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# Complex Chern–Simons Theory: Knot Invariants and Mapping Class Group Representations

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## Abstract

In this thesis we investigate some properties of *Complex Chern–Simons theory*. Contrary to the compact situation which was the object of a lot of focus for more than 20 years, not much was known until few years ago, regarding rigorous computations of invariants via Chern–Simons theory with complex gauge group. In the recent years, in parallel to an increased interest from physics, the works of Andersen and Kashaev, and of Andersen and Gammelgaard opened the way to a rigorous mathematical investigation of such theories. Andersen and Kashaev provided the theory to compute invariants of knot complements (and actually a more general class of cusped 3 manifolds) starting from the quantization of the Teichmüller space. The work of Andersen and Gammelgaard provides a general differential geometric setting for the ideas of Witten [Wit91], regarding techniques to quantize the 2 dimensional part of Chern–Simons theory with gauge group $\mathrm{SL}(n, \mathbb{C})$. In general we are still missing $2+1$ functorial interpretation, like the Witten-Reshetikhin-Turaev TQFT for the compact theory. In this thesis we try to have a closer look to some of the most elementary aspects of these constructions. We focused particularly in computing and studying explicit expressions for the simplest examples of knot invariants and mapping class group representations. We first construct invariants of hyperbolic knots, showing their relation with some new representations of Quantum Teichmller Theory. Then we focus in a couple of examples. The study of the asymptotic behavior of such knot invariants requires a generalization of the theory of Andersen and Kashaev to a non obviously unitary one, the existence of which was claimed again by Witten in [Wit91]. In this setting some parallel with other previously known invariants is discussed. Afterwards we follow the approach of Andersen and Gammelgaard in the example of a genus 1 surface, and study the mapping class group representations that this quantum theory defines. We give explicit formulas for the representations and show how the representations from Chern–Simons theory with gauge group $\mathrm{SU}(2)$ appear in these.



## Dansk Résumé

I denne afhandling undersøger vi nogle egenskaber ved *kompleks Chern–Simons teori*. I modsætning til den kompakte situation, der var genstand for megen fokus i mere end 20 år, var der indtil for få år siden ikke meget kendt om rigoristiske beregninger af invarianter via Chern - Simons teori med kompleks gauge gruppe. I de seneste år, sideløbende med en øget interesse fra fysik, har resultater af Andersen og Kashaev, og Andersen og Gammelgaard åbnet vejen for en streng matematisk undersøgelse af sådanne teorier. Andersen og Kashaev grundlagde teorien til at beregne invarianter af knude komplementer (og faktisk også en mere generel klasse af "cusped" 3 mangfoldigheder) med udgangspunkt i kvantisering af Teichmüller rummet. Resultater af Andersen og Gammelgaard giver en generel differential geometrisk baggrundsramme for ideer af Witten [Wit91], om teknikker til at kvantisere den 2 dimensionelle del af Chern–Simons teori med gauge gruppe $\mathrm{SL}(n, \mathbb{C})$. Generelt set mangler stadig den $2+1$ funktorielle tolkning, som kendes fra Witten-Reshetikhin-Turaev TQFT'en for den kompakte teori. I denne afhandling forsøger vi at få et nærmere kig på nogle af de mest elementære aspekter af disse konstruktioner. Vi fokuserede især på beregning af eksplicitte udtryk for de simpleste eksempler på knude invarianter og afbildningsklassegruppe repræsentationer. Vi konstruerer først invarianter

af hyperbolske knuder, og beskriver deres relation til nogle nye repræsentationer af Quantum Teichmller Theory. Dernæst ser vi nærmere på en række eksempler.

Studiet af den asymptotiske opførsel af sådanne knude invarianter krævede en generalisering af Andersens og Kashaevs teori til en ikke triviel unitær teori, hvis eksistens blev formodet af Witten i [Wit91]. I lyset af dette diskuteres nogle paraleller med andre tidligere kendte invarianter. Bagefter benytter vi Andersens og Gammelgaards tilgang i eksemplet med en genus 1 flade, og studerer de afbildningsklassegruppe repræsentationer, som denne kvanteteori definerer. Vi giver eksplicitte formler for repræsentationerne og viser, hvordan repræsentationer fra Chern–Simons teori med gauge gruppe SU(2) optræder i disse.

# Contents

# Preface

This manuscript is the culmination of my 3 years of PhD studies at the Centre for Quantum Geometry of Moduli Spaces at Aarhus University. I would like to thank the Department of Mathematics and the QGM for the inspiring environment that host me in these years.
I am especially grateful to my supervisor Jørgen Ellegaard Andersen for introducing me to the problems I studied, guiding me through and suggesting how to solve them. I had many stimulating discussions with the many visitors and postdocs and I would like to particularly thank Professor Rinat Kashaev for explaining me a lot of his research during his visits. I also had the luck of sharing these years with many fellow PhD students that I thank them all, particularly my office mate Alessandro Malusà for sharing a lot of discussions on our respective research and for a lot of suggestions he gave me through the process.
Finally I also want to thank my wife Francesca for sharing this experience with me, and for all the patience and support she showed.

# Introduction

*Topological Quantum Field Theories* in dimension $2+1$ were axiomatized in [Ati88], [Wit88] and [Seg04]. Further, in his seminal paper [Wit89], Witten introduced the study of quantum Chern-Simons theory with non-abelian, compact gauge group $G$. For $G = \mathrm{SU}(n)$, to any positive integer $k$ and couple $(M, L)$, where $M$ is a 3-manifold and $L \subset M$ is a link embedded Witten associates a number, claimed to be a topological invariant. The process he used, usually called *path integral*, was mathematically ill-defined and it is still so at the time of writing this thesis. Nevertheless it is of crucial importance in modern physics and Witten was able to use it to show that such topological invariant, for $n = 2$ and $M = S^3$, corresponds to the colored *Jones polynomials* of the link $L$. As the Lagrangian of the theory is the Chern–Simons functional, which is a topological invariant, quantum Chern–Simons theory should indeed be a TQFT. A mathematical construction of the theory via combinatorial means followed shortly after from Reshetikhin and Turaev [RT90, RT91, Tur10]. This so called Witten-Reshetikhin-Turaev TQFT is based on the finite dimensional representation theory of the quantum group $U_q(\mathfrak{sl}(2,\mathbb{C}))$, where $q$ is a root of unity. Careful choice of $q$ with respect to $k$ leads this TQFT to analogous results to the ones expected from quantum Chern–Simons theory. A similar rigorous approach, but entirely topological using skein theory, was showed in [BHMV92, BHMV95, Bla00]. Recently Andersen and Ueno showed, in a series of four papers [AU07b, AU07a, AU12, AU], that The Witten-Reshetikhin-Turaev TQFT is the same as the TQFT coming from Conformal Field Theory [TUY89, BK00] as was also proposed to be the case in Witten's original paper [Wit89]. Witten further suggested in the same paper that the geometric quantazation of moduli space of flat $\mathrm{SU}(n)$ connections should be related to this theory and he developed this approach further in his joint paper with Axelrod and Della Pietra [ADPW91]. Following shortly after, Hitchin gave a rigorous account of this work in [Hit90]. For a purely differential geometric account of the construction of this connection see [And12, AG11, AGL12]. By combining the work of Laszlo [Las98] with the above mentioned works of Andersen and Ueno, it has now been confirmed that one can use the geometric quantization of the moduli space of flat connections as an alternative construction of the Witten-Reshetikhin-Turaev TQFT. In the paper [Wit91] Witten also proposed a way to construct the mapping class groups representations of quantum Chern-Simons theory for the non-compact gauge group $\mathrm{SL}(n,\mathbb{C})$. This theory received less attention and it is much less developed. This thesis is indeed in the direction of understanding some aspects of it, as we will explain later. First we recall some of the relevant literature. In the physics literature, the complex quantum Chern-Simons theory has been discussed from a path integral point of view in a number of papers [Dim13, DGG14, DGLZ09, DG13, GM08, Guk05, Hik01, Hik07, Wit11, BNW91, Dim14] (see also references in these).

From a mathematical standpoint the problem of extending the quantization program of moduli spaces of flat connections to the non-compact group cases has been tackled into at least two different ways. One approach is to first consider the moduli space of flat $\mathrm{PSL}(2,\mathbb{R})$ connections over a surface, which is not connected, but has an important connected component, namely the Teichmüller space. This space is of primary interest in many areas of mathematics, and is fundamental in low dimensional topology. The quantization program of the Teichmüller space was independently carried out, for punctured surfaces, by Kashaev [Kas98] and Chekhov and Fock [FC99]. Both the approaches are based on two ingredients, Penner coordinates for the Decorated Teichmüller space [Pen87],[Pen12], and Faddeev's Quantum Dilogarithm [Fad95],[FK94]. Quantum Teichmüller theory was further extended from a 2-dimensional theory to a $(2+1)$-dimensional theory by Andersen and Kashaev in [AK14b] and [AK13]. In particular Andersen and Kashaev were able to construct knots and links invariants together with some cobordism aspects of the axiomatized definition of TQFT. Usually it is referred as Teichmüller TQFT. Later in the work [AK14a] the same authors extended their previous constructions to a theory with an extra parameter, a positive odd integer $N$. In the same work they also gave evidences that this theory produces invariants for the level $N$, $\mathrm{PSL}(2,\,\mathbb{C})$ quantum Chern–Simons theory. This theory is similar in the formalism (but not in the nature) to the one developed in the physics literature by Dimofte [Dim14].
A different approach is to follow the original quantization program proposed by Witten [Wit91]. This was done in a mathematical formalism by Andersen and Gammelgaard [AG14]. In this setting the quantization of $\mathrm{SL}(2,\mathbb{C})$ moduli spaces is possible for closed and punctured surfaces, however no $2+1$–dimensional interpretation is developed as of now.
In this thesis we review both the approaches to complex Chern–Simons theory and we try to exploit some aspects of them. First we try to put together the theory presented in [AK14a] with the constructions of [AK14b] and [Kas98]. In subsection 3.1.2, following Kashaev's approach to quantize the Teichmüller space, we define a formal quantization dependent on a continuous parameter $\mathrm{b}$, constrained to have $\operatorname{Re}\mathrm{b}>0$ and $\operatorname{Im}(\mathrm{b})(1-|\mathrm{b}|)=0$, together with an odd positive level $N$. In section 3.2, we show with arguments extrapolated from [AK14a], that the quantization of the moduli space $\mathrm{PSL}(2,\,\mathbb{C})$ flat connections of a 4–punctured sphere is equal to the formal $(\mathrm{b},N)$–quantization of its Teichmüller space. The main ingredient to see this equivalence is the level $N$ *Weil-Gel'fand-Zak transform* and it is expressed in Proposition 43. Afterwards we extend this construction to a level $N$ Teichmüller TQFT strictly following the presentation for level 1 in [AK14b]. The Theorem 52 updates the functor from [AK14b] to this setting. It is interesting that the theory, for $N>1$, has different unitary properties if $\mathrm{b}$ is real or unitary. Namely if $\mathrm{b}>0$ and $N>1$, the canonical inner product do not make the theory unitary, see Definition 26 for details. This particular setting was not considered in the literature as far as we know. The phenomenon is parallel to what Witten noticed in [Wit91] that there are two possible unitary complex theories, one obvious and one *exotic*, depending on the nature of the quantization parameter. In particular, for knot invariant, the semi–classical limit $\mathrm{b}\to 0$ is possible only in the exotic theory.
Another part of this work regards Witten's approach to quantize the $\mathrm{SL}(2,\mathbb{C})$ moduli space of a genus 1 surface $\Sigma$ with no punctures. Following the lines of [Wit91] and [AG14] we quantize the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections on $\Sigma$, and compute

the explicit description of the quantum representations of the mapping class group of $\Sigma$. These are a family of infinite dimensional representations parameterized by a continuous unitary parameter b, with $\operatorname{Re} \mathrm{b} > 0$, and a positive integer $k \geq 1$, see Theorems 67 and 73 for their explicit expression. In particular we show a tensor product decomposition of the representations, with one factor equal to the representations obtained via SU(2) quantum Chern Simons theory at level $k-2$, see Remark 5.2.4. Again the Weil-Gel'fand-Zak Transform is the main technical tool used to simplify the explicit descriptions. Another central objects in the quantization is the *Hitchin–Witten connection* and the parallel transport associated to it, that we describe explicitly in this baby example of genus 1.

## Summary

In **Chapter 1** we list some more or less standard preliminary material of general interest throughout the thesis, such as geometric quantization and Teichmüller theory. In Section 1.2, we present the level–$N$ Weil-Gel'fand-Zak transform and its general properties. Its applications will be transversal to the rest of the thesis.
In Section 1.3 we present Penner's $\lambda$–lengths coordinates for the the Teichmüller space, Kashaev's *ratio coordinates* which are a generalizations of Penner's one and some of their symplectic properties. A description of change of coordinates and action of the mapping class group in terms of decorated Ptolemy groupoid is also provided. Finally we describe the *complexification* of such coordinates as a way to describe the PSL(2, $\mathbb{C}$) moduli spaces.
In **Chapter 2** we recall some basics on Faddeev's quantum dilogarithm $\Phi_{\mathrm{b}}$, together with the more recent level $N$ dilogarithm, $\mathrm{D}_{\mathrm{b}} : \mathbb{R} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$, introduced in [AK14a], that we describe here in more details, showing analogous properties to the ones known for $\Phi_{\mathrm{b}}$, together with a different behavior in Proposition 20 which mirrors the different unitary properties of the theory at level $N > 1$.
In **Chapter 3** the formal quantization of the Teichmüller space is carried out with representations in $\mathrm{L}^2(\mathbb{R} \times \mathbb{Z}/N\mathbb{Z})$, thanks to the dilogarithm $\mathrm{D}_{\mathrm{b}}$. In particular we define a tetrahedral operator acting on $\mathrm{L}^2(\mathbb{R} \times \mathbb{Z}/N\mathbb{Z})$, which provides representations for the Ptolemy groupoid. In Section 3.2, we show that such formal quantization is compatible with the quantization of the moduli space of PSL(2, $\mathbb{C}$) flat connections for the 4–punctured sphere.
In **Chapter 4** we upgrade the construction of the previous chapter to a level $N$ Teichmüller TQFT. Following [AK14b] and [AK13] a *charged* version of the tetrahedral operator is introduced. In this way it is possible to compute partition functions of cusped 3 dimensional objects that admits an ideal triangulation with ideal tetrahedra and some extra admissibility condition. Hyperbolic knot complements are among them and we show a couple of examples together with some asymptotic property in the exotic unitary setting, i.e. when $\mathrm{b} > 0$ and $\mathrm{b} \to 0$. In the figure–eight knot we remark the appearance in this limit of another knot invariant, of similar nature, the Baseilhac–Benedetti invariant from [BB07], see subsection 4.4.4.
**Chapter 5** Is divided in two sections. The first is a general recall of classical and complex Chern–Simons theory, the quantization of them, and motivations for the Hichin and Hitchin–Witten connections. For the purposes of this thesis this section only serves to put the second one in a bigger and more general framework of

research. In the second section, we describe the quantization of complex quantum Chern–Simons thoery in genus 1 from the scratch, giving all the details and computations. From this we compute the representations of the mapping class group explicitly as integral operators.

# Chapter 1

# Preliminaries

## 1.1 Geometric Quantization

Let $(M, \omega)$ be a fixed manifold $M$ with symplectic 2 form $\omega \in \Omega^2(M, \mathbb{R})$ fixed for the rest of this section. We are going to summarize the process of *geometric quantization* following the two step approach of doing a so called *pre-quantization* first, and then concluding the process with the choice of a *polarization*. A general reference is [Woo92].
The first step consist in finding a *pre-quantum line bundle*

**Definition 1.** A pre-quantum line bundle $(\mathcal{L}, \nabla, (\cdot, \cdot))$ consists in an Hermitian line bundle $\mathcal{L} \to M$, $(\cdot, \cdot)$ being the Hermitian structure, together with a compatible connection $\nabla$ on the bundle such that the curvature satisfies

$$F_\nabla = -i\omega. \tag{1.1}$$

$M$ is said *pre-quantizable* if a pre-quantum line bundle exists.

If this is possible to be found, we can assign to every observables $f \in \mathrm{C}^\infty(M, \mathbb{R})$ a *pre-quantum operator*

$$\hat{f} = -i\nabla_{X_f} + f \tag{1.2}$$

where $X_f$ is the Hamiltonian vector field associated to $f$, i.e. the vector field satisfying

$$\omega(X_f, \cdot) = -\mathrm{d}f$$

and get the commutator

$$[\hat{f}, \hat{g}] = -i\widehat{\{f, g\}} \tag{1.3}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket induced by $\omega$.
Now we ask ourselves when is it $M$ pre-quantizable. It is well known that given a line bundle $\mathcal{L}$ with connection $\nabla$, then the cohomology class of the curvature satisfies

$$\frac{i}{2\pi}[F_\nabla] = c_1(\mathcal{L}) \in \mathrm{H}^2(M, \mathbb{Z})$$

where $c_1(\mathcal{L})$ is the Chern class of the line bundle. It follows that having $\nabla$ satisfying (1.1), imposes the condition on $\omega$

$$\left[\frac{\omega}{2\pi}\right] \in \mathrm{H}^2(M, \mathbb{Z}). \tag{1.4}$$

This condition turns out to be both necessary and sufficient for the existence of a pre-quantization.

Regarding the (non)-uniqueness of the pre-quantization we remark that inequivalent choices of $\nabla$ satisfying (1.1) are precisely parametrized by $\mathrm{H}^1(M, \mathrm{U}(1))$. See [Woo92] for a detailed discussion.

**Example 1.1.1** (Canonical (pre-)Quantization)**.** Fix a positive real $\hbar > 0$. Suppose $M = \mathbb{R}^{2n}$ and $\omega = 2\pi\hbar^{-1}\sum_{j=1}^{n} \mathrm{d}q_j \wedge \mathrm{d}p_j$. Then $[\omega] = 0 \in \mathrm{H}^2(\mathbb{R}^2, \mathbb{Z}) = \{0\}$, so the trivial line bundle $\mathcal{L} = M \times \mathbb{C}$ is the only possible one. We fix the gauge representative of the pre-quantum connection as $\nabla = \mathrm{d} + 2\pi i\hbar^{-1}\sum_{j=1}^{n} p_j \mathrm{d}q_j$ satisfying (1.1). The Hermitian structure is the canonical one on $\mathbb{R}^{2n}$. The pre-Quantum operators associated to the coordinate functions are

$$\hat{p}_j = -\frac{\hbar}{2\pi i}\frac{\partial}{\partial q_j} \qquad\qquad \hat{q}_j = \frac{\hbar}{2\pi i}\frac{\partial}{\partial p_j} + q_j$$

**Example 1.1.2** (Torus)**.** Consider the torus $M = \mathbb{T} \times \mathbb{T} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{C}/\mathbb{Z}^2$. A line bundle $\mathcal{L} \to M$ can be pulled back via $\pi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ to a trivializable line bundle $\varphi\colon \pi^*\mathcal{L} \xrightarrow{\sim} \mathbb{R}^2 \times \mathbb{C}$. More precisely, for every $p \in \mathbb{R}^2$, the trivialization $\varphi$ gives an identification of the fiber

$$\varphi_p\colon \left(\pi^*\mathcal{L}\right)_p \xrightarrow{\sim} \mathbb{C}.$$

Since the line bundle $\mathcal{L}$ is defined in the quotient, for every $\lambda \in \mathbb{Z}^2$ we should have

$$\left(\pi^*\mathcal{L}\right)_{p+\lambda} = \left(\pi^*\mathcal{L}\right)_p,$$

so the composition $\varphi_p \circ \varphi_{p+\lambda}^{-1}\colon \mathbb{C} \to \mathbb{C}$ is a well defined change of trivialization of the line bundle $\mathcal{L}$. Being an isomorphism of the complex plane it reduces to multiplying by a complex number $e_\lambda(p)$ named *multiplier*. The consistency condition that a choice of multipliers $e$ have to satisfy is the following

$$e_\lambda(p)e_{\lambda'}(p+\lambda) = e_{\lambda'}(p)e_\lambda(p+\lambda') \tag{1.5}$$

The hermitian lines bundles over $M$ are classified by their first Chern class $c_1(\mathcal{L}) \in \mathrm{H}^2(M, \mathbb{Z}) \simeq \mathbb{Z}$, and they satisfy the properties

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}'), \qquad c_1(\mathcal{L}^*) = -c_1(\mathcal{L}).$$

It follows that once we know the line bundle so that $c_1(\mathcal{L}) = 1$, we can generate all the other via these group properties.

**Fact 1.** *Let $(x, y)$ coordinates over $\mathbb{R}^2$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^2/\mathbb{Z}^2, \mathbb{R})$. The line bundle $\mathcal{L} \to M$ with multipliers*

$$e_{(1,0)}(x, y) = e^{\pi i y} \qquad\qquad e_{(0,1)}(x, y) = e^{-\pi i x}$$

*has Chern class $c_1(\mathcal{L}) = [\omega] = 1 \in \mathrm{H}^2(M, \mathbb{Z})$*

Any section $\psi \in \mathrm{C}^\infty(M, \mathcal{L})$ can be pulled back to $s = \pi^*\psi \in \mathrm{C}^\infty(\mathbb{R}^2, \pi^*\mathcal{L})$, which correspond to the space of quasi periodic functions satisfying

$$s(x+1, y) = e_{(1,0)}(x, y)s(x, y) \qquad\qquad s(x, y+1) = e_{(0,1)}(x, y)s(x, y)$$

We will carry out the quantization in this example in many details at several places in this thesis and with different techniques.

### 1.1.1 Polarizations and Quantization

Given a pre-quantum line bundle $(\mathcal{L}, \nabla, (\cdot,\cdot))$ for $(M, \omega)$, the natural Hilbert space to consider is $\mathrm{L}^2(M, \mathcal{L})$ where the square integrability is with respect to the following inner product

$$\langle \psi, \phi \rangle \equiv \int_M (\psi, \phi) \frac{\omega^n}{n!}. \tag{1.6}$$

However this space turns out to be wrong for the simplest applications in physics, as the 'states' should depends on half the number of variables. This can be obtained by choosing a polarization in $M$ and considering only sections covariantly constant in the directions that it determines. Precisely

**Definition 2.** A *complex Lagrangian Polarization* on $(M, \omega)$ is a distribution $P$ in $T_{\mathbb{C}}M$ satisfying

1. $\omega(X, Y) = 0$ for all $X$ and $Y \in P$ ($P$ is *Lagrangian*),
2. $[X, Y] \in P$ for all $X$ and $Y \in P$ ($P$ is *Involutive*)
3. $\dim(P_x \cap \overline{P}_x \cap T_xM)$ is constant for all $x \in M$.

A polarization is said *real* if $P = \overline{P}$.

Usually, with a polarization at hand, the Hilbert space of the quantization becomes

$$\mathcal{H} \equiv \{\psi \in \mathrm{L}^2(M, \mathcal{L}) \,:\, \nabla_X \psi = 0 \;\forall X \in P\}. \tag{1.7}$$

**Example 1.1.3** (Canonical Quantization)**.** Recall example 1.1.1. Let us conclude it with a particular choice of polarization, i.e. $P = \mathrm{Span}\langle \frac{\partial}{\partial p_j},\ j = 0, \dots n\rangle$. The Hilbert space is $\mathcal{H} \simeq \mathrm{L}^2(\mathbb{R})$, while the quantum operator reduce to the Schrodinger representation

$$\hat{p}_j = -\frac{\hbar}{2\pi i}\frac{\partial}{\partial q_j} \qquad\qquad \hat{q}_j = q_j \tag{1.8}$$

## 1.2 Weil-Gel'fand-Zak Transform

Let $S(\mathbb{R})$ be the space of Scwartz functions over the reals, $k$ be a positive integer. The space $\mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^k$ has an $\mathrm{L}^2$ inner product defined by equation (A.8) in Appendix A.2. Let $\mathcal{L}^k$ be the line bundle over the real 2–torus $\mathbb{T} \times \mathbb{T}$ with sections identified with quasi-periodic functions on $\mathbb{R}^2$ satisfying

$$s(u + 1, v) = e^{\pi i v} s(u, v) \qquad\qquad s(u, v + 1) = e^{-\pi i u} s(u, v). \tag{1.9}$$

On $C^\infty(\mathbb{T} \times \mathbb{T}, \mathcal{L}^k)$ consider the following inner product

$$\langle \psi, \phi \rangle = \int_{[0,1]^2} \psi(u, v)\overline{\phi(u, v)} \mathrm{d}u\mathrm{d}u \tag{1.10}$$

We now present the so-called level $k$ Weil-Gel'fand-Zak transform, introduced in [AK14a]. The level 1 transform is well known and detailed treatment can be found, for example, [Ner11].

**Proposition 2.** *We have an isomorphism*

$$W^{(k)} \colon S(\mathbb{R}) \otimes \mathbb{C}^k \longrightarrow C^\infty(\mathbb{T} \times \mathbb{T}, \mathcal{L}^k)$$

*given by*

$$W^{(k)}(\mathsf{f})(u,v) = \frac{1}{\sqrt{k}} e^{-k\pi iuv} \sum_{j=0}^{k-1} \sum_{m\in\mathbb{Z}} f_j \left(u + \frac{m}{k}\right) e^{-2\pi imv} e^{-2\pi i \frac{jm}{k}}, \tag{1.11}$$

*with inverse*

$$\overline{W}^{(k)}(s)(x,j) = \frac{1}{\sqrt{k}} \sum_{l=0}^{k-1} e^{2\pi i \frac{lj}{k}} \int_0^1 s\left(x - \frac{l}{k}, v\right) e^{\pi ik(x+\frac{l}{k})v} \mathrm{d}v.$$

*It satisfies the following unitarity property*

$$\langle W^{(k)}(\mathsf{f}), W^{(k)}(\mathsf{g})\rangle = \langle \mathsf{f}, \mathsf{g}\rangle.$$

*Proof.* Let $\mathsf{f} \in \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^k$. We want to prove that $W^{(k)}(\mathsf{f}) \in \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T}, \mathcal{L}^k)$. It is very simple to see that $W^{(k)}(\mathsf{f})$ satisfies equations (1.9) using the explicit definition of $W^{(k)}$. We now proceed to show that $W^{(k)}(\mathsf{f})$ is of class $\mathrm{C}^\infty$. Suppose $k=1$ for simplicity, as the general case follows analogously. For every $p,q \in \mathbb{Z}_{\geq 0}$

$$\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial v^q} W^{(1)}(\mathsf{f})(u,v) = \sum_{m\in\mathbb{Z}} \frac{\partial^p}{\partial u^p}\left(\mathsf{f}(u+m)e^{-\pi iuv - \pi imv}(-\pi i)^q \left(u^q + (2m)^q\right)\right)$$

Since $\partial_u^p u^q \mathsf{f}(u) \in \mathcal{S}(\mathbb{R})$ the series in $m$ converges uniformly. Conversely, let $s \in \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T}, \mathcal{L})$, (again we suppose $k=1$ for simplicity) and consider $\overline{W}^{(1)}(s)(x)$. The function $h(x,v) \equiv s(x,v)e^{\pi ixv}$ is 1–periodic in $v$. For $N \in \mathbb{Z}_{>0}$, the derivative

$$\frac{\partial^p}{\partial x^p}\overline{W}^{(1)}(s)(x+N) = \int_0^1 \frac{\partial^p h}{\partial x^p}(x,v)e^{2\pi iNv}\mathrm{d}v$$

which is the $N$-th coefficient of the Fourier series of $\frac{\partial^p h}{\partial x^p}$ as 1–periodic function of $v$ and so it rapidly decrease as $N \to \pm\infty$. We now consider

$$\begin{aligned}
(x+N)^q \overline{W}^{(1)}(s)(x+N) &= \sum_{j=0}^{q} \binom{q}{j} x^{q-j} \int_0^1 N^j h(x+N,v)\mathrm{d}v \\
&= \sum_{j=0}^{q} \binom{q}{j} x^{q-j} \int_0^1 h(x,v) N^j e^{2\pi iNv}\mathrm{d}v \\
&= \sum_{j=0}^{q} \binom{q}{j} \frac{x^{q-j}}{(2\pi i)^j} \int_0^1 h(x,v) \frac{\partial^j}{\partial v^j}\left(e^{2\pi iNv}\right)\mathrm{d}v \\
&= \sum_{j=0}^{q} \binom{q}{j} \frac{x^{q-j}}{(-2\pi i)^j} \int_0^1 e^{2\pi iNv} \frac{\partial^j}{\partial v^j}\left(h(x,v)\right)\mathrm{d}v
\end{aligned}$$

the last being a finite sum of $N$-th Fourier coefficients for $q$ distinct functions $\frac{\partial^j h}{\partial v^j}$, $j = 0,\dots q$. So it rapidly decrease as well. This concludes the proof that $\overline{W}^{(1)}(s) \in \mathcal{S}(\mathbb{R})$.

Now we compute the composition of the two inverse formulas.

$$
\begin{aligned}
W^{(k)}(\overline{W}^{(k)}s)(u,v) &= \frac{1}{\sqrt{k}}e^{-k\pi iuv}\sum_{j=0}^{k-1}\sum_{m\in\mathbb{Z}}\overline{W}^{(k)}(s)\left(u+\frac{m}{k},j\right)e^{-2\pi imv}e^{-2\pi i\frac{jm}{k}}\\
&= \frac{1}{k}e^{-k\pi iuv}\sum_{l,\,j=0}^{k-1}\sum_{m\in\mathbb{Z}}e^{2\pi i\frac{lj}{k}}\int_0^1 s\left(u-\frac{l-m}{k},\tilde{v}\right)e^{\pi ik(u+\frac{l+m}{k})\tilde{v}}\mathrm{d}\tilde{v}e^{-2\pi imv}e^{-2\pi i\frac{jm}{k}}\\
&= \frac{1}{k}e^{-k\pi iuv}\sum_{r,\,l=0}^{k-1}\sum_{q\in\mathbb{Z}}\int_0^1 s\left(u-\frac{l-r}{k},\tilde{v}\right)e^{\pi ik(u+2q+\frac{l+r}{k})\tilde{v}}\mathrm{d}\tilde{v}\\
&\quad\times e^{-2\pi i(kq+r)v}\sum_{j=0}^{k-1}e^{2\pi i\frac{j(l-r)}{k}}, \qquad \text{(where } m=qk+r,\ 0\le r<k\text{ )}\\
&= \frac{1}{k}e^{-k\pi iuv}\sum_{r=0}^{k-1}\sum_{q\in\mathbb{Z}}\int_0^1 s(u,\tilde{v})e^{\pi ik(u+\frac{2}{k}(q+r))\tilde{v}}\mathrm{d}\tilde{v}\\
&\quad\times e^{-2\pi i(kq+r)v}\delta(r-l)k\\
&= e^{-k\pi iuv}\sum_{m\in\mathbb{Z}}e^{-2\pi imv}\int_0^1 s(u,\tilde{v})e^{\pi iku\tilde{v}}e^{2\pi im\tilde{v}}\mathrm{d}\tilde{v}\\
&= e^{-k\pi iuv}s(u,v)e^{\pi ikuv}=s(u,v)
\end{aligned}
$$

where the sum over $m$ is computed as a Fourier series in [0, 1] of the 1-periodic function $\psi(\tilde{v})\equiv s(u,\tilde{v})e^{\pi iu\tilde{v}}$. We now verify the unitary property

$$
\begin{aligned}
&\langle W^{(k)}(\mathsf{f}),W^{(k)}(\mathsf{g})\rangle =\\
&= \frac{1}{k}\int_0^1\int_0^1\sum_{j_1,\,j_2=0}^{k-1}\sum_{m_1,\,m_2\in\mathbb{Z}}\mathsf{f}\left(x+\frac{m_1}{k},j_1\right)\overline{\mathsf{g}\left(x+\frac{m_2}{k},j_2\right)}e^{-2\pi iy(m_1-m_2)}\mathrm{d}y\mathrm{d}x\\
&\quad\times e^{\frac{2\pi i}{k}(j_2m_2-j_1m_1)}\\
&= \frac{1}{k}\int_0^1\sum_{j_1,\,j_2=0}^{k-1}\sum_{m_1,\,m_2\in\mathbb{Z}}\mathsf{f}\left(x+\frac{m_1}{k},j_1\right)\overline{\mathsf{g}\left(x+\frac{m_2}{k},j_2\right)}\mathrm{d}x\,e^{\frac{2\pi i}{k}(j_2m_2-j_1m_1)}\\
&\quad\times\int_0^1 e^{-2\pi iy(m_1-m_2)}\mathrm{d}y\\
&= \frac{1}{k}\int_0^1\sum_{j_1,\,j_2=0}^{k-1}\sum_{m_1,\,m_2\in\mathbb{Z}}\mathsf{f}\left(x+\frac{m_1}{k},j_1\right)\overline{\mathsf{g}\left(x+\frac{m_2}{k},j_2\right)}\mathrm{d}x\\
&\quad\times e^{\frac{2\pi i}{k}(j_2m_2-j_1m_1)}\delta(m_1-m_2)\\
&= \frac{1}{k}\sum_{m\in\mathbb{Z}}\int_0^1\sum_{j_1=0}^{k-1}\mathsf{f}\left(x+\frac{m}{k},j_1\right)e^{-\frac{2\pi i}{k}mj_1}\sum_{j_2=0}^{k-1}\overline{\mathsf{g}\left(x+\frac{m}{k},j_2\right)e^{\frac{2\pi i}{k}mj_2}}\mathrm{d}x\\
&= \sum_{q\in\mathbb{Z}}\int_q^{q+1}\sum_{r=0}^{k-1}\mathsf{F}_k^{-1}(\mathsf{f})\left(x+\frac{r}{k},r\right)\overline{\mathsf{F}_k^{-1}(\mathsf{g})\left(x+\frac{r}{k},r\right)}\mathrm{d}x, \qquad (m\equiv qk+r)\\
&= \int_{-\infty}^{+\infty}\sum_{r=0}^{k-1}\mathsf{f}\left(x+\frac{r}{k},r\right)\overline{\mathsf{g}}\left(x+\frac{r}{k},r\right)\mathrm{d}x\\
&= \int_{-\infty}^{+\infty}\sum_{r=0}^{k-1}\mathsf{f}\left(x,r\right)\overline{\mathsf{g}\left(x,r\right)}\mathrm{d}x
\end{aligned}
$$

□

**Proposition 3.** *$W^{(k)}$ extend to an isometry from $\mathrm{L}^2(\mathbb{R})\otimes\mathbb{C}^k$ to $\mathrm{L}^2(\mathbb{T}\times\mathbb{T},\mathcal{L}^k)$.*

*Proof.* $W^{(k)}$ and its inverse are defined in open dense subspaces of the relevant Hilbert spaces. Moreover it is unitary with respect to the respective $L^2$-norms. So it will send $L^2$–convergent sequences in $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^k$ to $L^2$–convergent sequences in $C^\infty(\mathbb{T} \times \mathbb{T}, \mathcal{L}^k)$. This permits to extend the transforms $W^{(k)}$ and $\overline{W}^{(k)}$ in the $L^2$–completions of the two spaces. □

## 1.3 Teichmüller Theory

### 1.3.1 Teichmüller Theory

In this section we establish some basic facts about (Decorated) Teichmüller Theory. Most of the content here is standard and can be found e.g. in [Pen12] and [FM12]. Let $\Sigma_{g,s}$ be a topological surface of genus $g$ with $s$ punctures. Broadly speaking the Teichmüller space $\mathcal{T}_{g,s} \equiv \mathcal{T}(\Sigma)$ is the space of all the complex structures admitted by $\Sigma$ up to isotopy. However the first half of this thesis deals with cases where $\chi(S) \equiv 2 - 2g + s$ is negative. In this case there is a remarkable phenomena that makes the definition of Teichmüller space equivalent to the one given as space of all hyperbolic metric structures up to isotopy. In the last Chapter 5, we will look closer to the special non hyperbolic case of $g = 1$, and $s = 0$.

Suppose now $\chi(S) \equiv 2 - 2g + s < 0$. Let $\mathcal{H}$ be the space of all the possible complete hyperbolic Riemannian structures on $\Sigma_{g,s}$ and let $\mathrm{Diff}^+(\Sigma_{g,s})$ be the space of the orientation preserving diffeomorphisms of $\Sigma_{g,s}$. Define also $\mathrm{Diff}^+_0(\Sigma_{g,s})$ to be the connected component of $\mathrm{Diff}^+(\Sigma_{g,s})$ containing the identity map.

We call the space $\mathcal{M}_{g,s} := \mathcal{H}/\,\mathrm{Diff}^+(\Sigma_{g,s})$ the Riemann moduli space of $\Sigma_{g,s}$. The space

$$\mathcal{T}_{g,s} := \mathcal{H}/\,\mathrm{Diff}^+_0(\Sigma_{g,s})$$

is called Teichmüller space of $\Sigma_{g,s}$ while the group

$$\mathrm{MCG}_{g,s} := \mathrm{Diff}^+(\Sigma_{g,s})/\,\mathrm{Diff}^+_0(\Sigma_{g,s})$$

is called Mapping Class Group of $\Sigma_{g,s}$.

It follows that $\mathcal{M}_{g,s} = \mathcal{T}_{g,s}/\,\mathrm{MCG}_{g,s}$ and the study of the action of the Mapping Class Group on the Teichmüller space become of great interest in order to understand the topology of the Riemann moduli space.

There is an equivalent definition of Teichmüller space relevant for our purpose. Let $\mathrm{Hom}^{\mathrm{dfp}}(\pi_1(\Sigma_{g,s}), \mathrm{PSL}(2,\mathbb{R}))$ be the space of group representations of $\pi_1(\Sigma_{g,s})$ into $\mathrm{PSL}(2,\mathbb{R})$ that are *discrete*, *faithful* and that send *peripheral* homotopy class in $\pi_1(\Sigma_{g,s})$ to *parabolic* elements of $\mathrm{PSL}(2,\mathbb{R})$ (an homotopy class is *peripheral* if it represents a curve around a puncture). We have the following isomorphism:

$$\mathcal{T}_{g,s} \cong \mathrm{Hom}^{\mathrm{dfp}}(\pi_1(\Sigma_{g,s}), \mathrm{PSL}(2,\mathbb{R}))/\,\mathrm{PSL}(2,\mathbb{R})$$

where $\mathrm{PSL}(2,\mathbb{R})$ acts by conjugation.

From this point of view $\Gamma \in \mathcal{T}_{g,s}$ is the conjugacy class of a representation of $\pi_1(\Sigma_{g,s})$ satisfying some properties. The image of such a representation is called a *Fuchsian* group. The following properties of the Tichmuller space are well known

**Theorem 4.** *The Teichmüller Space $\mathcal{T}_{g,s}$ is homeomorphic to $\mathbb{R}^{6g-6+2s}$.*

For details about Teichmüller theory and generally hyperbolic geometry see, for example [FM12] [Pen12] and [BP92]. From now to the end of this Section we will concentrate on the punctured case , i.e. $s > 0$.

**Definition 3.** A *decorated hyperbolic structure* $\tilde{\Gamma}$ on $\Sigma_{g,s}$ is a conjugacy class of Fuchsian group $\Gamma$ together with an $s$-tuple of horocycles, one for each puncture.

The space of all the decorated hyperbolic structures $\widetilde{\mathcal{T}}_{g,s}$ is called *Decorated Teichmüller Space*. The specification of the hrocycles reduces to the specification of a positive real number for each of them (heuristically, its hyperbolic length on the surface) so we get

$$\dim \widetilde{\mathcal{T}}_{g,s} = 6g - 6 + 3s. \tag{1.12}$$

We want to put coordinates in this space. Let first define a combinatorial data on the topological surface, over which the coordinates will be defined.

**Definition 4.** An *ideal arc* $\alpha$ is the homotopy class relative to the endpoints of the embedding of a path in $\Sigma_{g,s}$, such that the endpoints are punctures of the surface.
An *ideal triangle* is a triangle with the vertices removed, such that the edges are ideal arcs.
An *ideal triangulation* $\tau$ of $\Sigma_{g,s}$ is a collection of disjoint ideal arcs such that $\Sigma_{g,s} \setminus \tau$ is a collection of the interior of ideal triangles. The set of all the ideal triangulation of $\Sigma_{g,s}$ will be denoted by $\Delta = \Delta(\Sigma_{g,s})$.

Given an ideal triangulation $\tau$, $\Delta_j(\tau)$ will denote the set of its $j$-dimensional cells. Fix a decorated hyperbolic structure $\tilde{\Gamma}$ for $\Sigma_{g,s}$. Consider two punctures $p_1$ and $p_2$ on $\Sigma_{g,s}$ and the two horocycles $h_1$ and $h_2$ associated with them. Let $\gamma$ be a geodesic representative of an ideal arc $\alpha$ between $p_1$ and $p_2$. The signed hyperbolic distance between $h_1$ and $h_2$ along $\gamma$ is well defined as follows: look at the universal cover $\mathbb{H}^2$ of $\Sigma_{g,s}$, then $\gamma$ will lift to a geodesic line $\tilde{\gamma}$ between the two ideal points in $S^1 = \partial\mathbb{H}^2$ corresponding to $p_1$ and $p_2$, and $h_1$ and $h_2$ will lift to horocycles $\tilde{h}_1$ and $\tilde{h}_2$ in $\mathbb{H}^2$ tangent to them. Let $\delta$ be the signed geodesic distance between $\tilde{\gamma} \cap \tilde{h}_1$ and $\tilde{\gamma} \cap \tilde{h}_2$ (take the sign positive if $\tilde{h}_1 \cap \tilde{h}_2 = \emptyset$, negative otherwise).

**Definition 5.** In the situation above we can define the *lambda length* of $\alpha$ as

$$\lambda(\alpha, \tilde{\Gamma}) := \sqrt{e^{\delta}}.$$

Now we can find coordinates on $\widetilde{\mathcal{T}}_{g,s}$ in terms of the lambda lengths.

**Theorem 5** (Penner [Pen87])**.** *Let $\tau$ be an ideal triangulation of $\Sigma_{g,s}$, with $s \geq 1$ and $2g - 2 + s > 0$. Then the natural mapping*

$$\Lambda_\tau : \widetilde{\mathcal{T}}_{g,s} \longrightarrow \mathbb{R}_{>0}^{\Delta_1(\tau)}, \qquad \tilde{\Gamma} \mapsto (\alpha \mapsto \lambda(\alpha, \tilde{\Gamma})).$$

*is a real analytic homeomorphism.*

With these coordinates we can explicitly see an action of $\mathbb{R}^s_{>0}$ on $\widetilde{\mathcal{T}}_{g,s}$ which gives the projection $\widetilde{\mathcal{T}}_{g,s} \longrightarrow \mathcal{T}_{g,s}$, namely if $\alpha \in \tau$ is an ideal arc between the punctures $p$ and $q$ and $f \in \mathbb{R}^s_{>0}$, then the action in coordinates is as follows

$$\lambda(\alpha, f \cdot \tilde{\Gamma}) = f(p) f(q) \lambda(\alpha, \tilde{\Gamma}). \tag{1.13}$$

**Lemma 6** (Ptolemy Relation)**.** *Consider an ideal quadrilateral with lambda lengths of the sides $a$, $b$, $c$ and $d$ in this cyclic order. Then there are two different ways to choose a diagonal. Let $e$ and $f$ be the lambda lengths of the two possible diagonals. Then the following relation holds:*

$$ac + bd = ef. \tag{1.14}$$

The move that exchange the diagonal in a quadrilateral is called *diagonal exchange.*

**Fact 7** (Whithehead's Classical Fact)**.** *Two ideal triangulations are related by a finite sequence of diagonal exchanges.*

Lemma 6 gives the change of coordinates between $\Lambda_\tau(\widetilde{\mathcal{T}}_{g,s})$ and $\Lambda_{\tau'}(\widetilde{\mathcal{T}}_{g,s})$.

The Teichmüller space $\mathcal{T}_{g,s}$ has a natural symplectic structure given by the Weil Petersson form. For a geometric description of it we refer to [Wol83]. When pulled back to $\widetilde{\mathcal{T}}_{g,s}$ it has a remarkably simple expression. Let $\tau$ be an ideal triangulation of $\Sigma_{g,s}$, let $\Delta_2(\tau)$ the set of ideal triangles in $\tau$, then the pull back of the Weil Petersson symplectic form on $\widetilde{\mathcal{T}}_{g,s}$ is the following Penner 2-form

$$\omega_P = \sum_{t\in\Delta_2(\tau)} \frac{\mathrm{d}a\wedge\mathrm{d}b}{ab} + \frac{\mathrm{d}b\wedge\mathrm{d}c}{bc} + \frac{\mathrm{d}c\wedge\mathrm{d}a}{ca} \tag{1.15}$$

where $a$, $b$ and $c$ are the lambda lengths of the three sides of $t$, in the cyclic order defined by the orientation on $\Sigma_{g,s}$.
For more details on the Decorated Teichmüller Space see the book [Pen12].

### 1.3.2 Coordinates for the $\mathrm{PSL}(2,\mathbb{C})$ Moduli Spaces

In this section we want to describe how *Complexified* Lambda Lengths can be used to describe local coordinates on the space

$$\mathcal{M} \equiv \mathrm{Hom}^{p,i}(\pi_1(\Sigma_{g,s}),\, \mathrm{PSL}(2,\,\mathbb{C}))/\mathrm{PSL}(2,\,\mathbb{C}), \tag{1.16}$$

where the action is by conjugation, the $p$ stays for *peripheral* and the $i$ for *irreducible*, i.e. $\mathcal{M}$ is the space of irreducible representations of $\pi_1(\Sigma_{g,s})$ up to conjugation with holonomy around punctures conjugate to a parabolic element of $\mathrm{PSL}(2,\mathbb{C})$. We will need to give a construction analogous to the decoration on the Teichmüller space to define coordinates. In this section we strictly follow the description given by Kashaev in [Kas05] for the coordinates of the $\mathrm{PSL}(2,\mathbb{R})$ moduli space, adapting to the complex case. The topic is also treated in the more general and systematic language of cluster coordinates in [FG06].
We have a fixed surface $\Sigma_{g,s}$ of genus $g$ with $s$ punctures. Let $\overline{\Sigma}$ be its closure and let $\kappa \equiv -\chi(\Sigma) = 2g-2+s > 0$. In this section by *unipotent subgroup* we mean a one parameter subgroup $U$ of $\mathrm{PSL}(2,\,\mathbb{C})$ generated by parabolic elements, i.e. a subgroup conjugated to the following one

$$U = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},\, \lambda\in\mathbb{C} \right\}. \tag{1.17}$$

Fix the conjugacy class of such $U$ as above in (1.17). The normalizer of $U$ will be the Borel subgroup $N(U) = B$ of upper triangular matrices and the one parameter group $T = B/U$ is identified with diagonal matrices. Let $N(T)/T \ni \theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which satisfies $\theta^2 = 1$ in $\mathrm{PSL}(2,\,\mathbb{C})$. We have the Bruhat decomposition

$$\mathrm{PSL}(2,\,\mathbb{C}) = B\theta B \sqcup B,$$

where the union is disjoint.

**Definition 6.** Let $a$ and $b$ be two paths starting at the same point $x \in \Sigma$ and ending respectively at the punctures $P$ and $Q$. Let $m \in \mathcal{M}$ be a fixed flat connection.
(i) We define the homotopy class $l(a)$ of the loop based at $x$ that goes along $a$, then makes a small loop around $P$ and then closes along $a^{-1}$.
(ii) We say that the homotopy class relative to the endpoints of $a^{-1}b$ is $m$-admissible if $l(a)$ and $l(b)$ belong to distinct unipotent subgroups (of parabolic elements) of $\mathrm{PSL}(2, \mathbb{C})$.
(iii) For any ideal triangulation $\tau$ of $\Sigma$, define $\mathcal{M}_\tau$ as the set of flat connections $m \in \mathcal{M}$ such that $\tau$ is $m$-admissible.

The following theorem was proved in [Kas05]. Let $\Delta = \Delta(\Sigma)$ be the set of ideal triangulations of $\Sigma$.

**Theorem 8.** *$\{\mathcal{M}_\tau\}_{\tau \in \Delta(\Sigma_{g,s})}$ is a covering of $\mathcal{M}$.*

In this section we want to give a description of a set of coordinates for $\mathcal{M}$ which are an analogous of Penner coordinates for the Teichmüller space.

**Definition 7** (Graph Connection)**.** Let $\Gamma$ be a graph. Let $G$ be a Lie Group. A *flat graph $G$-connection* is the assignment of an element $g_e \in G$ to every edge $e$ in $\Gamma$ such that $g_e^{-1} = g_{\overline{e}}$, where $\overline{e}$ is the reversely oriented edge corresponding to $e$
Two flat graph $G$ connections $(g_e)_e$ and $(h_e)_e$ are equivalent if there exists a $k_v \in G$ for every vertex $v$ of $\Gamma$ such that $g_e = k_v^{-1} h_e k_w$ for every edge $e$ connecting the vertices $v$ and $w$.
Given a graph $\Gamma \subset \Sigma_{g,s}$ embedded into a surface, a flat graph $G$ connection $h$ on $\Gamma$ and a (gauge equivalene class of) flat connection $m$ on $\Sigma_{g,s}$ we say that $h$ represents $m$ if there is a representative of $m$ inducing a parallel transport operator on $\Gamma$ equivalent, as graph connection, to $h$.

*Remark* 1.3.1. If $\Sigma_{g,s}$ retracts to an homotopically equivalent graph $\Gamma$ supporting a flat graph $G$ connection $A_\Gamma$, then $A_\Gamma$ defines a flat $G$ connection over $\Sigma_{g,s}$.

Let $\mathcal{A} = \mathcal{A}(\Sigma_{g,s})$ be the set homotopy class of ideal arcs of $\Sigma \equiv \Sigma_{g,s}$ relative to the endpoints. Let $\hat{\Sigma} \subset \overline{\Sigma}$ be the complement of a set of $s$ disjoint disks, each centred at one puncture. Then

$$\partial\Sigma = \bigcup_P L(P),$$

where $P$ varies among the punctures of $\Sigma$ and $L(P)$ is diffeomorphic to a circle.
Let $\alpha \subseteq \mathcal{A}$ be a collection of ideal arcs. We define the graph $\Gamma(\alpha) = \{(e, p(e), q(e))\}_{e \in \alpha}$ where $e$ connects the two punctures $P_e$ and $Q_e$, $p(e) \in L(P_e)$ and $q(e) \in L(Q_e)$ are chosen as follows. For any two distinct homotopy class of ideal arcs $e$ and $f \in \alpha$, there exist paths $a \in e$ and $b \in f$ so that $a \cap \hat{\Sigma}$ goes from $p(e)$ to $q(e)$, $b \cap \hat{\Sigma}$ goes from $p(f)$ to $q(f)$ and $|a \cap b|$ is minimal as $a$ and $b$ vary in their respective homotopy classes of ideal arcs. The set $\{(p(e), q(e))\}_{e \in \alpha}$ will be the set of vertices of $\Gamma(\alpha)$ and there will be two types of edges: the one defined by ideal arcs $e \in \alpha$ that we will call *long* edges, and the one which are paths in $\partial\hat{\Sigma}$ that we call *short* edges.
Given $\alpha \subseteq \mathcal{A}$ we can define $\mathcal{M}(\alpha)$ as the set of $m \in \mathcal{M}$ such that $\alpha$ is a maximal set of $m$–admissible ideal arcs. Then

$$\mathcal{M} = \bigcup_\alpha \mathcal{M}(\alpha) \tag{1.18}$$

$$\mathcal{M}_\tau = \bigcup_{\alpha \supseteq \Delta_1(\tau)} \mathcal{M}(\alpha). \tag{1.19}$$

**Proposition 9.** *There exist a principal $\mathbb{C}^s$–bundle $\pi\colon \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ and, for every ideal triangulation $\tau$ of $\Sigma$, a function $\phi_\tau\colon \mathbb{C}_{\neq 0}^{\Delta_1(\tau)} \longrightarrow \mathbb{C}_{\neq 0}$ such that:*
*(i) For every ideal triangulation $\tau$, $\pi^{-1}(\mathcal{M}_\tau) \simeq \mathbb{C}_{\neq 0}^{\Delta_1(\tau)} \setminus \phi_\tau^{-1}(0)$.*
*(ii) The structure group action on $\pi^{-1}(\mathcal{M}_\tau)$ is as follows:*

$$\mathbb{C}_{\neq 0}^{\tau} \times \mathbb{C}_{\neq 0}^{s} \ni (f, g) \mapsto f \cdot g \in \mathbb{C}_{\neq 0}^{\Delta_1(\tau)}$$

*Where $f \cdot g(e) := f(e)g(P)g(Q)$ for every ideal arc $e \in \tau$ running between the ideal vertices $P$ and $Q$.*
*(iii)Suppose that two ideal triangulations $\tau$ and $\tau'$ are related by one diagonal exchange $e \mapsto e'$ inside the quadrilateral with ideal edges $a$, $b$, $c$ and $d$. Suppose that $m \in \mathcal{M}_\tau$ and $f \in \mathbb{C}_{\neq 0}^{\Delta_1(\tau)} \setminus \phi_\tau^{-1}(0)$ such that*

$$\frac{f(a)f(b)}{f(c)f(d)} + 1 \neq 0, \tag{1.20}$$

*then there exists $f' \in \mathbb{C}_{\neq 0}^{\Delta_1(\tau')} \setminus \phi_{\tau'}^{-1}(0)$ corresponding to the same element of $\pi^{-1}(m)$ as $f$ such that $f(x) = f'(x)$ for every $x \neq e, e'$ and*

$$f'(e')f(e) = f(a)f(c) + f(b)f(d). \tag{1.21}$$

*Proof.* Let $\alpha \subset \mathcal{A}$. We want to construct a fiber bundle $\pi\colon \widetilde{\mathcal{M}}(\alpha) \longrightarrow \mathcal{M}(\alpha)$ for each $\alpha$ and then combine them to a global fiber bundle $\pi\colon \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ in analogy to the Decorated Teichmüller space. Let $m \in \mathcal{M}(\alpha)$, the fiber $\pi^{-1}(m)$ is defined as the space of equivalence classes of flat graph-connections on $\Gamma(\alpha)$ representing $m$ and such that all short edges have assigned unipotent elements, while long edges have assigned $\theta T$ elements.
Let $h$ be an $\Gamma(\alpha)$ flat graph cnnection representing $m$. Since $\alpha$ is $m$–admissible, the $h$-holonomies around the boundary components of $\hat{\Sigma}$ based at respective vertices in the components it selves are all is some unipotent subgroups. We can replace $h$ by an equivalent flat graph connection were all this holonomies are in the same unipotent subgroup $U \subset G$. This makes the parallel transport operators along single short edges to be in $B = N(U)$. Now the $m$-admissibility conditions for the long edges, implies that parallel transport along them has to conjugate the holonomy around a puncture to a different unipotent subgroup (or it will be the same of the puncture at the other end). This together with the Bruhat decomposition gives that parallel transports along long edges has to be in $B\theta B$. Up to conjugation with elements in $B$, we can make them be elements of $\theta T$ (indeed, as Möbius transformations, elements of $B\theta B$ are characterised by not preserving $\infty$, and conjugating by elements in $B$ preserves this property). We can still gauge transform by $T$ valued functions at the vertices of the graph, so to restrict the parallel transport along short edges to be in $U$. We are left with some freedom in the equivalence relation given by the choice of a $T$ valued function on the vertices of $\Gamma(\alpha)$ which is constant on vertices in the same boundary component. This argument shows that $\pi^{-1}(m)$ has the structure of a $T^s$-torsor.

Now we look at an ideal triangle $t$ in $\tau$ with ideal vertices $\{v_0, v_1, v_2\}$ and sides $e_i$ respectively, opposite to $v_i$ for $i = 0, 1, 2$. Restricted to $\hat{\Sigma}$, $t$ will represents an hexagon with short edges $f_i$, $i = 0$, 1, 2 opposite to the long edges $e_i$. We now know that parallel transport along the long edge $e_i$ will have the form $\begin{pmatrix} 0 & \lambda_i^{-1} \\ -\lambda_i & 0 \end{pmatrix}$, $\lambda_i \in \mathbb{C}_{\neq 0}$ while parallel transport along short edges will be of the form $\begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix}$,

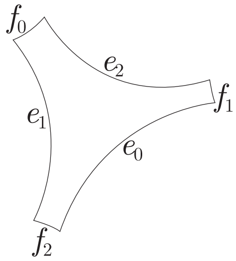



Figure 1.1: An ideal Triangle cut by short edges.

$u_i \in \mathbb{C}$. Since the parallel transport along the the whole hexagon has to be the identity, explicit computation gives condition

$$u_i = \frac{\lambda_i}{\lambda_j \lambda_k} \qquad \{i, j, k\} = \{0, 1, 2\}. \tag{1.22}$$

This permits to conclude that for every $\alpha \supseteq \Delta_1(\tau)$ the parallel transport in $\Gamma(\alpha)$ is completely determined by parallel transport in $\Gamma(\tau)$, since each time we add a long edge to $\Gamma(\tau)$ one of its opposite short edge and the two long edges around it will determine its parallel transport.
Moreover the formula gives a way to associate to each $h \in \pi^{-1}(\mathcal{M}_\tau)$ a value in the set $\mathbb{C}_{\neq 0}^{\Delta_1(\tau)}$. However one should be careful that the parallel transport along short edges compose to unipotent parabolic elements as holonomies around punctures. This can be state as follows. Given $f \in \mathbb{C}_{\neq 0}^{\tau}$ consider for any ideal triangle $t$ of $\tau$ having $P_i$, $i = 0, 1, 2$ as ideal vertices the numbers

$$u_{P_i}^t(f) = \frac{f(e_i)}{f(e_j) f(e_k)}$$

where $e_i$ is the ideal edge opposite to $P_i$. Define the following function

$$\phi_\tau(f) = \Pi_{P \in \Delta_0(\tau)} \phi_{\tau,\, P}(f), \qquad \phi_{\tau,\, P}(f) = \sum_{\Delta_3(\tau) \ni t \ni P} u_P^t(f) \tag{1.23}$$

Then $\phi_\tau(f) = 0$ if and only if there is a puncture with non parabolic holonomy.
This permits to conclude that $\pi^{-1}(\mathcal{M}_\tau) \simeq \mathbb{C}_{\neq 0}^{\Delta_1(\tau)} \setminus \phi_\tau^{-1}(0)$.
Finally we come back to the action of $T^s \simeq \mathbb{C}_{\neq 0}^s$ on $\pi^{-1}(\mathcal{M}_\tau)$. Let $f \in \mathbb{C}_{\neq 0}^{\Delta_1(\tau)} \setminus \phi_\tau^{-1}(0)$ and $g \in \mathbb{C}_{\neq 0}^s$. For every ideal arc $e \in \tau$ from the ideal verticex $P$ to the ideal vertex $Q$ we have associated a matrix $F(e) = \begin{pmatrix} 0 & f(e)^{-1} \\ -f(e) & 0 \end{pmatrix} \in \theta T$, and two matrices $G(P) = \begin{pmatrix} g(P) & 0 \\ 0 & g(P)^{-1} \end{pmatrix} \in T$ and $G(Q) = \begin{pmatrix} g(Q) & 0 \\ 0 & g(Q)^{-1} \end{pmatrix} \in T$. The action for every such arc $e$ is of the type $G(P)F(e)G(Q)^{-1}$, which in coordinates reads as $f \cdot g(e) = f(e) g(P) g(Q)$.

Now consider two ideal triangulations $\tau$ and $\tau'$ related by a unique diagonal exchange from the side $e$ to $e'$ inside the quadrilateral $q \subset \tau \cap \tau'$ with ideal boundary made of the four ideal arcs $a$, $b$, $c$ and $d$. Suppose that $e$ cuts the quadrilateral in the two triangle $\{a, b, e\}$ and $\{e, c, d\}$. Suppose now that there exists $m \in \mathcal{M}_\tau \cap \mathcal{M}_{\tau'}$ and let $f \in \mathbb{C}_{\neq 0}^{\Delta_1(\tau)} \setminus \phi_\tau^{-1}(0)$ and $f' \in \mathbb{C}_{\neq 0}^{\Delta_1(\tau')} \setminus \phi_{\tau'}^{-1}(0)$ both corresponding to the same element in $\pi^{-1}(m)$. Computing the parallel transport operators along $\Gamma(\tau) \cap \Gamma(\tau')$ gives the same values for $f$ and $f'$ along every edge except $e$ and $e'$. Now compute

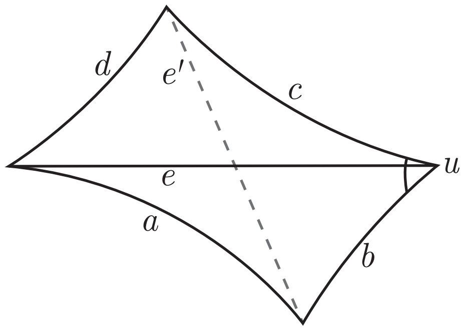



Figure 1.2: Diagonal exchange.

the parallel transport along one of the short edges of $\Gamma(\tau')$ opposite to $e'$ gives

$$u' = \frac{f'(e')}{f'(b)f'(c)} = \frac{f'(e')}{f(b)f(c)}.$$

On the other side the two short edges of $\Gamma(\tau)$ opposite to $a$ and $d$ give a combined parallel transport

$$u = \frac{f(a)}{f(b)f(e)} + \frac{f(d)}{f(c)f(e)}$$

Since $f$ and $f'$ represents the same element one has $u = u'$, which can be rewritten as

$$f'(e')f(e) = f(a)f(b) + f(c)f(d). \tag{1.24}$$

It is simple to see from this that a necessary condition for $m \in \mathcal{M}_\tau$ to be also in $\mathcal{M}_{\tau'}$ is that

$$\frac{f(a)f(b)}{f(c)f(d)} + 1 \neq 0.$$

This is also sufficient by construction. □

We will call the spaces $\Lambda_{\mathbb{C}}(\tau) := \pi^{-1}(\mathcal{M}_\tau)$ spaces of *complexified lambda length* coordinates in analogy with Penner coordinates for the Teichmüller space.

### 1.3.3 Ratio Coordinates

We are going to introduce the *ratio coordinates*, following the work of Kashaev [Kas98]. Ratio coordinates are a generalization of real Penner's lambda lengths. Later in section 1.3.5 we will discuss their complexification.

**Definition 8.** A *decorated ideal triangulation* of $\Sigma_{g,s}$ is an ideal triangulation $\tau$ up to isotopy relative to the punctures, together with the choice distinguished corner in each ideal triangle and a bijective ordering map

$$\overline{\tau} : \{1, \ldots, s\} \ni j \mapsto \overline{\tau}_j \in \Delta_2(\tau).$$

We denote the set of all decorated ideal triangulation as $\dot{\Delta} = \dot{\Delta}(\Sigma_{g,s})$.

When we say that $\tau$ is a decorated ideal triangulation we mean that $\tau$ is the set of decorated ideal triangles in the triangulation. For the remaining part of this subsection a decorated ideal triangulation $\tau$ of $\Sigma_{g,s}$ is considered fixed.

Let the *Ratio coordinates space* be defined as $\mathcal{R}(\tau) \cong \mathbb{R}^\tau_{>0} \times \mathbb{R}^\tau_{>0}$. There is an action of $\mathbb{P}(\mathbb{R}^s_{>0})$ on $\mathcal{R}(\tau)$, in any decorate ideal triangle of ratio coordinates

$x = (x_1, x_2)$ and ideal vertices $v_a$, $v_b$ and $v_c$, the last one being the distinguished one, we have

$$f \cdot x = \left( \frac{f(v_c)}{f(v_b)} x_1, \frac{f(v_c)}{f(v_a)} x_2 \right), \qquad \forall f \in \mathbb{P}(\mathbb{R}^s_{>0}) \tag{1.25}$$

We can map $\Lambda(\tau)$ into it via $\rho : \Lambda(\tau) \longrightarrow \mathcal{R}(\tau)$ which, for any triangle in $\tau$, sends the three lambda lengths $a$, $b$ and $c$ to the couple $(\frac{b}{c}, \frac{a}{c})$ where we are supposing the distinguished corner to be the one opposite to the edge of length $c$ and the cyclic ordering $a$, $b$, $c$ to be induced by the orientation on the surface. The map is compatible with the action of $\mathbb{R}^s_{>0}$ on $\Lambda(\tau)$ (1.13). Explicitly

$$\rho(f \cdot \lambda) = f \cdot \rho(\lambda) \qquad \text{for every } \lambda \in \Lambda(\tau),\ f \in \mathbb{R}^s_{>0}. \tag{1.26}$$

This makes $\rho$ a principal bundle morphism between $\Lambda(\tau)$ and $\mathcal{R}(\tau)$. Moreover there is a symplectic structure

$$\omega_{\mathcal{R}} = \sum_{x \in \tau} \frac{\mathrm{d}x_1}{x_1} \wedge \frac{\mathrm{d}x_2}{x_2}$$

which pulls back to the Penner form: $\omega_P = \rho^* \omega_{\mathcal{R}}$.

*Remark* 1.3.2. We can endow the sets $\Lambda(\tau)$ and $\mathcal{R}(\tau)$ with a multiplicative group structure defined by component–wise multiplication. This is artificial but $\rho$ is an homomorphism with respect to this structure and has kernel equal to the constant functions in $\Lambda(\tau)$

Given an homology class of a closed curve $\gamma$ in $\Sigma_{g,s}$ we want to define the *holonomy* $\mathrm{Hol}(x, \gamma)$ of such class, with respect to the ratio coordinates $x \in \mathcal{R}(\tau)$. First of all choose a representative for $\gamma$ such that, for every ideal triangle $t$, $\gamma \cap t$ is either empty or a simple arc which intersect $\partial t$ in exactly two distinct points in two distinct edges. In this way $\gamma$ decomposes in a finite number of oriented arcs $\gamma_i$, each contained in one ideal triangle. We define $\mathrm{Hol}(x, \gamma) := \prod_i \mathfrak{u}(\gamma_i)$, where $\mathfrak{u}$ is defined as follows. For every $i$, let $\gamma_i$ be the oriented arc from the point $p$ to the point $q$ laying inside the triangle with ratio coordinates $(x_1, x_2)$ and with edges labeled $a$, $b$, $c$ with the distinguished corner opposite to $c$. Then $\mathfrak{u}(\gamma_i)$ is defined as

$$\mathfrak{u}(\gamma_i) = \begin{cases} x_2 & \text{if } p \in a \text{ and } q \in c \\ x_1 & \text{if } p \in b \text{ and } q \in c \\ x_2/x_1 & \text{if } p \in a \text{ and } q \in b \end{cases} \tag{1.27}$$

and $\mathfrak{u}(-\gamma_i) := \mathfrak{u}(\gamma_i)^{-1}$. Notice that $\mathrm{Hol}(x, \partial R) = 1$, for every region $R \subseteq \Sigma_{g,s}$ with no punctures in its interior.

We also define a *moment map* $\mu_\tau : \mathcal{R}(\tau) \longrightarrow \mathrm{H}^1(\Sigma_{g,s}, \mathbb{R})$ by the rule

$$\langle \mu_\tau(x); \gamma \rangle = \log \mathrm{Hol}(x, \gamma).$$

Notice that the bracket $\langle \mu_\tau; \cdot \rangle$ is linear in the second argument and only depends on the homology class. In particular we remark that

$$\mu_\tau(x) = 0 \iff x = \rho(\lambda) \text{ for some } \lambda \in \Lambda(\tau)$$

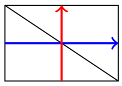
The statement can be easily seen noticing that the map $\mathfrak{u}(\gamma)$ for homologically non trivial $\gamma$ gives exactly the obstruction to construct lambda length on the triangles involved by $\gamma$. Doing this for all the loops around punctures is sufficient to involve all the ideal triangles.

Moreover $\mu_\tau$ is a group homomorphism with respect to the group structure of Remark 1.3.2. Recall that $\mathrm{H}_1(\Sigma_{g,s}, \mathbb{R})$ support an intersection form $\circ$.

**Proposition 10.** *The Poisson bracket induced by $\omega_{\mathcal{R}}$ is given by the intersection form $\circ$ in $\mathrm{H}_1(\Sigma_{g,s}, \mathbb{R})$, i.e.*

$$\frac{1}{2}\{\langle\mu_\tau;\gamma_1\rangle, \langle\mu_\tau;\gamma_2\rangle\} = \gamma_1 \circ \gamma_2.$$

*Proof.* The proof is an in-coordinate verification, that can be carried out deforming $\gamma_1$ and $\gamma_2$ so that they intersect only in points looking as follows



and recalling that every ideal triangle define a distinct symplectic leaf with respect to $\omega_{\mathcal{R}}$. □

Let $v_i$, $i = 1,\ldots,s$ be the punctures on the surface $\Sigma_{g,s}$, and le $\gamma_{v_i}$ be the homology class of curve around a puncture, oriented counterclockwise w.r.t. the orientation of $\Sigma_{g,s}$. For any $f \in \mathbb{P}(\mathbb{R}^s_{>0})$ define $\xi_f := \sum_{i=1}^s \gamma_{v_i} \log f(v_i) \in \mathrm{H}_1(\Sigma_{g,s}, \mathbb{R})$. This defines a group homomorphism $\mathbb{P}(\mathbb{R}^s_{>0}) \longrightarrow \mathrm{H}_1(\Sigma_{g,s}, \mathbb{R})$.

