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Abstract

An insurance company offers an insurance contract (p, K), consisting of a premium p and a deductible K. In this paper we consider the problem of choosing the premium optimally as a function of the deductible. The insurance company is facing a market of N customers, each characterized by their personal claim frequency, α, and risk aversion, β. When a customer is offered an insurance contract, she will based on these characteristics choose whether or not to insure. The decision process of the customer is analyzed in details. Since the customer characteristics are unknown to the company, it models them as iid random variables; $A_1, \ldots, A_N$ for the claim frequencies and $B_1, \ldots, B_N$ for the risk aversions. Depending on the distributions of A_i and B_i, expressions for the portfolio size $n(p; K) \in [0, N]$ and average claim frequency $\alpha(p; K)$ in the portfolio are obtained. Knowing these, the company can choose the premium optimally, mainly by minimizing the ruin probability.

Keywords: microeconomic insurance; customer characteristics; portfolio size; average claim frequency; ruin theory

1 Introduction.

An insurance company has several instruments for stochastic control at its disposal. Much studied are dividends, reinsurance and investment, see e.g. (Schmidli, 2007). The present paper concentrates on premiums and deductibles. These are also obvious instruments but have been somewhat less studied in the literature as such.

The standard model for the risk reserve R_t at time t is the Cramer-Lundberg process

$$R_t = r_0 + ct - A_t,$$

where r_0 is the initial reserve, c is the gross premium rate and $(A_t)_{t\geq 0}$ is a compound Poisson process with parameters λ and F. More explicitly, $A_t = \sum_{i=1}^{N_t} Z_i$ where $(N_t)_{t\geq 0}$ is a Poisson process counting the number of claims until time t and the Z_i's represents the (positive) claim sizes assumed to be i.i.d. and independent of $(N_t)_{t\geq 0}$, with common distribution F on $(0, \infty)$. Let $\overline{z} = \mathbb{E}[Z_i]$ and $\overline{z^2} = \mathbb{E}[Z_i^2]$. The Cramer-Lundberg process can then be approximated by the diffusion process

$$\mathrm{d}x_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \tag{1.1}$$

with $\mu = c + \lambda\overline{z}$ and $\sigma^2 = \lambda\overline{z^2}$ as in (Hipp and Taksar, 2010).

We will describe an insurance contract by a premium, p, and a deductible, K. The purpose, besides loss prevention and retention, of adding a deductible to a contract is to avoid administrating the numerous number of small claims. The deductible is therefore chosen to serve this purpose, and the premium will then be considered as a function of the chosen deductible. We will consider methods of how to find the premium optimally on a given market. For simplicity, we assume that all (potential) customers are offered the same insurance contract (p, K). We will also neglect the market effects by assuming that only one company supplies insurance.

A possible extension is to allow the insurance company to offer different deductibles, and let the premium in the contract be regulated correspondingly. A simple way of doing so is to say that the market can be divided into disjoint markets, for example; one with customers demanding a low deductible K_l, one with a medium deductible K_m, and one with a high deductible K_h. So the offered values of the deductibles are $\{K_l, K_m, K_h\}$ where there is a seperat market for each. It will then be possible to apply the same approach as considered here to each of the markets.

There are several types of deductibles. We will consider the very classical fixed amount deductible. With a fixed amount deductible the claims are truncated, so the loss for the insurance company in relation to a claim Z_i can be described by the random variable

$$X_i^{(K)} = (Z_i - K)^+ = \begin{cases} 0, & \text{if } Z_i < K, \\ Z_i - K, & \text{if } Z_i \geq K. \end{cases}$$

The risk reserve must therefore be modified

$$R_t = r_0 + ct - A_t^{(K)},$$

where $A_t^{(K)}$ is the compound Poisson process with losses $X_i^{(K)}$. Let $\overline{x^{n,(K)}}$ denote the n'th moment of $X_i^{(K)}$. Notice that $\overline{z} = \overline{x^{(0)}}$ and $\overline{z^2} = \overline{x^{2,(0)}}$, since the claims are assumed to be positive. The results here contains thus the no-deductible case.

Assume that the insurance company is facing a market consisting of N potential customers. Let $n(p; K) \in [0, N]$ denote the number of customers the insurance company attracts on the market when offering the contract (p, K). Obviously, increasing the premium should lead to a loss of customers, therefore it must be that $\frac{\partial n(p;K)}{\partial p} \leq 0$. Also, the average customer claim frequency in the portfolio will be denoted as $\alpha(p; K)$. Raising the premium will make it less attractive to insure for customers having low claim frequencies rates and so the average claim frequency of the portfolio increases, i.e. $\frac{\partial \alpha(p;K)}{\partial p} \geq 0$. This is commonly known as adverse selection. The gross premium c and the aggregate claim frequency λ will then depend on the chosen premium, p, and the given deductible, K, as follows

$$c(p; K) = n(p; K)p \quad \text{and} \quad \lambda(p; K) = n(p; K)\alpha(p; K).$$

The drift and variance of the diffusion process (1.1) can be modified accordingly

$$\begin{aligned} \mu(p; K) &= n(p; K)p - \lambda(p; K)\overline{x^{(K)}} - L = n(p; K)\big(p - \alpha(p; K)\overline{x^{(K)}}\big) - L, \\ \sigma^2(p; K) &= \lambda(p; K)\overline{x^{2,(K)}} = n(p; K)\alpha(p; K)\overline{x^{2,(K)}}. \end{aligned}$$

In order to avoid trivialities a fixed liability rate L is introduced. Otherwise, if the insurance chooses $p = \infty$ no customer will insure, and the reserve will remain constantly at r_0 yielding a ruin probability of zero. Let $\varphi(r_0)$ denote the ruin probability as a function of the initial reserve when the reserve is modelled by the diffusion process.

For a given deductible, changing the premium will have a double sided effect on the drift (and profit) of the insurance company. Raising the premium will increase the earnings per customer, but will also reduce the size of the portfolio and increase the average claim rate due to adverse selection, and vice versa for decreasing the premium. In order to say which effect is dominating, a specification of the portfolio characteristics is needed.

First of all, we need to gain insight into the decision process of a customer. In this context we introduce a risk aversion parameter β and motivate a method for incorporating risk aversion. For a microeconomic perspective of insurance, we refer to (Rees and Wambach, 2008).

In a naive setting the customers would have the same claim rate, implying the average claim frequency to be constant, $\alpha(p; K) = \alpha$, and $n(p; K)$ could be chosen on some ad hoc form. In a more realistic setting a market of potential customers is non-homogeneous in the sense that they have different characteristics, namely different α's and β's. A customer knows her own claim rate, but the insurer does not possess this information about the customers. The claim rates are therefore modelled as iid random variables $A_1, \ldots, A_N$ over the portfolio. Likewise for the risk aversion parameter, which will be considered as outcomes of the iid random variables $B_1, \ldots, B_N$. This results in some less ad hoc forms for the functions $\alpha(p; K)$ and $n(p; K)$ to characterize the portfolio. Once the portfolio characteristics are known, the insurance company can use these to choose the premium optimally. The main optimization problem considered is minimizing the ruin probability, but this only makes sense in case of a positive drift in the diffusion (1.1). In case of a negative drift, ruin is certain, so the premium will be chosen to maximize the time to ruin. For more background on ruin theory, we refer to (Asmussen and Albrecher, 2010).

The ideas here are very similar to the ones considered in Asmussen *et al.* (2013). The present paper takes however a different approach to risk aversion and, as said, incorporates deductibles. Burnecki *et al.* (2004) also finds the price of an insurance as a function of the deductible for different types of deductibles, though not exploiting the aspects of risk aversion. Højgaard (2002) controls the gross premium indirectly by controlling the safety loading. All in all, the contributions of this paper is threefold; analysing the customer's behaviour in Section 2, finding portfolio characteristics in Section 3, and choosing the optimal premium for the insurance company in Section 4. For examples and illustrations, see Section 5.

