Q_{opt}(\gamma) \geq \left(\frac{8\pi Q_{f^*}(\gamma)}{3}\right)^2 \geq \frac{9-4\sqrt{2}}{64\pi^2} > \frac{1}{80\pi} \geq \frac{1}{3} Q_{IUR}(\gamma)$$

for $\gamma \in [0, \pi]$. Likewise, $P_{opt}(\gamma) \geq \left(\frac{P_{f^*}(\gamma)}{M}\right)^2$. Elementary analysis shows that

$$\min_{0 \leq \gamma \leq \frac{\pi}{2}-\kappa} \left(\frac{P_{f^*}(\gamma)}{M}\right)^2 = \frac{25}{324\pi^2} > \frac{3}{160\pi} = \max_{0 \leq \gamma \leq \frac{\pi}{2}-\kappa} \frac{1}{3} P_{IUR}(\gamma),$$

and that

$$\left(\frac{P_{f^*}(\gamma)}{M}\right)^2 - \frac{1}{3} P_{IUR}(\gamma) \geq \left(\frac{P_{f^*}(\frac{\pi}{2})}{M}\right)^2 - \frac{1}{3} P_{IUR}\left(\frac{\pi}{2}\right) > 0$$

on $[\frac{\pi}{2} - \kappa, \frac{\pi}{2}]$. Hence $P_{opt} > \frac{1}{3} P_{IUR}$ on $[0, \pi]$, and the assertion is proved. □

This leads to the following conclusion: If one single component of the tensor $\Phi_{n-1,0,2}(K)$ is to be estimated for unknown K, the estimator (4.21) with $f = f^*$ is recommended, as its variance is strictly smaller than the one from isotropic sampling (where f is a constant). If all components are sought for, the estimator based on three *IUR* lines should be preferred.

5 Model based estimation

In this section we derive estimators of the specific surface tensors associated with a stationary process of convex particles based on linear sections. In [23], Schneider and Schuster treat the similar problem of estimating the area moment tensor ($s = 2$) associated with a stationary process of convex particles using planar sections.

Let X be a stationary process of convex particles in $\mathbb{R}^n$ with locally finite (and non-zero) intensity measure, intensity $\gamma > 0$ and grain distribution $\mathbb{Q}$ on $\mathcal{K}_0 := \{K \in \mathcal{K}^n \setminus \emptyset \mid c(K) = 0\}$; see, e.g., [24] for further information on this basic model of stochastic geometry. Here $c : \mathcal{K}^n \setminus \{\emptyset\} \to \mathbb{R}^n$ is the center of the circumball of K. Since X is a stationary process of convex particles, the intrinsic volumes $V_0, \ldots, V_n$ are $\mathbb{Q}$-integrable by [24, Theorem 4.1.2]. For $j \in \{0, \ldots, n-1\}$ and $s \in \mathbb{N}_0$ the tensor valuation $\Phi_{j,0,s}$ is measurable and translation invariant on $\mathcal{K}^n$, and since, by (2.1),

$$|\Phi_{j,0,s}(K)(e_{i_1}, \ldots, e_{i_s})| \leq \frac{\omega_{n-j}}{s!\omega_{n-j+s}} V_j(K),$$

it is coordinate-wise $\mathbb{Q}$-integrable. The *jth specific (translation invariant) tensor of rank s* can then be defined as

$$\overline{\Phi}_{j,0,s}(X) := \gamma \int_{\mathcal{K}_0} \Phi_{j,0,s}(K)\, \mathbb{Q}(dK) \tag{5.1}$$

for $j \in \{0, \ldots, n-1\}$ and $s \in \mathbb{N}_0$. For $j = n-1$, the specific tensors are called the specific surface tensors. Notice that $\overline{\Phi}_{n-1,0,2}(X) = \frac{1}{8\pi}\overline{T}(X)$, where $\overline{T}(X)$ is the mean area moment tensor described in [23]. By [24, Theorem 4.1.3] the specific tensors of X can be represented as

$$\overline{\Phi}_{j,0,s}(X) = \frac{1}{\lambda(B)}\, \mathbb{E} \sum_{\substack{K \in X \\ c(K) \in B}} \Phi_{j,0,s}(K), \tag{5.2}$$

where $B \in \mathcal{B}(\mathbb{R}^n)$ with $0 < \lambda(B) < \infty$.

We now restrict considerations to $j = n-1$ and discuss the estimation of $\overline{\Phi}_{n-1,0,s}(X)$ from linear sections of X. We assume from now on that $n \geq 2$. For $L \in \mathcal{L}_1^n$ we let $X \cap L := \{K \cap L \mid K \in X, K \cap L \neq \emptyset\}$ be the stationary process of convex particles in L induced by X. Let γ_L and $\mathbb{Q}_L$ denote the intensity and the grain distribution of $X \cap L$, respectively. The tensor valuation $\Phi_{0,0,s}^{(L)}$ is measurable and $\mathbb{Q}_L$-integrable on $\mathcal{K}_0^{(L)} := \{K \in \mathcal{K}_0 \mid K \subseteq L\}$. We can thus define

$$\overline{\Phi}_{0,0,s}^{(L)}(X \cap L) := \gamma_L \int_{\mathcal{K}_0^{(L)}} \Phi_{0,0,s}^{(L)}(K)\, \mathbb{Q}_L(dK).$$

This deviates in the special case $\overline{T}^{(L)}(X \cap L) = 8\pi\, \overline{\Phi}_{0,0,2}^{(L)}(X \cap L)$ from the definition in [23] due to a misprint there. An application of (3.4) yields,

$$\overline{\Phi}_{0,0,s}^{(L)}(X \cap L) = \frac{2}{s!\omega_{s+1}} Q(L)^{\frac{s}{2}} \gamma_L \tag{5.3}$$

for even s, and $\overline{\Phi}_{0,0,s}^{(L)}(X \cap L) = 0$ for odd s.

Theorem 5.1 *Let X be a stationary process of convex particles in $\mathbb{R}^n$ with positive intensity. If $s \in \mathbb{N}_0$ is even, then*

$$\int_{\mathcal{L}_1^n} \overline{\Phi}_{0,0,s}^{(L)}(X \cap L)\, \nu_1^n(dL) = \frac{2\omega_{n+s+1}}{\pi s!\omega_{s+1}^2\omega_n} \sum_{k=0}^{\frac{s}{2}} c_k^{(\frac{s}{2})} Q^{\frac{s}{2}-k}\, \overline{\Phi}_{n-1,0,2k}(X), \tag{5.4}$$

where the constants $c_k^{(\frac{s}{2})}$ for $k = 0, \ldots, \frac{s}{2}$ are given in Theorem 3.1.

P r o o f . Let $L \in \mathcal{L}_1^n$, and let γ_L be the intensity of the stationary process $X \cap L$. If $B \subseteq L$ is a Borel set with $\lambda_L(B) = 1$, then an application of Campbell's theorem and Fubini's theorem yields

$$\begin{aligned}\gamma_L &= \mathbb{E} \sum_{\substack{K \in X \\ K \cap L \neq \emptyset}} \mathbf{1}(c(K \cap L) \in B) \\ &= \gamma \int_{\mathcal{K}_0} \int_{L^\perp} V_0(K \cap (L + x))\, \lambda_{L^\perp}(dx)\, \mathbb{Q}(dK),\end{aligned}$$

where γ and $\mathbb{Q}$ are the intensity and the grain distribution of X. Then, (5.3) implies that

$$\overline{\Phi}_{0,0,s}^{(L)}(X \cap L) = \gamma \int_{\mathcal{K}_0} \int_{L^\perp} \Phi_{0,0,s}^{(L+x)}(K \cap (L + x))\, \lambda_{L^\perp}(dx)\, \mathbb{Q}(dK),$$

and by Fubini's theorem we get

$$\int_{\mathcal{L}_1^n} \overline{\Phi}_{0,0,s}^{(L)}(X \cap L)\, \nu_1^n(dL) = \gamma \int_{\mathcal{K}_0} \int_{\mathcal{E}_1^n} \Phi_{0,0,s}^{(E)}(K \cap E)\, \mu_1^n(dE)\, \mathbb{Q}(dK). \tag{5.5}$$

Now Theorem 3.1 yields the stated integral formula (5.4). □

A combination of Equation (5.5) and Equation (3.10) immediately gives the following Theorem 5.2, which suggests an estimation procedure of the specific surface tensor $\overline{\Phi}_{n-1,0,s}(X)$ of the stationary particle process X.

Theorem 5.2 *Let X be a stationary process of convex particles in $\mathbb{R}^n$ with positive intensity. If $s \in \mathbb{N}_0$ is even, then*

$$\int_{\mathcal{L}_1^n} \sum_{j=0}^{\frac{s}{2}} d_{\frac{s}{2}j} C_{2j} Q^{\frac{s}{2}-j} \overline{\Phi}_{0,0,2j}^{(L)}(X \cap L)\, \nu_1^n(dL) = \overline{\Phi}_{n-1,0,s}(X), \tag{5.6}$$

where $d_{\frac{s}{2}j}$ and C_{2j} for $j = 0, \ldots, \frac{s}{2}$ are given before Theorem 3.4.

Using (5.3), we can reformulate the integral formula (5.6) in the form

$$\int_{\mathcal{L}_1^n} G_s(L) \gamma_L\, \nu_1^n(dL) = \overline{\Phi}_{n-1,0,s}(X),$$

where G_s is given in Theorem 3.4.

Example 5.3 In the case where $s = 2$ formula (5.6) becomes

$$\int_{\mathcal{L}_1^n} \frac{2\pi^2 \omega_n}{\omega_{n+3}} \overline{\Phi}_{0,0,2}^{(L)}(X \cap L) - \frac{\omega_n}{4\omega_{n+1}} Q \overline{\Phi}_{0,0,0}^{(L)}(X \cap L)\, \nu_1^n(dL) = \overline{\Phi}_{n-1,0,2}(X).$$

Up to a normalizing factor 2π in the constant in front of $\overline{\Phi}_{0,0,2}^{(L)}$, this formula coincides with formula (7) in [23], when $n = 2$. Apparently the normalizing factor got lost, when Schneider and Schuster used [22, (36)], which is based on the spherical Lebesgue measure. In [23], Schneider and Schuster use the *normalized* spherical Lebesgue measure.

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Paper B

Reconstruction of convex bodies from surface tensors

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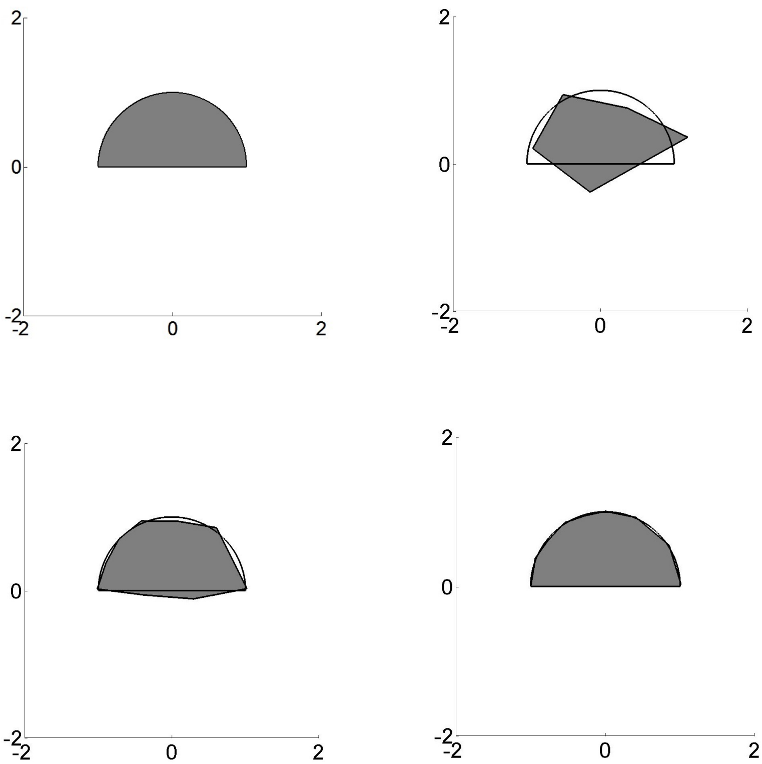

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Reconstruction of convex bodies from surface tensors

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ABSTRACT

We present two algorithms for reconstruction of the shape of convex bodies in the two-dimensional Euclidean space. The first reconstruction algorithm requires knowledge of the exact surface tensors of a convex body up to rank s for some natural number s. When only measurements subject to noise of surface tensors are available for reconstruction, we recommend to use certain values of the surface tensors, namely harmonic intrinsic volumes instead of the surface tensors evaluated at the standard basis. The second algorithm we present is based on harmonic intrinsic volumes and allows for noisy measurements. From a generalized version of Wirtinger's inequality, we derive stability results that are utilized to ensure consistency of both reconstruction procedures. Consistency of the reconstruction procedure based on measurements subject to noise is established under certain assumptions on the noise variables.

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1. Introduction

The problem of determining and reconstructing an unknown geometric object from indirect measurements is treated in a number of papers, see, e.g., [6]. In [19], a convex body is reconstructed from measurements of its support function. Measurements of the brightness function are used in [7], and in [4] it is shown that a convex body can be uniquely determined up to translation from measurements of its lightness function. Milanfar et al. [17] developed a reconstruction algorithm for planar polygons and quadrature domains from moments of the Lebesgue measure restricted to these sets. In particular, they showed that a non-degenerate convex polygon in $\mathbb{R}^2$ with k vertices is uniquely determined by its moments up to order $2k-3$. The reconstruction algorithm and the uniqueness result were generalized to convex polytopes in $\mathbb{R}^n$ by Gravin et al. in [10].

In continuation of the work in this area, we discuss reconstruction of convex bodies from a certain type of Minkowski tensors. In recent years, Minkowski tensors have been studied intensively. On the applied side, Minkowski tensors have been established as robust and versatile descriptors of shape and morphology of spatial patterns of physical systems, see e.g., [2,22] and the references given there. The importance of Minkowski tensors is further indicated by Alesker's characterization theorem, see [1], that states that products of Minkowski tensors and powers of the metric tensor span the space of tensor-valued valuations on convex bodies satisfying some natural conditions.

In the present work, we consider translation invariant Minkowski tensors, $\Phi_j^s(K)$ of rank s, which are tensors derived from the jth area measure $S_j(K,\cdot)$ of a convex body $K \subseteq \mathbb{R}^n$, $j = 0,\dots,n-1$. For details, see Sections 2 and 3. For a given $j = 1,\dots,n-1$, the set $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$ of all Minkowski tensors determines K up to translation. Calling the equivalence class of all translations of K the *shape* of K, we can say that $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$ determines the shape of K. When only Minkowski tensors $\Phi_j^s(K)$, $s \le s_o$ up to a certain rank s_o are given, this is, in general, no longer true. We establish a stability result (Theorem 4.9) stating that the shapes of two convex bodies are close to one another when the two convex bodies have coinciding Minkowski tensors $\Phi_1^s(K)$ of rank $s \le s_o$. The proof uses a generalization of Wirtinger's inequality (Corollary 4.7), which is different from existing generalizations in the literature (e.g. [5,8]) as it involves a higher order spherical harmonic expansion. We also show (Theorem 4.1) that there always exists a convex polytope P with the same surface tensors $\Phi_{n-1}^s(P)$ of rank $s \le s_o$ as a given convex body. The number of facets of P can be bounded by a polynomial of s_o of degree $n-1$. Using this result, we conclude (Corollary 4.2) that a convex body K is a polytope if the shape of K is uniquely determined by a finite number of surface tensors. In fact, the shape of a convex body K is uniquely determined by a finite number of its surface tensors if and only if K is a polytope (Theorem 4.3).

For actual reconstructions, we restrict considerations to the planar case. We consider two cases. Firstly, the case when the exact tensors are given, and secondly, the case when certain values of the tensors are measured with noise. *Algorithm Surface Tensor* in Section 5 allows to reconstruct an unknown convex body K_0 in $\mathbb{R}^2$ based on surface tensors $\Phi_1^s(K_0)$ up to rank s_o. The output of the reconstruction procedure is a polygon P with surface tensors identical to the surface tensors of K_0 up to rank s_o. Theorem 4.1 yields the existence of a polygon with the described property. Due to the bound on the number of facets of P and to the simple structure of surface tensors of polygons, the reconstruction problem can be solved by first finding the surface area measure of P using a least squares optimization, and then constructing P with the help of Algorithm MinkData in [6]. The consistency of the reconstruction procedure is established using the mentioned stability result.

Reconstruction algorithms for dimensions $n \geq 2$ could be developed along the same lines when surface tensors $\Phi_{n-1}^s(K_0)$, $s \leq s_o$ are used as input. However, the methods in this paper yield a stability result for $\Phi_1^s(K_0)$, $s \leq s_o$, and this is why we only consider the case $n = 2$. The higher dimensional situation will be discussed in future work.

Due to the structure of the stability result (Theorem 4.8), we recommend to use harmonic intrinsic volumes for reconstructions when only noisy measurements of surface tensors are available. The harmonic intrinsic volumes of a convex body in $\mathbb{R}^2$ are certain values of the surface tensors, and the harmonic intrinsic volumes up to degree s_o determine the surface tensors up to rank s_o. *Algorithm Harmonic Intrinsic Volumes LSQ* reconstructs an unknown convex body K_0 in $\mathbb{R}^2$ based on measurements of harmonic intrinsic volumes up to degree s_o, where the measurements are subject to noise. The output of the reconstruction is a polygon with surface tensors best fitting the measurements of the harmonic intrinsic volumes of K_0 in a least squares sense. As for the procedure for reconstruction of convex bodies from exact surface tensors, this reconstruction procedure is based on Theorem 4.1 and Algorithm MinkData. The consistency of the reconstruction algorithm is established using the stability result and requires that the variances of all measurements converge to zero sufficiently fast. When only noisy measurements are available, we use harmonic intrinsic volumes instead of surface tensors evaluated at the standard basis as this allows us to obtain stronger consistency results, see Section 6.4 for details.

The paper is organized as follows: After introducing notations and preliminaries in Sections 2 and 3, we present the main theoretical results in Section 4 in $\mathbb{R}^n$, $n \geq 2$: The existence of a polytope with finitely many surface tensors coinciding with those of a given convex body, the uniqueness result for shapes of polytopes, the generalized Wirtinger's inequality, and the derived stability result. In Section 5 Algorithm Surface Tensor and its properties are discussed, and Section 6 is devoted to the reconstruction from noisy measurements of harmonic intrinsic volumes.

2. Notation and preliminaries

We work in the n-dimensional Euclidean vector space $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$. As usual, S^{n-1} is the unit sphere in $\mathbb{R}^n$, and κ_n and ω_n denote the volume and the surface area of the unit ball B^n, respectively. The Borel σ-algebra of a topological space X is denoted by $\mathcal{B}(X)$. Further, let λ denote the Lebesgue measure on $\mathbb{R}^n$. The set $L^2(S^{n-1})$ of square integrable functions on S^{n-1} with respect to the spherical Lebesgue measure σ is equipped with the usual inner product $\langle\cdot,\cdot\rangle_2$ and the associated norm $\|\cdot\|$.

For a function F on the unit sphere S^{n-1}, we let $\check{F}$ denote the radial extension of F to $\mathbb{R}^n \setminus \{o\}$, that is,

$$\check{F}(x) = F\left(\frac{x}{|x|}\right)$$

for $x \in \mathbb{R}^n \setminus \{o\}$. Let $\nabla_S F$ denote the restriction of the gradient $\nabla\check{F}$ of $\check{F}$ to S^{n-1}, when the partial derivatives of $\check{F}$ exist. If further, $\check{F}$ has partial derivatives of second order, the Laplace–Beltrami operator $\Delta_S F$ of F is defined as the restriction of $\Delta\check{F}$ to S^{n-1}, where Δ denotes the Laplace operator on functions on $\mathbb{R}^n$.

2.1. Spherical harmonics

In the proofs of Lemma 4.6 and Theorem 4.8, spherical harmonics are a key ingredient. We use [11] as a general reference on the theory of spherical harmonics. A polynomial p on $\mathbb{R}^n$ is said to be harmonic if it is homogeneous and $\Delta p = 0$. A spherical harmonic of degree m is the restriction to S^{n-1} of a harmonic polynomial of degree m. Let $\mathcal{H}^n_m$ be the vector space of spherical harmonics of degree m on S^{n-1}, and let $N(n,m)$ denote the dimension of $\mathcal{H}^n_m$. For $m \in \mathbb{N}_0$, let $H_{m1},\dots,H_{mN(n,m)}$ be an orthogonal basis for $\mathcal{H}^n_m$. Then the condensed harmonic expansion of a function $F \in L^2(S^{n-1})$ is $\sum_{m=0}^{\infty} F_m$, where $F_m = \sum_{j=1}^{N(n,m)} \alpha_{mj} H_{mj}$ with

$$\alpha_{mj} = \frac{\langle F, H_{mj}\rangle_2}{\|H_{mj}\|^2}.$$

We write $F \sim \sum_{m=0}^{\infty} F_m$, when $\sum_{n=0}^{\infty} F_m$ is the condensed harmonic expansion of F. The condensed harmonic expansion of F is independent of the choice of bases of spherical harmonics used to derive it. The spherical harmonics are eigenfunctions of the Laplace–Beltrami operator as

$$\Delta_S H_m = -m(m+n-2)H_m$$

for $H_m \in \mathcal{H}^n_m$. We let γ_m denote the absolute value of the eigenvalues of Δ_S, that is $\gamma_m = m(m+n-2)$ for $m \in \mathbb{N}_0$.

As in [3], the Sobolev space W^α for $\alpha \geq 0$ is defined as the space of square integrable functions $F \sim \sum_{m=0}^{\infty} F_m$ on the sphere, for which

$$\sum_{m=0}^{\infty} \gamma_m^\alpha \|F_m\|^2 < \infty.$$

By definition $W^\alpha \subseteq L^2(S^{n-1})$ for $\alpha \geq 0$, and $W^0 = L^2(S^{n-1})$. For $F \in W^\alpha$, the sum

$$\sum_{m=0}^{\infty} (\gamma_m)^{\frac{\alpha}{2}} F_m$$

converges in the L^2-sense. The limit is denoted by $(-\Delta_S)^{\frac{\alpha}{2}} F$, and thus

$$\|(-\Delta_S)^{\frac{\alpha}{2}} F\|^2 = \sum_{m=0}^{\infty} \gamma_m^\alpha \|F_m\|^2. \tag{1}$$

The notation is explained by the fact that

$$\Delta_S F \sim -\sum_{m=0}^{\infty} \gamma_m F_m$$

for any $F \sim \sum_{m=0}^{\infty} F_m$ that is twice continuously differentiable.

In the two-dimensional setting we have $N(2,0) = 1$ and $N(2,m) = 2$ for $m \in \mathbb{N}$, and the spherical harmonic expansion is closely related to the classical Fourier expansion. We obtain an orthonormal sequence of spherical harmonics constituting a basis of $L^2(S^1)$ by letting $H_{01}(u_1,u_2) = (2\pi)^{-\frac{1}{2}}$,

$$H_{m1}(u_1,u_2) = \pi^{-\frac{1}{2}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \binom{m}{2i} u_1^{m-2i} u_2^{2i} \tag{2}$$

and

$$H_{m2}(u_1,u_2) = \pi^{-\frac{1}{2}} \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m}{2i+1} u_1^{m-2i-1} u_2^{2i+1}, \tag{3}$$

for $(u_1,u_2) \in S^1$ and $m \in \mathbb{N}$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. If the polynomials in (2) and (3) are considered as polynomials on $\mathbb{R}^2$, then due to homogeneity, the polynomials can be decomposed into linear factors. More precisely,

$$H_{m1}(u_1,u_2) = \pi^{-\frac{1}{2}} (u_1 - \zeta_1 u_2) \cdot \dots \cdot (u_1 - \zeta_m u_2), \tag{4}$$

where $\zeta_j = \frac{\cos(\frac{(2j-1)\pi}{2m})}{\sin(\frac{(2j-1)\pi}{2m})}$ for $j = 1, \dots, m$. The lines where H_{m1} vanishes (and herewith ζ_j, $j = 1, \dots, m$) are determined using the fact that

$$H_{m1}(\cos(\omega), \sin(\omega)) = \sqrt{\pi} \cos(m\omega)$$

for $\omega \in [0, 2\pi)$. Similarly, we can factorize H_{m2}. In this case, however, the factorization depends on the parity of m. This is due to the fact that a term involving u_1^m does not appear in H_{m2} and that the term involving u_2^m only appears, when m is odd. We get that

$$H_{m2}(u_1, u_2) = \begin{cases} \pi^{-\frac{1}{2}} m\, u_2\, (u_1 - \lambda_1 u_2) \cdot \cdots (u_1 - \lambda_{m-1} u_2) & \text{if m is even} \\ \pi^{-\frac{1}{2}} (-1)^{\frac{m-1}{2}} (u_2 - \rho_1 u_1) \cdot \cdots \cdot (u_2 - \rho_m u_1) & \text{if m is odd,} \end{cases} \tag{5}$$

where $\lambda_j = \frac{\cos(\frac{j\pi}{m})}{\sin(\frac{j\pi}{m})}$ for $j = 1, \ldots, m-1$ and $\rho_j = \frac{\sin(\frac{(j-1)\pi}{m})}{\cos(\frac{(j-1)\pi}{m})}$ for $j = 1, \ldots, m$. Here, we have used that

$$H_{m2}(\cos(\omega), \sin(\omega)) = \sqrt{\pi} \sin(m\omega)$$

for $\omega \in [0, 2\pi)$ in order to determine the lines where H_{m2} vanishes.

2.2. Convex bodies and area measures

As general reference on convex geometry, we use [20]. Let $\mathcal{K}^n$ denote the set of convex bodies (that is, compact, convex, non-empty sets) in $\mathbb{R}^n$, and let $\mathcal{K}^n_n$ denote the set of convex bodies with non-empty interior. We refer to convex polytopes and convex polygons as ‘polytopes’ and ‘polygons’, and let $\mathcal{P}^n_m$ denote the set of non-empty polytopes in $\mathbb{R}^n$ with at most m facets, $m \in \{n+1, n+2, \ldots\}$. The support function (restricted to S^{n-1}) of a convex body K is denoted by h_K. The set of support functions $\{h_K \mid K \in \mathcal{K}^n,\ K \subseteq RB^n\}$ for $R > 0$ is bounded in W^α for $0 < \alpha < \frac{3}{2}$, see [16, Prop. 2.1]. The set $\mathcal{K}^n$ of convex bodies is equipped with the Hausdorff metric δ, which can be expressed as the distance of support functions with respect to the supremum norm on S^{n-1}, i.e.

$$\delta(K, L) = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

In addition to the Hausdorff metric, we use the L^2-metric on $\mathcal{K}^n$. The L^2-distance between two convex bodies K and L is defined as the L^2-distance of their support functions, i.e.

$$\delta_2(K, L) = \|h_K - h_L\|.$$

The Hausdorff metric and the L^2-metric are equivalent and related by inequalities, see [11, Prop. 2.3.1]. This is used in Theorem 4.9 to transfer bounds on the L^2-distance to bounds on the Hausdorff distance between convex bodies satisfying certain conditions.

In the present work, two convex bodies are said to have the same shape if and only if they are translates. The position of a convex body has major influence on the above described distances, and as a measure of difference in shape only, we consider the translation invariant versions

$$\delta^t(K, L) = \inf_{x \in \mathbb{R}^n} \delta(K, L + x)$$

and

$$\delta_2^t(K, L) = \inf_{x\in\mathbb{R}^n} \delta_2(K, L + x).$$

If the support function h_K of a convex body K has condensed harmonic expansion $\sum_{m=0}^{\infty}(h_K)_m$, then $(h_K)_1 = \langle s(K), \cdot\rangle$, where $s(K)$ is the Steiner point of K,

$$s(K) = \frac{1}{\kappa_n} \int\limits_{S^{n-1}} h_K(u)\, u\, \sigma(du).$$

For convex bodies K and L, this implies that $\delta_2^t(K, L) = \delta_2(K, L)$ if and only if K and L have coinciding Steiner points, see [11, Prop. 5.1.2].

Let $p(K, x)$ be the metric projection of $x \in \mathbb{R}^n$ on a convex body K, and define $u(K, x) := \frac{x-p(K,x)}{|x-p(K,x)|}$ for $x \notin K$. For a Borel set $A \in \mathcal{B}(\mathbb{R}^n \times S^{n-1})$, the Lebesgue measure of the local parallel set

$$M_\epsilon(K, A) := \{x \in (K + \epsilon B^n) \setminus K \mid (p(K, x), u(K, x)) \in A\}$$

of K is a polynomial in $\epsilon \geq 0$, hence

$$\lambda(M_\epsilon(K, A)) = \sum_{k=0}^{n-1} \epsilon^{n-k}\kappa_{n-k}\Lambda_k(K, A).$$

This local version of the Steiner formula defines the support measures $\Lambda_0(K, \cdot)$, $\ldots, \Lambda_{n-1}(K, \cdot)$ of a convex body $K \in \mathcal{K}^n$. The intrinsic volumes of K appear as total masses of the support measures, $V_j(K) = \Lambda_j(K, \mathbb{R}^n \times S^{n-1})$ for $j = 0, \ldots, n-1$. The area measures $S_0(K, \cdot), \ldots, S_{n-1}(K, \cdot)$ of K are rescaled projections of the corresponding support measures on the second component. More explicitly, they are given by

$$\binom{n}{j} S_j(K, \omega) = n\kappa_{n-j}\Lambda_j(K, \mathbb{R}^n \times \omega)$$

for $\omega \in \mathcal{B}(S^{n-1})$ and $j = 0, \ldots, n-1$. The area measure of order $n-1$ is called the surface area measure, and for $K \in \mathcal{K}_n^n$ the surface area measure is the $(n-1)$-dimensional Hausdorff measure of the reverse spherical image of K. That is,

$$S_{n-1}(K, \omega) = \mathcal{H}^{n-1}(\tau(K, \omega)),$$

for $\omega \in \mathcal{B}(S^{n-1})$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and $\tau(K, \omega)$ is the set of all boundary points of K at which there exists an outer normal vector of K belonging to ω.

3. Minkowski tensors and harmonic intrinsic volumes

Let $\mathbb{T}^p$ be the vector space of symmetric tensors of rank p over $\mathbb{R}^n$, that is, the space of symmetric multilinear functions of p variables in $\mathbb{R}^n$. Due to linearity, a tensor $T \in \mathbb{T}^p$ can be identified with the array $\{T(e_{i_1}, \dots, e_{i_p})\}_{i_1,\dots,i_p=1}^n$, where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. We refer to the entries of the array as the components of T. For symmetric tensors $a \in \mathbb{T}^{p_1}$ and $b \in \mathbb{T}^{p_2}$, let $ab \in \mathbb{T}^{p_1+p_2}$ denote the symmetric tensor product of a and b. Identifying $x \in \mathbb{R}^n$ with the rank 1 tensor $z \mapsto \langle z, x\rangle$, we write $x^p \in \mathbb{T}^p$ for the p-fold symmetric tensor product of x. The metric tensor $Q \in \mathbb{T}^2$ is defined by $Q(x, y) = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$.

For a convex body $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$, and $j \in \{0, 1, \dots, n-1\}$, we define the *Minkowski tensors* of K as

$$\Phi_j^{r,s}(K) := \frac{\omega_{n-j}}{r!s!\omega_{n-j+s}} \int\limits_{\mathbb{R}^n \times S^{n-1}} x^r u^s \, \Lambda_j(K, d(x, u))$$

and supplement this definition by

$$\Phi_n^{r,0}(K) := \frac{1}{r!} \int\limits_K x^r \, \lambda(dx).$$

The tensor functionals $\Phi_j^{r,s}$ and $\Phi_n^{r,0}$ are motion covariant valuations on $\mathcal{K}^n$ and continuous with respect to the Hausdorff metric. In [15] the tensor functionals $Q^m\Phi_j^{r,s}$ with $m, r, s \in \mathbb{N}_0$ and either $j \in \{0, \dots, n-1\}$ or $(j, s) = (n, 0)$ are called the basic tensor valuations. Due to Alesker's characterization theorem, every motion covariant, continuous tensor-valued valuation is a linear combination of the basic tensor valuations. For further details, see [20] and the references given there.

In the present work, we only consider translation invariant Minkowski tensors, which are obtained by letting $r = 0$. We use the notation

$$\Phi_j^s(K) = \Phi_j^{0,s}(K) = \frac{\binom{n-1}{j}}{s!\,\omega_{n-j+s}} \int\limits_{S^{n-1}} u^s S_j(K, du)$$

for $j \in \{0, \dots, n-1\}$ and $s \in \mathbb{N}_0$. For $s \in \mathbb{N}_0$, the tensors $\Phi_{n-1}^s(K)$ derived from the surface area measure of a convex body K are called *surface tensors* of K. For later use, we mention that

$$\Phi_j^1(K) = 0 \tag{6}$$

for $j = 0, \dots, n-1$ and any $K \in \mathcal{K}^n$, which is a special case of [20, Eq. (5.30)]. For $s_o \in \mathbb{N}_0$, $j \in \{0, \dots, n-1\}$ and $K \in \mathcal{K}^n$, we let

$$\mathcal{M}_j^{s_o}(K) = \{L \in \mathcal{K}^n \mid \Phi_j^s(L) = \Phi_j^s(K),\ 0 \le s \le s_o\}.$$

As $S_0(K,\cdot) = \sigma$ independently of $K \in \mathcal{K}^n$, we have trivially $\mathcal{M}_0^{s_o}(K) = \mathcal{K}^n$. In the following, we will only consider these classes for $j = 1$ and $j = n-1$.

Remark 3.1. Let $K \in \mathcal{K}^n$ be given. By computing the trace of the tensor $\Phi_j^s(K)$, $j \in \{0, \dots, n-1\}$, $s \geq 2$, the rank of the tensor is reduced by 2, and the tensor $\frac{n-j+s-2}{2\pi s(s-1)}\Phi_j^{s-2}(K)$ is obtained. This follows from the identity

$$\sum_{k=1}^{n} \int_{S^{n-1}} u_{i_1} \cdots u_{i_{s-2}}\, u_k^2\, S_j(K, du) = \int_{S^{n-1}} u_{i_1} \cdots u_{i_{s-2}}\, S_j(K, du).$$

Therefore, the tensors $\Phi_j^s(K)$ and $\Phi_j^{s-1}(K)$ determine all tensors $\Phi_j^{s'}(K)$ of rank $s' \leq s$. More generally, the moments of order at most s of a measure μ on S^{n-1} are determined by the moments of μ of order $s-1$ and s.

For $s \in \mathbb{N}_0$ and a convex body K in $\mathbb{R}^2$, we let $\phi_{sj}(K)$ denote the different components of the surface tensor $\Phi_1^s(K)$ of rank s. That is,

$$\phi_{sj}(K) = \frac{1}{s!\omega_{s+1}} \int_{S^1} u_1^j u_2^{s-j} S_1(K, du) \tag{7}$$

for $j = 0, \dots, s$. For $s_o \in \mathbb{N}$, Remark 3.1 implies that it is sufficient to require knowledge of the $2s_o + 1$ different components of $\Phi_1^{s_o-1}(K)$ and $\Phi_1^{s_o}(K)$ in a reconstruction algorithm of shape based on surface tensors up to rank s_o as these components determine the surface tensors $\Phi_1^0(K), \dots, \Phi_1^{s_o}(K)$. This will be used in Section 5.

Instead of using only values of the surface tensors of rank $s_o - 1$ and s_o for the reconstruction, another option is to use the value of $\Phi_1^0(K)$ and two values of each surface tensor $\Phi_1^s(K)$ for $1 \leq s \leq s_o$. That this information is equivalent to the knowledge of $\Phi_1^s(K)$, $0 \leq s \leq s_o$, can be seen as follows. Due to the factorization into linear factors of the spherical harmonics in (4) and (5), there are vectors $(v_{s1}^i)_{i=1}^s, (v_{s2}^i)_{i=1}^s \subseteq \mathbb{R}^2$ for $s \in \mathbb{N}$ such that

$$\psi_{sj}(K) := \Phi_1^s(K)(v_{sj}^1, \dots, v_{sj}^s) = \int_{S^1} H_{sj}(u)\, S_1(K, du) \tag{8}$$

for $j = 1, 2$, where (H_{sj}) is the orthonormal sequence of spherical harmonics given by (2) and (3). Further, we have that

$$\psi_{01}(K) := \sqrt{\frac{2}{\pi}}\Phi_1^0(K) = \int_{S^1} H_{01}(u)\, S_1(K, du). \tag{9}$$

Equations (8) and (9) show that $\psi_{sj}(K)$ is a value of $\Phi_1^s(K)$ when $s \geq 1$ and that $\psi_{01}(K)$ is the value of $\Phi_1^0(K)$ up to a known constant. Thus trivially, the vector

$(\psi_{01}(K), \psi_{11}(K), \psi_{12}(K), \dots, \psi_{s_o 1}(K), \psi_{s_o 2}(K))$ is determined by $(\Phi_1^0(K), \dots, \Phi_1^{s_o}(K))$. The converse is also true, as polynomials on S^1 of degree at most s_o are linear combinations of the spherical harmonics of degree at most s_o, see [11, Cor. 3.2.6]. It follows that the knowledge of the $2s_o + 1$ values $\psi_{01}(K)$, $\psi_{s1}(K)$, $\psi_{s2}(K)$ for $1 \le s \le s_o$ is sufficient for a reconstruction algorithm based on surface tensors up to rank s_o.