**Proposition 11.** *For every $f \in \mathbb{P}(\mathbb{R}^s_{>0})$ the Hamiltonian vector field $X_{\langle\mu_\tau;\xi_f\rangle}$ of $\langle\mu_\tau;\xi_f\rangle \in C^\infty(\mathcal{R}(\tau))$ corresponds to the vector field induced by the infinitesimal action of $f \in \mathbb{P}(\mathbb{R}^s_{>0})$ on $\mathcal{R}(\tau)$, i.e. if $V_f \in \mathrm{Lie}(\mathbb{P}(\mathbb{R}^s_{>0}))$ is such that $\exp(V_f)\cdot x = f\cdot x$, then*

$$\left.\frac{d}{dt}\right|_{t=0} (\exp(tV_f)\cdot x) = X_{\langle\mu_\tau;\xi_f\rangle}|_x, \qquad \forall x \in \mathcal{R}(\tau)$$

Proposition 11 says that the action of $f \in \mathbb{P}(\mathbb{R}^s_{>0})$ on $\mathcal{R}(\tau)$ is Hamiltonian being $H_f := \langle\mu_\tau;\xi_f\rangle$ the Hamiltonian for the infinitesimal action associated to $f$. Explicitly it gives an identification between $\mathrm{Lie}(\mathbb{P}(\mathbb{R}^s_{>0}))$ and the subspace

$$\mathrm{span}\{\gamma_{v_i} : v_i \text{ is a puncture}\} \subseteq H_1(\Sigma_{g,s}, \mathbb{R}),$$

moreover the association $V_f \mapsto H_f$ is trivially an Lie algebra homomorphism, being the bracket on the right trivial due to Proposition 10, and $\mathrm{Lie}(\mathbb{P}(\mathbb{R}^s_{>0}))$ abelian.
In the end we have the following exact sequence of group homomorphisms:

$$1 \longrightarrow \mathbb{R}_{>0} \xrightarrow{i} \Lambda_\tau(\widetilde{\mathcal{T}}_{g,s}) \xrightarrow{\rho} \mathcal{R}(\tau) \xrightarrow{\mu_\tau} \mathrm{H}^1(\Sigma_{g,s}, \mathbb{R}) \longrightarrow 0,$$

where $i(a) \in \Lambda_\tau(\widetilde{\mathcal{T}}_{g,s})$ is the structure with lambda length equal to $a$ in all the edges, 1 stand for the trivial multiplicative group while 0 for the trivial additive group.
In particular

$$\mathcal{T}_{g,s} \cong \widetilde{\mathcal{T}}_{g,s}/\mathbb{R}^s_{>0} \cong \mu_\tau^{-1}(0)/\mathbb{P}(\mathbb{R}^s_{>0}),$$

where the quotient is an Hamiltonian reduction.
The space $\mathcal{R}(\tau)$ was introduced by Kashaev in [Kas98].

### 1.3.4 Ptolemy Groupoid Representations

One of the main interests in quantizing moduli spaces is the consequent construction of representations of (central extensions of) the mapping class group of the surfaces. Quantum Teichmüller theory produce instead representations of a bigger object called (decorated) *Ptolemy Groupoid* that we are going to introduce now. Recall that, given a group $G$ acting freely on a set $X$, we can define an associated groupoid $\mathcal{G}$ as follows. The objects of $\mathcal{G}$ are $G$-orbits in $X$ while morphisms are

$G$-orbits in $X \times X$ with respect to the diagonal action. Then for any $x \in X$ we can consider the object $[x]$ and for any pair $(x, y) \in X \times X$ we can consider the morphism $[x, y]$. When $[y] = [u]$ there will be a $g \in G$ so that $gu = y$ and we can define the composition $[x, y][u, v] = [x, gv]$. The unit for $[x]$ is given by $[x, x]$. If the action of $G$ is transitive, we would get an actual group. We will abbreviate $[x_1, x_2][x_2, x_3] \cdots [x_{n-1}, x_n]$ with $[x_1, x_2, \ldots, x_n]$.
We define the *decorated Ptolemy groupoid* $\mathcal{G}(\Sigma_{g,s})$ of a punctured surface $\Sigma_{g,s}$ following the above recipe. The set we consider is the set $\dot{\Delta}$ of decorated triangulations $\tau$ of $\Sigma_{g,s}$. The group free action is the one of the mapping class group $\mathrm{MCG}_{g,s}$ on $\dot{\Delta}$. This action is not transitive, meaning that not all pairs of decorated ideal triangulations can be related by a mapping class group element. However in the language of groupoids we can still describe generators and relations for the morphism groups. For $\tau \in \dot{\Delta}$ there are three kind of generators $[\tau, \tau^\sigma]$, $[\tau, \rho_i\tau]$ and $[\tau, \omega_{i,j}\tau]$, where $\tau^\sigma$ is obtained by applying the permutation $\sigma \in \mathbb{S}_{|\tau|}$ to the ordering of triangles in $\tau$, $\rho_i\tau$ is obtained by changing the distinguished corner in the triangle $\bar{\tau}_i \in \tau$ as in Figure 1.3, and $\omega_{i,j}$ is obtained by applying a decorated diagonal exchange to the quadrilateral made of the two decorated ideal triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ as in Figure 1.4.

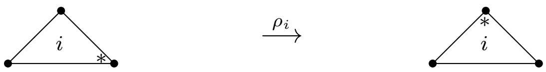



Figure 1.3: Transformation $\rho_i$.

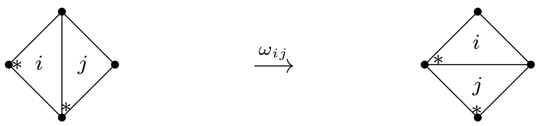



Figure 1.4: Transformation $\omega_{ij}$.

The relations are usually grouped in two sets, the first being:

$$[\tau, \tau^\alpha, (\tau^\alpha)^\beta] = [\tau, \tau^{\alpha\beta}], \quad \alpha, \beta \in \mathbb{S}_\tau, \tag{1.28}$$

$$[\tau, \rho_i\tau, \rho_i\rho_i\tau, \rho_i\rho_i\rho_i\tau] = \mathrm{id}_{[\tau]}, \tag{1.29}$$

$$[\tau, \omega_{i,j}\tau, \omega_{i,k}\omega_{i,j}\tau, \omega_{j,k}\omega_{i,k}\omega_{i,j}\tau] = [\tau, \omega_{j,k}\tau, \omega_{i,j}\omega_{j,k}\tau] \tag{1.30}$$

$$[\tau, \omega_{i,j}\tau, \rho_i\omega_{i,j}\tau, \omega_{j,i}\rho_i\omega_{i,j}\tau] = [\tau, \tau^{(i,j)}, \rho_j\tau^{(i,j)}, \rho_i\rho_j\tau^{(i,j)}] \tag{1.31}$$

The first two relations are obvious, the third is called the Pentagon Relation and it is explained in Figure 1.5 while the fourth is explained in Figure 1.6.

The second set of relations, are commutation relations.

$$[\tau, \rho_i\tau, \rho_i\tau^\sigma] = [\tau, \tau^\sigma, \rho_{\sigma^{-1}(i)}\tau^\sigma], \tag{1.32}$$

$$[\tau, \omega_{i,j}\tau, (\omega_{i,j}\tau)^\sigma] = [\tau, \tau^\sigma, \omega_{\sigma^{-1}(i)\sigma^{-1}(j)}\tau^\sigma], \tag{1.33}$$

$$[\tau, \rho_j\tau, \rho_j\rho_i\tau] = [\tau, \rho_i\tau, \rho_i\rho_j\tau], \tag{1.34}$$

$$[\tau, \rho_i\tau, \omega_{j,k}\rho_i\tau] = [\tau, \omega_{j,k}\tau, \rho_i\omega_{j,k}\tau],\ i \notin \{j, k\}, \tag{1.35}$$

$$[\tau, \omega_{i,j}\tau, \omega_{k,l}\omega_{i,j}\tau] = [\tau, \omega_{k,l}\tau, \omega_{i,j}\omega_{k,l}\tau],\ \{i, j\} \cap \{k, l\} = \emptyset, \tag{1.36}$$

The fact that the one we listed are all the possible transformations is Whitehead's Classical Fact 7 in the context of decorated triangulations.

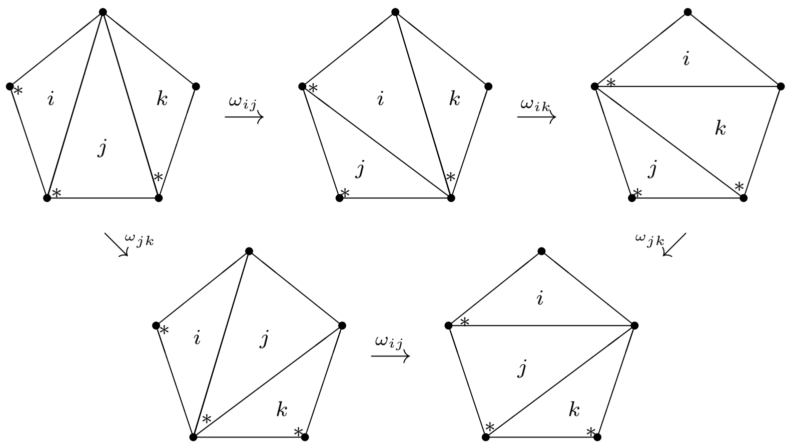



Figure 1.5: Pentagon relation (1.30).

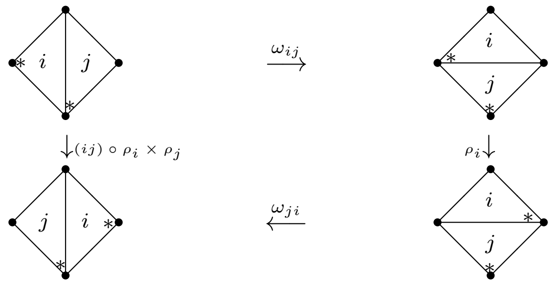



Figure 1.6: Inversion relation (1.31).

Recall that in the previous subsection we associated to every decorated ideal triangulation $\tau \in \dot{\Delta}$ the coordinate space $\mathcal{R}(\tau)$. Now we want to describe the action of $\mathcal{G}(\Sigma_{g,s})$ as morphisms between these spaces. The morphisms $[\tau, \tau^\sigma]$ act by permuting the coordinates in $\mathcal{R}(\tau)$. The morphism $[\tau, \rho_i \tau]$ acts as the identity on any pair $x = (x_1, x_2)$ corresponding to ideal triangles different from $\bar{\tau}_i$ and as $(x_1, x_2) = x \mapsto y = (\frac{x_2}{x_1}, \frac{1}{x_1})$ for the pair of coordinates corresponding to $\bar{\tau}_i$. Finally the action of $[\tau, \omega_{i,j}\tau]$ is the identity on $\bar{\tau}_k$ for $k \neq i, j$ while letting $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be the coordinates corresponding to the triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ respectively, and letting $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be the coordinates of the triangles $\overline{\omega_{i,j}\tau}_i$ and $\overline{\omega_{i,j}\tau}_j$, then we have $u = x \bullet y$ and $v = x * y$ where

$$\begin{aligned} x \bullet y &:= (x_1 y_1, x_1 y_2 + x_2) \\ x * y &:= (\frac{y_1 x_2}{x_1 y_2 + x_2}, \frac{y_2}{x_1 y_2 + x_2}). \end{aligned} \tag{1.37}$$

Let $\widetilde{\Delta}$ be the set of pairs $(\tau, \mathcal{R}(\tau))$, $\tau \in \dot{\Delta}$, and recall that $\mathcal{R}(\tau)$ carries a symplectic structure and an action of $\mathbb{P}(\mathbb{R}^s_{>0})$.

**Proposition 12.** *The action of $\mathcal{G}(\Sigma_{g,s})$ on $\widetilde{\Delta}$ is compatible with the action of $\mathbb{P}(\mathbb{R}^s_{>0})$, and preserves the symplectic structure $\omega_{\mathcal{R}}$.*

This means that we can consider the space $\mathcal{R}(\Sigma_{g,s})$ defined as the quotient of $\widetilde{\Delta}$ by the action of $\mathcal{G}(\Sigma_{g,s})$. This space of coordinates is now independent from the triangulation. For more details on the (decorated or not) Ptolemy groupoid see [Pen12],[FK14],[Kas12][Kas01].

### 1.3.5 Complexified Ratio Coordinates

In a similar way to the previous section, one can use the description in Proposition 9 of the spaces $\Lambda_{\mathbb{C}}(\tau)$, together with a non decorated version of $\mathcal{G}(\Sigma_{g,s})$ to describe the moduli space $\mathcal{M}$ of flat irreducible $\mathrm{PSL}(2,\mathbb{C})$ connections in terms of a $(\mathbb{C}^*)^s$ gauge action on the complexified decorated Teichmüller space $\widetilde{\mathcal{T}}^{\mathbb{C}}_{g,s}$ defined as the quotient of the set of couples $(\tau, \Lambda_{\mathbb{C}}(\tau))$ by the Ptolemy Groupoid [AK14a].

There exists a complex analogue of $\mathcal{R}(\Sigma_{g,s})$, presented in [AK14a], which proceed analogously to define *complexified ratio coordinates* starting from the complexified lambda coordinates $\Lambda_{\mathbb{C}}(\tau)$ we defined in Section 1.3.2.
Fix a surface $\Sigma_{g,s}$, with a decorated ideal triangulation $\tau$. We can define space of *complex ratio coordinates* $\mathcal{R}_{\mathbb{C}}(\tau)$ as the space $(\mathbb{C}^*)^{\Delta_2(\tau)} \times (\mathbb{C}^*)^{\Delta_2(\tau)}$ with the action of $\mathbb{P}((\mathbb{C}^*)^s)$ defined in direct analogy with (1.25). Define $\widetilde{\Delta}_{\mathbb{C}}$ to be the set of pairs $(\tau, \mathcal{R}_{\mathbb{C}}(\tau))$. The Ptolemy Groupoid gives us again a way to get a space $\mathcal{R}_{\mathbb{C}}(\Sigma_{g,s})$ identifying different couples of $\widetilde{\Delta}_{\mathbb{C}}$. We remark that in this process of identifications we need to use equations (1.37) to relate different ideal triangulations and such equations could produce a division by 0 when we use complex number. However this happen only in a closed algebraic subspace of $\mathcal{R}_{\mathbb{C}}(\tau)$ so the change of coordinates is well defined on an open dense subset. This phenomenon was explicated for complex lambda coordinates in Proposition 9 in equation (1.20). We can moreover map $\rho_{\mathbb{C}} : \widetilde{\mathcal{T}}^{\mathbb{C}}_{g,s} \longrightarrow \mathcal{R}_{\mathbb{C}}(\Sigma_{g,s})$ as we did with the real coordinates mapping $(a,b,c) \mapsto (\frac{b}{c}, \frac{a}{c})$ for a triangle of lambda lengths $a$, $b$, and $c$ and decoration opposite to $c$ (this map has to be thought as the quotient map of the maps defined in coordinates in each $\Lambda_{\mathbb{C}}(\tau)$ for every decorated ideal triangulation). The exactly same expression of the 2-forms $\omega_P$ and $\omega_{\mathcal{R}}$, gives here complex valued 2-forms in $\Lambda_{\mathbb{C}}$ and $\mathcal{R}_{\mathbb{C}}(\tau)$ respectively. These forms are $\mathcal{G}(\Sigma_{g,s})$–invariant so they define well posed two forms on $\widetilde{\mathcal{T}}^{\mathbb{C}}_{g,s}$ and $\mathcal{R}_{\mathbb{C}}(\Sigma_{g,s})$ respectively, again related by pull-back $\rho^*_{\mathbb{C}}\omega_{\mathcal{R}} = \omega_P$.
We can define a combinatorial map $\delta_{\mathbb{C}} : \mathcal{R}_{\mathbb{C}}(\Sigma_{g,s}) \longrightarrow \mathrm{H}^1(\Sigma_{g,s}, \mathbb{C}_{\neq 0})$ by the formula $\langle \delta_{\mathbb{C}}(x), \gamma \rangle = \prod_i \mathfrak{u}(\gamma_i)$, where $\mathfrak{u}$ is analogously defined to the real case, in each $\mathcal{R}_{\mathbb{C}}(\tau)$, see equation (1.27). We have an exact sequence of group homomorphisms like (1.3.3):

$$1 \longrightarrow \mathbb{C}^* \xrightarrow{\ i\ } \widetilde{\mathcal{T}}^{\mathbb{C}}_{g,s} \xrightarrow{\rho_{\mathbb{C}}} \mathcal{R}_{\mathbb{C}}(\Sigma_{g,s}) \xrightarrow{\delta_{\mathbb{C}}} \mathrm{H}^1(\Sigma_{g,s}, \mathbb{C}^*) \longrightarrow 1.$$

Here we lose the symplectic reduction interpretation, however we still have an interpretation of the $\mathrm{PSL}(2,\,\mathbb{C})$ moduli space $\mathcal{M}$ as a gauge group action over $\mathcal{R}_{\mathbb{C}}(\Sigma_{g,s})$. A choice of a decorated ideal triangulation still gives us an explicit simple formula for the symplectic form $\omega_{\mathcal{R}}$. Moreover we can still describe the action of the mapping class group explicitly in terms of decorated Ptolemy groupoid.

**Example 1.3.1** (Four punctured sphere)**.** Consider the sphere with 4 punctures $\{v_0, v_1, v_2, v_3\}$. Choose the ideal triangulation where all the vertices are trivalent. A point in $\widetilde{\mathcal{T}} \equiv \widetilde{\mathcal{T}}_{0,4}$ is given by a 6-upla $\mathsf{x} \equiv (x_{ij})_{0 \le i < j \le 3}$ of lambda lengths. We quotient by the gauge group $\mathbb{R}^4_{>0}$ acting as $\mathbb{R}^4_{>0} \ni f \cdot \mathsf{x} = (f_i f_j x_{ij})_{0 \le i < j \le 3}$. With the choice

$$f = \left( (\frac{x_{01}x_{02}}{x_{12}})^{\frac{1}{2}}, (x_{01}f_0)^{-1}, (x_{02}f_0)^{-1}, (x_{03}f_0)^{-1} \right) \tag{1.38}$$

we have

$$f \cdot \mathsf{x} = \left( 1, 1, 1, 1, \frac{x_{13}}{x_{12}}, \frac{x_{23}}{x_{12}} \right) \tag{1.39}$$

We remark that this choice of gauge is compatible with the map $\rho$ to $\mathcal{R}(\tau)$ for some appropriate choice of decoration. Under this gauge the Teichmüller space $\mathcal{T}$ is parametrized by two positive real number $(y_1, y_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ and the symplectic form $\omega_P$ takes the form

$$\omega_P = \frac{\mathrm{d}y_2}{y_2} \wedge \frac{\mathrm{d}y_1}{y_1} \tag{1.40}$$

Thanks to Proposition 9 all the coordinates described above remain valid, substituting $\mathbb{R}_{>0}$ with $\mathbb{C}^*$, for the moduli space $\mathcal{M}$ of flat $\mathrm{PSL}(2,\mathbb{C})$ connections, except that the so obtained parametrization and the formula (1.40) are valid only in an open dense subset $\mathbb{C}^* \times \mathbb{C}^*$ of $\mathcal{M}$.

# Chapter 2

# Quantum Dilogarithm

## 2.1 Quantum Dilogarithm

In this section we introduce several instances of a special function, the *quantum dilogarithm*. It was first introduced in the works of Faddeev [Fad95] and Kashaev and Faddeev [FK94]. We recall the properties of the so called Faddeev's quantum dilogarithm in Section 2.2. Standard references for the quantum dilogarithm are [Vol05], [FKV01] and the original work [FK94]. Recently, Andersen and Kashaev [AK14a] introduced al level $N$ quantum dilogarithm as a function on $\mathbb{A}_N \equiv \mathbb{R}\times\mathbb{Z}/N\mathbb{Z}$, which corresponds to Faddeev's quantum dilogarithm when $N = 1$. In Section 2.3 we introduce this level $N$ dilogarithm and we prove some of its properties, going by analogy to the level 1.

## 2.2 Faddeev's Quantum Dilogarithm

In this section we recall the properties of Faddeev' quantum dilogarithm. A general reference for the following statements is [FKV01], see also the thesis [Nis14] for some more detailed proofs.

**Definition 9** ($q$-Pochammer Symbol)**.** Let $x, q \in \mathbb{C}$, such that $|q| < 1$. Define the *$q$-Pochammer Symbol* of $x$ as

$$(x;q)_\infty := \prod_{i=0}^{\infty}(1 - xq^i)$$

The convergence of the infinite product is guaranteed by the hypothesis $|q| < 1$, so we can substitute the complex number $x$ with a formal variable. In particular it can be proved that (see [FK94])

**Theorem 13.** *Let $X$, $Y$ be two variables satisfying $XY = qYX$. Then the following five-term relation holds true*

$$(Y;q)_\infty (X;q)_\infty = (X;q)_\infty (-YX;q)_\infty (Y;q)_\infty . \tag{2.1}$$

**Definition 10** (Faddeev's Quantum Dilogarithm [Fad95])**.** Let $z, \mathrm{b} \in \mathbb{C}$ be such that $\mathrm{Re}\,\mathrm{b} \neq 0$, $|\,\mathrm{Im}(z)| < |\,\mathrm{Im}(c_\mathrm{b})|$, where $c_\mathrm{b} := i(\mathrm{b}+\mathrm{b}^{-1})/2$. Let $C \subset \mathbb{C}$, $C = \mathbb{R} + i0$ be a contour equal to the the real line outside a neighborhood of the origin that avoid the singularity in 0 going in the upper half plane. The *Faddeev's quantum dilogarithm* is defined to be

$$\Phi_{\mathrm{b}}(z) = \exp\left(\int_C \frac{e^{-2izw}\mathrm{d}w}{4\sinh(w\mathrm{b})\sinh(w\mathrm{b}^{-1})w}\right). \tag{2.2}$$

It is evident that $\Phi_{\mathrm{b}}$ is invariant under the following changes of parameter

$$\mathrm{b} \leftrightarrow \mathrm{b}^{-1} \leftrightarrow -\mathrm{b}, \tag{2.3}$$

so that our choice of b can be restricted to the first quadrant

$$\operatorname{Re}\mathrm{b} > 0, \qquad \operatorname{Im}\mathrm{b} \geq 0 \tag{2.4}$$

which implies

$$\operatorname{Im}(\mathrm{b}^2) \geq 0 \tag{2.5}$$

**Lemma 14.** *Suppose that* $\operatorname{Im}(\mathrm{b}^2) > 0$. *Define the constants* $q := e^{i\pi \mathrm{b}^2}$ *and* $\tilde{q} := e^{-i\pi\mathrm{b}^{-2}}$. *Then the following equality holds true:*

$$\Phi_{\mathrm{b}}(z) = \frac{\left(e^{2\pi(z+c_{\mathrm{b}})\mathrm{b}};q^2\right)_\infty}{\left(e^{2\pi(z-c_{\mathrm{b}})\mathrm{b}^{-1}};\tilde{q}^2\right)_\infty}$$

**Proposition 15.** *The Faddevv's quantum dilogarithm satisfies the following two functional equations*

$$\Phi_{\mathrm{b}}(z - i\mathrm{b}^{\pm1}/2) = \Phi_{\mathrm{b}}(z + i\mathrm{b}^{\pm1}/2)(1 + e^{2\pi\mathrm{b}^{\pm1}z}) \tag{2.6}$$

$$\Phi_{\mathrm{b}}(z)\Phi_{\mathrm{b}}(-z) = \zeta_{inv}^{-1}\, e^{i\pi z^2} \tag{2.7}$$

*where* $\zeta_{inv} = e^{i\pi(1+2c_{\mathrm{b}}^2)/6}$

**Theorem 16** (Pentagon Relation)**.** *Let* $p$ *and* $q$ *be two self-adjoint operators on* $\mathrm{L}^2(\mathbb{R})$ *satisfying the Heisenberg relation* $[p,q] = \frac{1}{2\pi i}$. *Then,* $\Phi_{\mathrm{b}}(p)$ *and* $\Phi_{\mathrm{b}}(q)$ *are well defined by use of the spectral theorem and the following five terms relation holds*

$$\Phi_{\mathrm{b}}(p)\Phi_{\mathrm{b}}(q) = \Phi_{\mathrm{b}}(q)\Phi_{\mathrm{b}}(p+q)\Phi_{\mathrm{b}}(p). \tag{2.8}$$

Before we look at the asymptotic behavior of $\Phi_{\mathrm{b}}$ let us recall the classical dilogarithm function, defined on $|z| < 1$ by

$$\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \tag{2.9}$$

and recall that it admits analytic continuation to $\mathbb{C} \setminus [1,\infty]$ through the following integral formula

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\mathrm{d}u. \tag{2.10}$$

**Proposition 17.** *For* $\mathrm{b} \to 0$ *and fixed* $x$, *one has the following asymptotic expansion*

$$\log\Phi_{\mathrm{b}}\left(\frac{x}{2\pi b}\right) = \sum_{n=0}^{\infty}(2\pi i b^2)^{2n-1}\frac{B_{2n}}{(2n)!}\frac{\partial^{2n}}{\partial x^{2n}}\mathrm{Li}_2(-e^x) \tag{2.11}$$

*where* $B_{2n}(1/2)$ *is the evaluation at* $\frac{1}{2}$ *of the Bernoulli polynomial.*

**Lemma 18.**

$$\Phi_{\mathrm{b}}(z)\Big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg z| > \frac{\pi}{2} + \arg \mathrm{b} \\ \zeta_{inv}^{-1} e^{i\pi z^2} & |\arg z| < \frac{\pi}{2} - \arg \mathrm{b} \\ \frac{(\tilde{q}^2;\tilde{q}^2)_\infty}{\Theta(i\mathrm{b}^{-1}z;-\mathrm{b}^{-2})} & |\arg z - \frac{\pi}{2}| < \arg \mathrm{b} \\ \frac{\Theta(i\mathrm{b}z;\mathrm{b}^2)}{(q^2;q^2)_\infty} & |\arg z + \frac{\pi}{2}| < \arg \mathrm{b} \end{cases} \tag{2.12}$$

*where*

$$\Theta(z;\tau) \equiv \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2 + 2\pi i z n}, \quad \operatorname{Im}\tau > 0$$

Faddeev's quantum dilogarithm has a lot of other interesting properties and applications, see for example [Fad95],[FK94] and [Vol05].

## 2.3 Quantum Dilogarithm on $\mathbb{A}_N \equiv \mathbb{R} \times (\mathbb{Z}/N\mathbb{Z})$

Let $N \geq 1$ be a positive *odd* integer. Consider $\mathbb{A}_N \equiv \mathbb{R} \times (\mathbb{Z}/N\mathbb{Z})$, which has the structure of a Locally Compact Abelian Group, with the normalized Haar measure $\mathrm{d}(x,n)$ defined as

$$\int_{\mathbb{A}_N} f(x,n)\mathrm{d}(x,n) := \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}} f(x,n)\mathrm{d}x$$

where $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ is an integrable function. Let $\mathrm{b} \in \mathbb{C}$ be such that $\mathrm{Re}(\mathrm{b}) > 0$ and $\mathrm{Im}(\mathrm{b}) \geq 0$ and define $c_{\mathrm{b}} := i(\mathrm{b} + \mathrm{b}^{-1})/2$. Then, following [AK14a], we can define a *quantum dilogarithm* over $\mathbb{A}_N$ as follows

$$\mathrm{D}_{\mathrm{b}}(x,n) := \prod_{j=0}^{N-1} \Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}^{-1}\frac{j}{N} - i\mathrm{b}\left\{\frac{j+n}{N}\right\}\right) \tag{2.13}$$

where $\{p\}$ is the fractional part of $p$, and $\Phi_{\mathrm{b}}$ is the Faddeev's quantum dilogarithm. Of course for $N = 1$ we have just $\Phi_{\mathrm{b}}(x)$. The function $\mathrm{D}_{\mathrm{b}}$ was introduced in [AK14a] only for $|\mathrm{b}| = 1$. In this thesis we review the constructions in [AK14a] also for $\mathrm{b} \in \mathbb{R}$. We will state and prove some properties of this quantum dilogarithm. Most of them are not in the literature but are just extensions of known properties of Faddeev's quantum dilogarithm $\Phi_{\mathrm{b}}$.

**Lemma 19** (Inversion Relation [AK14a])**.**

$$\mathrm{D}_{\mathrm{b}}(x,n)\mathrm{D}_{\mathrm{b}}(-x,-n) = e^{\pi i x^2} e^{-\pi i n(n+N)/N} \zeta_{N,\,inv}^{-1},$$

*where*

$$\zeta_{N,\,inv} = e^{\pi i (N + 2c_{\mathrm{b}}^2 N^{-1})/6}. \tag{2.14}$$

Unitarity properties are different in the two situations $|\mathrm{b}| = 1$ or $\mathrm{b} \in \mathbb{R}$.

**Lemma 20** (Unitarity)**.**

$$\overline{\mathrm{D}_{\mathrm{b}}(x,n)} = \mathrm{D}_{\mathrm{b}}(\bar{x},n)^{-1} \qquad \text{if } |\mathrm{b}| = 1, \tag{2.15}$$

$$\overline{\mathrm{D}_{\mathrm{b}}(x,n)} = \mathrm{D}_{\mathrm{b}}(\bar{x},-n)^{-1} \qquad \text{if } \mathrm{b} \in \mathbb{R}_{>0}. \tag{2.16}$$

*Proof.* First we remark that from the definition it is simple to see

$$\overline{\Phi_{\mathrm{b}}(x)} = \Phi_{\mathrm{b}}(\bar{x}) \qquad \text{when } \operatorname{Im}(\mathrm{b})(1-|\mathrm{b}|) = 0. \tag{2.17}$$

Suppose now $\mathrm{b} \in \mathbb{R}_{>0}$. Then

$$\begin{aligned}
&\overline{\mathrm{D}_{\mathrm{b}}(x,n)} = \\
&\quad = \overline{\prod_{j=0}^{N-1} \Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}^{-1}\frac{j}{N} - i\mathrm{b}\left\{\frac{j+n}{N}\right\}\right)} \\
&\quad = \prod_{j=0}^{N-1} \Phi_{\mathrm{b}}\left(\overline{\frac{x}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}^{-1}\frac{j}{N} - i\mathrm{b}\left\{\frac{j+n}{N}\right\}}\right)^{-1} \\
&\quad = \prod_{j=0}^{N-1} \Phi_{\mathrm{b}}\left(\frac{\bar{x}}{\sqrt{N}} - (1-N^{-1})c_{\mathrm{b}} + i\mathrm{b}^{-1}\frac{j}{N} + i\mathrm{b}\left\{\frac{j+n}{N}\right\}\right)^{-1} \\
&\quad = \prod_{j=0}^{N-1} \Phi_{\mathrm{b}}\left(\frac{\bar{x}}{\sqrt{N}} + i\mathrm{b}^{-1}\left(\frac{j}{N} - \frac{1}{2}\frac{N-1}{N}\right) + i\mathrm{b}\left(\left\{\frac{j+n}{N}\right\} - \frac{1}{2}\frac{N-1}{N}\right)\right)^{-1}
\end{aligned}$$

We can now write $j$ as $N-1-j'$ for $0 \le j' \le N-1$. Notice that $\{-\alpha\} = 1 - \{\alpha\}$ for any $\alpha > 0$. If $j' - n \ge 0$ we get

$$\begin{aligned}
\left\{\frac{j+n}{N}\right\} - \frac{1}{2}\frac{N-1}{N} &= \left\{\frac{N-1-j'+n}{N}\right\} - \frac{1}{2}\frac{N-1}{N} \\
&= 1 - \left\{\frac{1+j'-n}{N}\right\} - \frac{1}{2}\frac{N-1}{N} \\
&= \frac{N-1}{N} - \left\{\frac{j'-n}{N}\right\} - \frac{1}{2}\frac{N-1}{N} \\
&= \frac{1}{2}\frac{N-1}{N} - \left\{\frac{j'-n}{N}\right\},
\end{aligned}$$

while, if $j' - n < 0$, then $N-1 \ge n-j'-1 \ge 0$ and

$$\begin{aligned}
\left\{\frac{j+n}{N}\right\} - \frac{1}{2}\frac{N-1}{N} &= \left\{\frac{N-1-j'+n}{N}\right\} - \frac{1}{2}\frac{N-1}{N} \\
&= \left\{\frac{-1-j'+n}{N}\right\} - \frac{1}{2}\frac{N-1}{N} \\
&= \frac{n-j'}{N} - \frac{1}{N} - \frac{1}{2}\frac{N-1}{N} \\
&= 1 - \left\{\frac{N-n+j'}{N}\right\} - \frac{1}{N} - \frac{1}{2}\frac{N-1}{N} \\
&= \frac{1}{2}\frac{N-1}{N} - \left\{\frac{j'-n}{N}\right\}.
\end{aligned}$$

In the end we can write

$$\begin{aligned}
\overline{\mathrm{D}_{\mathrm{b}}(x,n)} &= \prod_{j'=0}^{N-1} \Phi_{\mathrm{b}}\left(\frac{\bar{x}}{\sqrt{N}} + i\mathrm{b}^{-1}\left(-\frac{j'}{N} + \frac{1}{2}\frac{N-1}{N}\right) + i\mathrm{b}\left(-\left\{\frac{j'-n}{N}\right\} + \frac{1}{2}\frac{N-1}{N}\right)\right)^{-1} \\
&= \prod_{j'=0}^{N-1} \Phi_{\mathrm{b}}\left(\frac{\bar{x}}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}^{-1}\frac{j'}{N} - i\mathrm{b}\left\{\frac{j'-n}{N}\right\}\right)^{-1} \\
&= \mathrm{D}_{\mathrm{b}}(\bar{x}, -n)^{-1}
\end{aligned}$$

The case $|\mathrm{b}| = 1$ is similar. $\square$

*Remark* 2.3.1. One can see that

$$\mathrm{D_b}(x,-n)=\mathrm{D_{b^{-1}}}(x,n) \tag{2.18}$$

just by the definition 2.13 for $\mathrm{D_{b^{-1}}}$ and carefully substituting $j+n\mapsto j'$. In particular the unitarity for $\mathrm{b}>0$ can be re-expressed as

$$\overline{\mathrm{D_b}(x,n)}=(\mathrm{D_{b^{-1}}}(x,n))^{-1} \tag{2.19}$$

**Lemma 21** (Faddeev's difference equations)**.** *Let*

$$\chi^{\pm}(x,n)\equiv e^{2\pi\frac{\mathrm{b}^{\pm1}}{\sqrt{N}}x}e^{\pm\frac{2\pi in}{N}}, \tag{2.20}$$

*for every $x$, $\mathrm{b}\in\mathbb{C}$, $\mathrm{Im(b)}\neq 0$ $n$, $N\in\mathbb{Z}$ we have*

$$\mathrm{D_b}\left(x+i\frac{\mathrm{b}^{\pm1}}{\sqrt{N}},n\pm1\right)=\mathrm{D_b}\left(x,n\right)\left(1+\chi^{\pm}(x,n)e^{-\pi i\frac{N-1}{N}}e^{\pi i\frac{\mathrm{b}^{\pm2}}{N}}\right)^{-1} \tag{2.21}$$

$$\mathrm{D_b}\left(x-i\frac{\mathrm{b}^{\pm1}}{\sqrt{N}},n\mp1\right)=\mathrm{D_b}(x,n)\left(1+\chi^{\pm}(x,n)e^{\pi i\frac{N-1}{N}}e^{-\pi i\frac{\mathrm{b}^{\pm2}}{N}}\right) \tag{2.22}$$

*Proof.*

$$\begin{aligned}&\mathrm{D_b}\left(x+i\frac{\mathrm{b}}{\sqrt{N}},n+1\right)=\\&\quad=\prod_{j=0}^{N-1}\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}+i\frac{\mathrm{b}}{N}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{j}{N}-i\mathrm{b}\{\frac{j+n+1}{N}\}\right)\end{aligned}$$

For the factors such that $j+n+1\neq 0 \mod N$ one simply has $\{\frac{j+n+1}{N}\}=\{\frac{j+n}{N}\}+\frac{1}{N}$, so they simplify back to $\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{j}{N}-i\mathrm{b}\{\frac{j+n}{N}\}\right)$. For $j=N-n-1$ one has, using equation (2.6)

$$\begin{aligned}&\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{N-n-1}{N}+i\frac{\mathrm{b}}{N}\right)\\&\quad=\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{N-n-1}{N}-i\mathrm{b}\frac{N-1}{N}+i\mathrm{b}\right)\\&\quad=\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{N-n-1}{N}-i\mathrm{b}\frac{N-1}{N}\right)\\&\quad\times\left(1+e^{2\pi\mathrm{b}\left(\frac{x}{\sqrt{N}}+\frac{N-1}{N}c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{N-n-1}{N}-i\mathrm{b}\frac{N-1}{N}\right)}e^{\pi i\mathrm{b}^2}\right)^{-1}\\&\quad=\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{N-n-1}{N}-i\mathrm{b}\frac{N-1}{N}\right)\\&\quad\times\left(1+\chi^{+}(x,n)e^{-\pi i\frac{N-1}{N}}e^{\pi i\frac{\mathrm{b}^2}{N}}\right)^{-1}.\end{aligned}$$

Putting all together we get the first equation. Next we do the one in the opposite spectrum

$$\begin{aligned}\mathrm{D_b}\left(x-i\frac{\mathrm{b}^{-1}}{\sqrt{N}},n+1\right)&=\prod_{j=0}^{N-1}\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}-i\frac{\mathrm{b}^{-1}}{N}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{j}{N}-i\mathrm{b}\{\frac{j+n+1}{N}\}\right)\\&=\prod_{j'=1}^{N}\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}}-i\frac{\mathrm{b}^{-1}}{N}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{j'}{N}+i\frac{\mathrm{b}^{-1}}{N}-i\mathrm{b}\{\frac{j'+n}{N}\}\right)\end{aligned}$$

For the factors such that $j' < N$ there is nothing needed. For $j' = N$ one has

$$\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}^{-1} - i\mathrm{b}\{\frac{n}{N}\}\right)$$
$$= \Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}\{\frac{n}{N}\}\right)\left(1 + e^{2\pi\mathrm{b}^{-1}\left(\frac{x}{\sqrt{N}}+\frac{N-1}{N}c_{\mathrm{b}}-i\mathrm{b}\frac{n}{N}\right)}e^{-\pi i\mathrm{b}^{-2}}\right)$$
$$= \Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}\{\frac{n}{N}\}\right)\left(1 + \chi^{-}(x,n)e^{\pi i\frac{N-1}{N}}e^{-\pi i\frac{\mathrm{b}^{-2}}{N}}\right).$$

The other cases are similar. □

**Proposition 22.** *If* $\mathrm{Im}(\mathrm{b}) > 0$ *and* $\mathrm{Re}(\mathrm{b}) > 0$ *we have*

$$\mathrm{D}_{\mathrm{b}}(x,n) = \frac{\left(\chi^{+}(x+\frac{c_{\mathrm{b}}}{\sqrt{N}},n);q^2\omega\right)_\infty}{\left(\chi^{-}(x-\frac{c_{\mathrm{b}}}{\sqrt{N}},n);\tilde{q}^2\overline{\omega}\right)_\infty} \tag{2.23}$$

*where* $q = e^{i\pi\frac{\mathrm{b}^2}{N}}$, $\tilde{q} = e^{-\pi i\frac{\mathrm{b}^{-2}}{N}}$, $\omega = e^{\frac{2\pi i}{N}}$ *and* $\chi^{\pm}(x,n) = e^{2\pi\frac{\mathrm{b}^{\pm1}}{\sqrt{N}}x}e^{\pm\frac{2\pi i n}{N}}$.

*Proof.*

$$\mathrm{D}_{\mathrm{b}}(x,n) = \prod_{j=0}^{N-1}\Phi_{\mathrm{b}}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_{\mathrm{b}} - i\mathrm{b}^{-1}\frac{j}{N} - i\mathrm{b}\{\frac{j+n}{N}\}\right)$$
$$= \prod_{j=0}^{N-1}\frac{\left(e^{2\pi\left(\frac{x}{\sqrt{N}}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{j}{N}-i\mathrm{b}\{\frac{j+n}{N}\}+c_{\mathrm{b}}\right)\mathrm{b}};e^{2\pi i\mathrm{b}^2}\right)_\infty}{\left(e^{2\pi\left(\frac{x}{\sqrt{N}}+(1-N^{-1})c_{\mathrm{b}}-i\mathrm{b}^{-1}\frac{j}{N}-i\mathrm{b}\{\frac{j+n}{N}\}-c_{\mathrm{b}}\right)\mathrm{b}^{-1}};e^{-2\pi i\mathrm{b}^{-2}}\right)_\infty}$$
$$= \frac{\prod_{k=0}^{N-1}\left(e^{2\pi\left(\frac{x}{\sqrt{N}}-\frac{c_{\mathrm{b}}}{N}+i\mathrm{b}\right)\mathrm{b}}\omega^n q^{-2k}\omega^{-k};e^{2\pi i\mathrm{b}^2}\right)_\infty}{\prod_{j=0}^{N-1}\left(e^{2\pi\left(\frac{x}{\sqrt{N}}-\frac{c_{\mathrm{b}}}{N}\right)\mathrm{b}^{-1}}\omega^{-n}\tilde{q}^{2j}\overline{\omega}^j;e^{-2\pi i\mathrm{b}^{-2}}\right)_\infty} \qquad (k := j+n \mod N)$$
$$= \frac{\prod_{k=0}^{N-1}\left(e^{2\pi\left(\frac{x}{\sqrt{N}}+\frac{c_{\mathrm{b}}}{N}\right)\mathrm{b}}q^{-2}\overline{\omega}\omega^n q^{2(N-k)}\omega^{N-k};e^{2\pi i\mathrm{b}^2}\right)_\infty}{\left(e^{2\pi\left(\frac{x}{\sqrt{N}}-\frac{c_{\mathrm{b}}}{N}\right)\mathrm{b}^{-1}}\omega^{-n};e^{-2\pi i\frac{\mathrm{b}^{-2}}{N}}\overline{\omega}\right)_\infty}$$
$$= \frac{\prod_{m=0}^{N-1}\left(e^{2\pi\left(\frac{x}{\sqrt{N}}+\frac{c_{\mathrm{b}}}{N}\right)\mathrm{b}}\omega^n q^{2m}\omega^m;e^{2\pi i\mathrm{b}^2}\right)_\infty}{\left(e^{2\pi\left(\frac{x}{\sqrt{N}}-\frac{c_{\mathrm{b}}}{N}\right)\mathrm{b}^{-1}}\omega^{-n};\tilde{q}^2\overline{\omega}\right)_\infty} \qquad (m = N-k-1),$$
$$= \frac{\left(e^{2\pi\left(\frac{x}{\sqrt{N}}+\frac{c_{\mathrm{b}}}{N}\right)\mathrm{b}};q^2\omega\right)_\infty}{\left(e^{2\pi\left(\frac{x}{\sqrt{N}}-\frac{c_{\mathrm{b}}}{N}\right)\mathrm{b}^{-1}}\omega^{-n};\tilde{q}^2\overline{\omega}\right)_\infty}$$

□

**Proposition 23.** *The quantum dilogarithm* $\mathrm{D}_{\mathrm{b}}(x,n)$, *for* $\mathrm{Im}(\mathrm{b}) > 0$ *has poles*

$$\begin{cases} x = \frac{c_{\mathrm{b}}}{\sqrt{N}} + i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l + i\frac{\mathrm{b}}{\sqrt{N}}m \\ n = m - l \mod N \end{cases}$$

*and zeros*

$$\begin{cases} x = -\frac{c_{\mathrm{b}}}{\sqrt{N}} - i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l - i\frac{\mathrm{b}}{\sqrt{N}}m \\ n = l - m \mod N \end{cases}$$

*for* $l,m \in \mathbb{Z}_{>0}$. *Moreover its residue at* $(x_{l,m}, n_{l,m}) = \left(\frac{c_{\mathrm{b}}}{\sqrt{N}} + i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l + i\frac{\mathrm{b}}{\sqrt{N}}m, m-l\right)$ *is*

$$\frac{\sqrt{N}}{2\pi\mathrm{b}^{-1}} \frac{(q^2\omega;q^2\omega)_\infty}{(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_\infty} \frac{(-\tilde{q}^2\overline{\omega})^l(\tilde{q}^2\overline{\omega})^{l(l-1)/2}}{(q^2\omega;q^2\omega)_m(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_l} \tag{2.24}$$

*Proof.* The set of zeros and poles are simply deduced from Proposition 22. For the residue we use the formula $\mathrm{Res}\left[\frac{g(x)}{h(x)}, x_0\right] = \frac{g(x_0)}{h'(x_0)}$ when $h'(x_0),\ g(x_0) \neq 0$.

$$\begin{aligned}
\mathrm{Res}\left[\mathrm{D_b}, (x_{l,m}, n_{l,m})\right] &= \mathrm{D_b}(x_{l,m}, n_{l,m}) \frac{\left(1 - \chi^-(x_{l,m} - c_{\mathrm{b}}/\sqrt{N}, m-l)\tilde{q}^{2l}\overline{\omega}^l\right)}{\frac{\partial}{\partial x}\left(1 - \chi^-(x - c_{\mathrm{b}}/\sqrt{N}, m-l)\tilde{q}^{2l}\overline{\omega}^l\right)|_{x=x_{l,m}}} \\
&= \frac{\sqrt{N}}{2\pi\mathrm{b}^{-1}} \frac{\left(e^{2\pi\left(2c_{\mathrm{b}}+i\mathrm{b}^{-1}l+i\mathrm{b}m\right)\frac{\mathrm{b}}{N}} e^{2\pi i\frac{m-l}{N}};q^2\omega\right)_\infty}{\prod_{j=0}^\infty\left(1 - e^{2\pi(i\mathrm{b}^{-1}l+i\mathrm{b}m)\frac{\mathrm{b}^{-1}}{N}}\overline{\omega}^{m-l}\tilde{q}^{2j}\overline{\omega}^j\right)} \qquad (j \neq l) \\
&= \frac{\sqrt{N}}{2\pi\mathrm{b}^{-1}} \frac{\left(q^2\omega q^{2m}\omega^m;q^2\omega\right)_\infty}{(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_\infty(\tilde{q}^{-2l}\overline{\omega}^{-l};\tilde{q}^2\overline{\omega})_l} \\
&= \frac{\sqrt{N}}{2\pi\mathrm{b}^{-1}} \frac{(q^2\omega;q^2\omega)_\infty}{(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_\infty} \frac{(-\tilde{q}^2\overline{\omega})^l(\tilde{q}^2\overline{\omega})^{l(l-1)/2}}{(q^2\omega;q^2\omega)_m(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_l}
\end{aligned}$$

where in the last step we used the following properties of the $q$-Pochammer symbol

$$\frac{1}{(q^{-l};q)_l} = \frac{(-q)^l q^{\frac{l(l-1)}{2}}}{(q;q)_l} \quad ; \qquad (aq^m;q)_\infty = \frac{(a;q)_\infty}{(a;q)_m} \tag{2.25}$$

□

The following Summation Formula is known for $N=1$, i.e. for $\Phi_{\mathrm{b}}$, and can be found in [FKV01] for example. Here we show that the proof works the same way for $N > 1$.

**Theorem 24** (Summation Formula)**.** *Suppose* $\mathrm{Im}(\mathrm{b}) > 0$ *and* $N$ *odd, and let* $u$, $v$, $w \in \mathbb{C}$ *and* $a,b,c \in \mathbb{Z}/N\mathbb{Z}$ *satisfy*

$$\mathrm{Im}\left(v + \frac{c_{\mathrm{b}}}{\sqrt{N}}\right) > 0, \quad \mathrm{Im}\left(-u + \frac{c_{\mathrm{b}}}{\sqrt{N}}\right) > 0, \quad \mathrm{Im}(v-u) < \mathrm{Im}(w) < 0. \tag{2.26}$$

*Define*

$$\Psi(u,v,w,a,b,c) \equiv \int_{\mathbb{A}_N} \frac{\mathrm{D_b}(x+u, a+d)}{\mathrm{D_b}(x+v, b+d)} e^{2\pi i w x} e^{-2\pi i\frac{cd}{N}} \mathrm{d}(x,d) \tag{2.27}$$

*Then we have*

$$\begin{aligned}
&\Psi(u,v,w,a,b,c) \\
&= \zeta_0 \frac{\mathrm{D_b}\left(v-u-w+\frac{c_{\mathrm{b}}}{\sqrt{N}}, b-a-c\right)}{\mathrm{D_b}\left(-w-\frac{c_{\mathrm{b}}}{\sqrt{N}}, -c\right)\mathrm{D_b}\left(v-u+\frac{c_{\mathrm{b}}}{\sqrt{N}}, b-a\right)} e^{2\pi i w\left(\frac{c_{\mathrm{b}}}{\sqrt{N}}-u\right)}\omega^{ac} \\
&= \zeta_0^{-1} \frac{\mathrm{D_b}\left(w+\frac{c_{\mathrm{b}}}{\sqrt{N}}, c\right)\mathrm{D_b}\left(-v+u-\frac{c_{\mathrm{b}}}{\sqrt{N}}, -b+a\right)}{\mathrm{D_b}\left(-v+u+w-\frac{c_{\mathrm{b}}}{\sqrt{N}}, -b+a+c\right)} e^{2\pi i w\left(-\frac{c_{\mathrm{b}}}{\sqrt{N}}-v\right)}\omega^{bc}
\end{aligned}$$

*where* $\zeta_0 = e^{-\pi i(N - 4c_{\mathrm{b}}^2N^{-1})/12}$.

*Proof.* Fix $d$ for a moment. The poles of $\frac{\mathrm{D_b}(x+u,a+d)}{\mathrm{D_b}(x+v,b+d)}e^{2\pi i w x}e^{-2\pi i \frac{cd}{N}}$ are grouped in two sets

$$x_{l,m,k} = -u+\frac{c_{\mathrm{b}}}{\sqrt{N}}+i\frac{\mathrm{b}^{-1}}{\sqrt{N}}(lN+k)+i\frac{\mathrm{b}}{\sqrt{N}}(mN+k+a+d) \qquad l,m\in\mathbb{Z}_{\geq 0},\ k\in\mathbb{Z}/N\mathbb{Z} \tag{2.28}$$

and

$$y_{l,m,k} = -v-\frac{c_{\mathrm{b}}}{\sqrt{N}}-i\frac{\mathrm{b}^{-1}}{\sqrt{N}}(lN+k+b+d)-i\frac{\mathrm{b}}{\sqrt{N}}(mN+k) \qquad l,m\in\mathbb{Z}_{\geq 0},\ k\in\mathbb{Z}/N\mathbb{Z} \tag{2.29}$$

The first two inequalities in (2.26) guarantee that this two sets lie in distinct components of $\mathbb{C}\setminus\mathbb{R}$. The other inequalities provide sufficient conditions for the absolute convergence of the integral. Indeed, from relations (2.12) we can see that

$$\lim_{x\to\infty}\mathrm{D_b}(x,n)=\zeta_{inv}^{-N}e^{-\pi i x^2}e^{\pi i Q(n)}$$

for some quadratic function $Q$, while

$$\lim_{x\to-\infty}\mathrm{D_b}(x,n)=1.$$

It follows that, for $x\to\infty$,

$$\left|\frac{\mathrm{D_b}(x+u,a+d)}{\mathrm{D_b}(x+v,b+d)}e^{2\pi i w x}e^{-2\pi i\frac{cd}{N}}\right| \sim \left|e^{-2\pi x(\operatorname{Im}u-\operatorname{Im}v+\operatorname{Im}w)}\right|\to 0$$

as $\operatorname{Im}(u-v+w)>0$. Similarly , for $x\to-\infty$,

$$\left|\frac{\mathrm{D_b}(x+u,a+d)}{\mathrm{D_b}(x+v,b+d)}e^{2\pi i w x}e^{-2\pi i\frac{cd}{N}}\right| \sim \left|e^{-2\pi x(\operatorname{Im}w)}\right|\to 0$$

as $\operatorname{Im}w<0$. We will now compute the sum of the residues in of the integrand in the first set and use it to get an explicit formula for the integral. First we remark that the totality of the poles can be counted as follows

$$\bigcup_{d\in\mathbb{Z}/N\mathbb{Z}}\{(x_{l,m,k},d)\in\mathbb{C}\times\mathbb{Z}/N\mathbb{Z}:\ l,m\in\mathbb{Z}_{\geq 0},\ k\in\mathbb{Z}/N\mathbb{Z}\}$$
$$=\{(x_{l,m},d_{l,m})\in\mathbb{C}\times\mathbb{Z}/N\mathbb{Z}:\ l,m\in\mathbb{Z}_{\geq 0}\}=:\mathcal{P}$$

where $x_{l,m}=-u+\frac{c_{\mathrm{b}}}{\sqrt{N}}+i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l+i\frac{\mathrm{b}}{\sqrt{N}}m$ and $d_{l,m}=m-l-a \mod N$. Proposition 23 gives

$$\operatorname{Res}\left[\mathrm{D_b}(x+u,a+d);\ x=x_{l,m},\ d=d_{l,m}\right]=\gamma\frac{(-\tilde{q}^2\overline{\omega})^l(\tilde{q}^2\overline{\omega})^{l(l-1)/2}}{(q^2\omega;q^2\omega)_m\,(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_l}$$

where

$$\gamma=\frac{\sqrt{N}}{2\pi\mathrm{b}^{-1}}\frac{(q^2\omega;q^2\omega)_\infty}{(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_\infty},\qquad q=e^{\pi i\frac{\mathrm{b}^2}{N}},\qquad \tilde{q}=e^{-\pi i\frac{\mathrm{b}^{-2}}{N}}. \tag{2.30}$$

Fix the notation $Z^{\pm}:=e^{\pi(v-u)\frac{\mathrm{b}^{\pm}}{\sqrt{N}}}\omega^{\pm(b-a)}$. We can compute

$$\mathrm{D_b}(v+x_{l,m},b+d_{l,m})=\frac{\left(e^{2\pi(v-u)\frac{\mathrm{b}}{\sqrt{N}}}q^2\omega q^{2m}\omega^m\omega^{b-a};q^2\omega\right)_\infty}{\left(e^{2\pi(v-u)\frac{\mathrm{b}^{-1}}{\sqrt{N}}}\tilde{q}^{-2l}\overline{\omega}^{-l}\overline{\omega}^{b-a};\tilde{q}^2\overline{\omega}\right)_\infty}$$
$$=\frac{(Z^+q^2\omega;q^2\omega)_\infty}{(Z^-(\tilde{q}\overline{\omega})^{-l};\tilde{q}^2\overline{\omega})_\infty\,(Z^+q^2\omega;q^2\omega)_m}$$

$$
\begin{aligned}
&= \frac{\left(Z^+q^2\omega;q^2\omega\right)_\infty}{\left(Z^+q^2\omega;q^2\omega\right)_m} \frac{1}{\left(Z^-(\tilde{q}^2\overline{\omega})^{-l};\tilde{q}^2\overline{\omega}\right)_l \left(Z^-;\tilde{q}^2\overline{\omega}\right)_\infty} \\
&= \frac{\left(Z^+q^2\omega;q^2\omega\right)_\infty}{\left(Z^+q^2\omega;q^2\omega\right)_m} \frac{(-\tilde{q}^2\overline{\omega}(Z^-)^{-1})^l (\tilde{q}^2\overline{\omega})^{\frac{l(l-1)}{2}}}{\left(Z^-;\tilde{q}^2\overline{\omega}\right)_\infty \left((Z^-)^{-1}\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega}\right)_l} \\
&= \mathrm{D_b}(v-u+\frac{c_\mathrm{b}}{\sqrt{N}}, b-a) \frac{(-\tilde{q}^2\overline{\omega}(Z^-)^{-1})^l (\tilde{q}^2\overline{\omega})^{\frac{l(l-1)}{2}}}{\left(Z^+q^2\omega;q^2\omega\right)_m \left((Z^-)^{-1}\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega}\right)_l}
\end{aligned}
$$

where we used the relation

$$
\frac{1}{(aq^{-l};q)_l} = \frac{(-qa^{-1})^l q^{l(l-1)/2}}{(qa^{-1};q)_l}. \tag{2.31}
$$

Putting everything together we have

$$
\begin{aligned}
&\mathrm{Res}\,[\Psi;\, x = x_{l,m},\, d = d_{l,m}] = \\
&\quad \gamma \left( \mathrm{D_b}\left( v-u+\frac{c_\mathrm{b}}{\sqrt{N}}, b-a \right) \right)^{-1} e^{2\pi i w\left(\frac{c_\mathrm{b}}{\sqrt{N}}-u\right)} \omega^{ac} \frac{\left(Z^+q^2\omega;q^2\omega\right)_m}{\left(q^2\omega;q^2\omega\right)_m} \\
&\qquad \times \left( e^{-2\pi\frac{\mathrm{b}}{\sqrt{N}}w}\omega^{-c} \right)^m \frac{\left((Z^-)^{-1}\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega}\right)_l}{\left(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega}\right)_l} \left( e^{2\pi\frac{\mathrm{b}^{-1}}{\sqrt{N}}(v-u-w)}\overline{\omega}^{b-a-c} \right)^l
\end{aligned}
$$

To compute

$$
2\pi i \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \mathrm{Res}\,[\Psi;\, x = x_{l,m},\, d = d_{l,m}] \tag{2.32}
$$

we can use the $q$-Binomial theorem

$$
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty} \tag{2.33}
$$

and get

$$
\gamma_0 \frac{\mathrm{D_b}(v-u-w+\frac{c_\mathrm{b}}{\sqrt{N}}, b-a-c)}{\mathrm{D_b}(-w-\frac{c_\mathrm{b}}{\sqrt{N}}, -c)\mathrm{D_b}(v-u+\frac{c_\mathrm{b}}{\sqrt{N}}, b-a)} e^{2\pi i w(\frac{c_\mathrm{b}}{\sqrt{N}}-u)} \omega^{ac} \tag{2.34}
$$

where $\gamma_0 \equiv 2\pi i\gamma$. This is the right hand side of the Summation Formula, except that we need a more explicit determination of $\gamma_0$ then the one in (2.30). We proceed as follows. In the limit where $u \to \frac{c_\mathrm{b}}{\sqrt{N}}$ and $a = 0$ we have just proved the following formula

$$
\int_{\mathbb{A}_N} \frac{\mathrm{D_b}(x+\frac{c_\mathrm{b}}{\sqrt{N}}, d)}{\mathrm{D_b}(x+v, d+b)} e^{2\pi i w x} \overline{\omega}^{cd} \mathrm{d}(x,d) = \gamma_0 \frac{\mathrm{D_b}(v-w, b-c)}{\mathrm{D_b}(-w-\frac{c_\mathrm{b}}{\sqrt{N}}, -c)\mathrm{D_b}(v,b)} \tag{2.35}
$$

After the change $(w,c) \mapsto (-w,-c)$ we can rewrite the left hand side as

$$
\begin{aligned}
&\int_{\mathbb{A}_N} \mathrm{D_b}(x-v-w, d-b-c)\mathrm{D_b}(-x+w, -d+c) e^{-\pi i x^2} e^{\pi i d(d+N)/N} \mathrm{d}(x,d) \\
&\qquad \times e^{\pi i w^2} e^{2\pi i v w} e^{-\pi i c(c+d)/N} \overline{\omega}^{bc} \zeta_{N,inv}^{-1}
\end{aligned}
$$

where $\zeta_{N,inv} = e^{\pi i (N+2c_\mathrm{b}^2 N^{-1})/6}$. At the same time we can rewrite the right hand side as

$$
\gamma_0 \frac{\mathrm{D_b}(-v,-b)}{\mathrm{D_b}(w-\frac{c_\mathrm{b}}{\sqrt{N}}, -c)\mathrm{D_b}(-w-v, -b-c)} e^{-\pi i w^2} e^{-2\pi i v w} e^{\pi i c(c+d)/N} \omega^{bc}
$$

changing variables $(y,k)\equiv(v+w,b+c)$ we get the equation

$$\int_{\mathbb{A}_N} \mathrm{D_b}(x-y,d-k)\mathrm{D_b}(-x+w,-d+c)e^{-\pi i x^2}e^{\pi i d(d+N)/N}\mathrm{d}(x,d) =$$
$$= \zeta_{N,inv}\gamma_0 \frac{\mathrm{D_b}(w-y,c-k)}{\mathrm{D_b}(-w-\frac{c_b}{\sqrt{N}},-c)\mathrm{D_b}(-y,-k)}$$

Now letting $y\to\infty$ and $w\to-\infty$ we get a limit formula involving the constant $\gamma$

$$\int_{\mathbb{A}_N} e^{-\pi i x^2}e^{\pi i d(d+N)/N}\mathrm{d}(x,d)=\zeta_{N,inv}\gamma_0 \tag{2.36}$$

We remark that

$$\sum_{d=0}^{N-1} e^{\pi i d(d+N)/N} = \sum_{d=0}^{N-1} e^{\pi i d^2(1+N)/N} = \sum_{d=0}^{N-1} e^{2\pi i d^2(k+1)/N} \tag{2.37}$$

where $N=2K+1$ and can be evaluated, using Gauss summation formula C.12, to be

$$\varepsilon_N\left(\frac{k+1}{N}\right)\sqrt{N}$$

where $\left(\frac{a}{N}\right)$ is the Legendre symbol and $\varepsilon_N = \begin{cases} 1, \text{ if } N=1 \mod 4 \\ i, \text{ if } N=3 \mod 4\end{cases}$

Noticing that $k+1$ is the inverse of $2 \mod N$, we can solve the Legendre symbol using known results about $\left(\frac{2}{N}\right)$ and get an explicit expression for the finite sum

$$e^{\frac{7}{4}\pi i(N-1)}\sqrt{N}.$$

We can also write

$$\int_{\mathbb{R}} e^{-\pi i x^2}\mathrm{d}x = e^{\frac{7}{4}\pi i} \tag{2.38}$$

to finally have

$$\gamma_0 = e^{-\pi i(N-4c_b^2N^{-1})/12}. \tag{2.39}$$

□

*Remark* 2.3.2. The Summation Formula is proved under the assumption $\mathrm{Im(b)}>0$. However both sides of the formula are well defined for $\mathrm{b}\in\mathbb{R}$, and the formula holds true in the limit $\mathrm{Im\,b}\mapsto 0$ by analytic continuation.

*Remark* 2.3.3. Assumptions (2.26) even though sufficient are not optimal. Indeed they guarantee the theorem to hold true when the integration is performed along the real line, however we can deform the integration contour as long as

$$|\arg(iz)|<\pi-\arg\mathrm{b} \qquad z \text{ being one of } \left\{w, v-u-w, u-v-2\frac{c_b}{\sqrt{N}}\right\} \tag{2.40}$$

We introduce here a bracket notation for Fourier coefficients and Gaussian exponentials in $\mathbb{A}_N$, following the notation introduced in [AK14a].

$$\langle(x,n),(y,m)\rangle \equiv e^{2\pi i x y}e^{-2\pi i n m/N} \qquad \langle(x,n)\rangle \equiv e^{\pi i x^2}e^{-\pi i n(n+N)/N} \tag{2.41}$$

For $(x,n)$ and $(y,m)$ in $\mathbb{A}_N$. Recall the following notation for the Fourier Transform

$$\mathcal{F}(f)(x,n) = \int_{-\infty}^{+\infty} f(y,n)e^{2\pi i x y}\mathrm{d}y \quad \mathsf{F}_N(f)(x,n) = \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} f(x,m)e^{2\pi i m n/N}$$

So that we have

$$\mathsf{F}_N^{-1}\circ\mathcal{F}(f)(x,n) = \int_{\mathbb{A}_N} f(y,m)\langle(x,n),(y,m)\rangle\mathrm{d}(y,m) \tag{2.42}$$

**Proposition 25** (Fourier Transformation Formula, [AK14a]). *For $N$ odd we have*

$$\int_{\mathbb{A}_N} \mathrm{D_b}(x,n)\langle(x,n);(w,c)\rangle \mathrm{d}(x,n) = \frac{e^{2\pi i w \frac{c_b}{\sqrt{N}}}}{\mathrm{D_b}\left(-w-\frac{c_b}{\sqrt{N}},-k\right)} e^{-\pi i(N-4c_b^2N^{-1})/12}$$

$$= \mathrm{D_b}\left(w+\frac{c_b}{\sqrt{N}},c\right)\overline{\langle(w,c)\rangle} e^{\pi i(N-4c_b^2N^{-1})/12}$$

$$\int_{\mathbb{A}_N} (\mathrm{D_b}(x,n))^{-1}\langle(x,n);(w,c)\rangle \mathrm{d}(x,n) = \frac{\langle(w,c)\rangle}{\mathrm{D_b}\left(-w-\frac{c_b}{\sqrt{N}},-k\right)} e^{-\pi i(N-4c_b^2N^{-1})/12}$$

$$= \mathrm{D_b}\left(w+\frac{c_b}{\sqrt{N}},c\right) e^{-2\pi i w\frac{c_b}{\sqrt{N}}} e^{\pi i(N-4c_b^2N^{-1})/12}$$

*Proof.* Just apply the Summation Formula to the limits

$$\lim_{v\to-\infty}\Psi(0,v,w,0,0,c)$$

$$\lim_{u\to-\infty}\Psi(u,0,w,0,0,c)$$

We remark that the Fourier integrals here are only conditional convergent, as we take $\Psi$ to a limit point. □

**Proposition 26** (Integral Pentagon Relation). *Let* $\widetilde{\mathrm{D_b}}(x,n) \equiv \mathsf{F}_N \circ \mathcal{F}^{-1}(\mathrm{D_b})(x,n)$. *We have the following integral relation*

$$\langle(x,n);(y,m)\rangle\widetilde{\mathrm{D_b}}(x,n)\widetilde{\mathrm{D_b}}(y,m)$$
$$= \int_{\mathbb{A}_N} \widetilde{\mathrm{D_b}}(x-z,n-k)\widetilde{\mathrm{D_b}}(z,k)\widetilde{\mathrm{D_b}}(y-z,m-k)\langle(z,k)\rangle \mathrm{d}(z,k)$$

*Proof.* Multiplying both sides by $\overline{\langle(u,j);(y,m)\rangle}$ and integrating both sides in $\mathrm{d}(y,m)$ we get the equivalent equation

$$\widetilde{\mathrm{D_b}}(x,n)\mathrm{D_b}(x-u,n-j)$$
$$= \mathrm{D_b}(-u,-j)\int_{\mathbb{A}_N} \widetilde{\mathrm{D_b}}(x-z,n-k)\widetilde{\mathrm{D_b}}(z,k)\langle(z,k)\rangle\overline{\langle(u,j);(z,k)\rangle}.$$

Using the Fourier Transformation Formulas from Proposition 25 this can be rewritten as

$$\frac{\mathrm{D_b}(x-u,n-j)}{\mathrm{D_b}\left(x-\frac{c_b}{\sqrt{N}},n\right)\mathrm{D_b}(-u,-j)} e^{-\pi i(N-4c_b^2N^{-1})/12}$$

$$= \int_{\mathbb{A}_N} \frac{\mathrm{D_b}\left(z+\frac{c_b}{\sqrt{N}},k\right)}{\mathrm{D_b}\left(x+z-\frac{c_b}{\sqrt{N}},n+k\right)} e^{2\pi i z\left(u-\frac{c_b}{\sqrt{N}}\right)} e^{-2\pi i\frac{jk}{N}} \mathrm{d}(z,k)$$

$$= \Psi\left(\frac{c_b}{\sqrt{N}}, x-\frac{c_b}{\sqrt{N}}, u-\frac{c_b}{\sqrt{N}}, 0, n, j\right)$$

which is an instance of the Summation Formula Theorem 24. □

**Proposition 27.** *We have the following behaviour when* $\mathrm{b}>0$, $\mathrm{b}\to 0$ *and* $x$, $n$, $N$ *are fixed*

$$\mathrm{D_b}\left(\frac{x}{2\pi \mathrm{b}},n\right) = \mathrm{Exp}\left(\frac{\mathrm{Li}_2(-e^{x\sqrt{N}})}{2\pi i b^2 N}\right)\phi_x(n)(1+\mathcal{O}(b^2)) \tag{2.43}$$

*where* $\phi_x(n)$ *is defined by* $\begin{cases}\phi_x(n+1)=\phi_x(n)\frac{(1-e^{x/\sqrt{N}}\overline{\omega}^{n+\frac{1}{2}})}{(1+e^{x\sqrt{N}})^{1/N}}\\ \phi_x(0)=(1+e^{x\sqrt{N}})^{-\frac{N-1}{2N}}\prod_{j=0}^{N-1}(1-e^{xN^{-\frac{1}{2}}}\overline{\omega}^{j+\frac{1}{2}})^{\frac{j}{N}}\end{cases}$

*whenever* $N$ *is odd.*

*Remark* 2.3.4. The function $\phi_x$ on the finite set $\mathbb{Z}/N\mathbb{Z}$ is a cyclic quantum dilogarithm [FK94],[Kas98], [Kas94]. Precisely $\frac{1}{\phi_x}$ corresponds to the function $\Psi_\lambda$ from Proposition 10 on [Kas98] with $\lambda = e^{x/\sqrt{N}}$.