2 Customer's problem.

A potential customer has to make a decision on whether to insure or not. If the customer chooses to insure, she must pay a premium p at constant rate, but the customer can then report claims and get the amount above the threshold K covered. More specifically, if the customer experience a loss Z_i, she will only report it to the

insurance company if $Z_i \geq K$ in which case she has to pay K herself, otherwise if $Z_i < K$ there is no purpose of reporting it and she will have to cover the entire loss Z_i. On the other hand, if the customer chooses not to insure, she no longer has to pay p continuously. She will instead have to cover all the uninsured losses by herself.

For the moment risk aversion is ignored, and the decision is made solely by comparing the present values of the wealth generated by the two options. Later risk aversion will be incorporated by pricing the excess uncertainty when not insuring using the variance premium principle. Let V_i denote the present value of insuring and V_n of not insuring.

The customer is assumed to have infinite life length, a subjective discount rate d, and access to a risk free asset with interest rate r in which all her wealth is assumed to be invested. Also, it is assumed that $d > r$ to ensure finite asset valuation. Finally, the customer is characterised by initial wealth w_0 and claim frequency α. The problem of the customer is identical every period, and the decision will therefore not change over time.

We start by finding the present value of not insuring. As said, if the customer chooses to insure she will have to cover every loss herself. So her wealth will develope according to

$$\mathrm{d}w_t = rw_t\mathrm{d}t - \mathrm{d}A_t^\alpha, \tag{2.1}$$

where $(A_t^\alpha)_{t\geq 0}$ is a compound Poisson process with parameter α and F, representing the total amount of losses at time t for the potential customer. Notice that (2.1) is an Ornstein-Uhlenbeck process driven by a Levy process (namely, the compound Poisson process) and the solution is therefore explicitly known as

$$w_t = \exp(rt)\Big(w_0 - \int_0^t \exp(-rs)\mathrm{d}A_s^\alpha\Big) = \exp(rt)\Big(w_0 - \sum_{i=1}^{N_t^\alpha}\exp(-rT_i)Z_i\Big).$$

The present value of the wealth when not insuring is evaluated as

$$V_n = \mathbb{E}\Big[\int_0^\infty \exp(-dt)\mathrm{d}w_t\Big].$$

Calculations, given in Appendix A, show that this can be reduced to

$$V_n = \frac{rw_0}{d-r} - \frac{\overline{z}\alpha}{d-r}.$$

If the customer chooses to insure she will have to pay a premium continuously and cover the parts of the claims below the deductible. Her wealth will then have the following dynamics,

$$\mathrm{d}w_t = (rw_t - p)\mathrm{d}t - \mathrm{d}A_t^{\alpha,K}, \tag{2.2}$$

where $(A_t^{\alpha,K})_{t\geq 0}$ is the compound Poisson process representing the total loss associated with claims until time t for the customer when insuring, that is

$$A_t^{\alpha,K} = \sum_{i=1}^{N_t^\alpha}\min\{Z_i, K\} = \sum_{i=1}^{N_t^\alpha}\left(K\mathbb{1}_{Z_i\geq K} + Z_i\mathbb{1}_{Z_i<K}\right).$$

Once again, we are looking at an Ornstein-Uhlenbeck process driven by a Levy process. This can be solved in the same way as seen previously,

$$\begin{aligned} w_t &= \exp(rt)\Big(w_0 - \int_0^t \exp(-rs)p\mathrm{d}t - \int_0^t \exp(-rs)\mathrm{d}A_s^{\alpha,K}\Big) \\ &= \exp(rt)\left(w_0 - \frac{p}{r}\right) + \frac{p}{r} - \exp(rt)\sum_{i=1}^{N_t^\alpha} \exp(-rT_i)(K\mathbb{1}_{\{Z_i \geq K\}} + Z_i\mathbb{1}_{Z_i<K}). \end{aligned}$$

The present value of the wealth when insuring will therefore be

$$V_i = \mathbb{E}\Big[\int_0^\infty \exp(-dt)\mathrm{d}w_t\Big] = \frac{rw_0 - p}{d-r} - \frac{\alpha}{d-r}\mathbb{E}[K\mathbb{1}_{\{Z_i \geq K\}} + Z_i\mathbb{1}_{\{Z_i<K\}})].$$

Since the approach and calculations are very similar to the ones used when finding V_n, details are skipped. We can now state the following.

Corollary 1. *Disregarding risk aversion, the customer will insure if and only if $V_i \geq V_n$, which is equivalent to*

$$p \leq \alpha\mathbb{E}[(Z_i - K)\mathbb{1}_{\{Z_i \geq K\}}] = \alpha\mathbb{E}[X_i^{(K)}] = \alpha\overline{x^{(K)}}$$

Hence, a risk neutral customer will insure if the net premium (taking deductibles into account) exceeds the premium. She has no incentive to pay a loading in order to avoid the risk.

Of course, in order for insurance to make sense, the customer must have some degree of risk aversion. We want to find which excess risk the customer is exposed to when not insuring, and how to price this risk when risk aversion is essential.

First step is to notice that V_n also could have been derived in a more intuitive way. Recall that all wealth is assumed to be invested in the risk free asset. So wealth itself has the dynamics $\mathrm{d}w_t = rw_t\mathrm{d}t$ of a bank account. When accumulating interest, this has present value

$$\int_0^\infty \exp(-dt)\mathrm{d}w_t = \int_0^\infty \exp(-(d-r)t)rw_0dt = \frac{rw_0}{d-r}. \tag{2.3}$$

When a non-insured customer has to pay a loss Z_i at time T_i, she also loses a possible interest rate income. So the total loss of paying Z_i at time T_i can be calculated using the formula (2.3) with the discounted loss $\exp(-rT_i)Z_i$ as w_0. Hence,

$$V_n = \frac{rw_0}{d-r} - \frac{r}{d-r}\mathbb{E}\Big[\sum_{i=1}^\infty \exp(-rT_i)Z_i\Big].$$

Note that due to Campbell-Mecke's formula (see e.g. Van Lieshout (2000) for further details), this is in fact equal to the expression for V_n found previously. Also, we have the following alternative characterization of V_i,

$$V_i = \frac{rw_0 - p}{d-r} - \frac{r}{d-r}\mathbb{E}\Big[\sum_{i=1}^\infty \exp(-rT_i)(K\mathbb{1}_{\{Z_i \geq K\}} + Z_i\mathbb{1}_{\{Z_i<K\}})\Big]$$

which could have been explained with a similar intuitive approach using truncated losses.

The criteria of insuring in Corollary 1 when disregarding risk aversion can then be expressed as the inequality

$$p \leq \mathbb{E}\Big[r \sum_{i=1}^{\infty} \exp(-rT_i)(Z_i - K)\mathbb{1}_{\{Z_i \geq K\}}\Big].$$

The additional risk the customer is exposed to when not insuring can therefore be captured by the random variable

$$S^{(K)} = r \sum_{i=1}^{\infty} \exp(-rT_i)(Z_i - K)\mathbb{1}_{\{Z_i \geq K\}} = r \sum_{i=1}^{\infty} \exp(-rT_i)X_i^{(K)}.$$

Next step is to find the maximum premium the risk averse customer is willing to pay, also called her reservation price for this risk. Inspired by microeconomics one can let the customer's preferences be represented by a concave utility function $u(\cdot)$. Assume that the only risk the customer cares about when pricing is the additional risk $S^{(K)}$ she is carrying when not insuring. The losses less than or equal to the deductible is a risk that the customer is also facing, but cannot be insured against, see Remark 4 for further details. Inspired by (Gerber and Pafum, 1998), the maximal premium $P(K)$ the customer is willing pay for an insurance of a risk $S^{(K)}$ is the solution to the equation

$$\mathbb{E}[u(w - S^{(K)})] = u(w - P(K)). \tag{2.4}$$

So the customer's reservation price satisfies that she is indifferent between carrying the risk $S^{(K)}$ and paying the premium $P(K)$.