The described values $(\psi_{sj}(K))$ are moments of the surface area measure of $K \in \mathcal{K}^2$ with respect to an orthonormal sequence of spherical harmonics. In [14] such moments are called *harmonic intrinsic volumes*. In general, the harmonic intrinsic volumes associated to a convex body K in $\mathbb{R}^n$ are defined as

$$\psi_{jmk}(K) = \int_{S^{n-1}} H_{mk}(u)\, S_j(K, du)$$

for $j = 0, \dots, n-1$, $m \in \mathbb{N}_0$ and $k = 1, \dots, N(n, m)$. The spherical harmonic H_{mk} is of degree m, and we, therefore, refer to m as the degree of ψ_{jmk}. The harmonic intrinsic volumes depend on the choice of orthonormal bases for $\mathcal{H}_m^n$ for $m \in \mathbb{N}_0$. For $n = 2$, we use the bases given by (2) and (3). We remark however that

$$\sum_{k=1}^{N(n,m)} \psi_{jmk}(K)^2 \text{ and } \sum_{k=1}^{N(n,m)} \psi_{jmk}(K)\psi_{jmk}(M),$$

$K, M \in \mathcal{K}^n$, do not depend on the chosen basis of $\mathcal{H}_m^n$ due to the addition theorem for spherical harmonics [11, Thm. 3.3.3]. In particular condition (14) in Theorem 4.8 does not depend on the basis chosen.

As we mainly consider harmonic intrinsic volumes derived from the surface area measure, we refer to those as harmonic intrinsic volumes. When referring to harmonic intrinsic volumes derived from area measures of lower order, this is explicitly stated. For $n = 2$ and $j = 1$, we write $\psi_{mk} = \psi_{1mk}$. The notation is consistent with (8) and (9).

As described above, the surface tensors and the harmonic intrinsic volumes of a convex body K are closely related. For $s_o \in \mathbb{N}_0$, the surface tensors $\Phi_{n-1}^0(K), \dots, \Phi_{n-1}^{s_o}(K)$ are uniquely determined by $\psi_{(n-1)mk}(K)$ for $m = 0, \dots, s_o$ and $k = 1, \dots, N(n, m)$, see [11, Cor. 3.2.6], and vice versa. Due to the nice properties of spherical harmonics, the harmonic intrinsic volumes are beneficial in the establishment of stability results for surface tensors.

4. Uniqueness and stability results

The components of the Minkowski tensors $\Phi_j^s(K)$, $s \in \mathbb{N}_0$ are coinciding with the moments of $S_j(K, \cdot)$ up to known constants. As S^{n-1} is compact, an application of Stone–Weiserstrass's theorem implies that $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$ determines $S_j(K, \cdot)$. Hence, these tensors determine $K \in \mathcal{K}_n^n$ up to translation when $1 \le j \le n-1$ by the Aleksandrov–Fenchel–Jessen theorem [20, Thm. 8.1.1]. Hence, the shape (as defined in Section 2) of a convex body $K \in \mathcal{K}_n^n$ is uniquely determined by $\{\Phi_j^s(K) \mid s \in \mathbb{N}_0\}$. For $n = 2$, the tensors $\{\Phi_1^s(K) \mid s \in \mathbb{N}_0\}$ even determine the shape of K when K is lower-dimensional.

In order to investigate how different the shape of two convex bodies with identical surface tensors up to a certain rank can be, we discuss properties of the sets $\mathcal{M}_1^{s_o}(K)$ and $\mathcal{M}_{n-1}^{s_o}(K)$ for $s_o \in \mathbb{N}_0$ and $K \in \mathcal{K}^n$. In Theorem 4.1, it is shown that $\mathcal{M}_{n-1}^{s_o}(K)$ contains a polytope, and in Theorem 4.3 a uniqueness result is established stating that $\mathcal{M}_{n-1}^{2s_o}$ is the class of translates of K if K is a polytope with non-empty interior and at most s_o facets. In Theorem 4.9, we show that for large s_o the set $\mathcal{M}_1^{s_o}(K)$ contains only translations of convex bodies close to K in Hausdorff distance.

In the following, we let m_s denote the number of different components of the tensors u^{s-1} and u^s for $s \in \mathbb{N}$ and $u \in S^{n-1}$. Then

$$m_s = \binom{s+n-2}{n-1} + \binom{s+n-1}{n-1} = \big(2 + \frac{n-1}{s}\big)\binom{s+n-2}{n-1} = \mathcal{O}(s^{n-1})$$

for fixed $n \in \mathbb{N}$ as $s \to \infty$. For instance, $m_s = 2s+1$ for $n = 2$, and $m_s = (s+1)^2$ for $n = 3$. The number of different components of u^{s-1} and u^s is identical to the dimension of $\mathcal{H}_0^n \oplus \mathcal{H}_1^n \oplus \cdots \oplus \mathcal{H}_s^n$, that is, $m_s = \sum_{m=0}^{s} N(n,m)$.

Theorem 4.1. *Let $K \in \mathcal{K}^n$ and $s_o \in \mathbb{N}$. Then there exists a $P \in \mathcal{P}^n_{m_{s_o}}$, such that*

$$\Phi_{n-1}^s(K) = \Phi_{n-1}^s(P) \tag{10}$$

for $0 \leq s \leq s_o$.

The proof of Theorem 4.1 follows the lines of the proof of Lemma 6.9 in [4] (see also [23]). For the readers convenience, the proof of Theorem 4.1 is included.

Proof of Theorem 4.1. If the interior of K is empty, then either $S_{n-1}(K,\cdot) = 0$ or $S_{n-1}(K,\cdot) = \alpha(\delta_u + \delta_{-u})$ for some $u \in S^{n-1}$ and $\alpha > 0$. In the first case, let $P = \{o\}$. In the latter case, let P be a polytope contained in the orthogonal complement $u^\perp$ of u with surface area α.

We may from now on assume that $K \in \mathcal{K}_n^n$. If $s_o = 1$, we let P be a polytope with at most $m_1 = n+1$ facets with the same surface area as K. Then (10) is satisfied due to (6). Now assume $s_o \geq 2$. To prove the claim in this case, we construct a Borel measure μ on S^{n-1} with support containing at most m_{s_o} points, satisfying the assumptions of Minkowski's existence theorem, see [20, Thm. 8.2.2], and such that μ has the same moments as $S_{n-1}(K,\cdot)$ up to order s_o. Due to homogeneity of the surface area measure (and herewith of the surface tensors), we may assume that $S_{n-1}(K, S^{n-1}) = 1$.

Let $f_1, \ldots, f_{m_{s_o}}$ denote the different components of the tensors u^{s_o-1} and u^{s_o}. For a Borel probability measure ν on S^{n-1}, let

$$\Gamma(\nu) = \Big(\int\limits_{S^{n-1}} f_1(u)\,\nu(du), \ldots, \int\limits_{S^{n-1}} f_{m_{s_o}}(u)\,\nu(du)\Big).$$

Put

$$M := \left\{\Gamma(\nu) \mid \nu \text{ is a Borel probability measure on } S^{n-1}\right\}$$

and

$$N := \left\{\Gamma(\delta_u) \mid u \in S^{n-1}\right\} = \left\{(f_1(u), \dots, f_{m_{s_o}}(u)) \mid u \in S^{n-1}\right\},$$

where δ_u denotes the Dirac measure at $u \in S^{n-1}$. As $f_1, \dots, f_{m_{s_o}}$ are continuous, the set N is compact in $\mathbb{R}^{m_{s_o}}$, so the convex hull $\operatorname{conv} N$ of N is compact and, in particular, closed. The convex hull $\operatorname{conv} N$ of N is the image of the set of Borel probability measures on S^{n-1} with finite support under Γ. Hence, $M = \operatorname{conv} N$ as every Borel probability measure on S^{n-1} can be weakly approximated by such measures. This implies that $\Gamma(S_{n-1}(K, \cdot)) \in \operatorname{conv} N$. As S^{n-1} is connected and $f_1, \dots, f_{m_{s_o}}$ are continuous, the set N is connected. Then a version of Caratheodory's theorem due to Fenchel (see [12] and references given there) yields the existence of unit vectors $v_1, \dots, v_{m_{s_o}} \in S^{n-1}$ and $\alpha_1, \dots, \alpha_{m_{s_o}} \geq 0$ with $\sum_{i=1}^{m_{s_o}} \alpha_j = 1$ such that

$$\Gamma(S_{n-1}(K, \cdot)) = \sum_{i=1}^{m_{s_o}} \alpha_i \Gamma(\delta_{v_i}) = \Gamma(\mu), \tag{11}$$

where $\mu := \sum_{i=1}^{m_{s_o}} \alpha_i \delta_{v_i}$ is a probability measure with support containing at most m_{s_o} points. Remark 3.1, (6) and (11) yield that

$$\int_{S^{n-1}} u_i \, \mu(du) = \int_{S^{n-1}} u_i \, S_{n-1}(K, du) = 0$$

for $i = 1, \dots, n$, hence the centroid of μ is at the origin.

If the support of μ was concentrated on a great subsphere $v^\perp \cap S^{n-1}$ of S^{n-1} for some $v \in S^{n-1}$, then

$$\int_{S^{n-1}} \langle u, v\rangle^2 S_{n-1}(K, du) = \int_{S^{n-1}} \langle u, v\rangle^2 \mu(du) = 0$$

by Remark 3.1 and (11) as $s_o \geq 2$. This would imply that $S_{n-1}(K, \cdot)$ is concentrated on $v^\perp \cap S^{n-1}$, which is a contradiction as K has interior points. Hence, the measure μ has full-dimensional support.

Herewith, μ satisfies the assumptions in Minkowski's existence theorem, and there is a polytope P with interior points such that $S_{n-1}(P, \cdot) = \mu$. As the support of $S_{n-1}(P, \cdot)$ contains at most m_{s_o} points, the polytope P has at most m_{s_o} facets. Due to (11) and Remark 3.1, the measures $S_{n-1}(K, \cdot)$ and $S_{n-1}(P, \cdot)$ have identical moments up to order s_o, which ensures that equation (10) is satisfied. $\square$

Corollary 4.2. *If K is determined up to translation among all convex bodies in $\mathbb{R}^n$ by its surface tensors up to rank s_o then $K \in \mathcal{P}^n_{m_{s_o}}$.*

On the other hand, a polytope is determined up to translation by finitely many surface tensors.

Theorem 4.3. *Let $m \geq n+1$ be a natural number. The shape of any $P \in \mathcal{P}^n_m$ with non-empty interior is uniquely determined in $\mathcal{K}^n$ by its surface tensors up to rank $2m$. If $n = 2$ then the result holds for any $P \in \mathcal{P}^n_m$.*

Proof. Let $P \in \mathcal{P}^n_m$ be given. We may assume without loss of generality that P has m facets. The surface area measure of P is of the form

$$S_{n-1}(P, \cdot) = \sum_{i=1}^{m} \alpha_i \delta_{u_i}$$

with $\alpha_1, \ldots, \alpha_m > 0$ and pairwise different $u_1, \ldots, u_m \in S^{n-1}$.

Let $K \in \mathcal{K}^n$ be a convex body such that $\Phi^s_{n-1}(K) = \Phi^s_{n-1}(P)$ for all $s \leq 2m$. We first show that $\operatorname{supp} S_{n-1}(K, \cdot) \subseteq \{\pm u_1, \ldots, \pm u_m\}$. Assume that $w \notin \{\pm u_1, \ldots, \pm u_m\}$. Then there exists $v_j \in u_j^{\perp} \setminus w^{\perp}$, $j = 1, \ldots, m$. Hence, the polynomial

$$q_1(u) = \prod_{j=1}^{m} \langle v_j, u \rangle^2,$$

$u \in S^{n-1}$, vanishes at $\pm u_1, \ldots, \pm u_m$ but not at w. By the assumption on coinciding tensors and as q_1 has degree $2m$, we have

$$\int_{S^{n-1}} q_1(u) S_{n-1}(K, du) = \int_{S^{n-1}} q_1(u) S_{n-1}(P, du) = \sum_{i=1}^{m} \alpha_i q_1(u_i) = 0.$$

As $q_1 \geq 0$, this shows that q_1 is zero for $S_{n-1}(K, \cdot)$-almost all u. As q_1 is continuous,

$$\operatorname{supp} S_{n-1}(K, \cdot) \subseteq \{u \in S^{n-1} | q_1(u) = 0\} \subseteq S^{n-1} \setminus \{w\}.$$

Hence $w \notin \operatorname{supp} S_{n-1}(K, \cdot)$ and then $\operatorname{supp} S_{n-1}(K, \cdot) \subseteq \{\pm u_1, \ldots, \pm u_m\}$. In particular, K is a polytope. Its surface area measure is of the form

$$S_{n-1}(K, \cdot) = \sum_{i=1}^{m} \left(\beta_i^+ \delta_{u_i} + \beta_i^- \delta_{-u_i} \right)$$

with $\beta_1^+, \beta_1^-, \ldots, \beta_m^+, \beta_m^- \geq 0$, where we may assume $\beta_i^- = 0$ whenever $-u_i \in \{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m\}$.

Consider now two cases. If $-u_1 \notin \{u_2, \dots, u_m\}$, we can find $v_j \in u_j^\perp \backslash u_1^\perp$, $j = 2, \dots, m$, and thus we have $q_2(u_1) \neq 0 \neq q_3(u_1)$ for

$$q_2(u) = \prod_{j=2}^{m} \langle v_j, u \rangle^2, \qquad q_3(u) = \left(\prod_{j=2}^{m-1} \langle v_j, u \rangle^2 \right) \langle v_m, u \rangle.$$

By the assumption on coinciding tensors, q_2 gives the same value when integrated with respect to $S_{n-1}(K, \cdot)$ and $S_{n-1}(P, \cdot)$. The same is true for q_3. This gives

$$\beta_1^+ + \beta_1^- = \alpha_1, \qquad \beta_1^+ - \beta_1^- = \alpha_1,$$

so $\beta_1^+ = \alpha_1$ and $\beta_1^- = 0$. If $-u_1 \in \{u_2, \dots, u_m\}$ we may without loss of generality assume $-u_1 = u_2 \notin \{\pm u_3, \dots, \pm u_m\}$. In this case, we have $\beta_1^- = \beta_2^- = 0$, and the remaining two parameters β_1^+ and β_2^+ can be determined with arguments similar to the ones above using

$$q_2(u) = \prod_{j=3}^{m} \langle v_j, u \rangle^2, \qquad q_3(u) = \left(\prod_{j=3}^{m-1} \langle v_j, u \rangle^2 \right) \langle v_m, u \rangle.$$

These arguments can be applied to any index i showing that $S_{n-1}(K, \cdot) = S_{n-1}(P, \cdot)$. If P has non-empty interior or if $n = 2$, this implies that P and K are translates. $\square$

Theorem 4.4, below, is a version of Theorem 4.1 for centrally symmetric convex bodies. If $K \in \mathcal{K}^n$ is centrally symmetric its surface area measure is even on S^{n-1}, and hence $\Phi_{n-1}^s(K) = 0$ for all odd s. This simplifies the arguments in the proof of Theorem 4.1 as outlined in the following. Let $s_o \in \mathbb{N}$ be even. Let l_{s_o} denote the number of components of u^{s_o}, that is,

$$l_{s_o} = \binom{s_o + n - 1}{n - 1}.$$

In particular, $l_{s_o} = s_o + 1$ for $n = 2$. Let $h_1, \dots, h_{l_{s_o}}$ denote the different components of u^{s_o}. Following the proof of Theorem 4.1 with Γ, M and N replaced by

$$\Gamma_s(\nu) = \left(\int_{S^{n-1}} h_1(u)\nu(du), \dots, \int_{S^{n-1}} h_{l_{s_o}}(u)\nu(du) \right),$$
$$M_s = \left\{ \Gamma(\nu) \mid \nu \text{ is a symmetric Borel probability measure on } S^{n-1} \right\},$$

and

$$N_s = \left\{ \frac{1}{2}(\Gamma(\delta_u) + \Gamma(\delta_{-u})) \mid u \in S^{n-1} \right\},$$

we obtain an even probability measure $\mu_s = \sum_{j=1}^{l_{s_o}} \alpha_j(\delta_{u_j} + \delta_{-u_j})$ on S^{n-1}, such that

$$\Gamma_s(S_{n-1}(K,\cdot)) = \Gamma_s(\mu_s). \tag{12}$$

As μ_s and $S_{n-1}(K,\cdot)$ are even, equation (12) implies that $\Gamma(\mu_s) = \Gamma(S_{n-1}(K,\cdot))$ with the notation from the proof of Theorem 4.1, and the result of Theorem 4.4 follows.

Theorem 4.4. *Let $K \in \mathcal{K}^n$ be centrally symmetric and $s_o \in \mathbb{N}$ be even. Then there exists an origin-symmetric polytope $P \in \mathcal{P}^n_{2l_{s_o}}$, such that*

$$\Phi^s_{n-1}(K) = \Phi^s_{n-1}(P)$$

for $0 \le s \le s_o$.

Remark 4.5. For later use, we note that the polytope P and the convex body K in Theorems 4.1 and 4.4 have identical harmonic intrinsic volumes up to degree s_o, as they have identical surface tensors up to rank s_o.

The following lemma gives a generalized version of Wirtinger's inequality, which is used in Theorem 4.8 to establish stability estimates for harmonic intrinsic volumes derived from the area measure of order 1.

Recall that $F \sim \sum_{m=0}^{\infty} F_m$ is the condensed harmonic expansion of $F \in L^2(S^{n-1})$.

Lemma 4.6. *Let $n \ge 2$, $s \in \mathbb{N}$ and $F \sim \sum_{m=0}^{\infty} F_m \in W^\alpha$ be given for some $\alpha > 0$. For $\gamma_m = m(m+n-2)$ we have*

$$\|F\|^2 \le \gamma_s^{-\alpha} \|(-\Delta_S)^{\frac{\alpha}{2}} F\|^2 + \sum_{m=0}^{s-1} \left(1 - (\gamma_m \gamma_s^{-1})^\alpha\right) \|F_m\|^2$$

with equality if and only if $F \in \bigoplus_{m=0}^{s} \mathcal{H}^n_m$.

Proof. It follows from (1) that

$$\begin{aligned}
\|F\|^2 &= \sum_{m=0}^{s-1} \|F_m\|^2 + \sum_{m=s}^{\infty} \gamma_m^{-\alpha} \gamma_m^{\alpha} \|F_m\|^2 \\
&\le \sum_{m=0}^{s-1} \|F_m\|^2 + \gamma_s^{-\alpha} \sum_{m=s}^{\infty} \gamma_m^{\alpha} \|F_m\|^2 \\
&= \gamma_s^{-\alpha} \|(-\Delta_S)^{\frac{\alpha}{2}} F\|^2 + \sum_{m=0}^{s-1} (1 - (\gamma_m \gamma_s^{-1})^\alpha) \|F_m\|^2.
\end{aligned}$$

Equality holds in the above calculations if and only if $F = \sum_{m=0}^{s} F_m$. □

Lemma 4.6 immediately yields Corollary 4.7, where the second statement is a generalized version of Wirtinger's inequality.

Corollary 4.7 *(Generalized Wirtinger's inequality). Let $n \geq 2$, $s \in \mathbb{N}$ and $F \sim \sum_{m=0}^{\infty} F_m \in W^{\alpha}$ be given for some $\alpha > 0$. Then*

(i) $\|F\|^2 \leq \gamma_s^{-\alpha}\|(-\Delta_S)^{\frac{\alpha}{2}}F\|^2 + \sum_{m=0}^{s-1}\|F_m\|^2$,
(ii) *if* $F_0 = \cdots = F_{s-1} = 0$, *then* $\|F\|^2 \leq \gamma_s^{-\alpha}\|(-\Delta_S)^{\frac{\alpha}{2}}F\|^2$.

Equality holds in (i) and/or (ii) if and only if F is a spherical harmonic of degree s.

If F is twice continuously differentiable, then $F \in W^1$ by [11, Cor. 3.2.12]. Hence, Corollary 4.7 (ii) with $\alpha = 1$ can be applied to F if $F_0 = \cdots = F_{s-1} = 0$. For $s \in \{1, 2\}$, this yields the usual versions of Wirtinger's inequality of functions on S^{n-1}, see, e.g., [11, Thm. 5.4.1].

A convex body $K \in \mathcal{K}_n^n$ is said to be of class C_+^2 if the boundary of K is a regular submanifold of $\mathbb{R}^n$ of class C^2 with positive Gauss curvature at each point. If $n \geq 2$ and K is of class C_+^2, then the support function h_K is twice continuously differentiable (see [20, Sec. 2.5]), and the area measure $S_1(K, \cdot)$ of order 1 has density

$$s_1 = h_K + \frac{1}{n-1}\Delta_S h_K \tag{13}$$

with respect to the spherical Lebesgue measure on S^{n-1}, see [20, (2.56) and (4.26)]. This establishes a connection between the support function of K and the harmonic intrinsic volumes of K derived from the area measure of order one. In combination with the generalized version of Wirtinger's inequality, this connection can be used to show the stability results in Theorems 4.8 and 4.9.

Theorem 4.8. *Let $n \geq 2$, $s_o \in \mathbb{N}_0$ and $\rho \geq 0$. Let $K, L \in \mathcal{K}^n$ such that $K, L \subseteq RB^n$ for some $R > 0$. Assume that*

$$\frac{1}{(m \vee 1)^{3-\varepsilon}} \sum_{k=1}^{N(n,m)} \big(\psi_{1mk}(K) - \psi_{1mk}(L)\big)^2 \leq \rho \tag{14}$$

for $m = 0, \ldots, s_o$ and some $\varepsilon > 0$. Then

$$\delta_2^t(K, L)^2 \leq c_1\big((s_o + 1)(n + s_o - 1)\big)^{-\alpha} + \rho M(n, \varepsilon) \tag{15}$$

for $0 < \alpha < \frac{3}{2}$, where $c_1 = c_1(\alpha, n, R)$ is a constant depending only on n, α and R, and M is a constant depending only on n and ε.

Proof. By [20, Thm. 3.4.1 and subsequent remarks] there exists a sequence $(K_j)_{j\in\mathbb{N}}$ of convex bodies of class C_+^2 converging to K in the Hausdorff metric. For each $j \in \mathbb{N}$, the support function h_{K_j} is twice continuously differentiable, as K_j is of class C^2_+. Then an application of Green's formula (see, e.g., [11, (1.2.7)]), implies that

$$\langle H_m, \Delta_S h_{K_j}\rangle_2 = \langle \Delta_S H_m, h_{K_j}\rangle_2 = -\gamma_m \langle H_m, h_{K_j}\rangle_2$$

for $H_m \in \mathcal{H}^n_m$ as spherical harmonics are eigenfunctions of the Laplace–Beltrami operator. Thus, (13) yields that

$$\int\limits_{S^{n-1}} H_m(u)\, S_1(K_j, du) = \alpha_{nm}\langle H_m, h_{K_j}\rangle_2 \tag{16}$$

for $H_m \in \mathcal{H}^n_m$, where $\alpha_{nm} = 1 - (n-1)^{-1}\gamma_m$. Note that $\alpha_{nm} = 0$ if and only if $m = 1$. As $S_1(K_j, \cdot)$ converges weakly to $S_1(K, \cdot)$ (see [20, Thm. 4.2.1]), and h_{K_j} converges uniformly to h_K, equation (16) implies that

$$\int\limits_{S^{n-1}} H_m(u)\, S_1(K, du) = \alpha_{n,m}\langle H_m, h_K\rangle_2. \tag{17}$$

By the same arguments, equation (17) holds with K replaced by L.

Now let $F = h_K - h_L + \langle x, \cdot\rangle$, where $x = s(L) - s(K)$. Then $F_1 = 0$, and by equation (17), inequality (14), and the fact that $\langle x, \cdot\rangle \in \mathcal{H}^n_1$ we obtain that

$$\begin{aligned}\sum_{m=0}^{s_o} \|F_m\|^2 &= \sum_{\substack{m=0\\ m\neq 1}}^{s_o} \sum_{k=1}^{N(n,m)} \Bigg(\int\limits_{S^{n-1}} H_{mk}(u) F(u)\, \sigma(du)\Bigg)^2 \\ &= \sum_{\substack{m=0\\ m\neq 1}}^{s_o} \alpha_{nm}^{-2} \sum_{k=1}^{N(n,m)} (\psi_{1mk}(K) - \psi_{1mk}(L))^2 \le \rho\, M(n, \varepsilon),\end{aligned}$$

where $M(n, \varepsilon) = \sum_{m=2}^{\infty} \frac{m^{3-\varepsilon}}{\alpha^2_{n,m}} + 1 < \infty$. For $0 < \alpha < \frac{3}{2}$, we have that

$$\|(-\Delta_S)^{\frac{\alpha}{2}} F\| \le \|(-\Delta_S)^{\frac{\alpha}{2}} h_{K-s(K)}\| + \|(-\Delta_S)^{\frac{\alpha}{2}} h_{L-s(L)}\| \le c_1(\alpha, n, R)$$

due to [16, (2.12)]. This implies that $F \in W^\alpha$ for $0 < \alpha < \frac{3}{2}$. Then Corollary 4.7 (i) with s replaced by $s_o + 1$ can be applied to F, which yields that

$$\|F\|^2 \le \big((s_o + 1)(s_o + n - 1)\big)^{-\alpha} c_1(\alpha, n, R) + \rho M(n, \varepsilon)$$

for $0 < \alpha < \frac{3}{2}$. Then inequality (15) follows, since $\delta^t_2(K, L)^2 = \|F\|^2$. $\square$

The result of Theorem 4.8 can be transferred to a stability result for the Minkowski tensors Φ^s_1 (which are the surface tensors in the two-dimensional setting).

Theorem 4.9. *Let $n \geq 2$, $s_o \in \mathbb{N}_0$ and let $K, L \in \mathcal{K}^n$ such that $K, L \subseteq RB^n$ for some $R > 0$. If $\Phi_1^s(K) = \Phi_1^s(L)$ for $s \in \{(s_o - 1) \vee 0, s_o\}$, then*

$$\delta_2^t(K, L) \leq c_1\, s_o^{-\alpha} \tag{18}$$

and

$$\delta^t(K, L) \leq c_2\, s_o^{-\frac{2\alpha}{n+1}} \tag{19}$$

for $0 < \alpha < \frac{3}{2}$, where $c_1 = c_1(\alpha, n, R)$ and $c_2 = c_2(\alpha, n, R)$ are constants depending only on α, n and R.

Proof. Inequality (18) follows from Theorem 4.8, since equation (14) is satisfied with $\rho = 0$, as $\Phi_1^s(K) = \Phi_1^s(L)$ for $0 \leq s \leq s_o$, see Remark 3.1. Inequality (18) in combination with a known connection between the L^2-distance and the Hausdorff distance (see, [11, Prop. 2.3.1]) yields inequality (19). □

5. Reconstruction of shape from surface tensors

We assume throughout this section that $n = 2$. In arbitrary dimension n, the surface tensors determine the shape of a convex body with interior points. In the two-dimensional case, however, the assumption on interior points is redundant, see [20, Thm. 8.3.6]. In the attempt to reconstruct shape from surface tensors in $\mathbb{R}^2$, it is therefore natural to consider $K_0 \in \mathcal{K}^2$. We suppose that the convex body K_0 is unknown and that the surface tensors $\Phi_1^0(K_0), \ldots, \Phi_1^{s_o}(K_0)$ are known for some $s_o \in \mathbb{N}_0$. By Remark 3.1, this is equivalent to assuming that the components $\phi_{sj}(K_0)$ for $j = 0, \ldots, s$ of $\Phi_1^s(K_0)$ are known for $s = s_o - 1, s_o$ (if $s_o = 0$, only the value of $\Phi_1^0(K_0)$ is assumed to be known).

Section 5.1 presents a reconstruction procedure of the shape of K_0 based on the components of the surface tensors of rank $s_o - 1$ and s_o. The output of the reconstruction procedure is a polygon P, where the surface tensors of P are identical to the surface tensors of K_0 up to rank s_o. In Section 5.2, we use results from Section 4 to show consistency of the reconstruction algorithm developed in Section 5.1.

As described in Section 3, the harmonic intrinsic volumes of K_0 up to degree s_o constitute a set of values of surface tensors that contains the same shape information as the components of $\Phi_1^{s_o-1}(K_0)$ and $\Phi_1^{s_o}(K_0)$. It only requires minor adjustments of the reconstruction algorithm to obtain an algorithm based on harmonic intrinsic volumes.

5.1. Reconstruction

Assume that $s_o \geq 1$, and define $D_{s_o}\colon \mathcal{K}^2 \to [0, \infty)$ as the sum of squared deviations of the components of the surface tensors of K to the components of the surface tensors of K_0 of rank $s_o - 1$ and s_o. That is

$$D_{s_o}(K) = \sum_{s=s_o-1}^{s_o} \sum_{j=0}^{s} \big(\phi_{sj}(K_0) - \phi_{sj}(K)\big)^2. \tag{20}$$

By Remark 3.1, the surface tensors of a convex body K and the surface tensors of K_0 are identical up to rank s_o if and only if $D_{s_o}(K) = 0$. In order to reconstruct the shape of K_0 from the surface tensors, it therefore suffices to find a convex body that minimizes D_{s_o}. Due to Theorem 4.1, there exists a $P \in \mathcal{P}^2_{2s_o+1}$ satisfying this condition.

Let δ_u denote the Dirac measure at $u \in S^1$, and let

$$M = \{(\alpha, u) \in \mathbb{R}^{2s_o+1} \times (S^1)^{2s_o+1} \mid \alpha_i \geq 0, \sum_{i=1}^{2s_o+1} \alpha_i u_i = o\}.$$

Then the surface area measure of a polygon $P \in \mathcal{P}^2_{2s_o+1}$ is of the form

$$S_1(P, \cdot) = \sum_{i=1}^{2s_o+1} \alpha_i \delta_{u_i},$$

where $(\alpha, u) \in M$. The vectors $u_1, \dots, u_{2s_o+1}$ are the facet normals of P, and $\alpha_1, \dots, \alpha_{2s_o+1}$ are the corresponding facet lengths, see [20, (4.24) and (8.15)]. Conversely, if a Borel measure φ on S^1 is of the form

$$\varphi = \sum_{i=1}^{2s_o+1} \alpha_i \delta_{u_i}$$

for some $(\alpha, u) \in M$, then by Minkowski's existence theorem there is a $P \in \mathcal{P}^2_{2s_o+1}$, such that φ is the surface area measure of P, see [20, Thm. 8.2.1]. Notice that the assumption that φ is not concentrated on a great subsphere in Minkowski's existence theorem can be omitted as $n = 2$, see [20, Thm. 8.3.1]. The minimization of D_{s_o} can now be reduced to its minimization on $\mathcal{P}^2_{2s_o+1}$, and hence to the finite dimensional minimization problem

$$\min_{(\alpha,u)\in M} \sum_{s=s_o-1}^{s_o} \sum_{j=0}^{s} \left(\phi_{sj}(K_0) - \frac{1}{s!\,\omega_{s+1}} \sum_{i=1}^{2s_o+1} \alpha_i u_{i1}^j u_{i2}^{s-j} \right)^2. \tag{21}$$

This can be solved numerically.

A solution to the minimization problem (21) is a vector $(\alpha, u) \in M$, which describes the surface area measure of a polygon. The reconstruction of the polygon from the surface area measure can be executed by means of Algorithm MinkData, see [6, Sec. A.4]. For $n = 2$, the reconstruction algorithm is simple. The vectors $\alpha_1 u_1, \dots, \alpha_{2s_o+1} u_{2s_o+1}$ are sorted such that the polar angles are increasing, and hereafter, the vectors are positioned successively such that they form the boundary of a polygon $\tilde{P}$ with facets of length α_j parallel to u_j for $j = 1, \dots, 2s_o + 1$. The output polygon $\hat{K}_{s_o}$ of the algorithm is $\tilde{P}$ rotated $\frac{\pi}{2}$ around the origin. Then $\hat{K}_{s_o}$ minimizes D_{s_o}, and it follows that the convex bodies $\hat{K}_{s_o}$ and K_0 have identical surface tensors up to rank s_o.

If $s_o = 0$, let $\hat{K}_{s_o}$ be the line segment $[0, \phi_{00}(K_0)e_1]$, where e_1 is the first standard basis vector in $\mathbb{R}^2$. Then $\hat{K}_{s_o}$ is a polygon with 1 facet, and $\Phi_1^0(K_0) = \Phi_1^0(\hat{K}_{s_o})$.

The reconstruction algorithm can be summarized as follows.

Algorithm Surface Tensor

Input: A natural number $s_o \in \mathbb{N}_0$ and the components of the surface tensors $\Phi_1^{s_o}(K_0)$ and $\Phi_1^{(s_o-1)\vee 0}(K_0)$ of an unknown convex body $K_0 \in \mathcal{K}^2$.

Task: Construct a polygon $\hat{K}_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o+1$ facets such that $\hat{K}_{s_o}$ and K_0 have identical surface tensors up to rank s_o.

Action: If $s_o = 0$, let $\hat{K}_{s_o}$ be the line segment $[0, \phi_{00}(K_0)e_1]$. Otherwise:

Phase I: Find a vector $(\alpha, u) \in M$ that minimizes

$$\sum_{s=s_o-1}^{s_o} \sum_{j=0}^{s} \left(\phi_{sj}(K_0) - \frac{1}{s!\,\omega_{s+1}} \sum_{i=1}^{2s_o+1} \alpha_i u_{i1}^j u_{i2}^{s-j} \right)^2,$$

where $\phi_{s0}(K_0), \ldots, \phi_{ss}(K_0)$ denote the components of $\Phi_1^s(K_0)$.

Phase II: The vector (α, u) describes a polygon $\hat{K}_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o+1$ facets. Reconstruct $\hat{K}_{s_o}$ from (α, u) using Algorithm MinkData.

It is worth mentioning that certain a priori information on $K_0 \in \mathcal{K}^n$ can be included in the reconstruction algorithm by modifying the set M in (21). We give two examples.

Example 5.1. If K_0 is known to be centrally symmetric, M can be replaced by

$$\big\{(\alpha, u) \in \mathbb{R}^{2s_o+2} \times (S^1)^{2s_o+2} \mid \alpha_j = \alpha_{(s_o+1)+j} \geq 0,\ u_j = -u_{(s_o+1)+j}\big\},$$

due to Theorem 4.4. This ensures central symmetry of the output polygon $\hat{K}_{s_o}$ of the reconstruction algorithm.

Example 5.2. If K_0 is known to be a polygon with at most m facets, M can be replaced by

$$\tilde{M} = \big\{(\alpha, u) \in \mathbb{R}^m \times (S^1)^m \mid \alpha_j \geq 0, \sum_{j=1}^{m} \alpha_j u_j = 0\big\}.$$

The assumption on K_0 implies that the optimization of (21) with M replaced by $\tilde{M}$ still has a solution with objective function value zero. The uniqueness statement in Theorem 4.3 even implies that the output $\hat{K}_{s_o}$ of this modified Algorithm Surface Tensor is unique and has the same shape as K_0 if $s_o \geq 2m$.

