The Hitchin connection for the Quantization of the moduli space of parabolic bundles on surfaces with marked points

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Preface

This dissertation presents the work I have done during my five years as a PhD student at the Centre for Quantum Geometry of Moduli Spaces, QGM, at the Department of mathematics, Aarhus University.

I would like to say a big thanks to the QGM and the Department of Mathematics, Aarhus university, for making my time as a PhD student interesting, fun, full of new experiences and most of all full of an enormous amount of mathematics. Thank you to University of Maryland, in particular Richard Wentworth, for hosting me during the autumn of 2015.

In particular, I would like to say thank you to my advisor Jørgen Ellegaard Andersen, both for his many ideas, infectious enthusiasm and especially for his great patience with me and trust in my abilities.

Furthermore I would like to thank the everyone at QGM, both senior staff, fellow PhD students and secretaries, who have made my years as a PhD student amazing. Thank you to Kenneth, William and Andreas, who have been my office mates for the past year, and have been willing to answer any question I might have had.

Lastly an enormous thank you to my husband, I could not have done this without you – I love you.

Mette Bjerre, Aarhus, 2018

Contents

Abstract

The subject of my studies for the last five years as a PhD student at QGM, has been the moduli space of flat connections over a surface with punctures, each assigned with a weight. We define different moduli spaces, using Sobolev spaces and parabolic bundles, that are diffeomorphic on the smooth locus to the moduli space of flat connections. The aim of the thesis is to find a Hitchin connection in this setting, with specific constraints on the weights and the genus of the surface. We use the construction of the Hitchin connection with metaplectic correction by Andersen, Gammelgaard and Roed, to construct such a projectively flat Hitchin connection on the moduli space of parabolic bundles.

Resumé

I denne afhandling definerer vi modulirummet af flade konnektioner over en flade med punkteringer og en vægt i hver punktering. Vi definerer forskellige modulirum, blandt andet ved brug af Sobolev rum og parabolske bundter, som er diffeomorfe til modulirummet af flade konnektioner, på den glatte del. Formålet med denne thesis er at finde en Hitchin konnektion i denne setting, med så få antagelser på vægtene som muligt. Vi bruger den generelle konstruktion af Hitchin konnektionen i metaplektisk korrektion af Andersen, Gammelgård og Roed, til at finde en sådan Hitchin konnektion.

Chapter **1**

Introduction

The aim of this thesis is to construct a Hitchin connection in the context of quantization of moduli space of flat parabolic connections over a surface with marked points.

We recall that in general, the possible states of a quantum system are vectors in a Hilbert space, the so-called state space. Each observable is represented by a self-adjoint linear operator acting on the state space. Most quantum systems have a classical limit, and a way of relating the quantum mechanical observables to the observables of the corresponding classical system. It is however most often that the construction of the quantum system goes the other way, in the sense that one starts with a classical system and then by some process of "quantization" one obtains the quantum system.

In physics one typically uses canonical quantization, and for examples for the hydrogen atom, canonical quantization matches observations. In mathematics, the general quest is to make a well defined quantization scheme, that can be used on any phase space, and that reproduce canonical quantization on $(\mathbb{R}^{2n}, \omega)$.

Geometric quantization is an attempt at such a quantization scheme, which in its most complete form involves metaplectic quantization. This quantization scheme however depends on the choice of a so-called *polarization*, which in the case we will consider will simply be a complex structure compatible with the given symplectic form. This quantization scheme, in its current state of development, fails to establish the independence of the polarization in general.

In the context of quantization of moduli spaces of flat connections on a closed surface without punctures, one has a natural family of complex structures on this moduli space parametrized by Teichmüller space and the quantization procedure produces a vector bundle over Teichmüller space. Hitchin constructed a projectively flat connection in this bundle of quantizations over Teichmüller space ([29]), which in turn was inspired by Welters work on quantization of abelian varieties ([46]). Parallel transport from one point to another in Teichmüller space gives an identification of the fibers of this bundle, which is well defined up to a projective ambiguity, thus solving the independence of polarization for this particular family of complex structures.

This is precisely the setting which we shall seek to generalize as far as possible in case of moduli spaces of flat parabolic connections. Besides from the general motivation of establishing in as many cases as possible that the quantization is independent of the choice of polarization, there is a further motivation to settle this case of moduli spaces of flat parabolic connections, which stems from quantum Chern-Simons theory.

In $2+1$ dimensional quantum Chern-Simon theory, as proposed by Witten in [47], space is a two dimensional oriented surface, possibly with marked points. Let us briefly review the basics of this theory, starting with the case of a closed oriented surface Σ without marked points.

The space $\mathcal{A}$ of fields in this theory are connections in a principal $SU(n)$-bundles $P \to \Sigma$, which is unique up to isomorphism. Standard constructions in classical Chern-Simons theory associates a Hermitian line bundle $\tilde{\mathcal{L}}$ over $\mathcal{A}$ and a lift of the action of the gauge group $\mathcal{G}$ on $\mathcal{A}$ to $\tilde{\mathcal{L}}$ (see e.q. [24]). The construction further gives a $\mathcal{G}$-invariant connection ∇ in $\tilde{\mathcal{L}}$ with curvature given by the Atiyah-Bott symplectic form on $\mathcal{A}$. In this setting a conformal structure on Σ induces a complex structure on $\mathcal{A}$, which is compatible with the Atiyah-Bott symplectic form and thus combined with the connection ∇ induces a complex structure in $\tilde{\mathcal{L}}$. Witten argues in [47] that the state space for the surface, with the given conformal structure should be the holomorphic sections of $\mathcal{L}$ over $\mathcal{A}$, which are $\mathcal{G}$-invariant. We recall that the action of $\mathcal{G}$ is Hamiltonian and the moment map is given by the curvature. Using the principal that quantization commutes with reduction, Witten therefore further argued that one could first do the symplectic reduction of the space of connections $\mathcal{A}$ with respect to the $\mathcal{G}$-action and quantize the resulting quotient space, namely the moduli space of flat connections

$$\mathcal{M} = \mathcal{A}^F/\mathcal{G},$$

where $\mathcal{A}^F$ is the space of flat connections. Geometrically, one defines

$$\mathcal{L} = \tilde{\mathcal{L}}|_{\mathcal{A}^F}/\mathcal{G}.$$

The complex structure on $\mathcal{A}$ induces a complex structure on $\mathcal{M}$, which on this reduced space only depends on the equivalence class of the conformal structure on the surface Σ, which corresponds to a point in $\mathcal{T}_\Sigma$, the Teichmüller space of Σ. This way we see that the family of complex structures on $\mathcal{M}$ parametrized by Teichmüller space naturally arrises out of quantum Chern-Simons theory. According to Witten, the quantization commutes with reduction, which implies that the resulting bundle of quantum vector space $H^{(k)}$ is given by

$$H^{(k)}_\sigma = H^0(\mathcal{M}_\sigma, \mathcal{L}^k),$$

where $\mathcal{M}_\sigma$ is $\mathcal{M}$ endowed with the complex structure induced by $\sigma \in \mathcal{T}_\Sigma$. We observe that group of orientation preserving diffeomorphism of Σ acts on $H^{(k)}$ and further that the subgroup of diffeomorphisms which are isotopic to the identity acts trivial, thus there is a natural action of the mapping class group Γ on $H^{(k)}$ covering its natural action on $\mathcal{T}_\Sigma$.

In the paper [15], Axelrod, Della Pietra and Witten argue using the above mentioned infinite dimensional description of the bundle $H^{(k)}$, that it should support a Γ-invariant projectively flat connection. On the mathematical side, this connection, ∇^H was constructed by Hitchin in [29]. See also the paper of Andersen [3], where he proves that the differential geometric construction of [15] agrees with Hitchin's from [29].

In order to remedy the fact that this connection is not flat, but only projectively flat one can follow the constructions outlined in [39] and discussed in more detail in [45] and [11], which we shall very briefly review here. Over the Teichmüller space we have the determinant line bundle $\mathcal{L}_D$, whose fibers are given by

$$L_{D,\sigma} = \bigwedge^g (H^0(\Sigma_\sigma, \Omega^1)$$

for all $\sigma \in \mathcal{T}_\Sigma$. The mapping class group Γ of course also acts on L_D. As it is argued in [11], there exist a connection ∇^D in L_D such that

$$c(k) F_{\nabla^D} \otimes \mathrm{Id}_{H^{(k)}} = F_{\nabla^H},$$

where $c(k)$ a rational function of k. In [11] an explicit construction (depending on the choice of a Lagrangian subspace $L \subset H^1(\Sigma, \mathbb{R})$) of the $-c(k)$-power of $\mathcal{L}_D$ is given together with the construction of a natural connection in this power whose curvature is $c(k)F_{\nabla^D}$, even though $c(k)$ is in general is only a rational number for a given k. There is a central extension (also constructed explicitly given L in [45] and [11])

$$0 \to \mathbb{Z} \to \tilde{\Gamma} \to \Gamma \to 0,$$

such that $\tilde{\Gamma}$ acts on $L_D^{-c(k)}$ preserving the natural connection.

One then considers the bundle

$$\tilde{H}^{(k)} = H^{(k)} \otimes \mathcal{L}_D^{-c(k)},$$

which has a flat connection $\tilde{\nabla}^H$, which is $\tilde{\Gamma}$-invariant.

Definition 1.1. We define the vector space $Z^{(k)}(\Sigma, L)$ associated to the surface Σ and the Lagrangian subspace $L \subset H^1(\Sigma, \mathbb{R})$ to be the space of global covariant constant sections of $(\tilde{H}^{(k)}, \tilde{\nabla}^H)$ and the corresponding *Quantum Representation* of $\tilde{\Gamma}$

$$Z^{(k)} : \tilde{\Gamma} \to \mathrm{Aut}(Z^{(k)}(\Sigma, L)).$$

Let us remark that this gauge theoretic construction of the representations was used by Andersen to prove his asymptotic faithfulness results for these representations [2].