A possible choice is to let the customer's preferences be represented by an exponential utility function,

$$u_\gamma(x) = \frac{1}{\gamma}(1 - \exp(-\gamma x)),$$

where γ is a risk aversion parameter. The premium in (2.4) can then be solved explicitly as

$$P_{exp}(K) = \frac{1}{\gamma} \log\big(\mathbb{E}\big[\exp(\gamma S^{(K)})\big]\big).$$

This is the same premium as found in (Gerber, 1974) where it is the insurer, not the insured, pricing a risk. It is commonly known as the exponential premium principle, and is mostly used by the insurer in the literature.

The object is to obtain an analytic expression for the premium, and this is extremely complicated (if not impossible without having to make a lot of simplifying assumptions) to get for $P_{exp}(K)$. This is mainly due to $S^{(K)}$ being a sum of dependent random variables caused by the dependence structure in the T_i's. Thus, the additivity property that (Gerber, 1974) proves the exponential premium principle possesses cannot be used. To illustrate this complexity, a simple case example is presented in Appendix C. In order to obtain a more simple expression, a second order Taylor approximation is considered,

$$P_{exp}(K) \approx \mathbb{E}[S^{(K)}] + \frac{\gamma}{2}\mathbb{V}\mathrm{ar}[S^{(K)}].$$

Assessments about the premium will be based on the right hand side hereinafter. Let $\beta = \gamma/2$ and introduce the notation

$$P_{var}(K) = \mathbb{E}[S^{(K)}] + \beta \mathbb{V}\text{ar}[S^{(K)}]. \tag{2.5}$$

This the well-known variance premium principle. It expands the net premium principle by adding a risk loading that is proportional by a factor $\beta > 0$ to the variance of the risk. So the customer is now further characterized by the risk aversion coefficient β. Approximating the exponential premium principle by the variance premium principle is a common approach, see for example (Gerber and Pafum, 1998).

It is indeed possible to find an analytic expression for (2.5). Applying Campbell-Mecke's formula, it appears that the expectation term simply is

$$\mathbb{E}[S^{(K)}] = \alpha \overline{x^{(K)}}.$$

The variance term in (2.5) is calculated using the Total Law of Variance in Appendix B, giving the result

$$\mathbb{V}\text{ar}[S^{(K)}] = \frac{r\overline{x^{2,(K)}}\alpha}{2}.$$

The concluding premium is summarized below.

Corollary 2. *Given a deductible K the customer is facing the excess risk $S^{(K)} = r\sum_{i=1}^{\infty} \exp(-rT_i)X_i^{(K)}$ when not insuring and is willing to pay the following price for an insurance*

$$P_{var}(K) = \alpha\overline{x^{(K)}} + \beta\left(\frac{r\overline{x^{2,(K)}}\alpha}{2}\right) = \left(1 + \beta\frac{r\overline{x^{2,(K)}}}{2\overline{x^{(K)}}}\right)\alpha\overline{x^{(K)}}. \tag{2.6}$$

In Example 8, the premium (2.6) is calculated explicitly for lognormally distributed claim sizes.

Remark 3. The approach leads to a fairly classical premium calculation principle, which in the literature is mostly seen from the insurer's perspective. This motivates the use of other already developed premium calculation principles applied from the customer's point of view. An example of such is the very similar standard deviation principle, where the premium depends on the mean and the standard deviation of the risk in a linear structure,

$$P_{\text{std}}(K) = \mathbb{E}[S^{(K)}] + \beta\sqrt{\mathbb{V}\text{ar}[S^{(K)}]}.$$

Since we already have expressions for the mean and the variance, we can write this more explicitly as

$$P_{\text{std}}(K) = \alpha\overline{x^{(K)}} + \beta\sqrt{\left(\frac{r\overline{x^{2,(K)}}\alpha}{2}\right)}.$$

Another could simply be the pure premium principle with a safety loading depending on the individual's risk aversion

$$P_{\text{pure}}(K) = (1 + \omega(\beta))\mathbb{E}[S] = (1 + \omega(\beta))\alpha\overline{x^{(K)}}.$$

The latter gives a simple expression, even though it still has a nice intuitive interpretation. All evaluations in the following will be based on the variance premium principle. But do note that a similar approach can be used with other premium calculation principles.

Remark 4. The risk the customer is facing can be split into two. First of these, is the risk the customer cannot insure. This is the (part of the) losses that customer must pay regardless of insuring or not, and is the uncertainty that appears in both V_n and V_i. The monetary value at time 0 can be deduced to

$$S^{\leq K} = r \sum_{i=1}^{\infty} \exp(-rT_i)(Z_i \mathbb{1}_{\{Z_i<K\}} + K \mathbb{1}_{\{Z_i \geq K\}}).$$

Second of these, is the additional risk $S^{(K)}$ the customer can buy insurance to cover. The maximum premium that the customer is willing to pay in (2.4) will be altered as follows if she takes $S^{\leq K}$ into consideration,

$$\mathbb{E}[u(w_0 - S^{\leq K} - S^{(K)})] = \mathbb{E}[u(w - S^{\leq K} - P(K))].$$

Using the exponential utility and solving yields

$$P(K) = \frac{1}{\gamma} \log \left(\frac{\mathbb{E}[\exp(\gamma S^{\leq K} + \gamma S^{(K)})]}{\mathbb{E}[\exp(\gamma S^{\leq K})]} \right).$$

Proceeding to the Taylor approximation

$$P_{\exp}(K) \approx \mathbb{E}[S^{(K)}] + \frac{\gamma}{2} \mathbb{V}\mathrm{ar}[S^{(K)}] - \gamma \mathbb{C}\mathrm{ov}(S^{\leq K}, S^{(K)}),$$

it appears that in the final pricing formula the assumption about the customer only caring about pricing the excess risk $S^{(K)}$ regardless of $S^{\leq K}$ corresponds mathematically to assuming that $\mathbb{C}\mathrm{ov}(S^{\leq K}, S^{(K)}) \approx 0$. A similar comment is also made in (Gerber and Pafum, 1998)

3 Portfolio characteristics.

A customer is characterized by her claim frequency α and risk aversion β. As previously commented, the characteristics are most likely customer-dependent. So the insurance company consider them as random variables being iid on the market. The claim frequencies are represented by $A_1, \ldots, A_N$, and the risk aversions by $B_1, \ldots, B_N$. First, we will consider the claim frequency as being random and the risk aversion as constant. Next, we will reverse it, by modelling the risk aversion as random, and letting the claim frequency be constant. A third, more advanced, possibility is of course to let the customer characteristics (α, β) be represented by a random vectors $(A_1, B_1), \ldots, (A_N, B_N)$. This complicates the evaluations considerably, and is therefore left open by this paper.

In each case, we derive an expression for the portfolio size and for the average claim frequency in the portfolio. These expressions will become explicit functions when assuming a concrete distribution.

3.1 Stochastic claim frequencies.

Consider the first case mentioned above, where the risk aversion is constant, and the claim frequency of the customer is unknown to the insurance company, and therefore modelled by a random variable denoted by A. As said, we want to find the expected size and average claim rate of the portfolio.

From the reservation price in (2.6), it follows that a customer with characteristics (α, β) will insure if the offered insurance contract (p, K) satisfies the inequality

$$p \leq \alpha\overline{x^{(K)}} + \frac{\beta\alpha r}{2}\overline{x^{2,(K)}}.$$

Since the claim frequency is modelled by a random variable, A, to the insurer, this translates to the relation

$$A \geq \frac{2p}{2\overline{x^{(K)}} + \beta r\overline{x^{2,(K)}}}.$$

The expected portfolio size will then be the probability of this event happening multiplied by the size of the market, namely

$$n(p; K) = \mathbb{P}\left(A \geq \frac{2p}{2\overline{x^{(K)}} + \beta r\overline{x^{2,(K)}}}\right) N.$$

This is the mean demand curve as a function of the premium and deductible. In the following it is assumed that N is large and that there is a continuum of customers such that the deviation from the actual demand curve is negligible.