*Proof.* First we want to prove the following expansion for $\Phi_{\mathrm{b}}$, let $\alpha$, $\beta$ be any two numbers

$$\Phi_{\mathrm{b}}(\frac{x+2\pi i\mathrm{b}^2\alpha+2\pi i\beta}{2\pi\mathrm{b}}) = \mathrm{Exp}\left\{\frac{\mathrm{Li}_2(-e^{x+2\pi i\beta})}{2\pi i\mathrm{b}^2}\right\}(1+e^{x+2\pi i\beta})^{-\alpha}(1+\mathcal{O}(\mathrm{b}^2)) \quad (2.44)$$

Indeed, let $x' := x+2\pi i\beta$, thanks to Proposition 17 we have

$$\begin{aligned}
\log\Phi_{\mathrm{b}}(\frac{x'+2\pi ib^2\alpha}{2\pi\mathrm{b}}) &= e^{2\pi i\mathrm{b}^2\alpha\frac{\partial}{\partial x}}\log\Phi_{\mathrm{b}}(\frac{x'}{2\pi\mathrm{b}}) \\
&= \sum_{l=0}^{\infty}\frac{(2\pi i\alpha)^l}{l!}\frac{\partial^l}{\partial x^l}\sum_{n=0}^{\infty}(2\pi i\mathrm{b}^2)^{2n-1}\frac{\mathrm{B}_{2n}(\frac{1}{2})}{(2n)!}\frac{\partial^{2n}}{\partial x^{2n}}\mathrm{Li}_2(-e^{x'}) \\
&= \sum_{m=0}^{\infty}(2\pi i\mathrm{b}^2)^{m-1}\left(\sum_{n=0}^{\lfloor\frac{m}{2}\rfloor}\alpha^{m-2n}\frac{\mathrm{B}_{2n}(\frac{1}{2})}{(2n)!(m-2n)!}\right)\frac{\partial^m}{\partial x^m}\mathrm{Li}_2(-e^{x'})
\end{aligned}$$

The summand for $m=0$ is then

$$\frac{1}{2\pi ib^2}\mathrm{Li}_2(-e^{x'})$$

while for $m=1$

$$\alpha\frac{\partial}{\partial x}\mathrm{Li}_2(-e^{x'}) = -\alpha\log(1+e^{x'})$$

so the equation (2.44) is proved.
Now we come back to $\mathrm{D}_{\mathrm{b}}$

$$\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi\mathrm{b}},n\right) = \prod_{j=0}^{N-1}\Phi_{\mathrm{b}}\left(\frac{1}{2\pi\mathrm{b}}\left(x/\sqrt{N}+2\pi i\mathrm{b}^2r_j/N+2\pi is_j/N\right)\right)$$

where $r_j = \frac{N-1}{2}-j$ and $s_j = \frac{N-1}{2}-N\{\frac{j+n}{N}\}$. Using approximation (2.44) we get

$$\prod_{j=0}^{N-1}\mathrm{Exp}\left(\frac{\mathrm{Li}_2(-e^{\frac{x}{\sqrt{N}}+2\pi is_j/N})}{2\pi i\mathrm{b}^2}\right)\left(1+e^{\frac{x}{\sqrt{N}}+2\pi is_j/N}\right)^{-\frac{r_j}{N}}(1+\mathcal{O}(\mathrm{b}^2)) \quad (2.45)$$

We can proceed to compute

$$\begin{aligned}
\log\prod_{j=0}^{N-1}\mathrm{Exp}\left(\frac{\mathrm{Li}_2(-e^{\frac{x}{\sqrt{N}}+2\pi is_j/N})}{2\pi i\mathrm{b}^2}\right) &= \sum_{j=0}^{N-1}\frac{\mathrm{Li}_2(-e^{\frac{x}{\sqrt{N}}+2\pi is_j/N})}{2\pi i\mathrm{b}^2} \\
&= \frac{1}{2\pi i\mathrm{b}^2}\sum_{j=0}^{N-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}e^{\frac{xk}{\sqrt{N}}}e^{2\pi iks_j/N} \\
&= \frac{1}{2\pi i\mathrm{b}^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}e^{\frac{xk}{\sqrt{N}}}e^{\pi ik\frac{N-1}{N}}\sum_{j=0}^{N-1}\omega^{k(j+n)} \\
&= \frac{1}{2\pi i\mathrm{b}^2N}\sum_{k=Nk_0,k_0\in\mathbb{Z}_{>0}}\frac{(-1)^{k_0}}{k_0^2}e^{xk_0\sqrt{N}}+ \\
&\quad+\frac{1}{2\pi i\mathrm{b}^2}\sum_{k\neq Nk_0}\frac{(-1)^k}{k^2}e^{\frac{xk}{\sqrt{N}}}e^{2\pi ik\frac{N-1}{N}}\delta(N-1) \\
&= \frac{1}{2\pi i\mathrm{b}^2N}\sum_{k_0=1}^{\infty}\frac{(-1)^{k_0}}{k_0^2}e^{xk_0\sqrt{N}}
\end{aligned}$$

$$= \frac{1}{2\pi i \mathrm{b}^2 N} \operatorname{Li}_2(-e^{x\sqrt{N}})$$

We are left to study the product

$$\begin{aligned}
\phi_x(n) &:= \prod_{j=0}^{N-1} \left(1 + e^{\frac{x}{\sqrt{N}} + 2\pi i s_j/N}\right)^{-\frac{r_j}{N}} \\
&= \prod_{j=0}^{N-1} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i (N-1)/N} \varpi^{j+n}\right)^{-\frac{N-1}{2N}} \prod_{j=0}^{N-1} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i (N-1)/N} \varpi^{j+n}\right)^{\frac{j}{N}}
\end{aligned}$$

we have

$$\prod_{j=0}^{N-1} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i (N-1)/N} \varpi^{j+n}\right)^{-\frac{N-1}{2N}} = (1 + e^{x\sqrt{N}})^{-\frac{N-1}{2N}}$$

thanks to the general decomposition $\prod_{j=0}^{N-1}(1 + y\omega^j) = (1 + y^N)$, whenever $N$ is odd. Computing $\phi_x(0)$ is now straightforward and we are left to verify the recursive property of $\phi_x$

$$\begin{aligned}
(\phi_x(n+1))^N &= (1 + e^{x\sqrt{N}})^{-\frac{N-1}{2N}} \prod_{j=0}^{N-1} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i \frac{N-1}{N}} \varpi^{j+1+n}\right)^j \\
&= (1 + e^{x\sqrt{N}})^{-\frac{N-1}{2N}} \prod_{j=1}^{N} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i \frac{N-1}{N}} \varpi^{j+n}\right)^{j-1} \\
&= (1 + e^{x\sqrt{N}})^{-\frac{N-1}{2N}} \frac{\prod_{j=1}^{N} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i \frac{N-1}{N}} \varpi^{j+n}\right)^j}{\prod_{j=1}^{N} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i \frac{N-1}{N}} \varpi^{j+n}\right)} \\
&= (\phi_x(n))^N \frac{\left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i \frac{N-1}{N}} \varpi^{n}\right)^N}{\prod_{j=1}^{N} \left(1 + e^{\frac{x}{\sqrt{N}}} e^{\pi i \frac{N-1}{N}} \varpi^{j+n}\right)} \\
&= (\phi_x(n))^N \frac{\left(1 - e^{\frac{x}{\sqrt{N}}} \varpi^{n+\frac{1}{2}}\right)^N}{(1 + e^{x\sqrt{N}})}
\end{aligned}$$

□

The Hilbert space $\mathrm{L}^2(\mathbb{A}_N)$ is naturally isomorphic to the tensor product $\mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \cong \mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^N$ (see Appendix A.2). Let $\mathsf{p}$ and $\mathsf{q}$ two self-adjoint operators on $\mathrm{L}^2(\mathbb{R})$ satisfying

$$[\mathsf{p}, \mathsf{q}] = \frac{1}{2\pi i} \tag{2.46}$$

and let $X$ and $Y$ unitary operators satisfying

$$YX = e^{2\pi i/N} XY, \quad X^N = Y^N = 1, \tag{2.47}$$

together with the cross relations

$$[\mathsf{p}, X] = [\mathsf{p}, Y] = [\mathsf{q}, X] = [\mathsf{q}, Y] = 0. \tag{2.48}$$

Equations (2.47) imply that $X$ and $Y$ will have finite and the same spectrum, and this will be a subset of the set $\mathbb{T}_N$ of all $N$-th complex roots of unity. Let

$$\mathrm{L}_N : \mathbb{T}_N \longrightarrow \mathbb{Z}/N\mathbb{Z}$$

be the natural group isomorphism. We can define $\mathrm{L}_N(A)$, by the spectral theorem, for any operator $A$ of order $N$, such that it formally satisfies

$$A = e^{2\pi i \,\mathrm{L}_N(A)/N}.$$

This permits to formally compute $[\mathrm{L}_N(X), \mathrm{L}_N(Y)] = \frac{(N-1)N}{2\pi i}$ from (2.47), which strictly does not have any meaning but, by Baker-Campbell-Hausdorff, gives the relation

$$XY = -e^{2\pi i(\mathrm{L}_N(X)+\mathrm{L}_N(Y))/N} e^{-\pi i/N}.$$

In particular this means that $-e^{\pi i/N}XY$ is of order $N$ and

$$\mathrm{L}_N(-e^{-\pi i/N}XY) = \mathrm{L}_N(X) + \mathrm{L}_N(Y). \tag{2.49}$$

We can define, for any function $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ the operator function $\not{f}(\mathsf{x}, A) \equiv f(\mathsf{x}, \mathrm{L}_N(A))$ for any commuting pair of operators $\mathsf{x}$ and $A$, where the former is self adjoint and the latter is of order $N$ (again, we are using the spectral theorem here). We have, for $\mathsf{x}$ and $A$ as above

$$\not{f}(\mathsf{x}, A) = \int_{\mathbb{A}_N} \tilde{f}(y, m) e^{2\pi i y \mathsf{x}} A^{-m} \mathrm{d}(y, m) \tag{2.50}$$

where

$$\tilde{f}(x, n) = \int_{\mathbb{A}_N} f(y, m) \overline{\langle (y, m); (x, n)\rangle} \mathrm{d}(y, m) = \mathsf{F}_N \circ \mathcal{F}^{-1}(f)(x, n) \tag{2.51}$$

The following Pentagon Identity for $\mathrm{D}_\mathrm{b}$ was first proved in [AK14a], where a projective ambiguity was undetermined and $|\mathrm{b}| = 1$. Here we show the same equation without projective ambiguity and for any $\mathrm{b}$ (with $\mathrm{Re}\,\mathrm{b} > 0$).

**Lemma 28** (Pentagon Equation)**.** *Let* $\mathsf{p}, \mathsf{q}$*,* $X$ *and* $Y$ *be as above, then the following five-term relation holds*

$$\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}, X)\not{\mathrm{D}}_\mathrm{b}(\mathsf{q}, Y) = \not{\mathrm{D}}_\mathrm{b}(\mathsf{q}, Y)\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}+\mathsf{q}, -e^{\pi i/N}XY)\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}, X). \tag{2.52}$$

*Proof.* This is equivalent to the Integral Pentagon equation of Proposition 26. To see this we need to use equation (2.50) an all the five terms. Then we compare the coefficients of $e^{2\pi i y \mathsf{q}} Y^m e^{2\pi i x \mathsf{p}} X^{-n}$ and get exactly the integral pentagon equation. An alternative proof follows from the q-Pochammer presentation of $\mathrm{D}_\mathrm{b}$ from Proposition 22. Indeed with the notation there

$$\chi^{\pm}\left(\mathsf{q} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(Y)\right) \chi^{\pm}\left(\mathsf{p} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(X)\right) = \\ = e^{2\pi i \mathrm{b}^{\pm 2}/N} e^{2\pi i/N} \chi^{\pm}\left(\mathsf{p} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(X)\right) \chi^{\pm}\left(\mathsf{q} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(Y)\right)$$

and

$$\chi^{\pm}\left(\mathsf{q} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(Y)\right) \chi^{\mp}\left(\mathsf{p} \mp \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(X)\right) = \\ = \chi^{\mp}\left(\mathsf{p} \mp \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(X)\right) \chi^{\pm}\left(\mathsf{q} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(Y)\right).$$

The first implies that we are in the hypothesis of Theorem 13, while the second implies that we can consider independently the nominator and denominator of the $q$-Pochammer presentation of $\mathrm{D}_\mathrm{b}$. Then one notice that

$$-\chi^{\pm}\left(\mathsf{p} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(X)\right) \chi^{\pm}\left(\mathsf{q} \pm \frac{c_\mathrm{b}}{\sqrt{N}}, \mathrm{L}_N(Y)\right) = \\ = -e^{2\pi \frac{\mathrm{b}^{\pm 2}}{\sqrt{N}}\left(\mathsf{p}+\mathsf{q}\pm 2\frac{c_\mathrm{b}}{\sqrt{N}}\right)} e^{(2\pi \mathrm{b}^{\pm 1})^2 [\mathsf{p},\mathsf{q}]/(2N)} X^{\pm 1} Y^{\pm 1} \\ = -e^{2\pi \frac{\mathrm{b}^{\pm 2}}{\sqrt{N}}\left(\mathsf{p}+\mathsf{q}\pm \frac{c_\mathrm{b}}{\sqrt{N}}\right)} e^{\pi i \mathrm{b}^{\pm 1}(\mathrm{b}+\mathrm{b}^{-1})/N} e^{-\pi i \mathrm{b}^{\pm 2}/N} X^{\pm 1} Y^{\pm 1}$$

$$= e^{2\pi \frac{\mathrm{b}^{\pm 2}}{\sqrt{N}}\left(\mathsf{p}+\mathsf{q}\pm\frac{c_{\mathrm{b}}}{\sqrt{N}}\right)}(-e^{\pi i/N})X^{\pm 1}Y^{\pm 1}$$
$$= \chi^{\pm}\left(\mathsf{p}+\mathsf{q}\pm\frac{c_{\mathrm{b}}}{\sqrt{N}}, \mathrm{L}_N(X)+\mathrm{L}_N(Y)\right).$$

For $\mathrm{Im}\,\mathrm{b} > 0$ the pentagon equation for $\mathrm{D}_{\mathrm{b}}$ then follows from Theorem 13 (and the inverse of the relation there) together with Proposition 22. By analytic continuation from the one just proved, one get the pentagon equation for $\mathrm{Im}\,\mathrm{b} = 0$. □

### 2.3.1 Charges

We are going to define a *charged* version of the dilogarithm. This charges will assume geometrical meaning in the construction of the partition function, however they already satisfy the purpose of turning all the conditional convergent integral relations of the the dilogarithm $\mathrm{D}_{\mathrm{b}}$ (e.g. Proposition 26 and 25) into absolutely convergent integrals .
Let $a$, $b$ and $c$ three real positive numbers such that $a+b+c = \frac{1}{\sqrt{N}}$. We define the charged quantum dilogarithm

$$\psi_{a,c}(x,n) := \frac{e^{-2\pi i c_{\mathrm{b}} a x}}{\mathrm{D}_{\mathrm{b}}(x - c_{\mathrm{b}}(a+c), n)} \tag{2.53}$$

From the Fourier transformation formula, Proposition 25, and the inversion formula in Lemma 19 we can deduce the following transformation formulas for $\psi_{a,c}$ (recall notation (2.51) for inverse Fourier transform)

**Lemma 29.** *Suppose* $\mathrm{Im}(\mathrm{b})(1-|\mathrm{b}|) = 0$, *then*

$$\tilde{\psi}_{a,c}(x,k) = \psi_{c,b}(x,k)\langle x,k\rangle e^{-\pi i c_{\mathrm{b}}^2 a(a+2c)}\zeta_0 \tag{2.54}$$
$$\overline{\psi_{a,c}(x,k)} = \psi_{c,a}(-x,\epsilon k)\langle x,k\rangle e^{\pi c_{\mathrm{b}}^2(a+c)^2}\zeta_{N,inv} \tag{2.55}$$
$$\overline{\tilde{\psi}_{a,c}(x,k)} = \psi_{b,c}(-x,\epsilon k)e^{-2\pi i c_{\mathrm{b}}^2 ab}\zeta_0 \tag{2.56}$$

*where* $\zeta_0 = e^{-\pi i(N-4c_{\mathrm{b}}^2N^{-1})/12}$ *and* $\zeta_{N,inv} = \zeta_0^2 e^{-\pi i c_{\mathrm{b}}^2/N}$ *and* $\epsilon = +1$ *if* $\mathrm{b} > 0$ *or* $\epsilon = -1$ *if* $|\mathrm{b}| = 1$.

*Remark* 2.3.5. The hypothesis on positivity of $a$, $b$ and $c$ assure that $\tilde{\psi}_{a,c}$ is absolutely convergent, as a simple computation using Proposition 17 can show.

*Proof.* Recall the Fourier transformation formula for $\mathrm{D}_{\mathrm{b}}$

$$\widetilde{\mathrm{D}_{\mathrm{b}}}(x,n) = \mathrm{D}_{\mathrm{b}}(-x+\frac{c_{\mathrm{b}}}{\sqrt{N}}, -n)\overline{\langle x,n\rangle}\zeta_0^{-1} = \frac{e^{-2\pi i x\frac{c_{\mathrm{b}}}{\sqrt{N}}}}{\mathrm{D}_{\mathrm{b}}(x-\frac{c_{\mathrm{b}}}{\sqrt{N}}, n)}\zeta_0$$

We use it to compute the following

$$
\begin{aligned}
(\tilde{\psi}_{a,c})(x,k) &= \int_{\mathbb{A}_N} \psi_{a,c}(y,m)\overline{\langle (x,k);(y,m)\rangle}\mathrm{d}(y,m) \\
&= \int_{\mathbb{A}_N} \frac{\langle (y,m);-(x+c_{\mathrm{b}}a,k)\rangle}{\mathrm{D}_{\mathrm{b}}(y-c_{\mathrm{b}}(a+c),m)}\mathrm{d}(y,m) \\
&= \int_{\mathbb{A}_N} \frac{\langle (y,m);-(x+c_{\mathrm{b}}a,k)\rangle}{\mathrm{D}_{\mathrm{b}}(y,m)}\mathrm{d}(y,m)\langle (c_{\mathrm{b}}(a+c),0);-(x+c_{\mathrm{b}}a,k)\rangle \\
&= \int_{\mathbb{A}_N} \frac{\langle (y+\frac{c_{\mathrm{b}}}{\sqrt{N}},0);-(\frac{c_{\mathrm{b}}}{\sqrt{N}},0)\rangle}{\mathrm{D}_{\mathrm{b}}(y,m)}\langle (y,m);-(x+c_{\mathrm{b}}(a-\frac{1}{\sqrt{N}}),k)\rangle\mathrm{d}(y,m)\times \\
&\quad\times \langle (c_{\mathrm{b}}(a+c),0);-(x+c_{\mathrm{b}}a,0)\rangle e^{2\pi i\frac{c_{\mathrm{b}}^2}{N}} \\
&= \int_{\mathbb{A}_N} \widetilde{\mathrm{D}_{\mathrm{b}}}(y+\frac{c_{\mathrm{b}}}{\sqrt{N}},m)\langle (y,m);-(x-c_{\mathrm{b}}(b+c),k)\rangle\mathrm{d}(y,m)\times \\
&\quad\times \langle (c_{\mathrm{b}}(a+c),0);-(x+c_{\mathrm{b}}a,0)\rangle e^{2\pi i\frac{c_{\mathrm{b}}^2}{N}}\zeta_0^{-1} \\
&= \int_{\mathbb{A}_N} \widetilde{\mathrm{D}_{\mathrm{b}}}(y,m)\langle (y,m);-(x-c_{\mathrm{b}}(b+c),k)\rangle\mathrm{d}(y,m)\times \\
&\quad\times \langle (\frac{c_{\mathrm{b}}}{\sqrt{N}},0);(x-c_{\mathrm{b}}(b+c),0)\rangle\langle (c_{\mathrm{b}}(a+c),0);-(x+c_{\mathrm{b}}a,0)\rangle e^{2\pi i\frac{c_{\mathrm{b}}^2}{N}}\zeta_0^{-1} \\
&= \mathrm{D}_{\mathrm{b}}(-x+c_{\mathrm{b}}(b+c),-k)\langle (x,k);(c_{\mathrm{b}}b,0)\rangle\times \\
&\quad\times \langle (\frac{c_{\mathrm{b}}^2}{\sqrt{N}},0);(b+c,0)\rangle^{-1}\langle (c_{\mathrm{b}}(a+c)a,0);(c_{\mathrm{b}},0)\rangle^{-1}e^{2\pi i\frac{c_{\mathrm{b}}^2}{N}}\zeta_0^{-1} \\
&= \frac{\langle (x-c_{\mathrm{b}}(b+c),k)\rangle}{\mathrm{D}_{\mathrm{b}}(x-c_{\mathrm{b}}(b+c),k)}\langle (x,k);(c_{\mathrm{b}}b,0)\rangle\times \\
&\quad\times \langle (\frac{c_{\mathrm{b}}^2}{\sqrt{N}},0);(b+c,0)\rangle^{-1}\langle (c_{\mathrm{b}}(a+c)a,0);(c_{\mathrm{b}},0)\rangle^{-1}e^{\pi i\frac{c_{\mathrm{b}}^2}{N}}\zeta_0 \\
&= \psi_{c,b}(x,k)\langle x,k\rangle e^{-\pi i c_{\mathrm{b}}^2 a(a+2c)}\zeta_0
\end{aligned}
$$

For the second one suppose $\mathrm{b}>0$. We have

$$
\begin{aligned}
\overline{\psi_{a,c}(x,k)} &= \langle (-c_{\mathrm{b}}x,0);(a,0)\rangle\mathrm{D}_{\mathrm{b}}(x+c_{\mathrm{b}}(a+c),-k) \\
&= \frac{\langle (-c_{\mathrm{b}}x,0);(a,0)\rangle\langle (x+c_{\mathrm{b}}(a+c),-k)\rangle}{\mathrm{D}_{\mathrm{b}}(-x-c_{\mathrm{b}}(a+c),k)}\zeta_{N,\mathrm{inv}} \\
&= \frac{\langle (-c_{\mathrm{b}}x,0);(a,0)\rangle\langle (x,k);(c_{\mathrm{b}}(a+c),0)\rangle}{\mathrm{D}_{\mathrm{b}}(-x-c_{\mathrm{b}}(a+c),k)}\langle (x,k)\rangle\langle (c_{\mathrm{b}}(a+c),0)\rangle\zeta_{N,\mathrm{inv}} \\
&= \psi_{c,a}(-x,k)\langle x,k\rangle e^{\pi c_{\mathrm{b}}^2(a+c)^2}\zeta_{N,\mathrm{inv}}
\end{aligned}
$$

The case $|\mathrm{b}|=1$ is similar. The third one is just a combination of the previous two. $\square$

**Theorem 30** (Charged Pentagon Equation)**.** *Let $a_j$, $c_j>0$ such that $\frac{1}{\sqrt{N}}-a_j-c_j>0$ for $j=0,\ 1,\ 2,\ 3$ or $4$. Define $\psi_j\equiv\psi_{a_j,b_j}$. Suppose the following relations hold true*

$$
a_1=a_0+a_2 \qquad a_3=a_2+a_4 \qquad c_1=c_0+a_4 \qquad c_3=a_0+c_4 \qquad c_2=c_1+c_3. \tag{2.57}
$$

*and consider the operators on $\mathrm{L}^2(\mathbb{A}_N)$ defined to satisfy (2.46 -2.47). We have the following charged pentagon relation*

$$
\begin{aligned}
&\psi_1(\mathsf{q},\mathrm{L}_N(X))\psi_3(\mathsf{p},\mathrm{L}_N(Y))\xi(a,c)= \\
&\qquad =\psi_4(\mathsf{p},\mathrm{L}_N(X))\psi_2(\mathsf{p}+\mathsf{q},\mathrm{L}_N(X)+\mathrm{L}_N(Y))\psi_0(\mathsf{q},\mathrm{L}_N(Y))
\end{aligned} \tag{2.58}
$$

*where $\xi(a,c)=e^{2\pi i c_{\mathrm{b}}^2(a_0a_2+a_0a_4+a_2a_4)}e^{\pi i c_{\mathrm{b}}^2 a_2^2}$.*

*Proof.* For brevity let us write

$$\hat{p} = (\mathsf{p}, \mathrm{L}_N(X)) \qquad\qquad \hat{q} = (\mathsf{q}, \mathrm{L}_N(Y))$$

Recall that the pentagon equation (2.52) can be rewritten as

$$\Psi_{\mathrm{b}}(\hat{q})\Psi_{\mathrm{b}}(\hat{p}) = \Psi_{\mathrm{b}}(\hat{p})\Psi_{\mathrm{b}}(\hat{p}+\hat{q})\Psi_{\mathrm{b}}(\hat{q}) \tag{2.59}$$

where $\Psi_{\mathrm{b}}(x,n) \equiv \frac{1}{\mathrm{D}_{\mathrm{b}}(x,n)}$. The left hand side of (2.58) can be rewritten as

$$\begin{aligned} &e^{-2\pi i c_{\mathrm{b}} a_1 \mathsf{q}}\Psi_{\mathrm{b}}(\hat{q}c_{\mathrm{b}}(a_1+c_1))e^{-2\pi i c_{\mathrm{b}} a_3 \mathsf{p}}\Psi_{\mathrm{b}}(\hat{p}-c_{\mathrm{b}}(a_3+c_3)) = \\ &\qquad\qquad = e^{-2\pi i c_{\mathrm{b}} a_1 \mathsf{q}}e^{-2\pi i c_{\mathrm{b}} a_3 \mathsf{p}}\Psi_{\mathrm{b}}(\hat{q}-c_{\mathrm{b}}(a_1+c_1-a_3))\Psi_{\mathrm{b}}(\hat{p}-c_{\mathrm{b}}(a_3+c_3)) \end{aligned}$$

On the right hand side we get

$$\begin{aligned} &e^{-2\pi i c_{\mathrm{b}} a_4 \mathsf{p}}\Psi_{\mathrm{b}}(\hat{p}-c_{\mathrm{b}}(a_4+c_4))e^{-2\pi i c_{\mathrm{b}} a_2 (\mathsf{p}+\mathsf{q})} \times \\ &\qquad \times \Psi_{\mathrm{b}}(\hat{p}+\hat{q}-c_{\mathrm{b}}(a_2+c_2))e^{-2\pi i c_{\mathrm{b}} a_0 \mathsf{q}}\Psi_{\mathrm{b}}(\hat{q}-c_{\mathrm{b}}(a_0+c_0)) \\ &= e^{-2\pi i c_{\mathrm{b}} a_4 \mathsf{p}}e^{-2\pi i c_{\mathrm{b}} a_2 (\mathsf{p}+\mathsf{q})}e^{-2\pi i c_{\mathrm{b}} a_0 \mathsf{q}} \times \\ &\qquad \times \Psi_{\mathrm{b}}(\hat{p}-c_{\mathrm{b}}(a_4+c_4+a_2+a_0))\Psi_{\mathrm{b}}(\hat{p}+\hat{q}-c_{\mathrm{b}}(a_2+c_2+a_0))\Psi_{\mathrm{b}}(\hat{q}-c_{\mathrm{b}}(a_0+c_0)) \\ &= e^{-2\pi i c_{\mathrm{b}} a_4 \mathsf{p}}e^{-2\pi i c_{\mathrm{b}} a_2 (\mathsf{p}+\mathsf{q})}e^{-2\pi i c_{\mathrm{b}} a_0 \mathsf{q}} \times \\ &\qquad \times \Psi_{\mathrm{b}}(\hat{q}-c_{\mathrm{b}}(a_0+c_0))\Psi_{\mathrm{b}}(\hat{p}-c_{\mathrm{b}}(a_4+c_4+a_2+a_0)) \end{aligned}$$

where in the last step we used the Pentagon relation (2.59) and the last three equations of (2.57). The two remaining equations give

$$\begin{aligned} &\Psi_{\mathrm{b}}(\hat{q}-c_{\mathrm{b}}(a_0+c_0))\Psi_{\mathrm{b}}(\hat{p}-c_{\mathrm{b}}(a_4+c_4+a_2+a_0)) = \\ &\qquad = \Psi_{\mathrm{b}}(\hat{q}-c_{\mathrm{b}}(a_1+c_1-a_3))\Psi_{\mathrm{b}}(\hat{p}-c_{\mathrm{b}}(a_3+c_3)). \end{aligned}$$

Finally we take care of the exponentials

$$\begin{aligned} &e^{-2\pi i c_{\mathrm{b}} a_4 \mathsf{p}}e^{-2\pi i c_{\mathrm{b}} a_2 (\mathsf{p}+\mathsf{q})}e^{-2\pi i c_{\mathrm{b}} a_0 \mathsf{q}} \\ &= e^{-2\pi i c_{\mathrm{b}} a_2((\mathsf{p}+\mathsf{q})-c_{\mathrm{b}}a_4)}e^{-2\pi i c_{\mathrm{b}} a_0(\mathsf{q}-c_{\mathrm{b}}a_4)}e^{-2\pi i c_{\mathrm{b}} a_4 \mathsf{p}} \\ &= e^{-2\pi i c_{\mathrm{b}} a_0 \mathsf{q}}e^{-2\pi i c_{\mathrm{b}} a_2((\mathsf{p}+\mathsf{q})-c_{\mathrm{b}}a_0)}e^{-2\pi i c_{\mathrm{b}} a_4 \mathsf{p}}e^{2\pi c_{\mathrm{b}}^2 a_4(a_2+a_0)} \\ &= e^{-2\pi i c_{\mathrm{b}} \mathsf{q}(a_0+a_2)}e^{\pi i c_{\mathrm{b}} a_2^2}e^{-2\pi i c_{\mathrm{b}}(a_2+a_4)\mathsf{p}}e^{2\pi i c_{\mathrm{b}}^2(a_0a_2+a_0a_4+a_2a_0)} \\ &= e^{-2\pi i c_{\mathrm{b}} a_1 \mathsf{q}}e^{-2\pi i c_{\mathrm{b}} a_3 \mathsf{p}}\xi(a,c). \end{aligned}$$

□

# Chapter 3

# Quantum Teichmüller Theory

## 3.1 Quantization of $\mathcal{R}(\Sigma_{g,s})$

In this section we are going to quantize the space $\mathcal{R}(\Sigma_{g,s})$ following [Kas98]. Recall that $\mathcal{R}(\tau)$ explicitly depends on the triangulation $\tau$, and $\mathcal{R}(\Sigma_{g,s})$ was defined as the quotient of all the couples $(\mathcal{R}(\tau),\, \tau)$ by the action of the decorated Ptolemy groupoid $\mathcal{G}(\Sigma_{g,s})$. For any fixed $\tau$ the quantization of $\mathcal{R}(\tau)$ is just the canonical quantization in exponential coordinates of the space $\mathbb{R}^M_{>0} \times \mathbb{R}^M_{>0}$, where $M = 2g - 2 + s$, with simplectic form $\omega_\tau = \sum_{j=0}^{M-1} \mathrm{d}\log u_j \wedge \mathrm{d}\log v_j$. Formally, following the expectations from the canonical quantization of $\mathbb{R}^{2M}$, we can quantize $\mathcal{R}(\tau)$ associating to it an algebra of operator

$$\mathcal{X}(\tau) \text{ generated by } \{\hat{u}_j,\, \hat{v}_j\}\,, \text{ where } 0 \le j < M, \text{ subject tot the relations}$$
$$\hat{u}_j \hat{v}_l = q^{\delta(j-l)} \hat{v}_l \hat{u}_j \quad \hat{u}_j \hat{u}_l = \hat{u}_l \hat{u}_j \quad \hat{v}_j \hat{v}_l = \hat{v}_l \hat{v}_j$$

where $q \in \mathbb{C}^*$. Here by algebra given as sets of generators and relations, we mean the associative algebra of non commutative fractions of non commutative polynomials generated such given of generators.
In order to obtain a quantization of $\mathcal{R}(\Sigma_{g,s})$ (i.e. triangulation independent) we have to look at the action of the $\mathcal{G}(\Sigma_{g,s})$ generators on coordinates and translate it into an action on the algebras $\mathcal{X}(\tau)$. Precisely consider the set of the couples $(\tau, \mathcal{X}(\tau))$ and let the generators $[\tau, \tau^\sigma]$, $[\tau, \rho_i \tau]$ and $[\tau, \omega_{i,j}\tau]$ act on them. The action on the operator algebras is as follows. The elements $[\tau, \tau^\sigma]$ just permutes the indexes of the generators according to the permutation $\sigma$. The change of decoration $[\tau, \rho_i \tau]$ acts trivially on the operators $(\hat{u}_j, \hat{v}_j)$ such that $j \neq i$ and as follows on $(\hat{u}_i, \hat{v}_i)$

$$(\hat{u}_i, \hat{v}_i) \mapsto (q^{-1/2} \hat{v}_i \hat{u}_i^{-1}, \hat{u}_i^{-1}). \tag{3.1}$$

The most interesting generator $[\tau, \omega_{i,j}\tau]$, is again trivial in the triangles not involved in the diagonal exchange while maps the two couples of operators $(\hat{u}_i, \hat{v}_i)$ and $(\hat{u}_j, \hat{v}_j)$ to the two new couples (following formulas (1.37))

$$(\hat{u}_i, \hat{v}_i) \bullet (\hat{u}_j, \hat{v}_j) \equiv (\hat{u}_i \hat{u}_j, \hat{u}_i \hat{v}_j + \hat{v}_i) \tag{3.2}$$
$$(\hat{u}_i, \hat{v}_i) * (\hat{u}_j, \hat{v}_j) \equiv (\hat{u}_j \hat{v}_i (\hat{u}_i \hat{v}_j + \hat{v}_i)^{-1}, \hat{v}_j (\hat{u}_i \hat{v}_j + \hat{v}_i)^{-1}). \tag{3.3}$$

The quantization $\mathcal{X}(\Sigma_{g,s})$ of $\mathcal{R}(\Sigma_{g,s})$ will then be the the quotient of the couples $(\tau, \mathcal{X}(\tau))$ by the generators above. This is all completely abstract. In order to get an actual quantization we need to provide a representation of $\mathcal{X}(\Sigma_{g,s})$ as endomorphisms of some vector space $\mathcal{H}$. In the original paper [Kas98], Kashaev proposed representations on the vector spaces $\mathrm{L}^2(\mathbb{R})$ and $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ for $N$ odd. The former was used to construct the Andersen Kashaev invariants in [AK14b], while the latter are related to the colored Jones polynomials ([Kas94], [MM01]) and the Volume Conjecture [Kas97]. In the more recent work [AK14a] a representation on the vector space $\mathrm{L}^2(\mathbb{A}_N) \equiv \mathrm{L}^2(\mathbb{R} \times \mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^N$ was implicitly proposed, or at least all the basics elements to construct it were presented. After Recalling the quantization at level $N = 1$ (i.e. representations in $\mathrm{L}^2(\mathbb{R})$) we will describe the representations in $\mathrm{L}^2(\mathbb{A}_N)$. Further, in Section 3.2, we will show how $\mathrm{L}^2(\mathbb{A}_N)$ representations of Quantum Teichmüller Theory can be related to Complex Quantum Chern Simons Theory. Later in Chapter 4 we will extend this representations to a 3 dimensional theory, following the construction of the Teichmulle TQFT by Andersen and Kashaev at level $N = 1$ [AK14b].

### 3.1.1 $\mathrm{L}^2(\mathbb{R})$ Representations

To each decorated ideal triangle $\overline{\tau}_j \in \tau$ we associate the Hilbert space $\mathrm{L}^2(\mathbb{R})$. Then the Hilbert space associated to $\mathcal{R}(\tau)$ will be $\mathcal{H} = \mathrm{L}^2(\mathbb{R})^{\otimes M} \cong \mathrm{L}^2(\mathbb{R}^M)$ where $M = 2g - 2 + s$ is the number of triangles in $\tau$. For any $j = 0, \dots, M$ let $\mathsf{p}_j$ and $\mathsf{q}_j$ be the two canonical Heisenberg operators acting on $\mathcal{H}$ as follows

$$\mathsf{q}_j(f)(x) = x_j f(x), \quad \mathsf{p}_j(f)(x) = \frac{1}{2\pi i}\frac{\partial}{\partial x_j}(f)(x), \quad \forall x \in \mathbb{R}^M,\ f \in \mathcal{H}. \tag{3.4}$$

We have the following canonical commutation relations

$$[p_k, p_j] = [q_k, q_j] = 0, \quad [p_k, q_j] = \frac{\delta_{k,j}}{2\pi i}.$$

Let $\mathrm{b} \in \mathbb{C}$ be such that $\mathrm{Re(b)} > 0$, $\mathrm{Im(b)} \geq 0$. Recall that $\omega_{\mathcal{R}} = \sum \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} = \sum \mathrm{d}(\log x) \wedge \mathrm{d}(\log y)$, so we need to consider exponentiated operators. Since the two operators $\mathsf{p}_j$ and $\mathsf{q}_j$ are self-adjoint, unbounded, and with spectrum $\mathbb{R}$ the following operators, thanks to the spectral theorem, will be well defined

$$\mathsf{u}_j = e^{2\pi \mathrm{b} \mathsf{q}_j}, \quad \mathsf{v}_j = e^{2\pi \mathrm{b} \mathsf{p}_j}. \tag{3.5}$$

satisfying the Weil commutation relations

$$\mathsf{u}_j \mathsf{v}_k = e^{2\pi i \mathrm{b}^2 \delta_{j,k}} \mathsf{v}_k \mathsf{u}_j, \quad [u_j, u_k] = [v_j, v_k] = 0. \tag{3.6}$$

To each $\overline{\tau}_j \in \tau$ consider the couple of operators $\mathsf{w}_j = (\mathsf{u}_j, \mathsf{v}_j)$. Following the formulas (1.37) we define the following operations

$$\mathsf{w}_1 \bullet \mathsf{w}_2 := (\mathsf{u}_1\mathsf{u}_2, \mathsf{u}_1\mathsf{v}_2 + \mathsf{v}_1) \tag{3.7}$$
$$\mathsf{w}_1 * \mathsf{w}_2 := \left(\frac{\mathsf{v}_1\mathsf{u}_2}{\mathsf{u}_1\mathsf{v}_2 + \mathsf{v}_1}, \frac{\mathsf{v}_2}{\mathsf{u}_1\mathsf{v}_2 + \mathsf{v}_1}\right).$$

**Theorem 31** (Kashaev, R. [Kas98]). *Let $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ be a solution of the functional equation*

$$\psi(z + i\mathrm{b}/2) = \psi(z - i\mathrm{b}/2)(1 + e^{2\pi \mathrm{b} z}) \tag{3.8}$$

*Then the operator*

$$\mathsf{T} = \mathsf{T}_{12} := e^{2\pi i \mathsf{p}_1 \mathsf{q}_2} \psi(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2) = \psi(\mathsf{q}_1 - \mathsf{p}_1 + \mathsf{p}_2) e^{2\pi i \mathsf{p}_1 \mathsf{q}_2} \tag{3.9}$$

*defines a linear operator on $\mathrm{L}^2(\mathbb{R}^2)$ satisfying*

$$\mathsf{T}\mathsf{q}_1 = (\mathsf{q}_1 + \mathsf{q}_2)\mathsf{T} \tag{3.10}$$

$$\mathsf{T}(\mathsf{p}_1+\mathsf{p}_2) = \mathsf{p}_2\mathsf{T} \tag{3.11}$$
$$\mathsf{T}(\mathsf{p}1+\mathsf{q}_2) = (\mathsf{p}_1+\mathsf{q}_2)\mathsf{T} \tag{3.12}$$
$$\mathsf{T}e^{2\pi \mathsf{b}\mathsf{p}_1} = (e^{2\pi\mathsf{b}(\mathsf{q}_1+\mathsf{p}_2)} + e^{2\pi\mathsf{b}\mathsf{p}_1})\mathsf{T} \tag{3.13}$$

*which will imply, in particular*

$$\mathsf{w}_1 \bullet \mathsf{w}_2\mathsf{T} = \mathsf{T}\mathsf{w}_1, \quad \mathsf{w}_1 * \mathsf{w}_2\mathsf{T} = \mathsf{T}\mathsf{w}_2. \tag{3.14}$$

*Proof.* Using commutation rules from Lemma 88 we can prove

$$\begin{aligned}
\mathsf{T}\mathsf{q}_1 &= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\mathsf{q}_1\\
&= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)\mathsf{q}_1e^{2\pi i\mathsf{p}_1\mathsf{q}_2} + \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)\mathsf{q}_2e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= (\mathsf{q}_1+\mathsf{q}_2)\mathsf{T} - \frac{1}{2\pi i}\psi'(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2) + \frac{1}{2\pi i}\psi'(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)\\
&= (\mathsf{q}_1+\mathsf{q}_2)\mathsf{T}.
\end{aligned}$$

$$\begin{aligned}
\mathsf{T}(\mathsf{p}_1+\mathsf{q}_2) &= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}(\mathsf{p}_1+\mathsf{q}_2)\\
&= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)(\mathsf{p}_1+x+2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= (\mathsf{p}_1+\mathsf{q}_2)\psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&\quad + \frac{1}{2\pi i}(-\psi'(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)+\psi'(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2))e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= (\mathsf{p}_1+\mathsf{q}_2)\psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= (\mathsf{p}_1+\mathsf{q}_2)\mathsf{T}
\end{aligned}$$

$$\begin{aligned}
\mathsf{T}(\mathsf{p}_1+\mathsf{p}_2) &= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}(\mathsf{p}_1+\mathsf{p}_2)\\
&= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)(\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2} - \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)\mathsf{p}_1e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= (\mathsf{p}_1+\mathsf{p}_2)\mathsf{T} - \mathsf{p}_1\mathsf{T} + \frac{1}{2\pi i}(\psi'(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)-\psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2))e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= \mathsf{p}_2\mathsf{T}
\end{aligned}$$

$$\begin{aligned}
\mathsf{T}e^{2\pi b\mathsf{p}_1} &= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}e^{2\pi b\mathsf{p}_1}\\
&= \psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi b\mathsf{p}_1}e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= e^{2\pi b\mathsf{p}_1}\psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2-\frac{b}{i})e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= e^{2\pi b\mathsf{p}_1}(1+e^{2\pi b(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2+\frac{ib}{2})})\psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= (e^{2\pi b\mathsf{p}_1}+e^{2\pi b(\mathsf{q}_1+\mathsf{p}_2-\frac{ib}{2}+\frac{ib}{2})})\psi(\mathsf{q}_1-\mathsf{p}_1+\mathsf{p}_2)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&= (e^{2\pi b\mathsf{p}_1}+e^{2\pi b(\mathsf{q}_1+\mathsf{p}_2)})\mathsf{T}
\end{aligned}$$

The equation (3.8) was used only in the last computation, where we also used the Baker-Cambell-Hausdorff Formula. □

Thanks to Faddeev's difference equation (2.6), the choice $\psi = 1/\Phi_\mathsf{b}$ satisfies (3.8).

**Proposition 32.** *Let*

$$\mathcal{A} \equiv e^{3\pi i\mathsf{q}^2}e^{\pi i(\mathsf{p}+\mathsf{q})^2}. \tag{3.15}$$

*It satisfies*

$$\mathcal{A}\mathsf{p} = -\mathsf{q}\mathcal{A} \qquad\qquad \mathcal{A}\mathsf{q} = (\mathsf{p}-\mathsf{q})\mathcal{A}. \tag{3.16}$$

*In particular*

$$\mathcal{A}(\mathsf{u},\mathsf{v}) = (\mathsf{v}\mathsf{u}^{-1}e^{-\pi i\mathsf{b}^2}, \mathsf{u}^{-1}) \tag{3.17}$$

*Proof.*

$$\mathcal{A}\mathsf{p} = e^{3\pi i \mathsf{q}^2} e^{\pi i(\mathsf{p}+\mathsf{q})^2}\mathsf{p} \tag{3.18}$$
$$= e^{3\pi i \mathsf{q}^2}\left(\mathsf{p} + [\mathsf{p}+\mathsf{q},\mathsf{p}]\, 2\pi i(\mathsf{p}+\mathsf{q})\right) e^{\pi i(\mathsf{p}+\mathsf{q})^2} \tag{3.19}$$
$$= -\mathsf{q}\mathcal{A} \tag{3.20}$$
$$\mathcal{A}\mathsf{q} = e^{3\pi i \mathsf{q}^2} e^{\pi i(\mathsf{p}+\mathsf{q})^2}\mathsf{q} \tag{3.21}$$
$$= e^{3\pi i \mathsf{q}^2}\left(\mathsf{q} + 2\pi i\,[\mathsf{p}+\mathsf{q},\mathsf{q}]\,(\mathsf{p}+\mathsf{q})\right) e^{\pi i(\mathsf{p}+\mathsf{q})^2} \tag{3.22}$$
$$= \left((\mathsf{p}+2\mathsf{q}) + 6\pi i\,[\mathsf{q},2\mathsf{q}+\mathsf{p}]\,\mathsf{q}\right) e^{3\pi i \mathsf{q}^2} e^{\pi i(\mathsf{p}+\mathsf{q})^2} \tag{3.23}$$
$$= \left((\mathsf{p}+2\mathsf{q}) - 3\mathsf{q}\right) e^{3\pi i \mathsf{q}^2} e^{\pi i(\mathsf{p}+\mathsf{q})^2} \tag{3.24}$$

□

### 3.1.2 $\mathrm{L}^2(\mathbb{A}_N)$ Representations

For conventions and notation on the space $\mathrm{L}^2(\mathbb{A}_N)$ see Appendix A.2. Recall $N$ odd positive integer. For every $i = 0,\ldots,M$ let $\mathsf{p}_i$, $\mathsf{q}_i$ be self adjoint operator in $\mathrm{L}^2(\mathbb{R})$ and $X_i$, $Y_i$ unitary operators in $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ such that, for $\omega = e^{\frac{2\pi i}{N}}$ we have

$$[p_i, q_j] = \frac{\delta_{ij}}{2\pi i}, \qquad Y_i X_j = \omega^{\delta_{ij}} X_j Y_i, \qquad X_i^N = Y_i^N = 1. \tag{3.25}$$

Fix $\mathrm{b} \in \mathbb{C}$, such that $\mathrm{Re}\,\mathrm{b} > 0$. We can define the operators

$$\mathsf{u}_i = e^{2\pi \frac{\mathrm{b}}{\sqrt{N}}\mathsf{q}_i} Y_i \qquad\qquad \mathsf{u}_i^* = e^{2\pi \frac{\mathrm{b}^{-1}}{\sqrt{N}}\mathsf{q}_i} Y_i^{-1} \tag{3.26}$$
$$\mathsf{v}_i = e^{2\pi \frac{\mathrm{b}}{\sqrt{N}}\mathsf{p}_i} X_i \qquad\qquad \mathsf{v}_i^* = e^{2\pi \frac{\mathrm{b}^{-1}}{\sqrt{N}}\mathsf{p}_i} X_i^{-1} \tag{3.27}$$

satisfying

$$\mathsf{u}_i\mathsf{v}_j = q^{\delta_{ij}}\mathsf{v}_j\mathsf{u}_i \qquad\qquad \mathsf{u}_i^*\mathsf{v}_j^* = \tilde{q}^{\delta_{ij}}\mathsf{v}_j^*\mathsf{u}_i^* \tag{3.28}$$
$$\mathsf{u}_i\mathsf{v}_j^* = \mathsf{v}_j^*\mathsf{u}_i \qquad\qquad \mathsf{u}_i^*\mathsf{v}_j = \mathsf{v}_j\mathsf{u}_i^* \tag{3.29}$$
$$q = e^{2\pi i \frac{\mathrm{b}^2}{N}}\omega \qquad\qquad \tilde{q} = e^{2\pi i \frac{\mathrm{b}^{-2}}{N}}\omega \tag{3.30}$$

The Quantum algebra $\mathcal{X}(\tau)$ is generated by the $\mathsf{u}_j$, $\mathsf{v}_j$ for $j = 0,\ldots M$, and has a $*$–algebra structure when extended to include $\mathsf{u}_j^*$ and $\mathsf{v}_j^*$. We remark that the $*$ operator we are using here is the standard hermitian conjugation only if $|\mathrm{b}| = 1$.
Explicitly let $X_j$, $Y_j$, $\mathsf{p}_j$, $\mathsf{q}_j$ , $j = 1,2$ be operators acting on $\mathcal{H} := \mathrm{L}^2(\mathbb{A}_N^2)$ as follow

$$\mathsf{p}_j f(\mathsf{x},\mathsf{m}) = \frac{1}{2\pi i}\frac{\partial}{\partial x_j} f(\mathsf{x},\mathsf{m}), \qquad \mathsf{q}_j f(\mathsf{x},\mathsf{m}) = x_j f(\mathsf{x},\mathsf{m}) \tag{3.31}$$
$$X_1 f(\mathsf{x},\mathsf{m}) = f(\mathsf{x},(m_1+1,m_2)), \quad X_2 f(\mathsf{x},\mathsf{m}) = f(\mathsf{x},(m_1,m_2+1)) \tag{3.32}$$
$$Y_j f(\mathsf{x},\mathsf{m}) = \overline{\omega}^{m_j}, \tag{3.33}$$

where $\mathsf{m} = (m_1,m_2) \in \mathbb{Z}_N^2$, $\mathsf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $\omega = e^{2\pi i/N}$.
These operators satisfy conditions (3.25). Let $\psi_{\mathrm{b}}(x,n) \equiv \frac{1}{\mathrm{D}_{\mathrm{b}}(x,n)}$ and consider the operators:

$$\mathsf{S}_{12} \equiv \sum_{j,k=0}^{N-1} \omega^{jk} Y_2^j X_1^k \tag{3.34}$$
$$\mathsf{D}_{12} \equiv e^{2\pi i \mathsf{q}_2\mathsf{p}_1}\mathsf{S}_{12} \tag{3.35}$$
$$\Psi_{12} \equiv \Psi_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) \tag{3.36}$$
$$\mathsf{T}_{12} \equiv \mathsf{D}_{12}\Psi_{12} \tag{3.37}$$

One has

**Lemma 33** (Tetrahedral Equations)**.**

$$\mathsf{T}_{12}\mathsf{u}_1 = \mathsf{u}_1\mathsf{u}_2\mathsf{T}_{12} \tag{3.38}$$

$$\mathsf{T}_{12}\mathsf{v}_1\mathsf{v}_2 = \mathsf{v}_2\mathsf{T}_{12} \tag{3.39}$$

$$\mathsf{T}_{12}\mathsf{v}_1\mathsf{u}_2 = \mathsf{v}_1\mathsf{u}_2\mathsf{T}_{12} \tag{3.40}$$

$$\mathsf{T}_{12}\mathsf{v}_1 = (\mathsf{u}_1\mathsf{v}_2 + \mathsf{v}_1)\mathsf{T}_{12} \tag{3.41}$$

$$\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12} \tag{3.42}$$

*Remark* 3.1.1. If we define $\tilde{\mathsf{T}}_{12} = \mathsf{D}_{12}\tilde{\Psi}_{12}$ where

$$\tilde{\Psi} \equiv \not{\Psi}_{\mathrm{b}^{-1}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}}\overline{Y}_1\overline{X}_2Y_2) \tag{3.43}$$

then $\tilde{\mathsf{T}}$ satisfies equations (3.38 – 3.42) with $\mathsf{u}_i$ and $\mathsf{v}_i$ substituted by $\mathsf{u}_i^*$ and $\mathsf{v}_i^*$. However from Remark 2.3.1 we know that $\Psi_{\mathrm{b}^{-1}}(x,n) = \Psi_{\mathrm{b}}(x,-n)$, and

$$\left(-e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y_2}\right)^{-1} = -e^{\frac{\pi i}{N}}\overline{Y}_1Y_2\overline{X}_2 = -e^{-\frac{\pi i}{N}}\overline{Y}_1\overline{X}_2Y_2$$

so that

$$\tilde{\mathsf{T}} = \mathsf{T}.$$

*Proof.* It is simple to check that

$$\begin{aligned}
\Psi_{12}\mathsf{u}_1 &= \mathsf{u}_1\Psi_{12}\\
\Psi_{12}\mathsf{v}_1\mathsf{v}_2 &= \mathsf{v}_1\mathsf{v}_2\Psi_{12}\\
\Psi_{12}\mathsf{v}_1\mathsf{u}_2 &= \mathsf{v}_1\mathsf{u}_2\Psi_{12}\\
\mathsf{D}_{12}\mathsf{u}_1 &= \mathsf{u}_1\mathsf{u}_2\mathsf{D}_{12}\\
\mathsf{D}_{12}\mathsf{v}_1\mathsf{v}_2 &= \mathsf{v}_2\mathsf{D}_{12}\\
\mathsf{D}_{12}\mathsf{v}_1\mathsf{u}_2 &= \mathsf{v}_1\mathsf{u}_2\mathsf{D}_{12}
\end{aligned}$$

which can be put together to give the first three equations. Define the function $E(x,n) = e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}x}e^{\frac{2\pi i}{N}n}$ Using Faddeev's difference equation, and shortening $\xi := e^{\pi i\frac{N-1}{N}}e^{\pi i\frac{\mathrm{b}^2}{N}}$, we compute

$$\begin{aligned}
\Psi_{12}\mathsf{v}_1 &= \not{\Psi}_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y}_2)e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}\mathsf{p}_1}X_1\\
&= e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}\mathsf{p}_1}X_1\not{\Psi}_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2 + i\frac{\mathrm{b}}{\sqrt{N}}, -\omega e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y}_2)\\
&= e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}\mathsf{p}_1}X_1(1 + \not{E}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y}_2)\xi)\Psi_{12},
\end{aligned}$$

together with

$$\mathsf{D}_{12}\mathsf{v}_1 = \mathsf{v}_1\mathsf{D}_{12}$$

and then

$$\begin{aligned}
&\mathsf{v}_1\mathsf{D}_{12}(1 + \not{E}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y}_2)\xi)\\
&=\mathsf{v}_1\mathsf{S}_{12}(1 + \not{E}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2 + 2\pi i\mathsf{q}_2[\mathsf{p}_1,\mathsf{q}_1] + 2\pi i\mathsf{p}_1[\mathsf{q}_2,\mathsf{p}_2], -e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y}_2)\xi)e^{2\pi i\mathsf{p}_1\mathsf{q}_2}\\
&=\mathsf{v}_1(1 + \not{E}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{p}_1, -\omega e^{-\frac{\pi i}{N}}Y_1X_2\overline{Y}_2Y_2\overline{X_1})\xi)\mathsf{D}_{12}\\
&=\mathsf{v}_1(1 + \not{E}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{p}_1, -e^{\frac{\pi i}{N}}Y_1X_2\overline{X_1})\xi)\mathsf{D}_{12}\\
&=\mathsf{v}_1(1 + e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}(\mathsf{q}_1-\mathsf{p}_1)}(-e^{\frac{\pi i}{N}})Y_1\overline{X}_1\mathsf{v}_2\xi)\mathsf{D}_{12}\\
&=(\mathsf{v}_1 + \mathsf{v}_1\overline{\mathsf{v}}_1\mathsf{u}_1\mathsf{v}_2e^{(\frac{2\pi\mathrm{b}}{\sqrt{N}})^2[\mathsf{p}_1,\mathsf{q}_2]/2}(-e^{\frac{\pi i}{N}})\overline{\omega}\xi)\mathsf{D}_{12}\\
&=(\mathsf{v}_1 + \mathsf{u}_1\mathsf{v}_2)\mathsf{D}_{12},
\end{aligned}$$

All together these computations give us the fourth equation.
For the last equation (3.42) we start remarking that the pentagon equation for $\not{D}_{\mathrm{b}}$ translate in the following pentagon equation for $\not{\Psi}_{\mathrm{b}}$

$$\not{\Psi}_{\mathrm{b}}(\mathsf{q},Y)\not{\Psi}_{\mathrm{b}}(\mathsf{p},X)=\not{\Psi}_{\mathrm{b}}(\mathsf{p},X)\not{\Psi}_{\mathrm{b}}(p+q,-e^{\frac{\pi i}{N}}XY)\not{\Psi}_{\mathrm{b}}(\mathsf{q},Y) \tag{3.44}$$

and that $\mathsf{D}_{lk}$ satisfy the last equation himself, indeed notice that

$$\begin{aligned}\mathsf{S}_{13}\mathsf{S}_{23}&=\sum_{j,k,l,m}\omega^{jk}\omega^{lm}Y_3^{j+l}X_1^kX_2^m\\&=\sum_{p,k,m}\omega^{pk}Y_3^pX_1^kX_2^m\sum_l\omega^{l(m-k)}\\&=\sum_{p,k}\omega^{pk}Y_3^pX_1^kX_2^k,\end{aligned}$$

then

$$\begin{aligned}&\mathsf{D}_{12}\mathsf{D}_{13}\mathsf{D}_{23}=\\&=e^{2\pi i\mathsf{q}_3\mathsf{p}_1}e^{2\pi i\mathsf{q}_2\mathsf{p}_1}e^{2\pi i\mathsf{q}_3\mathsf{p}_2}\sum_{j,l}\omega^{jl}Y_2^jX_1^l\sum_{p,k}\omega^{pk}Y_3^pX_1^kX_2^k\\&=e^{2\pi i\mathsf{q}_3\mathsf{p}_1}e^{-2\pi i\mathsf{q}_3\mathsf{p}_1}e^{2\pi i\mathsf{q}_3\mathsf{p}_2}e^{2\pi i\mathsf{q}_2\mathsf{p}_1}\sum_{j,l,p,k}\omega^{jl}\omega^{pk}\omega^{jk}Y_3^pX_2^kX_1^kY_2^jX_1^l\\&=e^{2\pi i\mathsf{q}_3\mathsf{p}_2}e^{2\pi i\mathsf{q}_2\mathsf{p}_1}\sum_{p,k}\omega^{pk}Y_3^pX_2^k\sum_{j,l}\omega^{j(l+k)}Y_2^jX_1^{l+k}\\&=\mathsf{D}_{23}\mathsf{D}_{12}.\end{aligned}$$

Now observe that

$$\begin{aligned}\Psi_{13}\mathsf{D}_{23}&=\not{\Psi}_{\mathrm{b}}(\mathsf{q}_1+\mathsf{p}_3-\mathsf{q}_3,-e^{-\frac{\pi i}{N}}Y_1X_3\overline{Y}_3)e^{2\pi i\mathsf{q}_3\mathsf{p}_2}\mathsf{S}_{23}\\&=e^{2\pi i\mathsf{q}_3\mathsf{p}_2}\mathsf{S}_{23}\not{\Psi}_{\mathrm{b}}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_3+\mathsf{p}_3,-e^{-\frac{\pi i}{N}}Y_1X_3\overline{Y}_3X_2)\end{aligned}$$

and that

$$\Psi_{12}\mathsf{D}_{13}\mathsf{D}_{23}=\mathsf{D}_{13}\mathsf{D}_{23}\Psi_{12}$$

which can now be put together

$$\begin{aligned}\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23}&=\mathsf{D}_{12}\Psi_{12}\mathsf{D}_{13}\Psi_{13}\mathsf{D}_{23}\Psi_{23}\\&=\mathsf{D}_{12}\mathsf{D}_{13}\mathsf{D}_{23}\Psi_{12}\not{\Psi}_{\mathrm{b}}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_3+\mathsf{p}_3,-e^{-\frac{\pi i}{N}}Y_1X_3\overline{Y}_3X_2)\Psi_{23}\\&=\mathsf{D}_{23}\mathsf{D}_{12}\Psi_{12}\not{\Psi}_{\mathrm{b}}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_3+\mathsf{p}_3,-e^{-\frac{\pi i}{N}}Y_1X_3\overline{Y}_3X_2)\Psi_{23}\end{aligned}$$

It then remain to prove that

$$\Psi_{12}\not{\Psi}_{\mathrm{b}}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_3+\mathsf{p}_3,-e^{-\frac{\pi i}{N}}Y_1X_3\overline{Y}_3X_2)\Psi_{23}=\Psi_{23}\Psi_{12} \tag{3.45}$$

which can be deduced from the pentagon equation (3.44). □

**Proposition 34.** *Let* $\mathsf{w}_i\equiv(\mathsf{u}_i,\mathsf{v}_i)$ *and* $\mathsf{w}_i^*=(\mathsf{u}_i^*,\mathsf{v}_i^*)$. *Then we have*

$$\mathsf{w}_1\bullet\mathsf{w}_2\mathsf{T}_{12}=\mathsf{T}_{12}\mathsf{w}_1,\quad \mathsf{w}_1*\mathsf{w}_2\mathsf{T}_{12}=\mathsf{T}_{12}\mathsf{w}_2, \tag{3.46}$$

$$\mathsf{w}_1^*\bullet\mathsf{w}_2^*\mathsf{T}_{12}=\mathsf{T}_{12}\mathsf{w}_1^*,\quad \mathsf{w}_1^**\mathsf{w}_2^*\mathsf{T}_{12}=\mathsf{T}_{12}\mathsf{w}_2^*. \tag{3.47}$$

**Proposition 35.** *Let*

$$\mathsf{A}_N\equiv\sum_{j=0}^{N-1}\overline{\langle j\rangle}^3Y^{3j}\sum_{l=0}^{N-1}\overline{\langle l\rangle}(-e^{-\pi i/N}YX)^l \tag{3.48}$$

*where* $\langle n\rangle = e^{-\pi i n(n+N)/N}$ *and* $Y$ *and* $X$ *are as above. Let* $\mathcal{A}$ *defined as in Proposition* 32*. Define* $\mathsf{A} = \mathcal{A}\circ \mathsf{A}_N$*. Then*

$$\mathsf{A}(\mathsf{u},\mathsf{v}) = (q^{-1/2}\mathsf{v}\mathsf{u}^{-1},\, \mathsf{u}^{-1}) \qquad \mathsf{A}(\mathsf{u}^*,\mathsf{v}^*) = (\tilde{q}^{-1/2}\mathsf{v}^*(\mathsf{u}^*)^{-1},\, (\mathsf{u}^*)^{-1}) \tag{3.49}$$

*where* $q$ *and* $\tilde{q}$ *are defined by equation* (3.30)*.*

*Proof.*

$$\begin{aligned}
\sum_{l=0}^{N-1}\overline{\langle l\rangle}\left(-e^{-\pi i/N}\right)^l (YX)^l X &= \sum_{l=0}^{N-1}\overline{\langle l\rangle}\left(-e^{-\pi i/N}\right)^l (YX)^l Y^{-1}YX\\
&= Y^{-1}\sum_{l=0}^{N-1}\overline{\langle l\rangle}\left(-e^{-\pi i/N}\right)^l (YX)^{l+1}\omega^l\\
&= Y^{-1}\sum_{l=0}^{N-1}\overline{\langle l+1\rangle}\overline{\omega}^l\left(-e^{-\pi i/N}\right)^{l+1} (YX)^{l+1}\omega^l\\
&= Y^{-1}\sum_{l=0}^{N-1}\overline{\langle l+1\rangle}\left(-e^{-\pi i/N}\right)^{l+1} (YX)^{l+1}
\end{aligned}$$

So that $\mathsf{A}_N X = Y^{-1}\mathsf{A}_N$. Now consider

$$\begin{aligned}
\sum_{l=0}^{N-1}\overline{\langle l\rangle}\left(-e^{-\pi i/N}\right)^l (YX)^l Y &= Y\sum_{l=0}^{N-1}\overline{\langle l\rangle}\left(-e^{-\pi i/N}\right)^l (YX)^l\overline{\omega}^l\\
&= Y^2X\sum_{l=0}^{N-1}\overline{\langle l\rangle}\left(-e^{-\pi i/N}\right)^l (YX)^{l-1}\overline{\omega}^l\\
&= Y^2X\sum_{l=0}^{N-1}\overline{\langle l-1\rangle}\omega^l\left(-e^{-\pi i/N}\right)^{l+1} (YX)^{l-1}\overline{\omega}^l\\
&= Y^2X\overline{\omega}\sum_{l=0}^{N-1}\overline{\langle l-1\rangle}\left(-e^{-\pi i/N}YX\right)^{l-1}
\end{aligned}$$

and also

$$\begin{aligned}
\sum_{j=0}^{N-1}\overline{\langle j\rangle}^3 Y^{3j}Y^2X\overline{\omega} &= XY^{-1}\sum_{j=0}^{N-1}\overline{\langle j\rangle}^3 Y^{3(j+1)}\omega^{3j+1}\\
&= XY^{-1}\sum_{j=0}^{N-1}\overline{\langle j+1\rangle}^3 Y^{3(j+1)}\overline{\omega}^{3j}(-e^{-\pi i/N})^3\omega^{3j+1}\\
&= XY^{-1}(-e^{-\pi i/N})\sum_{j=0}^{N-1}\overline{\langle j+1\rangle}^3 Y^{3(j+1)}
\end{aligned}$$

so that we have $\mathsf{A}_N Y = \omega^{(N-1)/2}XY^{-1}\mathsf{A}_N$. The rest of the statement follows from Proposition 32. □

## 3.2 Quantization of the Model Space for Complex Chern-Simons Theory

In this Section we want to quantize the space $\mathbb{C}^*\times\mathbb{C}^*$ with the complex differential form

$$\omega_{\mathbb{C}} = \frac{\mathrm{d}x\wedge \mathrm{d}y}{xy}.$$

We think of it as a model space for Complex Chern-Simons Theory because it is an open dense of the $\mathrm{PSL}(2,\mathbb{C})$ moduli space of flat connections on a four punctured sphere, with unipotent holonomy around the punctures. Moreover its quotient by $\mathbb{S}_2$ is the moduli space of semi–simple $\mathrm{SL}(2,\mathbb{C})$ flat connections of a genus 1 surface. We will look at this second case closer further in this thesis. For now we are interested in the first interpretation: tetrahedral operators are supposedly related to states in the quantization of the four punctured sphere. Since we want to construct knot invariants starting from tetrahedral ideal triangulations this is the space we need to quantize. We will propose two different approach to quantize this space: the first following Dimofte [Dim14], via Bohr-Sommerfeld quantization, that will give us an operator algebra similar to the one coming from $\mathrm{L}^2(\mathbb{A}_N)$-representations in quantum Teichmüller Theory. The second following the ideas in Andersen and Kashaev [AK14a] using a real polarization with contractible leaves. We will further show that an appropriately chosen Weil-Gel'fand-Zak Transform relates this two quantization. To use this transform to relates the Andersen–Kashaev invariants to complex Chern–Simons Theory was already proposed in [AK14a]. However the relation between the two approaches was not as tight as here.

### 3.2.1 Pre–Quantization

Let $t = N + is \in \mathbb{C}^*$ be the quantization constant, for $N \in \mathbb{R}$ and $s \in \mathbb{R} \sqcup i\mathbb{R}$. Denote also $\tilde{t} = N - is$. Fix $\mathrm{b} \in \mathbb{C}$ such that $s = -iN\frac{1-\mathrm{b}^2}{1+\mathrm{b}^2}$ and $\mathrm{Re}\,\mathrm{b} > 0$. This substitution, for $s \in i\mathbb{R}$, is only possible when $-N < is < N$. Notice that

$$s \in \mathbb{R} \iff |\mathrm{b}| = 1 \text{ and } \mathrm{b} \neq \pm i, \qquad s \in i\mathbb{R} \iff \mathrm{Im}\,\mathrm{b} = 0 \tag{3.50}$$

Indeed

$$s \in i\mathbb{R} \implies \frac{1-\mathrm{b}^2}{1+\mathrm{b}^2} \in \mathbb{R} \implies (1-\mathrm{b}^2)(1+\overline{\mathrm{b}}^2) = (1+\mathrm{b}^2)(1-\overline{\mathrm{b}}^2) \implies \mathrm{b}^2 = \overline{\mathrm{b}}^2$$

The last statement together with $\mathrm{Re}\,\mathrm{b} > 0$ implies $\mathrm{b} \in \mathbb{R}$. The opposite implication is trivial. For the other situation

$$\begin{aligned} |\mathrm{b}| = 1 \quad &\implies \frac{1-\mathrm{b}^2}{1+\mathrm{b}^2} = \frac{\overline{\mathrm{b}}-\mathrm{b}}{\overline{\mathrm{b}}+\mathrm{b}} = \frac{-i\,\mathrm{Im}\,\mathrm{b}}{\mathrm{Re}\,\mathrm{b}}\ s \in \mathbb{R} \\ s \in \mathbb{R} \quad &\implies \frac{1-\mathrm{b}^2}{1+\mathrm{b}^2} \in i\mathbb{R} \implies (-1+\mathrm{b}^2)(1+\overline{\mathrm{b}}^2) = (1+\mathrm{b}^2)(1-\overline{\mathrm{b}}^2) \\ &\implies \left|\mathrm{b}^2\right|^2 = 1 \implies |\mathrm{b}| = 1 \end{aligned}$$

We also remark the following useful expressions

$$t = \frac{2N}{1+\mathrm{b}^2} \qquad \tilde{t} = \frac{2N}{1+\mathrm{b}^{-2}} \tag{3.51}$$

Consider the covering maps

$$\begin{aligned} \zeta^{\pm} \colon \mathbb{R}^2 &\longrightarrow \mathbb{C}^* \\ (z,n) &\mapsto \exp\left(2\pi \mathrm{b}^{\pm 1} z \pm 2\pi i n\right) \end{aligned} \tag{3.52}$$

and consequently

$$\pi^{\pm} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{C}^* \times \mathbb{C}^*, \qquad \pi^{\pm} = (\zeta^{\pm}, \zeta^{\pm}) \tag{3.53}$$

such that

$$\mathbb{C}^* \times \mathbb{C}^* \ni (x,y) = \pi^{+}((z,n),(w,m)), \tag{3.54}$$

$$\mathbb{C}^* \times \mathbb{C}^* \ni (\tilde{x}, \tilde{y}) = \pi^-((z,n),(w,m)) \quad \text{for } ((z,n),(w,m)) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

We remark that

$$\overline{\zeta^+(z,n)} = \zeta^-(z,n) \qquad \Longleftrightarrow \qquad |\mathrm{b}| = 1 \tag{3.55}$$

in this case $\pi^- = \overline{\pi^+}$ and $\tilde{x} = \overline{x}$, $\tilde{y} = \overline{y}$. In this sense $x$, $y$ $\tilde{x}$ and $\tilde{y}$ are natural coordinate functions to quantize in $\mathbb{C}^* \times \mathbb{C}^*$. If $\mathrm{b} \in \mathbb{R}$ they are still coordinates functions for the underlying real manifold, but we lose the complex conjugate interpretation. We will work the quantization in the cover $\mathbb{R}^2 \times \mathbb{R}^2$ of the space. Define the form

$$\omega_t \equiv \frac{t}{4\pi}(\pi^+)^*(\omega_{\mathbb{C}}) + \frac{\tilde{t}}{4\pi}(\pi^-)^*(\omega_{\mathbb{C}}) \tag{3.56}$$

**Lemma 36.**

$$\omega_t = 2\pi N(\mathrm{d}z \wedge \mathrm{d}w - \mathrm{d}n \wedge \mathrm{d}m). \tag{3.57}$$

*In particular it is a real symplectic 2 form on $\mathbb{R}^2 \times \mathbb{R}^2$, independent of* $\mathrm{b}$.