1.1 Topological Quantum Field Theory

Returning to Witten original proposals in his famous paper [47], we recall that Witten argue using path integrals that the above Quantum Representations of closed surfaces should actually fit into a whole $2 + 1$ dimensional Topological Quantum Field Theory (TQFT).

In outline, a $(2 + 1)$ dimensional TQFT can be seen as a symmetric monoidal functor from the category of closed oriented surfaces with morphisms compact and oriented cobordisms, to the category of vector spaces

$$Z\colon (\mathrm{Cob}(3), \sqcup, \emptyset) \to (\mathrm{Vect}(\mathbb{C}), \otimes, \mathbb{C}).$$

For Σ, a closed oriented surface, Z associates a finite dimensional vector space $Z(\Sigma)$, called the module of states of Σ. It should be multiplicative with respect to the disjoint union, so for $\Sigma = \Sigma_1 \sqcup \Sigma_2$, Z has to satisfy $Z(\Sigma) \simeq Z(\Sigma_1) \otimes Z(\Sigma_2)$. The empty set is sent to $\mathbb{C}$. A cobordism from a surface Σ to a surface $\tilde{\Sigma}$ is a compact oriented three-manifold M. This should be sent to a linear operator

$$Z(M)\colon Z(\Sigma) \to Z(\tilde{\Sigma}).$$

A closed three-manifold M is a cobordism between two empty surfaces, and hence should be sent to a linear map from $\mathbb{C}$ to $\mathbb{C}$, which is a complex number in $\mathbb{C}$, called the quantum invariant of M.

In its more complete form the source category of TQFT's are typically enhanced with more structure, e.g. one considers surfaces with some more structure, concretely our objects will be tuples $\boldsymbol{\Sigma} = (\Sigma, P, V, L)$, where Σ is a closed oriented surfaces, L is a Lagrangian subspace of the first cohomology of Σ with real coefficients, $\mathcal{P} \subset \Sigma$ is a finite subset and

$$V \in \bigtimes_{p \in \mathcal{P}} \mathbb{P}(T_{p^{(i)}}\Sigma),$$

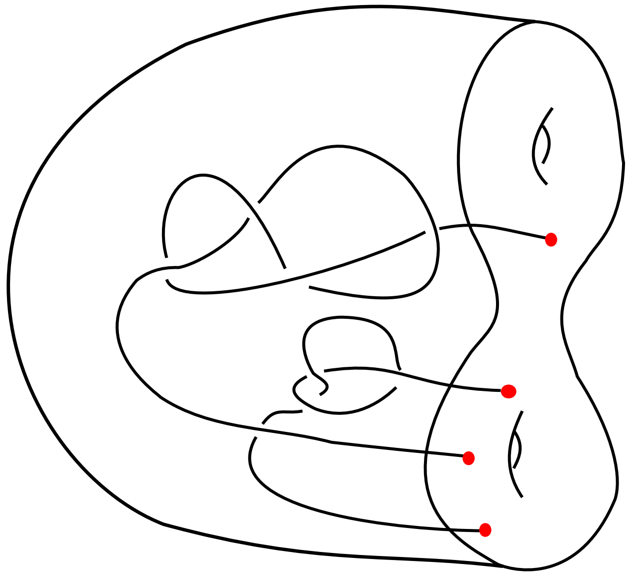


Figure 1.1: A cobordism from the empty surface to a pointed surface.

where $\mathbb{P}$ refers to the projectivization with respect to the real positive numbers. Furthermore, the TQFT comes with a specific finite *label* set Λ, and we require that each object is provided with a labelling of each of its marked points $\mathcal{P}$ by elements of Λ

$$\lambda\colon \mathcal{P} \to \Lambda.$$

Concerning the three dimensional part of the theory, a morphism from one object $(\mathbf{\Sigma}_1, \lambda_1)$ to another $(\mathbf{\Sigma}_2, \lambda_2)$ is a cobordism (X, K, V) from (Σ_1, P_1, V_1) to (Σ_2, P_2, V_2), where K is an oriented link in X with boundary $P_1 \cup P_2$ and V is framing of K which agrees with V_i over P_i (see Figure 1.1 for such an example of such a cobordism in the case where $\Sigma_1 = \emptyset$) together with a labelling of the components of K which agrees with λ_i at P_i and an integer n. See [44] and [9] for further details, in particular for the definition of composition of morphisms. A TQFT is a functor from the above cobordism category to the category of finite dimensional vector space of the complex numbers, as is described in detail in [44].

In 1990–91 Reshetikhin and Turaev ([38]) gave the first complete construction of such a TQFT using the representation theory of the quantum groups at root $U_q(\mathfrak{sl}(2, \mathbb{C})$, where $q = \exp(2\pi i/(k+2))$. A year later Blanchet, Habegger, Masbaum, and Vogel constructed, using Skein theory, isomorphic TQFTs in [18, 19, 20]. A few years later, the TQFT's for the whole A_n-series was constructed by Turaev and Wenzl in [43]. This family of TQFT now goes under the name of the *Witten-Reshetikhin-Turaev* TQFT's and for short WRT-TQFT's. The label set Λ at level k for the TQFT for the Lie-algebra $\mathfrak{sl}(n, \mathbb{C})$ given as a finite subset of the set of dominant positive weights $\mathcal{W}_+$ of the Lie-algebra $\mathfrak{sl}(n, \mathbb{C})$

$$\Lambda := \{\lambda \in \mathcal{W}_+ \mid 0 \leq \langle \theta, \lambda \rangle \leq k\},$$

where θ is the longest root and $\langle \cdot, \cdot \rangle$ is the Killing form normalized such that

$$\langle \theta, \theta \rangle = 2.$$

We shall not further expand on these 3 dimensional aspect of these TQFT and their combinatorial constructions, since we will not need it in this thesis. The point being that the two dimensional part of such a TQFT, which is called a *Modular Functor*, uniquely determines the TQFT. Please see [9] and [27] for details regarding this, where an explicit construction of the full TQFT from its modular functor is given.

The above TQFT constructions are all mathematically satisfactory, but they are however purely combinatorial. Witten however also proposed in [47] that these TQFT, in particular their underlying modular functors, should also be constructible using either conformal field theory or quantization of moduli spaces of flat parabolic connections.

In response to Witten's suggestion, Andersen and Ueno constructed modular functors using conformal field theory in [10] and [11], building on work by Tsuchikya, Ueno, and Yamada in [42]. Andersen and Ueno then proved in [12] and [13] that these modular functors are isomorphic to the modular functors due to Reshetikhin and Turaev [38] and further Turaev and Wenzl [43].

Laszlo proved in [33] that the projective representation of the mapping class group as defined above in Definition 1.1 is isomorphic to the projective representation of the mapping class group coming from conformal field theory from [42] and [11]. By combining this with the results of Andersen and Ueno above, we see that the Quantum Representations from Definition 1.1 above are projectively the same as the Witten-Reshetikhin-Turaev Quantum Representations which are part of the WRT-TQFT's.

In the case of surfaces with marked points Witten also conjectured that the quantization of the moduli spaces of flat parabolic connections should give rise to the same quantum representations of the corresponding mapping class groups (detailed below), which are part of the WRT-TQFT's – The mathematical status on the gauge side is however lacking. The first object is actually to extend Hitchin's construction of the projective flat connection discussed above to this case of the quantization of the moduli space of flat parabolic connections and this is precisely the focus of this thesis.

1.2 Quantization of moduli spaces of flat parabolic connections

We shall restrict attention to moduli spaces of parabolic connections for the compact Lie group $\mathrm{SU}(n)$. We need a few Lie theoretic notions for its Lie algebra $\mathfrak{su}(n)$ and its complexification $\mathfrak{sl}(n,\mathbb{C})$, which we briefly recall now.

The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$

The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ consists of all the traceless $n\times n$ complex matrices. The Cartan algebra $\mathfrak{h}$ is the subspace of diagonal traceless matrices, each of which we identify with an n-tuple $(a_1,\dots,a_n)$ with $\sum_i a_i = 0$. Let $L_i\colon \mathfrak{h}\to\mathbb{C}$ be defined by $L_i(a_1,\dots,a_n)=a_i$, then the dual to the Cartan algebra is

$$\mathfrak{h}^* = \mathbb{C}\langle L_1,\dots,L_n\rangle/\langle L_1+\cdots+L_n\rangle.$$

Define for $i\neq j$, $E_{ij}\in\mathfrak{sl}(n,\mathbb{C})$ to be the matrix which has a 1 in the (i,j)-entry and zeros otherwise. Then E_{ij} is an eigenvector for $\mathfrak{h}$ under the adjoint action with eigenvalue L_i-L_j. The weight lattice is $\mathcal{W}=\mathbb{Z}\langle L_1,\dots,L_n\rangle/\langle L_1+\cdots+L_n\rangle$ and the root lattice is $\mathcal{R}=\mathrm{span}_{\mathbb{Z}}\{L_i-L_j \mid i<j\}$. Note that we have an isomorphism $\mathcal{W}/\mathcal{R}\cong\mathbb{Z}/n\mathbb{Z}$ given by $\sum\alpha_i L_i\mapsto\sum\alpha_i\in\mathbb{Z}/n\mathbb{Z}$. We can define a set of positive roots by $R^+=\{L_i-L_j\mid i>j\}$. Then the simple roots are $\Pi=\{L_{i+1}-L_i\mid i=1,\dots,n-1\}$. The positive Weyl chamber is $\mathcal{C}^+=\{\sum_i a_iL_i\mid a_1\leq\cdots\leq a_n\}$. We define the positive weights to be $\mathcal{W}^+=\mathcal{W}\cap\mathcal{C}^+$. In general, when we have a positive weight $\lambda \in \mathcal{W}^+$, we get an n-tuple $[\lambda_1, \ldots, \lambda_n]$ of integers such that $\lambda_1 \leq \cdots \leq \lambda_n$. We observe that $\theta = L_n - L_1$, thus we see that

$$\Lambda = \{(0, \lambda_2, \ldots \lambda_n) \in \mathbb{Z}^n \mid 0 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq k\}.$$

Some of the entries in this n-vector might be equal, hence we get a reduced vector consisting of r different numbers, which we will denote $\tilde{\lambda}$,

$$\begin{aligned} \lambda_1 &= \ldots = \lambda_{k_1} \\ \lambda_{k_1+1} &= \ldots = \lambda_{k_1+k_2} \\ &\ldots \\ \lambda_{k_1+\cdots+k_{r-1}+1} &= \ldots = \lambda_n. \end{aligned}$$

Let $\tilde{\lambda} = [\lambda_{k_1}, \ldots, \lambda_{k_{r-1}}, k]$. This specifies a flag-type $(k_1, \ldots, k_r)$, where $\sum_i k_i = n$.