The average claim frequency rate in the portfolio is the expected claim frequency given that the customer chooses to insure, i.e.

$$\alpha(p; K) = \mathbb{E}\left[A \,\middle|\, A \geq \frac{2p}{2\overline{x^{(K)}} + \beta r\overline{x^{2,(K)}}}\right].$$

3.1.1 Exponentially distributed claim frequencies.

In classical empirical Bayes example of car insurance the claim frequency is assumed to be $\Gamma(s, b)$ distributed. In some cases there are empirical evidence of s being close to one. See e.g. (Bichsel, 1964) for more discussion. This motivates the assumption of an exponential distribution with parameter b of the claim frequency, A. Offering the contract (p, K) the customer will then insure with probability

$$\mathbb{P}\left(A \geq \frac{2p}{2\overline{x^{(K)}} + \beta r\overline{x^{2,(K)}}}\right) = \exp\left(-b\frac{2p}{2\overline{x^{(K)}} + \beta r\overline{x^{2,(K)}}}\right).$$

This yields a portfolio size of

$$n(p, K) = \exp\left(-b\frac{2p}{2\overline{x^{(K)}} + \beta r\overline{x^{2,(K)}}}\right) N.$$

The exponential distribution has a memoryless property, which implies $\mathbb{E}[A|A \geq a] = 1/b + a$. The expected claim frequency of an insured customer will therefore be

$$\alpha(p, K) = \frac{2p}{2\overline{x^{(K)}} + \beta r\overline{x^{2,(K)}}} + \frac{1}{b}.$$

This equation reflects adverse selection very clearly since $\alpha(p, K)$ is linearly increasing in p.

Remark 5. If the customer's reservation price was defined by the pure premium principle with a factor loading the portfolio characteristics would then be

$$n(p;K) = \mathbb{P}\left(A \leq \frac{p}{(1+\omega(\beta))\overline{x^{(K)}}}\right) N, \quad \alpha(p;K) = \mathbb{E}\left[A \,\middle|\, A \leq \frac{p}{(1+\omega(\beta))\overline{x^{(K)}}}\right].$$

Assuming an exponentially distributed claim frequency, the characteristics become

$$n(p;K) = \exp\left(-b\frac{p}{(1+\omega(\beta))\overline{x^{(K)}}}\right) N, \quad \alpha(p;K) = \frac{p}{(1+\omega(\beta))\overline{x^{(K)}}} + \frac{1}{b}.$$

This is slightly easier to work with, and has the advantage that $\omega(\beta)$ can be chosen on a simple form.

3.2 Stochastic risk aversions.

Instead of letting a customer being represented by a stochastic claim frequency and constant risk aversion, we now turn it around. Assume that the customer now has a constant claim frequency α. This is indeed relevant to consider. For example, everyone could be equally disposed to disaster caused by nature. Furthermore, assume that the risk aversion is represented by a random variable B. The criterion (2.6) of insuring for a given contract (p, K) can then be stated as

$$B \geq \frac{2p - 2\alpha\overline{x^{(K)}}}{r\overline{x^{2,(K)}}\alpha}.$$

So the portfolio size will be characterized by

$$n(p) = \mathbb{P}\left(B \geq \frac{2p - 2\alpha\overline{x^{(K)}}}{r\overline{x^{2,(K)}}\alpha}\right) N.$$

3.2.1 Gamma distributed risk aversions.

Risk aversion is somewhat an abstract concept. It is therefore difficult to suggest a distribution. Since the Gamma distribution is a popular choice for the claim frequency, we choose to consider the same distribution for the risk aversion. Assuming that the risk aversion has an $\Gamma(w, \nu)$ distribution, the demand curve will have the form

$$n(p) = \frac{\Gamma(w, \nu(2p - 2\alpha\overline{x^{(K)}})/(r\overline{x^{2,(K)}}\alpha))}{\Gamma(w)} N.$$

In case $w \approx 1$, the distribution approximately reduces to an exponential distribution with parameter ν, and the portolio size will then simply be

$$n(p) = \exp\left(-\nu\frac{2p - 2\alpha\overline{x^{(K)}}}{r\overline{x^{2,(K)}}\alpha}\right) N.$$

4 Ruin probability.

In this section we consider the optimization problem of the insurance company. It mainly wants to minimize the ruin probability, but as this only makes sense for a positive drift, the expected time to ruin is considered in case of a negative drift. The overall aim is to find the optimal premium as a function of the deductible.

When the controlled reserve develops according to a Brownian motion with positive drift, (Hipp and Taksar, 2010) shows that minimizing the ruin probability is equivalent to maximizing the following ratio μ/σ between the drift and volatility. In out setting this means maximizing

$$\frac{\mu(p;K)}{\sigma^2(p;K)} = \frac{p - \alpha(p;K)\overline{x^{(K)}}}{\alpha(p;K)\overline{x^{2,(K)}}} - \frac{L}{n(p;K)\alpha(p;K)\overline{x^{2,(K)}}}. \tag{4.1}$$

This also follows from the relation $\varphi(r_0) = \exp(-2r_0\mu(p,;K)/\sigma^2(p;K))$. If the drift is negative, then $\varphi(r_0) = 1$.

As previously implied, if the market consists of risk neutral customers, then the drift of the diffusion process will be negative making ruin certain. In order to avoid this, it is assumed in the following that there is a sufficiently large degree of risk aversion among the (potential) customers to satisfy the net profit condition $p > \alpha(p;K)\overline{x^{(K)}}$ for some p. Otherwise, there will be no motivation for selling insurance.

Notice that if $p = 0$ the insurance company will offer insurance for free, and so $\mu(0;K) < 0$ for any given K. Furthermore, due to the liability rate, L, the diffusion will also have a negative drift when the premium becomes very large, since no customer will then be interested in insuring, hence $\lim_{p\to\infty}\mu(p;K) < 0$. So ruin will be certain for both $p = 0$ and $p \to \infty$.

Now assume for some K that the premium is zero. What will then happen if the premium was raised marginally? The insurance company will have nearly the same amount of customers and therefore also the same amount of claims, but the firm will get a small revenue collecting premiums. Hence, $\mu(p,K)$ will increase. Note, that this is under the assumption that the effects from portfolio size decrease and adverse selection are smaller than the effect of raising the premium. Conversely, what if the premium was so large that no customer is interested in insuring? If the company lowers the premium such that it receives few customers, then under the condition of $p > \alpha(p,K)\overline{x^{(K)}}$ it will obtain revenue to cover at least some of the liability cost. So $\mu(p,K)$ will also increase when lowering the premium for very large values of it. This tells us intuitively that if there is an optimal premium, then it is not obtained for $p = 0$ nor in the limit $p \to \infty$, and we thereby avoid trivialities when finding the optimal premium.