Remark 5.3. If K_0 is a polygon with at most $m \in \mathbb{N}$ facets and known surface tensors of rank $2m-1$ and $2m-2$, then an alternative reconstruction procedure similar to methods for reconstruction of planar polygons from complex moments described in [17] and [9] can be applied. We let $k \leq m$ denote the number of facets of K_0, let $u_1, \ldots, u_k$ denote the facet normals and $\alpha_1, \ldots, \alpha_k$ denote the corresponding facet lengths. The facet normals are identified with complex numbers in the natural way (in particular, u^s denotes complex multiplication and not tensor multiplication in this remark). For $s = 0, \dots, 2m-1$, we let

$$\tau_s = \sum_{j=1}^{k} \alpha_j u_j^s = s!\omega_{s+1} \sum_{j=0}^{s} \binom{s}{j} i^{s-j} \phi_{sj}(K_0)$$

and define the Hankel matrix

$$H = \begin{pmatrix} \tau_0 & \cdots & \tau_{m-1} \\ \vdots & \ddots & \vdots \\ \tau_{m-1} & \cdots & \tau_{2m-2} \end{pmatrix}.$$

As

$$H = V \operatorname{diag}(\alpha_1, \dots, \alpha_k) V^\top$$

where V is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & \cdots & 1 \\ u_1 & \cdot & u_k \\ \vdots & & \vdots \\ u_1^{2m-1} & \cdots & u_k^{2m-1} \end{pmatrix} \in \mathbb{C}^{2m \times k},$$

the rank of H is the number k of facets of K_0. The facet normals and facet lengths of K_0 can be restored from H (or a submatrix of H, if $k < m$) using Prony's method, see [17] or [13]. The shape of the polygon K_0 can then be reconstructed from the facet normals and facet lengths by means of Algorithm MinkData. The facet normals and facet lengths can also be obtained by solving the generalized eigenvalue problem $Hx = \lambda H_1 x$ where H_1 is defined as H but its entries start with τ_1 and end with τ_{2m-1}, see [9].

5.2. Consistency of the reconstruction algorithm

Algorithm Surface Tensor described in Section 5.1 is consistent. This follows from Theorem 5.4.

Theorem 5.4. *Let $K_0 \in \mathcal{K}^2$, $s_o \in \mathbb{N}_0$ and $\varepsilon > 0$. If $K_{s_o} \in \mathcal{K}^2$ and K_0 have identical surface tensors up to rank s_o then*

$$\delta^t(K_0, K_{s_o}) \leq c_3\, s_o^{-1+\varepsilon}$$

where $c_3 = c_3(K_0, \varepsilon)$ only depends on K_0 and ε. Hence, if $K_{s_o}, s_o = 0, 1, 2, \dots,$ is a sequence of such bodies then the shape of K_{s_o} converges to the shape of K_0.

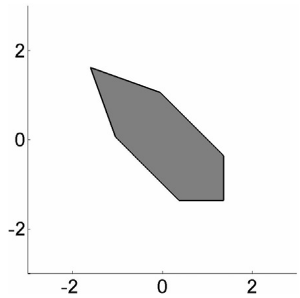

Fig. 1. Polygon with six facets.

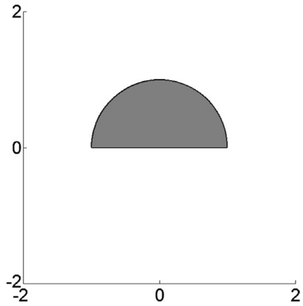

Fig. 2. Half disc.

Proof. As K_0 is compact, there is an $R > 0$ such that $K_0 \subseteq RB^2$. Let $s_o \in \mathbb{N}_0$, and let $x, y \in K_{s_o}$. The line segment $[x, y]$ with endpoints x and y satisfies

$$|x - y| = V_1([x, y]) \leq V_1(K_{s_o}) = V_1(K_0) \leq \pi R$$

by monotonicity of the intrinsic volumes on $\mathcal{K}^2$, see, e.g., [21]. It follows that there is a translate $K_{s_o} + x_{s_o}$ of K_{s_o} which is a subset of πRB^2. For each $s_o \in \mathbb{N}_0$, Theorem 4.9 with R replaced by πR can now be applied to K_0 and $K_{s_o} + x_{s_o}$, and we obtain that

$$\delta^t(K_0, K_{s_o}) \leq c_2(\alpha, 2, \pi R) s_o^{-\frac{2\alpha}{3}}$$

for $0 < \alpha < \frac{3}{2}$. This yields the result. $\square$

5.3. Examples of reconstructions

This section consists of two examples where Algorithm Surface Tensor is used to reconstruct a polygon (see Fig. 1) and a half disc (see Fig. 2). For each two of the convex bodies, the reconstruction is executed for $s_o = 2, 4, 6$. The minimization (21) is performed by use of the procedure *fmincon* provided by MatLab. As initial values for this procedure, we use regular polygons with $2s_o + 1$ facets. The reconstructions are illustrated in Fig. 3 and Fig. 4.

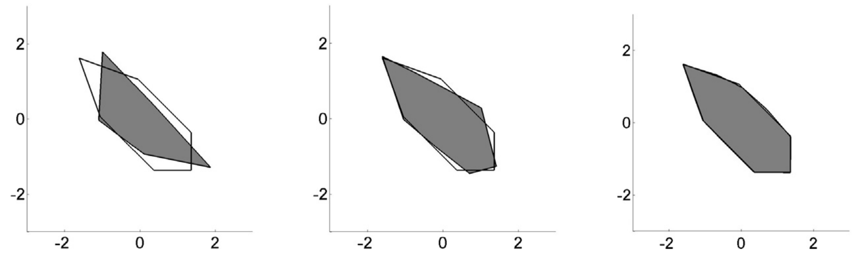

Fig. 3. Reconstructions of the polygon in Fig. 1 based on surface tensors up to rank $s_o = 2, 4, 6$.

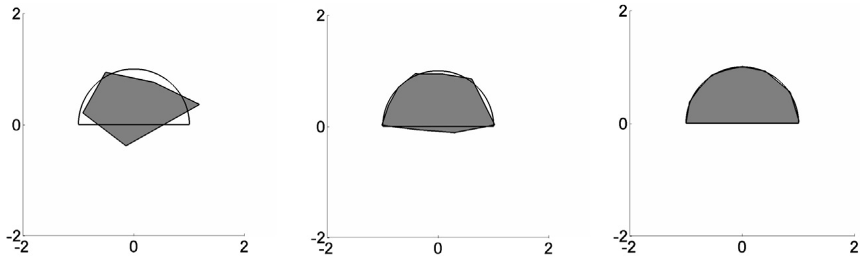

Fig. 4. Reconstructions of the half disc in Fig. 2 based on surface tensors up to rank $s_o = 2, 4, 6$.

The reconstructions with $s_o = 2$ and the corresponding underlying convex bodies have identical surface tensors up to rank 2, so the reconstructions have, in particular, the same boundary length as the corresponding underlying bodies. Further, the reconstructions (in particular, the reconstruction of the polygon) seem to have the same orientation and degree of anisotropy as the corresponding underlying convex bodies. This is due to the influence of the surface tensor of rank 2. As expected, the reconstructions with $s_o = 4$ are more accurate than the reconstructions with $s_o = 2$. In the current two examples, the Algorithm Surface Tensor provides very precise approximations of the polygon and the half disc already for $s_o = 6$.

6. Reconstruction of shape from noisy measurements of surface tensors

In Section 5, the reconstruction of shape from surface tensors was treated. In this section, we consider the problem of reconstructing shape from *noisy* measurements of surface tensors. As in Section 5, we assume that $n = 2$. As described in Section 3, the harmonic intrinsic volumes up to degree s contain the same shape information of a convex body as all surface tensors up to rank s. When only noisy measurements of the surface tensors are available, the structure of the stability result Theorem 4.8 proposes to use the harmonic intrinsic volumes for the reconstruction in order to obtain consistency of the reconstruction algorithm. Therefore, the reconstruction algorithm in this section is based on harmonic intrinsic volumes instead of surface tensors evaluated at the standard basis. In Section 6.4, we briefly discuss the drawbacks of an algorithm based on measurements of surface tensors at the standard basis.

Let $s_o \in \mathbb{N}_0$, and suppose that $K_0 \in \mathcal{K}^2$ is an unknown convex body, where measurements of the harmonic intrinsic volumes up to degree s_o are known. To include noise, the measurements are assumed to be of the form

$$\lambda_{sj}(K_0) = \psi_{sj}(K_0) + \epsilon_{sj} \tag{22}$$

for $j = 1, \dots, N(2, s)$ and $s = 0, \dots, s_o$, where (ϵ_{sj}) are independent random variables with zero mean and finite variance. In the following, let

$$\psi_s(K) = (\psi_{01}(K), \psi_{11}(K), \psi_{12}(K), \dots, \psi_{s2}(K))$$

and similarly

$$\lambda_s(K) = (\lambda_{01}(K), \lambda_{11}(K), \lambda_{12}(K), \dots, \lambda_{s2}(K))$$

for $s \in \mathbb{N}_0$ and $K \in \mathcal{K}^2$.

Section 6.1 presents a reconstruction algorithm for the shape of K_0 based on the measurements (22). The output of the reconstruction procedure is a polygon, which fits the measurements (22) in a least squares sense. It is natural to consider least squares estimation as this is equivalent to maximum likelihood estimation when the noise terms (ϵ_{sj}) are independent, identically distributed normal random variables. The consistency of the least squares estimator is discussed in Section 6.2.

6.1. Reconstruction from measurements of harmonic intrinsic volumes

Assume that $s_o \geq 1$, and define $D^H_{s_o}: \mathcal{K}^2 \to [0, \infty)$ as the sum of squared deviations of the harmonic intrinsic volumes of a convex body K to the measurements (22). That is

$$D^H_{s_o}(K) = \sum_{s=0}^{s_o} \sum_{j=1}^{n_s} \big(\lambda_{sj}(K_0) - \psi_{sj}(K)\big)^2 = |\lambda_{s_o}(K_0) - \psi_{s_o}(K)|^2,$$

where $n_s = N(2, s)$ for $s = 0, \dots, s_o$ ($n_0 = 1$ and $n_s = 2$ for $s \geq 1$). In order to obtain a least squares estimator, the infimum of $D^H_{s_o}$ has to be attained. In contrast to the situation in Section 5.1, the convex body K_0 does not necessarily minimize $D^H_{s_o}$. However, Lemma 6.1 ensures the existence of a polygon that minimizes $D^H_{s_o}$.

Lemma 6.1. *There exists a $P \in \mathcal{P}^2_{2s_o+1}$ such that*

$$D^H_{s_o}(P) = \inf_{K \in \mathcal{K}^2} D^H_{s_o}(K). \tag{23}$$

Furthermore, if $K', K'' \in \mathcal{K}^2$ both are solutions of (23) then $\psi_{s_o}(K') = \psi_{s_o}(K'')$, i.e. K' and K'' have the same surface tensors of rank at most s_o.

Proof. Let $\mathcal{M}_{s_o} = \{\psi_{s_o}(K) \mid K \in \mathcal{K}^2\} \subseteq \mathbb{R}^{2s_o+1}$. Due to Minkowski linearity of the area measure of order one, see [20, Eq. (8.23)], $\mathcal{M}_{s_o}$ is convex.

We now show that $\mathcal{M}_{s_o}$ is closed in $\mathbb{R}^{2s_o+1}$. Let $(\psi_{s_o}(K_n))_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_{s_o}$, such that $\psi_{s_o}(K_n) \to \xi$ for some $\xi \in \mathbb{R}^{2s_o+1}$. For sufficiently large n we have

$$\sqrt{\frac{2}{\pi}}V_1(K_n) = \psi_{01}(K_n) \le |\xi_1 - \psi_{01}(K_n)| + |\xi_1|$$
$$\le |\xi - \psi_{s_o}(K_n)| + |\xi| \le 1 + |\xi|.$$

By monotonicity of the intrinsic volumes on $\mathcal{K}^2$ (see, e.g, [21]), we have

$$|x - y| = V_1([x, y]) \le V_1(K_n) \le \sqrt{\frac{\pi}{2}}(1 + |\xi|)$$

for $x, y \in K_n$. This implies that a translate of K_n is a subset of $\sqrt{\frac{\pi}{2}}(1 + |\xi|)B^2$ for n sufficiently large. By continuity of $K \mapsto \psi_{s_o}(K)$ (with respect to the Hausdorff metric), an application of Blaschke's selection theorem (see, e.g., [20, Thm. 1.8.7]), yields the existence of a subsequence $(n_l)_{l\in\mathbb{N}}$ and a convex body $K \in \mathcal{K}^2$ satisfying $\psi_{s_o}(K_{n_l}) \to \psi_{s_o}(K)$ for $l \to \infty$. Hence, $\xi = \psi_{s_o}(K) \in \mathcal{M}_{s_o}$, so $\mathcal{M}_{s_o}$ is closed. The optimization problem

$$\inf_{K\in\mathcal{K}^2} D^H_{s_o}(K) = \inf_{\psi\in\mathcal{M}_{s_o}} |\lambda_{s_o}(K_0) - \psi|^2$$

corresponds to finding the metric projection of $\lambda_{s_o}(K_0)$ to the non-empty closed and convex set $\mathcal{M}_{s_o}$. This metric projection $\psi_{s_o}(K') \in \mathcal{M}_{s_o}$ always exists and is unique; see, e.g., [20, Section 1.2]. Note that $K' \in \mathcal{K}^2$ is not uniquely determined here, but any two sets $K', K'' \in \mathcal{K}^2$ minimizing (23) must satisfy $\psi_{s_o}(K') = \psi_{s_o}(K'')$. By Theorem 4.1 (and Remark 4.5), this ensures the existence of a polygon P with at most $2s_o + 1$ facets satisfying (23). $\square$

Remark 6.2. It follows from Lemma 6.1 that the measurements (22) are the exact harmonic intrinsic volumes of a convex body if and only if

$$\inf_{K\in\mathcal{K}^2} D^H_{s_o}(K) = 0.$$

By Lemma 6.1 and considerations similar to those in Section 5.1, the minimization of $D^H_{s_o}$ can be reduced to the finite dimensional minimization problem

$$\min_{(\alpha,u)\in M} \sum_{s=0}^{s_o} \sum_{j=1}^{n_s} \left(\lambda_{sj}(K_0) - \sum_{i=1}^{2s_o+1} \alpha_i H_{sj}(u_i)\right)^2, \tag{24}$$

where M is defined as in Section 5.1. This finite minimization problem can be solved numerically. The solution to the minimization problem (24) is a vector (α, u) in M

that describes the surface area measure of a polygon. As described in Section 5.1, the MinkData Algorithm can be applied for the reconstruction of this polygon. The least squares estimator $\hat{K}^H_{s_o}$ of the shape of K_0 is defined to be the output polygon of this algorithm. Then $\hat{K}^H_{s_o}$ minimizes $D^H_{s_o}$, so the harmonic intrinsic volumes of $\hat{K}^H_{s_o}$ fit the measurements (22) in a least squares sense. For $s_o = 0$, the estimator $\hat{K}^H_{s_o}$ is defined as the line segment $[0, \sqrt{\frac{\pi}{2}}\lambda_{01}(K_0)e_1]$ if $\lambda_{01}(K_0) \geq 0$. Otherwise, $\hat{K}^H_{s_o}$ is defined as the singleton $\{0\}$.

The reconstruction algorithm can be summarized as follows.

Algorithm Harmonic Intrinsic Volume LSQ

Input: A natural number $s_o \in \mathbb{N}_0$ and noisy measurements $\lambda_{sj}(K_0)$, $j = 1, \ldots, n_s$, $s = 0, \ldots, s_o$ of the harmonic intrinsic volumes up to degree s_o of an unknown convex body $K_0 \in \mathcal{K}^2$.

Task: Construct a polygon $\hat{K}^H_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o+1$ facets such that the harmonic intrinsic volumes of $\hat{K}^H_{s_o}$ fit the measurements of the harmonic intrinsic volumes of K_0 in a least squares sense.

Action: If $s_o = 0$, let $\hat{K}^H_{s_o}$ be the line segment (or singleton) $[0, (\sqrt{\frac{\pi}{2}}\lambda_{01}(K_0) \vee 0)e_1]$. Otherwise:

Phase I: Find a vector $(\alpha, u) \in M$ that minimizes

$$\sum_{s=0}^{s_o}\sum_{j=1}^{n_s}\left(\lambda_{sj}(K_0) - \sum_{i=1}^{2s_o+1}\alpha_i H_{sj}(u_i)\right)^2.$$

Phase II: The vector (α, u) describes a polygon $\hat{K}^H_{s_o}$ in $\mathbb{R}^2$ with at most $2s_o + 1$ facets. Reconstruct $\hat{K}^H_{s_o}$ from (α, u) using the MinkData Algorithm.

As described in Examples 5.1 and 5.2, additional information on the unknown convex body K_0 can be included in the reconstruction algorithm by modifying the set M in a suitable way.

6.2. *Consistency of the least squares estimator*

So far, we have oppressed the dependence of the noise terms in the notation of $D^H_{s_o}$. In the following, for $s_o \in \mathbb{N}$, we write

$$D^H_{s_o}(K, x) = |\psi_{s_o}(K_0) + x - \psi_{s_o}(K)|^2$$

where $K \in \mathcal{K}^2$ and $x \in \mathbb{R}^{2s_o+1}$. Further, we let

$$\mathbb{K}_{s_o}(x) = \{K \in \mathcal{K}^2 \mid D^H_{s_o}(K, x) = \inf_{L\in\mathcal{K}^2} D^H_{s_o}(L, x)\}.$$

If $\epsilon_{s_o} = (\epsilon_{01}, \epsilon_{11}, \epsilon_{12}, \ldots, \epsilon_{s_o 2})$ denotes the random vector of noise variables in the measurements (22), then $\mathbb{K}_{s_0}(\epsilon_{s_o})$ is the random set of solutions to the minimization (23).

Due to Lemma 6.1, the set $\mathbb{K}_{s_o}(\epsilon_{s_o})$ is non-empty for all $s_o \in \mathbb{N}$. We can without loss of generality assume that the noise variables are defined on a complete probability space.

In the following, we show that $\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K)$ is measurable. To this end, we use the notion of permissible sets, see [18, App. C]. For $K \in \mathcal{K}^2$ and $x \in \mathbb{R}^{2s_o+1}$, define

$$f(K, x) = \delta^t(K_0, K)\mathbf{1}_{\{0\}}(g(K, x))$$

where $g(K, x) = \inf_{L \in \mathcal{K}^2} D^H_{s_o}(L, x) - D^H_{s_o}(K, x)$, and let $\mathcal{F} = \{f(K, \cdot) \mid K \in \mathcal{K}^2\}$. Then

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) = \sup_{K \in \mathcal{K}^2} f(K, \epsilon_{s_o}).$$

As $D^H_{s_o}$ is continuous in the first variable and is measurable as a function of two variables, the mapping g is measurable as $\mathcal{K}^2$ is separable. As $\delta^t(K_0, \cdot)$ is continuous, this implies that f is measurable.

Let $\mathcal{F}_2$ denote the family of closed subsets of $\mathbb{R}^2$ equipped with the Fell topology, see, e.g., [21, Chapter 12.2]. Then, $\mathcal{F}_2$ is compact and metrizable, and the set of convex bodies $\mathcal{K}^2$ is an analytic subset of $\mathcal{F}_2$ as $\mathcal{K}^2 \in \mathcal{B}(\mathcal{F}_2)$, see, e.g., [21, Thm. 12.2.1, the subsequent remark and Thm. 2.4.2]. Further, the topology on the separable set $\mathcal{K}^2$ induced by the Fell topology and the topology on $\mathcal{K}^2$ induced by the Hausdorff metric coincide, see, e.g, [21, Thm. 12.3.4], so the set $\mathcal{F}$ is permissible. Due to [18, App. C, p. 197], this implies that $\sup_{K \in \mathcal{K}^2} f(K, \epsilon_{s_o})$ is measurable.

For $s_o \in \mathbb{N}$, the noise variables $\epsilon_{01}, \epsilon_{11}, \ldots, \epsilon_{s_o 2}$ are assumed to be independent with zero mean and finite variance bounded by a constant $\sigma^2_{s_o} < \infty$.

Theorem 6.3. *If $\sigma^2_{s_o} = \mathcal{O}(\frac{1}{{s_o}^{1+\varepsilon}})$ for some $\varepsilon > 0$, then*

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) \to 0$$

in probability as $s_o \to \infty$. If $\sigma^2_{s_o} = \mathcal{O}(\frac{1}{{s_o}^{2+\varepsilon}})$, then the convergence is almost surely.

Proof. Let $\delta > 0$, and let $0 < \rho < \frac{\delta}{2M} \wedge 1$ where $M = M(2, 3)$ is defined in Theorem 4.8. Let $s_o \in \mathbb{N}$, $K \in \mathbb{K}_{s_o}(\epsilon_{s_o})$, and assume first that $D^H_{s_o}(K_0, \epsilon_{s_o}) < \frac{\rho}{8}$. Then,

$$\begin{aligned}
&\max_{s=0,\ldots,s_o} \sum_{j=1}^{n_s} \big(\psi_{sj}(K_0) - \psi_{sj}(K)\big)^2 \\
&\leq 4 \max_{s=0,\ldots,s_o} \sum_{j=1}^{n_s} \Big(\epsilon^2_{sj} + \big(\lambda_{sj}(K_0) - \psi_{sj}(K)\big)^2\Big) \\
&\leq 8 D^H_{s_o}(K_0, \epsilon_{s_o}) < \rho.
\end{aligned}$$

In particular, $(\psi_{01}(K_0) - \psi_{01}(K))^2 < \rho$ which implies that

$$V_1(K) < \frac{\pi}{2} + V_1(K_0) =: R(K_0).$$

By arguments similar to those in the proof of Theorem 5.4, this implies that there are translates of K and K_0 contained in RB^2. As R is independent of s_o and K, we obtain by Theorem 4.8 that

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta_2^t(K_0, K) \leq c_1(1,2,R)(s_o+1)^{-2} + \rho M < \delta$$

for s_o sufficiently large. Due to the connection between the Hausdorff metric and L_2-metric, see, e.g., [11, Prop. 2.3.1], we obtain

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) < (3R\delta^2)^{\frac{1}{3}}. \tag{25}$$

As $D_{s_o}^H(K_0, \epsilon_{s_o}) = \sum_{s=0}^{s_o} \sum_{j=1}^{n_s} \epsilon_{sj}^2$, the assumption on the convergence rate, $\sigma_{s_o}^2 = \mathcal{O}(\frac{1}{{s_o}^{1+\varepsilon}})$ for some $\varepsilon > 0$, implies that $D_{s_o}^H(K_0, \epsilon_{s_o})$ converges to zero in mean and then in probability, when s_o increases. If $\sigma_{s_o}^2 = \mathcal{O}(\frac{1}{{s_o}^{2+\varepsilon}})$, then $\sum_{s_o=1}^{\infty} \mathbb{E}\, D_{s_o}^H(K_0, \epsilon_{s_o}) < \infty$, which ensures that $D_{s_o}^H(K_0)$ convergences to zero almost surely. In combination with inequality (25), this yields the convergence results. □

As $\hat{K}_{s_o}^H \in \mathbb{K}_{s_o}(\epsilon_{s_o})$ for $s_o \in \mathbb{N}$, Theorem 6.3 yields consistency of Algorithm Harmonic Intrinsic Volume LSQ.

6.3. Example on reconstruction from harmonic intrinsic volumes

This section is an example where Algorithm Harmonic Intrinsic Volume LSQ is used to reconstruct a half disc K_0 from noisy measurements of the harmonic intrinsic volumes. The reconstruction of the half disc is executed for $s_o = 2, 4, \ldots, 12$. The noise terms (ϵ_{sj}) are independent and normally distributed with zero mean. For the reconstruction based on harmonic intrinsic volumes up to degree s_o, the variance of the noise terms is $\sigma_{s_o}^2 = \frac{1}{{s_o}^{2.1}}$. Due to Theorem 6.3 this ensures that $\delta^t(K_0, \hat{K}_{s_o}) \to 0$ almost surely for $s_o \to \infty$. The minimization (24) is carried out by use of the procedure *fmincon* provided by MatLab. As initial values for the minimization procedure, we use regular polygons with $2s_o + 1$ facets. The reconstructions are plotted in Fig. 5.

For the reconstruction based on exact surface tensors, the values of $D_{s_o}(K_0)$ and $D_{s_o}(\hat{K}_{s_o})$ are always zero. This is not the case when the reconstruction is based on measurements subject to noise. In Fig. 6, the values of $D_{s_o}^H(K_0)$ and $D_{s_o}^H(\hat{K}_{s_o}^H)$ are plotted for $s_o = 2, 4, \ldots, 12$. As $\hat{K}_{s_o}^H$ minimizes $D_{s_o}^H$, the value of $D_{s_o}^H(\hat{K}_{s_o})$ is smaller than the value of $D_{s_o}^H(K_0)$ for each s_o. As the variance of the noise terms converges to zero sufficiently fast, the values of $D_{s_o}^H(K_0)$ and hence also the values of $D_{s_o}^H(\hat{K}_{s_o}^H)$ tend to zero, when s_o increases.

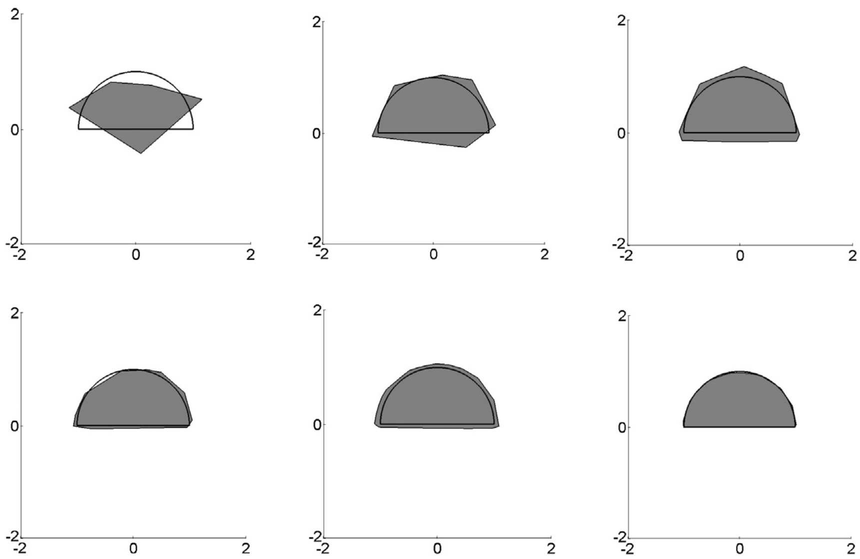

Fig. 5. Reconstruction of the half disc in Fig. 2 based on measurements of harmonic intrinsic volumes up to degree $s_o = 2, 4, 6, 8, 10, 12$. The noise variables are normally distributed with zero mean and variance $\frac{1}{s_o^{2.1}}$.

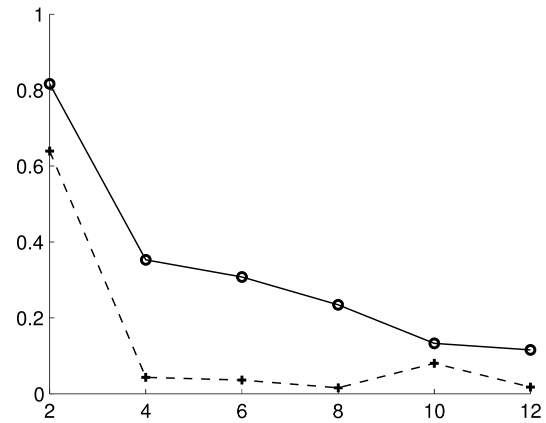

Fig. 6. $D^H_{s_o}(K_0)$ ('o') and $D^H_{s_o}(\hat{K}^H_{s_o})$ ('+') plotted for $s_o = 2, 4, \ldots, 12$.

6.4. Reconstruction from measurements of surface tensors at the standard basis

In this section, we briefly discuss reconstruction based on measurements of surface tensors evaluated at the standard basis. A reconstruction algorithm based on noisy surface tensors can be developed combining ideas from Section 5.1 and Section 6.1. We let $K_0 \in \mathcal{K}^2$ be an unknown convex body and assume that noisy measurements of the surface tensors of K_0 at the standard basis are available for two consecutive ranks $s_o - 1$ and $s_o \in \mathbb{N}$. We then consider the minimization problem

$$\min_{K \in \mathcal{K}^2} \tilde{D}_{s_o}(K) \tag{26}$$

with

$$\tilde{D}_{s_o}(K) = \sum_{s=s_o-1}^{s_o} \sum_{j=0}^{s} \Big(\int_{S^1} u_1^j u_2^{s-j} S_1(K_0, du) + \epsilon_{sj} - \int_{S^1} u_1^j u_2^{s-j} S_1(K_0, du) \Big)^2$$

for $K \in \mathcal{K}^2$, where $\epsilon_{(s_o-1)0}, \dots, \epsilon_{s_o s_o}$ are independent noise terms with zero mean and variance bounded by $\sigma_{s_o}^2 > 0$. By arguments as in Lemma 6.1, the minimum (26) is attained. Notice, that the constants $(s!\,\omega_s)^{-1}$ for $s \in \{s_o - 1, s_o\}$ are omitted in the expression of $\tilde{D}_{s_o}$, so we in fact consider measurements of rescaled versions of the surface tensors (compare with (7) and (20)). We choose to do so, as the constants are artificial (they are introduced in order to simplify certain formulas containing Minkowski tensors), and further, as the constants are decreasing rapidly, so that the surface tensors vanish compared to the noise when s_o increases. For the reconstruction based on exact surface tensors in Section 5, the output polygon of the reconstruction algorithm has surface tensors identical to the unknown convex body and is then independent of the presence of the constants in the objective function.

By arguments as in Sections 5.1 and 6.1, the minimization problem is reduced to

$$\min_{(\alpha,u)\in M} \sum_{s=s_o-1}^{s_o} \sum_{j=0}^{s} \Big(\int_{S^1} u_1^j u_2^{s-j} S_1(K_0, du) + \epsilon_{sj} - \sum_{i=1}^{2s_o+1} \alpha_i u_{i1}^j u_{i2}^{s-j} \Big)^2, \tag{27}$$

where M is defined as in Section 5.1. A solution to the minimization problem (27) corresponds to a polygon that minimizes $\tilde{D}_{s_o}$. This polygon can be reconstructed using Algorithm MinkData.

Using the same procedure as in Theorem 6.3, the stability result Theorem 4.8 can be applied to obtain consistency of the above described reconstruction algorithm. Let $K_{s_o} \in \mathcal{K}^2$ be a convex body that minimizes $\tilde{D}_{s_o}$. Then

$$(s!\omega_{s+1})^2 \sum_{j=0}^{s} (\phi_{sj}(K_0) - \phi_{sj}(K_{s_o}))^2 \le 8\tilde{D}_{s_o}(K_0) \tag{28}$$

for $s \in \{s_o - 1, s_o\}$. In order to apply the stability result Theorem 4.8, it is required to transform the bound (28) into an upper bound on the differences of the harmonic intrinsic volumes of K_0 and K_{s_o} of each degree up to s_o, see (14). We now consider only the harmonic intrinsic volumes of degree 0. According to Remark 3.1, we have

$$\psi_{01}(K) = \sqrt{\frac{2}{\pi}}\phi_{00}(K) = \frac{(2k)!\omega_{2k+1}}{\sqrt{2\pi}} \sum_{j=0}^{k} \binom{k}{j} \phi_{2k,2j}(K)$$

for $K \in \mathcal{K}^2$ and $k \in \mathbb{N}_0$, so

$$(\psi_{01}(K_0) - \psi_{01}(K_{s_o}))^2 = \frac{1}{2\pi}(s!\omega_{s+1})^2 \langle A_s, \phi_s(K_0) - \phi_s(K_{s_o})\rangle^2$$

where $s \in \{s_o - 1, s_o\}$ is even, $\phi_s(K) = (\phi_{s0}(K), \ldots, \phi_{ss}(K))$ and

$$A_s = \left(\binom{\frac{s}{2}}{0}, 0, \binom{\frac{s}{2}}{1}, 0, \ldots, \binom{\frac{s}{2}}{\frac{s}{2}} \right).$$

Since we have no detailed information on the vector $\phi_s(K_0) - \phi_s(K_{s_o})$, we apply the Cauchy–Schwarz inequality combined with (28) and obtain the bound

$$(\psi_{01}(K_0) - \psi_{01}(K_{s_o}))^2 \leq \frac{4}{\pi} |A_s|^2 \tilde{D}_{s_o}(K_0) = \frac{4}{\pi} \binom{s}{\frac{s}{2}} \tilde{D}_{s_o}(K_0).$$

Note that

$$\binom{s}{\frac{s}{2}} 2^{-s} \sqrt{s} \to \sqrt{\frac{2}{\pi}}$$

as $s \to \infty$. To apply the stability result Theorem 4.8 as in Theorem 6.3, we then (at least) have to ensure that

$$2^{s_o} s_o^{-\frac{1}{2}} \tilde{D}_{s_o}(K_0) \to 0$$

in probability or almost surely for $s_o \to \infty$. This requires a very restrictive assumption (compared to the assumption in Theorem 6.3) on the convergence rate of $\sigma_{s_o}^2$. Hence, the structure of the stability result yields a much weaker consistency result for the reconstruction algorithm based on surface tensors at the standard basis, than for the algorithm based on harmonic intrinsic volumes. Therefore, we recommend the use of harmonic intrinsic volumes when only measurements subject to noise are available.

In the example in Section 6.3, we reconstructed a half disc based on measurements of harmonic intrinsic volumes. For comparison, we reconstruct the same half disc from measurements of the surface tensors measured at the standard basis. For the reconstruction based on surface tensors of rank $s_o - 1$ and s_o, we let the noise terms $\epsilon_{(s_o-1)0}, \ldots, \epsilon_{s_o s_o}$ be independent and normally distributed with zero mean, and as in the above example, we let the variance $\sigma_{s_o}^2$ of the noise terms be $\sigma_{s_o}^2 = \frac{1}{{s_o}^{2.1}}$.

In Fig. 7, the reconstructions $\hat{K}_8, \ldots, \hat{K}_{18}$ of the half disc based on noisy surface tensors of rank $s_o - 1$ and s_o for $s_o = 8, 10, \ldots, 18$ are plotted. In Fig. 8, the values of $\tilde{D}_{s_o}(\hat{K}_{s_o})$ and $\tilde{D}_{s_o}(K_0)$ are plotted for $s_o = 8, \ldots, 18$.

The reconstructions $\hat{K}_8^H, \hat{K}_{10}^H$ and $\hat{K}_{12}^H$ based on measurements of harmonic intrinsic volumes in the example in Section 6.3 are better approximations of the half disc than the reconstructions $\hat{K}_8, \hat{K}_{10}$ and $\hat{K}_{12}$ based on measurements of surface tensors in this example. In particular, $\hat{K}_{12}^H$ is a very accurate reconstruction of the half disc. This is not the case for $\hat{K}_{12}$. When s_o increases, the reconstructions $\hat{K}_{14}, \hat{K}_{16}$ and $\hat{K}_{18}$ become more accurate, but even when the noise terms become very small (see Fig. 8), the reconstructions are, as expected, not as precise as $\hat{K}_{12}^H$.

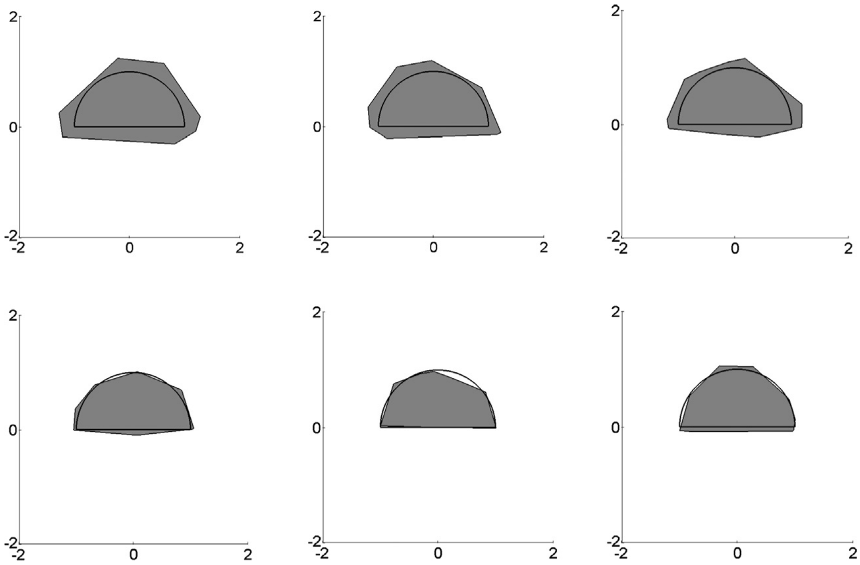

Fig. 7. Reconstruction of the half disc in Fig. 2 based on measurements of surface tensors at the standard basis of rank $s_o - 1$ and s_o for $s_o = 8, 10, 12, 14, 16, 18$. The noise variables are normally distributed with zero mean and variance $\frac{1}{s_o^{2.1}}$.