We have an isomorphism $i \colon \mathfrak{h} \to \mathfrak{h}^*$ given by $i(\xi)(\xi') = \langle \xi, \xi' \rangle$, such that $\frac{1}{k} i^{-1}$ is an inclusion of Λ in $\mathfrak{U} := \{\xi \in \mathfrak{h}_+ \mid \theta(\xi) \leq 1\} \subset \mathfrak{h}$, which in turn is in bijective correspondence with the set of conjugacy classes of $SU(n)$.

Explicitly take a weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$. Since $i \colon \mathfrak{h} \to \mathfrak{h}^*$ is the map $i(\xi)(\xi') = \langle \xi, \xi' \rangle$ we get

$$i^{-1}(\lambda_1 L_1 + \cdots + \lambda_n L_n) = (\lambda_1 - \frac{1}{n} \sum_i \lambda_i, \ldots, \lambda_n - \frac{1}{n} \sum_i \lambda_i).$$

Divide this with k to get

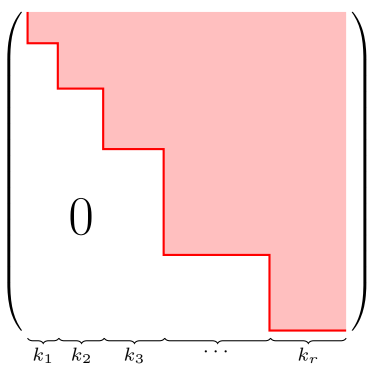
$$(\frac{1}{k}\lambda_1 - \frac{1}{kn} \sum_i \lambda_i, \ldots, \frac{1}{k}\lambda_n - \frac{1}{kn} \sum_i \lambda_i)$$

Now take the exponential

$$\exp(\tfrac{1}{k} i^{-1}(\lambda)) = \begin{pmatrix} e^{\frac{2\pi i}{k}(\lambda_1 - \frac{1}{n}\sum_i \lambda_i)} & 0 & \ldots & 0 \\ 0 & e^{\frac{2\pi i}{k}(\lambda_2 - \frac{1}{n}\sum_i \lambda_i)} & \ldots & 0 \\ \vdots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & e^{\frac{2\pi i}{k}(\lambda_n - \frac{1}{n}\sum_i \lambda_i)} \end{pmatrix}$$

Let $C_\lambda^{(k)}$ be the conjugacy class of $\exp(\frac{1}{k} i^{-1}(\lambda))$.

For each weight $\lambda \in \Lambda$ we have a corresponding parabolic subgroup P of $\mathrm{SL}(n, \mathbb{C})$, which is the maximal such stabilizing λ. Let W_λ to be the stabilizer of λ in the Weyl group W and let I be the subset of simple roots of $\mathrm{SL}(n, \mathbb{C})$ which are also roots of P. The parabolic subgroup P only depends on the tuple $(k_1, \ldots, k_r)$ and it consists of the following matrices



and $W_\lambda = S_{k_1} \times \cdots \times S_{k_r}$.

The moduli space of flat connections

Let us now describe precisely which moduli spaces we shall consider here. Let Σ be a closed oriented 2-manifold of genus ≥ 2. On Σ we have b marked points $\mathcal{P} = \{p^{(1)}, \ldots, p^{(b)}\}$. In each point there should be a chosen direction, that is for each i let

$$v^{(i)} \in \mathbb{P}(T_{p^{(i)}}\Sigma) := (T_{p^{(i)}}\Sigma \backslash \{0\})/\mathbb{R}_+.$$

Let V denote this set of directions. In each marked point, there should also be a given a weight $\lambda^{(i)} \in \Lambda$, $\overline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(b)})$. Let $\tilde{\Sigma}$ denote the punctured surface $\Sigma \backslash \mathcal{P}$.

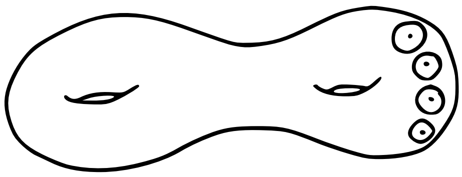


Figure 1.2: Surface with marked point and small embedded discs around each puncture, along which we compute the holonomy around each puncture in the direction induced from the orientation of the surface.

Let $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ be the moduli space of flat connections on $\tilde{\Sigma}$, whose holonomy around $p^{(i)}$ lies in the conjugacy classes $C^{(k)}_{\lambda^{(i)}}$, and let $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ denote the locus of this moduli space consisting of irreducible such flat connections. See Chapter 6 for further details. In particular we recall that if $b \neq 0$ or if $b = 0$, but $(g, n) \neq (2, 2)$, then the compliment of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ has real co-dimension at least 4.

Let $\mathcal{T}_{(\Sigma, \mathcal{P}, V)}$ denote the Teichmüller space of $(\Sigma, \mathcal{P}, V)$ (for the precise definition of this Teichmüller space please see Chapter 7).

We recall that $\mathcal{T}_{(\Sigma, \mathcal{P}, V)}$ parametrizes complex structures on $(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', \omega)$

$$\begin{aligned} I \colon \mathcal{T}_{(\Sigma, \mathcal{P}, V)} &\to \text{Complex structures on } (\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', \omega) \\ \sigma &\mapsto I_\sigma, \end{aligned}$$

such that (ω, I_σ) is Kähler for all $\sigma \in \mathcal{T}_{(\Sigma, \mathcal{P}, V)}$. We shall use the notation $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$ for the complex manifold $(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', I_\sigma)$.

In order to construct this family I of complex structures we use Sobolev spaces completions of certain kinds of connections on $\tilde{\Sigma}$ (see Chapter 6 for details) to construct a moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)_\sigma$, whose smooth part is shown by Daskalapoulos and Wentworth to be naturally an almost complex manifold in [22] which in fact they also show is integrable. For small enough ϵ the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)_\sigma$ is homeomorphic (diffeomorphic on the irreducible locus) to $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$.

By the Mehta and Seshadri Theorem [35], the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ is homeomorphic (diffeomorphic on the stable locus and in fact bi-holomorphic when we consider $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$) to the moduli space $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$ of semi-stable parabolic bundles with trivial determinant and weights determined by the $\lambda^{(i)}$'s for each $\sigma \in \mathcal{T}_{(\Sigma, \mathcal{P}, V)}$. However the Sobolev construction of this complex structure allows us to understand its variation with respect to $\sigma \in \mathcal{T}_{(\Sigma, \mathcal{P}, V)}$ better and further is in our favor when we need to identify to the pre-quantum line bundle we need.

Let $\omega \in \Omega^2(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})')$ be the natural symplectic form on the smooth part of this moduli space which is recalled in Chapter 6.

Note that the co-dimension of the strictly semi-stable locus is at least two, except for the case $(g, n) = (2, 2)$ and no marked points, which we exclude from this discussion. See [8] or the co-dimension estimates made in Section 6.2. Hence we can use Hartog's extension Theorem whenever needed.

Andersen, Himpel, Jørgensen, Martens, and McLellan constructed in [8] a prequantum line bundle $(\mathcal{L}_{k,\overline{\lambda}}, \nabla, \langle\cdot,\cdot\rangle)$ over $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ using classical Chern-Simons theory under the assumption that each $\lambda^{(i)}$ is in the interior of the positive Weyl chamber $\mathcal{C}^+$, however we will see below that this will actually not be a restriction for us. The construction is recalled in Chapter 7. Let $\mathcal{V}^{(k)}_{\overline{\lambda}}$ denote the bundle over $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$ with fibers

$$\mathcal{V}^{(k)}_{\overline{\lambda},\sigma} := H^0(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma, \mathcal{L}^{(k)}_{\overline{\lambda},\sigma}).$$

Let $\Gamma_{(\Sigma,\mathcal{P},V)}$ denote the mapping class group of $(\Sigma, \mathcal{P}, V)$. Then $\Gamma_{(\Sigma,\mathcal{P},V)}$ acts on $(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', \omega)$, it acts on $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$ and I is equivariant for this action. Furthermore in [8] an explicit construction of a lift of the action of $\Gamma_{(\Sigma,\mathcal{P},V)}$ to $(\mathcal{L}_{k,\overline{\lambda}}, \nabla, \langle\cdot,\cdot\rangle)$ is provided, and using this we see that $\Gamma_{(\Sigma,\mathcal{P},V)}$ naturally acts on $\mathcal{V}^{(k)}_{\overline{\lambda}}$ covering its action on $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$.