Breviate notation by $\mu_p(p,K) = \frac{\partial\mu(p;K)}{\partial p}$, and similarly for $\alpha_p(p,K)$ and $n_p(p,K)$. Assume that there is an unique $\tilde{p}$ that maximizes the drift, i.e. satisfies the first order conditions

$$\mu_p(\tilde{p};K) = n_p(\tilde{p};K)\big(\tilde{p} - \alpha(\tilde{p};K)\overline{x^{(K)}}\big) + n(\tilde{p};K)\big(1 - \alpha_p(\tilde{p};K)\overline{x^{(K)}}\big) = 0. \tag{4.2}$$

This is equivalent to saying that $\tilde{p}$ must satisfy

$$\frac{-n_p(\tilde{p};K)}{n(\tilde{p};K)} = \frac{1-\alpha_p(\tilde{p};K)\overline{x^{(K)}}}{\tilde{p}-\alpha(\tilde{p};K)\overline{x^{(K)}}}.$$

The intuitive interpretation is that the relative marginal change in the demand curve must equal the relative marginal change in average net revenue per customer due to a change in premium. Two cases then arise:

(i) If $\mu(\tilde{p};K) > 0$, then $\varphi(r_0) < 1$ for all $r_0 > 0$, $\tilde{K} > 0$ and p in some bounded open interval $I \subset (0,\infty)$ containing $\tilde{p}$.

(ii) If $\mu(\tilde{p};K) \leq 0$, then $\varphi(r_0) = 1$ for all $r_0 > 0$ and $p > 0$.

Notice that the drift is positive if the net profit per customer is greater than the liability cost per customer, that is $p - \alpha(p;K) > L/n(p;K)$. This is more strict than the net profit condition.

Since the current framework is very general, so will the results be. In every concrete application the existence of an unique solution must be verified. In case i) an optimization criterion is given and proved in Theorem 6.

Theorem 6. *When $\mu(\tilde{p};K) > 0$, the optimal premium minimizing the ruin probability must be a solution to the equation*

$$1 - p\frac{\alpha_p(p;K)}{\alpha(p;K)} + \frac{L}{n(p;K)}\left(\frac{n_p(p;K)}{n(p;K)} + \frac{\alpha_p(p;K)}{\alpha(p;K)}\right) = 0.$$

Proof. Differentiating (4.1) with respect to the premium,

$$\begin{aligned}\frac{\partial\left(\frac{\mu(p;K)}{\sigma^2(p;K)}\right)}{\partial p} &= \frac{1-\alpha_p(p;K)\overline{x^{(K)}}}{\alpha(p;K)\overline{x^{2,(K)}}} - \frac{\left(p-\alpha(p;K)\overline{x^{(K)}}\right)\alpha_p(p;K)\overline{x^{2,(K)}}}{\left(\alpha(p;K)\overline{x^{2,(K)}}\right)^2} \\ &\quad + \frac{L(n_p(p;K)\alpha(p;K)+n(p;K)\alpha_p(p;K))\overline{x^{2,(K)}}}{\left(n(p;K)\alpha(p;K)\overline{x^{2,(K)}}\right)^2} \\ &= \frac{\alpha(p;K)-\alpha_p(p;K)p}{\alpha^2(p;K)\overline{x^{2,(K)}}} + \frac{L(n_p(p;K)\alpha(p;K)+n(p;K)\alpha_p(p;K))}{n^2(p;K)\alpha^2(p;K)\overline{x^{2,(K)}}}\end{aligned}$$

yields the first order condition

$$\frac{\alpha(p;K)-\alpha_p(p;K)p}{\alpha^2(p;K)\overline{x^{2,(K)}}} + \frac{L(n_p(p;K)\alpha(p;K)+n(p;K)\alpha_p(p;K))}{n^2(p;K)\alpha^2(p;K)\overline{x^{2,(K)}}} = 0.$$

Reducing we obtain the following optimality criterion

$$1 - \frac{\alpha_p(p;K)}{\alpha(p;K)}p + \frac{L}{n(p;K)}\left(\frac{n_p(p;K)}{n(p;K)} + \frac{\alpha_p(p;K)}{\alpha(p;K)}\right) = 0.$$

□

In case ii), it no longer makes sense to minimize ruin probability since it is constantly one. Instead, we suggest to choose the control to maximize the expected time to ruin, $\mathbb{E}[\tau]$, and thereby extend the expected lifetime of the company as much as possible. This is a non-standard objective function. Since the controlled reserve is a Brownian motion with drift, we are able to obtain a very simple expression.

Theorem 7. *When $\mu(\tilde{p}; K) \leq 0$, then $\tilde{p}$ will optimize the exptected time to ruin.*

Proof. Recall that we have a Brownian motion with negative drift

$$X_t = r_0 + \mu(p; K)t + \sigma(p; K)B_t$$

and we want to consider the stopping time $\tau = \inf\{t \geq 0 : X_t \leq 0\}$. Since $(X_t)_{t\geq 0}$ is a continuous process it must be that $X_\tau = 0$. Furthermore, $\{X_t - \mu(p; K)t\}$ is a martingale with mean r_0. This implies

$$\mathbb{E}[X_\tau - \mu(p; K)\tau] = -\mu(p; K)\mathbb{E}[\tau] = r_0$$

Hence $\mathbb{E}[\tau] = -r_0/\mu(p; K)$. So the expected time to ruin is maximized when the drift is maximized. Due to the assumption of an unique $\tilde{p}$ such that the first order condition (4.2) is satisfied, then this must be the optimal choice. □

In Example 9 and 10 the optimal premium is calculated explicitly for the portfolio characteristics in Section 3.1.1 and 3.2.1, respectively.

5 Examples and Illustrations.

Example 8. Assume that the claim sizes are lognormally distributed, that is $Z_i \sim \log\mathcal{N}(\mu, \sigma^2)$. This has distribution has tail function $\overline{F}(x) = 1 - \Phi\left((\log(x) - \mu)/\sigma\right)$, where Φ denotes the standard normal distribution function. The k'th moment is given by $\mathbb{E}[Z_i^k] = \exp(k\mu + k^2\sigma^2/2)$. We seek to find a closed form solution to the premium $P_{var}(K)$. The challenge obviously is $\overline{x^{(K)}}$ and $\overline{x^{2,(K)}}$. Altering these yields

$$\begin{aligned}\overline{x^{(K)}} &= \mathbb{E}[(Z_i - K)\mathbb{1}_{\{Z_i \leq K\}}] = \mathbb{E}[(Z_i - K) \mid Z_i \geq K]\overline{F}(K) \\ &= (\mathbb{E}[Z_i \mid Z_i \geq K] - K)\overline{F}(K),\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\overline{x^{2,(K)}} &= \mathbb{E}[((Z_i - K)\mathbb{1}_{\{Z_i \geq K\}})^2] = \mathbb{E}[(Z_i - K)^2\mathbb{1}_{\{Z_i \leq K\}}] \\ &= \mathbb{E}[(Z_i^2 + K^2 - 2Z_iK)\mathbb{1}_{\{Z_i \geq K\}}] \\ &= K^2\overline{F}(K) + \mathbb{E}[Z_i^2 \mid Z_i \geq K]\overline{F}(K) - 2K\mathbb{E}[Z_i \mid Z_i \geq K]\overline{F}(K).\end{aligned} \tag{5.2}$$

In (Benckert and Jung, 1974) it is shown that the k'th moment of the truncated random variable is

$$\mathbb{E}[Z_i^k \mid Z_i \geq K] = \mathbb{E}[Z_i^k]\frac{\Phi\left(\frac{\mu + k\sigma^2 - \log(K)}{\sigma}\right)}{\overline{F}(K)}.$$

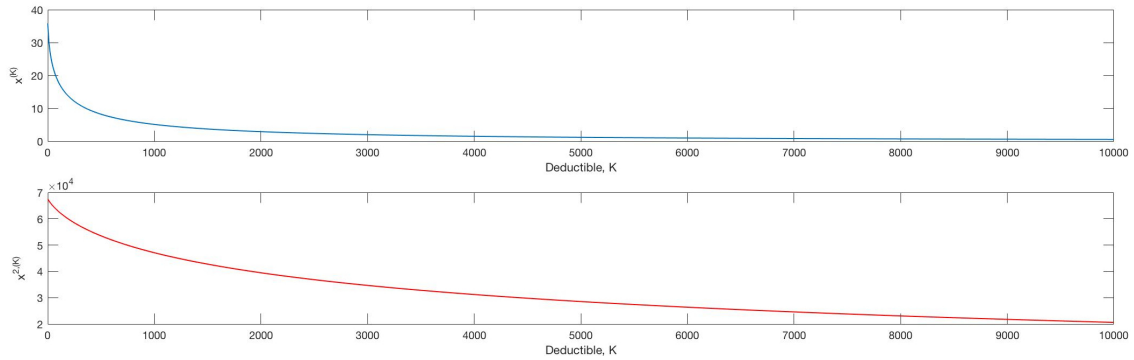

Figure 1: Plot of $x^{(K)}$ (top, blue) and $x^{2,(K)}$ (buttom, red) as functions of K for a lognormal claim size distribution with $\hat{\mu} = 1.6$ and $\hat{\sigma} = 1.99$.