*Proof.*

$$\begin{aligned}
(\pi^\pm)^*(\omega_{\mathbb{C}}) &= \frac{\mathrm{d}\zeta^\pm(z,n)}{\zeta^\pm(z,n)} \wedge \frac{\mathrm{d}\zeta^\pm(w,m)}{\zeta^\pm(w,m)} \\
&= \mathrm{d}\left(2\pi \mathrm{b}^{\pm 1} z \pm 2\pi i n\right) \wedge \mathrm{d}\left(2\pi \mathrm{b}^{\pm 1} w \pm 2\pi i m\right) \\
&= 4\pi^2 \mathrm{b}^{\pm 2} \mathrm{d}z \wedge \mathrm{d}w \pm 4\pi^2 \mathrm{b}^{\pm 1}(\mathrm{d}z \wedge \mathrm{d}m - \mathrm{d}w \wedge \mathrm{d}n) - 4\pi^2 \mathrm{d}n \wedge \mathrm{d}m
\end{aligned}$$

We can then compute

$$\begin{aligned}
\omega_t = 2\pi N &\left(\frac{1}{1+\mathrm{b}^2} + \frac{1}{1+\mathrm{b}^{-2}}\right) \mathrm{d}z \wedge \mathrm{d}w + \\
&2\pi i N \left(\frac{\mathrm{b}}{1+\mathrm{b}^2} - \frac{\mathrm{b}^{-1}}{1+\mathrm{b}^{-2}}\right)(\mathrm{d}z \wedge \mathrm{d}m - \mathrm{d}w \wedge \mathrm{d}n) + \\
&- 2\pi N \left(\frac{1}{1+\mathrm{b}^2} + \frac{1}{1+\mathrm{b}^{-2}}\right) \mathrm{d}n \wedge \mathrm{d}m \\
&= 2\pi N(\mathrm{d}z \wedge \mathrm{d}w - \mathrm{d}n \wedge \mathrm{d}m).
\end{aligned}$$

□

Over $\mathbb{R}^2 \times \mathbb{R}^2$ we take the trivial line bundle $\widetilde{\mathcal{L}} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{C}$. On the $N$-th tensor power of this line bundle $\widetilde{\mathcal{L}}^N$ we consider the connection

$$\nabla^{(t)} \equiv \mathrm{d} - i\alpha_t \tag{3.58}$$

where

$$\alpha_t \equiv \frac{t}{4\pi}\alpha_{\mathbb{C}}^+ + \frac{\tilde{t}}{4\pi}\alpha_{\mathbb{C}}^-, \tag{3.59}$$

$$\alpha_{\mathbb{C}}^\pm \equiv 2\pi^2(\mathrm{b}^{\pm 1} z \pm i n)\mathrm{d}(\mathrm{b}^{\pm 1} w \pm i m) - 2\pi^2(\mathrm{b}^{\pm 1} w \pm i m)\mathrm{d}(\mathrm{b}^{\pm 1} z \pm i n) \tag{3.60}$$

Computations similar to the one in Lemma 36 give

$$\alpha_t = \pi N(z\mathrm{d}w - w\mathrm{d}z - n\mathrm{d}m + m\mathrm{d}n). \tag{3.61}$$

It is clear, then, that

$$\mathrm{d}\alpha_{\mathbb{C}}^\pm = (\pi^\pm)^*(\omega_{\mathbb{C}}), \qquad \text{which implies} \tag{3.62}$$

$$F_{\nabla^{(t)}} = -i\omega_t. \tag{3.63}$$

Further, on $\mathbb{R}^2 \times \mathbb{R}^2$ we have an action of $\mathbb{Z} \times \mathbb{Z}$ compatible with the projection $\pi^+$, i.e.

$$(\mathbb{Z} \times \mathbb{Z}) \times \left(\mathbb{R}^2 \times \mathbb{R}^2\right) \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2 \tag{3.64}$$

$$(\lambda_1, \lambda_2) \cdot ((z, n), (w, m)) \mapsto ((z, n + \lambda_1), (w, m + \lambda_2)) \tag{3.65}$$

that satisfies

$$\pi^{\pm}((z, n + \lambda_1), (w, m + \lambda_2)) = \pi^{\pm}((z, n), (w, m)) \tag{3.66}$$

This action can be lifted to an action $\widetilde{\mathcal{L}}^N$ in such a way that the quotient bundle $\mathcal{L}^N \equiv \widetilde{\mathcal{L}}^N\ /(\mathbb{Z})^2 \to \mathbb{R}^4/\mathbb{Z}^2$ has first chern class $c_1(\mathcal{L}^N) = \frac{1}{2\pi}\,[\omega_t]$ ($\omega_t$ is evidently $\mathbb{Z}^2$–invariant). Such condition give the requirement (which is in fact the pre-quantum condition) $\frac{1}{2\pi}\,[\omega_t] \in H^2((\mathbb{R}^2 \times \mathbb{R}^2)/(\mathbb{Z}^2), \mathbb{Z})$ which boils down to the requirement $N \in \mathbb{Z}$. Explicitly the action of $\mathbb{Z} \times \mathbb{Z}$ on $\widetilde{\mathcal{L}}^N$ is given by the following two multipliers

$$e_{(1,0)} = e^{-\pi Nim} \qquad\qquad e_{(0,1)} = e^{\pi Nin} \tag{3.67}$$

that means that we consider the space of sections

$$(\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2} \tag{3.68}$$

of $\mathbb{Z}^2$–invariant, smooth sections of $\widetilde{\mathcal{L}}^N$ Explicitly

$$\begin{aligned}
&s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2} \text{ if and only if } s \in \mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N) \text{ and satisfies}\\
&s((z, n + 1), (w, m)) = e^{-\pi iNm} s((z, n), (w, m)), && (3.69)\\
&s((z, n), (w, m + 1)) = e^{\pi iNn} s((z, n), (w, m)) && (3.70)
\end{aligned}$$

**Lemma 37.**

$$\nabla^{(t)} s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}, \qquad \textit{for any } s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}$$

*Proof.*

$$\begin{aligned}
\nabla^{(t)}_m s((z, n + 1), (w, m)) &= \frac{\partial}{\partial m} s((z, n + 1), (w, m))\\
&\qquad + \pi iN(n + 1) s((z, n + 1), (w, m))\\
&= e^{-\pi iNm} \frac{\partial s}{\partial m}((z, n), (w, m))\\
&\qquad + \pi iN(n + 1 - 1) e^{-\pi iNm} s((z, n), (w, m))\\
&= e^{-\pi iNm} \nabla^{(t)}_m s((z, n), (w, m))\\
\nabla^{(t)}_n s((z, n + 1), (w, m)) &= e^{-\pi iNm} \nabla^{(t)}_n s((z, n), (w, m))\\
\nabla^{(t)}_z s((z, n + 1), (w, m)) &= e^{-\pi iNm} \nabla^{(t)}_z s((z, n), (w, m))\\
\nabla^{(t)}_w s((z, n + 1), (w, m)) &= e^{-\pi iNm} \nabla^{(t)}_w s((z, n), (w, m))
\end{aligned}$$

and computations for $(m + 1)$ are analogous. $\square$

The following Hermitian structure on $\widetilde{\mathcal{L}}^N$ is $\mathbb{Z}^2$–invariant and parallel with respect to $\nabla^{(t)}$.

$$s \cdot s'(p) \equiv s(p)\overline{s'(p)}, \quad \text{ for any } p \in \mathbb{R}^2 \times \mathbb{R}^2 \tag{3.71}$$

Being parallel here means that

$$\mathrm{d}(s \cdot s') = (\nabla^{(t)} s) \cdot s' + s \cdot (\nabla^{(t)} s'), \tag{3.72}$$

and this is a simple consequence of $\alpha_t$ being a real 1-form. It follows that the following is a well defined inner product in the completion of $\left(\left(\mathrm{L}^2 \cap \mathrm{C}^\infty\right)(\mathbb{R}^4, \widetilde{\mathcal{L}}^N)\right)^{\mathbb{Z}^2}$

$$(s,s') \equiv \int_{\mathbb{R}} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}w \left(\int_0^1 \mathrm{d}n \int_0^1 \mathrm{d}m \; s \cdot s'\right) \tag{3.73}$$

**Lemma 38.** *We have the following Hamiltonian vector field for the coordinates functions on* $\mathbb{R}^2 \times \mathbb{R}^2$

$$X_z = \frac{1}{2\pi N}\frac{\partial}{\partial w} \qquad X_w = -\frac{1}{2\pi N}\frac{\partial}{\partial z}$$
$$X_n = -\frac{1}{2\pi N}\frac{\partial}{\partial m} \qquad X_m = \frac{1}{2\pi N}\frac{\partial}{\partial n}$$

*Proof.* This is an immediate verification of the definition

$$\omega_t(X_f, Y) = -\mathrm{d}f\,[Y]$$

**Lemma 39** (Pre–Quantum operators)**.** *The following are the pre–quantum operators for the coordinate functions or* $\mathbb{R}^2 \times \mathbb{R}^2$

$$\hat{z} = \frac{-i}{2\pi N}\nabla^{(t)}_w + z \qquad \hat{w} = \frac{i}{2\pi N}\nabla^{(t)}_z + w$$
$$\hat{n} = \frac{i}{2\pi N}\nabla^{(t)}_m + n \qquad \hat{m} = \frac{-i}{2\pi N}\nabla^{(t)}_n + m$$

*and they satisfy the following canonical commuting relations*

$$[\hat{z},\ \hat{w}] = \frac{1}{2\pi i N} \qquad [\hat{n},\ \hat{m}] = -\frac{1}{2\pi i N} \tag{3.74}$$
$$[\hat{z},\hat{n}] = [\hat{z},\hat{m}] = [\hat{w},\hat{n}] = [\hat{w},\hat{m}] = 0 \tag{3.75}$$

*Proof.* The definition of pre–Quantum operator, for $f$ a real smooth function is generally

$$\hat{f} = -i\nabla_{X_f} + f \tag{3.76}$$

where nabla is the pre–Quantum connection in the pre–Quantum line bundle. The commuting relations should be fixed by our choices however we verify them just in case

$$\left[\nabla^{(t)}_w, \nabla^{(t)}_z\right] = [\partial_w,\ \pi i N w] + [-\pi i N z, \partial_z] = 2\pi i N$$
$$[\hat{z},\hat{w}] = \frac{1}{(2\pi N)^2}\left[\nabla^{(t)}_w, \nabla^{(t)}_z\right] - \frac{i}{2\pi N}\left(\left[\nabla^{(t)}_w,\ w\right] - \left[z,\ \nabla^{(t)}_z\right]\right)$$
$$= \frac{i}{2\pi N} - \frac{i}{\pi N} = \frac{1}{2\pi i N}$$

The other one is similar, while the trivial one are actually obvious. □

The Hermitian line bundle $\mathcal{L}^N \to \left(\mathbb{R}^2 \times \mathbb{R}^2/\mathbb{Z}^2\right)$ together with the connection $\nabla^{(t)}$ define a pre–Quantization of the theory. In order to finish the quantization program we need to choose a Lagrangian polarization. We will choose two different real polarizations in the following two subsections and then show how to relates the two different results

### 3.2.2 Bohr-Sommerfeld Quantization

Let us consider the following real Lagrangian polarization

$$\mathcal{P} \equiv \mathrm{Span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial w}, \frac{\partial}{\partial m} \right\}. \tag{3.77}$$

The space $\mathbb{R}^2 \times \mathbb{R}^2$ as an $\mathbb{Z}^2$–invariant decomposition into two distinct symplectic leaves

$$M_1 \equiv \{n = m = 0\}, \qquad M_2 \equiv \{z = w = 0\} \tag{3.78}$$

The symplectic form $\omega_t = 2\pi N(\mathrm{d}z \wedge \mathrm{d}w - \mathrm{d}n \wedge \mathrm{d}m)$ decompose accordingly into two symplectic spaces and the Lagrangian polarization $\mathcal{P}$ decomposes into two distinct Lagrangian Polarizations. Therefore we can compute the quantization $\mathcal{H}^{(N)}$ of $\mathbb{R}^2 \times \mathbb{R}^2/\mathbb{Z}^2$ as the tensor product of two distinct quantizations $\mathcal{H}^{(N)}(M_1) \otimes \left(\mathcal{H}^{(N)}(M_2)\right)^{\mathbb{Z}^2}$. For $M_1$ we are in the situation of a canonical quantization of $\mathbb{R}^2$. Indeed we are considering sections

$$\left\{ \psi \equiv \psi(z, w) \in \mathrm{C}^\infty(\mathbb{R}^2) \text{ such that } \nabla_w^{(t)} \psi \equiv 0 \right\} \tag{3.79}$$

which, restricted to the transversal $\{w = 0\}$ reduces to simply $\mathrm{C}^\infty(\mathbb{R})$. Since we will like to have an Hilbert space, we just take the square integrable functions, and then the completion of them

$$\mathcal{H}^{(N)}(M_1) = \mathrm{L}^2(\mathbb{R}).$$

The pre–Quantum operators that act non trivially on $M_1$ become

$$\hat{z} = z \qquad \hat{w} = \frac{i}{2\pi N} \frac{\partial}{\partial z} \tag{3.80}$$

Things are more complicated for $M_2$, and we are not going to give full details about this type of quantization, referring instead to the relevant literature (see, for example [Śni80, MP15, Ham10, Wei91]) and showing the main phenomena. The action of $\mathbb{Z}^2$ is not trivial here and the quotient is $M_2/\mathbb{Z}^2 = \mathbb{T} \times \mathbb{T}$, $\mathbb{T}$ being the unit circle. The line bundle $\mathcal{L}^N$ restricts to a non trivial line bundle over $\mathbb{T} \times \mathbb{T}$ determined by the multipliers (3.67). Define the sheaf $\mathcal{J}$ as follows, for any open $U \subseteq \mathbb{T} \times \mathbb{T}$ the space $\mathcal{J}(U)$ is the space of local sections $s \in \mathrm{C}^\infty(U, \mathcal{L}^N)$, that are polarized, meaning $\nabla_m^{(t)} s = 0$. By $H^*(\mathbb{T} \times \mathbb{T}, \mathcal{J})$ we mean the usual sheaf cohomology (see for example [BT82]). The space $\mathcal{H}^{(N)}(M_2)^{\mathbb{Z}^2}$, if defined to be the the space of global polarized sections $s \in \mathrm{C}^\infty(\mathbb{T} \times \mathbb{T}, \mathcal{L}^N) = H^0(\mathbb{T} \times \mathbb{T}, \mathcal{J})$ would be empty. One, instead, define the quantization in terms of the whole cohomology

$$\mathcal{H}^{(N)}(\mathbb{T} \times \mathbb{T}) \equiv \bigoplus_{j=0}^{2} H^j(\mathbb{T} \times \mathbb{T}, \mathcal{J}). \tag{3.81}$$

We are not going to discuss the relation between this definition and usual one, however we just remark that for our simple situation, results from definition (3.81) are well known and equivalent to results coming from Kahler quantization (see [Wei91]). Let us define another one more concept

**Definition 11.** Let $P$ be a Lagrangian real polarization for the symplectic manifold $M$ such that the leaf space $B$ is smooth, and let $\pi \colon M \longrightarrow B$ be the associated projection. Let $L \to M$ be a pre–quantum line bundle. We say that the point $b \in B$ is Bohr-Sommerfeld for $P$ if $\pi^{-1}(b)$ admits a covariant constant section of $L|_{\pi^{-1}(b)}$. Let $\mathcal{BS} \subset B$ denotes the set of Bohr-Sommerfeld points.

Then the following theorem of Sniatycki (see [Wei91] for an analogous statement) solve our situation

**Proposition 40** (Sniatycki, [Śni80]). *(i)* $H^0(\mathbb{T}\times\mathbb{T},\,\mathcal{J}) = H^2((\mathbb{T}\times\mathbb{T},\,\mathcal{J})) = \{0\}$ *(ii)* $\dim H^1(\mathbb{T}\times\mathbb{T},\,\mathcal{J}) = \#\mathcal{BS} = N$

We will just prove the following Lemma to be able to count the Bohr-Sommerfeld points

**Lemma 41.** *There are no global section $s\in \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T},\mathcal{L}^N)$ such that $\nabla^{(t)}_m s = 0$. There are exactly $N$ Bohr-Sommerfeld points $b_j$, each of them supporting a one dimensional vector space $S_j\simeq\mathbb{C}$ of covariantly constant sections,*

$$S_j \equiv \mathrm{Span}_{\mathbb{C}}\left\{\delta(n-j/N)e^{-\pi i N mn}\right\} \tag{3.82}$$

*Proof.* The differential equation $\nabla^{(t)}_m s = 0$, on $M_2$, is explicitly

$$\frac{\partial}{\partial m}s(n,m) = -\pi i N n\, s(n,m) \quad \text{or equivalently} \tag{3.83}$$

$$s(n,m) = \lambda e^{-\pi i N nm},\ \text{for some } \lambda\in\mathbb{C}. \tag{3.84}$$

Globally this can not be a section, being $\lambda e^{-\pi i N n(m+1)} \neq e^{\pi i N n}\lambda e^{-\pi i N nm}$. Fixing $n = n_0$, we seek solutions to

$$e^{-\pi i N n_0(m+1)} = e^{\pi i N n_0} e^{-\pi i N n_0 m} \tag{3.85}$$

and this does happen exactly when $n_0 = j/N$ for $j = 0,\,\dots,\,N-1$. $\square$

We have seen that $\mathcal{H}^{(N)}(M_2)^{\mathbb{Z}^2}\simeq\bigoplus_{j=0}^{N-1}S_j$. The distributional description of the generators can be used to give an heuristic understanding of how the operators (defined in agreement to (3.52 - 3.53) )

$$X\equiv\exp(2\pi i\hat{n}) \qquad\qquad Y\equiv\exp(2\pi i\hat{m}) \tag{3.86}$$

acts on this space. This is formal and we are not addressing the well definiteness of the action on distributional sections. We have, after polarizing,

$$X = \exp(2\pi i n) \qquad\qquad Y = \exp\left(\frac{1}{N}\frac{\partial}{\partial n} + \pi i m\right), \tag{3.87}$$

and we let them act on generators $e_j\equiv\delta(n-j/N)e^{-\pi i N mn}$, $j = 0,\,\dots,\,N-1$

$$Ye_p = e^{\pi i m}\delta(n+\frac{1}{N}-\frac{p}{N})e^{-\pi i N m(n+1/N)} = \delta(n-\frac{p-1}{N})e^{-\pi i N mn} = e_{p-1} \tag{3.88}$$

$$Xe_p = e^{2\pi i p/N}\delta(n-p/N)e^{-\pi i N mn} = e^{2\pi i p/N}e_p. \tag{3.89}$$

Putting everything together we gave the quantization of $\mathbb{C}^*\times\mathbb{C}^*$ with complex symplectic form $\omega_{\mathbb{C}} = \frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}$ as the vector space

$$\mathcal{H}^{(N)} = \mathrm{L}^2(\mathbb{R})\otimes\mathbb{C}^N\simeq\mathrm{L}^2(\mathbb{A}_N) \tag{3.90}$$

The algebra of quantized observables is generated by the following 4 operators

$$\hat{x} = e^{2\pi \mathrm{b} z}X \qquad\qquad \hat{\tilde{x}} = e^{2\pi\mathrm{b}^{-1}z}X^{-1} \tag{3.91}$$

$$\hat{y} = e^{i\frac{\mathrm{b}}{N}\frac{\partial}{\partial z}}Y \qquad\qquad \hat{\tilde{y}} = e^{i\frac{\mathrm{b}^{-1}}{N}\frac{\partial}{\partial z}}Y^{-1} \tag{3.92}$$

which act on $\mathcal{H}^{(N)}$ as follows, for any $f \equiv f(z) \in \mathrm{L}^2(\mathbb{R})$, and $e_p \in \mathbb{C}^N$ as above

$$\hat{x}\,(f(z) \otimes e_p) = e^{2\pi \mathrm{b} z} e^{2\pi i p/N} f(z) \otimes e_p \tag{3.93}$$
$$\hat{\tilde{x}}\,(f(z) \otimes e_p) = e^{2\pi \mathrm{b}^{-1} z} e^{-2\pi i p/N} f(z) \otimes e_p \tag{3.94}$$
$$\hat{y}\,(f(z) \otimes e_p) = f(z + i\frac{\mathrm{b}}{N}) \otimes e_{p-1} \tag{3.95}$$
$$\hat{\tilde{y}}\,(f(z) \otimes e_p) = f(z + i\frac{\mathrm{b}^{-1}}{N}) \otimes e_{p+1} \tag{3.96}$$

If we identify $\mathbb{C}^N \simeq \mathrm{L}^2(\mathbb{Z}_N)$ we may think of $e_p(l) \equiv \delta(p-l)$ where the $\delta$ is defined modulo $N$. Then the operators will act on $\mathsf{f} \in \mathrm{L}^2(\mathbb{A}_N)$ as

$$\hat{x}\mathsf{f}(z,l) = e^{2\pi \mathrm{b} z} e^{2\pi i l/N} \mathsf{f}(z,l) \qquad \hat{\tilde{x}}\mathsf{f}(z,l) = e^{2\pi \mathrm{b}^{-1} z} e^{-2\pi i l/N} \mathsf{f}(z,l) \tag{3.97}$$
$$\hat{y}\mathsf{f}(z,l) = f(z + i\frac{\mathrm{b}}{N}, l+1) \qquad \hat{\tilde{y}}\mathsf{f}(z,l) = f(z + i\frac{\mathrm{b}^{-1}}{N}, l-1) \tag{3.98}$$

These above are analogous to the formulas on [Dim14]. We can see that

$$\hat{y}\hat{x} = q\hat{x}\hat{y} \qquad \hat{\tilde{y}}\hat{\tilde{x}} = \tilde{q}\hat{\tilde{x}}\hat{\tilde{y}} \tag{3.99}$$
$$q = e^{2\pi i(1+\mathrm{b}^2)/N} = e^{\frac{4\pi i}{t}} \qquad \tilde{q} = e^{2\pi i(1+bpar^{-2})/N} = e^{\frac{4\pi i}{\bar{t}}} \tag{3.100}$$

The relation between the algebra of observables we got here and the algebra of observables from Quantum Teichmüller Theory when represented in $\mathrm{L}^2(\mathbb{A}_N)$ is now evident. Explicitly recall the operators $\mathsf{u} = \mathsf{u}(\mathrm{b})$ and $\mathsf{v} = \mathsf{v}(\mathrm{b})$ from equations (3.25 – 3.33), and recall that they depend on on a parameter b as well. Define the rescaling operator

$$\begin{aligned} \mathcal{O}_{\sqrt{N}}\colon\ \mathrm{L}^2(\mathbb{A}_N) &\longrightarrow \mathrm{L}^2(\mathbb{A}_N) \\ (z,n) &\mapsto (\sqrt{N}z, n) \end{aligned} \tag{3.101}$$

then we have

$$\hat{\tilde{x}} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{u}(\mathrm{b}^{-1}) \circ \mathcal{O}_{\sqrt{N}}^{-1} \qquad \hat{\tilde{y}}^{-1} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{v}(\mathrm{b}^{-1}) \circ \mathcal{O}_{\sqrt{N}}^{-1} \tag{3.102}$$
$$\hat{x} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{u}^*(\mathrm{b}^{-1}) \circ \mathcal{O}_{\sqrt{N}}^{-1} \qquad \hat{y}^{-1} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{v}^*(\mathrm{b}^{-1}) \circ \mathcal{O}_{\sqrt{N}}^{-1} \tag{3.103}$$

The inner product $(\cdot,\, \cdot)$ from (3.73), once restricted becomes

$$\begin{aligned} (f \otimes e_p,\, g \otimes e_q) &= \int_{\mathbb{R}} f(z)\overline{g(z)}\mathrm{d}z \int_{\mathbb{T}} \mathrm{d}m \int_{\mathbb{T}} \mathrm{d}n \delta(n - p/N)\delta(n - q/N) e^{-\pi i N mn} e^{\pi i N mn} \\ &= \int_{\mathbb{R}} f(z)\overline{g(z)}\mathrm{d}z \delta_{p,q} \end{aligned}$$

In terms of functions $\mathsf{f}, \mathsf{g} \in \mathrm{L}^2(\mathbb{A}_N)$ the above formula take the simple expression

$$(\mathsf{f},\, \mathsf{g}) = \sum_{j=0}^{N-1} \int_{\mathbb{R}} \mathsf{f}(x,j)\overline{\mathsf{g}(x,j)}\mathrm{d}x \tag{3.104}$$

as we have in equation (A.8).

### 3.2.3 Real Polarization with Simply Connected Leaves

Consider the following complex coordinates on $\mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathbb{C} \times \mathbb{C}$,

$$u = 2\pi \mathrm{b} z + 2\pi i n \qquad \tilde{u} = 2\pi \mathrm{b}^{-1} z - 2\pi i n \tag{3.105}$$

$$v = 2\pi \mathrm{b} w + 2\pi i m \qquad\qquad \tilde{v} = 2\pi \mathrm{b}^{-1} w - 2\pi i m. \tag{3.106}$$

The corresponding pre–quantum operators are

$$\begin{aligned}
\hat{u} &= -i\frac{\mathrm{b}}{N}\nabla_w^{(t)} - \frac{1}{N}\nabla_m^{(t)} + 2\pi \mathrm{b} z + 2\pi i n \\
\hat{v} &= i\frac{\mathrm{b}}{N}\nabla_z^{(t)} + \frac{1}{N}\nabla_n^{(t)} + 2\pi \mathrm{b} w + 2\pi i m \\
\hat{\tilde{u}} &= -i\frac{\mathrm{b}^{-1}}{N}\nabla_w^{(t)} + \frac{1}{N}\nabla_m^{(t)} + 2\pi \mathrm{b}^{-1} z - 2\pi i n \\
\hat{\tilde{v}} &= i\frac{\mathrm{b}^{-1}}{N}\nabla_z^{(t)} - \frac{1}{N}\nabla_n^{(t)} + 2\pi \mathrm{b}^{-1} w - 2\pi i m
\end{aligned} \tag{3.107}$$

Choose the following real polarization

$$\widetilde{\mathcal{P}} \equiv \mathrm{Span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial w} + \frac{\partial}{\partial n}, \frac{\partial}{\partial z} - \frac{\partial}{\partial m} \right\} \tag{3.108}$$

and notice that it is Lagrangian

$$\begin{aligned}
\omega_t \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial m}, \frac{\partial}{\partial w} + \frac{\partial}{\partial n} \right) &= \omega_t \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right) + \omega_t \left( -\frac{\partial}{\partial m}, \frac{\partial}{\partial n} \right) \\
&= 2\pi N \left( \mathrm{d}z \wedge \mathrm{d}w \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right) - \mathrm{d}n \wedge \mathrm{d}m \left( -\frac{\partial}{\partial m}, \frac{\partial}{\partial n} \right) \right) \\
&= 2\pi N (1-1) = 0
\end{aligned}$$

The leaves of this polarization are all contractible after the action of $\mathbb{Z}^2$ on $\mathbb{R}^2 \times \mathbb{R}^2$, so we do have polarized global sections. In particular the space $T \subset \mathbb{R}^2 \times \mathbb{R}^2$

$$T \equiv \{z = w = 0\} \tag{3.109}$$

is a transversal for the polarization. For any $\psi \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}$ polarized by $\widetilde{\mathcal{P}}$, the following two differential equations will determine $\psi \equiv \psi((z,n),(w,m))$ by its value in $(n,m)$

$$\nabla_w^{(t)}\psi = -\nabla_n^{(t)}\psi \qquad\qquad \nabla_z^{(t)}\psi = \nabla_m^{(t)}\psi. \tag{3.110}$$

The space $T/\mathbb{Z}^2$ is again $\mathbb{T} \times \mathbb{T}$, and the line bundle $\mathcal{L}^N$ will restrict to a non trivial line bundle over $\mathbb{T} \times \mathbb{T}$ that we shall call $\mathcal{L}^N$ again. The quantum space that we obtain is then

$$\hat{\mathcal{H}}^{(N)} \equiv \mathrm{C}^\infty \left( \mathbb{T} \times \mathbb{T}, \mathcal{L}^N \right) \tag{3.111}$$

The inner product structure $(\cdot,\, \cdot)$ restricts to

$$(\psi,\, \phi) = \int_0^1 \int_0^1 \psi \overline{\phi}\, \mathrm{d}n \mathrm{d}m \tag{3.112}$$

that is the standard inner product on the completion $\mathrm{L}^2 \left( \mathbb{T} \times \mathbb{T}, \mathcal{L}^N \right)$. Finally the quantum operators acts on polarized sections as

$$e^{\hat{u}} = \exp \left( i\frac{\mathrm{b}}{N}\nabla_n^{(t)} - \frac{1}{N}\nabla_m^{(t)} + 2\pi i n \right) \tag{3.113}$$

$$e^{\hat{v}} = \exp \left( i\frac{\mathrm{b}}{N}\nabla_m^{(t)} + \frac{1}{N}\nabla_n^{(t)} + 2\pi i m \right) \tag{3.114}$$

$$e^{\hat{\tilde{u}}} = \exp \left( i\frac{\mathrm{b}^{-1}}{N}\nabla_n^{(t)} + \frac{1}{N}\nabla_m^{(t)} - 2\pi i n \right) \tag{3.115}$$

$$e^{\hat{\tilde{v}}} = \exp \left( i\frac{\mathrm{b}^{-1}}{N}\nabla_z^{(t)} - \frac{1}{N}\nabla_n^{(t)} - 2\pi i m \right) \tag{3.116}$$

Now we are going to connect the two quantizations

**Theorem 42.** *Recall the line bundle $\mathcal{L}^N$. The following map $Z^{(N)} \colon \mathcal{S}(\mathbb{A}_N) \longrightarrow \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T}, \mathcal{L}^N)$ is a an isomorphism*

$$Z^{(N)}(\mathsf{f})(n,m) = \frac{1}{\sqrt{N}} e^{\pi i N mn} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l\right) e^{2\pi i m p} e^{2\pi i l p/N} \tag{3.117}$$

*which preserves the inner product $(\cdot,\cdot)$, i.e.*

$$\left(Z^{(N)}\mathsf{f},\, Z^{(N)}\mathsf{g}\right) = (\mathsf{f},\, \mathsf{g})$$

*and so extends to an isometry between $\mathrm{L}^2(\mathbb{A}_N)$ and $\mathrm{L}^2(\mathbb{T}\times\mathbb{T}, \mathcal{L}^N)$.*

*Proof.* We have proved an almost identical statement in Section 1.2. The difference is due to the line bundle $\mathcal{L}$ here being dual to the one used there. $\square$

**Proposition 43.** *We have*

$$\begin{aligned}
Z^{(N)} \circ X \circ (Z^{(N)})^{-1} &= \exp\left(-\frac{1}{N}\nabla^{(t)}_n - 2\pi i m\right)\\
Z^{(N)} \circ Y \circ (Z^{(N)})^{-1} &= \exp\left(-\frac{1}{N}\nabla^{(t)}_m + 2\pi i n\right)\\
Z^{(N)} \circ e^{2\pi \mathrm{b}^{\pm 1}\hat{z}} \circ (Z^{(N)})^{-1} &= \exp\left(-i\frac{\mathrm{b}^{\pm 1}}{N}\nabla^{(t)}_m\right)\\
Z^{(N)} \circ e^{2\pi \mathrm{b}^{\pm 1}\hat{w}} \circ (Z^{(N)})^{-1} &= \exp\left(i\frac{\mathrm{b}^{\pm 1}}{N}\nabla^{(t)}_n\right)
\end{aligned}$$

*In particular*

$$Z^{(N)} \circ \hat{x} \circ (Z^{(N)})^{-1} = e^{-\hat{v}} \qquad Z^{(N)} \circ \hat{y} \circ (Z^{(N)})^{-1} = e^{\hat{u}}$$
$$Z^{(N)} \circ \hat{\tilde{x}} \circ (Z^{(N)})^{-1} = e^{-\hat{\tilde{v}}} \qquad Z^{(N)} \circ \hat{\tilde{y}} \circ (Z^{(N)})^{-1} = e^{\hat{\tilde{u}}}$$

*Proof.* For any $\mathsf{f} \in \mathrm{L}^2(\mathbb{A}_N)$ we have

$$\begin{aligned}
Z^{(N)} \circ X(\mathsf{f})(n,m) &= \frac{1}{\sqrt{N}} e^{\pi i N mn} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l\right) e^{2\pi i m p} e^{2\pi i l(p+1)/N}\\
&= \frac{1}{\sqrt{N}} e^{\pi i N m(n-\frac{1}{N})} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n - \frac{1}{N} + \frac{p+1}{N}, l\right)\\
&\qquad \times e^{2\pi i m(p+1)} e^{2\pi i l(p+1)/N} e^{-\pi i m}\\
&= e^{-\pi i m} Z^{(N)}(\mathsf{f})(n - 1/N, m)\\
Z^{(N)} \circ Y(\mathsf{f})(n,m) &= \frac{1}{\sqrt{N}} e^{\pi i N mn} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l+1\right) e^{2\pi i m p} e^{2\pi i l p/N}\\
&= \frac{1}{\sqrt{N}} e^{\pi i N(m-1/N)n} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l+1\right) e^{2\pi i (m-1/N)p}\\
&\qquad e^{2\pi i (l+1)p/N} e^{\pi i n}\\
&= e^{\pi i n} Z^{(N)}(\mathsf{f})(n, m - 1/N)
\end{aligned}$$

$$
\begin{aligned}
Z^{(N)} \circ \left(e^{2\pi \mathrm{b} \hat{z}}\right)(\mathsf{f})(n,m) &= \frac{1}{\sqrt{N}} e^{\pi i N m n} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l\right) \\
&\qquad e^{2\pi \mathrm{b}(n+p/N)} e^{2\pi i m p} e^{2\pi i l p/N} \\
&= \frac{1}{\sqrt{N}} e^{\pi i N(m - i\mathrm{b}/N)n} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l\right) \\
&\qquad e^{2\pi i(m - i\mathrm{b}/N)p} e^{2\pi i l p/N} e^{\pi \mathrm{b} n} \\
&= e^{\pi \mathrm{b} n} Z^{(N)}(\mathsf{f})(n, m - i\mathrm{b}/N) \\
Z^{(N)} \circ \left(e^{2\pi \mathrm{b} \hat{w}}\right)(\mathsf{f})(n,m) &= \frac{1}{\sqrt{N}} e^{\pi i N m n} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + i\frac{\mathrm{b}}{N} + \frac{p}{N}, l\right) e^{2\pi i m p} e^{2\pi i l p/N} \\
&= \frac{1}{\sqrt{N}} e^{\pi i N m(n + i\mathrm{b}/N)} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + i\frac{\mathrm{b}}{N} + \frac{p}{N}, l\right) \\
&\qquad e^{2\pi i m p} e^{2\pi i l p/N} e^{\pi \mathrm{b} m} \\
&= e^{\pi \mathrm{b} m} Z^{(N)}(\mathsf{f})(n + i\mathrm{b}/N, m)
\end{aligned}
$$

$\square$

All together we have showed that two possible quantizations for the model space of complex Chern-Simons theory are the same, both as Hilbert spaces and as quantum algebras. First this makes up for the heuristic argument regarding the Bohr–Sommerfeld quantization in Section 3.2.2. Second, since the two quantizations are equivalent to the $\mathrm{L}^2(\mathbb{A}_N)$ representations of the quantum algebra defined from Quantum Teichmüller Theory, we have a strict connection between them and Complex Quantum Chern–Simons Theory on a 4-punctured sphere. In the following Chapter 4 we will extend the $\mathrm{L}^2(\mathbb{A}_N)$ representations to knots invariants following the recipe given by Andersen and Kashaev in [AK14b]. The previous discussion on the different quantizations serves to link such invariants to Complex Quantum Chern–Simons Theory.

# Chapter 4

# Andersen–Kashaev's Teichmüller TQFT at Level $N$

In this Chapter we construct the Teichmüller TQFT at level $N$ for every $N \geq 1$ odd. This construction was presented in [AK14b] for level 1 and it emerges as an extension of $\mathrm{L}^2(\mathbb{R})$ representations on Quantum Teichmüller Theory to 3–dimensional ideal triangulations. In a similar way the older work [Kas94] could be seen as an extension of $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z})$ representations (see [Kas98]) to knot invariants. Here, following strictly [AK14b], and using theory mostly developed in [AK14a], we extend $\mathrm{L}^2(\mathbb{A}_N)$ representations 3.1.2 to three dimensional ideal triangulations, getting in particular knot invariants. Many of the things we will say are implicit in the work [AK14a], however the behavior of the theory when $\mathrm{b} \in \mathbb{R}$ and $N > 1$, was not considered before. This setting is important for two reasons: first, asymptotic expressions for $\mathrm{b} \to 0$ are not possible when $|\mathrm{b}| = 1$. We study them in the examples in section 4.4.4, where we update a conjecture presented in [AK14b] for $N = 1$. Secondly, $\mathrm{b} \in \mathbb{R}$ should correspond to the exotic unitary setting that Witten conjecture exists in [Wit91] when the quantum parameter $s$ is purely imaginary. For the rest of the chapter, $\mathrm{b} \in \mathbb{C}$ is a fixed parameter such that $\operatorname{Re}\mathrm{b} > 0$ and $\operatorname{Im}\mathrm{b}(|\mathrm{b}| - 1) = 0$. $N$ is an odd positive integer. In the first two Sections we will describe the source and the target categoroid for the Teichmüller Functor $F_{\mathrm{b}}^{(N)}$. The existence and properties of such functor are stated at Theorem 52 at page 72. In the last Section 4.4 we compute two examples of knot invariant, together with the asymptotic analysis of them and we state a conjecture for general hyperbolic knot. In particular equation (4.58) shows the appearance of an instance of the Baseilhac–Benedetti invariant [BB07] in the asymptotic expession for the hyperbolic knot $4_1$.

## 4.1 Angle Structures on 3-Manifolds

In this section we are going to describe shaped triangulated pseudo 3-manifolds, which are the combinatorial data underlying the Andersen-Kashaev construction of their invariant. Following strictly [AK14b] we will describe the *categoroid* of *admissible* oriented triangulated pseudo 3-manifolds, where the words admissible and categoroid go together because admissibility is what will obstruct us to have a full category. See Appendix C.2 for a definition of categoroid.

**Definition 12** (Oriented Triangulated Pseudo 3-manifold)**.** An *Oriented Triangu-*

*lated Pseudo* 3*-manifold* $X$ is a finite collection of 3-simpleces (tetrahedra) with totally ordered vertices together with a collections of *gluing homeomorphisms* between some pairs of codimension 1 faces, so that every face is in, at most, one of such pairs. By gluing homeomorphism we mean a vertex order preserving, orientation reversing, affine homeomorphism between the two faces.
The quotient space by the glueing homeomorphism has the structure of CW-complex with oriented edges.

For $i \in \{0,1,2,3\}$ we denote by $\Delta_i(X)$ the collection of $i$-dimensional simpleces in $X$ and, for $i > j$, we denote

$$\Delta_i^j(X) = \{(a,b)|a \in \Delta_i(X), b \in \Delta_j(a)\}.$$

We have projection maps

$$\phi_{i,j} : \Delta_i^j(X) \longrightarrow \Delta_i(X), \quad \phi^{i,j} : \Delta_i^j(X) \longrightarrow \Delta_j(X),$$

and boundary maps

$$\partial_i : \Delta_j(X) \longrightarrow \Delta_{j-1}(X), \quad \partial_i[v_0,\dots,v_j] \mapsto [v_0,\dots,v_{i-1},v_{i+1},\dots,v_j]$$

where $[v_0,\dots,v_j]$ is the $j$-simplex with vertices $v_0,\dots,v_j$ and $i \leq j$.

**Definition 13** (Shape Structure)**.** Let $X$ be an oriented triangulated pseudo 3-manifold. A *Shape Structure* is a map

$$\alpha_X : \Delta_3^1(X) \longrightarrow \mathbb{R}_{>0},$$

so that, in every tetrahedron, the sum of the values of $\alpha_X$ along three incident edges is $\pi$.
The value of the map $\alpha_X$ in an edge $e$ inside a tetrahedron $T$ is called *dihedral angle* of $T$ at $e$. If we allow $\alpha_X$ to take values in $\mathbb{R}$ we will have a *Generalized Shape Structure.*
The set of shape structures supported by $X$ is denoted $\mathrm{S}(X)$. The space of generalized shape structures is denoted by $\widetilde{\mathrm{S}}(X)$. $X$ together with $\alpha_X$ is called *Shaped Pseudo* 3*-manifold.*

*Remark* 4.1.1 (Ideal Tetrahedron). A shape structure on a simplicial tetrahedron $T$ as above define an embedding of $T \setminus \Delta_0(T)$ in the hyperbolic 3–space $\mathbb{H}^3$ that extends to a map of $T$ to $\overline{\mathbb{H}^3}$. Namely we send the four vertices $(v_0,v_1,v_2,v_3)$ to the four points $(\infty,0,1,z) \in \mathbb{CP}^1 \simeq \partial\mathbb{H}^3$, where

$$z = \frac{\sin\alpha_T([v_0,v_2])}{\sin\alpha_T([v_0,v_3])}\exp\left(i\alpha_T([v_0,v_1])\right).$$

This four points in $\partial\mathbb{H}^3$ extend to a unique ideal tetrahedron in $\mathbb{H}^3$, by taking the geodesic convex hull, that has dihedral angles defined by $\alpha_T$.

*Remark* 4.1.2. In every tetrahedron, its orientation induces a cyclic ordering of any three edges meeting in a vertex. Such cyclic ordering descends to a cyclic ordering of the pairs of opposite edges of the whole tetrahedron. Moreover, it follows from the definition that opposite edges share the same dihedral angle. So it is well defined the cyclic order preserving projection $p : \Delta_3^1(X) \longrightarrow \Delta_3^{1/p}(X)$ which identifies opposite edges. $\alpha_X$ descends to a map from $\Delta_3^{1/p}(X)$ and we can consider the following skew-symmetric functions

$$\varepsilon_{a,b} \in \{0,1\}, \quad \varepsilon_{a,b} = -\varepsilon_{b,a}, \quad a,b \in \Delta_3^{1/p}(X),$$

defined to be $\varepsilon_{a,b} = 0$ if the underlying tetrahedra are distinct, and $\varepsilon_{a,b} = +1$ if the underlying tetrahedra coincides and $b$ cyclically follows $a$ in the order induced on $\Delta_3^{1/p}(X)$.

**Definition 14.** To any shaped pseudo 3-manifold $X$, we associate a *Weight* function

$$\omega_X : \Delta_1(X) \longrightarrow \mathbb{R}_{>0}, \quad \omega_X(e) = \sum_{a \in (\phi^{3,1})^{-1}(e)} \alpha_X(a).$$

An edge $e$ in $X$ is called *balanced* if $e$ is internal and $\omega_X(e) = 2\pi$. A shape structure is fully balanced if all its edges are balanced.

The shape structures of closed fully balanced 3-manifolds are called *Angle Structures* in the literature. For more details on them and their geometric admissibility see [Lac00] and [LT08].

**Definition 15.** A *leveled* (generalised) shaped pseudo 3-manifold is a pair $(X, l_X)$ consisting of a (generalized) shaped pseudo 3-manifold $X$ and a real number $l_X \in \mathbb{R}$, called the *level*. The set of all leveled (generalised) shaped pseudo 3-manifolds is denoted as $\mathrm{LS}(X)$ (resp. $\widetilde{\mathrm{LS}}(X)$).

There is a gauge action of $\mathbb{R}^{\Delta_1(X)}$ on $\widetilde{\mathrm{LS}}(X)$.

**Definition 16.** Let $(X, l_X)$ and $(Y, l_Y)$ be two (generalized) leveled shaped pseudo 3-manifolds. They are said to be *gauge equivalent* if there exists a an isomorphism $h : X \longrightarrow Y$ of the underlying cellular structures, and function $g : \Delta_1(X) \longrightarrow \mathbb{R}$ such that

$$\begin{aligned}
&\Delta_1(\partial X) \subset g^{-1}\{0\}, \\
&\alpha_Y(h(a)) = \alpha_X(a) + \pi \sum_{b \in \Delta_3^1(X)} \varepsilon_{p(a),p(b)} g(\phi^{3,1}(b)), \qquad \forall a \in \Delta_3^1(X), \text{ and} \\
&l_Y = l_X + \sum_{e \in \Delta_1(X)} g(e) \sum_{a \in (\phi^{3,1})^{-1}(e)} \left(\frac{1}{3} - \frac{\alpha_X(a)}{\pi}\right).
\end{aligned}$$

*Remark* 4.1.3. If we consider $g(a) = \delta_e(a)$ the Kronecker's delta for an internal edge $e$, we compute the gauge action as follows. Let $h : X \longrightarrow X$ be the identity. There is a finite number $k$ of tetrahedra $T_i$ sharing the edge $e$, together with other two pairs of opposite edges $\{a_i, \tilde{a}_i\}$ and $\{c_i, \tilde{c}_i\}$ so that $(p(a_i), p(e), p(c_i))$ is cyclically ordered for all $i = 1, \ldots, k$. Then

$$\varepsilon_{p(a_i),p(e)} = 1 = -\varepsilon_{p(c_i),p(e)}$$

and

$$\begin{aligned}
\alpha_Y(a_i) &= \alpha_X(a_i) + 2\pi, & i &= 1, \ldots k \\
\alpha_Y(c_i) &= \alpha_X(c_i) - 2\pi, & i &= 1, \ldots k \\
\alpha_Y(f) &= \alpha_X(f), & a_i &\neq f \neq c_i.
\end{aligned}$$

We remark that $\omega_X = \omega_Y \circ h$.

**Definition 17.** Let $(\alpha_X, l_X)$ and $(\alpha_{X'}, l_{X'})$ be two (generalized) leveled shape structures of the oriented pseudo 3-manifold $X$. They are said *based gauge equivalent* if they are gauge equivalent as in Definition 16 and the isomorphism $h : X \longrightarrow X$ is the identity.

Based gauge equivalence is an equivalence relation in the sets $\mathrm{S}(X)$, $\mathrm{LS}(X)$, $\widetilde{\mathrm{S}}(X)$, $\widetilde{\mathrm{LS}}(X)$ and the quotient sets are denoted (resp.) $\mathrm{S}_r(X)$, $\mathrm{LS}_r(X)$, $\widetilde{\mathrm{S}}_r(X)$, $\widetilde{\mathrm{LS}}_r(X)$. We remark that $\mathrm{S}_r(X)$ is an open convex (possibly empty) subset of the space $\widetilde{\mathrm{S}}_r(X)$. We will return to existence of shape structures later. Let us concentrate on $\widetilde{\mathrm{S}}(X)$ for now. Let

$$\tilde{\Omega}_X : \widetilde{\mathrm{S}}(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

be the map which sends the shape structure $\alpha_X$ to the corresponding weight function $\omega_X$. This map is gauge invariant, so it descends to a map

$$\tilde{\Omega}_{X,r} : \widetilde{\mathrm{S}}_r(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

For fixed $a \in \Delta_3^{1/p}(X)$ we can think of $\alpha_a := \alpha_X(a)$ as an element of $C^\infty\left(\widetilde{\mathrm{S}}(X)\right)$.

**Definition 18** ([NZ85])**.** The Neumann-Zagier symplectic structure on $\widetilde{\mathrm{S}}(X)$ is the unique symplectic structure which induces the Poisson bracket $\{\cdot,\cdot\}$ satisfying

$$\{\alpha_a, \alpha_b\} = \varepsilon_{a,b}$$

for all $a$, $b \in \Delta_3^{1/p}(X)$.

*Remark* 4.1.4. If we take $X = T$ a tetrahedron, then $\widetilde{\mathrm{S}}(T) \cong \mathbb{R}^2$ as affine symplectic spaces. Indeed on $\widetilde{\mathrm{S}}(T) = \{(a,b,c) \in \mathbb{R}^3 | a+b+c = \pi\}$ the symplectic structure $\omega = \mathrm{d}\alpha_a \wedge \mathrm{d}\alpha_b$ induces the desired Poisson bracket. It is clear that this make $\widetilde{\mathrm{S}}(T)$ into an affine copy of $\mathbb{R}^2$ with the standard symplectic structure.
For a general triangulated pseudo 3-manifold we have a symplectic decomposition

$$\widetilde{\mathrm{S}}(X) = \prod_{T \in \Delta_3(X)} \widetilde{\mathrm{S}}(T).$$

**Theorem 44** ([AK14b])**.** *The gauge action of $\mathbb{R}^{\Delta_1(X)}$ on $\widetilde{\mathrm{S}}(X)$ is symplectic and $\tilde{\Omega}_X$ is a moment map for this action. It follows that $\widetilde{\mathrm{S}}_r(X) = \widetilde{\mathrm{S}}(X)/\mathbb{R}^{\Delta_1(X)}$ is a Poisson manifold with symplectic leaves corresponding to the fibers of $\tilde{\Omega}_{X,r}$.*

Let $N_0(X)$ be a sufficiently small neighbourhood of $\Delta_0(X)$, then $\partial N_0(X)$ is a surface which inherits a triangulation from $X$, with a shape structure, if $X$ has a shape structure. Notice that this surface can have boundary if $\partial X \neq \emptyset$.

**Theorem 45** ([AK14b])**.** *The map*

$$\tilde{\Omega}_{X,r} : \widetilde{\mathrm{S}}_r(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

*is an affine $\mathrm{H}^1(\partial N_0(X), \mathbb{R})$-bundle. The Poisson structure of $\widetilde{\mathrm{S}}_r(X)$ coincide with the one induced by the $\mathrm{H}^1(\partial N_0(X), \mathbb{R})$-bundle structure.*
*If $h : X \longrightarrow Y$ is an isomorphisms of cellular structure, the induced morphism $h^* : \widetilde{\mathrm{S}}_r(Y) \longrightarrow \widetilde{\mathrm{S}}_r(X)$ is compatible with all this structures, i.e. it is a Poisson affine bundle morphism which fiberwise coincide with the naturally induced group morphism $h^* : \mathrm{H}^1(\partial N_0(Y), \mathbb{R}) \longrightarrow \mathrm{H}^1(\partial N_0(X), \mathbb{R})$. Moreover $h^*$ maps $\mathrm{S}_r(Y)$ to $\mathrm{S}_r(X)$.*

**Definition 19** (Shaped $3-2$ Pachner moves)**.** Let $X$ be a shaped pseudo 3 manifold and let $e$ be a balanced internal edge in it, shared exactly by three distinct tetrahedra $t_1$, $t_2$ and $t_3$ with dihedral angles in $e$ exactly $\alpha_1$, $\alpha_2$ and $\alpha_3$. Then the triangulated pseudo 3-manifold $X_e$ obtained by removing the edge $e$, and substituting the three tetrahedra $t_1$, $t_2$ and $t_3$ with other two new tetrahedra $t_4$ and $t_5$ glued along one face, is topologically the same space as $X$. In order to have the same weights of $X$

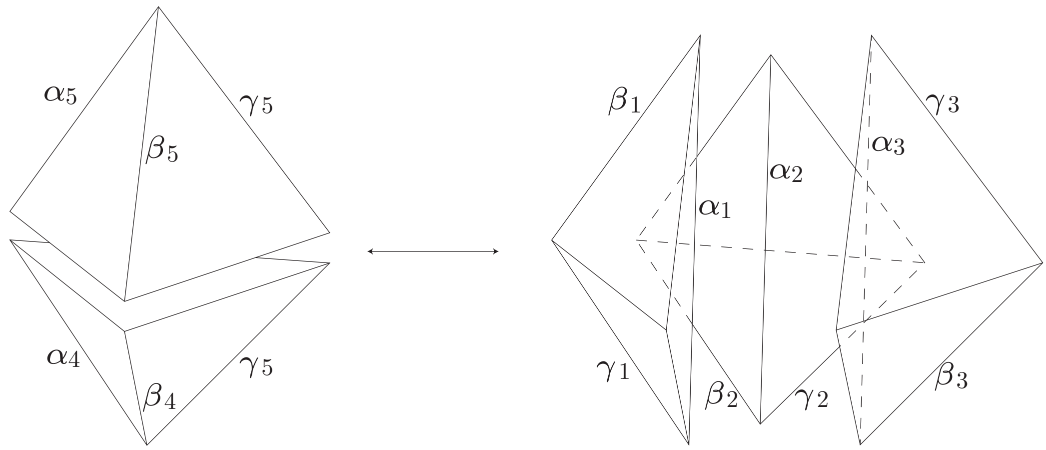



Figure 4.1: A $3--2$ Pachner move.

on $X_e$, the dihedral angles of $t_4$ and $t_5$ are uniquely determined by the ones of $t_1$, $t_2$ and $t_3$ as follows

$$\begin{array}{ll} \alpha_4 = \beta_2 + \gamma_1 & \alpha_5 = \beta_1 + \gamma_2 \\ \beta_4 = \beta_1 + \gamma_3 & \beta_5 = \beta_3 + \gamma_1 \\ \gamma_4 = \beta_3 + \gamma_2 & \gamma_5 = \beta_2 + \gamma_3. \end{array} \tag{4.1}$$

where $(\alpha_i, \beta_i, \gamma_i)$ are the dihedral angles of $t_i$. In this situation we say that $X_e$ is obtained from $X$ by a *shaped* $3-2$ *Pachner move.*

We remark that the linear system, together with $e$ being balanced, guaranties the positivity of the dihedral angles of $t_4$ and $t_5$ provided the positivity for $t_1$, $t_2$ and $t_3$ but it does not provide any guarantees on the converse, i.e. the positivity of a shaped $2-3$ Pachner moves. However, two different solutions for the angles for $t_1$, $t_2$ and $t_3$ from the same starting angles for $t_4$ and $t_5$ are always gauge equivalent. The system (4.1) define a map $P^e : \mathrm{S}(X) \longrightarrow \mathrm{S}(X_e)$, that extends to a map

$$\tilde{P}^e : \widetilde{\mathrm{S}}(X) \longrightarrow \widetilde{\mathrm{S}}(X_e).$$

For a balanced edge $e$, the latter restricts to the map

$$\tilde{P}_r : \tilde{\Omega}_{X,r}(e)^{-1}(2\pi) \longrightarrow \widetilde{\mathrm{S}}_r(X_e),$$

and it can be noticed that $\tilde{P}_r(\tilde{\Omega}_{X,r}(e)^{-1}(2\pi) \cap \mathrm{S}_r(x)) \subset \mathrm{S}_r(Y)$.

We also say that a leveled shaped pseudo 3-manifold $(X, l_X)$ is obtained from $(Y, l_Y)$ by a *leveled shaped* 3-2 *Pachner move* if, for some balanced $e \in \Delta_1(X)$, $Y = X_e$ as above and

$$l_Y = l_X + \frac{1}{12\pi} \sum_{a \in (\phi^{3,1})^{-1}(e)} \sum_{b \in \Delta_3^1(X)} \varepsilon_{p(a),p(b)} \alpha_X(b).$$

**Definition 20.** A (leveled) shaped pseudo 3-manifold $X$ is called a *Pachner refinement* of a (leveled) shaped pseudo 3-manifold $Y$ if there exists a finite sequence of (leveled) shaped pseudo 3-manifolds

$$X = X_1, X_2, \ldots, X_n = Y$$

such that for any $i \in \{1, \ldots, n-1\}$, $X_{i+1}$ is obtained from $X_i$ by a (leveled) shaped $3-2$ Pachner move. Two (leveled) shaped pseudo 3-manifolds $X$ and $Y$ are called *equivalent* if there exist gauge equivalent (leveled) shaped pseudo 3-manifolds $X'$ and $Y'$ which are respective Pachner refinements of $X$ and $Y$.

**Theorem 46** ([AK14b]). *Suppose that a (leveled) shaped pseudo 3-manifold $Y$ is obtained from a (leveled) shaped pseudo 3-manifold $X$ by a (leveled) shaped $3-2$ Pachner move. Then the map $\tilde{P}_r$ is a Poisson isomorphism. ($\tilde{P}_r$ is covered by an affine $\mathbb{R}$-bundle isomorphism from $\widetilde{\mathrm{LS}}_r(X)|_{\tilde{\Omega}_{X,r}(e)^{-1}(2\pi)}$ to $\widetilde{\mathrm{LS}}_r(Y)$.)*

□

As we said, technical reasons that we will discuss later impose the Category of triangulated $2+1$ cobordisms discussed so far to be restricted to a sub-categoroid. This means that we will remove some morphisms as the following definition imposes.

**Definition 21** (Admissibility). An oriented triangulated pseudo 3-manifold is called *admissible* if

$$\mathrm{S}_r(X) \neq \emptyset,$$

and

$$\mathrm{H}_2(X - \Delta_0(X), \mathbb{Z}) = 0.$$

**Definition 22.** Two (leveled) admissible shaped pseudo 3-manifolds $X$ and $Y$ are said *admissibly equivalent* if there exists a gauge equivalence $h' : X' \longrightarrow Y'$ of (leveled) shaped 3-manifolds $X'$ and $Y'$ which are respective Pachner refinements of $X$ and $Y$ such that $\Delta_1(X') = \Delta_1(X) \cup D_X$ and $\Delta_1(Y') = \Delta_1(Y) \cup D_Y$ and the following holds

$$\left[h(\mathrm{S}_r(X) \cap \tilde{\Omega}_{X',r}(D_X)^{-1}(2\pi))\right] \cap \left[\tilde{\Omega}_{Y',r}(D_Y)^{-1}(2\pi)\right] \neq \emptyset.$$

The following is a consequence of Theorems 45 and 46.

**Theorem 47** ([AK14b]). *Suppose two (leveled) shaped pseudo 3-manifolds $X$ and $Y$ are equivalent. Then there exist $D \subset \Delta_1(X)$ and $D' \subset \Delta_1(Y)$ and a bijection*

$$i : \Delta_1(X) - D \to \Delta_1(Y) - D'$$

*and a Poisson isomorphism*

$$R : \tilde{\Omega}_{X,r}(D)^{-1}(2\pi) \to \tilde{\Omega}_{Y,r}(D')^{-1}(2\pi),$$

*which is covered by an affine $\mathbb{R}$-bundle isomorphism from $\widetilde{\mathrm{LS}}_r(X)|_{\tilde{\Omega}_{X,r}(D)^{-1}(2\pi)}$ to $\widetilde{\mathrm{LS}}_r(Y)_{\tilde{\Omega}_{X,r}(D')^{-1}(2\pi)}$ and such that we get the following commutative diagram*

$$\begin{array}{ccc} \tilde{\Omega}_{X,r}(D)^{-1}(2\pi) & \xrightarrow{R} & \tilde{\Omega}_{Y,r}(D')^{-1}(2\pi) \\ \Big\downarrow \mathrm{proj} \circ \tilde{\Omega}_{X,r} & & \Big\downarrow \mathrm{proj} \circ \tilde{\Omega}_{Y,r} \\ \mathbb{R}^{\Delta_1(X)-D} & \xrightarrow{i^*} & \mathbb{R}^{\Delta_1(Y)-D'}. \end{array}$$

*Moreover, if $X$ and $Y$ are admissible and admissibly equivalent, the isomorphism $R$ takes an open convex subset $U$ of $S_r(X) \cap \tilde{\Omega}_{X,r}(D)^{-1}(2\pi)$ onto an open convex subset $U'$ of $S_r(Y) \cap \tilde{\Omega}_{Y,r}(D)^{-1}(2\pi)$.*

We remark that in the previous notation $D = \Delta_1(X) \cap h^{-1}(D_Y)$ and $D' = \Delta_1(Y) \cap h(D_X)$.

For a tetrahedron $T = [v_0, v_1, v_2, v_3]$ in $\mathbb{R}^3$ with ordered vertices $v_0, v_1, v_2, v_3$, we define its sign

$$\mathrm{sign}(T) = \mathrm{sign}(\det(v_1 - v_0, v_2 - v_0, v_3 - v_0)),$$

as well as the signs of its faces

$$\operatorname{sign}(\partial_i T) = (-1)^i \operatorname{sign}(T), \text{ for } i \in \{0, \dots, 3\}.$$

For a pseudo 3-manifold $X$, the signs of faces of the tetrahedra of $X$ induce a sign function on the faces of the boundary of $X$, $\operatorname{sign}_X : \Delta_2(\partial X) \to \{\pm 1\}$, which permits to split the boundary of $X$ into two components, $\partial X = \partial_+ X \cup \partial_- X$, where $\Delta_2(\partial_\pm X) = \operatorname{sign}_X^{-1}(\pm 1)$. Notice that $|\Delta_2(\partial_+ X)| = |\Delta_2(\partial_- X)|$.

**Definition 23** (Cobordism Categoroid)**.** The category $\mathcal{B}$ is the category that has triangulated surfaces as objects,equivalence classes of (leveled) shaped pseudo 3-manifolds $X$ as morphisms such that $X \in \operatorname{Hom}_{\mathcal{B}}(\partial_- X, \partial_+ X)$ and composition given by glueing along boundary components, through edge orientation preserving and face orientation reversing CW-homeomorphisms.
The Categoroid $\mathcal{B}_a$ is the subcategoroid of $\mathcal{B}$ whose morphisms are restricted to be admissible equivalence classes of admissible (leveled) shaped pseudo 3-manifolds. In particular composition is possible only if the gluing gives an (leveled) admissible pseudo 3-manifold.

*Remark* 4.1.5. *Admissible Shaped Pseudo* 3*-Manifolds in the real world.*
Even though we will discuss the whole Andersen Kashaev construction of the Teichmüller TQFT Functor in the general setting of cobordism categoroid, the two main products we want to put our hands on are mapping class group representations and invariants of links and 3-manifolds. For the former we already have a description in term of Quantum Teichmüller Theory and Ptolemy Groupoid. The latter is what we want to construct here. We interpret Triangulated Pseudo 3-manifolds $X$ as ideal triangulations of the (non closed) manifold $X \setminus \Delta_0(X)$. This interpretation is enlighten in Remark 4.1.1. We should ask ourself when a cusped 3-manifold (cusped means non compact with finite volume here) admits a positive fully balanced shape structure. This requirement is weaker the asking for a full geometric structure on the manifold, and in our language this can be expressed by the fact that we did not required a precise gauge to be fixed. The problem of finding positive or generalized angle structures has been studied in [LT08], where necessary and sufficient conditions for their existence are given. In the work [HRS12] it is proved, among other things, that a particular class of manifolds $M$ supporting positive shape structures are complements in $S^3$ of hyperbolic links. However the admissibility conditions kicks in here and further restrict us to just complements of hyperbolic knots. So, at the least, we know that the Andersen Kashaev construction will work on complement of hyperbolic knots, and that are the examples we will look a bit closer in this thesis. Now we should clarify the equivalence relation in $\mathcal{B}_a$, in the context of knot complements. Combinatorially speaking, any two ideal triangulations of a knot complement are related by finite sequences of 3–2 or 2–3 Pachner moves. This is true because on ideal triangulations, creation and destruction of vertices is forbidden. On the other hand it is not known (at least to the author) that any such sequence of moves can be realised as a sequence of shaped Pachner moves. For sure we know that 3–2 shaped Pachner moves are well defined in the category $\mathcal{B}_a$ as we remarked when we defined them, and if a shaped Pachner 2–3 move is possible in some particular case, than it is an equivalence in the category $\mathcal{B}_a$. So the knot invariants that we will define starting from $\mathcal{B}_a$ are not guaranteed to be topological invariants. There is however another construction of the Andersen–Kashaev invariant [AK13], that avoid

this problem with analytic continuation properties of the partition function. The equivalence of the two constructions is still conjectural though.
There are other things that we can try in $\mathcal{B}_a$ other then complements of hyperbolic knots. For example the authors in [AK14b] managed to define an invariant for the (non hyperbolic) trefoil knot. But these are usually singular and we have nothing of interesting to add about them. Again in [AK14b] is suggested another way to define knot invariants, by taking one vertex Hamiltonian triangulations of knots, that is, one vertex triangulations of $S^3$ (or a general manifold $M$) where the knot is represented by a unique edge with a degenerating shape structure, meaning that we take a limit on the shapes, sending all the weights to be balanced except the weight of the knot that is sent to 0. The partition function is actually divergent but a residue can be computed as an invariant. We will show this in a couple of examples in Section 4.4.

## 4.2 The target Categoroid $\mathcal{D}_N$

Recall al the relevant things regarding tempered distributions and the space $\mathcal{S}(\mathbb{A}_N)$ from Appendix A. In this Chapter, as in most of this thesis, $N$ is an odd positive integer and $\mathrm{b} \in \mathbb{C}$ is fixed to satisfy $\mathrm{Re}(\mathrm{b}) > 0$ and $\mathrm{Im}\,\mathrm{b}(1-|\mathrm{b}|) = 0$.

**Definition 24.** The categoroid $\mathcal{D}_N$ has as objects finite sets and for two finite sets $n, m$ the set of morphisms from $n$ to $m$ is

$$\mathrm{Hom}_{\mathcal{D}_N}(n,m) = \mathcal{S}'(\mathbb{A}_N^{n\sqcup m}) \simeq \mathcal{S}'(\mathbb{R}^{n\sqcup m}) \otimes \mathcal{S}((\mathbb{Z}/N\mathbb{Z})^{n\sqcup m}).$$

**Definition 25.** For $\mathcal{A} \otimes A_N \in \mathrm{Hom}_{\mathcal{D}_N}(n,m)$ and $\mathcal{B} \otimes B_N \in \mathrm{Hom}_{\mathcal{D}_N}(m,l)$, such that $\mathcal{A}$ and $\mathcal{B}$ satisfy condition (A.3) and $\pi^*_{n,m}(\mathcal{A})\pi^*_{m,l}(\mathcal{B})$ continuously extends to $S(\mathbb{R}^{n\sqcup m\sqcup l})_m$, we define

$$(\mathcal{A} \otimes A_N) \circ (\mathcal{B} \otimes B_N) = (\pi_{n,l})_*(\pi^*_{n,m}(\mathcal{A})\pi^*_{m,l}(\mathcal{B})) \otimes A_N B_N \in \mathrm{Hom}_{\mathcal{D}_N}(n,l).$$

Where the product $A_N B_N$ is just a matrix product.
We will frequently use the following notation in what follows: for any $a \in \mathbb{A}_N$ , $a = (x,n) \in \mathbb{R} \times \mathbb{Z}/N\mathbb{Z}$ we will consider the b–dependent operator $\varepsilon \equiv \varepsilon(\mathrm{b}) \colon \mathbb{A}_N \longrightarrow \mathbb{A}_N$ as

$$\varepsilon(x,n) \equiv \begin{cases} (x,n) & \text{if } |\mathrm{b}| = 1, \\ (x,-n) & \text{if } \mathrm{b} \in \mathbb{R} \end{cases} \tag{4.2}$$

For any $\mathcal{A} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, we have unique adjoint $\mathcal{A}^* \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^n))$ defined by the formula

$$\mathcal{A}^*(f)(g) = \overline{\bar{f}(\mathcal{A}(\bar{g}))}$$

for all $f \in \mathcal{S}(\mathbb{R}^m)$ and all $g \in \mathcal{S}(\mathbb{R}^n)$.

**Definition 26** ($\star_{\mathrm{b}}$ structure)**.** Consider $\mathrm{b} \in \mathbb{C}$ fixed as above and $N \in \mathbb{Z}_{>0}$ odd. Let $A_N \in \mathrm{Hom}(\mathcal{S}((\mathbb{Z}/N\mathbb{Z})^m), \mathcal{S}((\mathbb{Z}/N\mathbb{Z})^n)$. Recall the involution $\varepsilon$ on $\mathbb{Z}/N\mathbb{Z}$ form equation (4.2). Define $A_N^{\star_{\mathrm{b}}}$ as

$$\langle j_1, \dots j_m | A_N^{\star_{\mathrm{b}}} | p_1, \dots, p_n \rangle = \overline{\langle \varepsilon p_1, \dots, \varepsilon p_n | A_N | \varepsilon j_1, \dots \varepsilon j_m \rangle} \tag{4.3}$$

We can finally define the $*_{\mathrm{b}}$ operator as

$$(\mathcal{A} \otimes A_N)^{*_{\mathrm{b}}} = \mathcal{A}^* \otimes A_N^{\star_{\mathrm{b}}} \tag{4.4}$$

## 4.3 Tetrahedral Partition Function

Recall the operators from Section 3.1.2, $X_j$, $Y_j$, $\mathsf{p}_j$, $\mathsf{q}_j$ , $j = 1,2$ acting on $\mathcal{H} := \mathcal{S}(\mathbb{A}_N^2)$ as follow

$$\begin{aligned} \mathsf{p}_j f(\mathsf{x},\mathsf{m}) &= \frac{1}{2\pi i}\frac{\partial}{\partial x_j} f(\mathsf{x},\mathsf{m}) & \mathsf{q}_j f(\mathsf{x},\mathsf{m}) &= x_j f(\mathsf{x},\mathsf{m}) \\ X_1 f(\mathsf{x},\mathsf{m}) &= f(\mathsf{x},(m_1+1,m_2)) & X_2 f(\mathsf{x},\mathsf{m}) &= f(\mathsf{x},(m_1,m_2+1)) \\ Y_j f(\mathsf{x},\mathsf{m}) &= \overline{\omega}^{m_j}, \end{aligned}$$

where $\mathsf{m} = (m_1,m_2) \in \mathbb{Z}_N^2$, $\mathsf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $\omega = e^{2\pi i/N}$.
These operators satisfy conditions (3.25), i.e. for $\omega = e^{\frac{2\pi i}{N}}$ we have

$$[p_i, q_j] = \frac{\delta_{ij}}{2\pi i}, \qquad Y_i X_j = \omega^{\delta_{ij}} X_j Y_i, \qquad X_i^N = Y_i^N = 1. \tag{4.5}$$

we can further define the operators $\mathsf{u}_i = e^{2\pi \frac{\mathrm{b}}{\sqrt{N}}\mathsf{q}_i} Y_i$ and $\mathsf{v}_i = e^{2\pi \frac{\mathrm{b}}{\sqrt{N}}\mathsf{p}_i} X_i$ satisfying

$$\mathsf{u}_i \mathsf{v}_j = e^{2\pi i \frac{\mathrm{b}^2}{N}} \omega \mathsf{v}_j \mathsf{u}_i \tag{4.6}$$

Define the *Charged Tetrahedral Operator* as follows

**Definition 27.** Let $a$, $b$, $c > 0$ such that $a + b + c = \frac{1}{\sqrt{N}}$. Recall the Tetrahedral operator $\mathsf{T}$ defined in (3.37). Define the charged tetrahedral operator $\mathsf{T}(a,c)$ as follows

$$\mathsf{T}(a,c) \equiv e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}}\left(2(a-c)+\frac{1}{\sqrt{N}}\right)/6} e^{2\pi i c_{\mathrm{b}}(c\mathsf{q}_2 - a\mathsf{q}_1)} \mathsf{T}_{12} e^{-2\pi i c_{\mathrm{b}}(a\mathsf{p}_2 + c\mathsf{q}_2)} \tag{4.7}$$

**Lemma 48.** *We have*

$$\mathsf{T}(a,c) = e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}}\left(2(a-c)+\frac{1}{\sqrt{N}}\right)/6} e^{\pi i c_{\mathrm{b}}^2 a(a+c)} \mathsf{D}_{12} \psi_{a,c}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) \tag{4.8}$$

*were $\psi_{a,c}(x,n)$ is the charged quantum dilogarithm from* (2.53)

*Proof.*

$$\begin{aligned} \mathsf{T}(a,c) &= e^{2\pi i c_{\mathrm{b}}(c\mathsf{q}_2 - a\mathsf{q}_1)} e^{2\pi i \mathsf{q}_2 \mathsf{p}_1} \mathsf{S}_{12} \Psi_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) e^{-2\pi i c_{\mathrm{b}}(a\mathsf{p}_2 + c\mathsf{q}_2)} \\ &= e^{2\pi i c_{\mathrm{b}}(c\mathsf{q}_2 - a\mathsf{q}_1)} e^{2\pi i \mathsf{q}_2 \mathsf{p}_1} e^{-2\pi i c_{\mathrm{b}} a \mathsf{p}_2} e^{-2\pi i c_{\mathrm{b}} c \mathsf{q}_2} \mathsf{S}_{12} \times \\ &\qquad \times \Psi_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2 - c_{\mathrm{b}}(a+c), -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) e^{-\pi i c_{\mathrm{b}}^2 ac} \\ &= e^{2\pi i \mathsf{q}_2 \mathsf{p}_1} \mathsf{S}_{12} e^{2\pi i c_{\mathrm{b}} c \mathsf{q}_2} e^{-2\pi i c_{\mathrm{b}} a \mathsf{q}_1} e^{2\pi i c_{\mathrm{b}} a \mathsf{q}_2} e^{-2\pi i c_{\mathrm{b}} a \mathsf{p}_2} e^{-2\pi i c_{\mathrm{b}} c \mathsf{q}_2} \times \\ &\qquad \times \Psi_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2 - c_{\mathrm{b}}(a+c), -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) e^{-\pi i c_{\mathrm{b}}^2 ac} \\ &= e^{2\pi i \mathsf{q}_2 \mathsf{p}_1} \mathsf{S}_{12} e^{2\pi i c_{\mathrm{b}} c \mathsf{q}_2} e^{-2\pi i c_{\mathrm{b}} c \mathsf{q}_2} e^{-2\pi i c_{\mathrm{b}} a(\mathsf{q}_1 - \mathsf{q}_2 + \mathsf{p}_2)} e^{\pi i c_{\mathrm{b}}^2 a^2} \times \\ &\qquad \times \Psi_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2 - c_{\mathrm{b}}(a+c), -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) e^{-\pi i c_{\mathrm{b}}^2 ac} \\ &= \mathsf{D}_{12} \psi_{a,c}(q_1 + p_2 - q_2, -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) e^{\pi i c_{\mathrm{b}}^2 a(a+c)} \end{aligned}$$

□

*Extra Notation* Recall the notation for Fourier coefficients and Gaussian exponentials in $\mathbb{A}_N$. For $a = (x,n)$ and $b = (y,m)$ in $\mathbb{A}_N$ we write

$$\langle a, b \rangle \equiv e^{2\pi i x y} e^{-2\pi i n m/N} \qquad \langle a \rangle \equiv e^{\pi i x^2} e^{-\pi i n(n+N)/N}$$

For $a = (x, n) \in \mathbb{A}_N$, define $\delta(a) \equiv \delta(x)\delta(n)$ where $\delta(x)$ is Dirac's delta distribution while $\delta(n)$ is the Kronecker delta $\delta_{0,n}$ between 0 and $n$ mod $N$. Define

$$\tilde{f}(x, n) \equiv \mathsf{F}_{2k} \circ \mathcal{F}^{-1}(f)(x, n) = \frac{1}{\sqrt{N}} \sum_{m=0}^{k-1} \int_{-\infty}^{+\infty} f(y, m)\overline{\langle (x, n),\, (y, m)\rangle}\mathrm{d}y,$$

to denote the Fourier transform, as well the following notation

$$\varphi_{a,c}(x, n) \equiv \psi_{a,c}(x, -n). \tag{4.9}$$

Denote, for $x, y \in \mathbb{R}$ and $z \in \mathbb{A}_N$

$$\nu(x) \equiv e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}}\left(2x + \frac{1}{\sqrt{N}}\right)/6} \qquad\qquad \nu_{x,y} = \nu(x - y)e^{\pi i c_{\mathrm{b}}^2 x(x+y)} \tag{4.10}$$

Equations from Lemma 29 can be upgraded to

$$\nu_{a,c}\tilde{\varphi}_{a,c}(z) = \nu_{c,b}\varphi_{c,b}(z)\langle z\rangle e^{-\pi i N/12} \tag{4.11}$$

$$\nu_{a,c}\overline{\varphi_{a,c}}(z) = \nu_{c,a}\varphi_{c,a}(-\varepsilon z)\langle z\rangle e^{-\pi i N/6} \tag{4.12}$$

$$\nu_{a,c}\overline{\tilde{\varphi}_{a,c}}(z) = \nu_{b,c}\varphi_{b,c}(-\varepsilon z) e^{-\pi i N/12} \tag{4.13}$$

Where $\varepsilon$ was defined in (4.2).