The first step towards the construction of a Hitchin connection in the bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$ is to compute the first Chern class of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$. To do this we consider the moduli stacks instead of directly working with the moduli spaces, since it turns out the calculations are much more straight forward in this language.

Let $\overline{P} = (P^{(1)}, \ldots, P^{(b)})$ be the parabolic subgroups corresponding to $\overline{\lambda}$. We recall that a parabolic holomorphic $\mathfrak{sl}(n, \mathbb{C})$-bundle on Σ_σ with parabolic structures at $\{p^{(1)}, \ldots, p^{(b)}\}$ is an $\mathfrak{sl}(n, \mathbb{C})$-bundle $\mathcal{E}$ with a reduction of structure $\varphi^{(i)} \in \mathcal{E}_{p^{(i)}}/P^{(i)}$ for each $i = 1, \ldots b$.

Let $\mathfrak{M}_{\Sigma_\sigma}$ denote the moduli stack of $\mathfrak{sl}(n, \mathbb{C})$-bundles over Σ_σ and $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ the stack of parabolic $\mathfrak{sl}(n, \mathbb{C})$-bundles determined by $\overline{P}$. We note that there is a natural projection

$$\pi \colon \mathfrak{B}_{\Sigma_\sigma,\overline{P}} \to \mathfrak{M}_{\Sigma_\sigma}.$$

From [34] we know that the Picard group of the stack $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ is $\mathbb{Z}$ direct sum the Picard group of each $G/P^{(i)}$ for each of the $P^{(i)}$'s. Now the Picard group of $G/P^{(i)}$ is the character group $\mathcal{X}(P^{(i)}) \subset \mathcal{W}$ of $P^{(i)}$, which is precisely the sub-lattice

$$\mathcal{X}(P^{(i)}) = \{\mu \in \mathcal{W} \mid \langle \mu, \alpha \rangle = 0 \,\forall\, \alpha \in I^{(i)}\}.$$

Theorem 1.2. *The Picard group of* $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ *is*

$$\mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma,\overline{P}}) = \mathbb{Z} \oplus \bigoplus_{i=1}^{b} \mathcal{X}(P^{(i)}).$$

We remark that there is a morphism from the sub-stack of semi-stable bundles $\mathfrak{B}^{ss}_{\Sigma_\sigma,\overline{P}}$ to the moduli space of semi-stable parabolic bundles $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$, which induces an injection on the level of Picard groups. As it is argued in [37], the line bundle $\mathcal{L}_{k,\overline{\mu}}$ associated to $(k, \overline{\mu}) \in \mathbb{Z} \oplus \bigoplus_{i=1}^{b} \mathcal{X}(P^{(i)})$ descends to $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$ if and only if $\exp(\sum_i \mu^{(i)})$ acts trivial on the center of $SU(n)$, e.g. if $\sum_i \mu^{(i)} \in \mathcal{R}$. Which means that when we write each $\mu^{(i)}$ in the $\mathbb{Z}$-basis L_i of $\mathcal{W}$, then the total sum of all coefficients must be divisible by n. One can remark (see e.q. [37, 9]) that if we consider $(k, \overline{\lambda}) \in \mathbb{Z} \oplus \bigoplus_{i=1}^{b} \mathcal{X}(P^{(i)})$, with $\lambda^{(i)} \in \Lambda$ for each i, then if $\sum_i \lambda^{(i)} \notin \mathcal{R}$ then $\mathcal{L}_{k,\overline{\lambda}}$ has no holomorphic section over the stack $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ and the space of vacua associated to $(\Sigma, \mathcal{P}, V, \overline{\lambda})$ is also zero, so we really do not need to consider this case and we will from now on assume that our labelling $\overline{\lambda}$ satisfies that $\sum_i \lambda^{(i)} \in \mathcal{R}$.

We prove that the canonical bundle of $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ is the tensor product of the pullback bundle of the canonical bundle of $\mathfrak{M}_{\Sigma_\sigma}$ and a line bundle for each marked point in Chapter 9. We actually do this in the case of any simple complex Lie group $G^{\mathbb{C}}$ and the result is

Theorem 1.3. *The canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ has the form*

$$K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} = \otimes_{i=1}^n \mathcal{L}_{\kappa^{(i)}} \otimes \pi^* K_{\mathfrak{M}_{\Sigma_\sigma}}.$$

where $\kappa^{(i)} \in \mathcal{X}(P^{(i)})$ is the element $\kappa^{(i)} = -\sum_{\alpha\in R(\mathfrak{g}/\mathfrak{p}^{(i)})} \alpha$.

When we look at the specific Lie group $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$, we can write the $\kappa^{(i)}$s in our chosen $\mathbb{Z}$-basis L_i of $\mathcal{W}$ as

$$\kappa^{(i)} = [n - k_1^{(i)}, n - (2k_1^{(i)} + k_2^{(i)}), \dots, -(k_1^{(i)} + \cdots + k_{r-1}^{(i)})],$$

see Chapter 9, Corollary 9.15.

Combining this with the explicit description of the Picard group, we get the following result

Corollary 1.4. *The canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ correspond to the element*

$$K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} \simeq (-2\check{h}, \kappa^{(i)}, \dots, \kappa^{(b)}) \in \mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma,\overline{P}})$$

where $\check{h}$ is the dual Coxeter number, which for $SU(n)$ is n and for $G = SU(n)$ and the above specified parabolic sub-groups $\overline{P}$, we have that

$$\kappa^{(i)} = [(n - k_1^{(i)}), (n - (2k_1^{(i)} + k_2^{(i)})), \dots, (n - (2k_1^{(i)} + \cdots + 2k_{r-2}^{(i)} + k_{r-1}^{(i)})), -(k_1^{(i)} + \cdots + k_{r-1}^{(i)})]$$

This result gives us the first Chern class of the moduli stack of parabolic bundles $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$. In the cases where the first Chern class is proportional to the symplectic form ω we can use Andersen's general construction of the Hitchin connection, which we recall in details in Chapter 4.1. Let us here briefly recall the result. Suppose I is a rigid family, parametrized by a complex manifold $\mathcal{T}$, of Kähler structures on the symplectic prequantizable compact manifold (M, ω), which satisfies that there exists $l \in \mathbb{Q}$ such that the first Chern class of (M, ω) is $l[\omega] \in H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{R}) = 0$. Let $\mathcal{H}^{(k)}$ denote the trivial bundle $C^\infty(M, L^k)$. Assume the bundle $H^{(k)}$ with fibers

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k) = \left\{ s \in C^\infty(M, \mathcal{L}^k) \;\middle|\; \nabla_\sigma^{0,1} s = 0 \right\}$$

is a sub-bundle of $\mathcal{H}^{k)}$. Then

Theorem 1.5 ([3])**.** *There exists a Hitchin connection $\hat{\nabla}$ in the trivial $C^\infty(M, \mathcal{L}^k)$-bundle, which preserves the sub-bundle $H^{(k)}$. It is given by*

$$\hat{\nabla}_V = \hat{\nabla}_V^t - u(V),$$

where $\hat{\nabla}_V^t$ is the trivial connection in $\mathcal{H}^{(k)}$, V is any smooth vector field on $\mathcal{T}$, $u(V)$ is the second order differential operator given by

$$u(V) = \frac{1}{2k + l} \left(\frac{1}{2} \Delta_{G(V)}(s) + \nabla_{G(V)dF}(s) + 2kV'[F]s \right)$$

where $\Delta_{G(V)}$ is a certain second order operator depending linearly and smoothly on V defined by Equation (4.3)*. Further V' denotes the* $(1,0)$*-part of V on $\mathcal{T}$ and $F\colon \mathcal{T} \to C_0^\infty(M)$ is determined by $F_\sigma \in C^\infty(M)$ being the Ricci potential for (M, I_σ) for all $\sigma \in \mathcal{T}$.*

We remark that the compactness assumption on M is only used to obtained a smooth family of Ricci potentials F (using Hodge theory) and that this assumption therefore can be dropped if we are given a smooth family of Ricci potentials F by some other means. We note that for our moduli spaces such a smooth family of Ricci potentials were defined by Zograf and Takhtajan in [40]. Andersen and Gammelgaard prove in [4] that the connection $\hat{\nabla}$ is projectively flat, when there are no holomorphic sections.

Theorem 1.6 ([4])**.** *The connection $\hat{\nabla}$ defined in Theorem 1.5 is projectively flat, provided $H^0(M_\sigma, T_\sigma) = 0$ for all $\sigma \in \mathcal{T}$.*

In Chapter 10 we then conclude that we can use Andersen's construction of a projectively flat Hitchin connection in the bundle $\mathcal{V}_{\overline{\lambda}}^{(k)}$ when the first Chern class of the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ is $l[\omega]$ for some $l \in \mathbb{Q}$. We know from [8] that the class of ω is given by

$$(k, \lambda^{(1)}, \dots, \lambda^{(b)}) \in \mathbb{Z} \oplus \bigoplus_i \mathcal{X}(P^{(i)}),$$

hence we can figure out when the first Chern class of the canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma, \overline{P}}}$ and the class of the symplectic form ω are proportional for the stack of parabolic bundles. When they are indeed proportional on the stack they will also be so the for moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$. We conclude

Theorem 1.7. *If for all $i = 1, \dots, b$ there exists an $l \in \mathbb{Q}$ such that*

$$\begin{aligned} -2\check{h} \cdot l &= k \\ \lambda^{(i)} \cdot l &= \kappa_1^{(i)} = -(n - k_1^{(i)}) \\ &\dots \\ \lambda^{(i)} \cdot l &= \kappa_r^{(i)} = k_1^{(i)} + \dots + k_{r-1}^{(i)}, \end{aligned}$$

up to adding an integer $m^{(i)} \in \mathbb{Z}$ to each equation. Then we can apply Theorem 1.5 and Theorem 1.6 to construct a Hitchin connection in $\mathcal{V}_{\overline{\lambda}}^{(k)}$, which is projectively flat.