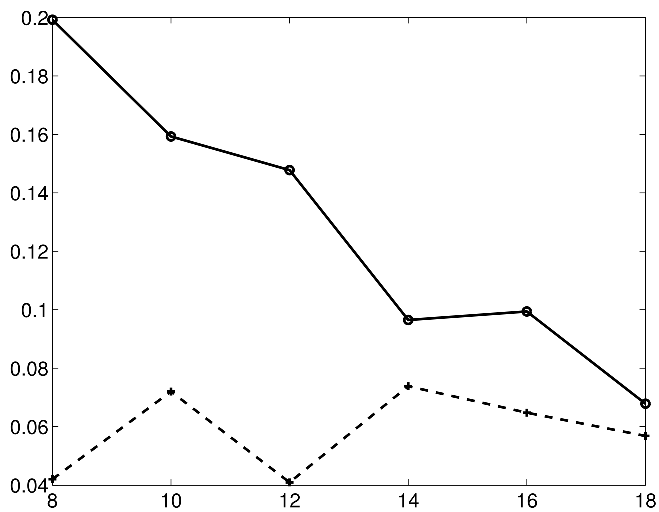

Fig. 8. $\tilde{D}_{s_o}(K_0)$ ('o') and $\tilde{D}_{s_o}(\hat{K}_{s_o})$ ('+') plotted for $s_o = 2, 4, \ldots, 12$.

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Paper C

Reconstruction of n-dimensional convex bodies from surface tensors

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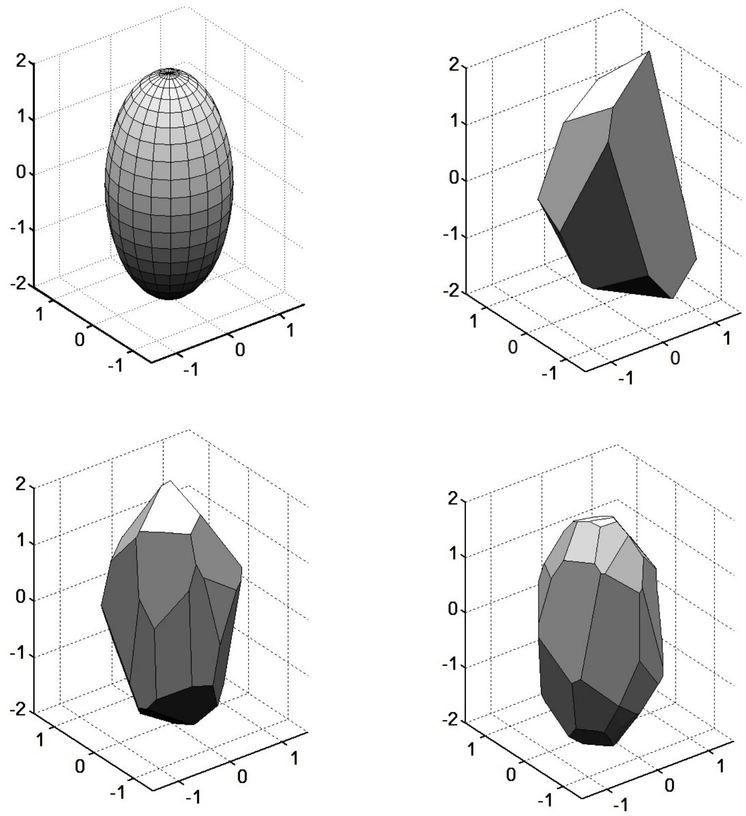

Reconstruction of n-dimensional convex bodies from surface tensors

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Abstract

In this paper, we derive uniqueness and stability results for surface tensors. Further, we develop two algorithms that reconstruct shape of n-dimensional convex bodies. One algorithm requires knowledge of a finite number of surface tensors, whereas the other algorithm is based on noisy measurements of a finite number of harmonic intrinsic volumes. The derived stability results ensure consistency of the two algorithms. Examples that illustrate the feasibility of the algorithms are presented.

Keywords: Convex body, surface tensor, harmonic intrinsic volume, uniqueness, stability, reconstruction algorithm
2010 MSC: 52A20, 44A60, 60D05

1. Introduction

Recently, Minkowski tensors have succesfully been used as shape descriptors of spatial structures in materials science, see, e.g., [3, 15, 16]. Surface tensors are translation invariant Minkowski tensors derived from surface area measures, and the shape of a convex body K with nonempty interior in $\mathbb{R}^n$ is uniquely determined by the surface tensors of K. In this context, the shape of K is defined as the equivalence class of all translations of K.

In [10], Kousholt and Kiderlen develop reconstruction algorithms that approximate the shape of convex bodies in $\mathbb{R}^2$ from a finite number of surface tensors. Two algorithms are described. One algorithm requires knowledge of exact surface tensors, and one allows for noisy measurements of surface tensors. For the latter algorithm, it is argued that it is preferable to use harmonic intrinsic volumes instead of surface tensors evaluated at the standard basis. The similar problem of reconstructing a convex body K from its volume tensors (moments of the Lebesgue measure restricted to K) has received considerable attention and can be applied in X-ray tomography, see, e.g., [12, 8].

The purpose of this paper is threefold. Firstly, the reconstruction algorithms developed in [10] are generalized to an n-dimensional setting. Secondly, stabil-

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ity and uniqueness results for surface tensors are established, and the stability results are used to ensure consistency of the generalized algorithms. Thirdly, we illustrate the feasibility of the reconstruction algorithms by examples. The generalizations of the reconstruction algorithms are developed along the same lines as the algorithms for convex bodies in $\mathbb{R}^2$. However, there are several non-trivial obstacles on the way. In particular, essentially different stability results are needed to ensure consistency.

The input of the first generalized algorithm is exact surface tensors up to a certain rank of an unknown convex body in $\mathbb{R}^n$. The output is a polytope with surface tensors identical to the given surface tensors of the unknown convex body. The input of the second generalized algorithm is (possibly noisy) measurements of harmonic intrinsic volumes of an unknown convex body in $\mathbb{R}^n$, and the output is a polytope with harmonic intrinsic volumes that fit the given measurements in a least squares sense. When $n \geq 3$, a convex body that fits the noisy input measurements of harmonic intrinsic volumes may not exist, and in this case, the output of the algorithm based on harmonic intrinsic volumes is a message stating that there is no solution to the given task. However, this situation only occurs when the measurements are too noisy, see Lemma 6.3.

The consistency of the algorithms described in [10] is established using the stability result [10, Thm. 4.8] for harmonic intrinsic volumes derived from the first order area measure. This result can be applied as the first order area measure and the surface area measure coincide for $n = 2$. However, for $n \geq 3$, the stability result is not applicable. Therefore, we establish stability results for surface tensors and for harmonic intrinsic volumes derived from surface area measures. More precisely, first we derive an upper bound of the Dudley distance between surface area measures of two convex bodies. This bound is small, when the distance between the harmonic intrinsic volumes up to degree s of the convex bodies is small for some large $s \in \mathbb{N}_0$ (Theorem 4.3). From this result and a known connection between the Dudley distance and the translative Hausdorff distance, we obtain that the translative Hausdorff distance between convex bodies with identical surface tensors up to rank s becomes small, when s is large (Corollary 4.4). The structures of the two stability result differ. The first result allows the difference between the harmonic intrinsic volumes to be nonzero, whereas the latter result requires that the surface tensors are *identical* up to a certain rank. This explains the use of harmonic intrinsic volumes instead of surface tensors when only noisy measurements are available. The stability result for surface tensors and the fact that the rank 2 surface tensor of a convex body K determines the radii of a ball containing K and a ball contained in K (Lemma 5.4) ensure consistency of the generalized reconstruction algorithm based on exact surface tensors (Theorem 5.5). The consistency of the reconstruction algorithm based on measurements of harmonic intrinsic volumes is ensured by the stability result for harmonic intrinsic volumes under certain assumptions on the variance of the noise variables (Theorems 6.4 and 6.5).

The described algorithms and stability results show that a finite number of surface tensors can be used to approximate the shape of a convex body, but in general, all surface tensors are required to uniquely determine the shape. However, there are convex bodies where a finite number of surface tensors contain full shape information. More precisely, in [10], it is shown that the shape of a convex body in $\mathbb{R}^n$ with nonempty interior is uniquely determined by a finite number of surface tensors if only if the convex body is a polytope. Further, the shape of a polytope with $m \geq n+1$ facets is uniquely determined by surface tensors up to rank $2m$. We strengthen and complete the uniqueness results from [10] by showing an optimal version of the latter statement, namely that the shape of a polytope with $m \geq n+1$ facets is uniquely determined by the surface tensors up to rank $m-n+2$. This result is optimal in the sense that for each $m \geq n+1$ there is a polytope P with m facets and a convex body K that is not a polytope, such that P and K have identical surface tensors up to rank $m-n+1$. This implies that the rank $m-n+2$ cannot be reduced.

The paper is organized as follows. General notation, surface tensors and harmonic intrinsic volumes are introduced in Section 2. The uniqueness results are derived in Section 3 and are followed by the stability results in Section 4. The two reconstruction algorithms are described in Sections 5 and 6.

2. Notation and preliminaries

We work in the n-dimensional Euclidean vector space $\mathbb{R}^n$, $n \geq 2$ with standard inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}, and the surface area and volume of the unit ball B^n in $\mathbb{R}^n$ are denoted by ω_n and κ_n, respectively.

In the following, we give a brief introduction to the concepts of convex bodies, surface area measures, surface tensors and harmonic intrinsic volumes. For further details, we refer to [14] and [10]. We let $\mathcal{K}^n$ denote the set of convex bodies (convex, compact and nonempty sets) in $\mathbb{R}^n$, and let $\mathcal{K}^n_n$ be the set of convex bodies with nonempty interior. Further, $\mathcal{K}^n(R)$ is the set of convex bodies contained in a ball of radius $R > 0$, and likewise, $\mathcal{K}^n(r,R)$ is the set of convex bodies that contain a ball of radius $r > 0$ and are contained in a concentric ball of radius $R > r$. The set of convex bodies $\mathcal{K}^n$ is equipped with the Hausdorff metric δ. The Hausdorff distance between two convex bodies can be expressed as the supremum norm of the difference of the support functions of the convex bodies, i.e.

$$\delta(K,L) = \|h_K - h_L\|_\infty = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|$$

for $K, L \in \mathcal{K}^n$.

In the present work, we call the equivalence class of translations of a convex body K the shape of K. Hence, two convex bodies are of the same shape exactly if they are translates. As a measure of distance in shape, we use the translative Hausdorff distance,

$$\delta^t(K,L) = \inf_{x \in \mathbb{R}^n} \delta(K, L+x)$$

for $K, L \in \mathcal{K}^n$. The translative Hausdorff distance is a metric on the set of shapes of convex bodies, see [6, p. 165].

For a convex body $K \in \mathcal{K}_n^n$, the surface area measure $S_{n-1}(K, \cdot)$ of K is defined as

$$S_{n-1}(K, \omega) = \mathcal{H}^{n-1}(\tau(K, \omega))$$

for a Borel set $\omega \subseteq S^{n-1}$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and $\tau(K, \omega)$ is the set of boundary points of K with an outer normal belonging to ω. In particular, for a convex polytope P with $m \geq n+1$ facets, the surface area is of the form

$$S_{n-1}(P, \cdot) = \sum_{j=1}^{m} \alpha_j \delta_{u_j}$$

where δ_v is the Dirac measure at $v \in S^{n-1}$, $u_1, \dots, u_m \in S^{n-1}$ are the outer normals of the facets of P, and $\alpha_1, \dots, \alpha_m > 0$ are the corresponding $(n-1)$-dimensional volumes of the facets.

For a convex body $K \in \mathcal{K}^n \setminus \mathcal{K}_n^n$ there is a unit vector $u \in S^{n-1}$ and an $x \in \mathbb{R}^n$, such that K is contained in the hyperplane $u^\perp + x$. The surface area measure of K is defined as

$$S_{n-1}(K, \cdot) = S(K)(\delta_u + \delta_{-u}),$$

where $S(K)$ is the surface area of K. Notice that $S_{n-1}(K, S^{n-1}) = S(K)$ for $K \in \mathcal{K}_n^n$, and $S^{n-1}(K, S^{n-1}) = 2S(K)$ for $K \in \mathcal{K}^n \setminus \mathcal{K}_n^n$.

Minkowski's existence theorem is a fundamental result stating that a finite Borel measure μ on S^{n-1} is the surface area measure of a convex body $K \in \mathcal{K}_n^n$ if and only if

$$\int_{S^{n-1}} u \, \mu(du) = 0$$

and the support of μ is full-dimensional (meaning that the support is not contained in any great subsphere of S^{n-1}), see, e.g., [14, Thm. 8.2.2]. Another important result is that the shape of a convex body $K \in \mathcal{K}_n^n$ is uniquely determined by $S_{n-1}(K, \cdot)$, see, e.g., [14, Thm. 8.1.1].

Translation invariant Minkowski tensors derived from surface area measures are called *surface tensors*. Hence for $s \in \mathbb{N}_0$ and $K \in \mathcal{K}^n$, the surface tensor of K of rank s is given as

$$\Phi_{n-1}^s(K) = \frac{1}{s! \, \omega_{s+1}} \int_{S^{n-1}} u^s \, S_{n-1}(K, du)$$

where $u^s \colon (\mathbb{R}^n)^s \to \mathbb{R}$ is the s-fold symmetric tensor product of $u \in S^{n-1}$ when u is identified with the rank 1 tensor $v \mapsto \langle u, v \rangle$. Since the shape of a convex body $K \in \mathcal{K}_n^n$ is uniquely determined by $S_{n-1}(K, \cdot)$, the shape of K is likewise uniquely determined by the set of surface tensors $\{\Phi_{n-1}^s(K) \mid s \in \mathbb{N}_0\}$, see [10, Sec. 4, p. 10].

Due to multilinearity, the surface tensor of rank s can be identified with the array $\{\Phi_{n-1}^s(K)(e_{i_1}, \dots, e_{i_s})\}_{i_1, \dots, i_s = 1}^n$ of components of $\Phi_{n-1}^s(K)$, where

$(e_1,\dots,e_n)$ is the standard basis of $\mathbb{R}^n$. Notice that for $i_1,\dots,i_s \in \{1,\dots,n\}$, we have

$$\Phi^s_{n-1}(K)(e_{i_1},\dots,e_{i_s}) = \frac{1}{s!\,\omega_{s+1}} \int_{S^{n-1}} u_{i_1}\cdots u_{i_s}\; S_{n-1}(K,du).$$

Hence, the components of $\Phi^s_{n-1}(K)$ are scaled versions of the moments of $S_{n-1}(K,\cdot)$, where the moments of order $s \in \mathbb{N}_0$ of a Borel measure μ on S^{n-1} are given by

$$\int_{S^{n-1}} u_1^{j_1}\cdots u_n^{j_n}\; \mu(du)$$

for $j_1,\dots,j_n \in \{0,\dots,s\}$ with $\sum_{k=1}^n j_k = s$.

By [10, Remark 3.1], the surface tensors $\Phi^0_{n-1}(K),\dots,\Phi^s_{n-1}(K)$ of K are uniquely determined by $\Phi^{s-1}_{n-1}(K)$ and $\Phi^s_{n-1}(K)$ for $s \geq 2$. More precisely, if $s_o \in \mathbb{N}_0$ and $s \in \{0,\dots,s_o\}$ have same parity, then Φ^s_{n-1} can be calculated from $\Phi^{s_o}_{n-1}$ by taking the trace consecutively and multiplying with the constant

$$c_{s,s_o} = \frac{s_o!\,\omega_{s_o+1}}{s!\,\omega_{s+1}}. \tag{1}$$

The trace $\mathrm{Tr}(T)$ of a symmetric tensor T of rank $s \geq 2$ is the symmetric tensor of rank $s-2$ given by

$$\mathrm{Tr}(T)(e_{i_1},\dots,e_{i_{s-2}}) = \sum_{j=1}^n T(e_{i_1},\dots,e_{i_{s-2}},e_j,e_j)$$

for $i_1,\dots,i_{s-2} \in \{1,\dots,n\}$.

For $s \geq 2$, the tensors $\Phi^{s-1}_{n-1}(K)$ and $\Phi^s_{n-1}(K)$ have

$$m_s = \binom{s+n-2}{n-1} + \binom{s+n-1}{n-1}$$

components, when we, for $j \in \{s-1,s\}$, let the identical components

$$\{\Phi^j_{n-1}(K)(e_{\sigma(i_1)},\dots,e_{\sigma(i_j)}) \mid \sigma \text{ permutation of } \{i_1,\dots,i_j\}\}$$

count as one. We use the notation $\phi^s_{n-1}(K)$ for the m_s-dimensional vector of different components of the surface tensors of K of rank $s-1$ and s.

To a convex body $K \in \mathcal{K}^n$, we further associate the harmonic intrinsic volumes (not to be confused with harmonic quermaß integrals). Harmonic intrinsic volumes are the moments of $S_{n-1}(K,\cdot)$ with respect to an orthonormal sequence of spherical harmonics (for details on spherical harmonics, see [9]). More precisely, for $k \in \mathbb{N}_0$, let $\mathcal{H}^n_k$ be the vector space of spherical harmonics of degree k on S^{n-1}. The dimension of $\mathcal{H}^n_k$ is denoted by $N(n,k)$, and $\sum_{k=0}^s N(n,k) = m_s$. We let $H_{k1},\dots,H_{kN(n,k)}$ be an orthonormal basis of $\mathcal{H}^n_k$. Then, the *harmonic intrinsic volumes* of K of degree k are given by

$$\psi_{(n-1)kj}(K) = \int_{S^{n-1}} H_{kj}(u)\, S_{n-1}(K,du)$$

for $j = 1, \dots, N(n,k)$. For a convex body $K \in \mathcal{K}^n$, we let $\psi^s_{n-1}(K)$ be the m_s-dimensional vector of harmonic intrinsic volumes of K up to degree s. The vector $\psi^s_{n-1}(K)$ only depends on K through the surface area measure $S_{n-1}(K,\cdot)$ of K, and we can write $\psi^s_{n-1}(S_{n-1}(K,\cdot)) = \psi^s_{n-1}(K)$. Likewise, for an arbitrary Borel measure μ on S^{n-1}, we write $\psi^s_{n-1}(\mu)$ for the vector of harmonic intrinsic volumes of μ up to degree s, that is the vector of moments of μ up to order s with respect to the given orthonormal basis of spherical harmonics. The harmonic intrinsic volumes and the surface tensors of a convex body K are closely related as there is an invertible linear mapping $f_s\colon \mathbb{R}^{m_s} \to \mathbb{R}^{m_s}$ such that $f_s(\phi^s_{n-1}(K)) = \psi^s_{n-1}(K)$. This follows as every polynomial $p\colon S^{n-1} \to \mathbb{R}$ of degree d can be written as a sum of spherical harmonics $p = H_1 + \dots + H_d$ where $H_j \in \mathcal{H}^n_j$, see [9, Cor. 3.2.6].

3. Uniqueness results

The shape of a convex body is uniquely determined by a finite number of surface tensors only if the convex body is a polytope, see [10, Cor. 4.2]. Further, in [10, Thm. 4.3] it is shown that a polytope in $\mathbb{R}^n$ with nonempty interior and $m \geq n+1$ facets is uniquely determined up to translation in $\mathcal{K}^n$ by its surface tensors up to rank $2m$. In Theorem 3.2, we replace $2m$ with $m-n+2$, and in addition, we show that the rank $m-n+2$ cannot be reduced.

We let $\mathcal{M}$ denote the cone of finite Borel measures on S^{n-1}. Further, we let $\mathcal{P}_m$ be the set of convex polytopes in $\mathbb{R}^n$ with at most $m \geq n+1$ facets. The proof of Lemma 3.1 is an improved version of the proof of [10, Thm. 4.3].

Lemma 3.1. *Let $m \in \mathbb{N}$ and $\mu \in \mathcal{M}$ have finite support* $\operatorname{supp}\mu = \{u_1, \dots, u_m\} \subseteq S^{n-1}$.

(i) The measure μ is uniquely determined in $\mathcal{M}$ by its moments up to order m.

(ii) If the affine hull $\operatorname{aff}\{u_1, \dots, u_m\}$ *of* $\operatorname{supp}\mu$ *is $\mathbb{R}^n$, then μ is uniquely determined in $\mathcal{M}$ by its moments up to order $m-n+2$.*

Proof. We first prove (ii). Since $\operatorname{aff}\{u_1, \dots, u_m\} = \mathbb{R}^n$, we have $m \geq n+1$ and the support of μ can be pared down to $n+1$ vectors, say $u_1, \dots, u_{n+1}$, such that $\operatorname{aff}\{u_1, \dots, u_{n+1}\} = \mathbb{R}^n$. For each $j = 1, \dots, n+1$, the affine hull

$$A_j = \operatorname{aff}(\{u_1, \dots, u_{n+1}\} \setminus \{u_j\})$$

is a hyperplane in $\mathbb{R}^n$, so there is a $v^j \in S^{n-1}$ and a $\beta_j \in \mathbb{R}$ such that

$$A_j = \{x \in \mathbb{R}^n \mid \langle x, v^j \rangle = \beta_j\}.$$

Now define the polynomial

$$p(u) = \sum_{j=1}^{n+1} (\langle u, v^j \rangle - \beta_j)^2 (1 - \langle u, u_j \rangle)(1 - \langle u, u_{n+2} \rangle) \dots (1 - \langle u, u_m \rangle)$$

for $u \in S^{n-1}$. The degree of p is $m-n+2$, and $p(u_j) = 0$ for $j = 1, \ldots, m$. Let $w \in S^{n-1} \setminus \{u_1, \ldots, u_m\}$ and assume that $p(w) = 0$. Then $w \in A_j$ for $j = 1, \ldots, n+1$, so in particular $w = \sum_{j=1}^n \gamma_j u_j$ where $\sum_{j=1}^n \gamma_j = 1$. We may assume that $\gamma_1 \neq 0$. Since $w \in A_1$, this implies that u_1 is an affine combination of $u_2, \ldots, u_{n+1}$, so

$$A_1 = \operatorname{aff}\{u_1, \ldots, u_{n+1}\} = \mathbb{R}^n.$$

This is a contradiction, and we conclude that $p(w) > 0$.

Now let $\nu \in \mathcal{M}$ and assume that μ and ν have identical moments up to order $m-n+2$. Since the polynomial p is of degree $m-n+2$, we obtain that

$$\int_{S^{n-1}} p(u)\, \nu(du) = \int_{S^{n-1}} p(u)\, \mu(du) = \sum_{j=1}^m \alpha_j\, p(u_j) = 0, \tag{2}$$

where we have used that μ is of the form

$$\mu = \sum_{j=1}^m \alpha_j \delta_{u_j}$$

for some $\alpha_1, \ldots, \alpha_m > 0$. Equation (2) yields that $p(u) = 0$ for ν-almost all $u \in S^{n-1}$ as the polynomial p is non-negative. Then, the continuity of p implies that

$$\operatorname{supp} \nu \subseteq \{u \in S^{n-1} \mid p(u) = 0\} = \{u_1, \ldots, u_m\},$$

so ν is of the form

$$\nu = \sum_{j=1}^m \beta_j \delta_{u_j} \tag{3}$$

with $\beta_j \geq 0$ for $j = 1, \ldots, m$.

For $i = 1, \ldots, n+1$, define the polynomial

$$p_i(u) = (\langle u, v^i \rangle - \beta_i)^2 (1 - \langle u, u_{n+2} \rangle) \ldots (1 - \langle u, u_m \rangle)$$

for $u \in S^{n-1}$. Then p_i is of degree $m-n+1$ and $p_i(u_j) = 0$ for $j \neq i$. If $p_i(u_i) = 0$, then $u_i \in A_i$ and we obtain a contradiction as before. Hence $p_i(u_i) > 0$. Due to (3) and the assumption on coinciding moments, we obtain that

$$\alpha_i p_i(u_i) = \sum_{j=1}^m \alpha_j p_i(u_j) = \sum_{j=1}^m \beta_j p_i(u_j) = \beta_i p_i(u_i). \tag{4}$$

Since $p_i(u_i) > 0$, Equation (4) implies that $\alpha_i = \beta_i$ for $i = 1, \ldots, n+1$.

For $i = n+2, \ldots, m$, define the polynomial

$$p_i(u) = \frac{p(u)}{(1 - \langle u, u_i \rangle)}$$

for $u \in S^{n-1}$. Then p_i is of degree $m-n+1$ and $p_i(u_j) = 0$ for $j \neq i$. If $p_i(u_i) = 0$, then $u_i \in A_j$ for $j = 1, \ldots, n+1$, which is a contradiction. Hence,

$p_i(u_i) > 0$. By arguments as before, we obtain that $\alpha_i = \beta_i$ for $i = n+2, \dots, m$. Hence $\nu = \mu$, which yields (ii).

The statement (i) can be proved in a similar manner using the polynomials

$$p(u) = \prod_{j=1}^{m} (1 - \langle u, u_j \rangle)$$

and

$$p_i(u) = \frac{p(u)}{1 - \langle u, u_i \rangle}$$

for $u \in S^{n-1}$ and $i = 1, \dots, m$. □

Theorem 3.2. *Let $m \geq n + 1$. Up to translation, a polytope $P \in \mathcal{P}_m$ with nonempty interior is uniquely determined in $\mathcal{K}^n$ by its surface tensors up to rank $m - n + 2$. If $n = 2$, then the result holds for any $P \in \mathcal{P}_m$.*

The rank $m - n + 2$ is optimal as there is a polytope $P_m \in \mathcal{P}_m$ and a convex body $K_m \notin \mathcal{P}_m$ having identical surface tensors up to rank $m - n + 1$.

Proof. Let $P \in \mathcal{P}_m$ have facet normals $u_1, \dots, u_m \in S^{n-1}$ and nonempty interior. Then, $\operatorname{supp} S_{n-1}(P, \cdot) = \{u_1, \dots, u_m\}$ and $\operatorname{aff}\{u_1, \dots, u_m\} = \mathbb{R}^n$, so $S_{n-1}(P, \cdot)$ is uniquely determined in $\{S_{n-1}(K, \cdot) \mid K \in \mathcal{K}^n\} \subseteq \mathcal{M}$ by its moments up to order $m - n + 2$ due to Lemma 3.1 (ii). Since the surface tensors of P are rescaled versions of the moments of $S_{n-1}(P, \cdot)$, the first part of the statement follows as a convex body in $\mathbb{R}^n$ with nonempty interior is uniquely determined up to translation by its surface area measure. Now assume that $P \subseteq \mathbb{R}^2$ is a polytope in $\mathcal{P}_m$ with empty interior. Then P is a line segment and $S_1(P, \cdot) = S(P)(\delta_u + \delta_{-u})$ for some $u \in S^1$. By Lemma 3.1 (i), the surface area measure of P is uniquely determined by its moments up to second order. The second part of the statement then follows since any convex body in $\mathbb{R}^2$ is uniquely determined up to translation by its surface area measure.

To show that the rank $m - n + 2$ cannot be reduced, we first consider the case $n = 2$. For $m \geq 3$, let P_m be a regular polytope in $\mathbb{R}^2$ with outer normals $u_j = (\cos(j\frac{2\pi}{m}), \sin(j\frac{2\pi}{m}))$ for $j = 0, \dots m - 1$ and facet lengths $\alpha_j = \frac{2\pi}{m}$ for $j = 0, \dots, m - 1$. Then, P_m and the unit disc B^2 in $\mathbb{R}^2$ have identical surface tensors up to rank $m - 1$. This is easily seen by calculating and comparing the harmonic intrinsic volumes of P_m and B^2.

Now, counterexamples in $\mathbb{R}^n, n \geq 3$ can be constructed inductively. Essentially, if P'_{m-1} and K'_{m-1} are counterexamples in $\mathbb{R}^{n-1}$, counterexamples P_m and K_m in $\mathbb{R}^n$ are obtained as bounded cones with scaled versions of P'_{m-1} and K'_{m-1} as bases. More precisely, for a fixed $0 < \alpha < 1$, define $f_\alpha : S^{n-2} \to S^{n-1}$ by $f_\alpha(u) = (\sqrt{1 - \alpha^2}\, u, \alpha)$ for $u \in S^{n-2}$, and let

$$\mu_m = f_\alpha(S_{n-1}(P'_{m-1}, \cdot)) + \alpha S(P'_{m-1})\delta_{-e_n}$$

and

$$\nu_m = f_\alpha(S_{n-1}(K'_{m-1}, \cdot)) + \alpha S(K'_{m-1})\delta_{-e_n}.$$

By Minkowski's existence theorem, the measures μ_m and ν_m are surface area measures of convex bodies $P_m \in \mathcal{P}_m$ and $K_m \in \mathcal{K}^n$, respectively. Direct calculations show that if P'_{m-1} and K'_{m-1} have identical surface tensors in $\mathbb{R}^{n-1}$ up to rank $(m-1)-(n-1)+1 = m-n+1$, then P_m and K_m have identical surface tensors in $\mathbb{R}^n$ up to the same rank. Thus, we obtain that the rank $m-n+2$ is optimal in the sense that it cannot be reduced. $\square$

Due to the one-to-one correspondence between surface tensors up to rank s and harmonic intrinsic volumes up to degree s of a convex body, the uniqueness result in Theorem 3.2 also holds if surface tensors are replaced by harmonic intrinsic volumes.

4. Stability results

The shape of a convex body $K \in \mathcal{K}^n_n$ is uniquely determined by the set of surface tensors $\{\Phi^s_{n-1}(K) \mid s \in \mathbb{N}_0\}$ of K, but as described in the previous section, only the shape of polytopes are determined by a finite number of surface tensors. However, for an arbitrary convex body, a finite number of its surface tensors still contain information about its shape. This statement is quantified in this section, where we derive an upper bound of the translative Hausdorff distance between two convex bodies with a finite number of coinciding surface tensors.

The cone of finite Borel measures $\mathcal{M}$ on S^{n-1} is equipped with the Dudley metric

$$d_D(\mu,\nu) = \sup\left\{ \left| \int_{S^{n-1}} f \, d(\mu-\nu) \right| \;\middle|\; \|f\|_{BL} \le 1 \right\}$$

for $\mu, \nu \in \mathcal{M}$, where

$$\|f\|_{BL} = \|f\|_\infty + \|f\|_L \quad \text{and} \quad \|f\|_L = \sup_{u \neq v} \frac{|f(u)-f(v)|}{\|u-v\|}$$

for any function $f\colon S^{n-1} \to \mathbb{R}$. It can be shown that the Dudley metric induces the weak topology on $\mathcal{M}$ (the case of probability measures is treated in [5, Sec. 11.3] and is easily generalized to finite measures on S^{n-1}) The set of real-valued functions on S^{n-1} with $\|f\|_{BL} < \infty$ is denoted by $BL(S^{n-1})$. Further, we let the vector space $L^2(S^{n-1})$ of square integrable functions on S^{n-1} with respect to the spherical Lebesgue measure σ be equipped with the usual inner product $\langle \cdot, \cdot \rangle_2$ and norm $\|\cdot\|_2$.

As in [1, Chap. 2.8.1], for $k \in \mathbb{N}$, we define the operator $\Pi_{\mathrm{k}}\colon L^2(S^{n-1}) \to L^2(S^{n-1})$ by

$$(\Pi_{\mathrm{k}} f)(u) = E_k \int_{S^{n-1}} \left(\frac{1+\langle u,v\rangle}{2} \right)^k f(v)\sigma(dv) \tag{5}$$

for $f \in L^2(S^{n-1})$ where the constant

$$E_k = \frac{(k+n-2)!}{(4\pi)^{\frac{n-1}{2}} \Gamma(k+\frac{n-1}{2})}$$

satisfies

$$E_k \int_{S^{n-1}} \left(\frac{1+\langle u,v\rangle}{2}\right)^k \sigma(du) = 1. \tag{6}$$

As $(1+\langle u,v\rangle)^k$ is a polynomial in $\langle u,v\rangle$ of order k, it follows from the addition theorem for spherical harmonics (see, e.g., [9, Thm. 3.3.3]) that the function $\Pi_{\mathrm{k}} f$ for $f \in L^2(S^{n-1})$ can be expressed as a linear combination of spherical harmonics of degree k or less, see also [1, pp. 61-62]. More precisely, there are real constants (a_{kj}) such that

$$\Pi_{\mathrm{k}} f = \sum_{j=0}^{k} a_{kj} P_j f, \tag{7}$$

where $P_j f$ is the projection of f onto the space $\mathcal{H}_j^n$ of spherical harmonics of degree j. The constants in the linear combination (7) are given by

$$a_{kj} = \frac{k!(k+n-2)!}{(k-j)!(k+n+j-2)!},$$

see [1, p. 62]. By [1, Thm. 2.30], the sequence $(\Pi_{\mathrm{k}} f)_{k\in\mathbb{N}}$ converges uniformly to f when $k \to \infty$ for any continuous function $f\colon S^{n-1} \to \mathbb{R}$. When $f \in BL(S^{n-1})$, Lemma 4.1 provides an upper bound for the convergence rate in terms of $\|f\|_L$ and $\|f\|_\infty$.

Lemma 4.1. *Let $0 < \varepsilon < 1$ and $k \in \mathbb{N}$. For $f \in BL(S^{n-1})$, we have*

$$\|\Pi_{\mathrm{k}} f - f\|_\infty \le k^{-\frac{1-\varepsilon}{2}} \|f\|_L + 2\omega_n E_k \exp(-\frac{1}{4}k^\varepsilon)\|f\|_\infty. \tag{8}$$

Proof. We proceed as in the proof of [1, Thm. 2.30]. Let $f \in BL(S^{n-1})$. Using (5) and (6), we obtain that

$$\begin{aligned} |(\Pi_{\mathrm{k}} f)(u) - f(u)| &\le E_k \int_{S^{n-1}} \left(\frac{1+\langle u,v\rangle}{2}\right)^k |f(u)-f(v)|\,\sigma(dv) \\ &\le I_1(\delta,u) + I_2(\delta,u) \end{aligned}$$

for $u \in S^{n-1}$ and $0 < \delta < 2$, where

$$I_1(\delta,u) = E_k \int_{\{v\in S^{n-1} : \|u-v\|\le\delta\}} \left(\frac{1+\langle u,v\rangle}{2}\right)^k |f(u)-f(v)|\,\sigma(dv)$$

and

$$I_2(\delta,u) = E_k \int_{\{v\in S^{n-1} : \|u-v\|>\delta\}} \left(\frac{1+\langle u,v\rangle}{2}\right)^k |f(u)-f(v)|\,\sigma(dv).$$

Since $I_1(\delta,u) \le \delta\|f\|_L$ and

$$I_2(\delta,u) \le 2\omega_n E_k \left(1 - \frac{\delta^2}{4}\right)^k \|f\|_\infty,$$

we obtain that

$$|(\Pi_{\mathrm{k}} f)(u) - f(u)| \leq \delta \|f\|_L + 2\omega_n E_k \left(1 - \frac{\delta^2}{4}\right)^k \|f\|_\infty. \tag{9}$$

To derive the upper bound on I_2, we have used that $\langle u, v\rangle = 1 - \frac{\|u-v\|^2}{2}$ for $u, v \in S^{n-1}$.

Now let $\delta = k^{-\frac{1-\varepsilon}{2}}$. From the mean value theorem, we obtain that

$$\ln\left(1 - \frac{\delta^2}{4}\right)^k = -\frac{1}{4} k^\varepsilon \frac{\ln(1) - \ln(1 - \frac{1}{4}k^{\varepsilon-1})}{\frac{1}{4}k^{\varepsilon-1}} = -\frac{1}{4}k^\varepsilon \xi_k^{-1}$$

for some $\xi_k \in [1 - \frac{1}{4}k^{\varepsilon-1}, 1]$. Hence,

$$\left(1 - \frac{\delta^2}{4}\right)^k \leq \exp(-\frac{1}{4}k^\varepsilon). \tag{10}$$

Combining (9) and (10) yields the assertion. □

Remark 4.2. Stirling's formula, $\Gamma(x) \sim \sqrt{2\pi} x^{x-\frac{1}{2}} \mathrm{e}^{-x}$ for $x \to \infty$, implies that

$$E_k \sim \left(\frac{k}{4\pi}\right)^{\frac{n-1}{2}}$$

for $k \to \infty$. Hence, the upper bound in (8) converges to zero for $k \to \infty$. The choice of δ in the proof of Lemma 4.1 is optimal in the sense that if we use $0 < \delta \leq \frac{c}{\sqrt{k}}$ with a constant $c > 0$, then the derived upper bound in (9) does not converge to zero. This follows as

$$1 \geq \left(1 - \frac{\delta^2}{4}\right)^k \geq \left(1 - \frac{c}{4k}\right)^k \to e^{-\frac{c}{4}}$$

for $k \to \infty$, when $0 < \delta \leq \frac{c}{\sqrt{k}}$.