**Proposition 49** (Charged Tetrahedral Pentagon equation)**.** *Let $a_j$, $c_j > 0$ such that $\frac{1}{\sqrt{N}} - a_j - c_j > 0$ for $j = 0, 1, 2, 3$ or $4$ satisfying the following relations hold true*

$$a_1 = a_0 + a_2 \qquad a_3 = a_2 + a_4 \qquad c_1 = c_0 + a_4 \qquad c_3 = a_0 + c_4 \qquad c_2 = c_1 + c_3. \tag{4.14}$$

*Then we have*

$$\mathsf{T}_{12}(a_4, c_4)\mathsf{T}_{1,3}(a_2, c_2)\mathsf{T}_{23}(a_0, c_0) = \mu\mathsf{T}_{23}(a_1, c_1)\mathsf{T}_{12}(a_3, c_3) \tag{4.15}$$

*wehere*

$$\mu = \exp \pi i \frac{c_{\mathrm{b}}^2}{6\sqrt{N}}\left(2(c_0 + a_2 + c_4) - \frac{1}{\sqrt{N}}\right)$$

*Proof.* The Charged Pentagon Equation (2.58) provides the equation

$$\Psi_{12}(a_4, c_4)\Psi_{1,3}(a_2, c_2)\Psi_{23}(a_0, c_0) = \xi\Psi_{23}(a_1, c_1)\Psi_{12}(a_3, c_3) \tag{4.16}$$

where

$$\Psi_{jl}(a, c) \equiv \psi_{a,c}(q_+p_l - q_l, -e^{-\frac{\pi i}{N}}Y_j X_l \overline{Y_l}) \tag{4.17}$$

$$\xi = e^{2\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}}(a_0a_2 + a_0a_4 + a_2a_4)} e^{\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}} a_2^2} \tag{4.18}$$

Following step by step the proof of the (un-charged) tetrahedral pentagon equation (3.42), together with the expression for $\mathsf{T}(a, c)$ from Lemma 48, we get a proof for (4.15) induced by the relation (4.16), up to a constant factor $\mu$ that we are going to determine. First notice

$$\mu = \frac{\nu(a_4 - c_4)\nu(a_2 - c_2)\nu(a_0 - c_0)}{\nu(a_1 - c_1)\nu(a_3 - c_3)} e^{\pi i c_{\mathrm{b}}^2 \theta}$$

where

$$\theta = a_0(a_0 + 2a_2 + c_0) + a_4(a_4 + 2a_2 + c_4) + 2a_2^2 + 2a_0a_4 + a_2c_2 +$$

$$
\begin{aligned}
&\quad - a_1(a_1+c_1) - a_3(a_3+c_3) \qquad \text{(using (4.14))}\\
&= a_0(a_0+2a_2+c_0) + a_4(a_4+2a_2+c_4) + 2a_2^2 + 2a_0a_4 + a_2c_2 +\\
&\quad - a_0(a_0+a_2+a_4+c_0) - a_4(a_2+a_4+a_0+c_4) - a_2(2a_0+2a_2+2a_4+c_0+c_4)\\
&= a_2(c_2 - a_0 - c_0 - c_4 - a_4) = 0
\end{aligned}
$$

while

$$
\frac{\nu(a_4-c_4)\nu(a_2-c_2)\nu(a_0-c_0)}{\nu(a_1-c_1)\nu(a_3-c_3)} = \nu\left(-c_0-a_2-c_4\right)
$$

□

We are now able to provide an integral kernel description for the charged tetrahedral operator. We use the Dirac Bra-Ket notation to denote integral kernels, see Appendix A.1.

**Proposition 50.** *Let* $\overline{\mathsf{T}}(a,c) \equiv (\mathsf{T}(a,c))^{*_{\mathrm{b}}}$.

$$
\begin{aligned}
&\langle a_0,\, a_2|\, \mathsf{T}_{12}(a,c)\, |a_1,\, a_3\rangle\\
&\quad = \nu(a-c) e^{\pi i c_{\mathrm{b}}^2 a(a+c)} \langle a_3-a_2,\, a_0\rangle \overline{\langle a_3-a_2\rangle} \delta(a_0+a_2-a_1)\tilde{\varphi}_{a,c}(a_3-a_2)\\
&\langle a_0,\, a_2|\overline{\mathsf{T}}(a,c)|a_1,\, a_3\rangle\\
&\quad = \nu(b-c) e^{\pi i c_{\mathrm{b}}^2 b(b+c)} e^{-\pi i N/12} \langle a_3-a_2, a_1\rangle \langle a_3-a_2\rangle \delta(a_1+a_3-a_0)\varphi_{b,c}(a_3-a_2)
\end{aligned}
$$

*Proof.* The constant part dependent on the charges in the kernel formula is evident from Lemma 48 so we will ignore it below.

$$
\begin{aligned}
&\langle a_0,\, a_2|\mathsf{T}_{12}(a,c)|a_1,\, a_3\rangle\\
&\quad = \langle a_0,\, a_2|\mathsf{D}_{12}\psi_{a,c}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_2, -e^{-\pi i/N} Y_1 X_2 \overline{Y}_2)|a_1,\, a_3\rangle\\
&\quad = \langle a_0+a_2,\, a_2|\psi_{a,c}(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_2, -e^{-\pi i/N} Y_1 X_2 \overline{Y}_2)|a_1,\, a_3\rangle\\
&\quad = \int_{\mathbb{A}_N} \langle a_0+a_2,\, a_2|e^{2\pi i y(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_2)}(-1)^k e^{-\pi i k^2/N} Y_1^{-k} X_2^{-k} Y_2^k|a_1,\, a_3\rangle\\
&\qquad \times\, \tilde{\psi}_{a,c}(y,k)\mathrm{d}(y,k)\\
&\quad = \int_{\mathbb{A}_N} \langle a_0+a_2,\, a_2|e^{2\pi i y \mathsf{q}_1}\overline{Y}_1^k|a_1\rangle\langle a_2|e^{2\pi i y(\mathsf{p}_2-\mathsf{q}_2)}(-1)^k e^{-\pi i k^2/N}\overline{X}_2^k Y_2^k|a_3\rangle\\
&\qquad \times \tilde{\psi}_{a,c}(y,k)\mathrm{d}(y,k)
\end{aligned}
$$

Now compute separately:

$$
\begin{aligned}
&\langle a_0+a_2|e^{2\pi i y \mathsf{q}_1}\overline{Y}_1^k|a_1\rangle\\
&\quad = e^{2\pi i y(x_0+x_2)}\omega^{k(n_0+n_2)}\delta(x_0+x_2-x_1)\delta(n_0+n_2-n_1)\\
&= \quad \langle (y,-k),\, a_0+a_2\rangle \delta(a_0+a_2-a_1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle a_2|e^{2\pi i y(\mathsf{p}_2-\mathsf{q}_2)}(-1)^k e^{-\pi i k^2/N}\overline{X}_2^k Y_2^k|a_3\rangle\\
&\quad = \langle (y,k)\rangle\langle a_2+(y,-k), (y,-k)\rangle^{-1}\delta((y,-k)-a_3+a_2)
\end{aligned}
$$

putting back into the integral we get

$$\int_{\mathbb{A}_N} \langle (y,k)\rangle \langle a_2+(y,-k),(y,-k)\rangle^{-1}\langle (y,-k),\, a_0+a_2\rangle \delta((y,-k)-a_3+a_2)\times$$

$$\begin{aligned}
&\times\, \delta(a_0+a_2-a_1)\tilde{\psi}_{a,c}(x_3-x_2,n_2-n_3)\mathrm{d}(y,k)\\
&= \langle a_3-a_2\rangle\langle a_3,\, a_3-a_2\rangle^{-1}\langle a_3-a_2,\, a_0+a_2\rangle\times\\
&\times\, \delta(a_0+a_2-a_1)\tilde{\psi}_{a,c}(x_3-x_2,n_2-n_3)\\
&= \frac{\langle a_3\rangle\langle a_2\rangle}{\langle a_3,\, a_2\rangle}\frac{\langle a_3,\, a_2\rangle}{\langle a_3\rangle^2}\frac{\langle a_3,\, a_2\rangle}{\langle a_2\rangle^2}\langle a_3-a_2,\, a_0\rangle\times\\
&\times\, \delta(a_0+a_2-a_1)\tilde{\psi}_{a,c}(x_3-x_2,n_2-n_3)\\
&= \langle a_3-a_2,\, a_0\rangle\overline{\langle a_3-a_2\rangle}\delta(a_0+a_2-a_1)\tilde{\psi_{a,c}}(x_3-x_2,n_2-n_3)
\end{aligned}$$

For a negative tetrahedron we have

$$\begin{aligned}
\langle a_0,\, a_2|\overline{\mathsf{T}}(a,c)|a_1,\, a_3\rangle &= \langle a_0,\, a_2|\mathsf{T}(a,c)^{*_\mathrm{b}}|a_1,\, a_3\rangle\\
&= \overline{\langle \varepsilon a_1,\, \varepsilon a_3|\mathsf{T}(a,c)|\varepsilon a_0,\, \varepsilon a_2\rangle}\\
&= \nu(a-c)^{-1}e^{-\pi i c_\mathrm{b}^2 a(a+c)}\langle a_3-a_2,\, a_1\rangle\langle a_3-a_2\rangle\times\\
&\quad\times\, \delta(a_1+a_3-a_0)\overline{\tilde{\varphi}_{a,c}(\varepsilon(a_2-a_3))}\\
&= \nu(a-c)^{-1}e^{-\pi i c_\mathrm{b}^2 a(a+c)}e^{-\pi i(N-4c_\mathrm{b}^2N^{-1})/12}e^{-2\pi i c_\mathrm{b}^2 ab}\times\\
&\quad\times\, \langle a_3-a_2,\, a_1\rangle\langle a_3-a_2\rangle\delta(a_1+a_3-a_0)\varphi_{b,c}(-\varepsilon^2(a_2-a_3))\\
&= \nu(b-c)e^{\pi i c_\mathrm{b}^2 b(b+c)}e^{-\pi i N/12}\times\\
&\quad\times\, \langle a_3-a_2,\, a_1\rangle\langle a_3-a_2\rangle\delta(a_1+a_3-a_0)\varphi_{b,c}(a_3-a_2)
\end{aligned}$$

The appearance of $\varepsilon$ is due to the non-unitarity of the theory for $\mathrm{b}>0$ and $N>1$. $\square$

Let $\mathsf{A}$ and $\mathsf{B}$ two operators on $\mathrm{L}^2(\mathbb{A}_N)$ defined via their integral kernel

$$\langle a_1,a_2|\mathsf{A}\rangle = \delta(a_1+a_2)\langle a_1\rangle e^{\pi i N/12} \qquad \langle a_1,a_2|\mathsf{B}\rangle = \langle a_1-a_2\rangle \tag{4.19}$$

$$\langle\overline{\mathsf{A}}|a_1,a_2\rangle = \overline{\langle \varepsilon a_1,\varepsilon a_2|\mathsf{A}\rangle} \qquad \langle\overline{\mathsf{B}}|a_1,a_2\rangle = \overline{\langle \varepsilon a_1,\varepsilon a_2|\mathsf{B}\rangle} \tag{4.20}$$

**Lemma 51** (Fundamental Lemma)**.** *We have the following three relations*

$$\int_{\mathbb{A}_N^2}\langle\overline{\mathsf{A}}|v,s\rangle\langle x,s|\mathsf{T}(a,c)|u,t\rangle\langle t,y|\mathsf{A}\rangle\mathrm{d}s\mathrm{d}t = \langle x,y|\overline{\mathsf{T}}(a,b)\langle u,v\rangle \tag{4.21}$$

$$\int_{\mathbb{A}_N^2}\langle\overline{\mathsf{A}}|u,s\rangle\langle s,x|\mathsf{T}(a,c)|v,t\rangle\langle t,y|\mathsf{B}\rangle\mathrm{d}s\mathrm{d}t = \langle x,y|\overline{\mathsf{T}}(b,c)\langle u,v\rangle \tag{4.22}$$

$$\int_{\mathbb{A}_N^2}\langle\overline{\mathsf{B}}|u,s\rangle\langle s,y|\mathsf{T}(a,c)|t,v\rangle\langle t,x|\mathsf{B}\rangle\mathrm{d}s\mathrm{d}t = \langle x,y|\overline{\mathsf{T}}(a,b)\langle u,v\rangle \tag{4.23}$$

*Proof.* First:

$$\begin{aligned}
&\int_{\mathbb{A}_N^2}\langle\overline{\mathsf{A}}|v,s\rangle\langle x,s|\mathsf{T}(a,c)|u,t\rangle\langle t,y|\mathsf{A}\rangle\mathrm{d}s\mathrm{d}t\\
&= \nu(a-c)e^{\pi i c_\mathrm{b}^2 a(a+c)}\int_{\mathbb{A}_N^2}\langle t-s,\, x\rangle\overline{\langle t-s\rangle}\delta(x+s-u)\delta(t+y)\delta(v+s)\times\\
&\quad\times\, \langle t\rangle\overline{\langle v\rangle}\tilde{\varphi}_{a,c}(t-s)\mathrm{d}s\mathrm{d}t\\
&= \nu(a-c)e^{\pi i c_\mathrm{b}^2 a(a+c)}\int_{\mathbb{A}_N^2}\langle v-y,\, x\rangle\overline{\langle v-y\rangle}\delta(x-v-u)\langle y\rangle\overline{\langle v\rangle}\tilde{\varphi}_{a,c}(v-y)\mathrm{d}s\mathrm{d}t\\
&= \nu(a-c)e^{\pi i c_\mathrm{b}^2 a(a+c)}e^{-\pi i c_\mathrm{b}^2 a(a+2c)}e^{-\pi i(N-4c_\mathrm{b}^2N^{-1})/12}\varphi_{c,b}(v-y)\delta(x-v-u)\times\\
&\quad\times\, \overline{\langle v-y\rangle}\langle v-y\rangle\langle v-y,u+v\rangle\langle y\rangle\overline{\langle v\rangle}\\
&= \nu(c-b)e^{\pi i c_\mathrm{b}^2 c(c+cb)}e^{-\pi i N/12}\varphi_{c,b}(v-y)\delta(x-v-u)\langle v-y,u\rangle\langle v-y\rangle.
\end{aligned}$$

Second:

$$\int_{\mathbb{A}_N^2} \langle \overline{\mathsf{A}}|u,s\rangle\langle s,x|\mathsf{T}(a,c)|v,t\rangle\langle t,y|\mathsf{B}\rangle \mathrm{d}s\mathrm{d}t$$

$$= e^{-\pi i N/12}\overline{\langle u\rangle}\int_{\mathbb{A}_N^2} \langle -u,x|\mathsf{T}(a,c)|v,t\rangle\langle t-y\rangle \mathrm{d}s\mathrm{d}t$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}e^{-\pi i N/12}\int_{\mathbb{A}_N} \tilde{\varphi}_{a,c}(t-x)\delta(u+v-x)\times$$

$$\times\ \overline{\langle u\rangle\langle u,\, t-x\rangle\langle t-x\rangle}\langle t-y\rangle \mathrm{d}t \qquad\qquad t\mapsto t+x$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}e^{-\pi i N/12}\int_{\mathbb{A}_N} \tilde{\varphi}_{a,c}(t)\delta(u+v-x)\times$$

$$\times\ \overline{\langle u\rangle\langle u,\, t\rangle\langle t\rangle}\langle t+x-y\rangle \mathrm{d}t$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}e^{-\pi i N/12}\delta(u+v-x)\overline{\langle u\rangle}\langle x-y\rangle\times$$

$$\times\int_{\mathbb{A}_N} \tilde{\varphi}_{a,c}(t)\overline{\langle u,\, t\rangle}\langle t,\, x-y\rangle \mathrm{d}t$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}e^{-\pi i N/12}\delta(u+v-x)\overline{\langle u\rangle}\langle x-y\rangle\times$$

$$\times\int_{\mathbb{A}_N} \tilde{\varphi}_{a,c}(t)\langle t,\, v-y\rangle \mathrm{d}t$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}e^{-\pi i N/12}\delta(u+v-x)\overline{\langle u\rangle}\langle x-y\rangle\varphi_{a,c}(v-y)$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}e^{-\pi i N/12}\delta(u+v-x)\langle u,\, v-y\rangle\langle v-y\rangle\varphi_{a,c}(v-y).$$

Third:

$$\int_{\mathbb{A}_N^2} \langle \overline{\mathsf{B}}|s,y\rangle\langle u,s|\mathsf{T}(a,c)|t,v\rangle\langle t,x|\mathsf{B}\rangle \mathrm{d}s\mathrm{d}t$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}\tilde{\varphi}_{a,c}(v-y)\overline{\langle v-y\rangle}\int_{\mathbb{A}_N} \langle \overline{\mathsf{B}}|u,s\rangle\langle s,v-y\rangle\langle s+y,x|\mathsf{B}\rangle \mathrm{d}s$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}\tilde{\varphi}_{a,c}(v-y)\overline{\langle v-y\rangle}\int_{\mathbb{A}_N} \overline{\langle u-s\rangle}\langle s,v-y\rangle\langle s+y-x\rangle \mathrm{d}s$$

$$= \nu(a-c)e^{\pi i c_{\mathrm{b}}^2 a(a+c)}\tilde{\varphi}_{a,c}(v-y)\overline{\langle v-y\rangle}\langle y-x\rangle\overline{\langle u\rangle}\int_{\mathbb{A}_N} \langle s,v-x+u\rangle \mathrm{d}s$$

$$= \nu(c-b)e^{\pi i c_{\mathrm{b}}^2 c(c+b)}\varphi_{c,b}(v-y)\langle y-x\rangle\overline{\langle u\rangle}\delta(v-x+u)$$

$$= \nu(c-b)e^{\pi i c_{\mathrm{b}}^2 c(c+b)}\varphi_{c,b}(v-y)\langle v-y\rangle\langle v-y,\, u\rangle\delta(v-x+u)$$

□

### 4.3.1 TQFT Rules, Tetrahedral Symmetries and Gauge Invariance

We consider oriented surfaces with cellular structure such that all 2-cells are either bigons or triangles. Not all the edge orientations will be admitted: we forbid triangles cyclically oriented. For the bigons, we consider only the *essential* ones, the others being contractible to an edge. These essential bigons are precisely the ones with cellular structure isomorphic to the unit disk with vertices $\pm 1\in\mathbb{C}$ and edges $\{e_1 = e^{\pi i t};\ e_2 = -e^{\pi i t}$, for $t\in[0,1]\}$ or $\{e_1 = -e^{-\pi i t};\ e_2 = e^{-\pi i t}$, for $t\in[0,1]\}$. Given such an ideally triangulated surface $\Sigma$ we will associate a copy of $\mathbb{C}$ to any bigon and a copy of $\mathcal{S}'(\mathbb{A}_N)$ to any triangle. Globally we associate to the surface the space $\mathcal{S}'(\mathbb{A}_N^{\Delta_2(\Sigma)})$. To a shaped tetrahedron $T$ with ordered vertices $\{v_0,v_1,v_2,v_3\}$ we associate the partition function $Z_{\mathrm{b}}^{(N)}(T)$ through the Nuclear Theorem (A.6) as a ket distribution

$$\langle x|\widetilde{Z_{\mathrm{b}}^{(N)}(T)}\rangle = \begin{cases} \langle a_0, a_2|\mathsf{T}(c(v_0v_1), c(v_0v_3))|a_1, a_3\rangle & \text{if } \operatorname{sign}(T) = 1; \\ \langle a_1, a_3|\overline{\mathsf{T}}(c(v_0v_1), c(v_0v_3))|a_0, a_2\rangle & \text{if } \operatorname{sign}(T) = -1. \end{cases} \tag{4.24}$$

where

$$\mathbb{A}_N \ni a_i := a(\partial_i T), \quad i \in \{0,1,2,3\}$$

and

$$c := \frac{1}{\pi\sqrt{N}}\alpha_T : \Delta_1(T) \to \mathbb{R}_{>0}.$$

Having allowed bigons on triangulations of surfaces, we get cones on triangulations of cobordisms. From the 2 classes of bigons described above we have 4 isotopy classes of cellular structures of cones over them, described in the following as embedded in $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$. The bigon is identified with the unit disc embedded in $\mathbb{C}$. The apex of the cone will be the point $(0,1) \in \mathbb{C} \times \mathbb{R}$. The 1-cells will be either

$$\{e^1_{0\pm}(t) = (\pm e^{i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm(1-t), t)\}$$

or

$$\{e^1_{0\pm}(t) = (\mp e^{-i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm(1-t), t)\}$$

or

$$\{e^1_{0\pm}(t) = (\pm e^{i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm t, 1-t)\}$$

or

$$\{e^1_{0\pm}(t) = (\mp e^{-i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm t, 1-t)\}.$$

We name these types of cones $A_+$, $A_-$, $B_+$ and $B_-$ respectively. We need TQFT rules for the gluing of this cones. We just need to look at their gluing over a tetrahedra. We assign a partition function to the cones as follows

$$\langle a_1, a_2|\widetilde{Z_{\mathrm{b}}^{(N)}(A_\pm)}\rangle = \delta(a_1+a_2)\langle a_1\rangle^{\pm 1}e^{\pm\pi i N/12}, \quad \langle a_1, a_2|\widetilde{Z_{\mathrm{b}}^{(N)}(B_\pm)}\rangle = \langle a_1 - a_2\rangle^{\pm 1}. \tag{4.25}$$

Tetrahedral symmetries are generated by permutation of the ordered vertices. Indeed the group of tetrahedral symmetries is identified with the symmetric group $\mathbb{S}_4$ and is generated by three transpositions. The three equations of the Fundamental Lemma 51 gain an interpretation as glueing of cones on the faces of a tetrahedron through definitions (4.25). These three glueing generates all the symmetries of a tetrahedron, and through this interpretation, the Fundamental Lemma assure that the partition function $Z_{\mathrm{b}}^{(N)}$ satisfies all the tetrahedral symmetries. For a general approach to tetrahedral partition functions and symmetries see [GKT12].

We can now formulate the main Theorem for the Teichmüller TQFT. This theorem was proved by Andersen and Kashaev for the case $N = 1$, and any b in [AK14b]. The statement that we have here is for every $N$ odd, and it is not present as such in the literature.

**Theorem 52** (Teichmüller TQFT, Andersen and Kashaev)**.** *For any* $\mathrm{b} \in \mathbb{C}^*$ *such that* $\operatorname{Im}\mathrm{b}(|\mathrm{b}|-1) = 0$ *and* $\operatorname{Re}\mathrm{b} > 0$*, and for any* $N \in \mathbb{Z}_{>0}$ *odd there exists a unique* $*_{\mathrm{b}}$*-functor* $F_{\mathrm{b}}^{(N)}: \mathcal{B}_a \to \mathcal{D}_N$ *such that* $F_{\mathrm{b}}^{(N)}(A) = \Delta_2(A)$, $\forall A \in \operatorname{Ob}\mathcal{B}_a$, *and for any admissible leveled shaped pseudo 3-manifold* $(X, l_X)$*, the associated morphism in* $\mathcal{D}_N$ *takes the form*

$$F_{\mathrm{b}}^{(N)}(X, l_X) = Z_{\mathrm{b}}^{(N)}(X)e^{-\pi i\frac{l_X c_{\mathrm{b}}^2}{N}} \in \mathcal{S}'\left(\mathbb{A}_N^{\Delta_2(\partial X)}\right), \tag{4.26}$$

*where* $Z_{\mathrm{b}}^{(N)}$ *is defined in* (4.24) *for a tetrahedron.*

Here $*_{\mathrm{b}}$-functor means that $F(X^*) = F(X)^{*_{\mathrm{b}}}$, where $X^*$ is the oppositely oriented pseudo 3-manifold to $X$.

We still need to address the the gauge invariance and the convergence of the partition functions under glueings. We won't discuss the convergence here because it follows directly from the convergence in the case level $N = 1$, which was addressed in [AK14b]. For the gauge invariance consider the suspension of an $n$-gone $SP_n$ naturally triangulated into $n$ tetrahedra sharing the only internal edge $e$. Every gauge transformation can be decomposed in a sequence of gauges involving only one edge $e$, and every such guage transformation can be understood in the example of the suspension. Suppose all the tetrahedra to be positive, and having vertex order such that the two last two vertices are the endpoints of the internal common edge. After enumerating the tetrahedra in cyclic order, let $a_i$, $c_i$ be the two shape parameter of $T_i$, $i = 0, \dots, n$, and $\mathrm{a} = (a_0, \dots, a_n)$, $\mathrm{c} = (c_0, \dots, c_n)$. Notice that $\sqrt{N}\pi a_i$ is the dihedral angle corresponding to the edge $e$. So a gauge corresponding to $e$ will affect the partition function of $SP_N$

$$Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}) := \mathrm{Tr}_0(\mathsf{T}_{01}(a_1, c_1)\mathsf{T}_{02}(a_2, c_2) \cdots \mathsf{T}_{0n}(a_n, c_n))$$

shifting c of an amount $\lambda = (\lambda, \dots, \lambda)$. One can show from the definitions and the discussion above, that

$$\mathsf{T}(a, c + \lambda) = e^{-2\pi i c_{\mathrm{b}} \lambda \mathsf{p}_1} \mathsf{T}(a, c) e^{2\pi i c_{\mathrm{b}} \lambda \mathsf{p}_1} e^{\pi i c_{\mathrm{b}}^2 \left(\frac{1}{\sqrt{N}} - 6a\right)\lambda/3}$$

which, after tracing, leads to the following Proposition

**Proposition 53.** *[AK14b]*

$$Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c} + \lambda) = Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}) e^{\pi i c_{\mathrm{b}}^2 \left(\frac{n}{\sqrt{N}} - 6Q_e\right)\lambda/3}$$

*where*

$$Q_e = a_1 + a_2 + \dots a_n$$

## 4.4 Knot Invariants: Computations and Conjectures

*Notation* In the examples we are going to use the following notation for quantum dilogarithms

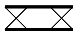
$$\varphi_{\mathrm{b}}(x, n) \equiv \mathrm{D}_{\mathrm{b}}(x, -n) \tag{4.27}$$

moreover we will often abuse of notation in favor of readability in the following ways. For $z = (x, n) \in \mathbb{A}_N$ the writing $e^{2\pi i c_{\mathrm{b}} z \alpha}$ will be sometimes used in place of $e^{2\pi i c_{\mathrm{b}} x \alpha}$. Moreover sums of the types $z + c_{\mathrm{b}} a$ will always mean $(x + c_{\mathrm{b}} a, n)$ . In the examples we encode an oriented triangulated pseudo 3-manifold $X$ into a diagram where a tetrahedron $T$ is represented by an element

where the vertical segments, ordered from left to right, correspond to the faces $\partial_0 T, \partial_1 T, \partial_2 T, \partial_3 T$ respectively. When we glue tetrahedron along faces, we illustrate this by joining the corresponding vertical segments.

### 4.4.1 Figure–Eight Knot $4_1$

Let $X$ be represented by the diagram

 (4.28)

Choosing an orientation, it consists of one positive tetrahedron $T_+$ and one negative tetrahedron $T_-$ with four identifications

$$\partial_{2i+j}T_+ \simeq \partial_{2-2i+j}T_-, \quad i, j \in \{0, 1\}.$$

Combinatorially, we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_0, e_1\}$, $\Delta_2(X) = \{f_0, f_1, f_2, f_3\}$, and $\Delta_3(X) = \{T_+, T_-\}$ with the boundary maps

$$f_{2i+j} = \partial_{2i+j}T_+ = \partial_{2-2i+j}T_-, \quad i, j \in \{0, 1\},$$

$$\partial_i f_j = \begin{cases} e_0, & \text{if } j - i \in \{0, 1\}; \\ e_1, & \text{otherwise,} \end{cases}$$

$$\partial_i e_j = *, \quad i, j \in \{0, 1\}.$$

The topological space $X \setminus \{*\}$ is homeomorphic to the complement of the figure–eight knot, and indeed $X \setminus \{*\}$ is an ideal triangulation of such cuspidal manifold. The set $\Delta_{3,1}(X)$ consists of elements $(T_\pm, e_{j,k})$ for $0 \le j < k \le 3$. We fix a shape structure

$$\alpha_X \colon \Delta_{3,1}(X) \to \mathbb{R}_{>0}$$

by the formulae

$$\alpha_X(T_\pm, e_{0,1}) = \pi\sqrt{N}a_\pm, \quad \alpha_X(T_\pm, e_{0,2}) = \pi\sqrt{N}b_\pm, \quad \alpha_X(T_\pm, e_{0,3}) = \pi\sqrt{N}c_\pm,$$

where $a_\pm + b_\pm + c_\pm = \frac{1}{\sqrt{N}}$. The weight function

$$\omega_X \colon \Delta_1(X) \to \mathbb{R}_{>0}$$

takes the values

$$\omega_X(e_0) = \sqrt{N}\pi(2a_+ + c_+ + 2b_- + c_-) =: 2\pi w, \quad \omega_X(e_1) = 2\pi(2 - w).$$

As the figure–eight knot is hyperbolic, the completely balanced case $w = 1$ is accessible directly. We can state the balancing condition $w = 1$ as

$$2b_+ + c_+ = 2b_- + c_-. \tag{4.29}$$

The kernel representations for the operators $\mathsf{T}(a_+, c_+)$ and $\mathsf{T}(a_-, c_-)$ are as follows. Let $z_j \in \mathbb{A}_N$ , $j = 0, 1, 2, 3$

$$\begin{aligned} &\langle z_0, z_2|\mathsf{T}(a_+, c_+)|z_1, z_3\rangle \qquad (4.30)\\ &\quad = \nu_{a_+,c_+}\langle z_3 - z_2, z_0\rangle\overline{\langle z_3 - z_2\rangle}\delta(z_0 + z_2 - z_1)\tilde{\varphi}_{a_+,c_+}(z_3 - z_2) \\ &\langle z_3, z_1|\overline{\mathsf{T}}(a_+, c_-)|z_2, z_0\rangle = \overline{\langle \varepsilon z_2, \varepsilon z_0|\mathsf{T}(a_+, c_-)|\varepsilon z_3, \varepsilon z_1\rangle} \qquad (4.31)\\ &\quad = \overline{\nu_{a_-,c_-}}\langle z_0 - z_1, z_2\rangle\langle z_1 - z_0\rangle\delta(z_0 + z_2 - z_3)\overline{\tilde{\varphi}_{a_-,c_-}(\varepsilon z_1 - \varepsilon z_0)} \end{aligned}$$

The Andersen–Kashaev invariant at level $N$ for the complement of the figure–eight knot is then

$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N^4}\langle z_0, z_2|\mathsf{T}(a_+, c_+)|z_1, z_3\rangle\langle z_3, z_1|\overline{\mathsf{T}}(a_+, c_-)|z_2, z_0\rangle \mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3$$

$$
\begin{aligned}
&= \int_{\mathbb{A}_N^4} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_3-z_2)\overline{\varphi_{c_-,b_-}(\varepsilon z_1-\varepsilon z_0)}\delta(z_0+z_2-z_1)\times \\
&\quad \times \delta(z_0+z_2-z_3)\langle z_3-z_2,z_0\rangle\langle z_0-z_1,z_2\rangle \mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\mathbb{A}_N^3} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_1-z_2)\overline{\varphi_{c_-,b_-}(\varepsilon z_1-\varepsilon z_0)}\delta(z_0+z_2-z_1)\times \\
&\quad \times \langle z_1-z_2,z_0\rangle\langle z_0-z_1,z_2\rangle \mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2 \\
&= \int_{\mathbb{A}_N^2} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_0)\overline{\varphi_{c_-,b_-}(\varepsilon z_2)}\langle z_0,z_0\rangle\langle -z_2,z_2\rangle \mathrm{d}z_0\mathrm{d}z_2 \\
&= \int_{\mathbb{A}_N} \nu_{c_+,b_+}\varphi_{c_+,b_+}(z_0)\langle z_0,z_0\rangle \mathrm{d}z_0 \overline{\int_{\mathbb{A}_N} \nu_{c_-,b_-}\varphi_{c_-,b_-}(\varepsilon z_2)\langle z_2,z_2\rangle \mathrm{d}z_2} \\
&= \sigma_{c_+,b_+}\overline{\sigma_{c_-,b_-}}
\end{aligned}
$$

We can compute

$$
\begin{aligned}
\sigma_{c_\pm,b_\pm} &= \nu_{c_\pm,b_\pm}\int_{\mathbb{A}_N} \frac{e^{-2\pi i c_{\mathrm{b}} z c_\pm}}{\varphi_{\mathrm{b}}(z-c_{\mathrm{b}}(b_\pm+c_\pm))}\langle z\rangle^2 \mathrm{d}z \\
&= \nu'_{c_\pm,b_\pm}\int_{\mathbb{A}_N+di} \frac{e^{4\pi i c_{\mathrm{b}} z(2b_\pm+c_\pm)}}{\varphi_{\mathrm{b}}(z)}\langle z\rangle^2 \mathrm{d}z
\end{aligned}
$$

where

$$
\nu'_{c_\pm,b_\pm} = \nu_{c_\pm,b_\pm}e^{4\pi i c_{\mathrm{b}}^2(c_\pm b_\pm - b_\pm^2)} \tag{4.32}
$$

and the domain of integration $\mathbb{A}_N + di = (\mathbb{R}+di)\times\mathbb{Z}/N\mathbb{Z})$ remarks that we have shifted the real integral to a contour integral in the complex plane, and $d\in\mathbb{R}$ is such that the integral converges absolutely. We sometimes omit the contour shift in the computations but we try to put it in the results. Defining

$$
\lambda \equiv 2b_+ + c_+ = 2b_- + c_-
$$

we have

$$
\begin{aligned}
Z_{\mathrm{b}}^{(N)}(X) &= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{e^{4\pi i c_{\mathrm{b}}\lambda(z_0+z_2)}}{\varphi_{\mathrm{b}}(z_0)\overline{\varphi_{\mathrm{b}}(\varepsilon z_2)}}\langle z_0\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2 \\
&= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{\varphi_{\mathrm{b}}(z_2)}{\varphi_{\mathrm{b}}(z_0)}e^{4\pi i c_{\mathrm{b}}\lambda(z_0+z_2)}\langle z_0\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2 \\
&= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{\varphi_{\mathrm{b}}(z_2-z_0)}{\varphi_{\mathrm{b}}(z_0)}e^{4\pi i c_{\mathrm{b}}\lambda z_2}\langle z_0,z_2\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2
\end{aligned}
$$

that has the structure

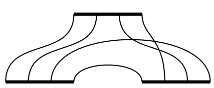
$$
Z_{\mathrm{b}}^{(N)}(X) = e^{i\phi}\int_{\mathbb{A}_N+i0}\chi_{4_1}^{(N)}(x,\lambda)\mathrm{d}x, \tag{4.33}
$$

$$
\chi_{4_1}^{(N)}(x,\lambda) = \chi_{4_1}^{(N)}(x)e^{4\pi i c_{\mathrm{b}}\lambda x}, \qquad \chi_{4_1}^{(N)}(x) = \int_{\mathbb{A}_N-i0}\frac{\varphi_{\mathrm{b}}(x-y)}{\varphi_{\mathrm{b}}(y)}\langle x,y\rangle^2\overline{\langle x\rangle^2}\mathrm{d}y \tag{4.34}
$$

where $\phi$ is some constant quadratic combination of dihedral angles.

### 4.4.2 The Complement of the Knot $5_2$

Let $X$ be the closed SOTP 3-manifold represented by the diagram



This triangulation has only one vertex $*$ and $X \setminus \{*\}$ is topologically the complement of the knot $5_2$. We denote $T_1, T_2, T_3$ the left, right, and top tetrahedra respectively. We choose the orientation so that all of them are positive. Balancing all the edges correspond to require the following equations to be true

$$2a_3 = a_1 + c_2, \quad b_3 = c_1 + b_2. \tag{4.35}$$

The three integral kernels reads

$$\begin{aligned}
&\langle z, w|\mathsf{T}(a_1, c_1)|u, x\rangle = \\
&\qquad = \nu_{a_1,c_1}\langle x - w, z\rangle\overline{\langle x - w\rangle}\delta(z + w - u)\tilde{\varphi}_{a_1,c_1}(x - w) \\
&\langle x, v|\mathsf{T}(a_2, c_2)|y, w\rangle = \\
&\qquad = \nu_{a_2,c_2}\langle w - v, x\rangle\overline{\langle w - v\rangle}\delta(x + v - y)\tilde{\varphi}_{a_2,c_2}(w - v) \\
&\langle y, u|\mathsf{T}(a_3, c_3)|v, z\rangle = \\
&\qquad = \nu_{a_3,c_3}\langle z - u, y\rangle\overline{\langle z - u\rangle}\delta(y + u - v)\tilde{\varphi}_{a_3,c_3}(z - u)
\end{aligned}$$

The three delta functions will give us the following substitutions inside the integrals

$$v = y - x \qquad\qquad u = -x \qquad\qquad w = -x - z$$

We can write

$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N} f_X(x)\mathrm{d}x, \tag{4.36}$$

and compute

$$\begin{aligned}
f_X(x) &= \nu_{a_1,c_1}\nu_{a_2,c_2}\nu_{a_3,c_3}\int_{\mathbb{A}_N^2}\tilde{\varphi}_{a_1,c_1}(2x + z)\tilde{\varphi}_{a_2,c_2}(-z - y)\tilde{\varphi}_{a_3,c_3}(z + x) \\
&\qquad\times\langle 2x + z, z\rangle\langle -z - y, x\rangle\langle z + x, y\rangle\overline{\langle 2x + z\rangle\langle z + y\rangle\langle z + x\rangle}\mathrm{d}z\mathrm{d}y \\
&= e^{-\pi iN/4}\nu_{c_1,b_1}\nu_{c_2,b_2}\nu_{c_3,b_3}\int_{\mathbb{A}_N^2}\varphi_{c_1,b_1}(2x + z)\varphi_{c_2,b_2}(-z - y)\varphi_{c_3,b_3}(z + x) \\
&\qquad\times\langle 2x + z, z\rangle\langle -z - y, x\rangle\langle z + x, y\rangle\mathrm{d}y\mathrm{d}z
\end{aligned}$$

Shifting $z \mapsto z - x$ we get

$$\begin{aligned}
f_X(x) = e^{-\pi iN/4}\nu_{c_1,b_1}\nu_{c_2,b_2}\nu_{c_3,b_3}\int_{\mathbb{A}_N^2}&\varphi_{c_1,b_1}(x + z)\varphi_{c_2,b_2}(x - z - y)\varphi_{c_3,b_3}(z) \\
&\times\langle x + z, z - x\rangle\langle x - z - y, x\rangle\langle z, y\rangle\mathrm{d}y\mathrm{d}z
\end{aligned}$$

and then $y \mapsto -y + x - z$

$$\begin{aligned}
f_X(x) &= e^{-\pi iN/4}\nu_{c_1,b_1}\nu_{c_2,b_2}\nu_{c_3,b_3}\int_{\mathbb{A}_N^2}\varphi_{c_1,b_1}(x + z)\varphi_{c_2,b_2}(y)\varphi_{c_3,b_3}(z) \\
&\qquad\times\langle x + z, z - x\rangle\langle y, x\rangle\langle z, x - z - y\rangle\mathrm{d}y\mathrm{d}z \\
&= e^{-\pi iN/4}\nu_{c_1,b_1}\nu_{c_2,b_2}\nu_{c_3,b_3}\int_{\mathbb{A}_N^2}\varphi_{c_1,b_1}(x + z)\varphi_{c_2,b_2}(y)\varphi_{c_3,b_3}(z)\langle x - y, z - x\rangle\mathrm{d}y\mathrm{d}z \\
&= e^{-\pi iN/4}\nu_{c_1,b_1}\nu_{c_2,b_2}\nu_{c_3,b_3}\int_{\mathbb{A}_N}\varphi_{c_1,b_1}(x + z)\tilde{\varphi}_{c_2,b_2}(z - x)\varphi_{c_3,b_3}(z)\langle x, z - x\rangle\mathrm{d}z \\
&= e^{-\pi iN/3}\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}\int_{\mathbb{A}_N}\varphi_{c_1,b_1}(x+z)\varphi_{b_2,a_2}(z-x)\varphi_{c_3,b_3}(z)\langle z-x\rangle\langle x, z-x\rangle\mathrm{d}z \\
&= e^{-\pi iN/3}\nu_{c_1,b_1}\nu_{b_2,a_2}\nu_{c_3,b_3}\int_{\mathbb{A}_N}\varphi_{c_1,b_1}(x + z)\varphi_{b_2,a_2}(z - x)\varphi_{c_3,b_3}(z)\langle z\rangle\overline{\langle x\rangle}\mathrm{d}z
\end{aligned}$$

we now make the complex shift $z \mapsto z + c_{\mathrm{b}}(c_3 + b_3)$ and get (we omit the constant factor in front for a moment, for clarity reasons)

$$f_X(x) = \int_{\mathbb{A}_N} \frac{\overline{\langle x \rangle} \langle z + c_{\mathrm{b}}(c_3 + b_3) \rangle e^{-2\pi i (z + c_{\mathrm{b}}(c_3+b_3))c_3}}{\varphi_{\mathrm{b}}(z + x - c_{\mathrm{b}}(c_1 + b_1 - c_3 - b_3)) \varphi_{\mathrm{b}}(z) \varphi_{\mathrm{b}}(z - x - c_{\mathrm{b}}(a_2 + b_2 - c_3 - b_3))} \times \\ \times e^{-2\pi i c_{\mathrm{b}}(x + z + c_{\mathrm{b}}(c_3+b_3))c_1} e^{-2\pi i c_{\mathrm{b}}(z - x + c_{\mathrm{b}}(c_3+b_3))b_2} \mathrm{d}z$$

From the balancing equations (4.35) we get the following

$$\begin{aligned} c_1 + b_1 - c_3 - b_3 &= \frac{1}{\sqrt{N}} - a_1 - c_3 - b_3 = \frac{1}{\sqrt{N}} + c_2 - 2a_3 - c_3 - b_3 \\ &= c_2 - a_3 \\ b_2 + a_2 - c_3 - b_3 &= \frac{1}{\sqrt{N}} - c_2 - \frac{1}{\sqrt{N}} + a_3 = a_3 - c_2, \end{aligned}$$

that together with a change of variable $x' = x - c_{\mathrm{b}}(c_2 - a_3)$ gives

$$f_X(x) = \int_{\mathbb{A}_N} \frac{\overline{\langle x' + c_{\mathrm{b}}(c_2 - a_3) \rangle} \langle z + c_{\mathrm{b}}(c_3 + b_3) \rangle e^{-2\pi i z c_3}}{\varphi_{\mathrm{b}}(z + x') \varphi_{\mathrm{b}}(z) \varphi_{\mathrm{b}}(z - x')} \times \\ \times e^{-2\pi i c_{\mathrm{b}}(x' + z)c_1} e^{-2\pi i c_{\mathrm{b}}(z - x')b_2} \mathrm{d}z$$

Finally, reincorporating the constants as

$$\mu = e^{-\pi i N/3} \nu_{c_1,b_1} \nu_{b_2,a_2} \nu_{c_3,b_3} e^{-2\pi i c_{\mathrm{b}}^2 (c_3(c_3+b_3) + b_2(c_2+b_2) + c_1(c_1+b_1))}.$$

we have

$$\begin{aligned} f_X(x) &= \mu \int_{\mathbb{A}_N} \frac{\overline{\langle x' \rangle} \langle z \rangle}{\varphi_{\mathrm{b}}(z + x') \varphi_{\mathrm{b}}(z) \varphi_{\mathrm{b}}(z - x')} \times \\ &\qquad \times e^{-2\pi i x' c_{\mathrm{b}}(c_1 - b_2 + c_2 - a_3)} e^{-2\pi i z c_{\mathrm{b}}(c_1 - b_3 + b_2)} \mathrm{d}z \\ &= \mu \int_{\mathbb{A}_N} \frac{\overline{\langle x' \rangle} \langle z \rangle}{\varphi_{\mathrm{b}}(z + x') \varphi_{\mathrm{b}}(z) \varphi_{\mathrm{b}}(z - x')} \times \\ &\qquad \times e^{-2\pi i x' c_{\mathrm{b}}(c_1 - b_2 + c_2 - a_3)} \mathrm{d}z \end{aligned}$$

So, after fixing $\lambda = -c_1 + b_2 - c_2 + a_3$ we can express

$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N + i0} \chi_{5_2}^{(N)}(x, \lambda) \mathrm{d}x, \qquad \chi_{5_2}^{(N)}(x, \lambda) = \chi_{5_2}^{(N)}(x) e^{2\pi i c_{\mathrm{b}} \lambda x} \tag{4.37}$$

$$\chi_{5_2}^{(N)}(x) = \int_{\mathbb{A}_N - i0} \frac{\overline{\langle x \rangle} \langle z \rangle}{\varphi_{\mathrm{b}}(z + x) \varphi_{\mathrm{b}}(z) \varphi_{\mathrm{b}}(z - x)} \mathrm{d}z \tag{4.38}$$

### 4.4.3 H-Triangulations

In this section we will look at one vertex H-triangulations of knots. Computation wise they are simple to deduce from the ideal triangulations (at least in the two examples at hand). The following small computation will be useful

Let $T_0$ consists of one tetrahedron with one face identification

Choosing a positive orientation we have

$$\langle x | Z_{\mathrm{b}}^{(N)}(T_0) | y \rangle = \int_{\mathbb{A}_N} \langle z, x | \mathsf{T}(a_0, c_0) | z, y \rangle \mathrm{d}z$$

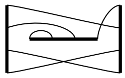
$$= \nu_{a_0,c_0}\tilde{\varphi}_{a_0,c_0}(y)\overline{\langle y\rangle}\delta(x)\int_{\mathbb{A}_N}\langle y,z\rangle \mathrm{d}z$$

$$= \nu_{a_0,c_0}\tilde{\varphi}_{a_0,c_0}(0)\delta(y)\delta(x) = e^{-\pi N/12}\nu_{c_0,b_0}\frac{\delta(y)\delta(x)}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}$$

The choice of $T_0$ negatively oriented gives

$$\langle x|Z_{\mathrm{b}}^{(N)}(T_0)|y\rangle = e^{-\pi N/12}\nu_{b,c}\frac{\delta(y)\delta(x)}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}$$

Now let $X$ be an H–Triangulation for the figure–eight knot, i.e. let $X$ be given by the diagram



where the figure-eight knot is represented by the edge of the central tetrahedron connecting the maximal and the next to maximal vertices. Choosing positive central tetrahedron ($T_0$), the left tetrahedron ($T_+$) will be positive and the right one ($T_-$) negative.The shape structure, in the limit $a_0 \to 0$ satisfies $2b_+ + c_+ = 2b_- + c_- =: \lambda$ The partition function takes the form

$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N^5}\langle z_0, z_2|\mathsf{T}(a_+,c_+)|x,z_3\rangle\langle x|Z_{\mathrm{b}}^{(N)}(T_0)|x'\rangle$$

$$\langle z_3, x'|\overline{\mathsf{T}}(a_+,c_-)|z_2,z_0\rangle \mathrm{d}z_0\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}x\mathrm{d}x'$$

$$= e^{-\pi N/12}\nu_{c_0,b_0}\frac{\delta(x')}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}$$

$$\int_{\mathbb{A}_N^3}\langle z_0, z_2|\mathsf{T}(a_+,c_+)|x,z_3\rangle\langle z_3,x|\overline{\mathsf{T}}(a_+,c_-)|z_2,z_0\rangle\delta(x)\mathrm{d}z_0\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}x$$

$$= \frac{e^{-\pi N/12}\nu_{c_0,b_0}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}\int_{\mathbb{A}_N^3}\nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(-z_2)\overline{\varphi_{c_-,b_-}(-\varepsilon z_0)}$$

$$\times\langle -z_2, z_0\rangle\langle z_0, z_2\rangle\delta(z_0+z_2)\mathrm{d}z_0\mathrm{d}z_2$$

$$= \frac{e^{-\pi N/12}\nu_{c_0,b_0}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}\int_{\mathbb{A}_N^3}\nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z)\overline{\varphi_{c_-,b_-}(-\varepsilon z)}\mathrm{d}z$$

$$= \frac{e^{-\pi N/12}\nu_{c_0,b_0}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}\nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}$$

$$\times\int_{\mathbb{A}_N^3}\frac{\varphi_{\mathrm{b}}(-z + c_{\mathrm{b}}(c_- + b_-))}{\varphi_{\mathrm{b}}(z - c_{\mathrm{b}}(c_+ + b_+))}e^{2\pi i c_{\mathrm{b}} z c_-}e^{-2\pi i c_{\mathrm{b}} z c_+}\mathrm{d}z$$

$$= \frac{e^{-\pi N/12}\nu_{c_0,b_0}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}\nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}$$

$$\times\int_{\mathbb{A}_N^3}\frac{\varphi_{\mathrm{b}}(c_{\mathrm{b}}(b_+ - b_-) - z)}{\varphi_{\mathrm{b}}(z)}e^{2\pi i c_{\mathrm{b}}(z + c_{\mathrm{b}}(c_+ + b_+))(c_- - c_+)}\mathrm{d}z$$

$$= \frac{e^{-\pi N/12}\nu_{c_0,b_0}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}\nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}e^{2\pi i c_{\mathrm{b}}^2(c_+ + b_+))(c_- - c_+)}$$

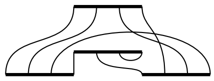
$$\times\int_{\mathbb{A}_N^3}\frac{\varphi_{\mathrm{b}}(c_{\mathrm{b}}(b_+ - b_-) - z)}{\varphi_{\mathrm{b}}(z)}e^{4\pi i c_{\mathrm{b}} z(b_+ - b_-)}\mathrm{d}z$$

$$= e^{2\pi i c_{\mathrm{b}}^2(c_+ + b_+))(c_- - c_+)}\frac{e^{-\pi N/12}\nu_{c_0,b_0}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})}\nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\chi_{4_1}^{(N)}(c_{\mathrm{b}}(b_+ - b_-))$$

Notice that in the limit $a_0 \to 0$ we have $b_+ = b_-$, so re-normalizing to remove the singularity we can write

$$\lim_{a_0\to 0}\varphi_{\mathrm{b}}(c_{\mathrm{b}}a_0 - c_{\mathrm{b}}/\sqrt{N})Z_{\mathrm{b}}^{(N)}(X) = \frac{e^{-\pi i N/12}}{\nu(c_0)}\chi_{4_1}^{(N)}(0) \tag{4.39}$$

Similarly let $X$ be represented by the diagram



that is, the H–triangulation for the $5_2$ knot. We denote $T_0, T_1, T_2, T_3$ the central, left, right, and top tetrahedra respectively and we choose the orientation so that the central tetrahedron $T_0$ is negative then all other tetrahedra are positive. The edge representing the knot $5_2$ connects the last two edges of $T_0$, so that the weight on the knot is given by $2\pi a_0$. In the limit $a_0 \to 0$, all edges, except for the knot, become balanced under the conditions

$$a_1 = c_2 = a_3, \quad b_3 = c_1 + b_2,$$

which in particular imply (4.35). It is really simple to see that the insertion of the partition function for $T_0$, i.e.

$$\langle x | Z_{\mathrm{b}}^{(N)}(T_0) | x' \rangle$$

into the expression of the function $f_X(x)$ defined in equation (4.36) produces

$$\begin{aligned} Z_{\mathrm{b}}^{(N)}(X) &= \frac{e^{-\pi i N/12}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}} a - c_{\mathrm{b}}\sqrt{N})} \nu_{b_0,c_0} f_X(0) && (4.40) \\ &= \Theta \frac{e^{-\pi i N/12}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}} a - c_{\mathrm{b}}\sqrt{N})} \chi_{5_2}^{(N)}(c_{\mathrm{b}}(a_1 - a_3)) && (4.41) \end{aligned}$$

For some constant phase factor $\Theta$

### 4.4.4 Asymptotic's of $\chi_{4_1}^{(N)}(0)$

In this section we want to study the asymptotic behavior of the invariant of the figure–8 knot

$$\begin{aligned} \chi_{4_1}^{(N)}(0) &= \int_{\mathbb{A}_N} \mathrm{D}_{\mathrm{b}}(-x, -k) \overline{\mathrm{D}_{\mathrm{b}}(x, -k)} \mathrm{d}(x, k) \\ &= \frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R} - id} \mathrm{D}_{\mathrm{b}}\left(\frac{-x}{2\pi \mathrm{b}}, -k\right) \overline{\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi \mathrm{b}}, -k\right)} \mathrm{d}x \end{aligned}$$

when $\mathrm{b} \to 0$. The analysis uses techniques similar to the one presented in [AK14b] for $N = 1$, however higher level gives new informations that we will show here.
The integration in the complex plane is a contour integral, where $d \in \mathbb{R}$ is so that the integral is absolutely convergent. Since $\mathrm{b} \in \mathbb{R}_{>0}$ we have

$$\left| \mathrm{D}_{\mathrm{b}}\left(\frac{-x}{2\pi \mathrm{b}}, -k\right) \overline{\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi \mathrm{b}}, -k\right)} \right| \approx e^{\mathrm{b}^{-1} |\operatorname{Re} x| \operatorname{Im} x}$$

as $x \to \pm\infty$, which requires $d$ to be strictly positive. By means of the asymptotic formula for the quantum dilogarithm (2.43) we have

$$\begin{aligned} \chi_{4_1}^{(N)}(0) = & \frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R} - id} \operatorname{Exp}\left[ \frac{\operatorname{Li}_2(-e^{-\sqrt{N}x}) - \operatorname{Li}_2(-e^{\sqrt{N}x})}{2\pi i \mathrm{b}^2 N} \right] \\ & \times \phi_{-x}(k) \overline{\phi_x(k)} (1 + \mathcal{O}(\mathrm{b}^2)) \mathrm{d}x \end{aligned}$$

We want to apply the steepest descent method to this integral to get an asymptotic formula for $\mathrm{b} \to 0$. First we show the computation for the exact integral,

$$\frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} \mathrm{Exp}\left[\frac{\mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})}{2\pi i \mathrm{b}^2 N}\right] \phi_{-x}(k)\overline{\phi_x(k)}\mathrm{d}x \quad (4.42)$$

and then we will argue that the former one can be approximated by the latter when $\mathrm{b} \to 0$.

Let $h(x) := \mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})$. Its critical points are solutions to

$$\begin{cases} h'(x) = 0 \\ h''(x) \neq 0 \end{cases}$$

are $\mathcal{S} = \{\pm\frac{2}{3}\frac{\pi i}{\sqrt{N}} + \frac{2\pi i k}{\sqrt{N}} : k \in \mathbb{Z}\}$. We can compute $\mathrm{Im}\, h$ at its critical points:

$$\mathrm{Im}\, h\left(\pm\frac{2}{3}\frac{\pi i}{\sqrt{N}} + \frac{2\pi i k}{\sqrt{N}}\right) = \pm 4\Lambda(\frac{\pi}{6}) \quad (4.43)$$

where $\Lambda$ is the Lobachevsky's function

$$\Lambda(\alpha) = -\int_0^\alpha \log|2\sin\varphi|\,\mathrm{d}\varphi \quad (4.44)$$

and we used the following equations for the computation (see [Kir95] for all this expressions)

$$\Lambda(2\theta) = 2\left(\Lambda(\theta) - \Lambda(\frac{\pi}{2} - \theta)\right) \quad (4.45)$$

$$\mathrm{Re}\,\mathrm{Li}_2(re^{i\alpha}) = -\frac{1}{2}\int_0^r \frac{\log(1 - 2x\cos(\alpha) + x^2)}{x}\mathrm{d}x \quad (4.46)$$

$$\mathrm{Im}\,\mathrm{Li}_2(re^{i\alpha}) = \beta\log r + \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\alpha+\beta) \quad (4.47)$$

where

$$\beta = \beta(r,\alpha) = \arctan\left(\frac{r\sin\alpha}{1-\cos\alpha}\right) \quad (4.48)$$

We remark here the well known fact that $4\Lambda(\frac{\pi}{6}) = \mathrm{Vol}(4_1)$, where by $\mathrm{Vol}(4_1)$ we mean the hyperbolic volume of knot complement $S^3 \setminus (4_1)$.

Fix $\mathbb{C} \ni x_0 = -\frac{2}{3}\frac{\pi i}{\sqrt{N}}$, which is accessible from the original contour without passing through other critical points, and consider the contour

$$\mathcal{C} = \{z \in \mathbb{C} : \mathrm{Re}(h(z)) = \mathrm{Re}(h(x_0)),\ \mathrm{Im}(h(z)) \leq \mathrm{Im}(h(x_0))\}$$

Using the following properties for the dilogarithm

$$\lim_{\mathrm{Re}(z)\to\infty} \mathrm{Li}_2(-e^z) = -\frac{z^2}{2} \quad (4.49)$$

$$\mathrm{Li}_2(-e^z) + \mathrm{Li}_2(-e^{-z}) = -\frac{1}{2}z^2 - \frac{\pi^2}{6} \qquad \text{for } z \in \mathbb{C}\setminus\{\mathrm{Re}(z) = 0, |\mathrm{Im}(z)| > \pi\} \quad (4.50)$$

one can see that the contour $\mathcal{C}$ is asymptotic to $\mathrm{Re}(z) + \mathrm{Im}(z) = 0$ for $\mathrm{Re}(z) \to \infty$ and to $\mathrm{Re}(z) - \mathrm{Im}(z) = 0$ for $\mathrm{Re}(z) \to -\infty$. Moreover the same computation shows

$$\lim_{\mathrm{Re}(z)\to\pm\infty} \mathrm{Im}(h(z)) = \lim_{\mathrm{Re}(z)\to\pm\infty} \pm\,\mathrm{Re}(z)\,\mathrm{Im}(z) = -\infty \quad (4.51)$$

All together we have found a contour $\mathcal{C}$ where integral (4.42 )can be computed with the steepest descent method (see [Won01]), giving as the following approximation for $\mathrm{b} \to 0$

$$e^{\frac{h(x_0)}{2\pi i \mathrm{b}^2 N}} \frac{g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)}{\sqrt{iN^{-1}h''(x_0)}}(1 + \mathcal{O}(\mathrm{b}^2) \quad (4.52)$$

where

$$g_{4_1}(x) := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \phi_{-x}(k)\overline{\phi}_x(k).$$

We now go back to $\chi_{4_1}^{(N)}(0)$, and we write it as the following integral

$$\chi_{4_1}^{(N)}(0) = \frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} f_{\mathrm{b}}(x,k)\mathrm{d}(x,k) \tag{4.53}$$

where

$$f_{\mathrm{b}}(x,k) = \mathrm{D}_{\mathrm{b}}\left(\frac{-x}{2\pi\mathrm{b}}, -k\right) \overline{\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi\mathrm{b}}, -k\right)}. \tag{4.54}$$

Then consider the contour

$$\mathcal{C}_{\mathrm{b}} = \{z \in \mathbb{C} : \arg f_{\mathrm{b}}(z) = \arg f_{\mathrm{b}}(z_{\mathrm{b}}),\ |f_{\mathrm{b}}(z)| = |f_{\mathrm{b}}(z_{\mathrm{b}})|\} \tag{4.55}$$

where $z_{\mathrm{b}}$ is defined as the solution to

$$\frac{\partial}{\partial x} \log f_{\mathrm{b}}(x) = 0 \tag{4.56}$$

which minimize the absolute value of $f_{\mathrm{b}}$. Using the asymptotic formula for $f_{\mathrm{b}}$ it is simple to show that the contours $\mathcal{C}_{\mathrm{b}}$ approximates $\mathcal{C}$ as $\mathrm{b} \to 0$ as well the points $z_{\mathrm{b}}$'s will converge to $x_0$. So, in the limit $\mathrm{b} \to 0$, integral (4.53) is approximated by the integral (4.42), for which we already have an asymptotic formula. We have proved the following

$$\chi_{4_1}^{(N)}(0) = e^{\frac{h(x_0)}{2\pi i \mathrm{b}^2 N}} \frac{g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)}{\sqrt{iN^{-1}h''(x_0)}}(1 + \mathcal{O}(\mathrm{b}^2)), \tag{4.57}$$

As we remarked above $\operatorname{Im} h(x_0) = -\operatorname{Vol}(4_1)$.
Next we look at the number $g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)$ which is a topological invariant of the knot in the formula above.

$$\begin{aligned}
\sqrt{N} g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right) &= \sum_{k=1}^{N} \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(k)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(k) \\
&= \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1) \sum_{k=1}^{N} \prod_{j=1}^{k-1} \frac{\left|1 - e^{-\frac{1}{3}\frac{\pi i}{N}} e^{\frac{-2\pi i j}{N}}\right|^2}{\left|1 + e^{\frac{2}{3}\pi i}\right|^{\frac{2}{N}}} \\
&= \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1) \sum_{k=0}^{N-1} \prod_{j=1}^{k} \frac{\left|1 - e^{-\frac{1}{3}\frac{\pi i}{N}} e^{\frac{-2\pi i j}{N}}\right|^2}{\left|1 + e^{\frac{2}{3}\pi i}\right|^{\frac{2}{N}}} \\
&= \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1) \sum_{k=0}^{N-1} \prod_{j=1}^{k} \frac{\left|1 - e^{-\frac{1}{3}\frac{\pi i}{N}} e^{\frac{-2\pi i j}{N}}\right|^2}{\left|e^{\pi i/3}\right|^{\frac{2}{N}}} \\
&= \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1) \sum_{k=0}^{N-1} \prod_{j=1}^{k} \left|1 - e^{\frac{1}{3}\frac{\pi i}{N}} e^{\frac{2\pi i j}{N}}\right|^2 \\
&= \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1) \frac{\left|1 - e^{\frac{\pi i}{3}}\right|^2}{\left|1 - e^{\frac{\pi i}{3N}}\right|^2} \sum_{k=0}^{N-1} \prod_{j=1}^{k} \frac{1}{\left|1 - e^{\frac{1}{3}\frac{-\pi i}{N}} e^{\frac{2\pi i j}{N}}\right|^2} \\
&= \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(1) \frac{1}{\left|1 - e^{\frac{\pi i}{3N}}\right|^2} \sum_{k=0}^{N-1} \prod_{j=1}^{k} \frac{1}{\left|1 - e^{\frac{1}{3}\frac{-\pi i}{N}} e^{\frac{2\pi i j}{N}}\right|^2}
\end{aligned}$$

$$= \left| \prod_{j=1}^{N-1} \left(1 - e^{-\frac{\pi i}{3N}} e^{-\frac{2\pi i j}{N}}\right)^{\frac{j}{N}} \right| \sum_{k=0}^{N-1} \prod_{j=1}^{k} \frac{1}{\left|1 - e^{\frac{1}{3}\frac{-\pi i}{N}} e^{\frac{2\pi i j}{N}}\right|^2}$$

The last expression makes possible the following remark

$$g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right) = \gamma_N \mathcal{H}_N^0(\overline{\rho_{comp}}) \tag{4.58}$$

where $H_N^0(\overline{\rho_{comp}})$ is the Baseilhac–Benedetti invariant for the figure–eight knot found in [BB07], computed at the conjugate of the complete hyperbolic structure (meaning that the holonomies of the structure are all complex conjugated) and $\gamma_N$ is a global rescaling

$$\gamma_N = \left| \prod_{j=1}^{N-1} \left(1 - e^{-\frac{2\pi i j}{N}}\right)^{\frac{j}{N}} \right| \tag{4.59}$$

*Remark* 4.4.1. The very same steps of the previous asymptotic computation for $\chi_{4_1}^{(N)}(0)$ can be applied to $\chi_{5_2}^{(N)}(0)$ up to the point of having an expression

$$\chi_{5_2}^{(N)}(0) = e^{\frac{\phi(x_{5_2})}{2\pi i \mathrm{b}^2 N})} \frac{g_{5_1}(x_{5_2})}{\sqrt{iN^{-1}h''_{5_2}(x_{5_2})}}(1 + \mathcal{O}(\mathrm{b}^2)), \tag{4.60}$$

where $x_{5_2}$ is the only critical point in the complex plane that contributes to the steepest descent and

$$g_{5_1}(x) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{\phi_{-x}}(j)\phi_x(j)\overline{\phi_{-x}}(j). \tag{4.61}$$

The fact that $\operatorname{Im}\phi(x_{5_2}) = -\operatorname{Vol}(5_2)$, can be seen directly, see for example [AK14b]. However this situation is already too much complicated to come up with some relations with other theories. The obvious guess is to look for the Baseilhac–Benedetti invariant, but no explicitly computed examples, other then $4_1$, of such invariant are known to the author of this thesis.

The following conjecture was originally stated in [AK14b] for $N = 1$. Here we restate it in the updated setting. See remark 4.4.2 for some less cautious conjecture.