As we can conclude from the above, it is not always the case that $[\omega]$ has the form as in Theorem (1.7), and when this is the case we can not use Andersen's general construction of the Hitchin connection. However, when the first Chern class is even, we can instead use metaplectic quantization and the following result from [5], which we present in Section 4.2. Let (M, ω) be a prequantizable symplectic manifold with vanishing second Stiefel Whitney class. Let J be a rigid family of Kähler structures on M parametrized by a smooth manifold $\mathcal{T}$, all satisfying $H^{0,1}(M_\sigma) = 0$, $\sigma \in \mathcal{T}$. Let $\mathcal{H}_\delta^{(k)}$ be the vector bundle with fibers

$$\mathcal{H}_{\delta,\sigma}^{(k)} := C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta),$$

and assume $H_\delta^{(k)}$, given by the spaces $H_{\delta,\sigma}^{(k)} := H^0(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) = \left\{ s \in \mathcal{H}_{\delta,\sigma}^{(k)} \;\middle|\; \nabla_\sigma^{0,1} s = 0 \right\}$ is indeed a sub-bundle of $\mathcal{H}_\delta^{(k)}$. Then

Theorem 1.8 ([5] Theorem 1.2)**.** *With the assumptions above there exists a one form $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$ satisfying $\overline{\partial}\beta(V) = -\frac{i}{2}\delta(\rho{\cdot}G(V))$ for all vector field V on $\mathcal{T}$ and the connection*

$$\hat{\nabla}^\delta_V = \hat{\nabla}_V + \frac{1}{4k}(\Delta_{G(V)} + \beta(V))$$

is a Hitchin connection on $\mathcal{H}^{(k)}_\delta$ over $\mathcal{T}$.

It was proven by Gammelgaard in his thesis (Theorem 6.22), that

Theorem 1.9 ([25])**.** *The connection $\hat{\nabla}$ defined in Theorem 1.8 is projectively flat, provided $H^0(M_\sigma, T_\sigma) = 0$ for all $\sigma \in \mathcal{T}$.*

We wish now to construct a Hitchin connection in the bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$ over $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$ using this metaplectic version of the construction. However, to use Theorem 1.8 to construct such a Hitchin connection in $\mathcal{V}^{(k)}_{\overline{\lambda}}$, we need that the canonical bundle $K_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda})'}$ has a square root, and there exists a fixed pre-quantum line bundle $\tilde{L}_{\overline{\lambda}}$ such that

$$K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda})'} \otimes \tilde{L}_{\overline{\lambda}} \simeq \mathcal{L}_{k,\overline{\lambda}}$$

as holomorphic line bundles for all $\sigma \in \mathcal{T}_{(\Sigma,\mathcal{P},V)}$.

In Theorem 1.4 we have written $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ as $(-2\check{h}, \kappa^{(1)}, \dots, \kappa^{(b)})$, where

$$\kappa^{(i)} = -\sum_{\alpha \in R(\mathfrak{g}/\mathfrak{p}^{(i)})} \alpha \in \mathcal{X}(P^{(i)}).$$

Hence we know the square root exists on the stack whenever each $\kappa^{(i)}$ is even in $\mathcal{X}(P^{(i)})$. For the special case $SU(n)$ this means that the canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ has a square root when the numbers

$$n - k^{(i)}_1, n - (2k^{(i)}_1 + k^{(i)}_2), \dots, n - (2k^{(i)}_1 + \dots + 2k^{(i)}_{r-2} + k^{(i)}_{r-1}), -(k^{(i)}_1 + \dots + k^{(i)}_{r-1})$$

have the same parity for each i. Let us assume this and then in this case we let $b_o(\overline{P})$ be the number of points $p^{(i)}$ where these numbers are odd. In Section 9.4 we do the little elementary computation which shows that

Proposition 1.10. *In the case where the $\kappa^{(i)}_j$'s have the same parity for each i, $\mathcal{M}(\tilde{\Sigma},\overline{\lambda})'_\sigma$ has a unique square root of its canonical bundle if and only if $b_o(\overline{P})$ is even.*

From this we immediately get

Corollary 1.11. *In the case where $k^{(i)}_j = 1$ for all $j = 1, \dots, r^{(i)}$, $i = 1, \dots b$, the moduli space $\mathcal{M}(\tilde{\Sigma},\overline{\lambda})'_\sigma$ has a unique square root of its canonical bundle if n or b is even.*

We will see that this is enough for us, since we can always arrange that b is even by propagation of vacua (see [42] or [11]), as we now detail. Suppose n is odd and suppose that b is odd. Then we will add to $\mathcal{P}$ a further point, with any tangent direction and label it with $0 \in \Lambda$. It is well know by the so called propagation of vacua that the space of conformal blocks for the surface with this extra point labeled by 0 is canonically isomorphic to the one without. Further the moduli space flat parabolic connection $\mathcal{M}(\tilde{\Sigma},\overline{\lambda})$ is also unchanged under this operation and further, under pull back of the natural projection from the Teichmüller space of the surface with one more marked point to the Teichmüller space of the original surface one gets an identification of the corresponding Verlinde bundles. Thus we will by abuse of notation also denote by $\overline{\lambda}$ the labelling where we have added one more point labeled by 0.

Now let $\overline{\rho} = (\rho, \ldots, \rho)$ and define

$$\overline{\lambda}' = \overline{\lambda} + \overline{\rho}.$$

Let $\overline{P}'$ be the vector of parabolic subgroups associated to $\overline{\lambda}'$. We observe that for $\overline{\lambda}'$ we have all $k_j^{(i)} = 1$, since all components of $\overline{\lambda}'$ are contained in the interior of $\mathcal{C}^+$ and thus $\overline{P}'$ is just b copies of the Borel subgroup. But then by Corollary 1.11, we have that $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'$ has a unique square root of its canonical bundle, which we denote $K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'}$. Let us define the bundle $\tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'}$ to be the bundle over $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$ whose fiber at $\sigma \in \mathcal{T}_{(\Sigma,\mathcal{P},V)}$ is

$$\tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'} = H^0(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma, \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'}).$$

As it is explained in Chapter 11, pulling back over the natural fibration

$$\pi' : \mathfrak{B}_{\Sigma_\sigma,\overline{P}'} \to \mathfrak{B}_{\Sigma_\sigma,\overline{P}}$$

first of all gives that

$$(\pi')^* \mathcal{L}_{k,\overline{\lambda}} = \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'}$$

and second of all induces a natural isomorphism of bundles

$$(\pi')^* : \tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'} \to \mathcal{V}^{(k)}_{\overline{\lambda}}$$

due to the fact that the strictly semi-stable locus has complex co-dimension at least two as argued in Section 6.2.

By the previously quoted result of [8], we know that $\mathcal{L}_{k+\check{h},\overline{\lambda}'}$ is a fixed pre-quantum line bundle over $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'$ since all components of $\overline{\lambda}'$ are contained in the interior of $\mathcal{C}^+$. Since we further have that $H^{0,1}(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma) = 0$ by Proposition 11.1 in Chapter 11, we conclude that Theorem 1.8 applies to provide a Hitchin connection in $\tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'}$. Since we also know that $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma$ has no holomorphic vector fields by Proposition 11.2 in Chapter 11, we can apply Theorem 1.9 (Theorem 6.22 in [25]) to conclude that this connection is projectively flat. Combining this with the isomorphism (1.2), we get the same conclusion for $\mathcal{V}^{(k)}_{\overline{\lambda}}$. We will further argue in Chapter 11, that such a Hitchin connection given by second order differential operators is unique.

We thus have the main result of this thesis.

Theorem 1.12. *The bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$ supports a projectively flat Hitchin connection which is mapping class group invariant and unique up to projective equivalence.*

This allows us to now give the gauge theory definition of the quantum representations of the mapping class group at least projectively, simply as the action of the mapping class group on the space of projectively covariant constant sections. The next step would then be to understand that the Pauly isomorphism is a projectively flat isomorphism (in analogy with Laszlo's result in the case of no marked points [33]) between $\mathcal{V}^{(k)}_{\overline{\lambda}}$ with the projectively flat connection constructed in this thesis and then the bundle of sheaf of vacua for the weights $\overline{\lambda}$ together with the TUY-connection in this bundle constructed in [42]. Once this has been done one would by combining with the work of Andersen and Ueno [10, 11, 12, 13] have the gauge theory construction of the WRT-modular function discussed in the beginning of this introduction. The flatness of Pauly's isomorphism however goes beyond the scope of this work.

Chapter 2

Complex geometry

Well known results about complex geometry are recalled in this chapter.