For functions $f \in BL(S^{n-1})$ satisfying $\|f\|_{BL} \leq 1$, Lemma 4.1 yields an uniform upper bound, only depending on k and the dimension n, of $\|\Pi_{\mathrm{k}} f - f\|_\infty$. In the following theorem, this is used to derive an upper bound of the Dudley distance between the surface area measures of two convex bodies where the harmonic intrinsic volumes up to a certain degree $s_o \in \mathbb{N}$ are close in $\mathbb{R}^{m_{s_o}}$.

Theorem 4.3. *Let $K, L \in \mathcal{K}^n(R)$ for some $R > 0$ and let $s_o \in \mathbb{N}$. Let $0 < \varepsilon < 1$ and $\delta > 0$. If*

$$\sqrt{\omega_n m_{s_o}} \|\psi_{n-1}^{s_o}(K) - \psi_{n-1}^{s_o}(L)\| \leq \delta \tag{11}$$

then

$$d_D(S_{n-1}(K, \cdot), S_{n-1}(L, \cdot)) \leq c(n, R, \varepsilon) s_o^{-\frac{1-\varepsilon}{2}} + \delta, \tag{12}$$

where $c > 0$ is a constant depending on n, R and ε.

Due to the addition theorem for spherical harmonics, the condition (11) is independent of the bases of $\mathcal{H}_k^n$, $k \in \mathbb{N}$ that are used to derive the harmonic intrinsic volumes.

Proof of Theorem 4.3. Let $f \in BL(S^{n-1})$ satisfy $\|f\|_{BL} \leq 1$ and define the signed Borel measure $\nu = S_{n-1}(K,\cdot) - S_{n-1}(L,\cdot)$. Then, by (7),

$$\Pi_{s_o} f = \sum_{j=0}^{s_o} a_{s_o j} \sum_{i=0}^{N(n,j)} \langle f, H_{ji}\rangle_2 H_{ji},$$

where $|a_{s_o j}| \leq 1$ and $H_{j1},\dots,H_{jN(n,j)}$ form the orthonormal basis of $\mathcal{H}_j^n$ used to derive the harmonic intrinsic volumes of degree $j \in \{0,\dots,s_o\}$. Since $\|f\|_2 \leq \sqrt{\omega_n}\|f\|_\infty \leq \sqrt{\omega_n}$, we obtain from Cauchy-Schwarz' inequality and a discrete version of Jensen's inequality that

$$\begin{aligned}
\left|\int_{S^{n-1}} \Pi_{s_o} f\, d\nu\right| &\leq \sqrt{\omega_n} \sum_{j=0}^{s_o} \sum_{i=0}^{N(n,j)} \left|\int_{S^{n-1}} H_{ji}\, d\nu\right| \\
&\leq \left(\omega_n \Big(\sum_{l=0}^{s_o} N(n,l)\Big) \sum_{j=0}^{s_o} \sum_{i=0}^{N(n,j)} \Big(\int_{S^{n-1}} H_{ji}\, d\nu\Big)^2\right)^{\frac{1}{2}} \\
&= \sqrt{\omega_n m_{s_o}}\, \|\psi_{n-1}^{s_o}(K) - \psi_{n-1}^{s_o}(L)\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\int_{S^{n-1}} f\, d\nu\right| &\leq \left|\int_{S^{n-1}} \Pi_{s_o} f - f\, d\nu\right| + \left|\int_{S^{n-1}} \Pi_{s_o} f\, d\nu\right| \\
&\leq 2R^{n-1}\omega_n (s_o^{\frac{\varepsilon-1}{2}} + 2\omega_n E_{s_o} \exp(-\frac{1}{4} s_o^{\varepsilon})) + \delta,
\end{aligned}$$

where we used Lemma 4.1 and that $\max\{S_{n-1}(K,S^{n-1}), S_{n-1}(L,S^{n-1})\} \leq R^{n-1}\omega_n$. For $k \to \infty$, the convergence of $E_k \exp(-\frac{1}{4}k^{\varepsilon})$ to zero is faster than the convergence of $k^{-\frac{1-\varepsilon}{2}}$, see Remark 4.2. This implies the existence of a constant c only depending on n, R and ε satisfying (12). ☐

Corollary 4.4. *Let $K, L \in \mathcal{K}^n(R)$ for some $R > 0$ and let $s_o \in \mathbb{N}_0$ and $0 < \varepsilon < 1$. If $\Phi_{n-1}^s(K) = \Phi_{n-1}^s(L)$ for $0 \leq s \leq s_o$, then*

$$d_D(S_{n-1}(K,\cdot), S_{n-1}(L,\cdot)) \leq c(n,R,\varepsilon) s_o^{-\frac{1-\varepsilon}{2}},$$

where $c > 0$ is a constant depending on n, R and ε.

Proof. The assumption that K and L have coinciding surface tensors up to rank s_o implies that $\|\psi_{n-1}^{s_o}(K) - \psi_{n-1}^{s_o}(L)\| = 0$. The result then follows from Theorem 4.3 with $\delta = 0$. ☐

The translative Hausdorff distance between two convex bodies in $\mathcal{K}^n(r, R)$ admits an upper bound expressed by the n'th root of the Prokhorov distance between their surface area measures, see [14, Thm. 8.5.3]. Further, the Prokhorov distance between two Borel measures on S^{n-1} can be bounded in terms of the square root of the Dudley distance between the measures. Therefore, Corollary 4.4 in combination with [14, Thm. 8.5.3] and [7, Lemma 9.5] yields the following stability result.

Theorem 4.5. *Let $K, L \in \mathcal{K}^n(r, R)$ for some $0 < r < R$ and let $s \in \mathbb{N}_0$ and $0 < \varepsilon < 1$. If $\Phi^s_{n-1}(K) = \Phi^s_{n-1}(L)$ for $0 \leq s \leq s_o$, then*

$$\delta^t(K, L) \leq c(n, r, R, \varepsilon) s_o^{-\frac{1-\varepsilon}{4n}}$$

for a constant $c > 0$ depending on n, r, R and ε.

5. Reconstruction of shape from surface tensors

In this section, we derive an algorithm that approximates the shape of an unknown convex body $K \in \mathcal{K}^n_n$ from a finite number of surface tensors $\{\Phi^s_{n-1}(K) \mid 0 \leq s \leq s_o\}$ of K for some $s_o \in \mathbb{N}$. The reconstruction algorithm is a generalization to higher dimension of Algorithm Surface Tensor in [10] that reconstructs convex bodies in $\mathbb{R}^2$ from surface tensors. The shape of a convex body K in $\mathbb{R}^n$ is uniquely determined by the surface tensors of K, when K has nonempty interior. For $n = 2$, the surface tensors of K determine the shape of K even when K is lower dimensional. Therefore, the algorithm in [10] can be used to approximate the shape of arbitrary convex bodies in $\mathbb{R}^2$, whereas the algorithm described in this section only allows for convex bodies in $\mathbb{R}^n$ with nonempty interior. A non-trivial difference between the algorithm in the two-dimensional setting and the generalized algorithm is that in higher dimension, it is crucial that the first and second order moments of a Borel measure μ on S^{n-1} determine if μ is the surface area measure of a convex body. Therefore, this is shown in Lemma 5.2, where statement (i) is a reformulation of Minkowski's existence theorem in terms of moments. The proof of Lemma 5.2 is based on the following remark.

Remark 5.1. Let μ be a finite Borel measure on the unit sphere S^{n-1}. Then,

$$\int_{S^{n-1}} \langle z, u\rangle^2 \mu(du) > 0 \tag{13}$$

for all $z \in S^{n-1}$ if and only if the support of μ is full-dimensional. As the integral in (13) is determined by the second order moments

$$m_{ij}(\mu) = \int_{S^{n-1}} u_i u_j \, \mu(du)$$

of μ, these moments determine if the support of μ is full-dimensional. More precisely, the support of μ is full-dimensional if and only if the matrix of second order moments $M(\mu) = \{m_{ij}(\mu)\}_{i,j=1}^n$ is positive definite as

$$z^\top M(\mu) z = \int_{S^{n-1}} \langle z, u\rangle^2 \mu(du)$$

for $z \in \mathbb{R}^n$.

Lemma 5.2. *Let μ be a finite Borel measure on S^{n-1} with $\mu(S^{n-1}) > 0$.*

(i) The measure μ is the surface area measure of a convex body $K \in \mathcal{K}_n^n$, if and only if the first order moments of μ vanish and the matrix $M(\mu)$ of second order moments of μ is positive definite.

(ii) The measure μ is the surface area measure of a convex body $K \in \mathcal{K}^n \setminus \mathcal{K}_n^n$ if and only if the first order moments of μ vanish and the matrix $M(\mu)$ of second order moments of μ has one positive eigenvalue and $n-1$ zero eigenvalues.

In the case, where (ii) *is satisfied, the measure μ is the surface area measure of every convex body K with surface area $\frac{1}{2}\mu(S^{n-1})$ contained in a hyperplane with normal vector u, where $u \in S^{n-1}$ is a unit eigenvector of $M(\mu)$ corresponding to the positive eigenvalue (u is unique up to sign).*

Proof. Remark 5.1 implies that the interior of a convex body K is nonempty if and only if the matrix of second order moments of $S_{n-1}(K,\cdot)$ is positive definite, so the statement (i) follows from Minkowski's existence theorem.

If μ is the surface area measure of $K \in \mathcal{K}^n \setminus \mathcal{K}_n^n$, then μ is of the form

$$\mu = \frac{\mu(S^{n-1})}{2}(\delta_u + \delta_{-u})$$

for some $u \in S^{n-1}$. Then, the first order moments of μ vanish, and the matrix $M(\mu)$ of second order moments of μ is $\mu(S^{n-1})u^2$. Hence, $M(\mu)$ has one positive eigenvalue $\mu(S^{n-1})$ with eigenvector u and $n-1$ zero eigenvalues.

If the matrix $M(\mu)$ is positive semidefinite with one positive eigenvalue $\alpha > 0$ and $n-1$ zero eigenvalues, then $M(\mu) = \alpha u^2$, where $u \in S^{n-1}$ is a unit eigenvector (unique up to sign) corresponding to the positive eigenvalue. Assume further that the first order moments of μ vanish, and define the measure $\nu = \frac{\alpha}{2}(\delta_u + \delta_{-u})$. Then μ and ν have identical moments up to order 2, and Lemma 3.1 (i) yields that $\mu = \nu$. Therefore, μ is the surface area measure of any convex body K with surface area α contained in a hyperplane with normal vector u. □

5.1. Reconstruction algorithm based on surface tensors

Let $K_0 \in \mathcal{K}_n^n$ be fixed. We consider K_0 as unknown and assume that the surface tensors $\Phi_{n-1}^0(K_0), \ldots, \Phi_{n-1}^{s_o}(K_0)$ of K_0 are known up to rank s_o for some natural number $s_o \geq 2$. The aim is to construct a convex body with surface tensors identical to the known surface tensors of K_0. We proceed as in [10, Sec. 5.1].

Let

$$M_{s_o} = \{(\alpha, \mathbf{u}) \in \mathbb{R}^{m_{s_o}} \times (S^{n-1})^{m_{s_o}} \mid \alpha_j \geq 0, \quad \sum_{j=1}^{m_{s_o}} \alpha_j u_j = 0\}, \tag{14}$$

and consider the minimization problem

$$\min_{(\alpha,\mathbf{u})\in M_{s_o}} \sum_{j=1}^{m_{s_o}} \left(\phi_{n-1}^{s_o}(K_0)_j - \sum_{i=1}^{m_{s_o}} \alpha_i g_{s_o j}(u_i) \right)^2, \tag{15}$$

where the polynomial $g_{s_o j} \colon S^{n-1} \to \mathbb{R}$ is defined such that $g_{s_o j}(u)$ is the component of the tensor $(s_o!\,\omega_{s_o+1})^{-1} u^{s_o}$ that corresponds to $\phi_{n-1}^{s_o}(K_0)_j$. Hence,

$$\int_{S^{n-1}} g_{s_o j}(u)\, S_{n-1}(K_0, du) = \phi_{n-1}^{s_o}(K_0)_j$$

for $j = 1, \ldots, m_{s_o}$. Notice, that the objective function in (15) is known, as the surface tensors $\Phi_{n-1}^{s_o-1}(K_0)$ and $\Phi_{n-1}^{s_o}(K_0)$ are assumed to be known. By [10, Thm. 4.1], there exists a polytope P (not necessarily unique) with at most m_{s_o} facets and surface tensors identical to the surface tensors of K_0 up to rank s_o. Now, let $v_1, \ldots, v_{m_{s_o}} \in S^{n-1}$ be the outer normals of the facets of such a polytope P and $a_1, \ldots, a_{m_{s_o}} \geq 0$ be the corresponding $(n-1)$-dimensional volumes of the facets. If P has $k < m_{s_o}$ facets, then $a_{k+1} = \cdots = a_{m_{s_o}} = 0$. Then $S_{n-1}(P, \cdot) = \sum_{j=1}^{m_{s_o}} a_j \delta_{v_j}$, and

$$\phi_{n-1}^{s_o}(P)_j = \sum_{i=1}^{m_{s_o}} a_i g_{s_o j}(v_i).$$

As P and K_0 has identical surface tensors up to rank s_o, this implies that

$$\sum_{j=1}^{m_{s_o}} \left(\phi_{n-1}^{s_o}(K_0)_j - \sum_{i=1}^{m_{s_o}} a_i g_{s_o j}(v_i) \right)^2 = 0. \tag{16}$$

Therefore, $(a, \mathbf{v}) = (a_1, \ldots, a_{m_{s_o}}, v_1, \ldots, v_{m_{s_o}}) \in M_{s_o}$ is a solution to the minimization problem (15).

Now, let $(\alpha, \mathbf{u}) \in M_{s_o}$ be an arbitrary solution to (15) and define the Borel measure $\varphi = \sum_{i=1}^{m_{s_o}} \alpha_i \delta_{u_i}$ on S^{n-1}. As the minimum value of the objective function is 0 due to (16), the moments of φ and $S_{n-1}(K_0, \cdot)$ of order $s_o - 1$ and s_o are identical. This implies that the moments of φ and $S_{n-1}(K_0, \cdot)$ of order 1 and 2 are identical as $s_o \geq 2$, see [10, Remark 3.1]. Then Lemma 5.2 (i) yields the existence of a polytope $Q \in \mathcal{P}_{m_{s_o}}$ with nonempty interior such that $S_{n-1}(Q, \cdot) = \varphi$. The surface tensors of Q are identical to the surface tensors of K_0 up to rank s_o.

In the two-dimensional setup in [10, Sec. 5.1], every vector in M_{s_o} corresponds to the surface area measure of a polytope. In the n-dimensional setting, this is not the case, as Minkowski's existence theorem requires that the linear hull of the vectors $\alpha_1 u_1, \ldots, \alpha_{m_{s_o}} u_{m_{s_o}}$ is $\mathbb{R}^n$, when $n > 2$. However, as the above considerations show, Lemma 5.2 ensures that every solution vector to the minimization problem (15), in fact, corresponds to the surface area measure of a polytope, which is sufficient to obtain a polytope with the required surface tensors.

The minimization problem (15) can be solved numerically, and a polytope corresponding to the obtained solution can be constructed using Algorithm MinkData described in [11], (see also [6, Sec. A.4]). This polytope has surface tensors identical to the surface tensors of K_0 up to rank s_o.

Algorithm Surface Tensor (n-dim).

Input: A natural number $s_o \geq 2$ and surface tensors $\Phi_{n-1}^{s_o-1}(K_0)$ and $\Phi_{n-1}^{s_o}(K_0)$ of an unknown convex body $K_0 \in \mathcal{K}_n^n$.

Task: Construct a polytope $\hat{K}_{s_o}$ in $\mathbb{R}^n$ such that $\hat{K}_{s_o}$ and K_0 have identical surface tensors up to rank s_o.

Action: Find a vector $(\alpha, \mathbf{u}) \in M_{s_o}$ that minimizes

$$\sum_{j=1}^{m_{s_o}} \left(\phi_{n-1}^{s_o}(K_0)_j - \sum_{i=1}^{m_{s_o}} \alpha_i g_{s_o j}(u_i) \right)^2 .$$

The vector $(\alpha, \mathbf{u})$ describes a polytope $\hat{K}_{s_o}$ in $\mathbb{R}^n$ with at most m_{s_o} facets. Reconstruct $\hat{K}_{s_o}$ from $(\alpha, \mathbf{u})$ using Algorithm MinkData.

Remark 5.3. Solving the minimization problem (15) numerically might introduce small errors, such that the surface tensors $\Phi_{n-1}^{s_o-1}(\hat{K}_{s_o})$ and $\Phi_{n-1}^{s_o}(\hat{K}_{s_o})$ are only approximations of the surface tensors $\Phi_{n-1}^{s_o-1}(K_0)$ and $\Phi_{n-1}^{s_o}(K_0)$. Small errors in the surface tensors of rank $s_o - 1$ and s_o imply the risk of huge errors in the surface tensors of rank less than s_o. This follows from the way the surface tensors Φ_{n-1}^{s}, $0 \leq s \leq s_o$ are related to the surface tensors $\Phi_{n-1}^{s_o-1}$ and $\Phi_{n-1}^{s_o}$ as described in Section 2, see (1). The main problem is the constant

$$c_{s,s_o} = \frac{s_o!\, \omega_{s_o+1}}{s!\, \omega_{s+1}}$$

that increases rapidly with s_o for fixed s and therefore might cause huge errors in, for instance, the surface area of $\hat{K}_{s_o}$. The algorithm can be made more robust to numerical errors by replacing the surface tensors with the scaled versions $(s!\omega_{s+1})^{-1}\Phi_{n-1}^{s}$ of the surface tensors. The two versions of the algorithm are theoretically equivalent.

5.2. Consistency of the reconstruction algorithm

The output of the algorithm described in the previous section is a polytope with surface tensors identical to the surface tensors of K_0 up to a given rank s_o. In this section, we show that for large s_o the shape of the output polytope is a good approximation of the shape of K_0.

For each $s_o \geq 2$, let $\hat{K}_{s_o}$ be an output of the algorithm based on surface tensors up to rank s_o. Then there exist radii $r_{s_o}, R_{s_o} > 0$ such that $\hat{K}_{s_o}, K_0 \in \mathcal{K}^n(r_{s_o}, R_{s_o})$ and by Theorem 4.5, we obtain

$$\delta^t(K_0, \hat{K}_{s_o}) \leq c(n, r_{s_o}, R_{s_o}, \epsilon) s_o^{-\frac{1-\varepsilon}{4n}}$$

for $\varepsilon > 0$. Notice that c depends on s_o through r_{s_o} and R_{s_o}, so even though the factor $s_o^{-1/(4n)+\varepsilon}$ converges to 0 when s_o increases, we do not immediately obtain the wanted consistency. To prevent the dependence of c on s_o, we show that there exist radii $r, R > 0$ such that $K_0, \hat{K}_{s_o} \in \mathcal{K}^n(r, R)$ for each $s_o \geq 2$.

For a convex body $K \in \mathcal{K}^n$, the coefficient matrix $\{\Phi^2_{n-1}(K)(e_i, e_j)\}_{i,j=1}^n$ of $\Phi^2_{n-1}(K)$ is a scaled version of the matrix $M(S_{n-1}(K, \cdot))$ of second order moments of $S_{n-1}(K, \cdot)$ defined in Remark 5.1. More precisely,

$$8\pi\{\Phi^2_{n-1}(K)(e_i, e_j)\}_{i,j=1}^n = M(S_{n-1}(K, \cdot)).$$

Therefore, Lemma 5.2 yields that the surface tensor $\Phi^2_{n-1}(K)$ determines if K has nonempty interior. In Lemma 5.4, we show that $\Phi^2_{n-1}(K)$ even determines the radius of a sphere contained in K and the radius of a sphere containing K, when K has nonempty interior.

For a convex body $K \in \mathcal{K}^n_n$, the coefficient matrix $\{\Phi^2_{n-1}(K)(e_i, e_j)\}_{i,j=1}^n$ is symmetric and positive definite, and has therefore n positive eigenvalues. In the following, we let $\lambda_{min}(K) > 0$ denote the smallest of these eigenvalues. The proof of Lemma 5.4 is inspired by the proof of [6, Lemma 4.4.6].

Lemma 5.4. *Let $K \in \mathcal{K}^n_n$ with centre of mass at the origin. Let*

$$R = \frac{S(K)}{4\pi\lambda_{min}(K)}\left(\frac{S(K)}{\omega_n}\right)^{\frac{1}{n-1}} \quad \text{and} \quad r = \frac{2\pi\lambda_{min}(K)}{(n+1)(4R)^{n-2}}. \tag{17}$$

Then $rB^n \subseteq K \subseteq RB^n$.

Proof. Let x be a point on the boundary ∂K of K. Then $\|x\| > 0$, so $v = \frac{x}{\|x\|} \in S^{n-1}$ is well-defined. By monotonicity and positive multilinearity of mixed volumes (see, e.g., [6, (A.16),(A.18)]) and the isoperimetric inequality (see, e.g., [6, (B.14)]), we obtain that

$$\|x\| V(K, n-1; [0, v]) = V(K, n-1; [0, x]) \leq V_n(K) \leq \left(\frac{S(K)}{\omega_n}\right)^{\frac{n}{n-1}} \kappa_n, \tag{18}$$

where V is the mixed volume, V_n is the n-dimensional volume and $[a, b]$ is the convex hull of $\{a, b\} \subseteq \mathbb{R}^n$. Further, we have that

$$\begin{aligned} V(K, n-1; [0, v]) &= \frac{1}{n}\int_{S^{n-1}} h_{[0,v]}(u)\, S_{n-1}(K, du) \\ &= \frac{1}{2n}\int_{S^{n-1}} |\langle u, v\rangle|\, S_{n-1}(K, du) \\ &\geq \frac{1}{2n}\int_{S^{n-1}} \langle u, v\rangle^2\, S_{n-1}(K, du) = \frac{4\pi}{n}\Phi^2_{n-1}(K)(v, v), \end{aligned}$$

where we have used [6, (A.11) and (A.12)] and that $S_{n-1}(K,\cdot)$ has centroid at the origin. Hence,

$$V(K,n-1;[0,v]) \geq \frac{4\pi}{n}\lambda_{min}(K). \tag{19}$$

Equations (18) and (19) yield that $\|x\| \leq R$, so $K \subseteq RB^n$.

As the centre of mass of K is at the origin, then [14, p. 320, note 6] and the references given there yield that

$$\frac{1}{n+1}w(K,u) \leq h_K(u)$$

for $u \in S^{n-1}$, where $w(K,\cdot)$ is the width function of K. Since

$$w(K,u) = h_K(u) + h_K(-u) = h_{K_s}(u)$$

where $K_s = K+(-K)$, it is sufficient to show that $r(n+1)B^n \subseteq K_s$ in order to obtain that $rB^n \subseteq K$. Due to origin-symmetry of K_s, we can proceed as in the proof of [6, Lemma 4.4.6]. Let $c = \sup\{a > 0 \mid aB^n \subseteq K_s\} > 0$. Then $cB^n \subseteq K_s$ and $\partial K_s \cap \partial cB^n \neq \emptyset$. As K_s and cB^n are origin-symmetric there are contact points $z, -z \in \partial K_s \cap \partial cB^n$ and common parallel supporting hyperplanes of K_s and cB^n in z and $-z$. By the first part of this proof, we have $K_s \subseteq 2RB^n$, so K_s is contained in a n-dimensional box with one edge of length $2c$ parallel to z and $n-1$ edges of length $4R$ orthogonal to z. More precisely,

$$K_s \subseteq \{x \in \mathbb{R}^n \mid |\langle x,z\rangle| \leq c\} \cap \bigcap_{j=2}^{n} \{x \in \mathbb{R}^n \mid |\langle x,u_j\rangle| \leq 2R\}$$

where z and $u_2,\ldots,u_n \in S^{n-1}$ form an orthogonal basis of $\mathbb{R}^n$. This implies that

$$V_{n-1}(K_s \mid (u_2)^\perp) \leq 2c(4R)^{n-2}, \tag{20}$$

where $K_s \mid (u_2)^\perp$ is the orthogonal projection of K_s onto $(u_2)^\perp$. Using [6, (A.37)] and that Equation (19) holds for any $v \in S^{n-1}$, we obtain

$$\begin{aligned} V_{n-1}(K_s \mid (u_2)^\perp) &\geq V_{n-1}(K \mid (u_2)^\perp) \\ &= nV(K,n-1;[0,u_2]) \geq 4\pi\lambda_{min}(K), \end{aligned}$$

so from (20) it follows that

$$c \geq \frac{2\pi\lambda_{min}(K)}{(4R)^{n-2}},$$

which yields that $r(n+1)B^n \subseteq K_s$. □

Theorem 5.5. *Let $K_0 \in \mathcal{K}_n^n$, $s_o \geq 2$ be a natural number and $0 < \varepsilon < 1$. If the surface tensors up to rank s_o of a convex body K_{s_o} coincide with the surface tensors of K_0, then*

$$\delta^t(K_0,K_{s_o}) \leq c(n,\varepsilon,\Phi_{n-1}^2(K_0))s_o^{-\frac{1-\varepsilon}{4n}}, \tag{21}$$

where $c > 0$ is a constant depending only on n, ε and $\Phi^2_{n-1}(K_0)$. Hence, if $(K_{s_o})_{s_o \in \mathbb{N}}$ is a sequence of convex bodies satisfying $\Phi^s_{n-1}(K_0) = \Phi^s_{n-1}(K_{s_o})$ for $0 \leq s \leq s_o$, then the shape of K_{s_o} converges to the shape of K_0 when $s_o \to \infty$.

Proof. When defined as in (17) with K replaced by K_0, the radii r and R are determined by $\Phi^2_{n-1}(K_0)$, and since $\Phi^2_{n-1}(K_0) = \Phi^2_{n-1}(K_{s_o})$, Lemma 5.2 and Lemma 5.4 yield that $K_{s_o}, K_0 \in \mathcal{K}^n(r, R)$. Then we obtain the bound (21) from Theorem 4.5. The constant c does not depend on s_o, so the stated convergence result is obtained from (21). □

The consistency of Algorithm Surface Tensor (n-dim) follows from Theorem 5.5.

5.3. Examples: Reconstruction of convex bodies in $\mathbb{R}^3$

In this section, we give two examples where Algorithm Surface Tensor (n-dim) is used to reconstruct the shape of a convex body in $\mathbb{R}^3$. Following Remark 5.3, the scaled surface tensors $s!\omega_{s+1}\Phi^s_2$ have been used in order to make the reconstructions more robust to numerical errors. In the first example, a prolate ellipsoid is reconstructed. The reconstructions of the ellipsoid are based on surface tensors up to rank $s_o = 2, 4, 6$, see Figure 1. In the second example, a pyramid is reconstructed. The reconstructions of the pyramid are executed with $s_o = 2, 3, 4$, see Figure 2.

The minimization problem (15) is solved by means of the *fmincon* procedure provided by MatLab, and a polytope corresponding to the solution to (15) is reconstructed using Algorithm MinkData. This algorithm has been implemented by Gardner and Milanfar for $n \leq 3$, see [6, Sec. A4], and for $n = 3$ the algorithm has recently become available on the website www.geometrictomography.com run by Richard Gardner.

The surface tensor of rank 2 of a convex body contains information of the main directions and the degree of anisotropy of the convex body. The effect of this is, in particular, visible in the plots in Figure 1 that show that the three reconstructions of the ellipsoid are elongated in the direction of the third axis. As expected, the reconstructions of the ellipsoid and the reconstructions of the pyramid become more accurate when s_o increases. The pyramid has 5 facets, so according to Theorem 3.2, the surface tensors up to rank 4 uniquely determine the shape of the pyramid. The last plot in Figure 2 shows that the reconstruction based on surface tensors up to rank 4 is indeed very precise. Deviation from the pyramid can be ascribed to numerical errors.

6. Reconstruction of shape from harmonic intrinsic volumes

Due to the correspondence between surface tensors and harmonic intrinsic volumes, a convex body $K \in \mathcal{K}^n_n$ is uniquely determined by the set of harmonic intrinsic volumes $\{\psi_{(n-1)sj}(K) \mid s \in \mathbb{N}_0, j = 1, \ldots, N(n, s)\}$ of K. In this section, we derive an algorithm that approximates the shape of an unknown convex body $K_0 \in \mathcal{K}^n_n$ from measurements subject to noise of a finite number

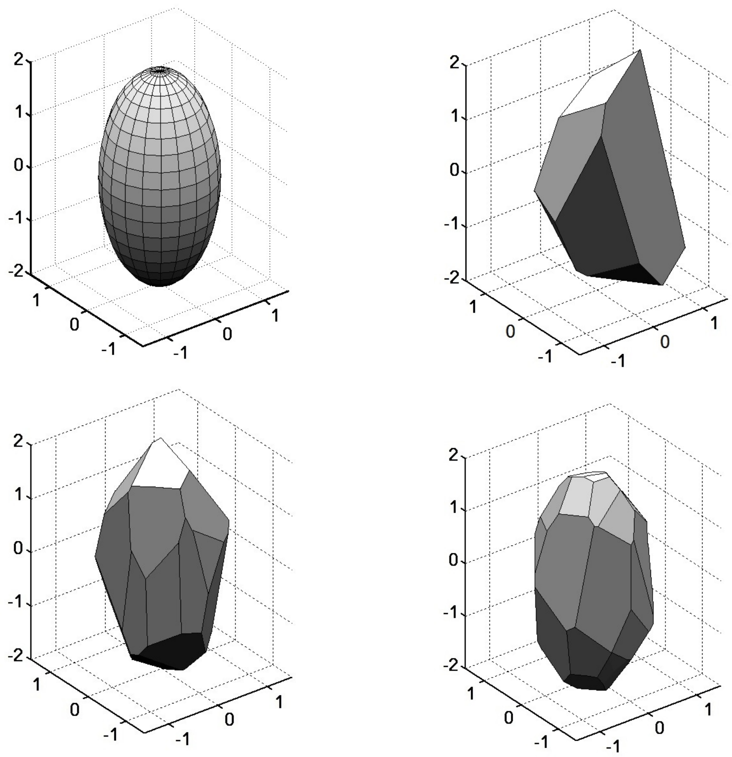

Figure 1: Plots 2, 3 and 4 show reconstructions of the ellipsoid in the first plot. The reconstructions are based on surface tensors up to rank $s_o = 2, 4, 6$.

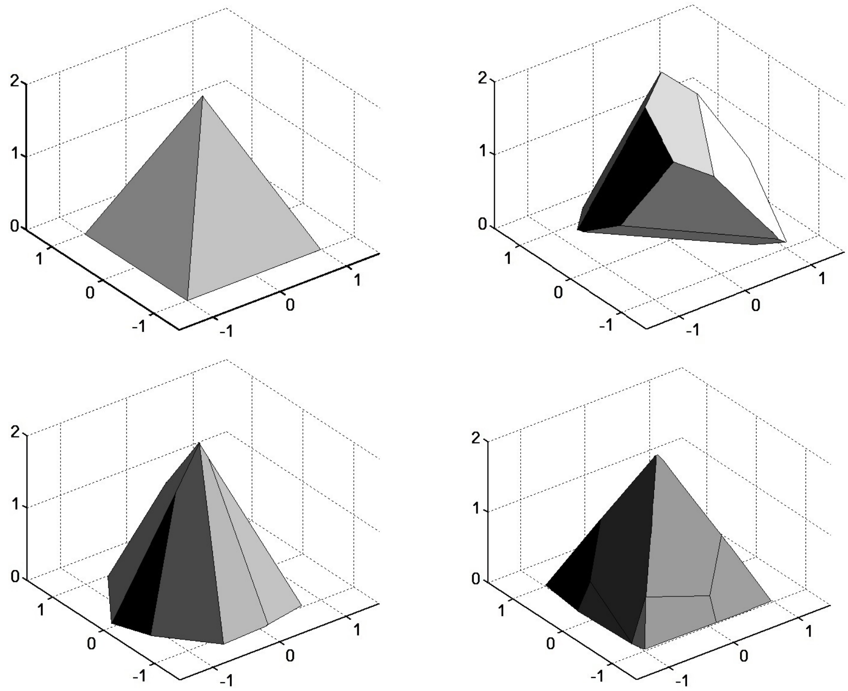

Figure 2: Plots 2, 3 and 4 show reconstructions of the pyramid in the first plot. The reconstructions are based on surface tensors up to rank $s_o = 2, 3, 4$.

of harmonic intrinsic volumes of K_0. The reconstruction algorithm we derive is a generalization to an n-dimensional setting of Algorithm Harmonic Intrinsic Volume LSQ described in [10].

6.1. Reconstruction algorithm based on measurements of harmonic intrinsic volumes

Let $K_0 \in \mathcal{K}_n^n$ be an unknown convex body where measurements of the harmonic intrinsic volumes of K_0 are available up to degree $s_o \geq 2$. Due to noise, the measurements are of the form $\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o}$, where ϵ_{s_o} is an m_{s_o}-dimensional vector of random variables with zero mean and finite variance. As the harmonic intrinsic volumes of degree 1 of K_0 are known to vanish, these should not be measured, so we let the corresponding noise variables be 0.

In Section 5, the exact surface tensors of K_0 were known. In that situation, we constructed a convex body with the same surface tensors as K_0. In this section, only *noisy measurements* of the harmonic intrinsic volumes are available, and it is typically no longer possible to construct a convex body with harmonic intrinsic volumes that fit the measurements exactly. Instead, the aim is to construct a convex body $\hat{K}_{s_o}^H \in \mathcal{K}^n$ such that the harmonic intrinsic volumes of $\hat{K}_{s_o}^H$ fit the measurements $\psi_{n-1}^{s_o}(K_0)+\epsilon_{s_o}$ of the harmonic intrinsic volumes of K_0 in a least squares sense. Hence, $\hat{K}_{s_o}^H$ should minimize the mapping $D_{s_o} \colon \mathcal{K}^n \to [0, \infty)$ defined as

$$D_{s_o}(K) = \|\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o} - \psi_{n-1}^{s_o}(K)\|^2$$

for $K \in \mathcal{K}^n$. In the 2-dimensional setup, [10, Lemma 6.1] yields the existence of a convex body that minimizes D_{s_o}. In the n-dimensional setting, however, the existence of such a convex body can not be ensured. This existence problem is overcome by extending the domain of D_{s_o} such that the mapping attains its infimum. This extension prevents the existence problem and thus establishes a natural framework for reconstruction in the n-dimensional setting.

First notice that $D_{s_o}(K)$ only depends on $K \in \mathcal{K}^n$ through $S_{n-1}(K, \cdot)$, so a version $\check{D}_{s_o}$ of D_{s_o} can be defined on the set $\{S_{n-1}(K, \cdot) \mid K \in \mathcal{K}^n\}$ letting $\check{D}_{s_o}(S_{n-1}(K, \cdot)) = D_{s_o}(K)$ for $K \in \mathcal{K}^n$. In the weak topology, the closure of $\{S_{n-1}(K, \cdot) \mid K \in \mathcal{K}^n\} \subseteq \mathcal{M}$ is the set

$$\mathcal{M}_0 = \left\{\mu \in \mathcal{M} \;\middle|\; \int_{S^{n-1}} u \, \mu(du) = 0\right\},$$

and the domain of $\check{D}_{s_o}$ is extended to $\mathcal{M}_0$ by defining

$$\check{D}_{s_o}(\mu) = \|\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o} - \psi_{n-1}^{s_o}(\mu)\|^2$$

for $\mu \in \mathcal{M}_0$. Then

$$\inf_{K \in \mathcal{K}^n} D_{s_o}(K) = \inf_{\mu \in \mathcal{M}_0} \check{D}_{s_o}(\mu) \tag{22}$$

since $\check{D}_{s_o}$ is continuous on $\mathcal{M}_0$.

The infimum of $\check{D}_{s_o}$ is attained on $\mathcal{M}_0$, and in addition, it can be shown that $\check{D}_{s_o}$ is minimized by a measure in $\mathcal{M}_{m_{s_o}}$, where

$$\mathcal{M}_k = \left\{ \mu \in \mathcal{M}_0 \mid \mu = \sum_{j=1}^{k} \alpha_j \delta_{u_j},\ \alpha_j \geq 0,\ u_j \in S^{n-1} \right\}$$

for $k \in \mathbb{N}$. This is the content of the following Lemmas 6.1 and 6.2. Due to the close connection between D_{s_o} and $\check{D}_{s_o}$, we write D_{s_o} for both versions of the mapping.