From this it follows that in the lognormal case, (5.1) can be written as

$$\overline{x^{(K)}} = \overline{z}\Phi\left(\frac{\mu + \sigma^2 - \log(K)}{\sigma}\right) - K\overline{F}(K),$$

and (5.2) as

$$\overline{x^{2,(K)}} = K^2\overline{F}(K) + \overline{z^2}\Phi\left(\frac{\mu + 2\sigma^2 - \log(K)}{\sigma}\right) - 2K\overline{z}\Phi\left(\frac{\mu + \sigma^2 - \log(K)}{\sigma}\right).$$

Hence, when the claims are lognormally distributed, the maximal premium that a customer with characteristics (α, β) is willing to pay for an insurance contract as a function of the deductible is

$$\begin{aligned} P_\beta^{var}(K) = \alpha K\overline{F}(K)\left(\frac{K\beta r}{2} - 1\right) + \alpha\overline{z}\Phi\left(\frac{\mu + \sigma^2 - \log(K)}{\sigma}\right)(1 - \beta r K) \\ + \frac{\beta r \alpha \overline{z^2}}{2}\Phi\left(\frac{\mu + 2\sigma^2 - \log(K)}{\sigma}\right). \end{aligned} \tag{5.3}$$

While this is not a straightforward expression, it is computationally easy to evaluate. Some combined data on claims in fire insurance reported 1958–1969 by Swedish fire insurance companies is studied in (Benckert and Jung, 1974), where the estimates $\hat{\mu} = 1.6$ and $\hat{\sigma} = 1.99$ are obtained. These are used in the following. In Figure 1 $\overline{x^{(K)}}$ and $\overline{x^{2,(K)}}$ are plotted as functions of the deductible. Notice in particular the very different regions of the functions. This implies that the premium the customer is willing to pay for a given deductible is very sensitive towards changes in the characterization parameters (α, β) of the customer. The smaller a deductible, the higher a sensitivity.

In Figure 2 the premium function (5.3) for different combinations of characteristics is illustrated. Here it is also very clear that the function is very sensitive towards changes in characteristics and most so for small deductibles. Notice the considerable change from the combination $(\alpha, \beta) = (1/10, 2)$ to $(\alpha, \beta) = (1/2, 3)$ where the premium gets approximately 7.4 times larger.

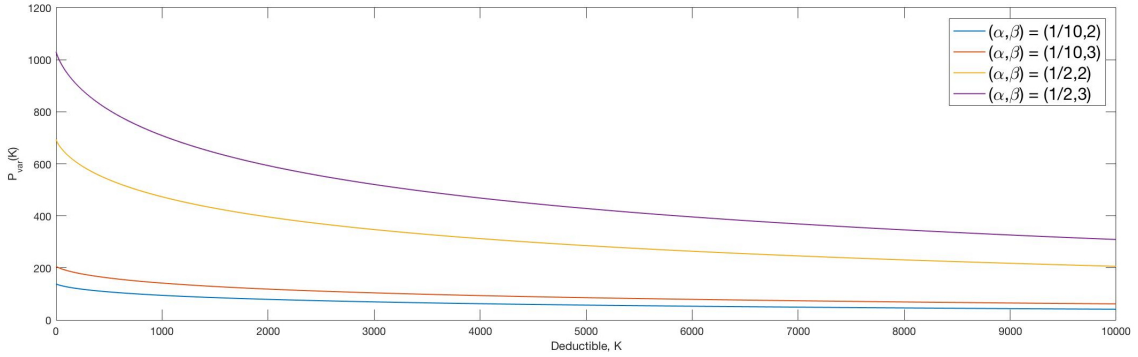

Figure 2: The premium (5.3) as function of K for different values of customer characteristics. The interest rate is chosen to be 2% and the estimates $\hat{\mu} = 1.6$ and $\hat{\sigma} = 1.99$ are used.

Example 9. An insurance company wants to supply fire insurance. It is entering a market where the customers have unknown, possibly different claim frequencies, and a constant risk aversion, β. The claim frequencies are once again assumed to be independent and identically exponentially distributed with parameter b. The portfolio can thus be characterized as in Section 3.1.1.

First, the insurer needs to see, which region of the premium it is profitable to even supply insurance. The criteria $p - \alpha(p, K)\overline{x^{(K)}} > 0$ translates into

$$p > \frac{2}{\beta r b}\frac{\overline{x^{(K)}}^2}{\overline{x^{2,(K)}}} + \frac{\overline{x^{(K)}}}{b}. \tag{5.4}$$

For a given deductible K, the considered price must exceed this threshold. Otherwise the insurance company should choose not to supply insurance.

Next, evaluating the drift is of interest. Knowing the portfolio characteristics, it can be written explicitly as

$$\mu(p; K) = N \exp\left(-b\frac{2p}{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}}\right)\left(p - \frac{2p\overline{x^{(K)}}}{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}} + \frac{\overline{x^{(K)}}}{b}\right) - L.$$

Solving the first order criteria (4.2) yields the solution

$$\tilde{p}(K) = \frac{(2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}})^2}{2\beta b r \overline{x^{2,(K)}}} = \frac{2}{\beta r b}\frac{\overline{x^{(K)}}^2}{\overline{x^{2,(K)}}} + \frac{\beta r \overline{x^{2,(K)}}}{2b} + \frac{2\overline{x^{(K)}}}{b}.$$

Notice that $\tilde{p}$ obviously satisfies being in the region of (5.4). We now seek to find the conditions under which the drift will be positive. Solving for $\mu(\tilde{p}; K) > 0$ yields

$$\frac{L}{N} < \frac{\beta r \overline{x^{2,(K)}}}{2b} \exp\left(-\frac{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}}{\beta r \overline{x^{2,(K)}}}\right).$$

Assuming that this inequality holds, then one must use Theorem 6 to find the optimal price. The optimality criterion in reduced form is

$$\frac{N}{L}\frac{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}}{2b} = \exp\left(\frac{2bp}{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}}\right)\frac{2bp}{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}}.$$

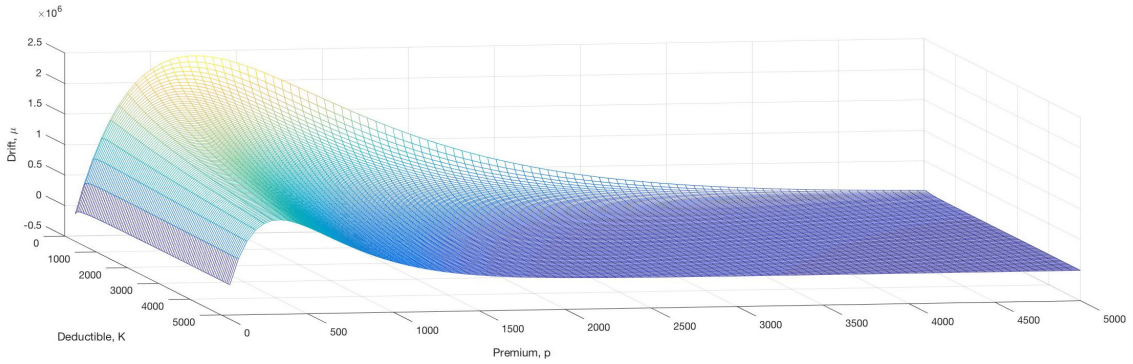

Figure 3: A mesh of the drift $\mu(p; K)$ for deductibles and premiums in the range $[0, 5000]$