2.1 Almost complex structure

Let M be a smooth manifold of dimension $2m$. An almost complex structure is a smooth section J of the endomorphism bundle $\mathrm{End}(TM) \to TM$, that satisfies $J^2 = -1$. When we have an almost complex structure, we get a splitting of the complexified tangent bundle into eigen-spaces of J corresponding to the eigenvalues $\pm i$.

$$TM_{\mathbb{C}} = T'M_J \oplus T''M_J,$$

where $T'M_J := \mathrm{Im}(\mathrm{Id} - iJ)$ and $T''M_J := \mathrm{Im}(\mathrm{Id} + iJ)$. The almost complex structure J acts on the cotangent bundle by $(J\alpha)X = \alpha(JX)$ for $\alpha \in TM^*_{\mathbb{C}}$ and $X \in TM_{\mathbb{C}}$. We then get a splitting of the cotangent bundle $TM^*_{\mathbb{C}} = T'M^*_J \oplus T''M^*_J$ into eigen-spaces of J. Note that $T'M^*_J$ consists exactly of the forms that vanish on $T''M_J$ and $T''M^*_J$ is the forms that vanish on $T'M_J$. The splitting of $TM_{\mathbb{C}}$ and $TM^*_{\mathbb{C}}$ induces splittings of tensor bundles of $TM_{\mathbb{C}}$ into direct sums of eigen-sub-bundles of $TM_{\mathbb{C}}$ and $TM^*_{\mathbb{C}}$. For example, if we look at $\bigwedge^k TM^*_{\mathbb{C}}$, then we get a decomposition

$$\bigwedge^k TM^*_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} TM^*_J,$$

where $\oplus_{p+q=k} \bigwedge^{p,q} TM^*_J := \bigwedge^p T'M^*_J \otimes \bigwedge^q T''M^*_J$. The splitting of $\bigwedge^k TM^*_{\mathbb{C}}$ induces a splitting of the complex valued differential forms. Let $\Omega^{p,q}_J(M) := C^\infty(M, \bigwedge^{p,q} TM^*_J)$, then

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}_J(M).$$

We have projections $\pi^{p,q}_J \colon \Omega^{p+q}(M) \to \Omega^{p,q}_J(M)$. From the exterior derivative d we get operators $\partial_J = \pi^{p+1,q}_J \circ d$, $\partial_J \colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial}^{p,q+1}_J \circ d$, $\overline{\partial}_J \colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$.

2.2 Complex structure

Any complex manifold has a naturally induced almost complex structure on its tangent bundle. For local holomorphic coordinates $z^k = x^k + iy^k$ with coordinate vector fields X^k and Y^k, the almost complex structure J is defined by

$$JX^k := Y^K \quad \text{and} \quad JY^k := -X^k.$$

Since the transition functions are holomorphic, it is proven that the definition is independent of the chosen coordinates. Therefore the tangent bundle becomes a complex vector bundle.

An almost complex structure that is induced by a complex structure, is called integrable. It was proven by Newlander and Nienberg that an almost complex structure is integrable if and only if the Nijenhuis tensor N_J vanishes, where

$$N_J := [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY].$$

An almost complex structure J is also integrable if $T'M_J$ is preserved by the Lie bracket, or if the exterior differential decomposes as $d = \partial_J + \overline{\partial}_J$. These three are equivalent.

Dolbeault Cohomology

Assume we have an integrable almost complex structure on M. Then $d = \partial_J + \overline{\partial}_J$ which implies $\partial_J^2 = 0$, $\overline{\partial}_J^2 = 0$ and $\partial_J \overline{\partial}_J = -\overline{\partial}_J \partial_J$. So we get a co-chain complex

$$\Omega^{0,p} \to^{\overline{\partial}} \Omega^{1,p} \to^{\overline{\partial}} \Omega^{2,p} \to^{\overline{\partial}} \dots$$

for each p. The cohomology of this complex is denoted by $H_J^{p,q}(M, \mathbb{C})$ and called the Dolbeault Cohomology.

2.3 Symplectic and Poisson structures

A symplectic structure on a smooth even dimensional manifold M is a closed non-degenerate differential 2-form ω. By closed we mean $d\omega = 0$ and by non-degenerate we mean, that if there exists $X \in T_pM$ such that $\omega(X, Y) = 0$ for all $Y \in T_pM$, then $X = 0$. A smooth manifold equipped with a symplectic form is called a symplectic manifold. It comes naturally with a volume form $\frac{\omega^m}{m!}$, when the manifold has dimension $2m$.

A Poisson structure on a smooth manifold M is a Lie bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ satisfying the Leibniz rule. So a Poisson bracket satisfies

$$\begin{aligned} \{fg, h\} &= f\{g, h\} + g\{f, h\}, \\ \{f, g\} &= -\{g, f\}, \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0. \end{aligned}$$

A symplectic structure gives rise to a Poisson structure by

$$\{f, g\} = -\omega(X_f, X_g).$$

Compatible almost complex structure

Assume a smooth manifold M is equipped with both an almost complex structure J and a symplectic structure ω. Then J and ω are said to be compatible if

$$g(X, Y) := \omega(X, JY) \tag{2.1}$$

defines a Riemannian metric on M, that is g has to be a symmetric and positive definite bilinear form. For J an almost complex structure, g a Riemannian metric and ω a symplectic form, the three are said to be compatible if Equation (2.1) is satisfied. Clearly any two of the three define the last. One can calculate that symmetry of g is equivalent to that ω is J-invariant, and hence g is J-invariant. So both g and ω have type $(1, 1)$.

First Chern class

Chern classes are characteristic classes. If the Chern classes of two vector bundles are not the same, then the vector bundles are different. For the integer Chern class, the first Chern class is a complete invariant for line bundles, so line bundles are defined by their first Chern class. There are several different ways to define Chern classes. In [21] one can see many of these definitions.

When we talk about the first Chern class of a manifold M, we mean the first Chern class of the tangent bundle TM. There is a close connection between the first Chern class of TM and the first Chern class of the canonical line bundle $K = \bigwedge^n T^*M$.

Lemma 2.1. *Let $K_M = \bigwedge^n T^*M$ be the canonical bundle of M. Then*

$$c_1(K_M) = -c_1(TM)$$

Remark 2.2. We will give a sketch of the proof. Let $K_M = \bigwedge^n T^*M$ be the canonical bundle. Let $E = L_1 \oplus \cdots \oplus L_n$ be a bundle. Then $\bigwedge^n E = L_1 \otimes \cdots \otimes L_n$.

$$c_1(E) = c_1(L_1 \oplus \cdots \oplus L_n) = \sum_i c_1(L_i).$$

Likewise, we can calculate

$$\mathrm{ch}(\wedge^n E) = \mathrm{ch}(L_1 \otimes \cdots \otimes L_n) = \prod_{i=1}^{n} \mathrm{ch}(L_i) = \prod_{i=1}^{n}(1 + c_1(L_i)) = 1 + \sum_{i=1}^{n} c_1(L_i) + \ldots$$

Which means

$$c_1(\wedge^n E) = \sum_{i=1}^{n} c_1(L_i) = c_1(E).$$

Therefore

$$c_1(K_M) = -c_1(K_M^*) = -c_1(\bigwedge^n TM) = -c_1(TM)$$

2.4 Metaplectic structure

Let (M, ω) be a symplectic manifold. For each point p in M there exists a compatible almost complex structure J_p on T_pM. We define L^+M as

$$L^+M := \{(p, J_p) \mid p \in M,\ J_P \text{ an almost complex structure on } T_pM \text{ compatible with } \omega\}.$$

Then $L^+M \to M$ is a smooth bundle. A section $J\colon M \to L^+M$ correspond to a compatible almost complex structure J on M. Which means the space of sections is contractible and the projection $L^+M \to M$ is a homotopy equivalence with any section a homotopy inverse.

At each point $(p, J_p) \in L^+M$ we can consider $K_{J_p} := \bigwedge^m T'M^*_{J_p}$. These form a smooth bundle K over L^+M. A pullback by an almost complex structure J yields the canonical line bundle $K_J \to M$ associated to the almost complex structure.

Definition 2.3 (Metaplectic structure)**.** A *metaplectic structure* on a symplectic manifold (M, ω) is a line bundle $\delta \to L^+M$ and a map $\psi^\delta \colon \delta^2 \to K$ which is an isomorphism of line bundles over L^+M.

Clearly a metaplectic structure exists if and only if $c_1(K) \in H^2(L^+M, \mathbb{Z})$ is even. Since $L^+M \to M$ is a homotopy equivalence $H^2(L^+M, \mathbb{Z})$ is canonically isomorphic to $H^2(M, \mathbb{Z})$. Any compatible almost complex structure $J\colon M \to L^+M$ will give a homotopy inverse, hence induces an isomorphism $J^*\colon H^2(L^+M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ which is independent of J. By naturality of the first Chern class we see $J^* c_1(K) = c_1(K_J)$. So the first Chern class $c_1(K)$ is even if and only if the first Chern class of the canonical bundle of M_J is even.

In conclusion: When the first Chern class of the canonical bundle of (M, ω) with a compatible almost complex structure is even, a metaplectic structure provides a canonical choice of square root of the canonical bundle.

2.5 Kähler manifolds

A Kähler manifold is a smooth manifold M equipped with a compatible triple (J, ω, g), where J is integrable. The Hermitian metric g on a Kähler manifold M is called a Kähler metric and the symplectic form ω is called a Kähler form.

On a symplectic manifold (M, ω), choosing a Kähler structure is the same as choosing a compatible integrable almost complex structure.

The Levi Civita connection is a unique connection on a Kähler manifold defined by the following.

$$\nabla g = 0 \quad \text{or} \quad X[g(Y,Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any vector fields X, Y and Z on M, and ∇ should be torsion free, i.e. for any vector fields X and Y

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

For a Kähler manifold, the almost complex structure J is parallel with respect to the Levi-Civita connection, that is

$$\nabla J = 0 \quad \text{or} \quad \nabla_X (JY) = J \nabla_X Y$$

for any vector fields X and Y on M.