Lemma 6.1. *Let $\mu \in \mathcal{M}_0$ and $s \in \mathbb{N}_0$. Then there exist a measure $\mu_s \in \mathcal{M}_{m_s}$ such that μ and μ_s have identical moments up to order s.*

The proof of Lemma 6.1 follows the lines of the proof of [10, Thm. 4.1]. The result also holds if $\mathcal{M}_0$ and $\mathcal{M}_{m_s}$ are replaced by the larger sets $\mathcal{M}$ and $\{\mu \in \mathcal{M} \mid \mu = \sum_{j=1}^{k} \alpha_j \delta_{u_j},\ \alpha_j \geq 0,\ u_j \in S^{n-1}\}$, respectively.

Lemma 6.2. *There exists a measure $\mu_{s_o} \in \mathcal{M}_{m_{s_o}}$ such that*

$$D_{s_o}(\mu_{s_o}) = \inf_{\mu \in \mathcal{M}_0} D_{s_o}(\mu). \tag{23}$$

If $\mu_1, \mu_2 \in \mathcal{M}_0$ minimize D_{s_o}, then μ_1 and μ_2 have identical moments up to order s_o.

Proof. Let $H = \{\psi_{n-1}^{s_o}(\mu) \mid \mu \in \mathcal{M}_0\} \subseteq \mathbb{R}^{m_{s_o}}$. Then

$$\inf_{\mu \in \mathcal{M}_0} D_{s_o}(\mu) = \inf_{x \in H} \|\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o} - x\|^2.$$

Let $\{\psi_{n-1}^{s}(\mu_k)\}_{k \in \mathbb{N}}$ be a convergent sequence in H. Then, $\sup_{k \in \mathbb{N}} \mu_k(S^{n-1}) < \infty$, since $\mu(S^{n-1}) = \sqrt{\omega_n}\psi_{(n-1)01}(\mu)$ for $\mu \in \mathcal{M}_0$. Since $\mathcal{M}_0$ is closed, this implies that there exists a subsequence $(\mu_{k_l})_{l \in \mathbb{N}}$ of $(\mu_k)_{k \in \mathbb{N}}$ that converges weakly to a measure $\mu \in \mathcal{M}_0$, see [2, Cor. 31.1]. Then $\psi_{n-1}^{s_o}(\mu_k) \to \psi_{n-1}^{s_o}(\mu)$ for $k \to \infty$ as spherical harmonics are continuous on S^{n-1}. Hence, H is closed in $\mathbb{R}^{m_{s_o}}$. Solving the minimization problem

$$\inf_{x \in H} \|\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o} - x\|^2$$

corresponds to finding the metric projection of $\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o}$ on the nonempty, convex and closed set H. This projection always exists and is unique, see [14, Sec. 1.2]. Then the existence of a measure $\mu_{s_o} \in \mathcal{M}_{s_o}$ that satisfies (23) follows from Lemma 6.1. The second statement of the lemma follows from the uniqueness of the projection. □

Due to Lemma 6.2 and the structure of $\mathcal{M}_{m_{s_o}}$, the minimization of D_{s_o} can be reduced to the finite minimization problem

$$\inf_{(\alpha, \mathbf{u}) \in M_{s_o}} \sum_{s=0}^{s_o} \sum_{j=1}^{N(n,s)} \left(\psi_{(n-1)sj}(K_0) + \epsilon_{sj} - \sum_{l=1}^{m_{s_o}} \alpha_l H_{sj}(u_l) \right)^2, \tag{24}$$

where M_{s_o} is defined in (14). In the following, we describe how a minimizer $\hat{K}^H_{s_o}$ of D_{s_o} can be derived from a solution of (24).

A solution $(\alpha, \mathbf{u}) \in M_{s_o}$ to the minimization problem (24) corresponds to the measure $\mu_{\alpha,\mathbf{u}} = \sum_{j=0}^{m_{s_o}} \alpha_j \delta_{u_j} \in \mathcal{M}_{m_{s_o}}$. It follows from Lemma 5.2 that the measure $\mu_{\alpha,\mathbf{u}}$ is a surface area measure of a convex body in $\mathcal{K}^n$ if and only if $\mu_{\alpha,\mathbf{u}}$ is of the form $a(\delta_v + \delta_{-v})$ for some $a \geq 0$ and $v \in S^{n-1}$ or if the matrix $M(\mu_{\alpha,\mathbf{u}})$ of second order moments of $\mu_{\alpha,\mathbf{u}}$ is positive definite. The assumption on $M(\mu_{\alpha,\mathbf{u}})$ can alternatively be replaced by the assumption that $\alpha_1 u_1, \ldots, \alpha_{m_{s_o}} u_{m_{s_o}}$ span $\mathbb{R}^n$.

Assume that $\mu_{\alpha,\mathbf{u}} = a(\delta_v + \delta_{-v})$ for some $v \in S^{n-1}$ and $a \geq 0$. If $a = 0$, we let $\hat{K}^H_{s_o}$ be the singleton $\{0\}$. If $a > 0$, we let $\hat{K}^H_{s_o}$ be a polytope in $u^\perp$ with surface area a. Now assume that $\alpha_1 u_1, \ldots, \alpha_{m_{s_o}} u_{m_{s_o}}$ span $\mathbb{R}^n$. Then $\mu_{\alpha,\mathbf{u}}$ is the surface area measure of a polytope with nonempty interior. We let $\hat{K}^H_{s_o}$ be the output polytope from Algorithm MinkData (see [6, Sec. A.4]) that reconstructs a polytope with surface area measure $\mu_{\alpha,\mathbf{u}}$ from $(\alpha, \mathbf{u})$. In all three cases, the surface area measure of $\hat{K}^H_{s_o}$ is $\mu_{\alpha,\mathbf{u}}$, so $\hat{K}^H_{s_o}$ minimizes D_{s_o}.

As $s_o \geq 2$, it follows from Lemma 5.2 and the uniqueness statement of Lemma 6.2 that if $\mu_{\alpha,\mathbf{u}}$ is not a surface area measure of a convex body, then the same holds for every measure in $\mathcal{M}_0$ that minimizes D_{s_o}. Hence, the mapping D_{s_o} does not attain its infimum on $\mathcal{K}^n$, and there does not exist a convex body with harmonic intrinsic volumes that fit the measurements $\psi^{s_o}_{n-1}(K_0) + \epsilon_{s_o}$ in a least squares sense. By Lemma 6.3 in Section 6.2, this situation only occurs when the measurements are too noisy. The reconstruction algorithm is summarized in the following.

Algorithm Harmonic Intrinsic Volume LSQ (n-dim).

Input: Measurements $\psi^{s_o}_{n-1}(K_0) + \epsilon_{s_o}$ of the harmonic intrinsic volumes up to degree $s_o \geq 2$ of an unknown convex body $K_0 \in \mathcal{K}^n_n$.

Task: Construct a polytope $\hat{K}^H_{s_o}$ such that the harmonic intrinsic volumes up to degree s_o of $\hat{K}^H_{s_o}$ fit the measurements $\psi^{s_o}_{n-1}(K_0) + \epsilon_{s_o}$ in a least squares sense.

Action: Let $(a, \mathbf{v})$ be a solution to the minimization problem

$$\inf_{(\alpha,\mathbf{u}) \in M_{s_o}} \sum_{s=0}^{s_o} \sum_{j=1}^{N(n,s)} \big(\psi_{(n-1)sj}(K_0) + \epsilon_{sj} - \sum_{l=1}^{m_{s_o}} \alpha_l H_{sj}(u_l)\big)^2.$$

Case 1: If $a = 0$, let $\hat{K}^H_{s_o} = \{0\}$.

Case 2: If $\mu_{a,\mathbf{v}} = \alpha(\delta_u + \delta_{-u})$ for some $\alpha > 0$ and $u \in S^{n-1}$, let $\hat{K}^H_{s_o}$ be a polytope in $u^\perp$ with surface area α.

Case 3: If $a_1 v_1, \ldots, a_{m_{s_o}} v_{m_{s_o}}$ span $\mathbb{R}^n$, then $\mu_{a,\mathbf{v}}$ is the surface area measure of a polytope $P \in \mathcal{K}^n_n$ with at most m_{s_o} facets. Use Algorithm MinkData to reconstruct P, and let $\hat{K}^H_{s_o} = P$.

Case 4: Otherwise, the solution $(a, \mathbf{v})$ does not correspond to a surface area measure of a convex body. The output of the algorithm is the message that there is no solution to the given task.

6.2. Consistency of the reconstruction algorithm

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space where the vectors of noise variables $(\epsilon_{s_o})_{s_o \geq 2}$ are defined. We assume that the noise variables are independent with zero mean and that the variance of $\epsilon_{s_o j}$ is bounded by $\sigma_{s_o}^2 > 0$ for $s_o \geq 2$ and $j = 1, \dots, m_{s_o}$. In the following, for $s_o \geq 2$, we write

$$D_{s_o}(\cdot, \epsilon_{s_o}) = \|\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o} - \psi_{n-1}^{s_o}(\cdot)\|^2$$

to emphasize the dependence of D_{s_o} on ϵ_{s_o}, and we let $r_{K_0} = \frac{r}{2}$ and $R_{K_0} = 2R$, where r and R are defined as in (17) with K replaced by K_0.

Lemma 6.3. *There exists a constant $c_{K_0} > 0$ such that any measure $\mu \in \mathcal{M}_0$ that minimizes $D_{s_o}(\cdot, \epsilon_{s_o})$ is the surface area measure of a convex body $K_\mu \in \mathcal{K}^n(r_{K_0}, R_{K_0})$ if $\|\epsilon_{s_o}\| < c_{K_0}$.*

Proof. If $\mu \in \mathcal{M}_0$ minimizes $D_{s_o}(\cdot, \epsilon_{s_o})$, then

$$\begin{aligned}\|\psi_{n-1}^2(K_0) - \psi_{n-1}^2(\mu)\| &\leq \|\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o} - \psi_{n-1}^{s_o}(\mu)\| + \|\epsilon_{s_o}\| \\ &\leq \sqrt{D_{s_o}(K_0, \epsilon_{s_o})} + \|\epsilon_{s_o}\| = 2\|\epsilon_{s_o}\|.\end{aligned}$$

The second order moments of μ depend linearly on $\psi_{n-1}^2(\mu)$, and the eigenvalues of the matrix of second order moments $M(\mu)$ of μ depend continuously on $M(\mu)$, see [17, Prop. 6.2], so for each $\alpha > 0$,

$$|\lambda_{min}(M(S_{n-1}(K_0, \cdot))) - \lambda_{min}(M(\mu))| < \alpha \tag{25}$$

if $\|\epsilon_{s_o}\|$ is sufficiently small. Here $\lambda_{min}(A)$ denotes the smallest eigenvalue of a symmetric matrix A. Due to Lemma 5.2 (i), we have $\lambda_{min}(M(S_{n-1}(K_0, \cdot))) > 0$ as K_0 has nonempty interior, so $M(\mu)$ is positive definite if $\|\epsilon_{s_o}\|$ is sufficiently small. Then μ is the surface area measure of a convex body $K_\mu \in \mathcal{K}_n^n$ by Lemma 5.2. By Lemma 5.4, (25) and the fact that

$$|S(K_0) - S(K_\mu)| = \sqrt{\omega_n}\|\psi_{n-1}^0(K_0) - \psi_{n-1}^0(\mu)\| \leq 2\sqrt{\omega_n}\|\epsilon_{s_o}\|,$$

we even have that $K_\mu \in \mathcal{K}^n(r_{K_0}, R_{K_0})$ if $\|\epsilon_{s_o}\| < c_{K_0}$, where $c_{K_0} > 0$ is chosen sufficiently small. $\square$

We let $\mathbb{K}_{s_o}(\epsilon_{s_o})$ be the random set of convex bodies that minimize $D_{s_o}(\cdot, \epsilon_{s_o})$, i.e.

$$\mathbb{K}_{s_o}(\epsilon_{s_o}) = \left\{K \in \mathcal{K}^n \mid D_{s_o}(K, \epsilon_{s_o}) = \inf_{L \in \mathcal{K}^n} D_{s_o}(L, \epsilon_{s_o})\right\}.$$

By Equation (22), the set $\mathbb{K}_{s_o}(\epsilon_{s_o})$ is nonempty if and only if Algorithm Harmonic Intrinsic Volume LSQ (n-dim) has an output polytope. Let the mapping

$g\colon \mathcal{K}^n \times \mathbb{R}^{m_{s_o}} \to \mathbb{R}$ be given as $g(K,x) = \inf_{L\in\mathcal{K}^n} D_{s_o}(L,x) - D_{s_o}(K,x)$ for $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^{m_{s_o}}$, then

$$\Big\{\mathbb{K}_{s_o}(\epsilon_{s_o}) \neq \emptyset\Big\} = \Big\{\sup_{K\in\mathcal{K}^n} \mathbf{1}_{\{0\}}(g(K,\epsilon_{s_o})) = 1\Big\} \subseteq \Omega,$$

and for $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}&\Big\{\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0,K) \le \alpha\Big\}\\ =&\Big\{\sup_{K\in\mathcal{K}^n} \delta^t(K_0,K)\,\mathbf{1}_{\{0\}}(g(K,\epsilon_{s_o})) \le \alpha\Big\} \cap \Big\{\sup_{K\in\mathcal{K}^n} \mathbf{1}_{\{0\}}(g(K,\epsilon_{s_o})) = 1\Big\},\end{aligned}$$

where the supremum over the empty set is defined to be ∞. Using the notation of permissible sets, see [13, App. C] and arguments as in [10, p. 27], we obtain that $\sup_{K\in\mathcal{K}^n} \delta^t(K_0,K)\,\mathbf{1}_{\{0\}}(g(K,\epsilon_{s_o}))$ and $\sup_{K\in\mathcal{K}^n} \mathbf{1}_{\{0\}}(g(K,\epsilon_{s_o}))$ are measurable. Then

$$\Big\{\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0,K) \le \alpha\Big\} \in \mathcal{F}$$

for $\alpha \in \mathbb{R}$, which implies that $\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0,K)$ is measurable.

Theorem 6.4. *Assume that $\sigma_{s_o}^2 = \mathcal{O}(s_o^{-(2n-1+\varepsilon)})$ for some $\varepsilon > 0$. Then*

$$\sup_{K\in\mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0,K) \to 0$$

almost surely for $s_o \to \infty$.

Proof. As $m_{s_o} = \mathcal{O}(s_o^{n-1})$, the assumption on $\sigma_{s_o}^2$ yields that

$$\mathbb{E}\sum_{s_o=2}^{\infty} m_{s_o}\|\epsilon_{s_o}\|^2 = \sum_{s_o=2}^{\infty} m_{s_o}\sum_{j=1}^{m_{s_o}} \mathbb{E}\epsilon_{s_oj}^2 \le \sum_{s_o=2}^{\infty} m_{s_o}^2\sigma_{s_o}^2 < \infty.$$

Then $\sum_{s_o=2}^{\infty} m_{s_o}\|\epsilon_{s_o}\|^2 < \infty$ almost surely, and it follows that $m_{s_o}\|\epsilon_{s_o}\|^2 \to 0$ almost surely for $s_o \to \infty$.

Now choose c_{K_0} according to Lemma 6.3 and fix a realization such that $m_{s_o}\|\epsilon_{s_o}\|^2 \to 0$ for $s_o \to \infty$. Then, there exists an $S \in \mathbb{N}$ satisfying that $\sqrt{m_{s_o}}\|\epsilon_{s_o}\| < c_{K_0}$ for $s_o > S$. In particular, $\|\epsilon_{s_o}\| < c_{K_0}$ for $s_o > S$, so by Lemma 6.2 and Lemma 6.3 there is an output polytope of Algorithm Harmonic Intrinsic Volume LSQ (n-dim). Then, for $s_o > S$, the set $\mathbb{K}_{s_o}(\epsilon_{s_o})$ is nonempty, and $K \in \mathcal{K}^n(r_{K_0}, R_{K_0})$ for $K \in \mathbb{K}_{s_o}(\epsilon_{s_o})$. Since

$$\begin{aligned}\|\psi_{n-1}^{s_o}(K_0) - \psi_{n-1}^{s_o}(K)\| &\le \|\psi_{n-1}^{s_o}(K_0) + \epsilon_{s_o} - \psi_{n-1}^{s_o}(K)\| + \|\epsilon_{s_o}\|\\ &\le \sqrt{D_{s_o}(K_0,\epsilon_{s_o})} + \|\epsilon_{s_o}\| = 2\|\epsilon_{s_o}\|\end{aligned}$$

for $K \in \mathbb{K}_{s_o}(\epsilon_{s_o})$, Theorem 4.3 yield that

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} d_D(S_{n-1}(K_0, \cdot), S_{n-1}(K, \cdot)) \\ \leq c(n, R_{K_0}, \frac{1}{3}) s_o^{-\frac{1}{3}} + 2\sqrt{\omega_n m_{s_o}} \|\epsilon_{s_o}\| \to 0$$

for $s_o \to \infty$. Hence, [7, Lemma 9.5] and [14, Thm. 8.5.3] imply that

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) \to 0$$

for $s_o \to \infty$. □

Theorem 6.5. *Assume that* $\sigma_{s_o}^2 = \mathcal{O}(s_o^{-(2n-2+\varepsilon)})$ *for some* $\varepsilon > 0$. *Then*

$$\sup_{K \in \mathbb{K}_{s_o}(\epsilon_{s_o})} \delta^t(K_0, K) \to 0$$

in probability for $s_o \to \infty$.

Proof. Markov's inequality and the assumption on $\sigma_{s_o}^2$ imply that $m_{s_o}\|\epsilon_{s_o}\|^2 \to 0$ in probability for $s_o \to \infty$. Then, Theorem 6.5 follows in the same way as Theorem 6.4. □

Theorems 6.4 and 6.5 yield that the reconstruction algorithm gives good approximations to the shape of K_0 for large s_o under certain assumptions on the variance of the noise variables. To test how noise affects the reconstructions for small s_o, the oblate ellipsoid in the first plot of Figure 3 is reconstructed from harmonic intrinsic volumes up to degree 6. For $k \in \mathbb{N}_0$, the dimension of $\mathcal{H}_k^3$ is $2k+1$, and to derive the harmonic intrinsic volumes, we use the orthonormal basis of $\mathcal{H}_k^3$ given by

$$H_{k(2j+1)}(u(\theta, \phi)) = \alpha_{kj} \sin^j(\theta) C_{k-j}^{j+\frac{1}{2}}(\cos(\theta)) \cos(j\phi), \qquad 0 \leq j \leq k$$

and

$$H_{k(2j)}(u(\theta, \phi)) = \alpha_{kj} \sin^j(\theta) C_{k-j}^{j+\frac{1}{2}}(\cos(\theta)) \sin(j\phi), \qquad 1 \leq j \leq k,$$

where $\alpha_{kj} \in \mathbb{R}$ is a normalizing constant, $C_l^\lambda, l \in \mathbb{N}_0, \lambda > 0$ are Gegenbauer polynomials and $u(\theta, \phi) = (\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), \cos(\theta))$ for $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, see [4, Sections 1.2 and 1.6.2].

The harmonic intrinsic volumes are subject to an increasing level of noise. The second plot in Figure 3 is a reconstruction based on exact harmonic intrinsic volumes, whereas the reconstructions in the third and fourth plot are based on harmonic intrinsic volumes disrupted by noise. The variance of the noise variables is $\sigma_2^2 = 1$ in the third plot and $\sigma_3^2 = 4$ in the fourth plot. Then the standard deviations σ_2 and σ_3 of the noise variables are approximately 5% and 10% of $\psi_{201}(K_0)$, respectively. For the three levels of noise, the minimization problem (24) is solved using the *fmincon* procedure provided by MatLab and

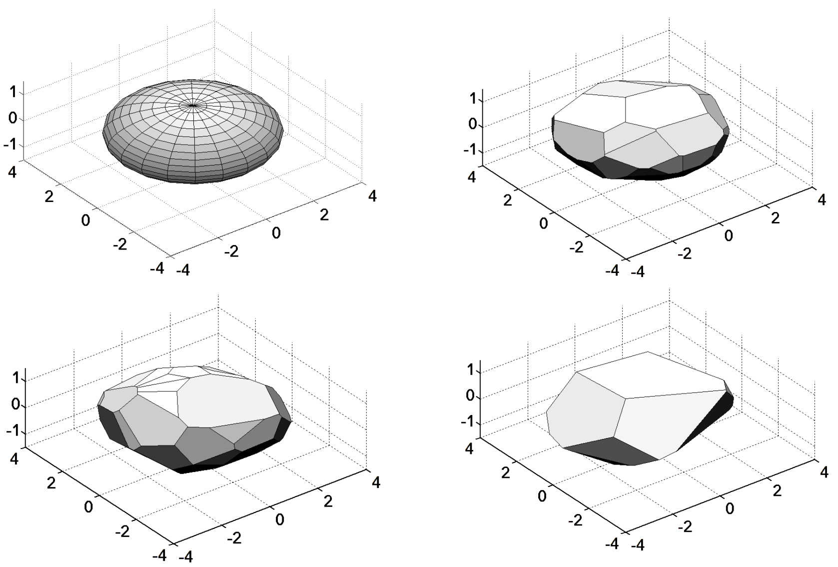

Figure 3: Plots $2, 3$ and 4 show reconstructions of the ellipsoid in the first plot based on noisy measurements of harmonic intrinsic volumes up to degree $s_o = 6$. In plots $2, 3$ and 4, the variances of the noise variables are $0, 1$ and 4.

Algorithm MinkData is applied to reconstruct a polytope corresponding to the solution.

Plots $2, 3$ and 4 in Figure 3 show how the reconstructions deviate increasingly from the ellipsoid as the variance of the noise variables increases. The reconstruction based on exact harmonic intrinsic volumes captures essential features of the ellipsoid. The reconstruction is approximately invariant under rotations around the third axis and has the same orientation and degree of anisotropy as the ellipsoid. Despite a noise level corresponding to 5% of $\psi_{201}(K_0)$, the reconstruction in the second plot captures to some extent the same features and provides a fairly good approximation of the ellipsoid. The reconstruction in the third plot is comparable to the ellipsoid. However, the effect of noise is clearly visible.

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Paper D

Reconstruction of convex bodies from moments

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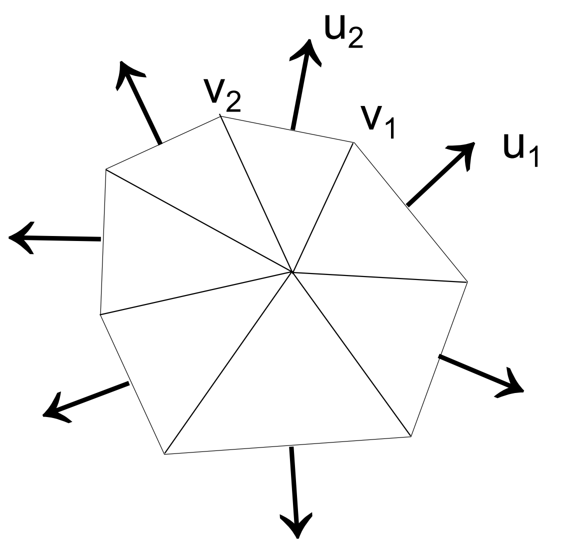

Reconstruction of convex bodies from moments

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Abstract

We investigate how much information about a convex body can be retrieved from a finite number of its geometric moments. We give a sufficient condition for a convex body to be uniquely determined by a finite number of its geometric moments, and we show that among all convex bodies, those which are uniquely determined by a finite number of moments form a dense set. Further, we derive a stability result for convex bodies based on geometric moments. It turns out that the stability result is improved considerably by using another set of moments, namely Legendre moments. We present a reconstruction algorithm that approximates a convex body using a finite number of its Legendre moments. The consistency of the algorithm is established using the stability result for Legendre moments. When only noisy measurements of Legendre moments are available, the consistency of the algorithm is established under certain assumptions on the variance of the noise variables.

Keywords: Convex body, geometric moment, Legendre moment, reconstruction, uniqueness, stability.

1 Introduction

Important characteristics of a compact set $K \subset \mathbb{R}^n$ are its *geometric moments* (sometimes only referred to as moments) where

$$\mu_\alpha(K) = \int_K x^\alpha dx$$

is the geometric moment of order $|\alpha|$ for a multi-index $\alpha \in \mathbb{N}_0^n$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. In the last two decades, the reconstruction of a geometric object from its moments has received considerable attention. Milanfar et al. developed in [17] an inversion algorithm for 2-dimensional polygons and presented a refined numerically stable version in [7]. Restricting to convex polygons they proved that every m-gon is uniquely determined by its complex moments up to order $2m-3$. Recently, Gravin et al. showed in [8] that an n-dimensional convex polygon P with m vertices is uniquely determined by its moments up to order $2m-n$. Apart from polytopes, an exact reconstruction from finitely many moments is known to be possible for so called quadrature domains in the complex plane, see [9]. In continuation of the work in this area, we investigate how much information can be retrieved from finitely many geometric moments of an arbitrary convex body in $\mathbb{R}^n$. Recently, a similar investigation of another set of moments, namely moments of surface area measures of convex bodies, was carried out in [11].

Using uniqueness results for functionals, see [13] and [20], applied to indicator functions, we show that if a convex body K is of the form $C \cap \{p \geq 0\}$, where C is a compact subset of $\mathbb{R}^n$ and p is a polynomial of degree N, then K is uniquely determined by its geometric moments up to degree N among all convex bodies in C. Further, any convex body in C can be approximated arbitrarily well in the Hausdorff distance by a convex body of the form $C \cap \{p \geq 0\}$. This result and the fact that the geometric moments up to order 2 of a convex body K determine an upper bound on the circumradius of K imply that among all convex bodies, those which are uniquely determined by finitely many geometric moments form a dense subset, see Theorem 3.8.

Restricting to convex bodies in the two-dimensional unit square, we derive an upper bound on the Nikodym distance between two convex bodies given finitely many of their geometric moments, see Theorem 4.2. The upper bound is derived using a stability result for absolutely continuous functions on the unit interval, see [23]. This result is extended to twice continuously differentiable functions on the two-dimensional unit square and applied to differences of indicator functions via an approximation argument. The upper bound depends on the number of moments used and also on the Euclidean distance between the moments of the two convex bodies. The upper bound decreases when the distance between the moments decreases, however, it increases exponentially in the number of moments. The method used to derive the upper bound of the Nikodym distance suggests that the geometric moments should be replaced by another set of moments, namely the Legendre moments, in order to remove the effect of the exponential factor. The Legendre moments of a convex body are defined like the usual geometric moments, but with the monomials replaced by products of Legendre polynomials, see Section 2. Using that these products of Legendre polynomials constitute an orthonormal basis of the square integrable functions on $[0,1]^2$ and that the Legendre polynomials satisfy a certain differential equation, we derive an upper bound of the Nikodym distance that becomes arbitrarily small when the distance between the Legendre moments decreases and the number of moments used increases, see Theorem 4.3.

In Section 5, we assume that the first $(N+1)^2$ Legendre moments of an unknown convex body K are available for some $N \in \mathbb{N}$. A polygon with at most $m \in \mathbb{N}$ vertices is called a least squares estimator of K if the Legendre moments of P fit the available Legendre moments of K in a least squares sense. We derive an upper bound of the Euclidean distance between the Legendre moments of K and the Legendre moments of an arbitrary least squares estimator P of K. In combination with the previously described stability result, this yields an upper bound of the Nikodym distance between K and P (Theorem 5.1). This upper bound of the Nikodym distance becomes arbitrarily small when N and m increase. For completeness, we further derive an upper bound for the Nikodym distance between K and a least squares estimator based on geometric moments. Due to the structure of the stability results, this upper bound increases exponentially when the number of available geometric moments increases.

In Section 6, we derive a reconstruction algorithm for convex bodies. The input of the algorithm is a finite number of Legendre moments of a convex body K, and the output of the algorithm is a polygon P with Legendre moments that fit the available Legendre moments of K in a least squares sense. The output polygon P has prescribed outer normals, which ensures that P can be found as the solution to a polynomial optimization problem. The consistency of the reconstruction algorithm is established in Corollary 6.5. In Section 6.3, the reconstruction algorithm is extended such that it allows for Legendre moments disrupted by noise. To ensure consistency of the algorithm in this case, the variances of the noise terms should decrease appropriately when the number of input moments increases, see Theorem 6.6.

The paper is organized as follows: Preliminaries and notations are introduced in Section 2. The uniqueness results are presented in Section 3, and the stability results are derived in Section 4. In Section 5, the least squares estimators based on geometric moments and Legendre moments are treated. Finally, the reconstruction algorithm is described and discussed in Section 6.

2 Notation and preliminaries

A convex body is a compact, convex subset of $\mathbb{R}^n$ with nonempty interior. The space of convex bodies contained in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$ and is equipped with the Hausdorff metric δ_{H}. On the set $\{K \in \mathcal{K}^2 \mid K \subset [0,1]^2\}$, we use the Nikodym metric δ_N in addition to the Hausdorff metric. The Nikodym distance of two convex bodies $K, L \subset [0,1]^2$ is the area of the symmetric difference of K and L, that is

$$\delta_N(K,L) = V_2((K \setminus L) \cup (L \setminus K)) = |\mathbf{1}_K - \mathbf{1}_L|^2_{L^2([0,1]^2)},$$

where $|\cdot|_{L^2([0,1]^2)}$ is the usual norm on the set $L^2([0,1]^2)$ of square integrable functions on $[0,1]^2$. On the set $\{K \in \mathcal{K}^2 \mid K \subset [0,1]^2\}$, the Hausdorff metric and the Nikodym metric induce the same topology, see [22]. The support function of a convex body K is denoted h_K, and for $K \in \mathcal{K}^2$ and $\theta \in [0, 2\pi)$, we write $h_K(\theta) := h_K((\cos\theta, \sin\theta))$.

In Section 4, we derive stability results for convex bodies in the unit square. In this context, it turns out to be natural and useful to introduce Legendre moments in addition to geometric moments. The shifted and normalized Legendre polynomials $L_i \colon [0,1] \to \mathbb{R}, i \in \mathbb{N}_0$, are obtained by applying the Gram-Schmidt orthonormalization to $1, x, x^2, \ldots$, and the products of Legendre polynomials

$$(x_1, x_2) \to L_i(x_1)L_j(x_2), \qquad i, j \in \mathbb{N}_0$$

form an orthonormal basis of $L^2([0,1]^2)$. For a convex body $K \subset [0,1]^2$, we define the Legendre moments of K as

$$\lambda_{ij}(K) = \int_K L_i(x_1)L_j(x_2)d(x_1,x_2)$$

for $i, j \in \mathbb{N}_0$.

The uniqueness and stability results we establish in Sections 3 and 4 are derived using uniqueness and stability results [13, 20, 23] for functionals. In the following, we introduce notation in relation to these results. For a compact set $C \subset \mathbb{R}^n$ with nonempty interior, we let $L^\infty(C)$ denote the space of essentially bounded measurable functions $\Phi : C \to \mathbb{R}$. The essential supremum for $\Phi \in L^\infty(C)$ in C is denoted by $\|\Phi\|_{\infty,C}$ and we define $\|\Phi\|_{1,C} := \int_C |\Phi(x)| dx$. Further, we let

$$\operatorname{sign}(\Phi)(x) = \begin{cases} 1, & \Phi(x) \geq 0, \\ -1, & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}^n$.

The signed distance function d_C of C is defined as in, e.g., [5, Section 5] or with opposite signs in [12, Chapter 4.4]. That is

$$d_C(x) := \begin{cases} -\inf_{y \in \partial C} \|x - y\|, & x \in C, \\ \inf_{y \in \partial C} \|x - y\|, & x \in \mathbb{R}^n \setminus C, \end{cases}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Then the ε-parallel set of C is defined as $C_\varepsilon := \{x : d_C(x) \leq \varepsilon\}$ for $\varepsilon \in \mathbb{R}$.

The geometric moments of a function $\Phi \in L^\infty(C)$ are given as

$$\mu_\alpha(\Phi) = \int_C \Phi(x)\, x^\alpha dx$$

for $\alpha \in \mathbb{N}_0^n$, and the Legendre moments of $\Psi \in L^\infty([0,1]^2)$ are defined as

$$\lambda_{ij}(\Psi) = \int_{[0,1]^2} \Psi(x)\, L_i(x_1) L_j(x_2) d(x_1, x_2)$$

for $i, j \in \mathbb{N}_0$. Notice that $\mu_\alpha(K) = \mu_\alpha(\mathbf{1}_K)$ for a convex body $K \subset C$, and $\lambda_{ij}(L) = \lambda_{ij}(\mathbf{1}_L)$ for a convex body $L \subset [0,1]^2$.

3 Uniqueness results

In this section, we present uniqueness results for convex bodies based on a finite number of geometric moments. We show that the convex bodies that are uniquely determined in $\mathcal{K}^n$ by a finite number of geometric moments form a dense subset of $\mathcal{K}^n$. This result is established using uniqueness results from [13] and [20] for functionals. The results from [13] and [20] are summarized in Section 3.1 and applied in Section 3.2 to derive uniqueness results for convex bodies.

3.1 Summary of results from [13] and [20]

Let $N \in \mathbb{N}_0$, $L > 0$ and $C \subset \mathbb{R}^n$ be compact. Further, let $m := (m_\alpha)_{|\alpha| \leq N}$, where $m_\alpha \in \mathbb{R}, \alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $\sum_{|\alpha| \leq N} m_\alpha^2 > 0$. A function $\Phi \in L^\infty(C)$ with $\|\Phi\|_{\infty,C} \leq L$ is called a solution of the *L-moment problem of order N* if

$$\mu_\alpha(\Phi) = m_\alpha, \qquad \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq N. \tag{3.1}$$

In [13], it is shown that the supremum

$$l(m) := \sup\Big\{ \Big\| \sum_{|\alpha|\le N} a_\alpha x^\alpha \Big\|_{1,C}^{-1} : a_\alpha \in \mathbb{R}, \alpha \in \mathbb{N}_0^n, |\alpha| \le N, \sum_{|\alpha|\le N} a_\alpha m_\alpha = 1 \Big\}$$

is attained. Thus, there exists an $\tilde{a} = (\tilde{a}_\alpha)_{|\alpha|\le N}$ with $\sum_{|\alpha|\le N} \tilde{a}_\alpha m_\alpha = 1$ and

$$l(m) = \Big\| \sum_{|\alpha|\le N} \tilde{a}_\alpha x^\alpha \Big\|_{1,C}^{-1}.$$

It follows from [13] that the L-moment problem (3.1) has a solution if and only if $L \ge l(m)$. Furthermore, (3.1) has a unique solution if and only if $L = l(m)$. If $L = l(m)$, then the unique solution is $\Phi = L\,\mathrm{sign}(p_m)$, where $p_m = \sum_{|\alpha|\le N} \tilde{a}_\alpha x^\alpha$.

For more details and proofs, we refer to [13, Section IX.1-2] and [20]. The one-dimensional case is proved in [13, Section IX.2, Thm. 2.2] by applying more general results from [13, Section IX.1] which are obtained in normed linear spaces with moments defined with respect to arbitrary linear independent functionals instead of monomials. The specialization of [13, Section IX.1] to the situation considered above is contained in [20, Section 2]. In particular, Putinar [20] formulates the following uniqueness result.

Lemma 3.1 ([20, Cor.2.3])**.** *A function $\Phi \in L^\infty(C)$ is uniquely determined in $\{\Psi \in L^\infty(C) : \|\Psi\|_{\infty,C} \le \|\Phi\|_{\infty,C}\}$ by its geometric moments $\mu_\alpha(\Phi)$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le N$ if and only if*

$$\Phi = \|\Phi\|_{\infty,C}\,\mathrm{sign}(p),$$

where $p \ne 0$ is a polynomial of degree at most N.

3.2 Consequences for convex bodies

Due to the relation between geometric moments of convex bodies and geometric moments of indicator functions, we conclude the following from Lemma 3.1.

Corollary 3.2. *A convex body $K \subset C$ is uniquely determined in $\{L \in \mathcal{K}^n : L \subset C\}$ by its geometric moments $\mu_\alpha(K)$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le N$ if*

$$K = C \cap \{p \ge 0\},$$

where $p \ne 0$ is a polynomial of degree at most N.

Proof. If $K = C \cap \{p \ge 0\}$, then

$$2\,\mathbf{1}_K(x) - 1 = \mathrm{sign}(p)(x), \quad x \in C.$$

Thus, we obtain from Lemma 3.1 that $2\,\mathbf{1}_K - 1$ is uniquely determined in

$$\{2\,\mathbf{1}_L - 1 : L \in \mathcal{K}, L \subset C\} \subset \{\Psi \in L^\infty(C) : \|\Psi\|_{\infty,C} \le 1\}$$

by its geometric moments $\mu_\alpha(2\,\mathbf{1}_K - 1) = 2\mu_\alpha(K) - \mu_\alpha(C), \alpha \in \mathbb{N}_0^n$ with $|\alpha| \le N$. This yields the assertion. $\square$

Example 1. An ellipsoid E is determined among all convex bodies by its geometric moments up to order 2 since $E = \{x \in \mathbb{R}^n : p(x) \geq 0\}$, where $p(x) := R - \|Tx\|^2$, $x \in \mathbb{R}^n$ with some invertible linear transformation T and some $R > 0$.