**Conjecture 54** ([AK14b]). *Let $M$ be a closed oriented compact 3-manifold. For any hyperbolic knot $K \subset M$, there exist a two paramters $(\mathrm{b}, N)$ family of smooth functions $J_{M,K}^{(\mathrm{b},N)}(x,j)$ on $\mathbb{R} \times \mathbb{Z}/N\mathbb{Z}$ which has the following properties.*

1. *For any fully balanced shaped ideal triangulation $X$ of the complement of $K$ in $M$, there exist a gauge invariant real linear combination of dihedral angles $\lambda$, a (gauge non-invariant) real quadratic polynomial of dihedral angles $\phi$ such that*

$$Z_{\mathrm{b}}^{(N)}(X) = e^{ic_{\mathrm{b}}^2\phi} \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \int_{\mathbb{R}} J_{M,K}^{(\mathrm{b},N)}(x,j) e^{ic_{\mathrm{b}}x\lambda} \mathrm{d}x$$

2. *For any one vertex shaped H-triangulation $Y$ of the pair $(M, K)$ there exists a real quadratic polynomial of dihedral angles $\varphi$ such that*

$$\lim_{\omega_Y \to \tau} \mathrm{D_b}\left(c_{\mathrm{b}} \frac{\omega_Y(K) - \pi}{\pi\sqrt{N}}, 0\right) Z_{\mathrm{b}}^{(N)}(Y) = e^{ic_{\mathrm{b}}^2\varphi - i\frac{\pi N}{12}} J_{M,K}^{(\mathrm{b},N)}(0,0),$$

*where $\tau \colon \Delta_1(Y) \to \mathbb{R}$ takes the value $0$ on the knot $K$ and the value $2\pi$ on all other edges.*

3. *The hyperbolic volume of the complement of $K$ in $M$ is recovered as the following limit:*
$$\lim_{\mathrm{b}\to 0} 2\pi \mathrm{b}^2 N \log |J_{M,K}^{(\mathrm{b},N)}(0,0)| = -\operatorname{Vol}(M \setminus K)$$

*Remark* 4.4.2. We have proved this extended conjecture for the knots $(S^3, 4_1)$ and $(S^3, 5_2)$, see formulas (4.58), (4.60) and (4.61). Moreover we gave a more explicit expansion, showing the appearance of an extra interesting therm $g_K$, and showed a precise relation between $g_{4_1}$ and a known invariant of hyperbolic knots, defined by Baseilhac–Benedetti in [BB07], see equation (4.58). We could have been less cautious and extend the conjecture declaring the appearance og $g_{(M,K)}$ to be general, and it to be proportional to the Baseilhac–Benedetti invariant. However we feel that there are not enough evidence to state it as general conjecture. We finally remark that the Baseilhac–Benedetti invariants are considered to be a generalization of the Kashaev invariants, in the sense that they depend also to choice of a flat $\mathrm{PSL}(2, \mathbb{C})$ connection. The level $N$ Andersen–Kashaev invariant could be interpreted as further quantization of this choice (meaning that we are quantizing also over the space of all possible $\mathrm{PSL}(2, \mathbb{R})$ flat connections). From this point of view, it is not unbelievable that the BB–invariant at the complete hyperbolic structure appear in the classical limit of the AK–invariant. The fact that appears as complex conjugate structure could be due to some different choice in the many conventions one needs to make along the process.

# Chapter 5

# Genus 1 $\mathbf{SL}(2,\mathbb{C})$ Chern-Simons Theory

The main purpose of this Chapter is to compute the representations of the Mapping Class Group of a genus 1 surface defined via *complex quantum Chern–Simons theory*, as suggested in [Wit91], and [AG14]. These are admittedly not interesting per se, as the mapping class group of a genus 1 surface is well known to be linear and isomorphic to $\mathrm{SL}(2,\mathbb{Z})$, however we hope this could be a first step to get an idea on what representations in higher genus look like. Indeed it is a general problem to find out if higher genus mapping class group are linear, and Chern–Simons theory already provided an asymptotic positive answer [And06].
On the other hand studying mapping class group representations is one of the first steps in order to get an interpretation of complex Chern–Simons theory similar to the modular functor formulation for compact Chern–Simons theory. This is still far from being accomplished.
In Section 5.1 we recall the construction for complex and compact Chern–Simons theory and their quantizations following the description and ideas in [AG14]and [Wit91].
In Section 5.2 we do all the constructions in elementary details in genus 1. This second section does not really need the previous one if not as motivation to use certain definitions, and many arguments are treated more elementarily then in the general setting as the genus 1 is indeed more simple. Some of the discussion follows ideas in [Wit91], but the computation of the mapping class group was not given there, while we are able to do so, thanks to an explicit description provided by the Weil-Gel'fand-Zak Transorm, see Section 1.2

## 5.1 The Hitchin–Witten Connection and Complex Chern–Simons Theory

In this section we want to recall the general setting of (Complex) Chern-Simons Theory, together with the main results and open problems related to its quantization. Most of the material here is taken from [ADPW91], [Wit91], [Fre95] and, mostly, [AG14].

### 5.1.1 Moduli Spaces of Flat Connections

Fix $n \geq 2$. Let $G$ be the compact Lie group $\mathrm{SU}(n)$, and its complexification $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. Denote with $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{C}}$) the Lie algebra of $G$ (resp. $G_{\mathbb{C}}$). Namely let $\mathfrak{g} = \mathfrak{su}(n)$ be the Lie algebra of traceless skew-Hermitian matrices and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ be the Lie algebra of traceless matrices. Let $\langle \cdot, \cdot \rangle \colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{R}$ be an invariant inner product on $\mathfrak{g}_{\mathbb{C}}$. In the setting we are discussing, after normalization, we can fix $\langle A, B \rangle \equiv -\frac{1}{8\pi^2} \mathrm{Tr}(AB)$. Fix an integer number $g \geq 2$ denoting the genus of a closed surface $\Sigma$, and fix a point $p \in \Sigma$. Denote by $\Sigma_p$ the surface punctured at $p$. Let $d \in \mathbb{Z}/n\mathbb{Z}$. Consider the following two moduli spaces of representations.

$$M \equiv \mathrm{Hom}_d(\pi_1(\Sigma_p), G)/\sim \tag{5.1}$$

$$M_{\mathbb{C}} \equiv \mathrm{Hom}^+_d(\pi_1(\Sigma_p), G_{\mathbb{C}})/\sim \tag{5.2}$$

where the $d$ means that we are mapping the representative of a small loop $\gamma$ around the puncture $p$ to the group element $e^{2\pi i d/n}\mathrm{Id}$. The $+$ in the definition of $M_{\mathbb{C}}$ indicates that only reductive representations are considered. The subspaces $M' \subset M'_{\mathbb{C}}$, i.e. the restriction of $M$ and $M_{\mathbb{C}}$ to the irreducible representations, are embedded one into the other. We have a gauge theoretic description of the spaces $M$ and $M_{\mathbb{C}}$, thanks to the Holonomy Theorem 86. Let $P$ (resp. $P_{\mathbb{C}}$) denote a trivial principal $G$ (resp. $G_{\mathbb{C}}$) bundle over $\Sigma_p$. Fix $a_d \in \mathfrak{g}$ such that $\exp(2\pi a_d) = e^{2\pi i d/n}\mathrm{Id}$ and denote by $\mathcal{F}$ (resp. $\mathcal{F}^+_{\mathbb{C}}$) the space of flat connections on $P$ (resp. reductive flat connections on $P_{\mathbb{C}}$) which have expression $a_d \mathrm{d}\theta$ near the puncture $p$, where $\theta$ is some fixed local angular coordinate centered in the puncture $p$. Let $\mathcal{G}$ (resp. $\mathcal{G}_{\mathbb{C}}$) be the group of gauge transformations of $P$ (resp. $P_{\mathbb{C}}$), which restricts to the identity near the puncture $p$. We have

$$M = \mathcal{F}/\mathcal{G} \tag{5.3}$$

$$M_{\mathbb{C}} = \mathcal{F}^+_{\mathbb{C}}/\mathcal{G}_{\mathbb{C}}. \tag{5.4}$$

With some cautious choice of gauge transformations and connections allowed near the puncture, see [DW97], $M'$ and $M'_{\mathbb{C}}$ have the structure of smooth manifolds. The tangent spaces to $M'$ and $M'_{\mathbb{C}}$ at an irreducible flat connection $[A]$, are

$$T_{[A]}M' = \mathrm{H}^1(\Sigma_p, \mathfrak{g}, \mathrm{d}_A) \qquad\qquad T_{[A]}M'_{\mathbb{C}} = \mathrm{H}^1_c(\Sigma_p, \mathfrak{g}_{\mathbb{C}}, \mathrm{d}_A) \tag{5.5}$$

where the differential $\mathrm{d}_A$ is defined by $\mathrm{d}_A\alpha = \mathrm{d}\alpha + [A \wedge \alpha]$ and $\mathrm{H}^*_c$ is the homology of differential forms with compact support.

### 5.1.2 Atiyah–Bott Symplectic Form and Pre–Quantization

Fix a complex number $t = k + is \in \mathbb{C}^*$, where $k, s \in \mathbb{R}$. Consider the following 2-form on $M'_{\mathbb{C}}$

$$\omega_{\mathbb{C}}([A], [B]) \equiv 4\pi \int_{\Sigma} \langle A \wedge B \rangle = -\frac{1}{2\pi} \int_{\Sigma_p} \mathrm{Tr}(A \wedge B) \tag{5.6}$$

for all $A$, $B \in \Omega(\Sigma_p, \mathfrak{g}_{\mathbb{C}})$. We will discuss the quantization of $M_{\mathbb{C}}$ with respect to the following real symplectic form

$$\omega_t \equiv \frac{t}{2}\omega_{\mathbb{C}} + \frac{\bar{t}}{2}\overline{\omega}_{\mathbb{C}}. \tag{5.7}$$

The same expression as in (5.6), on $M'$ instead of $M'_{\mathbb{C}}$, gives a real 2-form $\omega$. The analogous of formula (5.7) gives the symplectic form $\omega_k = k\omega$ on $M'$.

To construct the Chern–Simons line bundle first we introduce the Chern–Simons 3-form. Consider the cylinder $\Sigma_p \times [0,1]$ and the projection $\pi\colon \Sigma_p \times [0,1] \longrightarrow \Sigma_p$. Denote with $\tilde{A} \equiv \pi^* A$ the pull–back 1–form and, for every $g \in \mathcal{G}$ (resp. $\mathcal{G}_{\mathbb{C}}$), let $\tilde{g}\colon \Sigma_p \times [0,1] \longrightarrow G$ (resp. $G_{\mathbb{C}}$) be an homotopy from the identically trivial gauge map on $\Sigma_p$ to the gauge transformation $g$ (homotopies are required to keep identity values near the puncture $p$ fixed). We define the Chern–Simons 3-form

$$\alpha(\tilde{A}) \equiv \langle \tilde{A} \wedge F_{\tilde{A}} \rangle - \frac{1}{6} \langle \tilde{A} \wedge \left[ \tilde{A} \wedge \tilde{A} \right] \rangle \tag{5.8}$$

where $F_{\tilde{A}}$ is the curvature of $\tilde{A}$.

**Lemma 55.** *Let $\theta \in \Omega^1(G, \mathfrak{g})$ (or $\Omega^1(G_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$) be the Maurer–Cartan form, and define $\theta_{\tilde{g}} \equiv \tilde{g}^* \theta$. Recall the pull-back formula for a connection $\tilde{A}^{\tilde{g}} = \tilde{g}^{-1} \tilde{A} \tilde{g} + \theta_{\tilde{g}}$. The Chern–Simons form $\alpha$ satisfies the following relations*

$$\mathrm{d}\alpha(\tilde{A}) = \langle F_{\tilde{A}} \wedge F_{\tilde{A}} \rangle \tag{5.9}$$

$$\tilde{g}^* \alpha(\tilde{A}) = \alpha(\tilde{A}^{\tilde{g}}) = \alpha(\tilde{A}) + \mathrm{d}\langle \tilde{g}^{-1} \tilde{A} \tilde{g} \wedge \theta_{\tilde{g}} \rangle - \frac{1}{6} \langle \theta_{\tilde{g}} \wedge [\theta_{\tilde{g}} \wedge \theta_{\tilde{g}}] \rangle \tag{5.10}$$

*Remark* 5.1.1. With our choice of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$, the form $\theta_{\tilde{g}}$ is explicitly $\tilde{g}^{-1} \mathrm{d}\tilde{g}$.

In the complex theory (i.e. when we use $\mathfrak{g}_{\mathbb{C}}$ valued forms) we will need to take a real form from (5.8) as we did in (5.7) for $\omega_{\mathbb{C}}$

$$\alpha_t(\tilde{A}) \equiv \frac{t}{2} \alpha(\tilde{A}) + \frac{\bar{t}}{2} \overline{\alpha(\tilde{A})}. \tag{5.11}$$

Again in the real setting $\alpha_t = k\alpha$. We define the Chern–Simons cocycle as follows

$$\Theta_t(A, g) \equiv \exp\left( 2\pi i \int_{\Sigma \times [0,1]} \alpha_t(\tilde{A}^{\tilde{g}}) - \alpha_t(\tilde{A}) \right) \tag{5.12}$$

*Remark* 5.1.2. Thanks to (5.10) we have

$$\begin{aligned} \int_{\Sigma \times [0,1]} \alpha(\tilde{A}^{\tilde{g}}) - \alpha(\tilde{A}) = \\ &= \int_{\Sigma \times [0,1]} \mathrm{d}\langle \tilde{g}^{-1} \tilde{A} \tilde{g} \wedge \theta_{\tilde{g}} \rangle - \frac{1}{6} \langle \theta_{\tilde{g}} \wedge [\theta_{\tilde{g}} \wedge \theta_{\tilde{g}}] \rangle \\ &= \int_{\Sigma} \langle g^{-1} A g \wedge \theta_g \rangle - \frac{1}{6} \int_{\Sigma \times [0,1]} \langle \theta_{\tilde{g}} \wedge [\theta_{\tilde{g}} \wedge \theta_{\tilde{g}}] \rangle \end{aligned}$$

So, if we define

$$c(A, g) \equiv \int_{\Sigma} \langle g^{-1} A g \wedge \theta_g \rangle - \frac{1}{6} \int_{\Sigma \times [0,1]} \langle \theta_{\tilde{g}} \wedge [\theta_{\tilde{g}} \wedge \theta_{\tilde{g}}] \rangle, \tag{5.13}$$

we have

$$\Theta_t(A, g) = \exp 2\pi i \left( \frac{t}{2} c(A, g) + \overline{\frac{t}{2} c(A, g)} \right) \tag{5.14}$$

By analogy with previous notation, in the real theory we write

$$\Theta_k(A, g) \equiv \exp 2\pi i k c(A, g). \tag{5.15}$$

**Lemma 56** (Cocycle Condition)**.**

$$\Theta_t(A, gh) = \Theta_t(A^g, h) \Theta_t(A, g) \tag{5.16}$$

Define the trivial line bundle $\widetilde{\mathcal{L}} \to \mathcal{F}$ (resp. $\widetilde{\mathcal{L}}_{\mathbb{C}} \to \mathcal{F}_{\mathbb{C}}^{+}$). Using the cocycle $\Theta_k$ (resp. $\Theta_t$) we can lift the action of $\mathcal{G}$ (resp. $\mathcal{G}_{\mathbb{C}}$) to $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{L}}_{\mathbb{C}}$), as

$$(A, z) \cdot g \equiv (A^g, \Theta_t(A, g)z) \,. \tag{5.17}$$

This action defines an Hermitian line bundle $\mathcal{L}^k$ (resp. $\mathcal{L}^t$) over $M'$ (resp. $M'_{\mathbb{C}}$). Next we need a pre-quantum connection $\nabla^{(t)}$ on this bundle. This is given by the following formulas

$$\nabla^{(t)} \equiv \mathrm{d} - i\beta_t, \qquad \beta_t \in \Omega^1(\mathcal{F}_{\mathbb{C}}^{+}, \widetilde{\mathcal{L}}^t) \tag{5.18}$$

$$\beta_t(A') \equiv 2\pi \int_{\Sigma} t\langle A \wedge A'\rangle + \overline{t\langle A \wedge A'\rangle}, \qquad \forall\, A' \in T_A\mathcal{F}_{\mathbb{C}}^{+} \tag{5.19}$$

The definition of $\nabla^{(k)}$ on $\mathcal{L}^k$ is the straightforward analogous of the definitions above, which follows when $s = 0$. Finally we have

$$F_{\nabla^{(t)}} = -i\omega_t \qquad\qquad F_{\nabla^{(k)}} = -ik\omega \tag{5.20}$$

In both the situations, the pre-quantization condition produces the constraint $k \in \mathbb{Z}$. So we have finished the description of the pre-quantum line bundle with pre-quantum connection. In the next subsection we will describe how one could choose polarizations in the two cases.

### 5.1.3 Polarizations

For general reference to the complex geometry involved in the following discussion we refer to [Wel08]. We are not going into many details here, as we won't use the general results we present in our detailed discussion of the genus 1. Instead we will compute everything from scratches in that setting. At this point the discussion for the real and complex theory becomes different.
Let us start with $\mathcal{L}^k \to M'$. Choose a complex structure $\sigma \in \mathcal{T}$ on the surface. Here $\mathcal{T}$ is the Teichmüller space of the surface $\Sigma$ which parametrizes its complex structures. $\sigma$ defines an *Hodge star* operator $*\colon \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ as follows. Let $g_\sigma$ be the metric induced over $\Sigma$. For every $\alpha, \beta \in \Omega^1(\Sigma, \mathbb{C})$ define $*\alpha$ via the formula

$$*\alpha \wedge \beta = \langle \alpha, \beta\rangle_\sigma \mathrm{d}vol \tag{5.21}$$

where $\langle\cdot,\cdot\rangle_\sigma$ is an inner product in $T^*\Sigma$ induced by $g_\sigma$, and $\mathrm{d}vol$ is the volume form. In general, we have

$$*^2 = -1$$

Define

$$\begin{aligned} J_\sigma &\colon TM' \longrightarrow TM' \\ J_\sigma &= -* \end{aligned} \tag{5.22}$$

where we recall the identification of tangent spaces (5.5) so that an appropriately extended $*$ is a well defined complex structure on $TM'$. In this way one has a family of complex structures $J$ over $M'$, parametrized by the Teichmüller space $\mathcal{T}$, which can be all shown to satisfy the hypothesis of a Lagrangian polarization for $M'$. Taking $\mathcal{H}_\sigma^{(k)}$ as the space of holomorphic sections of the line bundle with respect to $J_\sigma$ one organizes this in a vector bundle

$$\mathcal{P}^{(k)} \times \mathcal{T} \longrightarrow \mathcal{T} \tag{5.23}$$

where $\mathcal{P}^{(k)} = \mathrm{C}^\infty(M', \mathcal{L}^k)$. It is a result of [ADPW91] and [Hit90] that this vector bundle admits a connection, usually referred as *Hitchin connection*, preserving the sub-bundle $\left(\mathcal{H}_\sigma^{(k)}, \sigma\right) \to \mathcal{T}$ and that is projectively flat (see also [And12] for a generalized setting). In this way one can define a unique quantum space of states $\mathcal{H}^{(k)}$ up to scalar factor multiplication, identifying the different $\mathcal{H}_\sigma^{(k)}$ by parallel transport through Hitchin connection.
We now switch our attention to $\mathcal{L}^t \to M'_{\mathbb{C}}$. The Hodge $*$ can be used to define a complex structure $J_\sigma$ on $T_{[A]}M'_{\mathbb{C}}$ in the same way as before. On the other hand $T_{[A]}M'_{\mathbb{C}} = \mathrm{H}^1_c\left(\Sigma_p, \mathfrak{g}_{\mathbb{C}}, \mathrm{d}_A\right)$ has another complex structure $I$ induced by the natural complex structure of $\mathfrak{g}_{\mathbb{C}}$. We don't want to define a complex polarization, but a real one here. We use $*$ to decompose the cohomology into its $(1,0)$ and $(0,1)$ parts, i.e.

$$\mathrm{H}^1_c\left(\Sigma_p, \mathfrak{g}_{\mathbb{C}}, \mathrm{d}_A\right) = \mathrm{H}^{(1,0)}_c\left(\Sigma_p, \mathfrak{g}_{\mathbb{C}}, \mathrm{d}_A\right) \oplus \mathrm{H}^{(0,1)}_c\left(\Sigma_p, \mathfrak{g}_{\mathbb{C}}, \mathrm{d}_A\right) \tag{5.24}$$

This decomposition follows from usual theory of Riemann surfaces and it depends on the choice of $\sigma$. One can take the space $\mathrm{H}^{(1,0)}_c\left(\Sigma_p, \mathfrak{g}_{\mathbb{C}}, \mathrm{d}_A\right)$ as real Lagrangian polarization. Such space at unitary (real) $A$ is transversal to $\mathrm{H}^1\left(\Sigma_p, \mathfrak{g}, \mathrm{d}_A\right)$ being 0 the only real 1-form of type $(1,0)$. So the polarized sections of $\mathcal{L}^t$ are determined by their value at $M'$. So we can define $\mathcal{H}_\sigma^{(t)} \equiv \mathrm{C}^\infty(M', \mathcal{L}^t)$. Such identification however depends again on the $\sigma \in \mathcal{T}$ so we shall proceed to consider the infinite rank vector bundle

$$\mathrm{C}^\infty_c(M'_{\mathbb{C}}, \mathcal{L}^t) \times \mathcal{T} \longrightarrow \mathcal{T}. \tag{5.25}$$

and its subbundle

$$\mathrm{C}^\infty(M', \mathcal{L}^t) \times \mathcal{T} \longrightarrow \mathcal{T}. \tag{5.26}$$

The first time these polarizations and bundle were considered was in [Wit91]. Witten, in the same work, proved with some infinite dimensional differential geometry arguments that there is an projectively flat connection on (5.25) preserving polarized sections. Such result was later proved in a finite dimensional differential geometry setting in [AG14]. As in the compact case we can use the parallel transport defined by this connection to identify different fibers $(\mathrm{C}^\infty(M', \mathcal{L}^t), \sigma)$ together. We refer to such connection as *Hitchin–Witten connection*, and we will construct it in genus 1 later.

### 5.1.4 Mapping Class Group Representations

The Mapping Class Group $\mathrm{MCG}(\Sigma) = \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^+_0(\Sigma)$, acts naturally on $\pi_1(\Sigma)$ by push froward. This induces an action by pull–backs on $\mathrm{Hom}(\pi_1(\Sigma), G)$. In particular we have a right action by pull-back of the mapping class group on $M'$ and $M'_{\mathbb{C}}$. Similarly $\mathrm{MCG}(\Sigma)$ acts on the right of $\mathcal{T}(\Sigma)$ via pull backs.
The action on $M'$ and $M'_{\mathbb{C}}$ can be lifted to an action on $\mathcal{L}^k$ and $\mathcal{L}^t$ respectively, as, for every $\gamma \in \mathrm{MCG}$,

$$\Theta_t(\gamma^* A, g \circ \gamma) = \Theta_t(A, g). \tag{5.27}$$

Globally this produces an action of the mapping class group on the vector bundle (5.23), so that $\gamma$ will map

$$\tilde{\gamma} \colon \mathcal{H}_\sigma^{(k)} \longrightarrow \mathcal{H}_{\gamma_*\sigma}^{(k)}.$$

In order to have representations of the mapping class group on $\mathcal{H}^{(k)}$ we need to parallel transport the sections in $\mathcal{H}_{\gamma_*\sigma}^{(k)}$ back to $\mathcal{H}_\sigma^{(k)}$. And this is possible thanks to the Hitchin connection. Similar considerations can be done in relation to the bundle (5.26). In the next Section we will look in details of this second situation for the case of a genus 1 surface. We will produce explicit descriptions of the representation into such scenario.

## 5.2 Genus 1

### 5.2.1 Moduli spaces

From now to the end of the section $\Sigma$ is a fixed topological surface of genus 1 identified with the quotient $\mathbb{R}^2/\mathbb{Z}^2$. The fundamental group of $\Sigma$ is well known to be

$$\pi_1(\Sigma, x_0) = \mathbb{Z} \times \mathbb{Z}, \tag{5.28}$$

We want to describe the following two spaces

$$\begin{aligned} M &\equiv \mathrm{Hom}\left(\pi_1(\Sigma),\ \mathrm{SU}(2)\right) / \,\mathrm{SU}(2) \\ M_{\mathbb{C}} &\equiv \mathrm{Hom}^+\left(\pi_1(\Sigma),\ \mathrm{SL}(2,\,\mathbb{C})\right) / \,\mathrm{SL}(2,\,\mathbb{C}) \end{aligned}$$

where the actions of the groups is by conjugation, and the $+$ stays for semi-simple representations. Let $A$ and $B$ be the two generators for $\pi_1(\Sigma) = \mathbb{Z}\times\mathbb{Z}$. Let $[\rho] \in M$. Up to conjugation we can suppose $\rho(A) \in \mathrm{U}(1)$. Since $\rho(A)\rho(B) = \rho(B)\rho(A)$, if $\rho(A) \neq \pm\mathrm{Id}$, it follows that $\rho(B) \in$U(1) as well. If $\rho(A) = \pm\mathrm{Id}$, then the conjugation action of SU(2) can be used to bring $\rho(B) \in \mathrm{U}(1)$. We can still conjugate by the element $R \in$SU(2) represented by the matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad R^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} R = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}. \tag{5.29}$$

We have just showed that

$$M = \left(\mathbb{T}\times\mathbb{T}\right)/\mathbb{S}_2 \tag{5.30}$$

where $\mathbb{S}_2$ denotes the group of order 2, $\mathbb{S}_2 = \{R, \mathrm{Id}\}$ and $\mathbb{T} \equiv \mathrm{U}(1)$. When we move from $M$ to $M_{\mathbb{C}}$ the main ingredient of the analysis, the commutativity of $\pi_1(\Sigma)$, remains and the discussion follows similarly. Let $[\eta] \in M_{\mathbb{C}}$. Up to conjugation $\eta(A)$ is either diagonal, or parabolic

$$\eta(A) = \begin{pmatrix} \lambda_A & 0 \\ 0 & \lambda_A^{-1} \end{pmatrix} \qquad \text{or} \qquad \eta(A) = \begin{pmatrix} 1 & \sigma_A \\ 0 & 1 \end{pmatrix}.$$

In the first case, $\eta(A)\eta(B) = \eta(B)\eta(A)$ implies that $\eta(B)$ is diagonal as well. In the second case we get $\eta(B)$ to be in the same unipotent subgroup of $\eta(A)$. This second case produces non semi-simple representations so we won't consider them. We have the following identification

$$M_{\mathbb{C}} = \left(\mathbb{C}^*\times\mathbb{C}^*\right)/\mathbb{S}_2 \tag{5.31}$$

where the action of $\mathbb{S}_2$ is as in (5.29).

*Remark* 5.2.1. There is a natural inclusion of $M \subset M_{\mathbb{C}}$. Obviously $\mathbb{T}\times\mathbb{T} \subset \mathbb{C}^*\times\mathbb{C}^*$. The action by conjugation of $\mathbb{S}_2$ has 4 fixed points on $\mathbb{C}^*\times\mathbb{C}^*$, namely $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$, which are all unitary points, and is a double cover on the rest which preserves unitarity.

Let $(x, y) \in \mathbb{R}^2$ be local coordinates on $\Sigma$ under the identification $\Sigma = \mathbb{R}^2/\mathbb{Z}^2$. The gauge theoretic description of $M_{\mathbb{C}}$ (and $M$) is as follows. Consider the space $\mathcal{W}_{\mathbb{C}}$ of $\mathfrak{sl}(2,\mathbb{C})$ valued connections on $\Sigma$, i.e.

$$\mathcal{W}_{\mathbb{C}} \ni A = A_1 \mathrm{d}x + A_2 \mathrm{d}y, \qquad A_1,\ A_2 \in \mathrm{C}^{\infty}(\Sigma, \mathfrak{sl}(2,\mathbb{C})).$$

Consider the subspace $\mathcal{A}_{\mathbb{C}} \subseteq \mathcal{W}_{\mathbb{C}}$ of connections

$$A = 2\pi i u T \mathrm{d}x + 2\pi i v T \mathrm{d}y, \qquad u,\ v \in \mathbb{C},\ T \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.32}$$

which are flat and constantly $\mathbb{C}$–valued. Every such $A \in \mathcal{A}_{\mathbb{C}}$ has holonomies

$$\begin{pmatrix} e^{2\pi i u} & 0 \\ 0 & e^{-2\pi i u} \end{pmatrix}, \qquad \begin{pmatrix} e^{2\pi i v} & 0 \\ 0 & e^{-2\pi i v} \end{pmatrix}, \tag{5.33}$$

which are representatives for elements of $M_{\mathbb{C}}$. Since every $[\rho] \in M_{\mathbb{C}}$ can be described in this way, we have an identification of its double cover

$$\mathcal{A}_{\mathbb{C}}/\mathbb{Z}^2 \simeq \mathbb{C}^* \times \mathbb{C}^* \tag{5.34}$$

In coordinates $(u, v)$ the action of $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ is by translations,

$$(u, v) \cdot (\lambda_1, \lambda_2) = (u + \lambda_1, v + \lambda_2) \tag{5.35}$$

while the action of $\mathbb{S}_2$ lifts to $\mathcal{A}_{\mathbb{C}}$ as

$$(u, v) \cdot R = (-u, -v). \tag{5.36}$$

The same kind of description can be given for $M$ by considering the quotient of the space $\mathcal{A}$

$$\mathcal{A} \ni A = 2\pi i u' T \mathrm{d}x + 2\pi i v' T \mathrm{d}y, \qquad u',\ v' \in \mathbb{R},\ T \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.37}$$

by analogous actions as in (5.35) and (5.36).
The coordinates $(u, v)$ on $\mathcal{A}_{\mathbb{C}}$ are complex, so if we want to treat it as a smooth manifold we need to use also their complex conjugate. That means we look at both

$$A = 2\pi i u T \mathrm{d}x + 2\pi i v T \mathrm{d}y \qquad \overline{A} = -2\pi i \overline{u} T \mathrm{d}x - 2\pi i \overline{v} T \mathrm{d}y \tag{5.38}$$

The Teichmüller space for $\Sigma$ is identified with the upper half space [FM12]

$$\mathbb{H} \equiv \{\sigma \in \mathbb{C}, \text{ such that } \operatorname{Im} \sigma > 0\}. \tag{5.39}$$

Let us briefly recall this identification. A rank 2 *lattice* $\Lambda \subset \mathbb{C}$ is an additive subgroup isomorphic to $\mathbb{Z}^2$. We say that the lattice $\Lambda$ is *marked* if it comes with an explicit choice of isomorphism with $\mathbb{Z}^2$ or, equivalently, an ordered set of two generators. The quotient space $\mathbb{C}/\Lambda$ is genus one surface with a complex structure induced by the standard one on $\mathbb{C}$. On the other hand every genus 1 surface $\Sigma = \mathbb{R}^2/\mathbb{Z}^2$ with a complex structure can be endowed with a metric compatible with the complex structure so that the covering $\mathbb{R}^2 \to \Sigma$ is metric. In particular this gives $\mathbb{R}^2$ a complex structure, i.e. an isomorphism $\mathbb{R}^2 \simeq \mathbb{C}$ and we take as lattice $\Lambda$ the group of deck transformations. It can be shown that two marked lattices $\Lambda$ and $\Lambda'$ induce two isotopic complex structures if and only if they are related by a finite sequence

of euclidean isometries or homotheties of $\mathbb{R}^2$. A way to choose a representative for equivalence classes of marked lattices is to fix the first generator of the lattice to be the number $1 \in \mathbb{C}$ and constrains the second generator $\sigma \in \mathbb{C}$ to have $\operatorname{Im}\sigma > 0$.
This way of associating complex structures can be expressed in the following way. Consider the map

$$\begin{aligned}\psi_\sigma \colon \mathbb{R}^2 &\longrightarrow \mathbb{C}\\ (x,y) &\mapsto x + \sigma y,\end{aligned}$$

the map descend to a map $\psi_\sigma \colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}/\Lambda_\sigma$ where $\Lambda_\sigma = \operatorname{Span}_{\mathbb{Z}}\langle 1, \sigma\rangle \subset \mathbb{C}$. Then the complex structure to $\Sigma$ is obtained by pull-back via $\psi_\sigma$.
However for our purposes it is more convenient to use the following alternative map to pull back complex structures

$$\begin{aligned}\phi_\sigma \colon \mathbb{R}^2 &\longrightarrow \mathbb{C}\\ (x,y) &\mapsto x - \sigma^{-1} y,\end{aligned} \tag{5.40}$$

that is, we are parameterizing complex structures with a biholomorphic copy of $\mathbb{H}$, obtained via the biholomorphism $\sigma \mapsto -\sigma^{-1}$.
In this way the complex coordinate $\tilde{w} = x - \sigma^{-1} y$ is holomorphic on the surface, as well any other complex rescaling. Indeed multiplying holomorphic coordinates by complex numbers is equivalent by applying rotations and homotheties to the lattice $\Lambda_{-\sigma^{-1}}$, which produces isotopic complex structures.
For $\sigma \in \mathbb{H}$, we let

$$w = \frac{1}{\overline{\sigma} - \sigma}(y - \sigma x)$$

be our choice of holomorphic coordinate on $\Sigma$, obtained as $w = \frac{\sigma}{\sigma - \overline{\sigma}}\tilde{w}$. We have the following relations

$$\mathrm{d}x = \mathrm{d}w + \mathrm{d}\overline{w} \qquad\qquad \mathrm{d}y = (\overline{\sigma}\mathrm{d}w + \sigma\mathrm{d}\overline{w}) \tag{5.41}$$

and an induced splitting of coordinates on $\mathcal{A}_{\mathbb{C}}$

$$A = A_w T\mathrm{d}w + A_{\overline{w}} T\mathrm{d}\overline{w}, \qquad \overline{A} = \overline{A}_{\overline{w}} T\mathrm{d}\overline{w} + \overline{A}_w T\mathrm{d}w, \tag{5.42}$$

$$A_w \equiv 2\pi i(u + \overline{\sigma} v), \qquad A_{\overline{w}} \equiv 2\pi i(u + \sigma v), \tag{5.43}$$

$$\overline{A}_{\overline{w}} \equiv -2\pi i(\overline{u} + \sigma\overline{v}), \qquad \overline{A}_w \equiv -2\pi i(\overline{u} + \overline{\sigma}\overline{v}), \tag{5.44}$$

and vector fields

$$\frac{\partial}{\partial A_w} = \frac{1}{2\pi i(\overline{\sigma} - \sigma)}\left(-\sigma\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial A_{\overline{w}}} = \frac{1}{2\pi i(\sigma - \overline{\sigma})}\left(-\overline{\sigma}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right), \tag{5.45}$$

$$\frac{\partial}{\partial \overline{A}_w} = \frac{-1}{2\pi i(\overline{\sigma} - \sigma)}\left(-\sigma\frac{\partial}{\partial \overline{u}} + \frac{\partial}{\partial \overline{v}}\right), \quad \frac{\partial}{\partial \overline{A}_{\overline{w}}} = \frac{-1}{2\pi i(\sigma - \overline{\sigma})}\left(-\overline{\sigma}\frac{\partial}{\partial \overline{u}} + \frac{\partial}{\partial \overline{v}}\right). \tag{5.46}$$

Writing $u = u' + iu''$ and $v = v' + iv''$, with $u', u'', v', v'' \in \mathbb{R}$, we get another splitting

$$A = P + iQ \tag{5.47}$$

$$\quad = 2\pi i\left(u'\mathrm{d}x + v'\mathrm{d}y\right) - 2\pi\left(u''\mathrm{d}x + v''\mathrm{d}y\right) \tag{5.48}$$

that shows that $P$ and $Q$ are unitary connections in $\mathcal{A}$ and we can combine the two splittings to get

$$A_w = (P_w + iQ_w) \qquad\qquad A_{\overline{w}} = (P_{\overline{w}} + iQ_{\overline{w}}) \tag{5.49}$$

$$\overline{A}_{\overline{w}} = -(P_{\overline{w}} - iQ_{\overline{w}}) \qquad\qquad \overline{A}_w = -(P_w - iQ_w) \tag{5.50}$$

and respective formulas for vector fields

$$\frac{\partial}{\partial A_w} = \frac{1}{2}\left(\frac{\partial}{\partial P_w} - i\frac{\partial}{\partial Q_w}\right) \qquad\qquad \frac{\partial}{\partial A_{\overline{w}}} = \frac{1}{2}\left(\frac{\partial}{\partial P_{\overline{w}}} - i\frac{\partial}{\partial Q_{\overline{w}}}\right) \tag{5.51}$$

$$\frac{\partial}{\partial \overline{A}_w} = -\frac{1}{2}\left(\frac{\partial}{\partial P_w} + i\frac{\partial}{\partial Q_w}\right) \qquad\qquad \frac{\partial}{\partial \overline{A}_{\overline{w}}} = -\frac{1}{2}\left(\frac{\partial}{\partial P_{\overline{w}}} + i\frac{\partial}{\partial Q_{\overline{w}}}\right) \tag{5.52}$$

$$\tag{5.53}$$

where

$$P_w = 2\pi i(u' + \overline{\sigma}v') \qquad\qquad P_{\overline{w}} = 2\pi i(u' + \sigma v') \tag{5.54}$$

$$Q_w = 2\pi i(u'' + \overline{\sigma}v'') \qquad\qquad Q_{\overline{w}} = 2\pi i(u'' + \sigma v'') \tag{5.55}$$

Let $t = k + is$, and $\overline{t} = k - is$, $k \in \mathbb{R}$, $s \in \mathbb{R}$ be fixed from now on. The Atiyah-Bott symplectic form, defined following (5.6), has the following explicit expressions in coordinates

$$\omega_{\mathbb{C}} = 4\pi \mathrm{d}u \wedge \mathrm{d}v \tag{5.56}$$

$$= \frac{1}{\pi(\sigma - \overline{\sigma})}\mathrm{d}A_{\overline{w}} \wedge \mathrm{d}A_w$$

$$= \frac{1}{\pi(\sigma - \overline{\sigma})}\mathrm{d}(P_{\overline{w}} + iQ_{\overline{w}}) \wedge \mathrm{d}(P_w + iQ_w)$$

$$\omega_t \equiv \frac{1}{2}(t\omega_{\mathbb{C}} + \overline{t\omega_{\mathbb{C}}}), \tag{5.57}$$

while on $M$ we have

$$\omega_k = 4\pi k \mathrm{d}u' \wedge \mathrm{d}v' \tag{5.58}$$

### 5.2.2 Pre-Quantization

Consider the following three gauge transformations in $\mathrm{C}^\infty(\Sigma, \mathrm{SL}(2,\mathbb{C}))$

$$g_u(x,y) \equiv e^{2\pi i T x} \qquad g_v(x,y) \equiv e^{2\pi i T y} \qquad g_R(x,y) \equiv R, \tag{5.59}$$

acting as follows on $A(u,v) = 2\pi i u T \mathrm{d}x + 2\pi i v T \mathrm{d}y \in \mathcal{A}_{\mathbb{C}}$

$$A^{g_u}(u,v) = A(u+1,v) \qquad A^{g_v}(u,v) = A(u,v+1) \qquad A^{g_R} = -A. \tag{5.60}$$

We want to compute $c(A,g)$ for the above $g$'s following (5.13). It is simple to see that $\langle \theta_{\tilde{g}} \wedge [\theta_{\tilde{g}} \wedge \theta_{\tilde{g}}]\rangle = 0$ for all the three gauge transformations above.
We then have

$$\begin{aligned}
c(A, g_u) &= \int_\Sigma \langle g_u^{-1} A g_u \wedge g_u^{-1} \mathrm{d}g_u \rangle \\
&= \int_\Sigma \langle A \wedge 2\pi i T \mathrm{d}x \rangle \\
&= \frac{-1}{8\pi^2} \int_\Sigma \mathrm{Tr}((2\pi i)^2 v T^2) \mathrm{d}y \wedge \mathrm{d}x \\
&= -v \\
c(A, g_v) &= u \\
c(A, g_R) &= 0
\end{aligned}$$

from which we can define the following cocycles

$$\Theta_t(A, g_u) = e^{-2\pi i(kv' - sv'')} \qquad \Theta_t(A, g_v) = e^{2\pi i(ku' - su'')} \qquad \Theta_t(A, g_R) = 1. \quad (5.61)$$

Let $\widetilde{\mathcal{L}} \to \mathcal{A}_{\mathbb{C}}$ be the trivial complex line bundle. Using the cocycles from (5.61) we can lift the actions of $\mathbb{Z}^2$ and $\mathbb{S}_2$ to $\widetilde{\mathcal{L}}$

$$(u, v, \zeta) \cdot (1, 0) = (u + 1, v, e^{-2\pi i(kv' - sv'')}\zeta) \quad (5.62)$$
$$(u, v, \zeta) \cdot (0, 1) = (u, v + 1, e^{+2\pi i(ku' - su'')}\zeta) \quad (5.63)$$
$$(u, v, \zeta) \cdot R = (-u, -v, \zeta). \quad (5.64)$$

This action induces a line bundle $\mathcal{L}^t \to M_{\mathbb{C}}$. Following formula (5.18) define the connection 1-form $\beta_{\mathbb{C}}$ on $\mathcal{L}^t$, given, for every $\tilde{A} = 2\pi i T(\tilde{u}\mathrm{d}x + \tilde{v}\mathrm{d}y) \in T_A M_{\mathbb{C}}$, by

$$\beta_{\mathbb{C}}(\tilde{A}) = 2\pi \int_\Sigma \langle A \wedge \tilde{A} \rangle \quad (5.65)$$
$$= -\frac{1}{4\pi} \int_\Sigma \mathrm{Tr}(A \wedge \tilde{A}) \quad (5.66)$$
$$= -\frac{(2\pi i)^2}{4\pi} \int_\Sigma 2(u\tilde{v} - v\tilde{u})\mathrm{d}x \wedge \mathrm{d}y \quad (5.67)$$
$$= 2\pi(u\tilde{v} - v\tilde{u}) \quad (5.68)$$

so that

$$\beta_{\mathbb{C}} = 2\pi(u\mathrm{d}v - v\mathrm{d}u), \quad (5.69)$$
$$\mathrm{d}\beta_{\mathbb{C}} = \omega_{\mathbb{C}} \quad (5.70)$$

and the level $t$ connection becomes

$$\nabla \equiv \mathrm{d} - i\beta_t \quad (5.71)$$
$$\beta_t \equiv \frac{1}{2}(t\beta_{\mathbb{C}} + \overline{t\beta_{\mathbb{C}}}) \quad (5.72)$$
$$F_\nabla = -i\omega_t. \quad (5.73)$$

The integral condition $\frac{[\omega_t]}{2\pi} \in \mathrm{H}^2(M_{\mathbb{C}}, \mathbb{Z})$, reduces to require $k \in \mathbb{Z}$. Then $(\mathcal{L}, \nabla)$ is the Chern-Simons pre-quantum line bundle for $(M_{\mathbb{C}}, \omega_t)$. The pre quantum Hilbert space is given by the smooth sections

$$\mathcal{P}^{(t)} = \mathrm{C}^\infty(M_{\mathbb{C}}, \mathcal{L}^t) \quad (5.74)$$

that are in correspondence to quasi-periodic functions $\psi \in \mathrm{C}^\infty(\mathbb{C}^2)$ such that

$$\psi(u + 1, v) = e^{2\pi i(kv' - sv'')}\psi(u, v) \quad (5.75)$$
$$\psi(u, v + 1) = e^{-2\pi i(ku' - su'')}\psi(u, v) \quad (5.76)$$
$$\psi(-u, -v) = \psi(u, v) \quad (5.77)$$

This is equivalently, and more conveniently, described as the space of $\mathbb{S}_2$ invariant section of the line bundle $\widetilde{\mathcal{L}}^{2t} \to \mathbb{C}^* \times \mathbb{C}^*$, where $\tilde{\mathcal{L}}^{2t}$ is the pull-back bundle of $\mathcal{L}^t$ through the projection $\pi\colon \mathbb{C}^* \times \mathbb{C}^* \to M_{\mathbb{C}}$. It has first Chern class $c_1(\tilde{\mathcal{L}}^{2t}) = [\pi^*\omega_t]/2\pi \in H^2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z})$. For $k = 1$ the class $c_1(\tilde{\mathcal{L}}^{2+2is})$ is the double of a generator for $H^2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z})$. We write

$$\mathcal{P}^{(t)} = \mathrm{C}^\infty(\mathbb{C}^* \times \mathbb{C}^*, \tilde{\mathcal{L}}^{2t})^{\mathbb{S}_2}. \quad (5.78)$$

From now on, as in other part of this thesis, we use the following notation. For any coordinate $z$ we write

$$\nabla_z = \nabla_{\frac{\partial}{\partial z}} \qquad\qquad \partial_z = \frac{\partial}{\partial z} \tag{5.79}$$

**Lemma 57.** *The connection $\nabla$ preserves $\mathcal{P}^{(t)}$*

*Proof.*

$$\begin{aligned}
\nabla_{v'}\psi(u+1,v) &= \partial_{v'}\psi(u+1,v) - i\beta_t\left[\partial_{v'}\right](u+1,v)\psi(u+1,v)\\
&= \partial_{v'}\psi(u+1,v) - i\beta_t\left[\partial_{v'}\right](u,v)\psi(u+1,v) - 2\pi i k\psi(u+1,v)\\
&= e^{2\pi i(kv'-sv'')}\left(\partial_{v'}\psi(u,v) - i\beta_t\left[\partial_{v'}\right](u,v)\psi(u,v)\right)\\
&\quad\ + 2\pi i k\psi(u+1,v) - 2\pi i k\psi(u+1,v)\\
&= e^{2\pi i(kv'-sv'')}\nabla_{v'}\psi(u,v)\\
\nabla_{v''}\psi(u+1,v) &= \partial_{v''}\psi(u+1,v) - i\beta_t\left[\partial_{v''}\right](u+1,v)\psi(u+1,v)\\
&= \partial_{v''}\psi(u+1,v) - i\beta_t\left[\partial_{v''}\right](u,v)\psi(u+1,v) + 2\pi i s\psi(u+1,v)\\
&= e^{2\pi i(kv'-sv'')}\left(\partial_{v''}\psi(u,v) - i\beta_t\left[\partial_{v''}\right](u,v)\psi(u,v)\right)\\
&\quad\ - 2\pi i s\psi(u+1,v) + 2\pi i s\psi(u+1,v)\\
&= e^{2\pi i(kv'-sv'')}\nabla_{v''}\psi(u,v)
\end{aligned}$$

The others are similar. □

A very close description gives the pre-quantum vector space for the level $k$, SU(2) Chern–Simons theory, that is the quantization of $(M, \omega_k)$, where $\omega_k$ is $k$ times the Atiyah-Bott symplectic form $\omega = 4\pi \mathrm{d}u' \wedge \mathrm{d}v'$. The pre–quantum space will be $\mathcal{P}^{(k)} = \mathrm{C}^\infty(M,\, \mathcal{L}^k)$ which can be described, after fixing $u'$ and $v'$ real local coordinates on $\mathbb{T}\times\mathbb{T}$ as in (5.37), as the space of quasi periodic functions $\psi \in \mathrm{C}^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned}
\psi(u'+1, v') &= e^{2\pi i k v'}\psi(u', v')\\
\psi(u', v'+1) &= e^{-2\pi i k u'}\psi(u', v')\\
\psi(-u', -v') &= \psi(u', v').
\end{aligned}$$

As before it is more convenient the description of $\mathcal{P}^{(k)}$ as $\mathbb{S}_2$ invariant sections

$$\mathcal{P}^{(k)} = \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T},\, \tilde{\mathcal{L}}^{2k})^{\mathbb{S}_2} \tag{5.80}$$

where $\tilde{\mathcal{L}}^{2k}$ it is the pull-back line bundle of $\mathcal{L}^k$ through $\pi\colon \mathbb{T}\times\mathbb{T} \to M$.
Actually every $\psi \in \mathcal{P}^{(t)}$ defines some $\psi' \in \mathcal{P}^{(k)}$ when restricted to the unitary connections through the inclusion $M \subseteq M_{\mathbb{C}}$.

### 5.2.3 Hitchin–Witten Connection

In order to go from a pre-quantization to a quantization we choose the following $\sigma$–dependent real polarization

$$P_\sigma = \mathrm{span}_{\mathbb{R}}\langle \frac{\partial}{\partial A_w},\ \frac{\partial}{\partial \overline{A}_{\overline{w}}}\rangle \tag{5.81}$$

That means that we are restricting to the Hilbert space

$$\mathcal{H}^{(k)}_\sigma = \{\psi \in \mathcal{P}^{(t)}\ :\ \nabla_{A_w}\psi = \nabla_{\overline{A}_{\overline{w}}}\psi = 0\} \tag{5.82}$$

**Definition 28.** An Hitchin-Witten Connection is a connection $\boldsymbol{\nabla}$ on the vector bundle

$$\mathcal{P}^{(t)} \times \mathbb{H} \to \mathbb{H},$$

such that $\boldsymbol{\nabla}$ preserves the subbundle

$$\hat{\mathcal{H}}^{(k)} = \left\{ \left( \mathcal{H}^{(k)}_{\sigma}, \sigma \right) : \sigma \in \mathbb{H} \right\} \subset \mathcal{P}^{(t)} \times \mathbb{H} \to \mathbb{H}. \tag{5.83}$$

The definition for $\boldsymbol{\nabla}$ reduces to verify two equations

$$\nabla_{A_w} \boldsymbol{\nabla} \psi = 0 \qquad\qquad \nabla_{\overline{A_{\overline{w}}}} \boldsymbol{\nabla} \psi = 0 \tag{5.84}$$

whenever $\nabla_{A_w} \psi = \nabla_{\overline{A_{\overline{w}}}} \psi = 0$.
Suppose

$$\boldsymbol{\nabla} = \mathrm{d} + \mathcal{U} \mathrm{d}\sigma + \tilde{\mathcal{U}} \mathrm{d}\overline{\sigma} \qquad\qquad \mathcal{U}, \tilde{\mathcal{U}} \in \mathrm{C}^{\infty} \left( \mathbb{H}, \mathrm{End}(\mathcal{P}^{(k)}) \right) \tag{5.85}$$

Computing

$$\left[ \frac{\partial}{\partial A_w}, \frac{\partial}{\partial \sigma} \right] = \frac{1}{\overline{\sigma} - \sigma} \frac{\partial}{\partial A_{\overline{w}}} \qquad\qquad \left[ \frac{\partial}{\partial A_w}, \frac{\partial}{\partial \overline{\sigma}} \right] = \frac{1}{\overline{\sigma} - \sigma} \frac{\partial}{\partial A_w} \tag{5.86}$$

$$\left[ \frac{\partial}{\partial \overline{A_{\overline{w}}}}, \frac{\partial}{\partial \sigma} \right] = \frac{1}{\sigma - \overline{\sigma}} \frac{\partial}{\partial \overline{A_{\overline{w}}}} \qquad\qquad \left[ \frac{\partial}{\partial \overline{A_{\overline{w}}}}, \frac{\partial}{\partial \overline{\sigma}} \right] = \frac{1}{\sigma - \overline{\sigma}} \frac{\partial}{\partial \overline{A_w}} \tag{5.87}$$

equations (5.84) are equivalent to

$$\nabla_{A_w} \mathcal{U} = -\frac{1}{\overline{\sigma} - \sigma} \nabla_{A_{\overline{w}}} \qquad\qquad \nabla_{A_w} \tilde{\mathcal{U}} = 0 \tag{5.88}$$

$$\nabla_{\overline{A_{\overline{w}}}} \tilde{\mathcal{U}} = -\frac{1}{\sigma - \overline{\sigma}} \nabla_{\overline{A_w}} \qquad\qquad \nabla_{\overline{A_{\overline{w}}}} \mathcal{U} = 0 \tag{5.89}$$

**Proposition 58.** *The two operators*

$$\mathcal{U} = \frac{\pi}{it} \nabla^2_{A_{\overline{w}}} \qquad\qquad \tilde{\mathcal{U}} = \frac{\pi}{i\tilde{t}} \nabla^2_{\overline{A_w}} \tag{5.90}$$

*satisfy equations (5.88 – 5.89).*

*Proof.*

$$\begin{aligned}
\nabla_{A_w} \nabla^2_{A_{\overline{w}}} \psi &= \nabla^2_{A_{\overline{w}}} \nabla_{A_w} \psi + 2 \left[ \nabla_{A_w}, \nabla_{A_{\overline{w}}} \right] \nabla_{A_{\overline{w}}} && (5.91)\\
&= -2i\omega_t \left( \frac{\partial}{\partial A_w}, \frac{\partial}{\partial A_{\overline{w}}} \right) \nabla_{A_{\overline{w}}} && (5.92)\\
&= \frac{it}{\pi(\sigma - \overline{\sigma})} \nabla_{A_{\overline{w}}} && (5.93)\\
\nabla_{A_w} \nabla^2_{\overline{A_w}} \psi &= \nabla^2_{\overline{A_w}} \nabla_{A_w} \psi + 2 \left[ \nabla_{A_w}, \nabla_{\overline{A_w}} \right] \nabla_{\overline{A_w}} && (5.94)\\
&= -2i\omega_t \left( \frac{\partial}{\partial A_w}, \frac{\partial}{\partial \overline{A_w}} \right) \nabla_{A_{\overline{w}}} = 0 && (5.95)
\end{aligned}$$

the other equations are analogous. □

The conditions for $\psi$ to be a polarized section can be rewritten as

$$\nabla_{Q_w} \psi = -i \nabla_{P_w} \psi \qquad\qquad \nabla_{Q_{\overline{w}}} \psi = i \nabla_{P_{\overline{w}}} \psi \tag{5.96}$$

The subspace

$$T_{\sigma} = \{ Q_w = Q_{\overline{w}} = 0 \} \subset \mathcal{A}_{\mathbb{C}} \tag{5.97}$$

is a transversal for this polarization and equations (5.96) determine $\psi$ from its restriction to $T_\sigma$ being the leafs contractible and $\mathcal{L}^t$ trivializable along the fibers.
We get an identification between the space $\mathcal{H}_\sigma^{(k)}$ and the pre-quantum space of SU(2) Chern-Simons Theory $\mathcal{P}^{(k)}$. Indeed one can easily see see that $T_\sigma = \{u'' = v'' = 0\}$ and is therefore independent on $\sigma$ and its quotient under the $\mathbb{Z}^2$ action coincides with $\mathbb{T}\times\mathbb{T}$.
The quantum space for the theory $\mathcal{H}^{(k)}$ will be the one obtained by identification of the different $\mathcal{H}_\sigma^{(k)}$. As we said this will be isomorphic to the SU(2) pre-quantum space, i.e.

$$\mathcal{H}^{(k)} \simeq \mathrm{C}^\infty\left(\mathbb{T}\times\mathbb{T}, \widetilde{\mathcal{L}}^{2k}\right)^{\mathbb{S}_2} \tag{5.98}$$

The inner product structure we put on it is the one induced by the following on $\mathbb{S}_2$ invariant sections

$$\langle\phi,\psi\rangle = \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{T}} \phi\overline{\psi}\tilde{\omega}, \qquad p\in\mathbb{T}\times\mathbb{T}, \quad \phi,\psi\in \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T},\widetilde{\mathcal{L}}^{2k}) \tag{5.99}$$

where $\tilde{\omega} = \pi^*\omega/2$, $\pi : \mathbb{T}\times\mathbb{T}\to M$ is the projection and $\omega$ is the Atiyah Bott form. To get an actual Hilbert space one should consider the metric completion of $\mathcal{H}^{(k)}$.
Recall the local expression for the pre-quantum connection restricted to $M$

$$\nabla = \mathrm{d} - 2\pi i k(u'\mathrm{d}v' - v'\mathrm{d}u') \tag{5.100}$$

Renaming

$$z = \frac{1}{2\pi i}P_{\overline{w}} = (u' + \sigma v') \qquad\qquad \overline{z} = \frac{1}{2\pi i}P_w = (u' + \overline{\sigma}v') \tag{5.101}$$

the Hitchin-Witten connection has the following expression

$$\boldsymbol{\nabla} = \mathrm{d} + \frac{i}{2\pi 2t}\nabla_z^2\mathrm{d}\sigma + \frac{i}{2\pi 2\overline{t}}\nabla_{\overline{z}}^2\mathrm{d}\overline{\sigma} \tag{5.102}$$

**Proposition 59.** *Suppose $s\in\mathbb{R}$, $\phi$, $\psi\in\mathcal{H}^{(k)}$ and $\langle\cdot,\cdot\rangle : \mathcal{H}^{(k)}\times\mathcal{H}^{(k)}\to\mathbb{C}$ as in (5.99). We have*

$$\mathrm{d}(\langle\phi,\psi\rangle) = \langle\boldsymbol{\nabla}\phi,\psi\rangle + \langle\phi,\boldsymbol{\nabla}\psi\rangle$$

*Proof.* By integration by parts one shows

$$\langle\phi,\nabla_z\psi\rangle = -\langle\nabla_{\overline{z}}\phi,\psi\rangle \quad \langle\phi,\nabla_{\overline{z}}\psi\rangle = -\langle\nabla_z\phi,\psi\rangle. \tag{5.103}$$

It follows that

$$\langle\phi,\frac{i}{2\pi 2t}\nabla_z^2\psi\rangle = \langle\frac{-i}{2\pi 2\overline{t}}\nabla_{\overline{z}}^2\phi,\psi\rangle \tag{5.104}$$

Now we have

$$\langle\boldsymbol{\nabla}\phi,\psi\rangle = \langle\mathrm{d}\phi,\psi\rangle + \langle\frac{i}{2\pi 2t}\nabla_z^2\phi,\psi\rangle\mathrm{d}\sigma + \langle\frac{i}{2\pi 2\overline{t}}\nabla_{\overline{z}}^2\phi,\psi\rangle\mathrm{d}\overline{\sigma} \tag{5.105}$$

$$= \langle\mathrm{d}\phi,\psi\rangle + \langle\phi,-\frac{i}{2\pi 2\overline{t}}\nabla_{\overline{z}}^2\psi\rangle\mathrm{d}\sigma + \langle\phi,-\frac{i}{2\pi 2t}\nabla_z^2\psi\rangle\mathrm{d}\overline{\sigma} \tag{5.106}$$

and analogously

$$\langle\phi,\boldsymbol{\nabla}\psi\rangle = \langle\phi,\mathrm{d}\psi\rangle + \langle\phi,\frac{i}{2\pi 2t}\nabla_z^2\psi\rangle\mathrm{d}\overline{\sigma} + \langle\phi,\frac{i}{2\pi 2\overline{t}}\nabla_{\overline{z}}^2\psi\rangle\mathrm{d}\sigma \tag{5.107}$$

$$= \langle\phi,\mathrm{d}\psi\rangle - \langle\phi,-\frac{i}{2\pi 2t}\nabla_z^2\psi\rangle\mathrm{d}\overline{\sigma} - \langle\phi,-\frac{i}{2\pi 2\overline{t}}\nabla_{\overline{z}}^2\psi\rangle\mathrm{d}\sigma \tag{5.108}$$

Summing up this two equations we get exactly

$$\mathrm{d}(\langle\phi,\psi\rangle) = \langle\boldsymbol{\nabla}\phi,\psi\rangle + \langle\phi,\boldsymbol{\nabla}\psi\rangle.$$

□

Let us define a new operator

$$\Delta_\sigma = \frac{i}{2\pi}\left(\sigma - \overline{\sigma}\right)\nabla_z \nabla_{\overline{z}} \tag{5.109}$$

which is dependent on the complex number $\sigma$. We are going to establish some property of the different differential operators we introduced, with the aim of computing parallel sections for the Hitchin–Witten connection. The following basic relations will be useful in the computations (deduced from equations (5.101) )

$$\frac{\partial}{\partial z} = \frac{1}{\sigma - \overline{\sigma}}\left(-\overline{\sigma}\frac{\partial}{\partial u'} + \frac{\partial}{\partial v'}\right) \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{\overline{\sigma} - \sigma}\left(-\sigma\frac{\partial}{\partial u'} + \frac{\partial}{\partial v'}\right)$$

**Lemma 60.** *On $\mathcal{H}^{(k)} \times \mathbb{H}$*

$$\left[\nabla_{v'}, \nabla_{u'}\right] = 4\pi i k, \tag{5.110}$$

$$\left[\nabla_z, \nabla_{\overline{z}}\right] = \frac{4\pi i k}{\sigma - \overline{\sigma}}, \tag{5.111}$$

$$\left[\partial_\sigma, \nabla_z\right] = -\left[\partial_\sigma, \nabla_{\overline{z}}\right] = \frac{-1}{\sigma - \overline{\sigma}}\nabla_z, \tag{5.112}$$

$$\left[\partial_{\overline{\sigma}}, \nabla_{\overline{z}}\right] = -\left[\partial_{\overline{\sigma}}, \nabla_z\right] = \frac{-1}{\overline{\sigma} - \sigma}\nabla_{\overline{z}} \tag{5.113}$$

$$\left[\partial_\sigma, \Delta_\sigma\right] = \frac{i}{2\pi}\nabla_z^2, \tag{5.114}$$

$$\left[\partial_{\overline{\sigma}}, \Delta_\sigma\right] = -\frac{i}{2\pi}\nabla_{\overline{z}}^2, \tag{5.115}$$

$$\left[\Delta_\sigma, \nabla_z^2\right] = 4k\nabla_z^2, \tag{5.116}$$

$$\left[\Delta_\sigma, \nabla_{\overline{z}}^2\right] = -4k\nabla_{\overline{z}}^2 \tag{5.117}$$

$$\left[\partial_\sigma, \nabla_{u'}\right] = \left[\partial_\sigma, \nabla_{v'}\right] = \left[\partial_{\overline{\sigma}}, \nabla_{u'}\right] = \left[\partial_{\overline{\sigma}}, \nabla_{v'}\right] = 0 \tag{5.118}$$

*Proof.*

$$\begin{aligned}
\left[\nabla_{v'}, \nabla_{u'}\right] &= -i\omega_k\left(\partial_{v'}, \partial_{u'}\right) \\
&= -4\pi i k \mathrm{d}u' \wedge \mathrm{d}v'\left(\partial_{v'}, \partial_{u'}\right) = 4\pi i k \\
\left[\nabla_z, \nabla_{\overline{z}}\right] &= \frac{-1}{(\sigma - \overline{\sigma})^2}\left(-\overline{\sigma}\left[\nabla_{u'}, \nabla_{v'}\right] + \sigma\left[\nabla_{u'}, \nabla_{v'}\right]\right) \\
&= -\frac{1}{(\sigma - \overline{\sigma})}\left[\nabla_{u'}, \nabla_{v'}\right] = \frac{4\pi i k}{\sigma - \overline{\sigma}} \\
\left[\partial_\sigma, \nabla_z\right] &= \left[\partial_\sigma, \frac{-\overline{\sigma}}{\sigma - \overline{\sigma}}\nabla_{u'}\right] + \left[\partial_\sigma, \frac{1}{\sigma - \overline{\sigma}}\nabla_{v'}\right] \\
&= \frac{1}{(\sigma - \overline{\sigma})^2}\left(\overline{\sigma}\nabla_{u'} - \nabla_{v'}\right) \\
&= -\frac{1}{(\sigma - \overline{\sigma})}\nabla_z \\
\left[\partial_\sigma, \nabla_{\overline{z}}\right] &= \left[\partial_\sigma, \frac{-\sigma}{\overline{\sigma} - \sigma}\nabla_{u'}\right] + \left[\partial_\sigma, \frac{1}{\overline{\sigma} - \sigma}\nabla_{v'}\right] \\
&= -\frac{\overline{\sigma}}{(\overline{\sigma} - \sigma)^2}\nabla_{u'} + \frac{1}{(\overline{\sigma} - \sigma)^2}\nabla_{v'} \\
&= \frac{1}{(\sigma - \overline{\sigma})}\nabla_z
\end{aligned}$$

$$
\begin{aligned}
[\partial_\sigma, \Delta_\sigma] &= \frac{i}{2\pi}\nabla_z\nabla_{\overline{z}} + \frac{i}{2\pi}(\sigma-\overline{\sigma})\left[\partial_\sigma, \nabla_z\nabla_{\overline{z}}\right] \\
&= \frac{i}{2\pi}\nabla_z\nabla_{\overline{z}} + \frac{i}{2\pi}(\sigma-\overline{\sigma})\left(\left[\partial_\sigma, \nabla_z\right]\nabla_{\overline{z}} + \nabla_z\left[\partial_\sigma, \nabla_{\overline{z}}\right]\right) \\
&= \frac{i}{2\pi}\nabla_z\nabla_{\overline{z}} + \frac{i}{2\pi}(\sigma-\overline{\sigma})\left(-\frac{1}{(\sigma-\overline{\sigma})}\nabla_z\nabla_{\overline{z}} + \frac{1}{(\sigma-\overline{\sigma})}\nabla_z\nabla_z\right) \\
&= \frac{1}{2\pi}\nabla_z^2 \\
\left[\Delta_\sigma, \nabla_z^2\right] &= i\frac{\sigma-\overline{\sigma}}{2\pi}\left(\nabla_z\left[\nabla_{\overline{z}}, \nabla_z^2\right] + \left[\nabla_z, \nabla_z^2\right]\nabla_{\overline{z}}\right) \\
&= i\frac{\sigma-\overline{\sigma}}{2\pi}\nabla_z\left(\nabla_z\left[\nabla_{\overline{z}}, \nabla_z\right] + \left[\nabla_{\overline{z}}, \nabla_z\right]\nabla_z\right) \\
&= i\frac{\sigma-\overline{\sigma}}{2\pi}\left(\frac{-8\pi ik}{\sigma-\overline{\sigma}}\right)\nabla_z^2 \\
&= 4k\nabla_z^2
\end{aligned}
$$

The missing equations are either specular to the one proved or completely trivial. We remark that we used equation (C.10) to compute the second order commutators. □

**Proposition 61.** *The genus* 1 *Hitchin Witten connection* $\boldsymbol{\nabla}$ *is flat*

*Proof.* We need to compute

$$
\begin{aligned}
F_{\boldsymbol{\nabla}} &= \boldsymbol{\nabla}_{\partial_\sigma}\boldsymbol{\nabla}_{\overline{\partial_\sigma}} - \boldsymbol{\nabla}_{\overline{\partial_\sigma}}\boldsymbol{\nabla}_{\partial_\sigma} &\quad (5.119)\\
&= \left[\mathcal{U}, \tilde{\mathcal{U}}\right] + \left[\partial_\sigma, \tilde{\mathcal{U}}\right] - \left[\partial_{\overline{\sigma}}, \mathcal{U}\right] &\quad (5.120)
\end{aligned}
$$

Using

$$
[\partial_{\overline{\sigma}}, \nabla_z] = \frac{-1}{\sigma-\overline{\sigma}}\nabla_{\overline{z}}, \qquad [\partial_\sigma, \nabla_{\overline{z}}] = \frac{1}{\sigma-\overline{\sigma}}\nabla_z, \qquad [\nabla_z, \nabla_{\overline{z}}] = \frac{4\pi ik}{\sigma-\overline{\sigma}}. \quad (5.121)
$$

we can compute

$$
\begin{aligned}
\left[\nabla_z^2, \nabla_{\overline{z}}^2\right] &= 2\left[\nabla_z, \nabla_{\overline{z}}\right]\left(\nabla_z\nabla_{\overline{z}} + \nabla_{\overline{z}}\nabla_z\right) &\quad (5.122)\\
\left[\overline{\partial_\sigma}, \nabla_z^2\right] &= \frac{-1}{\sigma-\overline{\sigma}}(\nabla_z\nabla_{\overline{z}} + \nabla_{\overline{z}}\nabla_z) &\quad (5.123)\\
\left[\partial_\sigma, \nabla_{\overline{z}}^2\right] &= \frac{1}{\sigma-\overline{\sigma}}(\nabla_z\nabla_{\overline{z}} + \nabla_{\overline{z}}\nabla_z) &\quad (5.124)
\end{aligned}
$$

Putting everything together we get

$$
\begin{aligned}
F_{\boldsymbol{\nabla}} &= \frac{-1}{16\pi^2 t\tilde{t}}\left[\nabla_z^2, \nabla_{\overline{z}}^2\right] + \frac{i}{4\pi\tilde{t}}\left[\partial_\sigma, \nabla_{\overline{z}}^2\right] - \frac{i}{4\pi t}\left[\overline{\partial_\sigma}, \nabla_z^2\right] &\quad (5.125)\\
&= \left(\frac{-2ik + it + i\tilde{t}}{4\pi t\tilde{t}(\sigma-\overline{\sigma})}\right)(\nabla_z\nabla_{\overline{z}} + \nabla_{\overline{z}}\nabla_z) &\quad (5.126)\\
&= 0 &\quad (5.127)
\end{aligned}
$$

□

As noticed by Witten in [Wit91], Lemma 90 together with Lemma 60 gives us the following conjugation rule

$$
e^{-r\Delta_\sigma} \circ \mathrm{d} \circ e^{r\Delta_\sigma} = \boldsymbol{\nabla} \quad (5.128)
$$

where $r$ is chosen so that

$$
e^{-4kr} = -\frac{k - is}{k + is}. \quad (5.129)
$$

In particular, equation (5.128) implies

**Proposition 62** ([Wit91]). *For every smooth section* $\mathrm{C}^\infty\left(\mathbb{T}\times\mathbb{T},\widetilde{\mathcal{L}}^{(2k)}\right)^{\mathbb{S}_2}$ *independent on the complex structure of* $M$*, the section*

$$e^{-r\Delta_\sigma}\psi$$

*of the vector bundle* $\hat{\mathcal{H}}^{(k)}$ *(defined in (5.83)) is parallel with respect to* $\boldsymbol{\nabla}$*.*

This permits to define a parallel transport operator $\mathcal{P}_{\sigma_0,\sigma_1}$ in the bundle $\hat{\mathcal{H}}^{(k)}$ from the fiber over $\sigma_1\in\mathbb{H}$ to the one over $\sigma_0$. We have

$$\mathcal{P}_{\sigma_0,\sigma_1}\psi_{\sigma_1}=e^{-r\Delta_{\sigma_0}}e^{r\Delta_{\sigma_1}}\psi_{\sigma_1}. \tag{5.130}$$

An explicit expression is possible but we need to use the Weil-Gel'fand-Zak Transform. Recall Proposition 2 and the transform $W^{(2k)}$ defined there. We need to use the level $2k$ WGZ transform instead of level $k$ because the line bundle over the torus that we have is $\tilde{\mathcal{L}}^{2k}$ and this has degree $2k$.
In our setting we have the isomorphism

$$W^{(2k)}\colon \mathcal{S}(\mathbb{R}\times\mathbb{Z}_{2k})\longrightarrow \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T},\widetilde{\mathcal{L}}^{2k})$$

Recall that $\mathcal{H}^{(k)}=\mathrm{C}^\infty\left(\mathbb{T}\times\mathbb{T},\widetilde{\mathcal{L}}^{2k}\right)^{\mathbb{S}_2}$ where $\mathbb{S}_2=\{\mathrm{id},R\}$ with $R(s)(u',v')=s(-u',-v')$. Let $\hat{R}(\mathsf{f})=\mathsf{f}(-x,2k-j)$, for $\mathsf{f}\in\mathcal{S}(\mathbb{R}\times\mathbb{Z}_{2k})$. We have

$$\hat{R}(\mathsf{f})(x,j)=\overline{W}^{(2k)}\circ R\circ W^{(2k)}(\mathsf{f}). \tag{5.131}$$

Therefore there is a well defined action of $\mathbb{S}_2$ on $S(\mathbb{R})\otimes\mathbb{C}^{2k}$, compatible with the WGZ transform $W^{(2k)}$, so that we have an induced unitary isomorphism

$$W^{(2k)}\colon \left(S(\mathbb{R})\otimes\mathbb{C}^{(2k)}\right)^{\mathbb{S}_2}\longrightarrow\mathcal{H}^{(k)}. \tag{5.132}$$

*Proof of Equation* (5.131)

$$\begin{aligned}
R\circ W^{(2k)}(\mathsf{f})(u',v')&=W^{(2k)}(\mathsf{f})(-u',-v')\\
&=e^{-2\pi iku'v'}\sum_{m\in\mathbb{Z}}\sum_{l=0}^{2k-1}\mathsf{f}\left(-u'+\frac{m}{2k},l\right)e^{2\pi imv'}e^{-2\pi ilm/(2k)}\\
&=e^{-2\pi iku'v'}\sum_{m\in\mathbb{Z}}\sum_{l=0}^{2k-1}\mathsf{f}\left(-u'-\frac{m}{2k},l\right)e^{-2\pi imv'}e^{2\pi ilm/(2k)}\\
&=e^{-2\pi iku'v'}\sum_{m\in\mathbb{Z}}\sum_{l=0}^{2k-1}\mathsf{f}\left(-u'-\frac{m}{2k},2k-l\right)e^{-2\pi imv'}e^{-2\pi i(2k-l)m/(2k)}\\
&=W^{(2k)}\circ\hat{R}(\mathsf{f})(u',v')
\end{aligned}$$

□

Let us define the following operator

$$\begin{aligned}
D_\sigma\colon S(\mathbb{R})\otimes\mathbb{C}^{2k}&\longrightarrow S(\mathbb{R})\otimes\mathbb{C}^{2k}\\
\mathsf{f}(x,j)&\mapsto\frac{1}{\overline{\sigma}-\sigma}\left(\overline{\sigma}\frac{\mathrm{d}}{\mathrm{d}x}+2\pi i2kx\right)\mathsf{f}(x,j)
\end{aligned}\tag{5.133}$$

*Remark* 5.2.2. The operators $D_\sigma$ is Hermitian anti-adjoint to $D_{\overline{\sigma}}$ with respect to the hermitian product (A.8).