2.6 Curvature

For a Riemannian manifold, we define the curvature as a 2-form with values in the endomorphism bundle $\mathrm{End}(TM)$. Let ∇ denote the Levi-Civita connection, then we define

$$R^\nabla(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Note that R^∇ is symmetric in X and Y. The curvature tensor satisfies several identities, among others the algebraic Bianchi identity, which we will often use

$$R^\nabla(X,Y)Z + R^\nabla(Y,Z)X + R^\nabla(Z,X)Y = 0.$$

On a Kähler manifold we call the curvature stated above the Kähler curvature and denote it by $R(X,Y)Z$. Let us now assume that we have a Kähler manifold (M, J, g, ω). We denote the inverse of g and ω respectively by $\tilde{g}$ and $\tilde{\omega}$. They are unique, symmetric respectively antisymmetric bi-vector fields satisfying

$$g \cdot \tilde{g} = \mathrm{Id} = \tilde{g} \cdot g \quad \text{and} \quad \omega \cdot \tilde{\omega} = \mathrm{Id} = \tilde{\omega} \cdot \omega$$

From the curvature tensor we can define the Ricci curvature as

$$r(X,Y) = \mathrm{tr}[Z \mapsto R(Z,X)Y].$$

The associated skew-symmetric $(1,1)$-form ρ is given by

$$\rho(X,Y) = r(JX,Y).$$

Abstract index notation

To make calculations easier, we introduce an abstract index notation as follows. When we write $\omega \cdot J$, it means that we contract the tensors. Sometimes we will need expressions where the entries to be contracted cannot be indicated simply by placing the tensors next to each other. Because of this, we introduce the following abstract index notation. A subscript means a covariant entry, a superscript a contravariant entry, repeated indices indicate contraction. With these conventions we get

$$g_{ab} = \omega_{au} J_b^u.$$

If the two contracted indices are both either sub- or superscript the Kähler metric is used for contraction. As an example of this we have the scalar curvature

$$s = r_{uu} = r_{uv}\tilde{g}^{uv}.$$

In abstract index notation we write the curvature tensor as R_{abc}^d. We can use the metric to lower the upper index, so we write $R_{abcd} = R_{abc}^u g_{ud}$. In this notation it is easy to write the Ricci curvature, since it becomes $r_{ab} = R_{uab}^u$. R_{abcd} is symmetric in the two first entries and in the two last, and antisymmetric when we switch the two first with the two last entries. We see that

$$r_{ab} = R_{uab}^u = R_{uabu} = R_{auub}.$$

We can write ρ as

$$\rho_{ab} = J_a^u r_{ub} = \frac{1}{2} R_{abuv}\tilde{\omega}^{uv},$$

where the last equality comes from using the Bianchi identity and the symmetries of R.

We should also note something about the curvature of the canonical line bundle. Since the canonical bundle is exactly the top exterior power, we get

$$R_{\nabla^K} = \mathrm{tr} R_{\nabla^{T^*}} = i\rho,$$

where the last equality comes from the calculation $\rho_{ab} = \frac{1}{2} R_{abuv}\tilde{\omega}^{uv}$.

The Ricci potential

On a complex manifold M, any closed form is locally exact with respect to the $\partial\overline{\partial}$-operator. That is for a closed form $\alpha \in \Omega^{p,q}(M)$, $U \subset M$ a contractible open subset, there exists a form $\beta \in \Omega^{p-1,q-1}(U)$ such that $\alpha|_U = \partial\overline{\partial}\beta$ provided $pq > 0$. On a compact Kähler manifold a similar, but global, version of this statement can be proven using Hodge Theory, see [16]. Let M denote a compact Kähler manifold.

Proposition 2.4 ([16])**.** *For any exact form $\alpha \in \Omega^{p,q}(M)$ there exists a $\beta \in \Omega^{p-1,q-1}(M)$ such that $\alpha = 2i\partial\overline{\partial}\beta$ provided $pq > 0$.*

We will apply this to the Ricci form ρ. It is a real, closed $(1,1)$ form on M, hence it differs from its harmonic part ρ^H by a real, exact $(1,1)$ form, which we can use the proposition on. Hence

$$\rho = \rho^H + 2i\partial\overline{\partial}F,$$

where $F \in C^\infty(M)$ is a real function, called the *Ricci potential*.

Divergence

Divergence of a vector field X on M is the function $\delta X \in C^\infty(M)$ defined in terms of the Lie derivative and volume form by the equation

$$\mathcal{L}_X \omega^m = (\delta X)\omega^m.$$

Note that δX only depends on the symplectic volume, not on the Kähler metric. By computation we can write δX in terms of the Levi-Civita connection, but then the independence of the Kähler structure is not as obvious

$$\delta X = \mathrm{tr}\nabla X = \nabla_a X^a.$$

Proof. This can be proven by simply writing out the two sides, and noticing that they are equal. We want to prove that $\delta X = \mathrm{tr}\nabla X$. To do this we calculate $(\mathcal{L}_X\omega^m)(v_1, \ldots, v_{2m})$

$$(\mathcal{L}_X\omega^m)(v_1, \ldots, v_{2m}) = \mathcal{L}_X(\omega^m(v_1, \ldots, v_{2m})) - \sum_i \omega^m(v_1, \ldots, \mathcal{L}_X v_i, \ldots, v_{2m}).$$

The first term $\mathcal{L}_X(\omega^m(v_1, \ldots, v_{2m}))$ will be a sum of X used on products of $\omega(v_i, v_j)$.

$$X(\omega(v_i, v_j)) = \omega(\nabla_X v_i, v_j) + \omega(v_i, \nabla_X v_j),$$

This means we get the following

$$\begin{aligned}\mathcal{L}_X(\omega^m(v_1, \ldots, v_{2m})) &= X(\omega^m(v_1, \ldots, v_{2m})) \\ &= \sum_i \omega^m(v_1, \ldots, \nabla_X v_i, \ldots, v_{2,}).\end{aligned}$$

For the last terms we have

$$\omega^m(v_1, \ldots, \mathcal{L}_X v_i, \ldots, v_{2m}) = \omega^m(v_1, \ldots, [X, v_i], \ldots, v_{2,}).$$

By combining Equation 2.6 and Equation 2.6 the result is

$$(\mathcal{L}_X\omega^m)(v_1, \ldots, v_{2m}) = \sum_i \omega^m(v_1, \ldots \nabla_{v_i} X, \ldots, v_{2m}),$$

for each i, all the components, except from the i'th, are zero, so we get trace as we wanted

$$= \mathrm{tr}\nabla X \omega^m$$

Hence we have proven $\delta X = \mathrm{tr}\nabla X$. ☐

The Laplace de Rahm operator on functions can be expressed in terms of the divergence by

$$\Delta f = -2i\delta X'_f$$

where $X'_f = \overline{\partial} f \cdot \tilde{\omega}$ is the $(1,0)$-part of the Hamiltonian vector field associated with $f \in C^\infty(M)$.

We can generalize the notion of divergence to tensors of higher degree. Let $X_1, \ldots, X_n$ be vector fields on M, then

$$\delta(X_1 \otimes \cdots \otimes X_n) = \delta(X_1)X_1 \otimes \cdots \otimes X_n + \sum_j X_2 \otimes \cdots \otimes \nabla_{X_1} X_j \otimes \cdots \otimes X_n$$

This defines a map $\delta\colon C^\infty(M, TM^{\otimes n}) \to C^\infty(M, TM^{\otimes(n-1)})$, which we also call the divergence. Note that this map is dependent on the Kähler structure.

We will also need a notion of divergence on sections of the endomorphism bundle of the tangent bundle. For $\alpha \in \Omega^1(M)$ a one form and X a vector field, we define

$$\delta(X \otimes \alpha) := \delta(X)\alpha + \nabla_X \alpha.$$

This gives us a map $\delta\colon C^\infty(M, \mathrm{End}(TM)) \to \Omega^1(M)$.

2.7 Families of Kähler structures

Assume that we have a smooth manifold $\mathcal{T}$ and a smooth map

$$I\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$$

such that (M, ω, I_σ) is a Kähler manifold for each σ in $\mathcal{T}$. We say $\mathcal{T}$ smoothly parametrizes Kähler structures on M, or that we have a family of Kähler structures on (M, ω), hence the following definition.

Definition 2.5 (Family of Kähler structures)**.** For (M, ω) a symplectic manifold, $\mathcal{T}$ a manifold and I a smooth map as above, we say I is a family of Kähler structures on (M, ω).

Note that $I\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ smooth means that it defines a smooth section of the pullback bundle $\pi_M^* \mathrm{End}(TM) \to \mathcal{T} \times M$ where $\pi_M\colon \mathcal{T} \times M \to M$ denotes the canonical projection.

We use the notation M_σ to denote the complex manifold (M, I_σ).

For each σ we get an almost complex structure I_σ, as we have seen earlier, we can use this to split the complexified tangent bundle $TM_{\mathbb{C}} = TM \otimes \mathbb{C}$ into the holomorphic and anti-holomorphic parts,

$$T' M_\sigma := \mathrm{Im}(\mathrm{id} - iI_\sigma) \quad \text{and} \quad T'' M_\sigma := \mathrm{Im}(\mathrm{id} + iI_\sigma).$$

Since ω is non-degenerate, the map

$$X \mapsto \omega(X, \cdot)$$

is injective and hence an isomorphism, since we map between vector bundles of the same rank. This map will be denoted i_ω. In exactly the same way we get a map we call i_{g_σ}.

$$i_\omega, i_{g_\sigma}\colon TM_{\mathbb{C}} \to T^* M_{\mathbb{C}}.$$

These two maps are related

$$\begin{aligned} i_{g_\sigma}(X) &= g_\sigma(X, \cdot) = \omega(X, I_\sigma \cdot) \\ &= \omega(I_\sigma X, I_\sigma^2 \cdot) = -\omega(I_\sigma X, \cdot) \\ &= -I_\sigma \cdot i_\omega(X) \end{aligned}$$

We know that $\Lambda^2(I_\sigma)\omega = \omega$ so we can calculate

$$(\Lambda^2 i_{g_\sigma})(\lambda^2 i_\omega)^{-1}\omega = (\Lambda^2 i_{g_\sigma})(\Lambda^2 i_\omega^{-1})\omega = \Lambda^2(i_{g_\sigma} \circ i_\omega^{-1})\omega = \Lambda^2(I_\sigma)\omega = \omega$$

which gives us

$$(\Lambda^2 i_{g_\sigma})^{-1}\omega = (\lambda^2 i_\omega)^{-1}\omega.$$

We see that this means the left side is independent of σ, even though it doesn't appear to be.