This yields the following optimal premium as a function of the deductible,

$$p^*(K) = \frac{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}}{2b} W\left(\frac{N}{L}\frac{2\overline{x^{(K)}} + \beta r \overline{x^{2,(K)}}}{2b}\right).$$

using the Lambert W function. The Lambert W function defined as the (multivalued) inverse of the function $w \mapsto \exp(w)w$. For more details, see (Corless *et al.*, 1996). In the case $\mu(\tilde{p}; K) \leq 0$, it follows from Theorem 7 that $\tilde{p}(K)$ is the optimal choice. Note that due to the Lambert W function being increasing for positive values, then p^* will be prefered to $\tilde{p}$ if $p^*(K) \geq \tilde{p}(K)$. So for a given deductible, K, it is preferable for the insurance firm to simply choose the maximum of $p^*(K)$ and $\tilde{p}(K)$.

Assume that the insurance company evaluates that the market consists of $N = 10\,000$ house owners considering to buy insurance, and that it calculated the liability costs to be $L = 5000$. It also assesses that $b = 3$ and $\beta = 3$. Assume furthermore that the company has some information about the distribution of the claims, and based on this it believes that the claims are lognormally distributed according to the estimates in (Benckert and Jung, 1974). It also knows how to reasonably choose a deductible K to serve the purpose described in the introduction. The interest rate applied is 2%.

In the Figure 3 a mesh of the drift is presented. The preliminary analysis of the drift in Section 4 is very well illustrated in this. The concavity in the premium is obvious. For all the considered deductibles in the range $[0, 5000]$, the drift will be positive in $\tilde{p}(K)$.

Next, a contour of the ratio μ/σ^2 is illustrated in Figure 4. The concavity in the premium also appears very clearly here. The ratio is at its highest within the region of approximately $K \in [0, 220]$, followed by the regions $K \in (220, 590]$ and then $K \in (590, 1070]$. In Figure 5 the premiums $\tilde{p}(K)$, $p^*(K)$ as functions of the deductible are plotted.

Notice that the model does not take the cost of processing an increasing number of claims into consideration. The company therefore evaluates that a deductible of $K = 1000$ is suitable. This yields the following values

$$\tilde{p}(K) = 474.2, \qquad p^*(K) = 2458.1,$$

Since $p^*(1000) > \tilde{p}(1000)$ then p^* is the optimal premium.

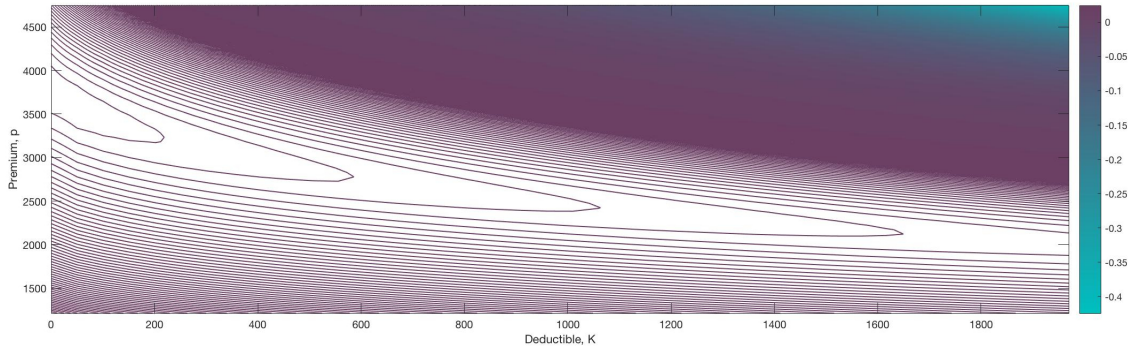

Figure 4: A contour of the relation $\mu(p;K)/\sigma(p;K)^2$ for deductibles in the range $[0, 2000]$ and premiums in $[1210, 4750]$.

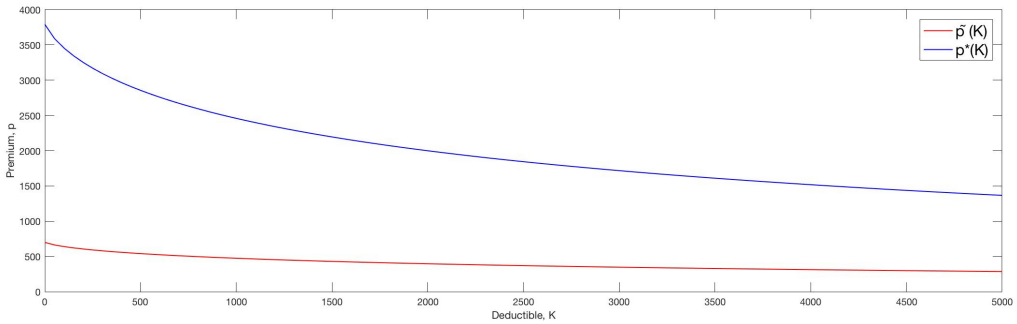

Figure 5: The premiums $p^*(K)$ and $\tilde{p}(K)$ as functions of the deductible K.

Example 10. Assume now that the insurance company beliefs that the claim frequency is constant in the portfolio, but does not possess any information about the customers' risk aversion. The risk aversion is therefore modelled by a random variable, which is assumed to have an exponential distribution with parameter ν. Portfolio characteristics are then as in Section 3.2.1. The existence criteria of the insurance company will be $p - \alpha\overline{x^{(K)}} > 0$, i.e. the premium must simply be larger than the net premium. First is to find the solution to (4.2), namely

$$\tilde{p}(K) = \frac{r\overline{x^{2,(K)}}\alpha}{2\nu} + \alpha\overline{x^{(K)}}.$$

Next, is to find the region of which the drift evaluated in $\tilde{p}$ is positive

$$\frac{L}{N} < \frac{r\overline{x^{2,(K)}}\alpha}{2\nu}\exp(-1).$$

In this region

$$p^*(K) = \frac{r\overline{x^{2,(K)}}\alpha}{2\nu}\log\left(\frac{N}{L}\frac{r\overline{x^{2,(K)}}\alpha}{2\nu}\right) + \alpha\overline{x^{(K)}}$$

is optimal. Otherwise $\tilde{p}$ is optimal. Again, due to the logarithm being an increasing function, one can simply state that the insurance company should choose the maximum of $\tilde{p}$ and p^*.

Acknowledgments

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A Calculations of the present value in Section 2.