We can read the connection $\nabla$ from equation (5.100) over $S(\mathbb{R}) \otimes \mathbb{C}^{2k}$. Let

$$\hat{\nabla}_X = \overline{W}^{(2k)} \circ \nabla_X \circ W^{(2k)}$$

for any $X \in \mathrm{C}^\infty(\mathbb{T} \times \mathbb{T}, T(\mathbb{T} \times \mathbb{T}))$

**Lemma 63.** *Let $u'$ and $v'$ be local coordinates on $\mathbb{T}\times\mathbb{T}$, let $\sigma \in \mathbb{H}$, and let $z = u'+\sigma v'$ be an holomorphic coordinate on $\mathbb{T} \times \mathbb{T}$ as above. Let $\mathsf{f} \in \mathcal{S}(\mathbb{A}_{2k})$. We have*

$$\hat{\nabla}_{u'} \mathsf{f}(x,j) = \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}(x,j) \tag{5.134}$$

$$\hat{\nabla}_{v'} \mathsf{f}(x,j) = -2\pi i 2kx\mathsf{f}(x,j) \tag{5.135}$$

$$\hat{\nabla}_{z} \mathsf{f}(x,j) = D_\sigma \mathsf{f}(x,j) \tag{5.136}$$

$$\hat{\nabla}_{\overline{z}} \mathsf{f}(x,j) = D_{\overline{\sigma}} \mathsf{f}(x,j) \tag{5.137}$$

$$\hat{\Delta}_\sigma = \overline{W}^{(2k)} \circ \Delta_\sigma \circ W^{(2k)} = \frac{i}{2\pi}\left(\sigma - \overline{\sigma}\right) D_\sigma D_{\overline{\sigma}} \tag{5.138}$$

$$\hat{\boldsymbol{\nabla}} = \overline{W}^{(2k)} \circ \boldsymbol{\nabla} \circ W^{(2k)} = \mathrm{d} + \frac{i}{2\pi t} D_\sigma^2 + \frac{i}{4\pi \overline{t}} D_{\overline{\sigma}}^2 \tag{5.139}$$

*Proof.*

$$\begin{aligned}
\nabla_{u'} W^{(2k)}(\mathsf{f})(u',v') &= \left(\frac{\partial}{\partial u'} + 2\pi i k v'\right) W^{(2k)}(\mathsf{f})(u',v') \\
&= \left(-2\pi i k v'\right) e^{-2\pi i k u' v'} \sum_{m\in\mathbb{Z}} \sum_{l=0}^{2k-1} \mathsf{f}\left(u' + \frac{m}{2k}, l\right) e^{-2\pi i m v'} e^{-2\pi i l m/(2k)} + \\
&\quad + e^{-2\pi i k u' v'} \sum_{m\in\mathbb{Z}} \sum_{l=0}^{2k-1} \frac{\mathrm{d}\mathsf{f}}{\mathrm{d}u'}\left(u' + \frac{m}{2k}, l\right) e^{-2\pi i m v'} e^{-2\pi i l m/(2k)} + \\
&\quad + 2\pi i k v' W^{(2k)}(\mathsf{f})(u',v') \\
&= e^{-2\pi i k u' v'} \sum_{m\in\mathbb{Z}} \sum_{l=0}^{2k-1} \frac{\mathrm{d}\mathsf{f}}{\mathrm{d}u'}\left(u' + \frac{m}{2k}, l\right) e^{-2\pi i m v'} e^{-2\pi i l m/(2k)} \\
&= W^{(2k)}(\tilde{\mathsf{f}'})(u',v')
\end{aligned}$$

where $\mathsf{f}'(x,n) = \frac{\mathrm{d}\mathsf{f}}{\mathrm{d}x}(x,n)$.

$$\begin{aligned}
\nabla_{v'} W^{(2k)}(\mathsf{f})(u',v') &= \left(\frac{\partial}{\partial v'} - 2\pi i k u'\right) W^{(2k)}(\mathsf{f})(u',v') \\
&= \left(-2\pi i k u' - 2\pi i m\right) e^{-2\pi i k u' v'} \sum_{m\in\mathbb{Z}} \sum_{l=0}^{2k-1} \mathsf{f}\left(u' + \frac{m}{2k}, l\right) e^{-2\pi i m v'} e^{-2\pi i l m/(2k)} + \\
&\quad - 2\pi i k u' W^{(2k)}(\mathsf{f})(u',v') \\
&= e^{-2\pi i k u' v'} \sum_{m\in\mathbb{Z}} \sum_{l=0}^{2k-1} \left(-2\pi i 2k\left(u' + \frac{m}{2k}\right) \mathsf{f}\left(u' + \frac{m}{2k}, l\right)\right) e^{-2\pi i m v'} e^{-2\pi i l m/(2k)} \\
&= W^{(2k)}(\tilde{\mathsf{f}})(u',v')
\end{aligned}$$

where $\tilde{\mathsf{f}}(x,l) = -2\pi i 2kx\mathsf{f}(x,l)$. The following two are combinations of the two above together with the definition of $z$. For $\hat{\Delta}_\sigma$ we have

$$\hat{\Delta}_\sigma = \frac{i}{2\pi}\left(\sigma - \overline{\sigma}\right) \hat{\nabla}_z \hat{\nabla}_{\overline{z}} = \frac{i}{2\pi}\left(\sigma - \overline{\sigma}\right) D_\sigma D_{\overline{\sigma}}$$

The last one for $\hat{\boldsymbol{\nabla}}$ follows in the same way. $\square$

From the expression for $\hat{\boldsymbol{\nabla}}$ and $\hat{\Delta}_\sigma$ we can see that their action on $\mathcal{S}(\mathbb{R})\otimes\mathbb{C}^{2k}$ is the identity in the second factor of the tensor product. In the next Proposition we consider the operator $\Delta_\sigma$ as an operator on $\mathcal{S}(\mathbb{R})$.

**Proposition 64.** *We have the following factorization*

$$\hat{\Delta}_\sigma = \frac{i}{2\pi}\frac{\sigma\overline{\sigma}}{\overline{\sigma}-\sigma}\hat{A}_\sigma \hat{A}_{\overline{\sigma}} \tag{5.140}$$

*where*

$$\hat{A}_\sigma = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{\sigma}2\pi i 2kx\right). \tag{5.141}$$

$$\left[\hat{A}_\sigma,\, \hat{A}_{\overline{\sigma}}\right] = 2\pi i 2k\frac{\overline{\sigma}-\sigma}{\sigma\overline{\sigma}} \tag{5.142}$$

*The set*

$$\mathcal{B}_\sigma = \left\{\hat{A}^n_\sigma v(x,\sigma)\, \in S(\mathbb{R})\, :\, n\in\mathbb{Z}_{\geq 0},\, v(x,\sigma) := e^{-2\pi i k x^2/\sigma}\right\}$$

*is a complete set of eigenvectors for $\hat{\Delta}_\sigma$ with corresponding eigenvalues*

$$\hat{\Delta}_\sigma \hat{A}^n_\sigma v(x,\sigma) = 2kn\hat{A}^n_\sigma v(x,\sigma).$$

*Remark* 5.2.3. Define

$$\alpha \equiv \frac{\overline{\sigma}-\sigma}{\sigma\overline{\sigma}}2\pi i k > 0 \qquad\qquad \beta\equiv\frac{2\pi i k}{\sigma} \tag{5.143}$$

Then, in the notation of Appendix B

$$v_n(x,\sigma) = \psi_{n,\beta} = H_{n,\alpha}(x)v(x,\sigma). \tag{5.144}$$

In particular this means that the $v_n$'s, for $\sigma$ fixed, are an orthogonal basis for $\mathrm{L}^2(\mathbb{R})$ in the sense of Hilbert basis. See Appendix B.

*Proof of Proposition 64.* Equations (5.140), (5.141) and (5.142) follow from equation (5.138) and definition (5.133). We can immediately see that

$$\begin{aligned}\hat{A}_{\overline{\sigma}}\varphi(x) = 0 \iff& \varphi'(x) = -\frac{2\pi i 2k}{\sigma}x\varphi(x)\\ \iff& \varphi(x) = \lambda e^{-2\pi i k x^2/\sigma},\text{ for any } \lambda\in\mathbb{C}\end{aligned}$$

So we have shown the first eigenfuction

$$\hat{\Delta}_\sigma v(x,\sigma) = 0,$$

Now

$$\begin{aligned}\hat{A}_\sigma\hat{A}_{\overline{\sigma}}\hat{A}^n_\sigma v(x,\sigma) &= \hat{A}_\sigma\left(\hat{A}^{n-1}_\sigma\hat{A}_{\overline{\sigma}} + \left[\hat{A}_{\overline{\sigma}},\, \hat{A}^{n-1}_\sigma\right]\right)\hat{A}_\sigma v(x,\sigma)\\ &= \hat{A}_\sigma\left(\hat{A}^{n-1}_\sigma\hat{A}_{\overline{\sigma}} - (n-1)\left[\hat{A}_\sigma,\, \hat{A}_{\overline{\sigma}}\right]\hat{A}^{n-2}_\sigma\right)\hat{A}_\sigma v(x,\sigma)\\ &= \left(\hat{A}^n_\sigma\hat{A}_{\overline{\sigma}}\hat{A}_\sigma - (n-1)\left[\hat{A}_\sigma,\, \hat{A}_{\overline{\sigma}}\right]\hat{A}^n_\sigma\right)v(x,\sigma)\\ &= \left(\hat{A}^n_\sigma\hat{A}_\sigma\hat{A}_{\overline{\sigma}} + \hat{A}^n_\sigma\left[\hat{A}_{\overline{\sigma}},\, \hat{A}_\sigma\right] - (n-1)\left[\hat{A}_\sigma,\, \hat{A}_{\overline{\sigma}}\right]\hat{A}^n_\sigma\right)v(x,\sigma)\\ &= \left(\hat{A}^n_\sigma\left[\hat{A}_{\overline{\sigma}},\, \hat{A}_\sigma\right] - (n-1)\left[\hat{A}_\sigma,\, \hat{A}_{\overline{\sigma}}\right]\hat{A}^n_\sigma\right)v(x,\sigma)\\ &= \left[\hat{A}_\sigma,\, \hat{A}_{\overline{\sigma}}\right](-1-(n-1))\hat{A}^n_\sigma v(x,\sigma)\\ &= -2\pi i 2k\frac{\overline{\sigma}-\sigma}{\sigma\overline{\sigma}}n\hat{A}^n_\sigma v(x,\sigma)\end{aligned}$$

$\square$

For computational reasons it is convenient to change the parametrization of the parameter $t$ in an analogous way as we did in Chapters 3 and 4. Choose $\mathrm{b} \in \mathbb{C}$ such that $|\mathrm{b}| = 1$, $\mathrm{Re}(\mathrm{b}) > 0$ and

$$is = k\frac{1-\mathrm{b}^2}{1+\mathrm{b}^2}, \tag{5.145}$$

then, from (5.129), we have (after choosing $\sqrt{-1} = i$)

$$e^{-4kr} = -\mathrm{b}^2 \qquad\qquad e^{-2kr} = i\mathrm{b} \tag{5.146}$$

From Proposition 64 we see that $\Delta_\sigma = 2kN_\sigma$ where $N_\sigma$ has spectrum equal to $\mathbb{Z}_{\geq 0}$. So, its exponential can be written as

$$e^{-r\Delta_\sigma} = e^{-2krN_\sigma} = (i\mathrm{b})^{N_\sigma}. \tag{5.147}$$

Finding an explicit expression for $e^{-r\hat{\Delta}}\psi = f$ can be solved trying to compute a kernel $k_{\sigma,\mathrm{b}}$ as follows

$$f(x,\sigma,\mathrm{b}) = \int_{\mathbb{R}} k_{\sigma,\mathrm{b}}(x,y)\psi(y)\mathrm{d}y. \tag{5.148}$$

Having an Hilbert bases for $\mathrm{L}^2(\mathbb{R})$ diagonalizing $\Delta_\sigma$ we can use the decomposition of the identity (B.9) to rewrite the kernel as

$$k_{\sigma,\mathrm{b}}(x,y) = \sum_{n\geq 0}\frac{(i\mathrm{b})^n}{\langle v_n, v_n\rangle} v_n(x,\sigma)\overline{v_n(y,\sigma)} \tag{5.149}$$

Since $\mathrm{Re}(-\mathrm{b}^2) = 1 - 2(\mathrm{Re}\,\mathrm{b})^2 < 1$, we can apply Mehler Formula (B.5) to get the following explicit kernel

**Lemma 65.** *For $i\mathrm{b} = e^{-2kr}$ we have*

$$e^{-r\Delta_\sigma}\psi(x) = \sqrt{\frac{\alpha}{\pi}}\frac{1}{\sqrt{1+\mathrm{b}^2}}\int_{\mathbb{R}}\exp\left(\alpha\frac{2i\mathrm{b}xy + \mathrm{b}^2(x^2+y^2)}{1+\mathrm{b}^2}\right)v(x,\sigma)\overline{v(y,\sigma)}\psi(y)\mathrm{d}y \tag{5.150}$$

*where*

$$\alpha = \frac{\overline{\sigma}-\sigma}{\sigma\overline{\sigma}}2\pi ik, \qquad v(x,\sigma) = e^{-2\pi ikx^2/\sigma}.$$

*Proof.* This is just a direct application of Mehler formula (B.5) to the orthonormal decomposition (5.149) using notation from Remark 5.2.3. $\square$

### 5.2.4 Mapping Class Group Action

The Mapping Class Group $\Gamma \equiv \mathrm{MCG}(\Sigma)$ of a genus one surface is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$. We briefly recall the identification, see [FM12] for a more detailed description. In genus 1 the first homology group $\mathrm{H}_1(\Sigma,\mathbb{Z})$ is ismorphic to the $\pi_1(\Sigma)$, so given a diffeomorphism $\phi \in \mathrm{Diff}^+$ its isotopy class $[\phi] \in \Gamma$ acts on $\mathrm{H}_1(\Sigma,\mathbb{Z})$ via the induced morphism $\phi_*$ and this action of $\Gamma$ is well defined and faithfull, meaning that $\phi_* = \psi_*$ if and only if $[\phi] = [\psi]$. The fact that $\mathrm{H}_1(\Sigma,\mathbb{Z}) \simeq \mathbb{Z}^2$ gives that $\Gamma \leq \mathrm{GL}(2,\mathbb{Z})$. The determinant on $\mathrm{GL}(2,\mathbb{Z})$ is either plus or minus 1, and the hypothesis on the diffeomorphism $\phi$ to be orientation preserving restrict the mapping class group to act as $\mathrm{SL}(2,\mathbb{Z})$ on the homology. The precise isomorphism $\mathrm{SL}(2,\mathbb{Z}) \simeq \Gamma$ depends on the precise identification $\mathrm{H}_1(\Sigma,\mathbb{Z}) \simeq \mathbb{Z}^2$. In the following we will always confuse the two isomorphic groups $\pi_1(\Sigma) \simeq \mathrm{H}_1(\Sigma,\mathbb{Z})$. Let us consider the following generators of $\Gamma$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which satisfy the full set of relations

$$S^2 = (ST)^3 \qquad\qquad S^4 = \mathrm{id},$$

and act on a fixed basis $\{A, B\}$ of $\mathrm{H}_1(\Sigma,\mathbb{Z}) \simeq \mathbb{Z}^2$. Under the identification $\Sigma \simeq \mathbb{R}^2/\,\mathrm{H}_1(\Sigma,\mathbb{Z})$ we can find representatives for $S$ and $T$ in $\mathrm{Diff}^+(\Sigma)$ (which we will call $S$ and $T$ again) acting as follows on coordinates $(x, y) \in \mathbb{R}^2$.

$$S(x,y) = (-y, x) \qquad\qquad T(x,y) = (x+y, y). \tag{5.151}$$

The action of $S$ and $T$ on $\mathcal{A}$ can also be computed in coordinates $(u', v')$ from the action in homology. The representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(2))$ is determined by its holonomies, that is by its values $\rho(A)$ and $\rho(B)$. So

$$S^*\rho(A) = \rho(S(A)) = \rho(B), \qquad\qquad S^*\rho(B) = \rho(S(B)) = \rho(-A) \tag{5.152}$$

$$T^*\rho(A) = \rho(T(A)) = \rho(A) \qquad T^*\rho(B) = \rho(T(B)) = \rho(A) + \rho(B). \tag{5.153}$$

Up to conjugation we can parametrize $\rho(A) = e^{2\pi i T u'}$ and $\rho(B) = e^{2\pi i T v'}$, so the action on $(u', v')$ coordinates for $\mathcal{A}$ is as follows

$$S^*(u', v') = (v', -u'), \qquad\qquad T^*(u', v') = (u', v' + u'). \tag{5.154}$$

The action of $\gamma \in \Gamma$ lifts trivially to the trivial line bundle $\widetilde{\mathcal{L}} \to \mathcal{A}$,

$$\gamma^*(A, \zeta) = (\gamma^* A, \zeta).$$

Let us verify equation (5.27) for the gauge transformation $g_{u'}$ (recall its definition from (5.59)).

$$\begin{aligned}
\Theta_k(S^*A(u',v'), g_{u'} \circ S) &= \Theta_k(A(v', -u'), g_{v'}^{-1}) \\
&= e^{-2\pi i k v'} = \Theta_k(A(u', v'), g_{u'}) \\
\Theta_k(T^*A(u',v'), g_{u'} \circ T) &= \Theta_k(A(u', u'+v'), g_{u'} \circ g_{v'}) \\
&= \Theta_k(A(u'+1, u'+v'), g_{v'})\Theta_k(A(u', u'+v'), g_{u'}) \\
&= e^{2\pi i k(u'+1)} e^{-2\pi i k(u'+v')} \\
&= e^{-2\pi i k v'} = \Theta_k(A(u', v'), g_{u'}).
\end{aligned}$$

The verification for $g_{v'}$ is similar. This permits to descend the action of $\Gamma$ from $\tilde{\mathcal{L}}$ to $\mathcal{L}^k \to M$ and then get the following action

$$\gamma \in \Gamma \text{ maps to } \tilde{\gamma} : \mathcal{H}^{(k)} \to \mathcal{H}^{(k)}. \tag{5.155}$$

This is not sufficient to define a complex quantum representation of $\Gamma$, indeed an identification of the $SL(2,\mathbb{C})$ quantum Hilbert space with $\mathcal{H}^{(k)}$ depends on a choice of complex structure $\sigma$. Such a choice is not invariant under the action of $\Gamma$, indeed $\gamma \in \Gamma$ acts on $\mathbb{H}$ via Möbius transformations. Nevertheless we can compute $\tilde{S}$ and $\tilde{T}$ as follows

$$\tilde{S}\psi(u', v') = \psi(v', -u') \qquad\qquad \tilde{T}\psi(u', v') = \psi(u', v' + u'). \tag{5.156}$$

We remark that $(\tilde{S}\psi)$ and $\tilde{T}\psi$ are sections of the line bundle $\widetilde{\mathcal{L}}^{2k} \to \mathbb{T}^2$ (recall the description on equation (5.80)) and satisfy

$$\left[R, \tilde{S}\right] = \left[R, \tilde{T}\right] = 0$$

**Lemma 66.**

$$(\tilde{S}\tilde{T})^3 = \tilde{S}^2 \qquad\qquad \tilde{S}^4 = \mathrm{id}\,.$$

A straightforward computation will verify these two formulas. We call this action the *pre-Quantum representation* $\rho_k$ of $\Gamma$, and it is obtained trivially identifying all the spaces $\mathcal{H}_\sigma^{(k)}$ with $\mathrm{C}^\infty(M, \mathcal{L}^k)$. What we are seeking, however, are the *quantum representations* $\eta_t$ which arise when we take into account the full action of $\Gamma$ on the bundle $\mathcal{H}^{(k)} \times \mathbb{H}$ composed with Hitchin-Witten parallel transport $\mathcal{P}_{\sigma_0,\sigma_1}\colon \left(\mathcal{H}^{(k)}, \sigma_1\right) \to \left(\mathcal{H}^{(k)}, \sigma_0\right)$. That is we first consider the pre-quantum action

$$\rho_k(\phi) = \tilde{\phi}\colon \left(\mathcal{H}^{(k)}, \sigma\right) \to \left(\mathcal{H}^{(k)}, \phi_*\sigma\right) \tag{5.157}$$

and then compose it with the parallel transport, to get a representation on $\left(\mathcal{H}^{(k)}, \sigma\right)$

$$\eta_t(\phi) \equiv \mathcal{P}_{\sigma,\phi_*\sigma} \circ \rho_k(\phi) : \left(\mathcal{H}^{(k)}, \sigma\right) \to \left(\mathcal{H}^{(k)}, \sigma\right) \tag{5.158}$$

Let us describe the action of $\gamma \in \Gamma$ on $\mathcal{H}^{(k)} \times \mathbb{H}$. In the first factor $\gamma$ acts on the left as described in (5.156). In the half plane $\mathbb{H}$, $\gamma$ acts on the left by pull-back via $\gamma^{-1}$. Recall the map $\phi_\sigma$ from (5.40), which defines the complex structure parametrized by $\sigma$ on $\Sigma$. Then the complex structure $\gamma_*\sigma$ is defined as the one such that $\phi_{\gamma_*\sigma}$ and $\phi_\sigma \circ \gamma^{-1}$ defines the same complex structures. We recall that this is the case if the two maps are proportional by a complex number. We can compute

$$\begin{aligned}
\phi_\sigma \circ S^{-1}(x,y) &= \phi_\sigma(y, -x) \\
&= y + \sigma^{-1}x \\
&= \sigma^{-1}(x + \sigma y) = \sigma^{-1}\phi_{-\sigma^{-1}}(x,y) \\
\phi_\sigma \circ T^{-1}(x,y) &= \phi_\sigma(x - y, y) \\
&= x - y - \sigma^{-1}y \\
&= x - \left(\frac{1+\sigma}{\sigma}\right)y = \phi_{\frac{\sigma}{1+\sigma}}(x,y).
\end{aligned}$$

Finally we can write

$$S_*\sigma = -\frac{1}{\sigma} \qquad\qquad T_*\sigma = \frac{\sigma}{\sigma+1}. \tag{5.159}$$

### 5.2.5 pre-Quantum Representations

Before we study the quantum representations $\eta_t$ let us concentrate more on the pre-quantum action (5.155). We define the following auxiliary operators on $S(\mathbb{R}) \otimes \mathbb{C}^{2k}$

$$\mathsf{F}_{2k}(\mathsf{f})(x,j) = \frac{1}{\sqrt{2k}} \sum_{l=0}^{2k-1} f_l(x) e^{2\pi i l j/2k} \qquad \mathsf{G}_{2k}(\mathsf{f})(x,j) = e^{\pi i j^2/2k}\mathsf{f}(x,j) \tag{5.160}$$

$$\mathcal{F}_{2k}(\mathsf{f})(x,j) = \int_{\mathbb{R}} f_j(y) e^{4\pi i k x y} \qquad \mathcal{G}_{2k}(\mathsf{f})(x,j) = e^{2\pi i k x^2}(\mathsf{f})(x,j). \tag{5.161}$$

Let us discuss the $\mathbb{S}_2$ action on $\mathcal{S}(\mathbb{R})\otimes\mathcal{S}(\mathbb{Z}_{2k})$ in more details. Consider the following elements of $\mathcal{S}(\mathbb{Z}_{2k})$

$$e_j := (\delta_j + \delta_{2k-j}) \qquad j = 0, \dots, k \tag{5.162}$$
$$\tilde{e}_j := (\delta_j - \delta_{2k-j}) \qquad j = 1, \dots, k-1 \tag{5.163}$$

satisfying

$$e_j(2k-l) = e_j(l) \qquad \tilde{e}_j(2k-l) = -\tilde{e}_j(l). \tag{5.164}$$

We have the following decomposition

$$\mathcal{S}(\mathbb{Z}_{2k}) = \mathcal{C}_{k+1} \oplus \tilde{\mathcal{C}}_{k-1} \tag{5.165}$$
$$\mathcal{C}_{k+1} = \mathrm{span}_{0\le j\le k}\{e_j\} \qquad \tilde{\mathcal{C}}_{k-1} = \mathrm{span}_{0<j<k}\{\tilde{e}_j\} \tag{5.166}$$

Consider also the decomposition of $S(\mathbb{R}) = S_{even}(\mathbb{R}) \oplus S_{odd}(\mathbb{R})$ into even and odd functions. We then have

$$\left(\mathcal{S}(\mathbb{R})\otimes\mathbb{C}^{2k}\right)^{\mathbb{S}_2} = \left(\mathcal{S}_{even}(\mathbb{R})\otimes\mathcal{C}_{k+1}\right) \oplus \left(\mathcal{S}_{odd}(\mathbb{R})\otimes\tilde{\mathcal{C}}_{k-1}\right) \tag{5.167}$$

We will work out the expression of the action (5.156) on $\left(\mathcal{S}(\mathbb{R})\otimes\mathbb{C}^{(2k)}\right)^{\mathbb{S}_2}$, using the WGZ transform. Define

$$\hat{\rho}_k(S) = (W^{(2k)}\circ\mathsf{F}_{2k})^{-1}\circ\widetilde{S}\circ(W^{(2k)}\circ\mathsf{F}_{2k}) \tag{5.168}$$
$$\hat{\rho}_k(T) = (W^{(2k)}\circ\mathsf{F}_{2k})^{-1}\circ\widetilde{T}\circ(W^{(2k)}\circ\mathsf{F}_{2k}) \tag{5.169}$$

The extra conjugation by the discrete Fourier transform $\mathsf{F}_{2k}$ is a technicality that will simplify the exposition, as we will see soon. We remark here that $\left[\hat{\boldsymbol{\nabla}},\mathsf{F}_{2k}\right] = \left[\hat{\Delta}_\sigma,\mathsf{F}_{2k}\right] = 0$, so this extra conjugation do not interfere with the parallel transport. Moreover we remark that $\left[\mathsf{F}_{2k},\, \hat{R}\right] = 0$.
In this subsection we want to prove the following result.

**Theorem 67** (pre-Quantum Representations)**.** *The pre-Quantum representation $\hat{\rho}_k$ splits into the direct sum of two representations, induced by the decomposition (5.167)*

$$\hat{\rho}_k = (\hat{\rho}'_{k,0}\otimes\hat{\rho}''_{k,0}) \oplus (\hat{\rho}'_{k,1}\otimes\hat{\rho}''_{k,1})$$

*where*

$$\hat{\rho}'_{k,0}(S)(f)(x) = -i\sqrt{2k}\int_{\mathbb{R}} f(y)\cos(4\pi kxy)\mathrm{d}y \qquad \hat{\rho}'_k(T)(f)(x) = e^{\pi i/4}e^{-2\pi ikx^2}f(x),$$
$$\hat{\rho}'_{k,1}(S)(f)(x) = \sqrt{2k}\int_{\mathbb{R}} f(y)\sin(4\pi kxy)\mathrm{d}y \qquad \hat{\rho}'_k(T)(f)(x) = e^{\pi i/4}e^{-2\pi ikx^2}f(x),$$
$$\hat{\rho}''_{k,0}(S)(e_j) = i\sqrt{\frac{2}{k}}\sum_{p=0}^{k}\cos\frac{\pi pj}{k}e_p \qquad \hat{\rho}''_{k,0}(T)(e_j) = e^{-\pi i/4}e^{\frac{\pi i}{2k}j^2}e_j,$$
$$\hat{\rho}''_{k,1}(S)(\tilde{e}_j) = \sqrt{\frac{2}{k}}\sum_{p=1}^{k-1}\sin\frac{\pi pj}{k}\tilde{e}_p \qquad \hat{\rho}''_{k,1}(T)(\tilde{e}_j) = e^{-\pi i/4}e^{\frac{\pi i}{2k}j^2}\tilde{e}_j.$$

*Remark* 5.2.4. Consider $\hat{\rho}'_k = \hat{\rho}'_{k,0}\oplus 1\hat{\rho}'_{k,1}$. Define the operator

$$\mathcal{O}_a f(x) = f(ax), \qquad \text{for any } a\in\mathbb{R}_{>0}, \text{ and } f\in\mathcal{S}(\mathbb{R}),$$

For every $\gamma \in \Gamma$ we have

$$\hat{\rho}'_k(\gamma) = \mathcal{O}_{\sqrt{2k}} \circ \hat{\rho}'_1(\gamma) \circ \mathcal{O}_{\frac{1}{\sqrt{2k}}} \tag{5.170}$$

The representation $\hat{\rho}''_{k,1}$, up to some phase, is the same representation computed for quantum SU(2) Chern-Simons theory at level $k$ in [Wei91], using a real polarization. In the work [Jef92] the same representation (with the same phases of ours) are computed with a level shift of 2, i.e. our level $k+2$ representations correspond to the one for level $r$ in [Jef92]. This level shift agrees with the different nature of the quantization in genus 1 for the complex and the unitary theories. As noted in [ADPW91] the level shift is needed in the unitary theory in order to make techniques valid in genus 1 generalize in higher genus. In the complex theory such tweak is not needed.

Let

$$\hat{S} = \overline{W}^{(2k)} \circ \tilde{S} \circ W^{(2k)} \qquad \text{and} \qquad \hat{T} = \overline{W}^{(2k)} \circ \tilde{T} \circ W^{(2k)} \tag{5.171}$$

we can compute the explicit actions as

**Proposition 68.**

$$\hat{S}(\mathsf{f})(x,j) = \sqrt{2k}\,\mathsf{F}_{2k}^{-1} \circ \mathcal{F}_{2k}(\mathsf{f})(x,j), \tag{5.172}$$

$$\hat{T}(\mathsf{f})(x,j) = e^{\pi i/4}\mathsf{G}_{2k}^{-1} \circ \mathcal{G}_{2k}^{-1} \circ \mathsf{F}_{2k} \circ \mathsf{G}_{2k}^{-1}(\mathsf{f})(x,j). \tag{5.173}$$

*Proof.* First we compute that

$$\begin{aligned}\tilde{S} \circ W^{(2k)}(\mathsf{f})(x,y) &= W^{(2k)}(\mathsf{f})(-y,x)\\ &= e^{2\pi ikxy} \sum_{r=0}^{2k-1} \sum_{m\in\mathbb{Z}} \mathsf{f}\left(y + \frac{m}{2k}\right) e^{2\pi imx} e^{-2\pi i\frac{mr}{2k}}\end{aligned}$$

hence we have that

$$\begin{aligned}
&\overline{W}^{(2k)} \circ \tilde{S} \circ W^{(2k)}(\mathsf{f})(u,j) =\\
&= \frac{1}{\sqrt{2k}} \sum_{l=0}^{2k-1} e^{2\pi i\frac{lj}{2k}} \int_0^1 \left(\tilde{S} \circ W^{(2k)}\right)(\mathsf{f})(u - \frac{l}{2k}, v) e^{2\pi ik(u+\frac{l}{k})v} \mathrm{d}v\\
&= \frac{1}{2k} \sum_{l,\,r=0}^{2k-1} \sum_{m\in\mathbb{Z}} e^{2\pi i\frac{lj}{2k}} \int_0^1 e^{2\pi ik\left(u-\frac{l}{2k}\right)v} \mathsf{f}\left(v + \frac{m}{2k}, r\right) e^{2\pi ik\left(u+\frac{l}{2k}\right)v} \mathrm{d}v\\
&\quad \times e^{2\pi im\left(u-\frac{l}{2k}\right)} e^{-2\pi i\frac{mr}{2k}}\\
&= \frac{1}{2k} \sum_{p,\,r=0}^{2k-1} \sum_{q\in\mathbb{Z}} \int_0^1 e^{2\pi ikuv} \mathsf{f}\left(v + q + \frac{p}{2k}, r\right) e^{2\pi ikuv} \mathrm{d}v\, e^{2\pi i(2kq+p)u}\\
&\qquad \times e^{-2\pi i\frac{pr}{2k}} \sum_{l=0}^{2k-1} e^{2\pi i\frac{l(j-p)}{2k}} \qquad (m = 2kq + p,\, 0 \le p < 2k)\\
&= \sum_{r=0}^{2k-1} e^{-2\pi i\frac{jr}{2k}} \sum_{q\in\mathbb{Z}} \int_0^1 e^{2\pi i2ku\left(v+q+\frac{j}{2k}\right)} \mathsf{f}\left(v + q + \frac{j}{2k}, r\right) \mathrm{d}v\\
&= \sum_{r=0}^{2k-1} e^{-2\pi i\frac{jr}{2k}} \sum_{q\in\mathbb{Z}} \int_{q+\frac{j}{2k}}^{q+\frac{j}{2k}+1} \mathsf{f}\left(v, r\right) e^{2\pi i2kuv} \mathrm{d}v
\end{aligned}$$

$$= \sum_{r=0}^{2k-1} e^{-2\pi i \frac{jr}{2k}} \int_{-\infty}^{+\infty} \mathsf{f}\left(v, r\right) e^{2\pi i 2kuv} \mathrm{d}v.$$

For the $\hat{T}$ operator we get that

$$\begin{aligned}
\tilde{T} \circ W^{(2k)}(\mathsf{f})(x,y) &= W^{(2k)}(\mathsf{f})(x, y+x) \\
&= e^{-2\pi i k(x+y)x} \sum_{r=0}^{2k-1} \sum_{m\in\mathbb{Z}} \mathsf{f}\left(x + \frac{m}{2k}\right) e^{-2\pi i m(x+y)} e^{-2\pi i \frac{mr}{2k}}
\end{aligned}$$

so we have that

$$\begin{aligned}
&\overline{W}^{(2k)} \circ \tilde{T} \circ W^{(2k)}(\mathsf{f})(u,j) = \\
&= \frac{1}{\sqrt{2k}} \sum_{l=0}^{2k-1} e^{2\pi i \frac{lj}{2k}} \int_0^1 \left(\tilde{T} \circ W^{(2k)}\right)(\mathsf{f}) \left(u - \frac{l}{2k}, v\right) e^{2\pi i k(u+\frac{l}{k})v} \mathrm{d}v \\
&= \frac{1}{2k} \sum_{r,\, l=0}^{2k-1} \sum_{m\in\mathbb{Z}} e^{2\pi i \frac{lj}{2k}} \int_0^1 e^{-2\pi i k\left(u-\frac{l}{2k}\right)\left(u-\frac{l}{2k}+v\right)} \mathsf{f}\left(u - \frac{l}{2k} + \frac{m}{2k}, r\right) \\
&\quad \times e^{-2\pi i m\left(v+u-\frac{l}{2k}\right)} e^{-2\pi i \frac{mr}{2k}} e^{2\pi i k\left(u+\frac{l}{2k}\right)v} \mathrm{d}v \\
&= \frac{1}{2k} \sum_{r,\, l=0}^{2k-1} \sum_{m\in\mathbb{Z}} e^{2\pi i \frac{lj}{2k}} e^{-2\pi i k\left(u-\frac{l}{2k}\right)\left(u-\frac{l}{2k}\right)} \mathsf{f}\left(u - \frac{l}{2k} + \frac{m}{2k}, r\right) \\
&\quad \times e^{-2\pi i m\left(u-\frac{l}{2k}\right)} e^{-2\pi i \frac{mr}{2k}} \int_0^1 e^{2\pi i k\left(2\frac{l}{2k}-2\frac{m}{2k}\right)v} \mathrm{d}v \\
&= \frac{1}{2k} \sum_{r,\, l=0}^{2k-1} \sum_{m\in\mathbb{Z}} e^{2\pi i \frac{lj}{2k}} e^{-2\pi i k\left(u-\frac{l}{2k}\right)^2} \mathsf{f}\left(u - \frac{l}{2k} + \frac{m}{2k}, r\right) \\
&\quad \times e^{-2\pi i m\left(u-\frac{l}{2k}\right)} e^{-2\pi i \frac{mr}{2k}} \delta(l-m) \\
&= \frac{1}{2k} \sum_{r,\, l=0}^{2k-1} e^{2\pi i \frac{lj}{2k}} e^{-2\pi i k\left(u-\frac{l}{2k}\right)^2} \mathsf{f}\left(u, r\right) e^{-2\pi i l\left(u-\frac{l}{2k}\right)} e^{-2\pi i \frac{lr}{2k}} \\
&= \frac{1}{2k} e^{-2\pi i k u^2} \sum_{r,\, l=0}^{2k-1} e^{\pi i \frac{l^2}{2k}} e^{2\pi i \frac{lj}{2k}} e^{-2\pi i \frac{lr}{2k}} \mathsf{f}\left(u, r\right) \\
&= \frac{e^{\pi i/4}}{2k} e^{-2\pi i k u^2} \sum_{r=0}^{2k-1} e^{-\pi i (j-r)^2/2k} \mathsf{f}\left(u, r\right) \qquad \text{(see Prop. 92)} \\
&= \frac{e^{\pi i/4}}{2k} e^{-2\pi i k u^2} e^{-\pi i j^2/2k} \sum_{r=0}^{2k-1} e^{2\pi i j r/2k} e^{-\pi i r^2/2k} \mathsf{f}\left(u, r\right).
\end{aligned}$$

□

**Lemma 69.** *It is simple to use the finite convolution theorem to see that*

$$\begin{aligned}
\zeta_{2k}(S)(x,j) &\equiv \mathsf{F}_{2k}^{-1} \circ \hat{S} \circ \mathsf{F}_{2k}(\mathsf{f})(x,j) = \sqrt{2k}\, \mathsf{F}_{2k}^{-1} \circ \mathcal{F}_{2k}(\mathsf{f})(x,j) \\
\zeta_{2k}(T)(x,j) &\equiv \mathsf{F}_{2k}^{-1} \circ \hat{T} \circ \mathsf{F}_{2k}(\mathsf{f})(x,j) = \mathcal{G}_{2k}^{-1} \circ \mathsf{G}_{2k}(\mathsf{f})(x,j)
\end{aligned}$$

*Proof.* First, we noticed that

$$\begin{aligned}
\mathsf{G}_{2k}^{-1} \circ \mathsf{F}_{2k} \circ \mathsf{G}_{2k}^{-1}(\mathsf{x})(j) &= e^{-\frac{\pi i j^2}{2k}} \sum_{p=0}^{2k-1} e^{-\frac{\pi i p^2}{2k}} \mathsf{x}(p)\, e^{2\pi i \frac{pj}{2k}} \\
&= \sum_{p=0}^{2k-1} e^{-\pi i \frac{(p-j)^2}{2k}} \mathsf{x}(p)
\end{aligned}$$

$$= (g_{2k} * \mathsf{x})(j)$$

where $g_{2k}(j) \equiv e^{-\pi i \frac{j^2}{2k}}$ and $*$ is the finite convolution $(\mathsf{y} * \mathsf{x})(j) = \sum_{p=0}^{2k-1} \mathsf{y}(p-j)\mathsf{x}(p)$. By the convolution theorem

$$\mathsf{F}_{2k}(\mathsf{x} \cdot \mathsf{y}) = \mathsf{F}_{2k}(\mathsf{x}) * \mathsf{F}_{2k}(\mathsf{y}), \tag{5.174}$$

where $(\mathsf{x} \cdot y)(j) \equiv \mathsf{x}(j)y(j)$. So we can write

$$\begin{aligned} \mathsf{F}_{2k}^{-1} \circ \mathsf{G}_{2k}^{-1} \circ \mathsf{F}_{2k} \circ \mathsf{G}_{2k}^{-1} \circ \mathsf{F}_{2k}(\mathsf{x})(j) &= \mathsf{F}_{2k}^{-1}\left(g_{2k} * \mathsf{F}_{2k}(\mathsf{x})\right)(j) \\ &= \mathsf{F}_{2k}^{-1}(g_{2k}) \cdot \mathsf{x}(j) \end{aligned}$$

We then evaluate the Gauss sum

$$\begin{aligned} \mathsf{F}_{2k}^{-1}(g_{2k})(j) &= \frac{1}{\sqrt{2k}} \sum_{p=0}^{2k-1} e^{-\pi i \frac{p^2}{2k}} e^{-2\pi i \frac{pj}{2k}} \\ &= e^{-\pi i/4} e^{\pi i \frac{j^2}{2k}} \end{aligned}$$

Noticing that $[\mathcal{G}_{2k}, \mathsf{F}_{2k}] = [\mathcal{G}_{2k}, \mathsf{G}_{2k}] = 0$ we have showed that

$$\begin{aligned} \zeta_{2k}(T) &\equiv \mathsf{F}_{2k}^{-1} \circ \hat{T} \circ \mathsf{F}_{2k}(\mathsf{f})(x,j) \\ &= e^{\pi i/4}\, \mathsf{F}_{2k}^{-1} \circ \mathsf{G}_{2k}^{-1} \circ \mathcal{G}_{2k}^{-1} \circ \mathsf{F}_{2k} \circ \mathsf{G}_{2k}^{-1} \circ \mathsf{F}_{2k}(\mathsf{f})(x,j) \\ &= e^{\pi i/4}\, \mathcal{G}_{2k}^{-1}\left(\mathsf{F}_{2k}^{-1}(g_{2k}) \cdot \mathsf{f}\right)(x,j) \\ &= \mathcal{G}_{2k}^{-1} \circ \mathsf{G}_{2k}\mathsf{f}(x,j) \end{aligned}$$

Computations for $\zeta_{2k}(S)$ are trivial, indeed

$$\begin{aligned} \zeta_{2k}(S)(x,j) &\equiv \mathsf{F}_{2k}^{-1} \circ \hat{S} \circ \mathsf{F}_{2k}(\mathsf{f})(x,j) \\ &= \sqrt{2k}\, \mathsf{F}_{2k}^{-1} \circ \mathsf{F}_{2k}^{-1} \circ \mathcal{F}_{2k} \circ \mathsf{F}_{2k}(\mathsf{f})(x,j) \\ &= \sqrt{2k}\, \mathsf{F}_{2k}^{-1} \circ \mathcal{F}_{2k}(\mathsf{f})(x,j) \end{aligned}$$

as $[\mathcal{F}_{2k}, \mathsf{F}_{2k}] = 0$.

□

**Proposition 70.** *The operators $\zeta_{2k}(S)$ and $\zeta_{2k}(T)$ have a tensor product decomposition on $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^{2k}$ as*

$$\zeta_{2k}(S) = \zeta_{2k}(S)' \otimes \zeta_{2k}(S)'' \equiv \sqrt{2k}\, \mathcal{F}_{2k} \otimes \mathsf{F}_{2k}^{-1} \tag{5.175}$$

$$\zeta_{2k}(T) = \zeta_{2k}(T)' \otimes \zeta_{2k}(T)'' \equiv \mathcal{G}_{2k}^{-1} \otimes \mathsf{G}_{2k}. \tag{5.176}$$

*The following two operators generates a representation of $SL(2,\mathbb{Z})$ on $\mathbb{C}^{2k}$.*

$$\zeta_{2k}(S)'' = i\mathsf{F}_{2k}^{-1} \qquad \zeta_{2k}(T)'' = e^{-\frac{\pi i}{4}}\, \mathsf{G}_{2k} \tag{5.177}$$

*while the following two generates a representation of $SL(2,\mathbb{Z})$ on $\mathcal{S}(\mathbb{R})$*

$$\zeta_{2k}(S)' = -i\sqrt{2k}\, \mathcal{F}_{2k} \qquad \zeta_{2k}(T)' = e^{\frac{\pi i}{4}}\, \mathcal{G}_{2k}^{-1}. \tag{5.178}$$

*Proof.* Let $\delta_j \in \mathbb{C}^{2k}$ be the $j$-th unit vector, that is $\delta_j(l) = \delta_{j,l}$ where $\delta_{j,l}$ is the Kronecker delta mod $2k$.

$$i\mathsf{F}_{2k}^{-1}(\delta_j)(l) = \frac{i}{\sqrt{2k}} e^{-2\pi i \frac{lj}{2k}}$$

$$
\begin{aligned}
i\mathsf{F}_{2k}^{-1} \circ i\mathsf{F}_{2k}^{-1}(\delta_j)(l) &= -\frac{1}{2k}\sum_{p=0}^{2k-1} e^{-\pi i \frac{pl}{2k}} e^{-2\pi i \frac{pj}{2k}} \\
&= -\delta_j(-l) \\
\left(i\mathsf{F}_{2k}^{-1}\right)^4(\delta_j)(l) &= \delta_j(l) \\
i\mathsf{F}_{2k}^{-1} \circ e^{-\pi i/4}\mathsf{G}_{2k}(\delta_j)(l) &= \frac{e^{\pi i/4}}{\sqrt{2k}}\sum_{p=0}^{2k-1} e^{\pi i p^2/2k} e^{-2\pi i pl/2k}\delta_j(p) \\
&= \frac{e^{\pi i/4}}{\sqrt{2k}} e^{\pi i j^2/2k} e^{-2\pi i jl/2k} \\
\left(i\mathsf{F}_{2k}^{-1} \circ e^{-\pi i/4}\mathsf{G}_{2k}\right)^2(\delta_j)(l) &= \frac{e^{\pi i/2}}{\sqrt{2k}}\sum_{p=0}^{2k-1} e^{\pi i p^2/2k} e^{-2\pi i pl/2k}\mathsf{F}_{2k}^{-1} \circ \mathsf{G}_{2k}(\delta_j)(p) \\
&= \frac{e^{\pi i/2}}{2k}\sum_{p=0}^{2k-1} e^{\pi i p^2/2k} e^{-2\pi i pl/2k} e^{\pi i j^2/2k} e^{-2\pi i jp/2k} \\
&= \frac{e^{\pi i/2}}{2k} e^{\pi i j^2/2k}\sum_{p=0}^{2k-1} e^{\frac{\pi i p}{2k}(p-2(l+j))} \\
&= \frac{e^{\pi i/2}}{\sqrt{2k}} e^{\pi i j^2/2k} e^{\frac{\pi i}{8k}(2k-4(l+j)^2)} \\
&= \frac{e^{3\pi i/4}}{\sqrt{2k}} e^{-\pi i l^2/2k} e^{-2\pi i jl/2k} \\
\left(i\mathsf{F}_{2k}^{-1} \circ e^{\pi i/4}\mathsf{G}_{2k}\right)^3(\delta_j)(l) &= \frac{e^{3\pi i/4}}{\sqrt{2k}}\sum_{p=0}^{2k-1} e^{\pi i p^2/2k} e^{-2\pi i pl/2k}\left(\mathsf{F}_{2k}^{-1} \circ \mathsf{G}_{2k}\right)^2(\delta_j)(p) \\
&= \frac{e^{4\pi i/4}}{2k}\sum_{p=0}^{2k-1} e^{\pi i p^2/2k} e^{-\pi i p^2/2k} e^{-2\pi i pl/2k} e^{-2\pi i pj/2k} \\
&= -\delta_j(-l)
\end{aligned}
$$

For $\mathcal{F}_{2k}$ and $\mathcal{G}_{2k}$ we will use the Bra-Ket notation to compute their kernels see Appendix A.1. We see immediately that

$$
\langle x|\mathcal{F}_{2k}|y\rangle = e^{4\pi i k x y} \qquad \langle x|\mathcal{G}_{2k}^{-1}|y\rangle = e^{-2\pi i k x^2}\delta(x-y) \tag{5.179}
$$

We can therefore compute

$$
\begin{aligned}
\langle x|\mathcal{F}_{2k}^2|y\rangle &= \int_{\mathbb{R}} \langle x|\mathcal{F}_{2k}|z\rangle\langle z|\mathcal{F}_{2k}|y\rangle \mathrm{d}z \\
&= \int_{\mathbb{R}} e^{4\pi i k z(x+y)}\mathrm{d}z = \frac{1}{2k}\delta(x+y) \\
\langle x|\mathcal{F}_{2k}\mathcal{G}_{2k}^{-1}|y\rangle &= \int_{\mathbb{R}} \langle x|\mathcal{F}_{2k}|z\rangle\langle z|\mathcal{G}_{2k}^{-1}|y\rangle \mathrm{d}z \\
&= e^{4\pi i k x y} e^{-2\pi i k y^2} \\
\langle x|\left(\mathcal{F}_{2k}\mathcal{G}_{2k}^{-1}\right)^2|y\rangle &= \int_{\mathbb{R}} \langle x|\mathcal{F}_{2k}\mathcal{G}_{2k}^{-1}|z\rangle\langle z|\mathcal{F}_{2k}\mathcal{G}_{2k}^{-1}|y\rangle \mathrm{d}z \\
&= \int_{\mathbb{R}} e^{4\pi i k z(x+y)} e^{-2\pi i k z^2}\mathrm{d}z e^{-2\pi i k y^2} \\
&= \frac{1}{\sqrt{2ki}} e^{2\pi i k x^2} e^{4\pi i k x y}
\end{aligned}
$$

$$
\begin{aligned}
\langle x|\left(\mathcal{F}_{2k}\mathcal{G}_{2k}^{-1}\right)^{3}|y\rangle &= \int_{\mathbb{R}}\langle x|\mathcal{F}_{2k}\mathcal{G}_{2k}^{-1}|z\rangle\langle z|\left(\mathcal{F}_{2k}\mathcal{G}_{2k}^{-1}\right)^{2}|y\rangle \mathrm{d}z \\
&= \frac{1}{\sqrt{2ik}}\int_{\mathbb{R}} e^{4\pi ikz(x+y)}\mathrm{d}z \\
&= e^{-\pi i/4}(2k)^{-\frac{3}{2}}\delta(x+y)
\end{aligned}
$$

□

Recall the following basis elements of $\mathbb{C}^{2k}$

$$
e_j := (\delta_j + \delta_{2k-j}) \qquad j = 0,\ \dots,\ k \tag{5.180}
$$
$$
\tilde{e}_j := (\delta_j - \delta_{2k-j}) \qquad j = 1,\ \dots,\ k-1 \tag{5.181}
$$

They satisfy

$$
\langle e_j, e_l\rangle = \delta_{j,\,l}, \qquad \langle \tilde{e}_j, \tilde{e}_l\rangle = \delta_{j,\,l}, \tag{5.182}
$$
$$
\langle \tilde{e}_j, e_l\rangle = \langle e_j, \tilde{e}_l\rangle = 0, \tag{5.183}
$$

where

$$
\langle x, y\rangle = \frac{1}{2}\sum_{p=0}^{2k-1} x_p\overline{y}_p \qquad x,\ y\in\mathbb{C}^{2k}. \tag{5.184}
$$

Indeed

$$
\begin{aligned}
\langle \tilde{e}_j,\ \tilde{e}_l\rangle &= \frac{1}{2}\sum_{p=0}^{2k-1}\left(\delta_j(p) - \delta_{2k-j}(p)\right)\left(\delta_l(p) - \delta_{2k-l}(p)\right) \\
&= \frac{1}{2}\left(\delta_{j,l} + \delta_{2k-j,2k-l} - \delta_{j,2k-l} - \delta_{2k-j,l}\right) \\
&= \delta_{j,l} \qquad\qquad \text{being } 1\le j,\ l\le k-1 \\
\langle e_j,\ \tilde{e}_l\rangle &= \frac{1}{2}\sum_{p=0}^{2k-1}\left(\delta_j(p) + \delta_{2k-j}(p)\right)\left(\delta_l(p) - \delta_{2k-l}(p)\right) \\
&= \frac{1}{2}\sum_{p=0}^{2k-1}\left(\delta_l(p)\delta_j(p) + \delta_l(p)\delta_{2k-j}(p) - \delta_{2k-l}(p)\delta_j(p) - \delta_{2k-l}(p)\delta_{2k-j}(p)\right) \\
&= \frac{1}{2}\left(\delta_{j,l} - \delta_{2k-j,2k-l} + \delta_{2k-j,l} - \delta_{j,2k-l}\right) = 0
\end{aligned}
$$

Recall also the induced splitting

$$
\mathbb{C}^{2k} = \mathcal{C}_{k+1}\oplus\tilde{\mathcal{C}}_{k-1} \tag{5.185}
$$
$$
\mathcal{C}_{k+1} = \mathrm{span}_{0\le j\le k}\{e_j\} \qquad \tilde{\mathcal{C}}_{k-1} = \mathrm{span}_{0<j<k}\{\tilde{e}_j\} \tag{5.186}
$$

We can compute

**Lemma 71.**

$$
\langle e_j, \mathsf{F}_{2k}e_l\rangle = \frac{2}{\sqrt{2k}}\cos\frac{\pi lj}{k} \qquad \langle \tilde{e}_j, \mathsf{F}_{2k}\tilde{e}_l\rangle = \frac{2i}{\sqrt{2k}}\sin\frac{\pi lj}{k} \tag{5.187}
$$
$$
\langle e_j, \mathsf{G}_{2k}e_l\rangle = e^{\frac{\pi i}{2k}j^2}\delta_{j,\,l} \qquad \langle \tilde{e}_j, \mathsf{G}_{2k}\tilde{e}_l\rangle = e^{\frac{\pi i}{2k}j^2}\delta_{j,\,l} \tag{5.188}
$$
$$
\langle e_j, \mathsf{G}_{2k}\tilde{e}_l\rangle = \langle e_j, \mathsf{F}_2\tilde{e}_l\rangle = 0. \tag{5.189}
$$

*Proof.*

$$\begin{aligned}
\langle \tilde{e}_l,\, \mathsf{F}_{2k}\tilde{e}_l\rangle &= \frac{1}{2\sqrt{2k}} \sum_{p=0}^{2k-1} \tilde{e}_l(p) \sum_{q=0}^{2k-1} \tilde{e}_j(q) e^{2\pi i \frac{pq}{2k}} \\
&= \frac{1}{2\sqrt{2k}} \sum_{q,\, p=0}^{2k-1} \left(\delta_l(p) - \delta_{2k-l}(p)\right)\left(\delta_j(q) - \delta_{2k-j}(q)\right) e^{2\pi i \frac{pq}{2k}} \\
&= \frac{1}{2\sqrt{2k}} \sum_{q,\, p=0}^{2k-1} \left(\delta_l(p)\delta_j(q) + \delta_{2k-l}(p)\delta_{2k-j}(q) - \delta_l(p)\delta_{2k-j}(q) - \delta_{2k-l}(p)\delta_j(q)\right) e^{2\pi i \frac{pq}{2k}} \\
&= \frac{1}{\sqrt{2k}} \left(e^{2\pi i \frac{lj}{2k}} - e^{-2\pi i \frac{lj}{2k}}\right) \\
&= \frac{2i}{\sqrt{2k}} \sin \pi lj/k \\
\langle \tilde{e}_l,\, \mathsf{G}_{2k}\tilde{e}_j\rangle &= \frac{1}{2} \sum_{p=0}^{2k-1} \tilde{e}_l(p) e^{\frac{\pi i p^2}{2k}} \tilde{e}_j(p) \\
&= \frac{1}{2} \sum_{p=0}^{2k-1} \left(\delta_l(p) - \delta_{2k-l}(p)\right) e^{\frac{\pi i p^2}{2k}} \left(\delta_j(p) - \delta_{2k-j}(p)\right) \\
&= \frac{1}{2} e^{\frac{\pi i j^2}{2k}} \left(\delta_{l,j} + \delta_{2k-l,2k-j}\right) - \frac{1}{2} e^{\frac{\pi i j^2}{2k}} \left(\delta_{l,2k-j} + \delta_{2k-l,j}\right) \\
&= e^{\frac{\pi i j^2}{2k}} \delta_{j,l} \qquad\qquad \text{being } 1 \le j,\, l \le k-1 \\
\langle \tilde{e}_l, \mathsf{F}_{2k} e_j\rangle &= \frac{1}{2\sqrt{2k}} \sum_{p=0}^{2k-1} \tilde{e}_l(p) \sum_{q=0}^{2k-1} e_j(q) e^{2\pi i \frac{pq}{2k}} \\
&= \frac{1}{2\sqrt{2k}} \sum_{q,\, p=0}^{2k-1} \left(\delta_l(p) - \delta_{2k-l}(p)\right)\left(\delta_j(q) + \delta_{2k-j}(q)\right) e^{2\pi i \frac{pq}{2k}} \\
&= \frac{1}{2\sqrt{2k}} \sum_{q,\, p=0}^{2k-1} \left(\delta_l(p)\delta_j(q) + \delta_l(p)\delta_{2k-j}(q) - \delta_{2k-l}(p)\delta_j(q) - \delta_{2k-l}(p)\delta_{2k-j}(q)\right) e^{2\pi i \frac{pq}{2k}} \\
&= \frac{1}{2\sqrt{2k}} \left(e^{2\pi i \frac{lj}{2k}} + e^{-2\pi i \frac{lj}{2k}} - e^{-2\pi i \frac{lj}{2k}} - e^{2\pi i \frac{lj}{2k}}\right) = 0.
\end{aligned}$$

The remaining are similar to the above. □

*Proof of Theorem 67* Lemma 71 together with Proposition 70 provides a proof of Theorem 67. Indeed we have all the representations explicated over $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^{(2k)}$ as we called it $\zeta_{2k}$ and we showed its compatibility with the tensor product in Proposition 70. All of the operators involved preserves odd and even decomposition of $\mathcal{S}(\mathbb{R})$, in particular the Fourier transform decomposes into Cosine and Sine transform. Similarly Lemma 71 gives the decomposition of the finite operators into invariant and anti-invariant parts. We can reconstruct $\hat{\rho}_k$ as follows

$$\begin{aligned}
\hat{\rho}''_k(S) &\equiv \zeta_{2k}(S)'' & \qquad \hat{\rho}''_k(T) &\equiv \zeta_{2k}(T)'' \\
\hat{\rho}'_k(S) &\equiv \zeta_{2k}(S)' & \qquad \hat{\rho}'_k(T) &\equiv \zeta_{2k}(T)'
\end{aligned}$$

then the restriction to even and respectively odd functions makes the decompositions

$$\hat{\rho}'_k = \hat{\rho}'_{k,0} \oplus \hat{\rho}'_{k,1} \qquad\qquad \hat{\rho}''_k = \hat{\rho}''_{k,0} \oplus \hat{\rho}''_{k,1} \tag{5.190}$$

□

### 5.2.6 Quantum Representations

Given $\gamma \in \Gamma$ a its pre-quantum action on $\mathcal{H}^{(k)}$ was defined in (5.156), however when we look at the action on the whole bundle $\mathcal{H}^{(k)} \times \mathbb{H} \to \mathbb{H}$, $\gamma$ acts on $\mathbb{H}$ as described in (5.159). We will then need to compose $\tilde{\gamma}$ with the parallel transport $\mathcal{P}_{\sigma,\gamma_*\sigma}$ of the pre-quantum action with the Hitchin-Witten connection from $\gamma_*\sigma$ back to $\sigma$. By the results in Proposition 62 we have

$$\mathcal{P}_{\sigma_0,\sigma_1}\psi_{\sigma_1} = e^{-r\Delta_{\sigma_0}} e^{r\Delta_{\sigma_1}}\psi_{\sigma_1}. \tag{5.191}$$

**Theorem 72.** *The operators*

$$\eta_t(S) = e^{-r\Delta_{\sigma_0}} \circ e^{r\Delta_{S_*\sigma_0}} \circ \tilde{S} \qquad\qquad \eta_t(T) = e^{-r\Delta_{\sigma_0}} \circ e^{r\Delta_{T_*\sigma_0}} \circ \tilde{T}$$

*generates a representation of* $\Gamma$, *that we call quantum representation.*

Actually, as we did for the pre-quantum representations $\rho_k$, we will study the WGZ–conjugated representations

$$\hat{\eta}_t(S) = (W^{(2k)} \circ \mathsf{F}_{2k})^{-1} \circ \eta_t(S) \circ (W^{(2k)} \circ \mathsf{F}_{2k}) \tag{5.192}$$
$$\hat{\eta}_t(T) = (W^{(2k)} \circ \mathsf{F}_{2k})^{-1} \circ \eta_t(T) \circ (W^{(2k)} \circ \mathsf{F}_{2k}). \tag{5.193}$$

The main theorem for this subsection

**Theorem 73** (Quantum Representations)**.** *Let* $\mathrm{b} \in \mathbb{C}$ *such that* $\mathrm{Re}(\mathrm{b}) > 0$, *and* $t = \frac{2k}{1+\mathrm{b}^2}$. *The Quantum representation* $\hat{\eta}_t$ *splits into the direct sum of two representations, induced by the decomposition (5.167)*

$$\hat{\eta}_t = (\hat{\eta}'_{t,0} \otimes \hat{\rho}''_{k,0}) \oplus (\hat{\eta}'_{t,1} \otimes \hat{\rho}''_{k,1}).$$

*The representation* $\hat{\eta}_t$ *is conjugate to* $\hat{\rho}_k$*.*

$$\hat{\eta}'_t = e^{-r\hat{\Delta}_\sigma} \circ \hat{\rho}'_k \circ e^{r\hat{\Delta}_\sigma}, \qquad \hat{\eta}_t = e^{-r\hat{\Delta}_\sigma} \circ \hat{\rho}_k \circ e^{r\hat{\Delta}_\sigma}.$$

*For the particular choice of complex structure* $\sigma = i\mathrm{b}$ *the representations take the following explicit integral form*

$$\hat{\eta}'_{t,0}(S)(f)(x) = -i\sqrt{2k}\; e^{2\pi k(\mathrm{b}-\bar{\mathrm{b}})x^2} \int_{\mathbb{R}} f(y)\cos(4\pi kyx) e^{-2\pi k(\mathrm{b}-\bar{\mathrm{b}})y^2}\mathrm{d}y,$$
$$\hat{\eta}'_{t,1}(S)(f)(x) = \sqrt{2k}\; e^{2\pi k(\mathrm{b}-\bar{\mathrm{b}})x^2} \int_{\mathbb{R}} f(y)\sin(4\pi ikyx) e^{-2\pi k(\mathrm{b}-\bar{\mathrm{b}})y^2}\mathrm{d}y,$$
$$\hat{\eta}'_{t,0}(T)(f)(x) = \hat{\eta}'_{t,1}(T)(f)(x) = e^{\pi i/4}\sqrt{\frac{2k}{i}}\; e^{2\pi k(\mathrm{b}-\bar{\mathrm{b}})x^2} \int_{\mathbb{R}} f(y) e^{2\pi ik(x-y)^2} e^{-2\pi k(\mathrm{b}-\bar{\mathrm{b}})y^2}\mathrm{d}y.$$

*Remark* 5.2.5. From equation (5.129) we can see that

$$\lim_{s\to\infty} e^{-2kr} = \pm 1.$$

It follows that $e^{-r\hat{\Delta}_\sigma} = e^{-2k\hat{N}_\sigma} \to (\pm 1)^{\hat{N}_\sigma}$ and the last operator is constant when restricted to the either even or odd elements $v_n$ of the Hilbert basis (recall it from Remark 5.2.3). So we can see that the conjugation from the previous Theorem trivialize in the limit $s \to \infty$, giving back the pre-quantum representations $\hat{\rho}_k$.

Let $\sigma_0$, $\sigma_1 \in \mathbb{H}$ and recall the spectral analysis of $\hat{\Delta}_{\sigma_0}$ and $\hat{\Delta}_{\sigma_1}$ from Proposition 64 and subsequent Remark 5.2.3. The first eigenvectors are related by the relation

$$v(x,\sigma_1) = \mathcal{Q}_{\sigma_1,\sigma_0}\, v(x,\sigma_0), \qquad \mathcal{Q}_{\sigma_1,\sigma_0} = e^{-2\pi ikx^2\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_0}\right)} \tag{5.194}$$

In general, defined the $n$-th eigenvectors for the the two operators as

$$v_n(x,\sigma_0) = \hat{\mathcal{A}}^n_{\sigma_0} v(x,\sigma_0) \qquad\qquad v_n(x,\sigma_1) = \hat{\mathcal{A}}^n_{\sigma_1} v(x,\sigma_1)$$

then the base change is as follows

$$v_n(x,\sigma_1) = \hat{\mathcal{A}}^n_{\sigma_1} \mathcal{Q}_{\sigma_1,\sigma_0} \hat{\mathcal{A}}^{-n}_{\sigma_0}\, v_n(x,\sigma_0) \tag{5.195}$$

If we look at the particular case when

$$\sigma_1 = T_*\sigma_0 = \frac{\sigma_0}{1+\sigma_0},$$

the relations simplify as follows

$$\mathcal{Q}_{T_*\sigma_0,\,\sigma_0} = e^{-2\pi ikx^2} = \mathcal{G}^{-1}_{2k} \qquad \hat{\mathcal{A}}_{T_*\sigma_0} = \left(\hat{\mathcal{A}}_{\sigma_0} - 4\pi ikx\right) \tag{5.196}$$

which give

$$\mathcal{B}_{\sigma_0} = \{(H_{n,\alpha}v)(x,\sigma_0), \text{ such that } n \in \mathbb{Z}_{\geq 0}\} \tag{5.197}$$

$$\mathcal{B}_{T_*\sigma_0} = \left\{\mathcal{G}^{-1}_{2k}(H_{n,\alpha}v)(x,\sigma_0), \text{ such that } n \in \mathbb{Z}_{\geq 0}\right\} \tag{5.198}$$

where $H_{n,\alpha}(x)$ is defined in Appendix B and

$$\alpha = \frac{\overline{\sigma}-\sigma}{\sigma\overline{\sigma}}2\pi ik \qquad\qquad v(x,\sigma_0) = e^{-2\pi kix^2/\sigma_0}. \tag{5.199}$$

In particular

$$\mathcal{G}^{-1}_{2k}\mathcal{B}_{\sigma_0} = \mathcal{B}_{T_*\sigma_0} \tag{5.200}$$

and the parallel transport operator

$$\hat{\mathcal{P}}_{\sigma_0,\,T_*\sigma_0} = e^{-r\hat{\Delta}_{\sigma_0}} \circ e^{r\hat{\Delta}_{T_*\sigma_0}}$$

satisfies

$$\hat{\mathcal{P}}_{\sigma_0,\,T_*\sigma_0} = e^{-r\hat{\Delta}_{\sigma_0}} \circ \mathcal{G}^{-1}_{2k} \circ e^{r\hat{\Delta}_{\sigma_0}} \circ \mathcal{G}_{2k}. \tag{5.201}$$

Since $\hat{\Delta}_\sigma$ acts trivially on the second factor of the tensor $\mathcal{S}(\mathbb{R})\otimes\mathbb{C}^{2k}$ it is not restrictive to discuss the parallel transport of the representation $\hat{\rho}'_k$ first. Recall that it is the direct sum of the *odd* and *even* representations on $\mathcal{S}(\mathbb{R})$ as said in equation (5.190). From Theorem 67 we have

$$\begin{aligned}
\hat{\eta}'_t(T) &= \hat{\mathcal{P}}_{\sigma_0,\,T_*\sigma_0} \circ \hat{\rho}'_k(T) && (5.202)\\
&= e^{\pi i/4} e^{-r\hat{\Delta}_{\sigma_0}} \circ \mathcal{G}^{-1}_{2k} \circ e^{r\hat{\Delta}_{\sigma_0}} \circ \mathcal{G}_{2k} \circ \mathcal{G}^{-1}_{2k} && (5.203)\\
&= e^{\pi i/4} e^{-r\hat{\Delta}_{\sigma_0}} \circ \mathcal{G}^{-1}_{2k} \circ e^{r\hat{\Delta}_{\sigma_0}} && (5.204)\\
&= e^{-r\hat{\Delta}_{\sigma_0}} \circ \hat{\rho}'_k(T) \circ e^{r\hat{\Delta}_{\sigma_0}} && (5.205)
\end{aligned}$$

Let now analyze the parallel transport of $\hat{\rho}'_k(S) = -i\sqrt{2k}\,\mathcal{F}_{2k}$. We have

$$S_*\sigma = -\sigma^{-1}.$$

Let us remark some commutation properties of $\mathcal{F}_{2k}$. First it is simple to see

$$\frac{\mathrm{d}}{\mathrm{d}x} \circ \mathcal{F}_{2k} = \mathcal{F}_{2k} \circ (4\pi ik\hat{x}) \qquad (4\pi ik\hat{x}) \circ \mathcal{F}_{2k} = -\mathcal{F}_{2k} \circ \frac{\mathrm{d}}{\mathrm{d}x} \tag{5.206}$$

where $\hat{x}f(x) = xf(x)$ (it is well known the analogous for the Fourier Transform). Recall from (5.133) and (5.138)

$$\hat{\Delta}_\sigma = \frac{i}{2\pi}(\sigma - \overline{\sigma})D_\sigma D_{\overline{\sigma}}$$

and

$$D_\sigma = \frac{1}{\overline{\sigma} - \sigma}\left(\overline{\sigma}\frac{\mathrm{d}}{\mathrm{d}x} + 2\pi i 2k\hat{x}\right)$$

We have

$$D_\sigma \circ \mathcal{F}_{2k} = \frac{1}{\sigma}\mathcal{F}_{2k} \circ D_{-\sigma^{-1}}$$

indeed

$$\begin{aligned}
D_\sigma \circ \mathcal{F}_{2k} &= \frac{1}{\overline{\sigma} - \sigma}\left(\overline{\sigma}\frac{\mathrm{d}}{\mathrm{d}x} + 2\pi i 2k\hat{x}\right) \circ \mathcal{F}_{2k} \\
&= \mathcal{F}_{2k} \circ \frac{1}{\overline{\sigma} - \sigma}\left(\overline{\sigma} 2\pi i 2k\hat{x} - \frac{\mathrm{d}}{\mathrm{d}x}\right) \\
&= \mathcal{F}_{2k} \circ \frac{\overline{\sigma}}{\overline{\sigma} - \sigma}\left(-\overline{\sigma}^{-1}\frac{\mathrm{d}}{\mathrm{d}x} + 2\pi i 2k\hat{x}\right) \\
&= \mathcal{F}_{2k} \circ \sigma^{-1}\frac{\sigma\overline{\sigma}}{\overline{\sigma} - \sigma}\left(-\overline{\sigma}^{-1}\frac{\mathrm{d}}{\mathrm{d}x} + 2\pi i 2k\hat{x}\right) \\
&= \mathcal{F}_{2k} \circ \sigma^{-1}\frac{1}{-\overline{\sigma}^{-1} + \sigma^{-1}}\left(-\overline{\sigma}^{-1}\frac{\mathrm{d}}{\mathrm{d}x} + 2\pi i 2k\hat{x}\right) = \mathcal{F}_{2k} \circ \sigma^{-1} D_{-\sigma^{-1}}
\end{aligned}$$

It follows that

$$\hat{\Delta}_\sigma \circ \mathcal{F}_{2k} = \mathcal{F}_{2k} \circ \frac{i}{2\pi}\frac{(\sigma - \overline{\sigma})}{\sigma\overline{\sigma}} D_{-\sigma^{-1}} D_{-\overline{\sigma}^{-1}} = \mathcal{F}_{2k} \circ \hat{\Delta}_{-\sigma^{-1}} \tag{5.207}$$

So we can write

$$\begin{aligned}
\hat{\eta}'_t(S) &= \hat{\mathcal{P}}_{\sigma_0, S_*\sigma_0} \circ \hat{\rho}'(S) \\
&= -i\sqrt{2k}\, e^{-r\hat{\Delta}_{\sigma_0}} \circ e^{r\hat{\Delta}_{-\sigma_0^{-1}}} \circ \mathcal{F}_{2k} \\
&= -i\sqrt{2k}\, e^{-r\hat{\Delta}_{\sigma_0}} \circ \mathcal{F}_{2k} \circ e^{r\hat{\Delta}_{\sigma_0}} \\
&= e^{-r\hat{\Delta}_{\sigma_0}} \circ \hat{\rho}'_k(S) \circ e^{r\hat{\Delta}_{\sigma_0}}
\end{aligned}$$

What we did so far can be phrased as

**Proposition 74.** *The representations $\hat{\eta}'_t$ is conjugate to the representation $\hat{\rho}'_k$ via the operator $e^{-r\hat{\Delta}_{\sigma_0}}$, i.e.*

$$\hat{\eta}'_t(\gamma) = e^{-r\hat{\Delta}_{\sigma_0}} \circ \hat{\rho}'_k(\gamma) \circ e^{r\hat{\Delta}_{\sigma_0}}, \qquad \forall \gamma \in \Gamma.$$

So we reduced the two parallel transport we need to understand the operator $e^{r\hat{\Delta}_{\sigma_0}}$ for a specific $\sigma_0$. Indeed once we write explicit representations for a specific $\sigma_0$ we do already know that the other are isomorphic, related by conjugation via parallel transport operator. The next Lemma will choose a a particularly simple $\sigma_0$.