Remark 3.3. Corollary 3.2 gives a sufficient condition for a convex body to be uniquely determined among convex bodies in a prescribed set by a finite number of moments. It is not clear if the condition is also necessary.

Remark 3.4. Let $m := (m_\alpha)_{|\alpha| \leq N}$ be a finite number of geometric moments of some unknown convex body $K \subset C$. Let $\tilde{m}_\alpha := 2m_\alpha - \mu_\alpha(C)$ and define $l(\tilde{m})$ and $p_{\tilde{m}}$ as in the previous section. Then it holds that $l(\tilde{m}) \leq 1$, and if $l(\tilde{m}) = 1$, then $K = C \cap \{p_{\tilde{m}} \geq 0\}$.

In Theorems 3.6 and 3.8, we show that the convex bodies which are uniquely determined among all convex bodies by finitely many geometric moments form a dense subset of $\mathcal{K}^n$ with respect to the Hausdorff metric δ_{H}. The ideas of the proofs are shortly summarized in the following. For a convex body $K \subset C$, a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $K = C \cap \{f \geq 0\}$ is constructed. The function f is approximated by a polynomial p_m of degree m in such a way that $K_m := C \cap \{p_m \geq 0\}$ is convex and $\delta_{\mathrm{H}}(K, K_m)$ is small. Then, it follows from Corollary 3.2 that K_m is uniquely determined by its geometric moments up to order m among all convex bodies contained in C. The circumradius of K_m admits an upper bound which can be expressed in terms of the geometric moments $\mu_\alpha(K_m), |\alpha| \leq 2$ of K_m. Therefore, K_m is uniquely determined by its geometric moments up to order m among all convex bodies if C is large enough.

At first observe that we can assume that K is of class C^∞_+, see [21, Thm. 3.4.1] and the subsequent discussion. Hence, the boundary of K is a regular submanifold of $\mathbb{R}^n$ of class C^k for all $k \in \mathbb{N}_0$. Further, the principal curvatures of K are strictly positive. By $\kappa_i(K, x), 1 \leq i \leq n-1$ we denote the principal curvatures of K in $x \in \partial K$. Since $K \in C^\infty_+$ there exist $m_K, M_K > 0$ such that

$$\kappa_i(K, x) \in (m_K, M_K), \qquad x \in \partial K, 1 \leq i \leq n-1.$$

For $\varepsilon < M_K^{-1}$ it follows from the inverse function theorem applied as in [6, Lemma 14.16] that the signed distance function d_K of K is a C^∞-function in $\mathbb{R}^n \setminus K_{-\varepsilon}$. As in [6, Sec. 14.6] we define for $y \in \partial K$ the principal coordinate system at y as the coordinate system with coordinate axes $x_1(y), \ldots, x_n(y)$, where $x_1(y), \ldots, x_{n-1}(y)$ are the principal directions and $x_n(y)$ is the inner unit normal vector of K at y. Then the following lemma is obtained by adapting [6, Lemma 14.17].

Lemma 3.5. *Let $K \in C^\infty_+$, $\varepsilon < M_K^{-1}$, $x_0 \in \mathbb{R}^n \setminus K_{-\varepsilon}$ and $y_0 := \operatorname{argmin}_{y \in \partial K} \|x_0 - y\|$. Then, with respect to the principal coordinate system at y_0, we have*

$$\nabla d_K(x_0) = (0, \ldots, 0, -1)^\top$$

and

$$\left(\frac{\partial^2}{\partial_i \partial_j} d_K(x_0)\right)_{i,j=1}^n = \operatorname{diag}\left(\frac{\kappa_1(K, y_0)}{1 + \kappa_1(K, y_0) d_K(x_0)}, \ldots, \frac{\kappa_{n-1}(K, y_0)}{1 + \kappa_{n-1}(K, y_0) d_K(x_0)}, 0\right).$$

By the described approximation argument and Lemma 3.5, we obtain the following result.

Theorem 3.6. *Let K, C be convex bodies with $K \subset \operatorname{int} C$. For $\varepsilon > 0$ there exists an $m \in \mathbb{N}$ and a convex body $K_m \subset C$ which is uniquely determined by its geometric moments up to order m among all convex bodies contained in C and fulfils*

$$\delta_{\mathrm{H}}(K, K_m) \leq \varepsilon.$$

Proof. We may assume $K \in C_+^\infty$, $2\varepsilon < M_K^{-1}$ and $K_\varepsilon \subset C$. We have $K = \{f \geq 0\}$ for the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1, & x \in K_{-2\varepsilon}, \\ -\frac{(d_K(x)+2\varepsilon)^4}{16\varepsilon^4} + 1, & x \in \mathbb{R}^n \setminus K_{-2\varepsilon}. \end{cases}$$

Observe that f is of class $C^3(\mathbb{R}^n)$ and

$$f(x) \in [-65/16, 15/16] \quad \Longleftrightarrow \quad x \in (\partial K)_\varepsilon. \tag{3.2}$$

The Hessian matrix $(\frac{\partial^2}{\partial_i \partial_j} f(x))_{1 \leq i,j \leq n}$ is negative definite for $x \in (\partial K)_\varepsilon$. Namely, let $x_0 \in (\partial K)_\varepsilon$ and $y_0 := \operatorname{argmin}_{y \in \partial K} \|x_0 - y\|$. Then it follows from Lemma 3.5 that, with respect to the principal coordinate system at y_0,

$$\frac{\partial^2}{\partial_i \partial_j} f(x_0) = \begin{cases} -\frac{(d_K(x_0)+2\varepsilon)^3}{4\varepsilon^4} \frac{\kappa_i(K,y_0)}{1+\kappa_i(K,y_0) d_K(x_0)}, & i = j < n, \\ -\frac{3(d_K(x_0)+2\varepsilon)^2}{4\varepsilon^4}, & i = j = n, \\ 0 & i \neq j. \end{cases}$$

Therefore, the eigenvalues of the Hessian matrix $\big(\frac{\partial^2}{\partial_i \partial_j} f(x)\big)$ for $x \in (\partial K)_\varepsilon$ are all negative and their absolute values are uniformly bounded from below by

$$\min \left\{ \frac{m_K}{4\varepsilon(1 + M_K \varepsilon)}, \frac{3}{4\varepsilon^2} \right\}. \tag{3.3}$$

From [2, Thm. 2], we obtain that for every $m \geq 2$ there exists a polynomial p_m of degree m such that

$$\left\| \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}} (f - p_m) \right\|_{\infty, C} \leq c(n, C, f) \frac{1}{m^{3-|\alpha|}}, \qquad \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq 2, \tag{3.4}$$

where $c(n, C, f) > 0$ depends on n, C and $\max_{|\alpha| \leq 3} \|\frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}} f\|_{\infty, C}$.

As the function that maps a symmetric matrix to its eigenvalues is Lipschitz continuous ([3, Thm. VI.2.1]), equation (3.3) and (3.4) imply that the Hessian matrix $(\frac{\partial^2}{\partial_i \partial_j} p_m)_{1 \leq i,j \leq n}$ of p_m is negative definite on $(\partial K)_\varepsilon$ if we choose $m \geq 2$ sufficiently large. Thus, by the well-known convexity criterion [21, Thm. 1.5.13], the polynomial p_m is concave on every convex subset of $(\partial K)_\varepsilon$, and in particular,

$$\big(p_m(x), p_m(y) \geq 0 \text{ for } x, y \in (\partial K)_\varepsilon \text{ with } [x, y] \subset (\partial K)_\varepsilon\big) \quad \Rightarrow \quad [x, y] \subset K_m, \tag{3.5}$$

where we define

$$K_m := C \cap \{p_m \geq 0\}.$$

Due to (3.4), we can furthermore assume that

$$\|f - p_m\|_{C,\infty} < 15/16.$$

Then it follows from (3.2) that $p_m \geq 0$ on $K_{-\varepsilon}$ and $p_m < 0$ on $C \setminus K_\varepsilon$. In other words, we have $K_{-\varepsilon} \subset K_m \subset K_\varepsilon$. This implies that $\delta_{\mathrm{H}}(K, K_m) \leq \varepsilon$ since $\varepsilon < M_K^{-1}$.

Furthermore, we can show that K_m is convex. Let $x, y \in K_m$. If $x, y \in K_{-\varepsilon}$, then $[x, y] \subset K_{-\varepsilon}$ since $K_{-\varepsilon}$ is convex and thus $[x, y] \subset K_m$. If $x \in K_{-\varepsilon}$ and $y \in (\partial K)_\varepsilon$, there is a $z \in [x, y]$ such that $[x, z] \subset K_{-\varepsilon}$ and $[z, y] \subset (\partial K)_\varepsilon$. Hence, it follows from (3.5) that $[x, y] \subset K_m$. If $x, y \in (\partial K)_\varepsilon$ and $[x, y] \subset (\partial K)_\varepsilon$, then $[x, y] \subset K_m$ because of (3.5). If $x, y \in (\partial K)_\varepsilon$ and $[x, y] \cap K_{-\varepsilon} \neq \emptyset$, there are $z_1, z_2 \in [x, y]$ such that $[x, z_1] \subset (\partial K)_\varepsilon$, $[z_1, z_2] \subset K_{-\varepsilon}$ and $[z_2, y] \subset (\partial K)_\varepsilon$. Then, it follows again from the convexity of $K_{-\varepsilon}$ and by (3.5) that $[x, y] \subset K_m$. $\square$

For a convex body K, let $s(K)$ denote the center of mass, $V_n(K)$ the volume, and $R(K)$ the circumradius of K. Then, we define

$$\tilde{K} := \big(V_n(K)\big)^{-1/n}\big(K - s(K)\big).$$

As $V_n(\tilde{K}) = 1$ and $s(\tilde{K}) = 0$, a special case of [18, Lem. 4.1] yields that

$$\left(\int_{\tilde{K}} |\langle x, u\rangle|^2 dx\right)^{1/2} \geq \left(\frac{\Gamma(3)\Gamma(n)}{2e\Gamma(n+3)}\right)^{1/2} \max\{h_{\tilde{K}}(u), h_{\tilde{K}}(-u)\}$$

for $u \in S^{n-1}$. Then Cauchy-Schwarz's inequality implies that

$$\tilde{K} \subset I_2(\tilde{K}) \left(\frac{\Gamma(3)\Gamma(n)}{2e\Gamma(n+3)}\right)^{-1/2} B_n,$$

where

$$I_2(L) := \left(\int_L \|x\|^2 dx\right)^{\frac{1}{2}}$$

for a convex body L. Since $R(K) = \big(V_n(K)\big)^{1/n} R(\tilde{K})$, we obtain an upper bound

$$R(K) \leq \left(\frac{2e\Gamma(n+3)}{\Gamma(3)\Gamma(n)}\right)^{1/2} V_n(K)^{1/n} I_2(\tilde{K}) \tag{3.6}$$

of the circumradius of K.

Remark 3.7. Observe that $I_2(\tilde{K})$ can be expressed in terms of the geometric moments of K up to order 2. More precisely,

$$I_2(\tilde{K}) = \mu_0(K)^{-\frac{1+n}{n}} \Big(\sum_{i=1}^{n} \mu_0(K)\mu_{2e_j}(K) - \mu_{e_j}(K)^2\Big)^{1/2},$$

where $\{e_1, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^n$.

The previous considerations allow us to formulate a strengthened version of Theorem 3.6 for the whole class of convex bodies and not only those contained in a prescribed compact set.

Theorem 3.8. *Let K be a convex body. For $\varepsilon > 0$ there exists an $m \in \mathbb{N}$ and a convex body K_m which is uniquely determined by its geometric moments up to order m among all convex bodies and fulfils*

$$\delta_{\mathrm{H}}(K, K_m) \leq \varepsilon.$$

Proof. Without loss of generality we may assume that $K \in C_+^\infty$ and $V_n(K_{-\varepsilon}) > 0$. Let

$$c(K,\varepsilon) := \left(\frac{e\Gamma(n+3)}{\Gamma(3)\Gamma(n)} \frac{2^{n+3}\,\omega_n}{n+2} V_n(K_{-\varepsilon})^{-1} R(K_\varepsilon)^{n+2} \right)^{1/2},$$

and choose

$$R > c(K,\varepsilon) \tag{3.7}$$

such that $K \subset RB^n$. By Theorem 3.6 there exists an $m \in \mathbb{N}$ and a convex body $K_m \subset (3R+\varepsilon)B^n$ which is uniquely determined by its geometric moments up to order m among all convex bodies contained in $(3R+\varepsilon)B^n$ and fulfils

$$\delta_{\mathrm{H}}(K, K_m) \leq \varepsilon. \tag{3.8}$$

Due to the proof of Theorem 3.6, we can assume that $m \geq 2$ and $K_{-\varepsilon} \subset K_m \subset K_\varepsilon$. Then, condition (3.7) ensures that K_m is uniquely determined among all convex bodies. Namely, let L be a convex body with

$$\mu_\alpha(L) = \mu_\alpha(K_m), \qquad \alpha \in \mathbb{N}_0^n, \text{ with } |\alpha| \leq m. \tag{3.9}$$

Then, it follows by Remark 3.7 and a simple calculation that

$$\begin{aligned} I_2(\tilde{L}) &= I_2(\tilde{K}_m) \leq I_2(2V_n(K_m)^{-1/n} R(K_m) B^n) \\ &= \left(\frac{2^{n+2}\,\omega_n}{n+2} V_n(K_m)^{-(2+n)/n} R(K_m)^{n+2} \right)^{1/2}, \end{aligned}$$

where we have used that $\tilde{K}_m \subset 2R(\tilde{K}_m)B^n$ as $s(\tilde{K}_m) = 0$. Thus, we obtain by (3.6) that

$$R(L) \leq \left(\frac{e\Gamma(n+3)}{\Gamma(3)\Gamma(n)} \frac{2^{n+3}\,\omega_n}{n+2} V_n(K_m)^{-1} R(K_m)^{n+2} \right)^{1/2} \leq c(K,\varepsilon).$$

Assumption (3.9) implies that $s(L) = s(K_m)$, so

$$\sup_{x\in L} \|x\| \leq \sup_{x \in L} \|x - s(L)\| + \|s(K_m)\| \leq 3R + \varepsilon$$

as $K_m \subset (R+\varepsilon)B^n$ by (3.8). Thus, $L \subset (3R+\varepsilon)B^n$, so $K_m = L$, and we obtain the assertion. □

Remark 3.9. Due to the one-to-one correspondence between the geometric moments up to order m and the Legendre moments up to order m of a convex body, the uniqueness results stated in this section hold if the geometric moments are replaced by Legendre moments in the two-dimensional case.

4 Stability results

In this section, we derive stability results for two-dimensional convex bodies contained in the unit square. We derive an upper bound for the Nikodym distance of convex bodies where the first $(N+1)^2$ moments are close in the Euclidean distance. The stability results are based on more general results for twice continuously differentiable functions on the unit square.

4.1 Stability results for functions on the unit square

The study in this section uses ideas from [23] (see also [1]), which considers the problem of recovering a real-valued function u defined on the interval $(0,1)$ from its first $N+1$ moments $\mu_0(u),\dots,\mu_N(u)$. In [23], it is shown that if $u,v:(0,1)\to\mathbb{R}$ are absolutely continuous functions satisfying

$$\sum_{k=0}^{N}|\mu_k(u)-\mu_k(v)|^2\leq\varepsilon^2$$

and

$$|u'(x)-v'(x)|^2_{L^2(0,1)}\leq E^2$$

for some $\varepsilon,E>0$, then

$$|u-v|^2_{L^2(0,1)}\leq\min\{\varepsilon^2e^{3.5(n+1)}+\tfrac{1}{4}(n+1)^{-2}:n=0,\dots,N\}.$$

Using the same ideas as [23], we deduce the following corresponding theorem in two dimensions.

Theorem 4.1. *If $v,w\in C^2([0,1]^2)$ are twice continuously differentiable functions satisfying*

$$\sum_{i,j=0}^{N}|\mu_{ij}(v)-\mu_{ij}(w)|^2\leq\varepsilon^2$$

and

$$\frac{1}{4}\left|\frac{d}{dx_1}(v-w)\right|^2_{L^2([0,1]^2)}+\frac{1}{4}\left|\frac{d}{dx_2}(v-w)\right|^2_{L^2([0,1]^2)}\leq E^2$$

for some $\varepsilon,E>0$, then

$$|v-w|^2_{L^2([0,1]^2)}\leq\min\{a_0(n+1)^2e^{7(n+1)}\varepsilon^2+(n+1)^{-2}E^2:n=0,\dots,N\}$$

where $a_0>0$.

Proof. Let h_N be the orthogonal projection of $u:=v-w$ on the linear hull $\operatorname{lin}\{x_1^ix_2^j: i,j=0,\dots,N\}$ with respect to the usual scalar product on $L^2([0,1]^2)$. Furthermore, let

$$t_N:=u-h_N$$

be the projection of u on the orthogonal complement of $\operatorname{lin}\{x_1^i x_2^j : i,j = 0,\dots,N\}$. Then

$$h_N(x_1,x_2) = \sum_{i,j=0}^{N} \lambda_{ij}(u) L_i(x_1) L_j(x_2)$$

and

$$t_N(x_1,x_2) = \sum_{\substack{i,j=0 \\ i\vee j>N}}^{\infty} \lambda_{ij}(u) L_i(x_1) L_j(x_2),$$

where $\lambda_{ij}(u), i,j \in \mathbb{N}_0$ are the Legendre moments of u. For $i \in \mathbb{N}_0$, the coefficients of the polynomial L_i are denoted by C_{ij}, $j = 0,\dots,i$, that is

$$L_i(x) = \sum_{j=0}^{i} C_{ij} x^j, \qquad x \in [0,1].$$

Then it follows for $i,j = 0,\dots,N$ that

$$\begin{aligned}
\lambda_{ij}(u) &= \sum_{k=0}^{i}\sum_{l=0}^{j} C_{ik}C_{jl} \int\limits_{[0,1]^2} u(x_1,x_2) x_1^k x_2^l d(x_1,x_2) \\
&= \sum_{k=0}^{i}\sum_{l=0}^{j} C_{ik}C_{jl}\mu_{kl}(u) \\
&= (CMC^\top)_{ij}, \qquad (4.1)
\end{aligned}$$

with

$$C := \begin{pmatrix} C_{00} & & & \\ C_{10} & C_{11} & & \\ \vdots & & \ddots & \\ C_{N0} & C_{N1} & \dots & C_{NN} \end{pmatrix}, \qquad \text{and} \qquad M := (\mu_{ij}(u))_{i,j=0,\dots,N}.$$

The Frobenius norm of a square matrix A is defined as $|A|_F := \sqrt{\operatorname{tr}(A^\top A)}$, and since this norm is submultiplicative, see [10, (3.3.4)], we obtain that

$$\begin{aligned}
|h_N|_{L^2([0,1]^2)} &= \sqrt{\sum_{i,j=0}^{N} \lambda_{ij}(u)^2} = \sqrt{\operatorname{tr}(L^\top L)} \\
&= |L|_F = |CMC^\top|_F \le |C|_F |M|_F |C^\top|_F \\
&= |C|_F^2 |M|_F \qquad (4.2)
\end{aligned}$$

where $L := (\lambda_{ij}(u))_{i,j=0,\dots,N}$. The matrix $C^\top C$ has $N+1$ non-negative eigenvalues, $0 \le l_0 \le l_1 \le \dots \le l_N$, and $C^\top C = H_N^{-1}$, where H_N is the Hilbert matrix

$$H_N := \left(\frac{1}{i+j+1}\right)_{i,j=0,\dots,N},$$

see [23, (22)]. Since $\|H_N e_1\| > 1$, the Hilbert matrix H_N has an eigenvalue larger than 1, so the smallest eigenvalue l_0 of H_N^{-1} is smaller than 1. This implies that

$$|C|_F^2 = \operatorname{tr}(C^\top C) = \sum_{i=0}^{N} l_i \leq (N+1)l_N \leq (N+1)\frac{l_N}{l_0} \approx a_0(N+1)e^{3.5(N+1)} \tag{4.3}$$

with a constant $a_0 > 0$, where we have used the approximation [23, (8)], see also [24, p. 111]. From equation (4.2) and (4.3), we obtain that

$$|h_N|_{L^2([0,1]^2)} \leq a_0(N+1)e^{3.5(N+1)} \sqrt{\sum_{i,j=0}^{N} \mu_{ij}(u)^2}. \tag{4.4}$$

The shifted Legendre polynomials satisfy the differential equation

$$-\frac{d}{dx_1}[x_1(1-x_1)L_i'(x_1)] = i(i+1)L_i(x_1), \qquad x_1 \in [0,1], i \in \mathbb{N}_0,$$

see [23, (25)]. From this differential equation, we obtain by multiplication with $u(x_1, x_2)$, integration over $[0,1]$ with respect to x_1 and twofold partial integration for all $x_2 \in (0,1)$ that

$$\begin{aligned} &-\int_{[0,1]} L_i(x_1)\frac{d}{dx_1}\left[x_1(1-x_1)\frac{d}{dx_1}u(x_1,x_2)\right]dx_1 \\ &\quad = i(i+1)\int_{[0,1]} L_i(x_1)u(x_1,x_2)dx_1. \end{aligned} \tag{4.5}$$

By multiplication with $L_j(x_2)$ and integration with respect to x_2, it follows from (4.5) that

$$\begin{aligned} &-\int_{[0,1]^2} L_i(x_1)L_j(x_2)\frac{d}{dx_1}\left[x_1(1-x_1)\frac{d}{dx_1}u(x_1,x_2)\right]dx_1dx_2 \\ &\quad = i(i+1)\int_{[0,1]^2} L_i(x_1)L_j(x_2)u(x_1,x_2)dx_1dx_2. \end{aligned}$$

This implies that the Legendre moments of the function

$$(x_1,x_2) \mapsto \frac{d}{dx_1}\left[x_1(1-x_1)\frac{d}{dx_1}u(x_1,x_2)\right]$$

are equal to $-i(i+1)\lambda_{ij}(u), i,j \in \mathbb{N}_0$. Thus, we obtain from the theory of Hilbert spaces and by partial integration that

$$\begin{aligned} \sum_{i,j=0}^{\infty} i(i+1)\lambda_{ij}(u)^2 &= \int_{[0,1]^2} \frac{d}{dx_1}\left[x_1(1-x_1)\frac{d}{dx_1}u(x_1,x_2)\right]\big(-u(x_1,x_2)\big)d(x_1,x_2) \\ &= \int_{[0,1]^2} x_1(1-x_1)\left(\frac{d}{dx_1}u(x_1,x_2)\right)^2 d(x_1,x_2) \\ &\leq \frac{1}{4}\left|\frac{d}{dx_1}u(x_1,x_2)\right|_{L^2([0,1]^2)}^2. \end{aligned} \tag{4.6}$$

In the same way, we conclude that

$$\sum_{i,j=0}^{\infty} j(j+1)\lambda_{ij}(u)^2 \le \frac{1}{4}\left|\frac{d}{dx_2}u(x_1,x_2)\right|^2_{L^2([0,1]^2)}. \tag{4.7}$$

The inequalities (4.6) and (4.7) imply that

$$\begin{aligned}
|t_N|^2_{L^2([0,1]^2)} &= \sum_{\substack{i,j=0\\ i\vee j>N}}^{\infty} \lambda_{ij}(u)^2 \le \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty}\lambda_{ij}(u)^2 + \sum_{i=0}^{\infty}\sum_{j=N+1}^{\infty}\lambda_{ij}(u)^2 \\
&\le \sum_{i,j=0}^{\infty}\frac{i(i+1)}{(N+1)^2}\lambda_{ij}(u)^2 + \sum_{i,j=0}^{\infty}\frac{j(j+1)}{(N+1)^2}\lambda_{ij}(u)^2 \\
&\le \frac{1}{(N+1)^2}\left(\frac{1}{4}\left|\frac{d}{dx_1}u(x_1,x_2)\right|^2_{L^2([0,1]^2)} + \frac{1}{4}\left|\frac{d}{dx_2}u(x_1,x_2)\right|^2_{L^2([0,1]^2)}\right),
\end{aligned} \tag{4.8}$$

and as a consequence we obtain that

$$|v-w|^2_{L^2([0,1]^2)} = |h_N|^2_{L^2([0,1]^2)} + |t_N|^2_{L^2([0,1]^2)} \le a_0(N+1)^2 e^{7(N+1)}\varepsilon^2 + \frac{1}{(N+1)^2}E^2.$$

□

4.2 Application to convex bodies

In this section, we approximate the indicator function $\mathbf{1}_K$ of a convex body K by a smooth function and apply the result from the previous section. In this way, we obtain an estimate for the Nikodym distance of two convex bodies in terms of the Euclidean distance of their first $(N+1)^2$ geometric moments.

Theorem 4.2. *If $K, L \subset [0,1]^2$ are convex bodies satisfying*

$$\sum_{i,j=0}^{N} |\mu_{ij}(K) - \mu_{ij}(L)|^2 \le \varepsilon^2,$$

for some $\varepsilon \ge 0$, then

$$\delta_N(K,L) \le \min\left\{a_0(n+1)^2 e^{7(n+1)}\varepsilon^2 + \frac{a_1}{(n+1)} : n = 0,\dots,N\right\},$$

with constants $a_0, a_1 > 0$.

Proof. Let $u := \mathbf{1}_K - \mathbf{1}_L$. As in the proof of Theorem 4.1, we let h_N denote the orthogonal projection of u on $\mathrm{lin}\{x_1^i x_2^j : i,j = 0,\dots,N\}$ and let t_N denote the projection on the orthogonal complement of $\mathrm{lin}\{x_1^i x_2^j : i,j = 0,\dots,N\}$. In the proof of Theorem 4.1, the smoothness of u is not used when the estimate (4.4) is derived. Therefore, we obtain in the same way that

$$|h_N|_{L^2([0,1]^2)} \le a_0(N+1)e^{3.5(N+1)}\sqrt{\sum_{i,j=0}^{N}\mu_{ij}(u)^2}. \tag{4.9}$$

Using a mollification, see [16, p. 110], we obtain for every $\rho > 0$ a differentiable function $u^{(\rho)} : [0,1]^2 \to \mathbb{R}$ approximating u in the L^1-norm. More precisely, we choose

$$u^{(\rho)}(x) := (J_\rho * u)(x) = \int\limits_{[0,1]^2} J_\rho(x-y)u(y)dy, \qquad x \in [0,1]^2,$$

where

$$J_\rho = \begin{cases} c_0\rho^{-2}\mathrm{e}^{-\frac{\rho^2}{\rho^2-\|x\|^2}}, & \text{for } \|x\| < \rho \\ 0, & \text{for } \|x\| \geq \rho \end{cases}$$

with a constant $c_0 > 0$ chosen such that $|J_\rho|_{L^1(\mathbb{R}^2)} = 1$. Notice that c_0 is independent of ρ and that $J_\rho \in C^\infty(\mathbb{R}^2)$. From the definition of the mollification, we obtain that

$$\|u - u^{(\rho)}\|_\infty \leq \|u\|_\infty + |J_\rho|_{L^1(\mathbb{R}^2)}\|u\|_\infty \leq 2$$

and

$$(u - u^{(\rho)})(x) = 0, \qquad x \in [K_{-\rho} \cup ([0,1]^2 \setminus K_\rho)] \cap [L_{-\rho} \cup ([0,1]^2 \setminus L_\rho)],$$

so

$$\begin{aligned} |u - u^{(\rho)}|^2_{L^2([0,1]^2)} &\leq \|u - u^{(\rho)}\|^2_\infty \, V_2([(K_\rho \setminus K) \cup (K \setminus K_{-\rho})] \cup [(L_\rho \setminus L) \cup (L \setminus L_{-\rho})]) \\ &\leq 4\,[V_2(K_\rho \setminus K) + V_2(K \setminus K_{-\rho}) + V_2(L_\rho \setminus L) + V_2(L \setminus L_{-\rho})] \end{aligned}$$

for $\rho \in (0,1)$. Then, the fact that

$$V_2(K \setminus K_{-\rho}) \leq V_2(K_\rho \setminus K),$$

the Steiner formula, and the monotonicity of the intrinsic volumes imply that

$$\begin{aligned} |u - u^{(\rho)}|^2_{L^2([0,1]^2)} &\leq 8[V_2(K_\rho \setminus K) + V_2(L_\rho \setminus L)] \\ &\leq 8[2\rho^2\pi + 2\rho(V_1(K) + V_1(L))] \\ &\leq (16\pi + 64)\rho \leq 11^2\rho, \end{aligned}$$

where V_1 is the intrinsic volume of order 1, so $V_1(M)$ is half the boundary length of a convex body M. For $\rho \in (0,1)$, let $t_N^{(\rho)}$ be the orthogonal projection of $u^{(\rho)}$ on the orthogonal complement of $\mathrm{lin}\{x_1^i x_2^j : i,j = 0,\ldots,N\}$. Then it follows from Pythagoras' theorem and (4.8) that

$$\begin{aligned} |t_N|_{L^2([0,1]^2)} &\leq |t_N - t_N^{(\rho)}|_{L^2([0,1]^2)} + |t_N^{(\rho)}|_{L^2([0,1]^2)} \\ &\leq |u - u^{(\rho)}|_{L^2([0,1]^2)} + \frac{1}{N+1}E_\rho \\ &\leq 11\sqrt{\rho} + \frac{1}{N+1}E_\rho, \end{aligned}$$

where $E_\rho > 0$ is some constant satisfying

$$\frac{1}{4}\left|\frac{d}{dx_2}u^{(\rho)}\right|^2_{L^2([0,1]^2)} + \frac{1}{4}\left|\frac{d}{dx_1}u^{(\rho)}\right|^2_{L^2([0,1]^2)} \leq E_\rho^2. \tag{4.10}$$

In order to obtain an expression for a constant E_ρ that satisfies (4.10), we at first observe that

$$\frac{d}{dx_2}u^{(\rho)}(y) = \frac{d}{dx_1}u^{(\rho)}(y) = 0, \qquad y \in [K_{-\rho} \cup ((0,1)^2 \setminus K_\rho)] \cap [L_{-\rho} \cup ((0,1)^2 \setminus L_\rho)].$$

Furthermore,

$$\begin{aligned}\frac{d}{dx_1}u^{(\rho)}(x) &= \left(\left[\frac{d}{dx_1}J_\rho\right] * u\right)(x) = \int_{[0,1]^2}\left[\frac{d}{dx_1}J_\rho\right](x-y)u(y)dy \\ &\leq \int_{\mathbb{R}^2}\left|\frac{d}{dx_1}J_\rho(y)\right|dy = \rho^{-3}\int_{\mathbb{R}^2}\left|\left[\frac{d}{dx_1}J_1\right]\left(\frac{y}{\rho}\right)\right|dy \\ &= \rho^{-1}\int_{\mathbb{R}^2}\left|\left[\frac{d}{dx_1}J_1\right](y)\right|dy \leq c_1\rho^{-1}\end{aligned}$$

for $x \in \mathbb{R}^2$ and a constant $c_1 > 0$ independent of ρ. It follows that

$$\begin{aligned}\left|\frac{d}{dx_1}u^{(\rho)}\right|^2_{L^2([0,1]^2)} &\leq c_1^2\rho^{-2}\big[V_2(K_\rho \setminus K) + V_2(K \setminus K_{-\rho}) + V_2(L_\rho \setminus L) + V_2(L \setminus L_{-\rho})\big] \\ &\leq \frac{c_1^2 11^2}{4\rho}.\end{aligned}$$

In the same way, we obtain

$$\left|\frac{d}{dx_2}u^{(\rho)}\right|^2_{L^2([0,1]^2)} \leq \frac{c_2^2 11^2}{4\rho}$$

for a suitable $c_2 > 0$ independent of ρ. Therefore, we can choose

$$E_\rho^2 := c_3\rho^{-1}$$

for $\rho \in (0,1)$ and some constant $c_3 > 0$ independent of ρ. Letting $\rho = (N+1)^{-1}$, we obtain that

$$\begin{aligned}|t_N|^2_{L^2([0,1]^2)} &\leq \left(11\sqrt{\rho} + \frac{1}{N+1}\sqrt{c_3}\rho^{-1/2}\right)^2 \\ &= 11^2\rho + 22\sqrt{c_3}\frac{1}{N+1} + \frac{c_3}{(N+1)^2\rho} \\ &= (11^2 + 22\sqrt{c_3} + c_3)\frac{1}{N+1},\end{aligned}$$

which leads to the assertion. □

The matrix C defined in the proof of Theorem 4.1 is ill-conditioned and introduces an exponential factor in the upper bound for the Nikodym distance derived in Theorem 4.2. If the geometric moments are replaced by Legendre moments, the use of the matrix C is avoided and the upper bound can be improved.

Theorem 4.3. *If $K, L \subset [0,1]^2$ are convex bodies satisfying*

$$\sum_{i,j=1}^{N} |\lambda_{ij}(K) - \lambda_{ij}(L)|^2 \leq \varepsilon^2 \tag{4.11}$$

for some $\varepsilon \geq 0$, then

$$\delta_N(K, L) \leq \varepsilon^2 + \frac{a_1}{N+1} \tag{4.12}$$

with a constant $a_1 > 0$.

The proof of Theorem 4.3 follows the lines of the proof of Theorem 4.2. Due to inequality (4.11), the upper bound on the L^2-norm of h_N in (4.9) can be replaced by ε. This yields the upper bound (4.12) of the Nikodym distance.

Remark 4.4. If the first $(N+1)^2$ geometric moments of two convex bodies $K, L \subset [0,1]^2$ are identical, then the first $(N+1)^2$ Legendre moments of K and L are identical. In this case, Theorem 4.2 (or Theorem 4.3) implies that $\delta_N(K, L) \leq \frac{a_1}{N+1}$.

Remark 4.5. The Nikodym metric δ_N is extended in the natural way to the set of convex, compact subsets of the unit square. It then defines a pseudo metric, which we also denote by δ_N. As the proofs of Theorems 4.2 and 4.3 do not use that the interior of the convex bodies are nonempty, the stability results hold for convex, compact subsets of the unit square and the pseudo metric δ_N. In the following sections, we repeatedly consider the distance $\delta_N(K, P_k)$ for a convex body $K \subset [0,1]^2$ and a sequence of polygons $(P_k)_{k\in\mathbb{N}}$ contained in $[0,1]^2$, see Theorems 5.1, 6.3 and 6.6. If $\delta_N(K, P_k) \to 0$ for $k \to \infty$, then $\operatorname{int} P_k \neq \emptyset$ for k sufficiently large. This implies that δ_N in the expression $\delta_N(K, P_k)$ is a proper metric for k sufficiently large.

5 Least squares estimators based on moments

Let $K \subset [0,1]^2$ be a convex body and assume that its geometric moments $\mu_{ij}(K)$ for $i, j \in \mathbb{N}_0$ are given. For $m \geq 3$, let $\mathcal{P}^{(m)}$ denote the set of convex polygons contained in $[0,1]^2$ with at most m vertices. Any polygon $\hat{P}_m \in \mathcal{P}^{(m)}$ satisfying

$$\hat{P}_m = \operatorname{argmin}\Big\{ \sum_{i,j=0}^{N} (\mu_{ij}(K) - \mu_{ij}(P))^2 : \quad P \in \mathcal{P}^{(m)} \Big\}$$

is called a least squares estimator of K with respect to the first $(N+1)^2$ geometric moments on the space $\mathcal{P}^{(m)}$, where $N \in \mathbb{N}_0$. Likewise, we define a least squares estimator based on the Legendre moments. Assume that the Legendre moments $\lambda_{ij}(K), i, j \in \mathbb{N}_0$ of K are given. Then, any polygon $\hat{Q}_m \in \mathcal{P}^{(m)}$ satisfying

$$\hat{Q}_m = \operatorname{argmin}\Big\{ \sum_{i,j=0}^{N} (\lambda_{ij}(K) - \lambda_{ij}(P))^2 : \quad P \in \mathcal{P}^{(m)} \Big\}$$

is called a least squares estimator of K with respect to the first $(N+1)^2$ Legendre moments on the space $\mathcal{P}^{(m)}$. Since the polygons in $\mathcal{P}^{(m)}$ are uniformly bounded, Blaschke's selection theorem ensures the existence of least squares estimators $\hat{P}_m$ and $\hat{Q}_m$.