The present value of not insuring can be split into three terms

$$\begin{aligned} V_n &= \mathbb{E}\Big[\int_0^\infty \exp(-dt)\mathrm{d}w_t\Big] = \mathbb{E}\Big[\int_0^\infty \exp(-dt)rw_t\mathrm{d}t - \int_0^\infty \exp(-dt)d\mathrm{d}A_t^\alpha\Big] \\ &= \mathbb{E}\Big[\int_0^\infty \exp(-(d-r)t)r\Big(w_0 - \sum_{i=1}^{N_t^\alpha}\exp(-rT_i)Z_i\Big)\mathrm{d}t - \int_0^\infty \exp(-dt)\mathrm{d}A_t^\alpha\Big] \\ &\equiv K_1 - K_2 - K_3. \end{aligned}$$

The three terms are defined and assessed below. The first is easily evaluated as

$$K_1 = \mathbb{E}\Big[\int_0^\infty \exp(-(d-r)t)rw_0\mathrm{d}t\Big] = \frac{rw_0}{d-r}.$$

The second and third term require a bit more effort. Using Campbell-Mecke's formula yields

$$\begin{aligned} K_2 &= \mathbb{E}\Big[\int_0^\infty r\exp(-(d-r)t)\Big(\sum_{i=1}^{N_t^\alpha}\exp(-rT_i)Z_i\Big)\mathrm{d}t\Big] \\ &= \int_0^\infty r\exp(-(d-r)t)\mathbb{E}\Big[\Big(\sum_{i=1}^{N_t^\alpha}\exp(-rT_i)Z_i\Big)\mathrm{d}t\Big] \\ &= \overline{z}r\int_0^\infty \exp(-(d-r)t)\Big(\int_0^t \exp(-rs)\alpha\mathrm{d}s\Big)\mathrm{d}t \\ &= \overline{z}r\int_0^\infty \exp(-(d-r)t)(1-\exp(-rt))\mathrm{d}t \\ &= \frac{\overline{z}\alpha}{d-r} - \frac{\overline{z}\alpha}{d}, \\ K_3 &= \mathbb{E}\Big[\int_0^\infty \exp(-dt)\mathrm{d}A_t^\alpha\Big] = \mathbb{E}\Big[\sum_{i=1}^{\infty}\exp(-dT_i)Z_i\Big] \\ &= \overline{z}\alpha\int_0^\infty \exp(-ds)\mathrm{d}s = \frac{\overline{z}\alpha}{d}. \end{aligned}$$

So to conclude, the present value of not insuring will be

$$V_n = \frac{rw_0}{d-r} - \frac{\overline{z}\alpha}{d-r}.$$

B Calculations of the variance in Section 2

The Total Law of Variance states that

$$\mathbb{V}\mathrm{ar}[S^{(K)}] = \mathbb{V}\mathrm{ar}\big[\mathbb{E}[S^{(K)} \mid \mathcal{F}]\big] + \mathbb{E}\big[\mathbb{V}\mathrm{ar}[S^{(K)} \mid \mathcal{F}]\big] \tag{B.1}$$

where the filtration, $\mathcal{F}$, is the sigma algebra generated by the T_i's. Consider the first term of the variance in (B.1). The conditional expectation is easily calculated as

$$\mathbb{E}[S^{(K)} \mid \mathcal{F}] = \mathbb{E}\Big[r\sum_{i=1}^{\infty}\exp(-rT_i)X_i^{(K)} \;\Big|\; \mathcal{F}\Big] = r\overline{x^{(K)}}\sum_{i=1}^{\infty}\exp(-rT_i).$$

The square of a sum is used for finding the variance of this expression

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}\Big[r\overline{x^{(K)}}\sum_{i=1}^{\infty}\exp(-rT_i)\Big] \\
&\quad= r^2\overline{x^{(K)}}^2\Big(\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\exp(-rT_i)\Big)^2\Big] - \mathbb{E}\Big[\sum_{i=1}^{\infty}\exp(-rT_i)\Big]^2\Big) \\
&\quad= r^2\overline{x^{(K)}}^2\Big(\mathbb{E}\Big[\sum_{i=1}^{\infty}\exp(-r2T_i) + \sum_{i,j=1\,:\,i\neq j}^{\infty}\exp(-rT_i)\exp(-rT_j)\Big] - \Big(\frac{\alpha}{r}\Big)^2\Big) \\
&\quad= r^2\overline{x^{(K)}}^2\left(\frac{\alpha}{2r} + \Big(\frac{\alpha}{r}\Big)^2 - \Big(\frac{\alpha}{r}\Big)^2\right) = \frac{r\overline{x^{(K)}}^2\alpha}{2}.
\end{aligned}$$

The square of a sum is also used for finding the conditional variance in the second term of (B.1),

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}[S^{(K)} \mid \mathcal{F}] \\
&\quad= r^2\Big(\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\exp(-rT_i)X_i^{(K)}\Big)^2 \;\Big|\; \mathcal{F}\Big] - \mathbb{E}\Big[\sum_{i=1}^{\infty}\exp(-rT_i)X_i^{(K)} \;\Big|\; \mathcal{F}\Big]^2\Big) \\
&\quad= r^2\mathbb{E}\Big[\sum_{i=1}^{\infty}\exp(-r2T_i)\left(X_i^{(K)}\right)^2 \\
&\qquad\qquad + \sum_{i,j=1\,:\,i\neq j}^{\infty}\exp(-rT_i)\exp(-rT_j)X_i^{(K)}X_j^{(K)} \;\Big|\; \mathcal{F}\Big] \\
&\qquad - r^2\overline{x^{(K)}}^2\Big(\sum_{i=1}^{\infty}\exp(-r2T_i) - \sum_{i,j=1\,:\,i\neq j}^{\infty}\exp(-rT_i)\exp(-rT_j)\Big) \\
&\quad= r^2(\overline{x^{2,(K)}} - \overline{x^{(K)}}^2)\sum_{i=1}^{\infty}\exp(-r2T_i).
\end{aligned}$$

Taking the expectation yields

$$\mathbb{E}\big[\mathbb{V}\mathrm{ar}[S^{(K)} \mid \mathcal{F}]\big] = \mathbb{E}\Big[r^2(\overline{x^{2,(K)}} - \overline{x^{(K)}}^2)\sum_{i=1}^{\infty}\exp(-r2T_i)\Big] = \frac{r(\overline{x^{2,(K)}} - \overline{x^{(K)}}^2)\alpha}{2}.$$

Concluding, the variance in (B.1) will be $\mathbb{V}\mathrm{ar}[S^{(K)}] = (r\overline{x^{2,(K)}}\alpha)/2$.

C Illustration of complexity of the exponential premium in Section 2

To illustrate the complexity of $P_{exp}(K)$ consider the simple case where $K = 0$ and the claim size distribution F is degenerate in 1, i.e. $Z_i = 1$ almost surely. Let

$$f(t) = \mathbb{E}\Big[\exp\Big(\beta r \sum_{i=1}^{\infty} \exp(-rT_i)\mathbb{1}_{\{T_i \leq t\}}\Big)\Big].$$

During a short time interval $\mathrm{d}t$ the function $f(t)$ might get an additional term to the sum (corresponding to a new claim) with probability $\alpha \mathrm{d}t$ or $f(t)$ will remain unchanged with probability $1 - \alpha \mathrm{d}t$, i.e.

$$f(t + \mathrm{d}t) = f(t)(1 - \alpha \mathrm{d}t) + f(t) \exp(\beta \exp(-rt))\alpha \mathrm{d}t.$$

Letting $\mathrm{d}t$ tend to zero an ordinary differential equation appears,

$$f'(t) = \alpha f(t)(\exp(\beta \exp(-rt)) - 1).$$

This can also be expressed as

$$\begin{aligned}
\log(f(t)) &= \alpha \int_0^t (\exp(\beta \exp(-rs)) - 1)\mathrm{d}s = \alpha r \int_{-\beta}^{-\beta \exp(-rt)} \frac{1}{u} \exp(-u)\mathrm{d}u - \alpha t \\
&= \alpha r \Big(\int_{-\beta}^{\infty} \frac{1}{u} \exp(-u)\mathrm{d}u - \int_{-\beta \exp(-rt)}^{\infty} \frac{1}{u} \exp(-u)\mathrm{d}u \Big) - \alpha t \\
&= \alpha r (Ei(\beta \exp(-rt)) - Ei(\beta)) - \alpha t,
\end{aligned}$$

where $Ei(.)$ denotes the exponential integral

$$Ei(x) = -\int_{-x}^{\infty} \frac{1}{u} \exp(-u)\mathrm{d}u \qquad \text{for } x > 0.$$

The exponential integral is known not to have a closed form solution. So even for the simple case of a claim size distribution an analytic solution to the premium cannot be obtained.