**Lemma 75.** *Suppose that $\sigma = i\mathrm{b}$. Then, for every $\psi \in \mathrm{L}^2(\mathbb{R})$,*

$$e^{-r\hat{\Delta}_{i\mathrm{b}}}\psi(x) = \sqrt{2k\mathrm{b}}\, e^{2\pi k(\mathrm{b} - \overline{\mathrm{b}})x^2}\mathcal{F}_{2k}(\psi)(x) \tag{5.208}$$

*Proof.* First we remark that $\mathrm{Re}\,\mathrm{b} > 0 \implies \mathrm{Im}\,\sigma > 0$. From Lemma 65 we have

$$e^{-r\Delta_{i\mathrm{b}}}\psi(x) = \sqrt{\frac{\alpha}{\pi}}\frac{1}{\sqrt{1 + \mathrm{b}^2}}\int_{\mathbb{R}} \exp\left(\alpha\frac{2i\mathrm{b}xy + \mathrm{b}^2(x^2 + y^2)}{1 + \mathrm{b}^2}\right) v(x, i\mathrm{b})\overline{v(y, i\mathrm{b})}\psi(y)\mathrm{d}y \tag{5.209}$$

where

$$\alpha = \frac{-i\overline{\mathrm{b}} - i\mathrm{b}}{i^2 \mathrm{b}\overline{\mathrm{b}}} 2\pi i k = 2\pi k(\mathrm{b} + \overline{\mathrm{b}}), \qquad v(x, i\mathrm{b}) = e^{-2\pi i k x^2/(i\mathrm{b})} = e^{-2\pi k \overline{\mathrm{b}} x^2}.$$

Notice that $\frac{\mathrm{b}+\overline{\mathrm{b}}}{1+\mathrm{b}^2} = \frac{1}{\mathrm{b}} = \overline{\mathrm{b}}$. The coefficient of the exponential of $x^2$ is

$$2\pi k\mathrm{b} - 2\pi k\overline{\mathrm{b}} = 2\pi k(\mathrm{b} - \overline{\mathrm{b}}),$$

the coefficient of $y^2$ is

$$2\pi k\mathrm{b} - 2\pi k\mathrm{b} = 0$$

and the one for $xy$ is $4\pi i k$. So the integral writes

$$e^{-r\Delta_{i\mathrm{b}}}\psi(x) = \sqrt{2k(\mathrm{b}+\overline{\mathrm{b}})} \frac{1}{\sqrt{1+\mathrm{b}^2}} \int_{\mathbb{R}} e^{2\pi k(\mathrm{b}-\overline{\mathrm{b}})x^2} e^{4\pi i k x y} \psi(y) \mathrm{d}y \tag{5.210}$$

$$= \sqrt{2k\mathrm{b}}\, e^{2\pi k(\mathrm{b}-\overline{\mathrm{b}})x^2} \mathcal{F}_{2k}(\psi)(x) \tag{5.211}$$

□

**Lemma 76.** *The $\Delta$ operator is mapping class group invariant, i. e. let $\hat{\gamma}$ be the pre quantum operator acting on $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^{2k}$ induced by $\gamma \in \Gamma$, we have*

$$\hat{\gamma}\hat{\Delta}_\sigma \psi = \hat{\Delta}_{\gamma_*\sigma}\hat{\gamma}\psi, \qquad \psi \in \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^{2k}. \tag{5.212}$$

*In particular the parallel transport is Mapping Class Group invariant.*

*Proof.* This is no new, as it is known that the Hitchin-Witten connection is mapping class group invariant. However we verify it explicitly here. First recall that the $\hat{\Delta}$ operator acts trivially on the second factor of the tensor decomposition of $\mathcal{S}(\mathbb{R})\otimes\mathbb{C}^{2k}$. So we forget about it and think only of equation 5.212 as on $\mathcal{S}(\mathbb{R})$. For the $\hat{S} = -i\sqrt{2k}\mathcal{F}_{2k}$ operator, equation (5.207) is equivalent to (5.212). For the $\hat{T} = e^{\pi i/4}\mathcal{G}_{2k}^{-1}$ operator, from the eigenfunctions properties we saw in (5.200), we have

$$\hat{\Delta}_{T_*\sigma} \circ \hat{T} = \mathcal{G}_{2k}^{-1} \circ \hat{\Delta}_\sigma \circ \mathcal{G}_{2k} \circ \hat{T} = e^{\pi i/4}\mathcal{G}_{2k}^{-1} \circ \hat{\Delta}_\sigma = \hat{T} \circ \hat{\Delta}_\sigma.$$

□

*Proof of Theorem 73*

The finite action on $\mathbb{C}^{2k}$ was already computed in Lemma 71 and the parallel transport is identical in such tensor factor so there is nothing going on.

Proposition 74 gives first half of Theorem 73. We define the following operators for simplifying the notation in the proof

$$\mathcal{O}_a\phi(x) \equiv \phi(ax) \qquad\qquad \mathcal{N}_{\mathrm{b}}\psi(x) = e^{2\pi k(\mathrm{b}-\overline{\mathrm{b}})x^2}\psi(x). \tag{5.213}$$

Then we have

$$\mathcal{F}_{2k} = \mathcal{O}_{2k} \circ \mathcal{F} = \frac{1}{2k}\mathcal{F} \circ \mathcal{O}_{1/2k} \tag{5.214}$$

where $\mathcal{F}$ is the Fourier Transform. Fixing b and consequently choosing $\sigma = i\mathrm{b}$ we have

$$e^{-r\hat{\Delta}_{i\mathrm{b}}} = \sqrt{2k\mathrm{b}}\,\mathcal{N}_{\mathrm{b}} \circ \mathcal{F}_{2k} = \sqrt{2k\mathrm{b}}\,\mathcal{N}_{\mathrm{b}} \circ \mathcal{O}_{2k} \circ \mathcal{F} \tag{5.215}$$

The quantum representation of $S$ takes the form

$$\begin{aligned}\hat{\eta}'_t(S) &= -i\sqrt{2k}\, e^{-r\hat{\Delta}_{i\mathrm{b}}} \circ \mathcal{F}_{2k} \circ e^{r\hat{\Delta}_{i\mathrm{b}}} \\ &= -i\sqrt{2k}\, e^{-r\hat{\Delta}_{i\mathrm{b}}} \circ \mathcal{F}_{2k} \circ \left(e^{-r\hat{\Delta}_{i\mathrm{b}}}\right)^{-1} \\ &= -i\sqrt{2k}\,\sqrt{2k\mathrm{b}}\,\mathcal{N}_{\mathrm{b}} \circ \mathcal{F}_{2k} \circ \mathcal{F}_{2k} \circ \left(\sqrt{2k\mathrm{b}}\,\mathcal{N}_{\mathrm{b}} \circ \mathcal{F}_{2k}\right)^{-1} \\ &= -i\sqrt{2k}\,\mathcal{N}_{\mathrm{b}} \circ \mathcal{F}_{2k} \circ \mathcal{N}_{\mathrm{b}}^{-1}\end{aligned}$$

This concludes the proof for the operator $S$. For the operator $T$, notice that

$$\mathcal{O}_{2k} \circ \mathcal{G}_{2k}^{-1}\psi(x) = e^{-2\pi ik(x/2k)^2}\psi(x/2k) = \mathcal{G}_{1/2k}^{-1} \circ \mathcal{O}_{2k}\psi(x) \tag{5.216}$$

so we have that the quantum representation of $T$ takes the form

$$\begin{aligned}\hat{\eta}'_t(T) &= e^{\pi i/4} e^{-r\hat{\Delta}_{\sigma_0}} \circ \mathcal{G}_{2k}^{-1} \circ \left(e^{r\hat{\Delta}_{\sigma_0}}\right)^{-1} \\ &= e^{\pi i/4}\,\mathcal{N}_{\mathrm{b}} \circ \mathcal{F} \circ \mathcal{O}_{1/2k} \circ \mathcal{G}_{2k}^{-1} \circ \mathcal{O}_{2k} \circ \mathcal{F}^{-1} \circ \mathcal{N}_{\mathrm{b}}^{-1} \\ &= e^{\pi i/4}\,\mathcal{N}_{\mathrm{b}} \circ \mathcal{F} \circ \mathcal{G}_{1/2k}^{-1} \circ \mathcal{F}^{-1} \circ \mathcal{N}_{\mathrm{b}}^{-1}\end{aligned}$$

The inner operator $\mathcal{F} \circ \mathcal{G}_{1/2k}^{-1} \circ \mathcal{F}^{-1}$ acts on a function $\psi \in \mathcal{S}(\mathbb{R})$ as

$$\mathcal{F}\left(g_{1/2k}(x) \cdot \mathcal{F}^{-1}(\psi)(x)\right), \tag{5.217}$$

where $g_{1/2k}(x) = e^{-\pi ix^2/2k}$ is a function multiplied. By the convolution theorem the operation in (5.217) is the convolution of $\psi$ with the function $\mathcal{F}(g_{1/2k})$. To get the explicit expression we need compute the Fresnel Integral

$$\int_{\mathbb{R}} e^{-\pi ix^2/2k} e^{2\pi ixy}\mathrm{d}y \tag{5.218}$$

which is not absolute convergent but an explicit formula as conditional convergent integral (or improper integral) is possible

$$\int_{\mathbb{R}} e^{-\pi ix^2/2k} e^{2\pi ixy}\mathrm{d}y = \sqrt{\frac{2k}{i}}\, e^{2k\pi iy^2}. \tag{5.219}$$

This concludes the proof of Theorem 73. □

**Proposition 77.** *The space* $\mathcal{H}^{(k)} = \left(\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^{2k}\right)^{\mathbb{S}_2}$ *decomposes into eigenspaces for the Laplace operator* $\hat{\Delta}_\sigma$*, that is*

$$\mathcal{H}^{(k)} = \bigoplus_{n\geq 0} \mathrm{Eig}_{2kn}(\hat{\Delta}_\sigma) \tag{5.220}$$

*induced by the decomposition on* $S(\mathbb{R}) = \bigoplus_{n\geq 0}\mathrm{span}\{v_n(x,\sigma)\}$*. The dimension of the eigenspaces is given by*

$$\dim \mathrm{Eig}_{2kn}(\hat{\Delta}_\sigma) = k + (-1)^n, \tag{5.221}$$

*and the decomposition is mapping class group invariant, i.e.*

$$\hat{\gamma}\left(\mathrm{Eig}_{2kn}(\hat{\Delta}_\sigma)\right) = \mathrm{Eig}_{2kn}(\hat{\Delta}_{\gamma_*\sigma}) \qquad \text{for any } \gamma \in \Gamma \tag{5.222}$$

*Proof.* The direct sum decomposition follows from Proposition 64 and the decomposition (5.167). Indeed

$$\mathrm{Eig}_{2kn}(\hat{\Delta}_\sigma) = \begin{cases} v_n(x,\sigma) \otimes \mathcal{C}_{k+1} & \text{if } n \text{ even} \\ v_n(x,\sigma) \otimes \widetilde{\mathcal{C}}_{k-1} & \text{if } n \text{ odd}\end{cases} \tag{5.223}$$

Equation (5.222) follows from equation (5.212). □

*Remark* 5.2.6 (The Representation $\hat{\eta}'_t$). The integral presentation at $\sigma = i\mathrm{b}$ of the representation $\hat{\eta}'_t = \hat{\eta}'_{t,0} \oplus \hat{\eta}'_{t,1}$ permits us to verify the relations between its generators directly computing their kernels. Up to conjugating everything by $\exp \pm 2\pi k(\mathrm{b} - \overline{\mathrm{b}})$ we have

$$\langle x|\hat{\eta}'_t(S)|y\rangle = -i\sqrt{2k}e^{4\pi ikxy} \qquad \langle x|\hat{\eta}'_t(T)|y\rangle = e^{\pi i/4}\sqrt{\frac{2k}{i}}e^{2\pi ik(x-y)^2} \tag{5.224}$$

So we can easily compute

$$\langle x|\left(\hat{\eta}'_t(S)\right)^2|y\rangle = -\delta(x+y) \tag{5.225}$$

$$\langle x|\hat{\eta}'_t(S)\,\hat{\eta}'_t(T)|y\rangle = \int_{\mathbb{R}} \langle x|\hat{\eta}'_t(S)|z\rangle\langle z|\hat{\eta}'_t(T)|y\rangle \mathrm{d}z \tag{5.226}$$

$$= \frac{-i}{\sqrt{i}}e^{\pi i/4}\,2k\,e^{2\pi iky^2}\int_{\mathbb{R}} e^{4\pi ikz(x-y)}e^{2\pi ikz^2}\mathrm{d}z \tag{5.227}$$

$$= -ie^{\pi i/4}\sqrt{2k}e^{-2\pi ikx^2}e^{4\pi ikxy} \tag{5.228}$$

$$\langle x|\left(\hat{\eta}'_t(S)\,\hat{\eta}'_t(T)\right)^2|y\rangle = \int_{\mathbb{R}} \langle x|\hat{\eta}'_t(S)\,\hat{\eta}'_t(T)|z\rangle\langle z|\hat{\eta}'_t(S)\,\hat{\eta}'_t(T)|y\rangle \mathrm{d}z \tag{5.229}$$

$$= -i2ke^{-2\pi ikx^2}\int_{\mathbb{R}} e^{4\pi ikz(x+y)}e^{-2\pi ikz^2}\mathrm{d}z \tag{5.230}$$

$$= -\sqrt{2ik}e^{2\pi iky^2}e^{4\pi ikxy} \tag{5.231}$$

$$\langle x|\left(\hat{\eta}'_t(S)\,\hat{\eta}'_t(T)\right)^3|y\rangle = \int_{\mathbb{R}} \langle x|\left(\hat{\eta}'_t(S)\,\hat{\eta}'_t(T)\right)^2|z\rangle\langle z|\hat{\eta}'_t(S)\,\hat{\eta}'_t(T)|y\rangle \mathrm{d}z \tag{5.232}$$

$$= 2ki\sqrt{i}e^{\pi i/4}\int_{\mathbb{R}} e^{4\pi ikz(x+y)}\mathrm{d}z \tag{5.233}$$

$$= -2k\frac{\delta(x+y)}{2k} \tag{5.234}$$

An interesting presentation we get is the following on Gaussian Wavelets

**Theorem 78** (Wavelet Presentation)**.** *Define the three complex parameter wavelet*

$$f_{a,c,d}(x) = \exp(2\pi k(\mathrm{b} - \overline{\mathrm{b}})x^2)\exp(-2\pi ika(x-c)^2 - 2\pi ikd) \in \mathcal{S}(\mathbb{R}) \tag{5.235}$$

*for* $\mathrm{Im}(a) < 0$*. Define* even *and* odd *wavelets as* $\psi^+(a,c,d) = f_{a,c,d} + f_{a,-c,d}$ *and* $\psi^-(a,c,d) = f_{a,c,d} - f_{a,-c,d}$ *respectively. The projective representation* $\hat{\eta}'_t$ *sends this two families of wavelets to them selves, acting on* $\psi^\pm(a,c,d)$ *as follows*

$$\hat{\eta}'_t(S)\colon \psi^\pm(a,c,d) \mapsto -i(ia)^{-\frac{1}{2}}\psi^\pm\left(-\frac{1}{a}, -ac, d+ac^2\right)$$

$$\hat{\eta}'_t(T)\colon \psi^\pm(a,c,d) \mapsto e^{\pi i/4}(1-a)^{-\frac{1}{2}}\psi^\pm\left(\frac{-a}{a-1}, c, d\right)$$

*Proof of Theorem 78* For semplicity suppose $d = 0$ and $\mathrm{b} = 1$. Due to our definition of $f_{a,c,d}$ this two hypothesis can be made without loss of generality, indeed we can always conjugate both operators $\hat{\eta}'_t(S)$ and $\hat{\eta}'_t(T)$ by $\exp -2\pi k(\mathrm{b} - \overline{\mathrm{b}})x^2$ to get to the $\mathrm{b} = 1$ situation

$$\begin{aligned}
\hat{\eta}'_t(S)(f_{a,c,0}) &= -i\sqrt{2k}\int_{\mathbb{R}} e^{-2\pi ika(y-c)^2}e^{4\pi ikyx}\mathrm{d}y \\
&= -i\sqrt{2k}e^{-2\pi ikac^2}\int_{\mathbb{R}} e^{-2\pi ikay^2}e^{4\pi iky(ac+x)}\mathrm{d}y \\
&= -\frac{i}{\sqrt{ia}}e^{-2\pi ikac^2}e^{2\pi ik\frac{(x+ac)^2}{a}} \\
\hat{\eta}'_t(T)(f_{a,c,0}) &= e^{\pi i/4}\sqrt{\frac{2k}{i}}\int_{\mathbb{R}} e^{-2\pi ika(y-c)^2}e^{2\pi ik(x-y)^2}\mathrm{d}y
\end{aligned}$$

$$= e^{\pi i/4}\sqrt{\frac{2k}{i}}e^{2\pi ikx^2}e^{-2\pi ikac^2}\int_{\mathbb{R}} e^{-2\pi iky^2(a-1)}e^{4\pi iky(ac-x)}\mathrm{d}y$$

$$= e^{\pi i/4}\sqrt{\frac{2k}{i}}\sqrt{\frac{1}{2ik(a-1)}}\,e^{2\pi ikx^2}e^{-2\pi ikac^2}e^{2\pi ik\frac{(ac-x)^2}{(a-1)}}$$

$$= e^{\pi i/4}\sqrt{\frac{2k}{i}}\sqrt{\frac{i}{2k(1-a)}}\,e^{2\pi ikx^2(1+\frac{1}{a-1})}e^{-4\pi ikx\frac{ac}{a-1}}e^{-2\pi ikac^2\left(1-\frac{a}{a-1}\right)}$$

$$= e^{\pi i/4}\frac{1}{\sqrt{1-a}}\,e^{2\pi ik\frac{a}{a-1}(x^2-2xc+c^2)}$$

$$= e^{\pi i/4}\frac{1}{\sqrt{1-a}}\,e^{-2\pi ik\frac{a}{1-a}(x-c)^2}$$

$\square$

# Appendices

# Appendix A

# Tempered Distributions

Standard references for the topics of this Appendix are [Hör90, Hör69] and [RS80, RS75].

**Definition 29.** The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all the functions $\phi \in C^\infty(\mathbb{R}^n, \mathbb{C})$ such that

$$||\phi||_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indeces $\alpha$, $\beta$.
The space of Tempered Distributions $\mathcal{S}'(\mathbb{R}^n)$ is the space of linear functionals on $\mathcal{S}(\mathbb{R}^n)$ which are continuous with respect to all these seminorms.

Both these spaces are stable under the action of the Fourier transform operator $\mathcal{F}$. Let $Z_n$ be the sero section set of $T^*(\mathbb{R}^n)$.

**Definition 30.** For a temperate distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, we define its *Wave Front Set* to be the following subset of the cotangent bundle of $\mathbb{R}^n$

$$\mathrm{WF}(u) = \{(x, \xi) \in T^*(\mathbb{R}^n) - Z_{\mathbb{R}^n} |\ \xi \in \Sigma_x(u)\}$$

where

$$\Sigma_x(u) = \cap_{\phi\in C_x^\infty(\mathbb{R}^n)} \Sigma(\phi u).$$

Here

$$C_x^\infty(\mathbb{R}^n) = \{\phi \in C_0^\infty(\mathbb{R}^n) | \phi(x) \neq 0\}$$

and $\Sigma(v)$ are all $\eta \in \mathbb{R}^n - \{0\}$ having no conic neighborhood $V$ such that

$$|\hat{v}(\xi)| \leq C_N(1 + |\xi|)^{-N},\ N \in \mathbb{Z}_{>0},\ \xi \in V.$$

**Lemma 79.** *Suppose $u$ is a bounded density on a $C^\infty$ sub-manifold $Y$ of $\mathbb{R}^n$, then $u \in \mathcal{S}'(\mathbb{R}^n)$ and*

$$\mathrm{WF}(u) = \{(x, \xi) \in T^*(\mathbb{R}^n) | x \in \operatorname{Supp} u,\ \xi \neq 0 \text{ and } \xi(T_x Y) = 0\}.$$

In particular if $\operatorname{Supp} u = Y$, then we see that $\mathrm{WF}(u)$ is the co-normal bundle of $Y$.

**Definition 31.** Let $u$ and $v$ be temperate distributions on $\mathbb{R}^n$. Then we define

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) = \{(x, \xi_1 + \xi_2) \in T^*(\mathbb{R}^n) | (x, \xi_1) \in \mathrm{WF}(u),\ (x, \xi_2) \in \mathrm{WF}(v)\}.$$

**Theorem 80.** *Let $u$ and $v$ be temperate distributions on $\mathbb{R}^n$. If*

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) \cap Z_n = \emptyset,$$

*then the product of $u$ and $v$ exists and $uv \in \mathcal{S}'(\mathbb{R}^n)$.*

**Definition 32.** We denote by $\mathcal{S}(\mathbb{R}^n)_m$ the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha(\phi)(x)| < \infty$$

for all multi-indices $\alpha$ and $\beta$ such that if $\alpha_i = 0$ then $\beta_i = 0$ for $n - m < i \leq n$. We define $\mathcal{S}'(\mathbb{R}^n)_m$ to be the continuous dual of $\mathcal{S}(\mathbb{R}^n)_m$ with respect to these semi-norms.

We observe that if $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$ is the projection onto the first $n - m$ coordinates, then $\pi^*(\mathcal{S}(\mathbb{R}^{n-m})) \subset \mathcal{S}(\mathbb{R}^n)_m$. This means we have a well defined push forward map

$$\pi_* : \mathcal{S}'(\mathbb{R}^n)_m \longrightarrow \mathcal{S}'(\mathbb{R}^{n-m}).$$

**Proposition 81.** *Suppose $Y$ is a linear subspace in $\mathbb{R}^n$, $u$ a density on $Y$ with exponential decay in all directions in $Y$. Suppose $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a projection for some $m < n$. Then $u \in \mathcal{S}'(\mathbb{R}^n)_m$ and $\pi_*(u)$ is a density on $\pi(Y)$ with exponential decay in all directions of the subspace $\pi(Y) \subset \mathbb{R}^m$.*

Tempered distributions can be thought of as functions of growth at most polynomial, thanks to the following theorem:

**Theorem 82.** *Let $T \in \mathcal{S}'(\mathbb{R}^n)$, then $T = \partial^\beta g$ for some polynomially bounded continuous function $g$ and some multi-index $\beta$. That is, for $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$T(f) = \int_{\mathbb{R}^n} (-1)^{|\beta|} g(x)(\partial^\beta f)(x)\mathrm{d}x$$

In particular it is possible to show that $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n) \ni f \mapsto T_f \in \mathcal{S}'(\mathbb{R}^n)$ with $T_f(g) = \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x$.

Denoting by $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$ the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^m)$, we remark that we have an isomorphism

$$\tilde{\cdot} : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m)) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m}) \tag{A.1}$$

determined by the formula

$$\varphi(f)(g) = \tilde{\varphi}(f \otimes g) \tag{A.2}$$

for all $\varphi \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, $f \in \mathcal{S}(\mathbb{R}^n)$, and $g \in \mathcal{S}(\mathbb{R}^m)$. This is the content of the Nuclear theorem, see e.g. [RS80]. Since we can not freely multiply distributions we end up with a categoroid instead of a category. The partially defined composition in this categroid is defined as follows. Let $n, m, l$ be three finite sets and $A \in \mathcal{S}'(\mathbb{R}^{n\sqcup m})$ and $B \in \mathcal{S}'(\mathbb{R}^{m\sqcup l})$. We have pull back maps

$$\pi^*_{n,m} : \mathcal{S}'(\mathbb{R}^{n\sqcup m}) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l}) \text{ and } \pi^*_{m,l} : \mathcal{S}'(\mathbb{R}^{m\sqcup l}) \to \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l}).$$

By what we summarised above, the product

$$\pi^*_{n,m}(A)\pi^*_{m,l}(B) \in \mathcal{S}'(\mathbb{R}^{n\sqcup m\sqcup l})$$

is well defined provided the wave front sets of $\pi^*_{n,m}(A)$ and $\pi^*_{m,l}(B)$ satisfy the condition

$$(\mathrm{WF}(\pi^*_{n,m}(A)) \oplus \mathrm{WF}(\pi^*_{m,l}(B))) \cap Z_{n\sqcup m\sqcup l} = \varnothing \tag{A.3}$$

If we now further assume that $\pi^*_{n,m}(A)\pi^*_{m,l}(B)$ continuously extends to $\mathcal{S}(\mathbb{R}^{n\sqcup m\sqcup l})_m$, then we obtain a well defined element

$$(\pi_{n,l})_*(\pi^*_{n,m}(A)\pi^*_{m,l}(B)) \in \mathcal{S}'(\mathbb{R}^{n\sqcup l}).$$

## A.1 Bra-Ket Notation

We often use the Bra-Ket notation to make computations with distributions. For $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ a density and $x \in \mathbb{R}^n$ we will write

$$\langle x|\varphi\rangle := \varphi(x),$$

with distributional meaning

$$\varphi(f) = \int_{\mathbb{R}^n} \langle x|\varphi\rangle f(x)\mathrm{d}x = \int_{\mathbb{R}^n} \varphi(x)f(x)\mathrm{d}x.$$

In particular if $\varphi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, then

$$\langle x|\varphi\rangle = \varphi(x) = \delta_x(\varphi)$$

We extend the notation defining

$$|y\rangle \equiv |\delta_y\rangle$$

so that formally $\langle x|y\rangle = \delta(x-y)$. Let $\mathsf{T}\colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ be a linear operator and $\psi \in \mathcal{S}(\mathbb{R}^n)$. The integral kernel of the operator $\mathsf{T}$, if it exists, is a distribution $k_{\mathsf{T}}$ such that

$$\mathsf{T}(\psi)(x) = \int_{\mathbb{R}^n} k_{\mathsf{T}}(x,y)\psi(y)\mathrm{d}y \tag{A.4}$$

Working with Schwartz functions, the nuclear theorem expressed by formula (A.2) guarantees that the kernel $k_T$ exists and that it is a tempered distribution. We will usually write the kernel from equation (A.4), in Bra-Ket notation as follows

$$\mathsf{T}(\psi)(x) = \int_{\mathbb{R}^n} \langle x|\mathsf{T}|y\rangle\psi(y)\mathrm{d}y \tag{A.5}$$

and the nuclear theorem morphism (A.2) can be read as

$$\langle x|\mathsf{T}|y\rangle = \langle x,\, y|\widetilde{T}\rangle. \tag{A.6}$$

## A.2 $\mathrm{L}^2(\mathbb{A}_N)$ and $\mathcal{S}(\mathbb{A}_N)$

The space $\mathrm{L}^2(\mathbb{A}_N) \equiv \mathrm{L}^2(\mathbb{R}\times\mathbb{Z}/N\mathbb{Z})$ is the main block to construct Hilbert spaces in this thesis. By definition it is the space of functions $\mathsf{f}\colon \mathbb{A}_N \longrightarrow \mathbb{C}$ such that

$$\int_{\mathbb{A}_N} |\mathsf{f}(a)|^2\,\mathrm{d}a \equiv \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\int_{\mathbb{R}} |\mathsf{f}(x,n)|^2\,\mathrm{d}x < \infty \tag{A.7}$$

with standard inner product

$$\langle \mathsf{f},\mathsf{g}\rangle \equiv \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\int_{\mathbb{R}} \mathsf{f}(x,n)\overline{\mathsf{g}(x,n)}\mathrm{d}(x,n) \tag{A.8}$$

Finite square integrable sequences are just a finite dimensional vector space

$$\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N,$$

with a preferred basis given by mod $N$ Kronecker delta functions

$$\delta_j(n) \equiv \begin{cases} 1 & \text{if } j = n \bmod N \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

There is a natural isomorphism

$$\mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{L}^2(\mathbb{A}_N) \tag{A.10}$$

defined by

$$f \otimes \delta_j\,(a) = f(x)\delta_j(n), \text{ for } a = (x,n) \in \mathbb{A}_N \tag{A.11}$$

with inverse

$$\mathbb{A}_N \ni \mathsf{f} \mapsto \sum_{j=0}^{N-1} \mathsf{f}(\cdot, j) \otimes \delta_j \in \mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \tag{A.12}$$

Everything just said holds true substituting $\mathrm{L}^2$ with $\mathcal{S}$, with the isomorphism $\mathcal{S}(\mathbb{A}_N) \simeq \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^N$ and even tempered distributions, defined as linear continuous functionals over $\mathcal{S}(\mathbb{A}_N)$, are simply $\mathcal{S}'(\mathbb{R}) \otimes \mathbb{C}^N$. All the Bra-Ket notation extends trivially to $\mathcal{S}(\mathbb{A}_N)$, including the nuclear theorem (A.6), substituting all the integrals over $\mathbb{R}$ with integrals over $\mathbb{A}_N$.

We can define $\mathrm{L}_N(A)$, by the spectral theorem, for any operator $A$ of order $N$, such that it formally satisfies

$$A = e^{2\pi i\, \mathrm{L}_N(A)/N}.$$

We can define, for any function $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ the operator function $f\!\!\!/(\mathsf{x}, A) \equiv f(\mathsf{x}, \mathrm{L}_N(A))$ for any commuting pair of operators $\mathsf{x}$ and $A$, where the former is self adjoint and the latter is of order $N$. We have, for $\mathsf{x}$ and $A$ as above

$$f\!\!\!/(\mathsf{x}, A) = \int_{\mathbb{A}_N} \tilde{f}(y,m) e^{2\pi i y \mathsf{x}} A^{-m} \mathrm{d}(y,m) \tag{A.13}$$

where

$$\tilde{f}(x,n) = \int_{\mathbb{A}_N} f(y,m) \overline{\langle (y,m); (x,n) \rangle} \mathrm{d}(y,m). \tag{A.14}$$

# Appendix B

# Hermite Polynomials

Let $\alpha > 0$ fixed. We define the $n$-th Hermite polynomial with weight $\alpha$ as

$$H_{n,\alpha}(x) = (-1)^n e^{\alpha x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\alpha x^2}. \tag{B.1}$$

They are the following rescaling of physicists Hermite polynomials $H_n$

$$H_{n,\alpha}(x) = \alpha^{\frac{n}{2}} H_n(\sqrt{\alpha}\, x) \tag{B.2}$$

and satisfy $\deg H_{n,\alpha}(x) = n$. Define also the following inner product,

$$\langle f, g \rangle_\alpha = \int_{\mathbb{R}} f(x)\overline{g(x)} e^{-\alpha x^2} \mathrm{d}x \tag{B.3}$$

which give Hilbert space structure to $\mathrm{L}^2(\mathbb{R}, e^{-\alpha x^2}\mathrm{d}x)$, the space of functions satisfying

$$\int_{\mathbb{R}} |f(x)|\, e^{-\alpha x^2} \mathrm{d}x < \infty. \tag{B.4}$$

The Hermite polynomials with weight $\alpha$ satisfy the following relations

$$\langle H_{n,\alpha}, H_{m,\alpha} \rangle_\alpha = \sqrt{\frac{\pi}{\alpha}} (2\alpha)^n (n!) \delta_{n,m} \tag*{(othogonality)}$$

$$H_{n+1,\alpha}(x) = 2\alpha x H_{n,\alpha}(x) - H'_{n,\alpha} \tag*{(differential relation)}$$

$$e^{-\alpha(t^2-2tx)} = \sum_{n\geq 0} H_{n,\alpha}(x) \frac{t^n}{n!} \tag*{(generating function)}$$

$$x^n = \frac{n!}{(2\alpha)^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{m!(n-2m)!} H_{n-2m,\alpha}(x) \tag*{(monomial expression)}$$

$$H_{n,\alpha}(-x) = (-1)^n H_{n,\alpha}(x) \tag*{(parity)}$$

$$(\langle f, H_{n,\alpha} \rangle = 0,\ \forall n \geq 0) \implies f = 0, \text{ for every } f \in \mathrm{L}^2(\mathbb{R}, e^{-\alpha x^2}\mathrm{d}x) \tag*{(completeness)}$$

and the following *Mehler Formula*

$$\sum_{n\geq 0} \frac{w^n}{\langle H_{n,\alpha}, H_{n,\alpha} \rangle_\alpha} H_{n,\alpha}(x) H_{n,\alpha}(y) = \sqrt{\frac{\alpha}{\pi}} \frac{1}{\sqrt{1-w^2}} \exp\left( \alpha \frac{2xyw - w^2(x^2+y^2)}{1-w^2} \right) \quad \text{if } \mathrm{Re}(w^2) < 1 \tag{B.5}$$

*Proof. Orthogonality*
Suppose $m < n$.

$$\begin{aligned}\langle H_{n,\alpha}, H_{m,\alpha}\rangle_\alpha &= \int_{\mathbb{R}} (-1)^n e^{\alpha x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-\alpha x^2}\right) \overline{H_{m,\alpha}(x)} e^{-\alpha x^2} \mathrm{d}x \\ &= (-1)^n \int_{\mathbb{R}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-\alpha x^2}\right) H_{m,\alpha}(x) \mathrm{d}x \\ &= \int_{\mathbb{R}} e^{-\alpha x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} H_{m,\alpha}(x) \mathrm{d}x = 0\end{aligned}$$

Suppose now $n = m$. We have $\frac{\mathrm{d}^n}{\mathrm{d}x^n} H_{n,\alpha} = n! \frac{\mathrm{d}^n}{\mathrm{d}x^n}|_{x=0} H_{n,\alpha} = (2\alpha^2)^n n!$. We have

$$\begin{aligned}\langle H_{n,\alpha}, H_{n,\alpha}\rangle_\alpha &= \int_{\mathbb{R}} e^{-\alpha x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} H_{m,\alpha}(x) \mathrm{d}x \\ &= (2\alpha)^n n! \int_{\mathbb{R}} e^{-\alpha x^2} \mathrm{d}x = (2\alpha)^n n! \sqrt{\frac{\pi}{\alpha}}.\end{aligned}$$

*Differential Relation*

$$\begin{aligned}H_{n+1,\alpha}(x) &= (-1)^{n+1} e^{\alpha x^2} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} e^{-\alpha x^2} \\ &= (-1)^{n+1} e^{\alpha x^2} \left((-1)^n \frac{\mathrm{d}}{\mathrm{d}x} H_{n,\alpha}(x) e^{-\alpha x^2}\right) \\ &= (-1) e^{\alpha x^2} \left(H'_{n,\alpha}(x) e^{-\alpha x^2} - 2\alpha x H_{n,\alpha}(x) e^{-\alpha x^2}\right) \\ &= 2\alpha x H_{n,\alpha}(x) - H'_{n,\alpha}(x)\end{aligned}$$

*Generating Function*

$$\begin{aligned}e^{-\alpha(x-t)^2} &= \sum_{n\geq 0} \frac{t^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\bigg|_{t=0} e^{-\alpha(x-t)^2} \\ &= \sum_{n\geq 0} \frac{t^n}{n!} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}u^n}\bigg|_{u=x} e^{-\alpha u^2} \\ &= e^{-\alpha x^2} \sum_{n\geq 0} \frac{t^n}{n!} H_{n,\alpha}(x)\end{aligned}$$

*Completeness*
Consider the function $F\colon \mathbb{C} \longrightarrow \mathbb{C}$

$$F(z) \equiv \int_{\mathbb{R}} f(x) e^{-\alpha(x^2 - 2xz)} \mathrm{d}x \tag{B.6}$$

associated to $f \in \mathrm{L}^2(\mathbb{R}, e^{-\alpha x^2}\mathrm{d}x)$. The function $F(z)$ is holomorphic, indeed we have the following series expansion

$$\begin{aligned}F(z) &= \int_{\mathbb{R}} f(x) e^{-\alpha(x^2 - 2xz)} \mathrm{d}x \\ &= \int_{\mathbb{R}} f(x) e^{-\alpha x^2} \sum_{n\geq 0} \frac{(2zx\alpha)^n}{n!} \mathrm{d}x \\ &= \sum_{n\geq 0} 2^n z^n \alpha^n \int_{\mathbb{R}} \frac{x^n}{n!} f(x) e^{-\alpha x^2} \mathrm{d}x\end{aligned}$$

and the integrand $\frac{x^n}{n!} f(x) e^{-\alpha x^2}$ is in $\mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$.

The derivatives in 0 can be evaluated directly

$$\begin{aligned}\left.\frac{\mathrm{d}^n}{\mathrm{d}z^n}\right|_{z=0} F(z) &= \int_{\mathbb{R}} f(x) \left.\frac{\mathrm{d}^n}{\mathrm{d}z^n}\right|_{z=0} e^{-\alpha(x^2-2xz)}\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)(2\alpha x)^n e^{-\alpha x^2}\mathrm{d}x \\ &= (2\alpha)^n \langle f, x^n\rangle_\alpha.\end{aligned}$$

If we start from the assumption

$$\langle f, x^n\rangle = 0, \, \forall n \geq 0$$

which is equivalent to the assumption

$$\langle f, H_{n,\alpha}\rangle = 0, \, \forall n \geq 0$$

we just showed that we will have

$$F^{(n)}(0) = 0, \, \forall n \geq 0$$

which, by analyticity is equivalent to

$$F(z) \equiv 0$$

However, for $t \in \mathbb{R}$

$$F(2\pi i t/\alpha) = \int_{\mathbb{R}} f(x) e^{-\alpha x^2} e^{2\pi i x t}\mathrm{d}x$$

and this is the Fourier transform of the function $f(x)e^{-\alpha x^2}$. In order to this to be identically 0 for any $t$, $f$ has to be identically 0.
*Mehler Formula*
From the Gaussian integral

$$e^{-\alpha x^2} = \sqrt{\frac{\alpha}{\pi}} \int_{\mathbb{R}} e^{-\pi^2 y^2/\alpha} e^{2\pi i x y}\mathrm{d}y \tag{B.7}$$

we can compute an integral formula for $H_{n,\alpha}$

$$\begin{aligned}H_{n,\alpha}(x) &= (-1)^n e^{\alpha x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\alpha x^2} \\ &= (-1)^n \sqrt{\frac{\pi}{\alpha}} e^{\alpha x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{\mathbb{R}} e^{-\pi^2 y^2/\alpha} e^{2\pi i x y}\mathrm{d}y \\ &= \sqrt{\frac{\pi}{\alpha}} \int_{\mathbb{R}} (-2\pi i y)^n e^{\alpha x^2} e^{-\pi^2 y^2/\alpha} e^{2\pi i x y}\mathrm{d}y \\ &= \sqrt{\frac{\pi}{\alpha}} \int_{\mathbb{R}} (-2\pi i y)^n e^{-(\pi y - i\alpha x)^2/\alpha}\mathrm{d}y\end{aligned}$$

Now we can compute expression (B.5) (recall the hypothesis $\operatorname{Re} w^2 < 1$)

$$\begin{aligned}
&\sum_{n\geq 0} \frac{w^n}{\langle H_{n,\alpha}, H_{n,\alpha}\rangle_\alpha} H_{n,\alpha}(x) H_{n,\alpha}(y) \\
&\quad = \sqrt{\frac{\pi}{\alpha}} \int_{\mathbb{R}^2} e^{-(\pi\eta - i\alpha y)^2/\alpha} e^{-(\pi\xi - i\alpha x)^2/\alpha} \sum_{n\geq 0} \frac{(2\pi)^{2n}(-\xi\eta w)^n}{(2\alpha)^n n!} \mathrm{d}\eta \mathrm{d}\xi \\
&\quad = \sqrt{\frac{\pi}{\alpha}} \int_{\mathbb{R}^2} e^{-(\pi\eta - i\alpha y)^2/\alpha} e^{-(\pi\xi - i\alpha x)^2/\alpha} e^{-2\pi^2\xi\eta w/\alpha} \mathrm{d}\eta \mathrm{d}\xi \\
&\quad = \sqrt{\frac{\pi}{\alpha}} \int_{\mathbb{R}} e^{-(\pi\eta - i\alpha y)^2/\alpha} e^{-2\pi i\eta w x} e^{\pi^2\eta^2 w^2/\alpha} \int_{\mathbb{R}} e^{-(\pi\xi - i\alpha x + \pi\eta w)^2/\alpha} \mathrm{d}\xi \mathrm{d}\eta \\
&\quad = \int_{\mathbb{R}} e^{-(\pi\eta - i\alpha y)^2/\alpha} e^{-2\pi i\eta w x} e^{\pi^2\eta^2 w^2/\alpha} \mathrm{d}\eta \\
&\quad = e^{\alpha y^2} \int_{\mathbb{R}} e^{-\pi^2\eta^2(1-w^2)/\alpha} e^{2\pi i\eta(y - wx)} \mathrm{d}\eta \\
&\quad = e^{\alpha y^2} \sqrt{\frac{\alpha}{\pi}} \frac{1}{\sqrt{1-w^2}} e^{-\alpha\frac{(y-wx)^2}{1-w^2}} \\
&\quad = \sqrt{\frac{\alpha}{\pi}} \frac{1}{\sqrt{1-w^2}} e^{\alpha\frac{2\alpha wxy - \alpha w^2(x^2+y^2)}{1-w^2}}
\end{aligned}$$

□

For every $\beta \in \mathbb{C}$ such that $\beta + \overline{\beta} = \alpha$ we can choose an Hilbert basis $\mathcal{B}_\beta = \{\Psi_{n,\beta}\}_{n\geq 0}$ for the inner product space $\mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$, setting

$$\Psi_{n,\beta}(x) \equiv H_{n,\alpha}(x) e^{-\beta x^2}. \tag{B.8}$$

From the Completeness and Orthogonality properties of the Hermite polynomials we get the following decomposition of the identity, for any $f \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$

$$\int_{\mathbb{R}} \sum_{n\geq 0} \frac{\Psi_{n,\beta}(x)\overline{\Psi_{n,\beta}(y)}}{\langle \Psi_{n,\beta}, \Psi_{n,\beta}\rangle} f(y)\mathrm{d}y = \int_{\mathbb{R}} \delta(x-y) f(y)\mathrm{d}y \tag{B.9}$$

Two different choices $\beta$ and $\beta'$ are related by a unitary transformation

$$\Psi_{n,\beta'} = \mathcal{Q}_{\beta',\beta}\Psi_{n,\beta} \qquad \mathcal{Q}_{\beta',\beta}\varphi(x) = e^{-(\beta'-\beta)x^2}\varphi(x) \tag{B.10}$$

Suppose that $\beta \in \mathbb{R}$, that is $\beta = \alpha/2$. Introduce the operator

$$\mathcal{F}_{\alpha/2\pi}(\varphi)(x) \equiv \int_{\mathbb{R}} e^{\alpha i x y}\varphi(y)\mathrm{d}y \tag{B.11}$$

**Lemma 83.**

$$\sqrt{\frac{\alpha}{2\pi}} \mathcal{F}_{\alpha/2\pi}(\Psi_{n,\alpha/2}) = i^n \Psi_{n,\alpha/2}$$

*Proof.* Consider the generating function $\Theta_\alpha(x,t) = e^{-\alpha(t^2 - 2tx)}$. We have

$$\sqrt{\frac{\alpha}{2\pi}} \mathcal{F}_{\alpha/2\pi}(e^{-\alpha x^2/2}\Theta_\alpha(x,t)) = \sum_{n\geq 0} \frac{t^n}{n!} \sqrt{\frac{\alpha}{2\pi}} \mathcal{F}_{\alpha/2\pi}(\Psi_{n,\alpha/2})(x)$$

On the other hand we can compute directly

$$
\begin{aligned}
\mathcal{F}_{\alpha/2\pi}(e^{-\alpha x^2/2}\Theta_\alpha(x,t)) &= e^{-\alpha t^2}\int_{\mathbb{R}} e^{\alpha i x y} e^{-\alpha y^2/2} e^{2\alpha t y}\mathrm{d}y \\
&= e^{-\alpha t^2}\sqrt{\frac{2\pi}{\alpha}}\int_{\mathbb{R}} e^{\alpha y(ix+2t)} e^{-\alpha y^2/2}\mathrm{d}y \\
&= e^{-\alpha t^2}\sqrt{\frac{2\pi}{\alpha}} e^{\alpha(ix+2t)^2/2} \\
&= \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha x^2/2} e^{-\alpha((it)^2-2itx)} \\
&= \sqrt{\frac{2\pi}{\alpha}} e^{-\alpha x^2/2}\Theta_\alpha(x,t) \\
&= \sum_{n\geq 0} i^n \frac{t^n}{n!}\sqrt{\frac{2\pi}{\alpha}}\Psi_{n,\alpha/2}(x)
\end{aligned}
$$

□

# Appendix C

# Miscellanea

## C.1 Principal Bundles, Flat Connections and Holonomy Representations

Let $G$ be a Lie group, and let $\mathfrak{g}$ be the Lie algebra of $G$. The *Maurer Cartan* form $\theta \in \Omega^1(G, \mathfrak{g})$ is defined as $\theta(v) = (\mathrm{d}l_{g^{-1}})v$, for every $v \in T_gG$, where $l_g \colon G \longrightarrow G$ is the left multiplication $l_g(h) = gh$. The map $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation, that is the differential of the map $\Psi_g \colon G \to G$, $h \mapsto ghg^{-1}$.

**Definition 33.** Let $B$ be a differentiable manifold and $G$ be a Lie group. A *Principal $G$-bundle* over $B$ is a manifold $P$ satisfying the following properties

1. There is a right action of $G$ over $P$ such that $B$ is the quotient under this action and the projection $\pi \colon P \longrightarrow P/G$ is smooth.

2. For every $b \in B$ there is an open neighborhood $U \subseteq B$ of $b$ such that $\pi^{-1}(U) \simeq U \times G$ via an equivariant diffeomorphism.

**Definition 34.** A principal bundle homomorphism between two principal $G$-bundles $P$ and $P'$ is a $G$-equivariant bundle homomorphism. If $P' = P$ it is called a *gauge transformation* of the bundle. We denote by $\mathcal{G}$ the group of gauge transformations of $P$.

*Remark* C.1.1. To every $G$-equivariant map $u \colon P \longrightarrow G$, $p \mapsto u_p \in G$ we associate a gauge transformation $\varphi \colon P \longrightarrow P$ via the rule $\varphi(p) = p \cdot u_p$. This association is a bijection. Here $G$-equvariance of $u$ is with respect to the right action $u \cdot g = g^{-1}ug$.

Let $\pi \colon P \longrightarrow B$ a fixed principal $G$-bundle for the rest of this subsection. We say that a vector field $V \in TP$ is *vertical* if $\mathrm{d}\pi(V) = 0$ and, for every $p \in P$, we define the vertical tangent subspace $V_p$ as $\ker(\mathrm{d}\pi)_p$.
Consider the two maps $i_p \colon G \longrightarrow P$, $i_p(g) = p \cdot g$ and $r_g \colon P \longrightarrow P$, $r_g(p) = p \cdot g$. Let $e \in G$ be the unit element. For every $X \in T_eG \simeq \mathfrak{g}$ we can define $X^* \in TP$ as the vector field given at every $p$ by the push forward $X^*_p \equiv (\mathrm{d}i_p)_e X$. All such $X^*$ are vertical.
The concept of being *horizontal* is not canonical from the bundle, as it is to be vertical. We need an extra object named connection

**Definition 35.** A *connection* on the principal $G$-bundle $P$ is a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ such that

1. $A(X^*) = X$, for all $X \in \mathfrak{g}$.
2. $r_g^*(A) = \mathrm{Ad}_{g^{-1}}A$, i.e. $A$ is $G$–equivariant

The space of connections on $P$ is denoted $\mathcal{A}$.

We define the horizontal tangent subspace at $p \in P$ as $H_p \equiv \ker(A_p)$. It follows that $T_pP = V_p \oplus H_p$. Let $A'$ a $\mathfrak{g}$-valued $k$-form, i.e. $A' \in \Omega^k(P, \mathfrak{g})$. $A'$ is said to be $G$-equivariant if $r_g^*A' = \mathrm{Ad}_{g^{-1}}A'$ for every $g \in G$. We denote with $\Omega^k(P, \mathfrak{g})^G$ the subspace of $\Omega^k(P, \mathfrak{g})$ of $G$-equivariant forms.

**Definition 36.** The *curvature* of a connection $A$ on $P$ is the 2-form $F_A \in \Omega^2(P, \mathfrak{g})^G$ defined by the formula

$$F_A = \mathrm{d}A + \frac{1}{2}\,[A \wedge A]$$

A connection is called *flat* if $F_A = 0$. The space of flat connections on $P$ is denoted $\mathcal{F}$.

The gauge group $\mathcal{G}$ acts on $\mathcal{A}$ on the right and preserves $\mathcal{F}$. The action is as follows. to the gauge map $\varphi\colon P \to P$ there is associated a map $g_\varphi\colon P \to G$ (see remark C.1.1). Let $\theta_\varphi = g_\varphi^*\theta$ be the pull-back of the Maurer–Cartan form. Then, for every $a \in \mathcal{A}$

$$\varphi^*A = \mathrm{Ad}_{g_\varphi^{-1}}A + \theta_\varphi \tag{C.1}$$

while the curvature pulls-back via gauge transformations as

$$\varphi^*F_A = \mathrm{Ad}_{g_\varphi^{-1}}A. \tag{C.2}$$

As notation, we sometimes write $A^{g_\varphi}$ in place of $\varphi^*A$.

**Lemma 84.** *Let $\alpha\colon [0,1] \longrightarrow B$ be a smooth curve on $B$. Let $p_0 \in \pi^{-1}(\alpha(0))$ and fix a connection $A$ on $P$. Then there exists an horizontal lift of $\alpha$, that is a smooth curve $\beta\colon [0,1] \longrightarrow P$ such that $\beta(0) = p_0$, $\mathrm{d}\pi(\dot\beta) = \dot\alpha$ and $A(\dot\beta) = 0$.*

If $\alpha[0,1] \longrightarrow B$ is a loop, i.e. $\alpha(0) = \alpha(1) = x_0 \in B$, then $\beta(0)$ and $\beta(1) \in \pi^{-1}(x_0)$. So there exists a $g \in G$ such that $\beta(0) \cdot g = \beta(1)$. This is called the *holonomy* of $A$ along $\alpha$ with respect to $p_0$. Denoting $g = \mathrm{Hol}_{A,p_0}(\alpha)$ we get an holonomy map

$$\mathrm{Hol}_{A,p_0}\colon \mathrm{Loop}(B, x_0) \longrightarrow G \tag{C.3}$$

Turns out that this association behaves very well. First let us define the *moduli space of flat connection*

$$M = \mathcal{F}/\mathcal{G} \tag{C.4}$$

as the set of gauge equivalence classes of flat connections on $P$.

**Proposition 85.** *Let $A$ be a flat connection on $P$, and assume that $B$ is connected. Let $x_0 \in B$ and let $p_0 \in \pi^{-1}(x_0)$. Let $\alpha$ be a loop in $B$ based at $x_0$. Up to conjugation in $G$, the association*

$$A \mapsto \mathrm{Hol}_{A,p_0}(\alpha)$$

*is independent of the base point $x_0$, the choice of lift $p_0$, the gauge tranformation class of the connection $A$, and the homotopy class of $\alpha$. In other words, we have a well-defined map*

$$\mathrm{Hol}\colon M \longrightarrow \mathrm{Hom}(\pi_1(B), G)/\sim$$

*where* $\sim$ *is the equivalence relation given by the action of* $G$ *on the right of* $\mathrm{Hom}(\pi_1(B), G)$ *by conjugation.*

**Definition 37.** A *flat principal* $G$*-bundle* is a pair $(P, A)$ where $P$ is a principal $G$-bundle and $A$ is a flat connection on it. Two flat principal $G$-bundles $(P, A)$ and $(P', A')$ are gauge equivalent if there is an isomorphism of principal $G$-bundles $\varphi : P \to P'$ such that $\varphi^* P' = P$. Denote the set of equivalence classes of flat principal $G$-bundles $[(P, A)]$ as $\mathcal{M}$.

**Theorem 86.** *The map*

$$\begin{aligned} \mathcal{M} &\longrightarrow \mathrm{Hom}(\pi_1(B), G)/\sim \\ [(P, A)] &\mapsto \mathrm{Hol}_A \end{aligned} \tag{C.5}$$

*is a bijection*

*Remark* C.1.2. In this thesis the Lie groups $G$ that we use are always matrix groups. In this situation some of the operations we described take a more explicit formulation. In particular let $\varphi$ a gauge transformation with associated $g_\varphi \in \mathrm{C}^\infty(P, G)$, and $A$ a flat connection, then the pull-back is explicitly

$$A^{g_\varphi} = g_\varphi^{-1} A g_\varphi + g_\varphi^{-1} \mathrm{d} g_\varphi \tag{C.6}$$

where all the multiplications are standard matrix multiplications.
Moreover we will restrict ourselves to have only trivial bundles $P \simeq B \times G$. As our base manifolds $B$ will be always 2 or 3 dimensional, this is not so restrictive, indeed

*Lemma* 87. *[Fre95] If* $G$ *is simply connected then every principal* $G$*-bundle over a manifold of dimension lesser or equal then* 3 *admits a global section, hence is trivializable.*

### C.1.1 SU(2), SL(2, ℂ) and Möbius Transformations

In this subsection we will list some useful properties of the Lie groups we use most often in this thesis.
The group $\mathrm{SU}(2)$ is the group of $2 \times 2$ matrices $X$ such that $X^\dagger X = X X^\dagger = \mathrm{Id}$, where $X^\dagger$ is the conjugate transpose of $X$.
The conjugacy class of $X \in \mathrm{SU}(2)$ is determined by its eigenvalues, since $X$ can be diagonalized in $\mathrm{SU}(2)$ to a matrix in $\mathrm{U}(1)$. They are determined by the characteristic polynomial, which, in $\mathrm{SU}(2)$, is determined by the trace. So, up to conjugation, every $X$ can by written as

$$X = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \qquad \text{and has } \mathrm{Tr}(X) = 2\cos(\theta).$$

The map Tr gives a bijection $[-2, 2] \simeq \mathrm{SU}(2)/\sim$. The fiber $\mathrm{Tr}^{-1}(t)$ is diffeomorphic to $S^2$, the unit sphere, for every $t \neq \pm 2$, whereas for $t = \pm 2$ the fiber is just one matrix, respectively $\pm\mathrm{Id}$.
Globally $\mathrm{SU}(2)$ is diffeomorphic to $S^3$ and is in particular simply connected.
The Lie algebra $\mathfrak{su}(2)$, by definition the tangent space to $\mathrm{SU}(2)$ at the identity, is described as the space of $2 \times 2$ traceless, anti-Hermitian matrices.
The group $\mathrm{SL}(2, \mathbb{C})$ is the group of $2 \times 2$ complex valued matrices with determinant 1. Its quotient

$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm\mathrm{Id}\}$$

is called group of Möbius transformations. It naturally acts on the Riemann sphere $\mathbb{C}\mathrm{P}^1 \simeq \mathbb{C} \cup \{\infty\}$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \equiv \frac{az+b}{cz+d}, \qquad z \in \mathbb{C}\mathrm{P}^1 \tag{C.7}$$

Given three points $z_0, z_1$ and $z_2 \in \mathbb{C}\mathrm{P}^1$ a Möbius transformation $\mu$ is completely determined by its values on such three points. Conversely given three values $w_0$, $w_1$ and $w_2 \in \mathbb{C}\mathrm{P}^1$ there exists exactly one $\mu \in \mathrm{PSL}(2, \mathbb{C})$ such that $\mu(z_i) = w_i$ for $i = 0, 1, 2$. In particular it is common to choose 0, 1 and $\infty$ as special points on $\mathbb{C}\mathrm{P}^1$ and say that a Möbius $\mu$ is determined by its values at them.
Elements $\mu \in \mathrm{PSL}(2, \mathbb{C})$ are usually classified in terms of of conjugacy classes or squared trace. Given $\mu \in \mathrm{PSL}(2, \mathbb{C})$ there are exactly two matrices $A$,$A' \in \mathrm{SL}(2,\mathbb{C})$ in its equivalence class. They satisfies $(\mathrm{Tr}(A))^2 = (\mathrm{Tr}(A'))^2$ so the squared trace $\mathrm{Tr}^2$ is well defined in $\mathrm{PSL}(2, \mathbb{C})$.
Two transformations $\mu$ and $\mu' \in \mathrm{PSL}(2, \mathbb{C})$ different from the identity, are conjugated if and only if they have the same squared trace $\mathrm{Tr}^2(\mu) = \mathrm{Tr}^(\mu')$.
There is some standard terminology here that we will use sometimes:
(i) $\mu$ is said *parabolic* if $\mathrm{Tr}^2(\mu) = 4$, $\mu \neq \mathrm{Id}$.
(ii) $\mu$ is said *elliptic* if $\mathrm{Tr}^2(\mu) < 4$.
(iii) $\mu$ is said *loxodromic* if $\mathrm{Tr}^2(\mu) \in \mathbb{C} \setminus [0, 4]$.
Loxodromic transformations with positive real squared trace are usually called *hyperbolic*. This classification can be reformulated in terms of fixed points on $\mathbb{C}\mathrm{P}^1$:
(i) $\mu$ is *parabolic* if has exactly one fixed point.
(ii) $\mu$ is said *elliptic* if it has no fixed points.
(iii) $\mu$ is said *loxodromic* if it has precisely two fixed points.
All the parabolic transformations are conjugate each other and, in particular, can be conjugated to the element associated to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is so that it preserves the point $\infty \in \mathbb{C}\mathrm{P}^1$.
The description of $\mathrm{SL}(2,\mathbb{C})$ follows from the one we gave for $\mathrm{PSL}(2, \mathbb{C})$, as double cover of it. A conjugacy class in $\mathrm{PSL}(2, \mathbb{C})$ lifts to two distinct conjugacy classes in $\mathrm{SL}(2,\mathbb{C})$. Topologically $\mathrm{SL}(2,\mathbb{C})$ is simply connected while $\mathrm{PSL}(2, \mathbb{C})$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$.

## C.2 Categroids

We need a notion which is slightly more general than categories to define the Teichmüller TQFT functor.

**Definition 38.** [AK14b]
A *Categroid* $\mathcal{C}$ consist of a family of objects $\mathrm{Obj}(\mathcal{C})$ and for any pair of objects $A, B$ from $\mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Mor}_{\mathcal{C}}(A, B)$ such that the following holds

**A** For any three objects $A, B, C$ there is a subset $K^{\mathcal{C}}_{A,B,C} \subset \mathrm{Mor}_{\mathcal{C}}(A, B)\times\mathrm{Mor}_{\mathcal{C}}(B, C)$, called the composable morphisms and a *composition* map

$$\circ : K^{\mathcal{C}}_{A,B,C} \to \mathrm{Mor}_{\mathcal{C}}(A, C).$$

such that composition of composable morphisms is associative.

**B** For any object $A$ we have an identity morphism $1_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ which is composable with any morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ or $g \in \mathrm{Mor}_{\mathcal{C}}(B, A)$ and we have the equation

$$1_A \circ f = f, \text{ and } g \circ 1_A = g.$$

## C.3 Operator Identities

**Lemma 88.** *Let $x, y$ and $z$ be three operators such that $[x, y] = z$ and $[z, x] = 0$. Let $f$ a power series. Then we have*

$$f(x)y = yf(x) + zf'(x) \tag{C.8}$$

$$e^x f(y) = f(y + z)e^x, \tag{C.9}$$

*whenever all the expressions involved make sense in the relevant operator algebra.*

*Proof.* Let

$$f(x) = \sum_{j=0}^{\infty} a_j x^j.$$

Then

$$\begin{aligned}
[f(x), y] &= \sum_j a_j [x^j, y] \\
&= \sum_j a_j \sum_{k=0}^{j-1} x^k [x, y] x^{j-k-1} \\
&= \sum_j a_j j z x^{j-1} \\
&= z f'(x)
\end{aligned}$$

which proves the first equation. For the second one

$$\begin{aligned}
e^x y^l &= y e^x y^{l-1} + z e^x y^{l-1} \\
&= (y + z) e^x y^{l-1} \\
&= (y + z)^l e^x.
\end{aligned}$$

So

$$\begin{aligned}
e^x f(y) &= e^x \sum_{j=0}^{\infty} a_j y^j \\
&= \sum_{j=0}^{\infty} a_j (y + z)^j e^x \\
&= f(y + z) e^x
\end{aligned}$$

□

**Lemma 89.** *Let $A$, $B$ and $C$ three elements non commuting elements of an associative algebra, and $[\cdot, \cdot]$ the commutator induced by the algebra structure. The following identity is true*

$$[A, BC] = B\,[A, C] + [A, B]\,C \tag{C.10}$$

*Proof.*

$$
\begin{aligned}
[A, BC] &= (ABC - BCA) = BAC + [A,B]\,C - BAC - B\,[C,A] \\
&= B\,[A,C] + [A,B]\,C
\end{aligned}
$$

$\square$

**Lemma 90.** *Let $x$, $y$ and $z$ be three elements of a Lie algebra satisfying the relations $[x,y] = z$ and $[y,z] = az$ with $a$ central. Suppose that the formal exponential $e^y = \sum_{k\geq 0} \frac{y^k}{k!}$ is well defined. Then we have*

$$[x, e^y] = \frac{z}{a}\left(e^{y+a} - e^y\right)$$

*Proof.* First we prove the following equation by induction

$$y^n z = \sum_{j=0}^{n} \binom{n}{j} a^j z y^{n-j}$$

For $n = 0$ it is trivial. Suppose the equation valid for some $n$, the $n+1$ expression on the left hand side becomes

$$
\begin{aligned}
y^{n+1}z &= y^n(zy + [y,z]) = y^n(zy + az) \\
&= \sum_{j=0}^{n} \binom{n}{j} a^j z y^{n-j} y + a \sum_{j=0}^{n} \binom{n}{j} a^j z y^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} a^j z y^{n+1-j} + \sum_{j=1}^{n+1} \binom{n}{j-1} a^j z y^{n+1-j} \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} a^j z y^{n+1-j}
\end{aligned}
$$

Next we prove the main statement

$$
\begin{aligned}
[x, e^y] &= \sum_{k\geq 0} \frac{1}{k!}\left[x, y^k\right] \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{n=0}^{k-1} y^n\,[x,y]\,y^{k-n-1} \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{n=0}^{k-1} y^n z y^{k-n-1} \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{n=0}^{k-1} \sum_{j=0}^{n} \binom{n}{j} a^j z y^{n-j} y^{k-n-1} \\
&= \sum_{k\geq 0} \frac{1}{k!} z \sum_{n=0}^{k-1} (a+y)^n y^{k-n-1} \\
&= \sum_{k\geq 0} \frac{1}{k!} z((a+y)^k - y^k)(y + \alpha - y)^{-1} \\
&= \frac{z}{\alpha} \sum_{k\geq 0} \frac{1}{k!}(a+y)^k - \sum_{k\geq 0} \frac{1}{k!} y^k
\end{aligned}
$$

$\square$

## C.4 Quadratic Gauss Sum

The topic that we briefly recall here can be looked up on [BEW98].

**Theorem 91** (Gauss Quadratic Sum)**.** *Let $\left(\frac{n}{k}\right)$ be the Legendre–Jacobi Symbol, and define. for $m \in \mathbb{N}$*

$$\varepsilon_m = \begin{cases} 1 & \textit{if } m \equiv 1 \textit{ mod } 4 \\ i & \textit{if } m \equiv 3 \textit{ mod } 4 \end{cases} \tag{C.11}$$

*the Gauss Quadratic Sum can be evaluated as follows*

$$\sum_{n=0}^{k-1} e^{2\pi i a \frac{n^2}{k}} = \begin{cases} 0 & \textit{if } k \equiv 2 \textit{ mod } 4 \\ \varepsilon_k \sqrt{k} \left(\frac{a}{k}\right) & \textit{if } k \textit{ odd} \\ (1+i)\sqrt{k}\left(\frac{k}{a}\right)\varepsilon_a^{-1} & \textit{if } a \textit{ odd and } k \equiv 0 \textit{ mod } 4 \end{cases} \tag{C.12}$$

We have the following small application by completing squares

**Proposition 92.** *Suppose $k$, $b \in \mathbb{Z}$, $k \geq 1$*

$$\frac{1}{\sqrt{2k}} \sum_{n=0}^{2k-1} e^{\pi i (n^2+2bn)/2k} = e^{\pi i (2k-4b^2)/8k}$$

*Proof.*

$$\begin{aligned}
\frac{1}{\sqrt{2k}} e^{-\pi i (2k-4b^2)/8k} \sum_{n=0}^{2k-1} e^{\pi i (n^2+2bn)/2k} &= e^{-\pi i/4} \frac{1}{\sqrt{2k}} \sum_{n=0}^{2k-1} e^{\frac{\pi i}{2k}(n^2+2bn+b^2)} \\
&= e^{-\pi i/4} \frac{1}{\sqrt{2k}} \sum_{n=0}^{2k-1} e^{\frac{\pi i}{2k}(n+b)^2} \\
&= e^{-\pi i/4} \frac{1}{\sqrt{2k}} \left( \sum_{n=b}^{2k-1} e^{\frac{\pi i}{2k} n^2} + \sum_{n=2k}^{2k+b-1} e^{\frac{\pi i}{2k} n^2} \right) \\
&= e^{-\pi i/4} \frac{1}{\sqrt{2k}} \left( \sum_{n=b}^{2k-1} e^{\frac{\pi i}{2k} n^2} + \sum_{n=0}^{b-1} e^{\frac{\pi i}{2k} n^2} e^{2\pi i n} e^{2\pi i k} \right) \\
&= e^{-\pi i/4} \frac{1}{\sqrt{2k}} \left( \sum_{n=0}^{2k-1} e^{\frac{\pi i}{2k} n^2} \right) = 1
\end{aligned}$$

Where the last equality follows from the following computations

$$\sum_{n=0}^{2k-1} e^{\frac{\pi i}{2k} n^2} = \sum_{n=0}^{4k-1} e^{\frac{2\pi i}{4k} n^2} - \sum_{n=2k}^{4k-1} e^{\frac{2\pi i}{4k} n^2} = \sum_{n=0}^{4k-1} e^{\frac{2\pi i}{4k} n^2} - \sum_{n=0}^{2k-1} e^{\frac{\pi i}{2k} n^2} \tag{C.13}$$

$$\implies \sum_{n=0}^{2k-1} e^{\frac{\pi i}{2k} n^2} = \frac{\sqrt{4k}}{2}(1+i) = \sqrt{k}\left(\frac{1}{\sqrt{k}} + \frac{i}{\sqrt{k}}\right) = \sqrt{2k} e^{\pi i/4} \tag{C.14}$$

□

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