Theorem 5.1. *Let $\hat{P}_m$ and $\hat{Q}_m$ be least squares estimators of K on the space $\mathcal{P}^{(m)}$ with respect to the first $(N+1)^2$ geometric moments and the first $(N+1)^2$ Legendre moments. Then*

$$\delta_N(\hat{P}_m, K) \leq \left(a_0(n+1)^2 e^{7(n+1)} \left(1 + \tfrac{1}{2}\ln(2n+1)\right)^2 \frac{8\pi^3 + 16\pi}{m^2} + \frac{a_1}{(n+1)}\right)$$

for $n = 0, \dots, N$ and

$$\delta_N(\hat{Q}_m, K) \leq \frac{8\pi^3 + 16\pi}{m^2} + \frac{a_1}{(N+1)}.$$

Proof. Let $P \in \mathcal{P}^{(m)}$ and define $u := \mathbf{1}_P - \mathbf{1}_K$. Using the notation h_N, C, L, M and H_N from the proof of Theorem 4.1, we obtain that

$$\begin{aligned}
&\sqrt{\sum_{i,j=0}^{N} (\mu_{ij}(P) - \mu_{ij}(K))^2} \\
&\quad = |M|_F = |C^{-1}L(C^{-1})^\top|_F \leq |C^{-1}|_F^2 |L|_F \\
&\quad \leq \left(1 + \tfrac{1}{2}\ln(2N+1)\right)|h_N|_{L^2([0,1]^2)} \leq \left(1 + \tfrac{1}{2}\ln(2N+1)\right)|u|_{L^2([0,1]^2)} \\
&\quad = \left(1 + \tfrac{1}{2}\ln(2N+1)\right)\sqrt{\delta_N(K, P)},
\end{aligned}$$

where we have used that

$$\begin{aligned}
|C^{-1}|_F^2 &= \mathrm{tr}(C^{-1}(C^{-1})^\top) = \mathrm{tr}(H_N) \\
&= \sum_{i=0}^{N} \frac{1}{2i+1} \leq 1 + \int_0^N \frac{1}{2x+1} dx = 1 + \tfrac{1}{2}\ln(2N+1)
\end{aligned}$$

by the definition of the Hilbert matrix H_N. From [4, p. 730], the monotonicity of the intrinsic volumes, and the fact that $\sin(x) \leq x$ for $x \geq 0$, we obtain that

$$\min_{P \in \mathcal{P}^{(m)}} \delta_{\mathrm{H}}(K, P) \leq \frac{V_1(K)\sin(\frac{\pi}{m})}{m(1 + \cos(\frac{\pi}{m}))} \leq \frac{2\pi}{m^2}.$$

Further, the definition of the Hausdorff distance and the Steiner formula yield that

$$\begin{aligned}
\delta_N(K, P) &\leq V_2((K + \delta_{\mathrm{H}}(K, P)B^2) \setminus K) + V_2((P + \delta_{\mathrm{H}}(K, P)B^2) \setminus P) \\
&\leq 8\,\delta_{\mathrm{H}}(K, P) + 2\pi\,\delta_{\mathrm{H}}(K, P)^2
\end{aligned}$$

for $P \in \mathcal{P}^{(m)}$, so

$$\min_{P \in \mathcal{P}^{(m)}} \delta_N(K, P) \leq \frac{8\pi^3 + 16\pi}{m^2}. \tag{5.1}$$

Therefore,

$$\begin{aligned}
\min_{P \in \mathcal{P}^{(m)}} \sum_{i,j=0}^{N} (\mu_{ij}(P) - \mu_{ij}(K))^2 &\leq \left(1 + \tfrac{1}{2}\ln(2N+1)\right)^2 \min_{P \in \mathcal{P}^{(m)}} \delta_N(K, P) \\
&\leq \left(1 + \tfrac{1}{2}\ln(2N+1)\right)^2 \frac{8\pi^3 + 16\pi}{m^2}.
\end{aligned}$$

Then, Theorem 4.2 and Remark 4.5 yields that

$$\delta_N(\hat{P}_m, K) \leq a_0(n+1)^2 e^{7(n+1)} \left(1 + \tfrac{1}{2}\ln(2n+1)\right)^2 \frac{8\pi^3 + 16\pi}{m^2} + \frac{a_1}{(n+1)}$$

for $n = 0, \ldots, N$. For $P \in \mathcal{P}^{(m)}$, Parseval's identity yields that

$$\sum_{i,j=0}^{N} (\lambda_{ij}(K) - \lambda_{ij}(P))^2 \leq |\mathbf{1}_K - \mathbf{1}_P|^2_{L^2([0,1]^2)} = \delta_N(K, P),$$

so we obtain from (5.1), Theorem 4.3 and Remark 4.5 that

$$\delta_N(\hat{Q}_m, K) \leq \frac{8\pi^3 + 16\pi}{m^2} + \frac{a_1}{N+1}$$

for $P \in \mathcal{P}^{(m)}$ □

In Theorem 5.1, the upper bound on the distance between the convex body K and the least squares estimator $\hat{P}_m$ based on geometric moments decreases polynomially in the number of vertices m, but increases exponentially in the number of moments N. However, for the least squares estimator $\hat{Q}_m$ based on Legendre moments, the upper bound decreases polynomially in both N and m. Therefore, we concentrate on reconstruction from Legendre moments in Section 6.

6 Reconstruction based on Legendre moments

In this section, we develop a reconstruction algorithm for a convex body $K \subset [0,1]^2$ based on Legendre moments. To simplify an optimization problem, we approximate K by a polygon with prescribed outer normals. Thus, the input of the algorithm is the first $(N+1)^2$ Legendre moments of K for some $N \in \mathbb{N}_0$, and the output is a polygon $P \subset [0,1]^2$ with prescribed outer normals satisfying that the Euclidean distance between the first $(N+1)^2$ Legendre moments of P and K is minimal.

6.1 Reconstruction algorithm

Let $0 \leq \theta_1 < \cdots < \theta_n < 2\pi$, and let $c_i := \cos(\theta_i)$, $s_i := \sin(\theta_i)$ and $u_i := [c_i, s_i]^\top$ for $1 \leq i \leq n$. We assume that

$$\left\{\sum_{i=1}^{n} \lambda_i u_i : \lambda_i \geq 0, 1 \leq i \leq n\right\} = \mathbb{R}^2. \tag{6.1}$$

For $h_1, \ldots, h_n \in (-\infty, \infty)$, let

$$P(h_1, \ldots, h_n) := \bigcap_{i=1}^{n} \{x \in \mathbb{R}^2 : \langle x, u_i \rangle \leq h_i\}.$$

A vector $(h_1, \ldots, h_n) \in (-\infty, \infty)^n$ is called consistent with respect to $(\theta_1, \ldots, \theta_n)$ if the polygon $P(h_1, \ldots, h_n)$ has support function value h_i in the direction u_i for $1 \leq i \leq n$. In [15, p. 1696], it is shown that $(h_1, \ldots, h_n)$ is consistent if and only if

$$h_{i-1}(s_{i+1}c_i - c_{i+1}s_i) - h_i(s_{i+1}c_{i-1} - c_{i+1}s_{i-1}) + h_{i+1}(s_i c_{i-1} - c_i s_{i-1}) \geq 0$$

for $1 \leq i \leq n$, where we define $h_0 := h_n$ and $h_{n+1} := h_1$. We let $\mathcal{P}(\theta_1, \ldots, \theta_n)$ denote the set of polygons $P(h_1, \ldots, h_n) \subset [0,1]^2$ where $(h_1, \ldots, h_n) \in (-\infty, \infty)^n$ is consistent with respect to $(\theta_1, \ldots, \theta_n)$.

Now let $K \subset [0,1]^2$ be a convex body. Any polygon $\hat{P}_{N,n} \in \mathcal{P}(\theta_1, \ldots, \theta_n)$ satisfying

$$\hat{P}_{N,n} = \operatorname{argmin}\Big\{ \sum_{k,l=0}^{N} (\lambda_{kl}(K) - \lambda_{kl}(P))^2 : \quad P \in \mathcal{P}(\theta_1, \ldots, \theta_n) \Big\}$$

is called a least squares estimator of K with respect to the first $(N+1)^2$ moments on the space $\mathcal{P}(\theta_1, \ldots, \theta_n)$. As $\mathcal{P}(\theta_1, \ldots, \theta_n)$ is closed in the Hausdorff metric, Blaschke's selection theorem ensures the existence of a least squares estimator.

In the following, we let the directions $0 \leq \theta_1 < \cdots < \theta_n < 2\pi$ be fixed. We use the notation s_i, c_i and u_i as introduced above and assume that condition (6.1) is satisfied. When $(h_1, \ldots, h_n) \in (-\infty, \infty)^n$ is consistent with respect to $(\theta_1, \ldots, \theta_n)$, we write

$$v_i := H(u_i, h_i) \cap H(u_{i+1}, h_{i+1}), \qquad 1 \leq i \leq n$$

for the vertices of $P(h_1, \ldots, h_n)$, see Figure 1.

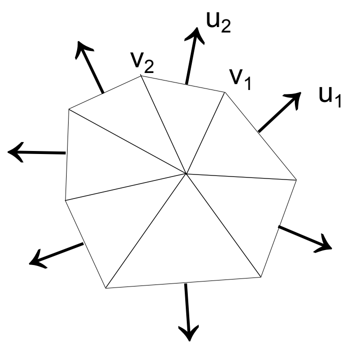

Figure 1: Polygons with normals $u_1, \ldots, u_n$.

In Lemma 6.1, the geometric moments and the Legendre moments of polygons of the form $P(h_1, \ldots, h_n)$ are expressed by means of $(h_1, \ldots, h_n)$.

Lemma 6.1. *Let $(h_1, \ldots, h_n) \in (-\infty, \infty)^n$ be consistent with respect to $(\theta_1, \ldots, \theta_n)$. Then the geometric moments and the Legendre moments of $P(h_1, \ldots, h_n)$ are polynomials in $(h_1, \ldots, h_n)$. More precisely,*

$$\mu_{kl}(P(h_1, \ldots, h_n)) = \sum_{i=1}^{n} \sum_{q_1=0}^{k+l+1} \sum_{q_2=0}^{k+l+2-q_1} M_{kl}(i, q_1, q_2) h_i^{q_1} h_{i+1}^{q_2} h_{i+2}^{k+l+2-q_1-q_2} \tag{6.2}$$

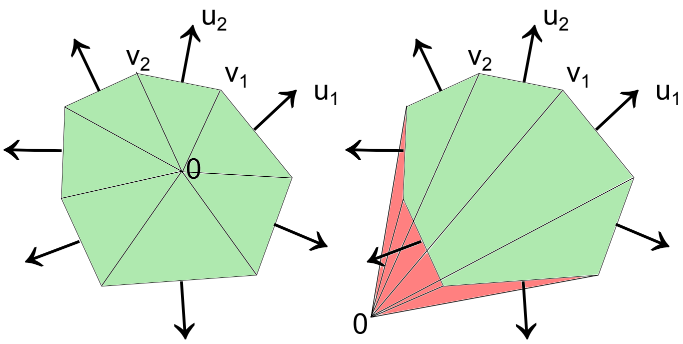

Figure 2: Representation of polygons $P(h_1,\dots,h_n)$ as difference of the sets A (red and green) and B (red).

and

$$\lambda_{kl}(P(h_1,\dots,h_n)) = \sum_{i=1}^{n}\sum_{s=0}^{k+l}\sum_{q_1=0}^{s+1}\sum_{q_2=0}^{s+2-q_1} L_{kl}(i,s,q_1,q_2) h_i^{q_1} h_{i+1}^{q_2} h_{i+2}^{s+2-q_1-q_2},$$

for $k,l \in \mathbb{N}_0$ and known real constants $M_{kl}(i,q_1,q_2)$ and $L_{kl}(i,s,q_1,q_2)$.

Proof. Observe that

$$P(h_1,\dots,h_n) = \mathrm{cl}(A \setminus B),$$

where

$$A := \bigcup_{\substack{1\le i\le n\\ h_{i+1}\ge 0}} \mathrm{conv}\{0,v_i,v_{i+1}\} \text{ and } B := \bigcup_{\substack{1\le i\le n\\ h_{i+1}< 0}} \mathrm{conv}\{0,v_i,v_{i+1}\}$$

and $v_1,\dots,v_n$ are the vertices of $P(h_1,\dots,h_n)$, see Figure 2. In particular, we have $B \subset A$, so the moments of $P(h_1,\dots,h_n)$ are equal to the sum

$$\mu_{kl}(P(h_1,\dots,h_n)) = \sum_{i=1}^{n} \mathrm{sign}(h_{i+1})\mu_{kl}(\mathrm{conv}\{0,v_i,v_{i+1}\}). \tag{6.3}$$

For $i = 1,\dots,n$, let $\overline{u}_i := (-s_i, c_i)^\top$. Then there exist unique $t_i, \bar{t}_i \in \mathbb{R}$ with

$$v_i = h_i u_i + t_i \overline{u}_i = h_{i+1}u_{i+1} - \bar{t}_i \overline{u}_{i+1}. \tag{6.4}$$

This implies that

$$(\overline{u_i}, \overline{u_{i+1}}) \begin{pmatrix} t_i \\ \bar{t}_i \end{pmatrix} = h_{i+1}u_{i+1} - h_i u_i,$$

and thus

$$\begin{aligned}\begin{pmatrix} t_i \\ \bar{t}_i \end{pmatrix} &= \frac{1}{-s_i c_{i+1} + c_i s_{i+1}} \begin{pmatrix} c_{i+1} & s_{i+1} \\ -c_i & -s_i \end{pmatrix} \begin{pmatrix} h_{i+1}c_{i+1} - h_i c_i \\ h_{i+1}s_{i+1} - h_i s_i \end{pmatrix} \\ &= \frac{1}{-s_i c_{i+1} + c_i s_{i+1}} \begin{pmatrix} h_{i+1} - h_i(c_i c_{i+1} + s_i s_{i+1}) \\ h_i - h_{i+1}(c_i c_{i+1} + s_i s_{i+1}) \end{pmatrix}.\end{aligned}$$

Substituting this expression of $(t_i, \bar{t}_i)^\top$ into (6.4), the vertex v_i can be expressed by (h_i, h_{i+1}) and (u_i, u_{i+1}). We obtain that

$$v_i = \frac{1}{c_i s_{i+1} - s_i c_{i+1}} \begin{pmatrix} h_i s_{i+1} - h_{i+1} s_i \\ h_{i+1} c_i - h_i c_{i+1} \end{pmatrix}. \tag{6.5}$$

Now define $T_i(x_1, x_2) := \begin{pmatrix} v_i, v_{i+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v_{i,1}x_1 + v_{i+1,1}x_2 \\ v_{i,2}x_1 + v_{i+1,2}x_2 \end{pmatrix}$. Integration by substitution then yields that

$$\begin{aligned} &\mu_{kl}(\mathrm{conv}\{0, v_i, v_{i+1}\}) \\ &\quad = \int_{\mathrm{conv}\{0,v_i,v_{i+1}\}} x_1^k x_2^l \, d(x_1, x_2) \\ &\quad = \int_{\mathrm{conv}\{0,e_1,e_2\}} (v_{i,1}x_1 + v_{i+1,1}x_2)^k (v_{i,2}x_1 + v_{i+1,2}x_2)^l \\ &\qquad\qquad \times |v_{i,1}v_{i+1,2} - v_{i,2}v_{i+1,1}| \, d(x_1, x_2). \end{aligned}$$

Using (6.4), the Jacobian determinant of T_i can be expressed as $h_{i+1}(\bar{t}_i + t_{i+1})$, and since $\bar{t}_i + t_{i+1}$ is the length of the facet of $P(h_1, \ldots, h_n)$ bounded by v_i and v_{i+1}, it follows that

$$\mathrm{sign}(v_{i,1}v_{i+1,2} - v_{i,2}v_{i+1,1}) = \mathrm{sign}(h_{i+1}).$$

This implies that

$$\begin{aligned} &\mathrm{sign}(h_{i+1})\mu_{kl}(\mathrm{conv}\{0, v_i, v_{i+1}\}) \\ &\quad = \int_0^1 \int_0^{x_2} (v_{i,1}x_1 + v_{i+1,1}x_2)^k (v_{i,2}x_1 + v_{i+1,2}x_2)^l (v_{i,1}v_{i+1,2} - v_{i,2}v_{i+1,1}) dx_1 dx_2 \\ &\quad = \int_0^1 \int_0^{x_2} \sum_{p=0}^{k} \sum_{q=0}^{l} \binom{k}{p} \binom{l}{q} v_{i,1}^p v_{i+1,1}^{k-p} v_{i,2}^q v_{i+1,2}^{l-q} x_1^{p+q} x_2^{k+l-p-q} \\ &\qquad\qquad \times (v_{i,1}v_{i+1,2} - v_{i,2}v_{i+1,1}) dx_1 dx_2 \\ &\quad = \sum_{p=0}^{k} \sum_{q=0}^{l} \binom{k}{p} \binom{l}{q} v_{i,1}^p v_{i+1,1}^{k-p} v_{i,2}^q v_{i+1,2}^{l-q} \\ &\qquad\qquad \times \frac{1}{(p+q+1)(k+l+2)} (v_{i,1}v_{i+1,2} - v_{i,2}v_{i+1,1}) \\ &\quad = \sum_{p=0}^{k} \sum_{q=0}^{l} \sum_{q_1=0}^{p+q+1} \sum_{q_2=p+q+1-q_1}^{k+l+2-q_1} \tilde{M}_{kl}(i, q_1, q_2) h_i^{q_1} h_{i+1}^{q_2} h_{i+2}^{k+l+2-q_1-q_2} \\ &\quad = \sum_{q_1=0}^{k+l+1} \sum_{q_2=0}^{k+l+2-q_1} M_{kl}(i, q_1, q_2) h_i^{q_1} h_{i+1}^{q_2} h_{i+2}^{k+l+2-q_1-q_2}, \end{aligned}$$

where the constants $\tilde{M}_{kl}(i, q_1, q_2)$ and $M_{kl}(i, q_1, q_2)$ can be derived using (6.5). In combination with (6.3), this yields (6.2). Furthermore, we obtain from formula (4.1) for the Legendre moments that

$$\begin{aligned}
\lambda_{kl}(P(h_1,\dots,h_n)) &= \sum_{p=0}^{k}\sum_{q=0}^{l} C_{kp}C_{lq}\mu_{pq}(P(h_1,\dots,h_n)) \\
&= \sum_{s=0}^{k+l}\sum_{q=s-k\vee 0}^{s\wedge l} C_{k,s-q}C_{lq}\sum_{i=1}^{n}\sum_{q_1=0}^{s+1}\sum_{q_2=0}^{s+2-q_1} M_{s-q,q}(i,q_1,q_2)h_i^{q_1}h_{i+1}^{q_2}h_{i+2}^{s+2-q_1-q_2} \\
&= \sum_{i=1}^{n}\sum_{s=0}^{k+l}\sum_{q_1=0}^{s+1}\sum_{q_2=0}^{s+2-q_1} L_{kl}(i,s,q_1,q_2)h_i^{q_1}h_{i+1}^{q_2}h_{i+2}^{s+2-q_1-q_2},
\end{aligned}$$

where

$$L_{kl}(i,s,q_1,q_2) := \sum_{q=s-k\vee 0}^{s\wedge l} C_{k,s-q}C_{lq}M_{i,s-q,q}(q_1,q_2).$$

□

The structure of $\mathcal{P}(\theta_1,\dots,\theta_n)$ ensures that a least squares estimator can be reconstructed using polynomial optimization. This follows as Lemma 6.1 yields that $\hat{P}_{N,n} = P(\hat{h}_1,\dots,\hat{h}_n)$ is a least squares estimator of K, where $(\hat{h}_1,\dots,\hat{h}_n)$ is the solution of the polynomial optimization problem

$$(\hat{h}_1,\dots,\hat{h}_n) = \operatorname{argmin}\{f(h_1,\dots,h_n) : (h_1,\dots,h_n)\in A_n\} \tag{6.6}$$

where the objective function $f : (-\infty,\infty)^n \to [0,\infty)$ is defined by

$$f(h_1,\dots,h_n) = \sum_{k,l=0}^{N}\Big(\lambda_{kl}(K) - \sum_{i=1}^{n}\sum_{s=0}^{k+l}\sum_{q_1=0}^{s+1}\sum_{q_2=0}^{s+2-q_1} L_{kl}(i,s,q_1,q_2)h_i^{q_1}h_{i+1}^{q_2}h_{i+2}^{s+2-q_1-q_2}\Big)^2$$

and the feasible set A_n is the set of vectors $(h_1,\dots,h_n)\in(-\infty,\infty)^n$ which fulfil the inequalities

$$\begin{aligned}
&0 \le h_{i-1}(s_{i+1}c_i - c_{i+1}s_i) - h_i(s_{i+1}c_{i-1} - c_{i+1}s_{i-1}) + h_{i+1}(s_ic_{i-1} - c_is_{i-1}), \\
&0 \le \frac{1}{c_is_{i+1} - s_ic_{i+1}}(h_is_{i+1} - h_{i+1}s_i) \le 1, \\
&0 \le \frac{1}{c_is_{i+1} - s_ic_{i+1}}(h_{i+1}c_i - h_ic_{i+1}) \le 1
\end{aligned}$$

for $1 \le i \le n$. Algorithms for solving polynomial optimization problems like (6.6) have been developed only recently. We mention the software GloptiPoly, see [14], which is recommended for small-scale problems. Another possible choice for solving a problem like (6.6) seems to be the software SparsePop, see [25], which is designed for problems with a special sparse structure.

6.2 Convergence of the reconstruction algorithm

In this section, we use the stability result Theorem 4.3 to show that the output polygon of the reconstruction algorithm described in the previous section converges to K in the Nikodym distance when the number n of outer normals of the polygon and the number N of moments increase.

Lemma 6.2. *Let $K \subset [0,1]^2$ be a convex body, $0 \le \theta_1 < \cdots < \theta_n < 2\pi$ and $\theta_{n+1} = \theta_1$. Assume that condition (6.1) is satisfied. Then*

$$\begin{aligned}\delta_N\big(P(h_K(\theta_1),\ldots,h_K(\theta_n)),K\big) &\le \frac{1}{\sqrt{2}} V_1(K) \max_{1\le i\le n} \tan\left(\frac{\theta_{i+1}-\theta_i}{2}\right)\\ &\le \sqrt{2} \max_{1\le i\le n} \tan\left(\frac{\theta_{i+1}-\theta_i}{2}\right).\end{aligned}$$

Proof. Choose $x_1,\ldots,x_n \in \partial K$ such that $(\cos(\theta_i),\sin(\theta_i))^\top$ is an outer normal of K in x_i. Let

$$P_{in} := \mathrm{conv}\{x_1,\ldots,x_n\}.$$

Note that $P_{in} \subset K$. Recall that the vertices of $P(h_K(\theta_1),\ldots,h_K(\theta_n))$ are denoted by $v_1,\ldots,v_n$ and let $T_i := \mathrm{conv}\{x_i,x_{i+1},v_i\}$, $c_i := \|x_{i+1}-x_i\|$ and $\gamma_i := \pi - (\theta_{i+1}-\theta_i)$. Then, it holds obviously

$$P(h_K(\theta_1),\ldots,h_K(\theta_n)) \setminus \mathrm{int}\, P_{in} = \bigcup_{i=1}^n T_i, \tag{6.7}$$

see Figure 3. Observe that the area of a triangle where one angle and the length of

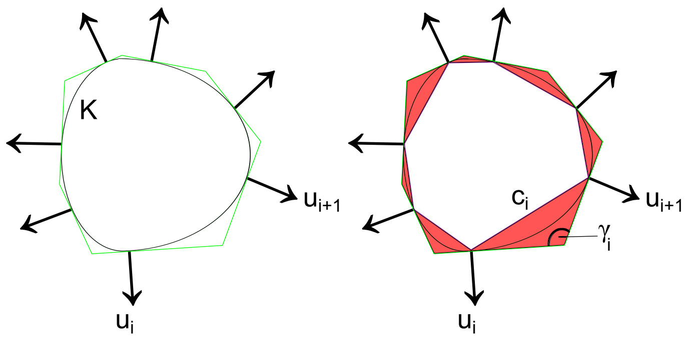

Figure 3: On the left, K and the polytope $P(h_K(u_1),\ldots,h_K(u_n))$. On the right, $P(h_K(u_1),\ldots,h_K(u_n)) \setminus P_{in}$ coloured in red.

the side opposite to the angle are prescribed is maximal if the remaining angles are equal. Thus,

$$V_2(T_i) \le \tfrac{1}{4}c_i^2 \cot(\gamma_i/2) = \tfrac{1}{4}c_i^2 \tan\left(\frac{\theta_{i+1}-\theta_i}{2}\right). \tag{6.8}$$

Equations (6.7) and (6.8) imply that

$$\begin{aligned}\delta_N\big(P(h_K(\theta_1),\ldots,h_K(\theta_n)),K\big) &\le V_2(P(h_K(\theta_1),\ldots,h_K(\theta_n)) \setminus P_{in})\\ &\le \tfrac{1}{4} \max_{1\le i\le n} \tan\left(\frac{\theta_{i+1}-\theta_i}{2}\right) \sum_{i=1}^n c_i^2,\end{aligned}$$

and since $c_i^2/2 \le c_i/\sqrt{2}$ and $\sum_{i=1}^n c_i = 2V_1(P_{in})$, we arrive at

$$\delta_N\big(P(h_K(\theta_1),\ldots,h_K(\theta_n)),K\big) \le \frac{1}{\sqrt{2}} V_1(P_{in}) \max_{1\le i\le n} \tan\left(\frac{\theta_{i+1}-\theta_i}{2}\right).$$

The monotonicity of intrinsic volumes with respect to set inclusion then yields the assertion. □

Theorem 6.3. *Let $K \subset [0,1]^2$ be a convex body, $0 \le \theta_1 < \cdots < \theta_n < 2\pi$, $\theta_{n+1} = \theta_1$ and assume that $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \in \{\theta_1, \ldots, \theta_n\}$. Any least squares estimator $\hat{P}_{N,n}$ of K on $\mathcal{P}(\theta_1, \ldots, \theta_n)$ satisfies that*

$$\delta_N(\hat{P}_{N,n}, K) \le \sqrt{2} \max_{1 \le i \le n} \tan\left(\frac{\theta_{i+1} - \theta_i}{2}\right) + \frac{a_1}{N+1},$$

where $a_1 > 0$ is a constant.

Proof. Since $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \in \{\theta_1, \ldots, \theta_n\}$ and $K \subset [0,1]^2$ it follows that

$$P(h_K(\theta_1), \ldots, h_K(\theta_n)) \subset [0,1]^2$$

and thus $P(h_K(\theta_1), \ldots, h_K(\theta_n)) \in \mathcal{P}(\theta_1, \ldots, \theta_n)$. Then the definition of $\hat{P}_{N,n}$ and Parseval's identity yield that

$$\begin{aligned}\sum_{i,j=0}^{N} (\lambda_{ij}(K) - \lambda_{ij}(\hat{P}_{N,n}))^2 &\le \sum_{i,j=0}^{N} [\lambda_{ij}(K) - \lambda_{ij}(P(h_K(\theta_1), \ldots, h_K(\theta_n)))]^2 \\ &\le \delta_N(P(h_K(\theta_1), \ldots, h_K(\theta_n)), K).\end{aligned}$$

Thus, an application of Lemma 6.2 implies that

$$\sum_{i,j=0}^{N} (\lambda_{ij}(K) - \lambda_{ij}(\hat{P}_{N,n}))^2 \le \sqrt{2} \max_{1 \le i \le n} \tan\left(\frac{\theta_{i+1} - \theta_i}{2}\right). \tag{6.9}$$

Then the result follows from Theorem 4.3 and Remark 4.5. □

Remark 6.4. If we choose $n = 4m$ for some $m \in \mathbb{N}$ and equidistant angles $\theta_i := 2\pi\left(\frac{i-1}{n}\right)$ for $1 \le i \le n$, then $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \in \{\theta_1, \ldots, \theta_n\}$ and we obtain

$$\sqrt{2} \max_{1 \le i \le n} \tan\left(\frac{\theta_{i+1} - \theta_i}{2}\right) \approx \frac{\sqrt{2}\pi}{n} \approx \begin{cases} 0.05, & n = 100, \\ 0.005, & n = 1000, \\ 0.0025, & n = 2000. \end{cases}$$

In the following, we write $\theta_{(1)}, \ldots, \theta_{(n)}$ for a permutation of $\theta_i \in [0, 2\pi), 1 \le i \le n$ satisfying $\theta_{(1)} \le \cdots \le \theta_{(n)}$. From Theorem 6.3, we then obtain Corollary 6.5.

Corollary 6.5. *Let $K \subset [0,1]^2$ be a convex body and let $(\theta_i)_{i \in \mathbb{N}}$ be a dense sequence in $[0, 2\pi)$ such that $\theta_i \ne \theta_j$ for $i \ne j$ and $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, \frac{\pi}{2}, \pi, \frac{3\pi}{2})$. For $n, N \in \mathbb{N}$, let $\hat{P}_{N,n}$ be a least squares estimator of K with respect to the $(N+1)^2$ first Legendre moments on the space $\mathcal{P}(\theta_{(1)}, \ldots, \theta_{(n)})$. Then*

$$\delta_N(K, \hat{P}_{N,n}) \to 0 \qquad \text{for } n, N \to \infty.$$

6.3 Reconstruction from noisy measurements

The reconstruction algorithm described in Section 6.1 requires knowledge of exact Legendre moments of a convex body. The reconstruction algorithm can be modified such that it allows for noisy measurements of Legendre moments. Let $N \in \mathbb{N}_0$, and assume that $K \subset [0,1]^2$ is a convex body where measurements of the first $(N+1)^2$ Legendre moments are known. To include noise, we assume that the measurements are of the form

$$\tilde{\lambda}_{kl}(K) = \lambda_{kl}(K) + \epsilon_{Nkl} \tag{6.10}$$

for $k,l = 0,\dots,N$, where ϵ_{Nkl}, $k,l = 0,\dots,N$ are random variables with zero means and finite variances bounded by a constant σ_N^2. Let $0 \le \theta_1 < \dots < \theta_n < 2\pi$ satisfy condition (6.1). Any polygon $\tilde{P}_{N,n} \in \mathcal{P}(\theta_1,\dots,\theta_n)$ satisfying

$$\tilde{P}_{N,n} = \operatorname{argmin}\Big\{ \sum_{k,l=0}^{N} (\tilde{\lambda}_{kl}(K) - \lambda_{kl}(P))^2 : \quad P \in \mathcal{P}(\theta_1,\dots,\theta_n)\Big\}$$

is called a least squares estimator of K with respect to the measurements (6.10) on the space $\mathcal{P}(\theta_1,\dots,\theta_n)$. As the set $\mathcal{P}(\theta_1,\dots,\theta_n)$ is closed in the Hausdorff metric, Blaschke's selection theorem ensures the existence of a least squares estimator.

As in Section 6.1, a least squares estimator can be found using polynomial optimization. Let $(\tilde{h}_1,\dots,\tilde{h}_n)$ be a solution to the polynomial optimization problem (6.6) with the Legendre moments $\lambda_{kl}(K)$ of K replaced by the measurements $\tilde{\lambda}_{kl}(K)$ of the Legendre moments in the objective function f. Then $P(\tilde{h}_1,\dots,\tilde{h}_n) \in \mathcal{P}(\theta_1,\dots,\theta_n)$ is a least squares estimator of K with respect to the measurements (6.10).

Now, let $\mathbb{P}_{N,n}(\epsilon)$ denote the random set of least squares estimators of K with respect to the measurements (6.10) on the space $\mathcal{P}(\theta_1,\dots,\theta_n)$. When the noise variables are defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, it follows by arguments as in [11, p. 27] (see also [19, App. C]) that $\sup_{P\in\mathbb{P}_{N,n}(\epsilon)} \delta_N(K,P)$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable. We can then formulate the following theorem, which ensures consistency of the reconstruction algorithm under certain assumptions on the variances of the noise variables.

Theorem 6.6. *Let $(\theta_i)_{i\in\mathbb{N}}$ be a dense sequence in $[0,2\pi)$ such that $\theta_i \neq \theta_j$ for $i \neq j$ and $(\theta_1,\theta_2,\theta_3,\theta_4) = (0,\frac{\pi}{2},\pi,\frac{3\pi}{2})$.*

- *(i) If $\sigma_N^2 = \mathcal{O}(\frac{1}{N^{2+\varepsilon}})$ for some $\varepsilon > 0$, then $\sup_{P\in\mathbb{P}_{N,n}(\epsilon)} \delta_N(K,P) \to 0$ in mean and in probability for $n,N \to \infty$.*
- *(ii) If $\sigma_N^2 = \mathcal{O}(\frac{1}{N^{3+\varepsilon}})$ for some $\varepsilon > 0$, then $\sup_{P\in\mathbb{P}_{N,n}(\epsilon)} \delta_N(K,P) \to 0$ almost surely for $n,N \to \infty$.*

Proof. Let $P \in \mathbb{P}_{N,n}(\epsilon)$ and let $\hat{P}_{N,n} \in \mathcal{P}(\theta_{(1)},\dots,\theta_{(n)})$ denote a least squares estimator of K with respect to the exact Legendre moments. By using the inequality

$(x+y)^2 \le 2(x^2+y^2)$ for $x, y \in \mathbb{R}$ and properties of P and $\hat{P}_{N,n}$, we obtain that

$$\begin{aligned}\sum_{k,l=0}^{N}(\lambda_{kl}(K)-\lambda_{kl}(P))^2 &\le 2\sum_{k,l=0}^{N}\big((\tilde{\lambda}_{kl}(K)-\lambda_{kl}(P))^2+\epsilon_{Nkl}^2\big)\\ &\le 2\sum_{k,l=0}^{N}(\tilde{\lambda}_{kl}(K)-\lambda_{kl}(\hat{P}_{N,n}))^2+2\sum_{k,l=0}^{N}\epsilon_{Nkl}^2\\ &\le 4\sum_{k,l=0}^{N}(\lambda_{kl}(K)-\lambda_{kl}(\hat{P}_{N,n}))^2+6\sum_{k,l=0}^{N}\epsilon_{Nkl}^2.\end{aligned}$$

Using the upper bound (6.9) on $\sum_{k,l=0}^{N}(\lambda_{kl}(K)-\lambda_{kl}(\hat{P}_{N,n}))^2$ derived in the proof of Theorem 6.3, we arrive at

$$\sum_{k,l=0}^{N}(\lambda_{kl}(K)-\lambda_{kl}(P))^2 \le 4\sqrt{2}\max_{1\le i\le n}\left|\tan\left(\frac{\theta_{(i)}-\theta_{(i+1)}}{2}\right)\right|+6\sum_{k,l=0}^{N}\epsilon_{Nkl}^2,$$

where $\theta_{(1)} < \cdots < \theta_{(n)}$ is an ordering of $\theta_1, \dots, \theta_n$ and $\theta_{(n+1)} := \theta_1$. In the notation, we suppress that the ordering depends on n. Then it follows from Theorem 4.3 and Remark 4.5 that

$$\sup_{P\in\mathbb{P}_{N,n}(\epsilon)}\delta_N(K,P) \le 4\sqrt{2}\max_{1\le i\le n}\left|\tan\left(\frac{\theta_{(i)}-\theta_{(i+1)}}{2}\right)\right|+6\sum_{k,l=0}^{N}\epsilon_{Nkl}^2+\frac{a_1}{N+1}.$$

The mean of the sum of the squared error terms are bounded by $(N+1)^2\sigma_N^2$, and the assumption that $\sigma_N^2 = \mathcal{O}(\frac{1}{N^{2+\varepsilon}})$ ensures that $(N+1)^2\sigma_N^2 \to 0$ for $N \to \infty$. As the sequence $(\theta_i)_{i\in\mathbb{N}}$ is dense in $[0, 2\pi)$, we further have that

$$\max_{1\le i\le n}\left|\tan\left(\frac{\theta_{(i)}-\theta_{(i+1)}}{2}\right)\right| \to 0$$

for $n \to \infty$. Hence, $\sup_{P\in\mathbb{P}_{N,n}(\epsilon)}\delta_N(K,P) \to 0$ in mean and in probability for $n, N \to \infty$.

If $\sigma_N^2 = \mathcal{O}(\frac{1}{N^{3+\varepsilon}})$, then $\sum_{N=0}^{\infty}(N+1)^2\sigma_N^2 < \infty$, which ensures that $\sum_{k,l=0}^{N}\epsilon_{Nkl}^2 \to 0$ almost surely for $N \to \infty$. Then, $\sup_{P\in\mathbb{P}_{N,n}(\epsilon)}\delta_N(K,P) \to 0$ almost surely for $N, n \to \infty$. □

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