Infinitesimal calculations

Suppose V is a vector field in $\mathcal{T}$. We can differentiate I, the family of Kähler structures, along V. We denote this derivative $V[I]$

$$V[I] \colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM)).$$

I_σ defines an almost complex structure for any σ in $\mathcal{T}$, hence $I_\sigma^2 = -\mathrm{id}$. When we differentiate this we get

$$V[I]_\sigma I_\sigma + I_\sigma V[I]_\sigma = 0,$$

which means $V[I]$ anti commutes with I, hence it switches types of vectors on M_σ. So

$$V[I]_\sigma \in C^\infty(M, (T'M_\sigma^* \otimes T''M_\sigma) \oplus (T''M_\sigma^* \otimes T'M_\sigma)).$$

Because of this we can decompose the vector space as

$$V[I]_\sigma = V[I]'_\sigma + V[I]''_\sigma$$

where $V[I]'_\sigma \in C^\infty(M, T'M_\sigma^* \otimes T''M_\sigma)$ and $V[I]''_\sigma \in C^\infty(M, T''M_\sigma^* \otimes T'M_\sigma)$.

Like we differentiated I along V, we can also differentiate g along V, we will use the same notation and denote it $V[g]$. Since we know how g depends on ω we get

$$V[g](X, Y) = \omega(X, V[I]Y).$$

So ω is of type $(1, 1)$ and g is symmetric, hence

$$V[g] \in C^\infty(M, S^2(T'M^*) \oplus S^2(T''M^*)).$$

Now define $\widetilde{G}(V) \in C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$ by the contraction

$$V[I] = \widetilde{G}(V) \cdot \omega = \widetilde{G}(V)^{au} \omega_{ub}.$$

Construction of $G(V)$

Define $G(V) \in C^\infty(M, T'M_\sigma \otimes T'M_\sigma)$ such that

$$\widetilde{G}(V) = G(V) + \overline{G}(V)$$

where $\overline{G}(V) \in C^\infty(M, T''M_\sigma \otimes T''M_\sigma)$. This is possible since $\tilde{G}(V)$ has no $(1, 1)$-part.

We observe that when we have a holomorphic family of Kähler structures, which we will introduce in Section 2.7, we get

$$V'[I] = G(V) \cdot \omega \text{ and } G(V) = G(V').$$

Since $V[g] = \omega \cdot V[I]$ we have

$$V[g] = \omega \cdot V[I] = \omega \cdot \widetilde{G}(V) \cdot \omega = -(i_\omega \otimes i_\omega)\widetilde{G}(V)$$

so $V[g] = -(i_\omega \otimes i_\omega)\widetilde{G}(V)$. From this it is clear that $\widetilde{G}(V) \in C^\infty(M, S^2(TM_{\mathbb{C}}))$, and then $G(V)$ takes values in $C^\infty(M, S^2(T_\sigma))$.

We will also need the variation of the Levi-Civita connection $V[\nabla] \in C^\infty(M, S^2(TM^*) \otimes TM)$. In [16] Theorem 1.174 we have the following formula

$$2g(V[\nabla]_X Y, Z) = \nabla_X(V[g])(Y, Z) + \nabla_Y(V[g])(X, Z) - \nabla_Z(V[g])(X, Y). \tag{2.2}$$

By using $V[g] = g \cdot \tilde{G}(V) \cdot g$ and switching to abstract index notation we get

$$2V[\nabla]^c_{ab} = \nabla_a \tilde{G}(V)^{cu} g_{ub} + g_{au} \nabla_b \tilde{G}(V)^{uc} - g_{au} \tilde{g}^{cw} \nabla_w \tilde{G}(V)^{uv} g_{vb}.$$

By a calculation we see that the trace of this tensor is zero

$$\begin{aligned} 2V[\nabla]^x_{xb} &= \nabla_x \tilde{G}(V)^{xu} g_{ub} + g_{xu} \nabla_b \tilde{G}(V)^{ux} - g_{xu} \tilde{g}^{xw} \nabla_w \tilde{G}(V)^{uv} g_{vb} \\ &= \nabla_x \tilde{G}(V)^{xu} g_{ub} + g_{xu} \nabla_b \tilde{G}(V)^{ux} - \nabla_u \tilde{G}(V)^{uv} g_{vb} \\ &= g_{xu} \nabla_b \tilde{G}(V)^{ux} = 0, \end{aligned}$$

where the last equality follows since $\tilde{G}(V)$ has no $(1,1)$-part, which is the type of g.

The canonical line bundle of a family

As before let I be a family of Kähler structures. Now consider the vector bundle

$$\hat{T}'M \to \mathcal{T} \times M$$

with fibers $\hat{T}'M_{\sigma,p} = T'_p M_\sigma$ given by the holomorphic tangent spaces of M. We will use the hat whenever we are working over $\mathcal{T} \times M$ instead of M.

We know that the Kähler metric induces a Hermitian structure $\hat{h}^{T'M}$ on $\hat{T}'M$. The Levi-Civita connection gives us a partial connection along the directions of M. We can extend this to a full connection on $\hat{\nabla}^{T'M}$ on $\hat{T}'M$ in the following way: Let $Z \in C^\infty(\mathcal{T} \times M, \hat{T}'M)$ be a smooth family of sections of the holomorphic tangent bundle. Let V be a vector field on $\mathcal{T}$. We can regard Z as a smooth family of sections of the complexified tangent bundle $TM_{\mathbb{C}}$, then we differentiate Z along V in this bundle. We then project the result back onto the holomorphic tangent bundle:

$$\hat{\nabla}_V Z := \pi^{1,0} V[Z].$$

Since $\hat{\nabla}^{T'M}$ is induced by the Levi-Civita connection it preserves the Hermitian structure in the directions of M. By a simple calculation we see that it preserves the Hermitian structure on $\hat{T}'M$. Let V be a vector field on $\mathcal{T}$, X, Y sections of $\hat{T}'M$. We get the following

$$\begin{aligned} V[\hat{h}^{T'M}(X,Y)] &= V[g(X,\overline{Y})] \\ &= V[g](X,\overline{Y}) + g(V[X],\overline{Y}) + g(X, V[\overline{Y}]) \\ &= h(\hat{\nabla}_V X, Y) + h(X, \hat{\nabla}_V Y), \end{aligned}$$

where the last equality is because the $(1,1)$-part of $V[g]$ vanishes. We see that the Hermitian structure is preserved.

Let us now define the canonical line bundle of a family of Kähler structures as the top exterior power of $\hat{T}'M^*$

Definition 2.6 (The canonical bundle of a family)**.** We define *the canonical bundle* of a family of Kähler structures to be $\hat{K} = \bigwedge^m \hat{T}'M^* \to \mathcal{T} \times M$.

The Hermitian structure and connections we just defined on $\hat{T}'M$ induce a Hermitian structure $\hat{h}^K$ and a compatible connection $\hat{\nabla}^K$ on $\hat{K}$.

Definition of Θ and θ

For any vector fields V, W on $\mathcal{T}$ we define $\Theta \in \Omega^2(\mathcal{T}, S^2(TM))$

$$\Theta(V,W) = S(\tilde{G}(V) \cdot \omega \cdot \tilde{G}(W)),$$

where S denotes the symmetrization. From this we can define a real two-form $\theta \in \Omega^2(\mathcal{T}, C^\infty(M))$ as

$$\theta(V,W) := -\frac{1}{4} g(\Theta(V,W)) = -\frac{1}{4} g_{uv} \Theta(V,W)^{uv}.$$

We can prove that Θ is exact. To do this observe that $G(V) = \pi^{2,0}\tilde{G}(V) = (\pi^{1,0} \otimes \pi^{1,0})\tilde{G}(V)$. Let V and W be commuting vector fields on $\mathcal{T}$. Now calculate the variation of $G(V)$ along W using a Leibniz rule

$$\begin{aligned} W[G(V)] &= W[(\pi^{1,0} \otimes \pi^{1,0})\tilde{G}(V)] \\ &= (W[\pi^{1,0}] \otimes \pi^{1,0})(\tilde{G}(V)) + (\pi^{1,0} \otimes W[\pi^{1,0}])(\tilde{G}(V)) + (\pi^{1,0} \otimes \pi^{1,0}) W[\tilde{G}(V)] \\ &= -\frac{i}{2}\tilde{G}(W) \cdot \omega \cdot G(V) + \frac{i}{2} G(V) \cdot \omega \cdot \tilde{G}(W) - \pi^{2,0}(WV[\tilde{g}]) \\ &= -\frac{i}{2}\overline{G}(W) \cdot \omega \cdot G(V) + \frac{i}{2} G(V) \cdot \omega \cdot \overline{G}(W) - \pi^{2,0}(WV[\tilde{g}]) \\ &= iS(G(V) \cdot \omega \cdot \overline{G}(W)) - \pi^{2,0}(WV[\tilde{g}]), \end{aligned}$$

we have used the fact that $G(V) \cdot \omega \cdot G(W) = 0$.

This shows that

$$\begin{aligned} V[G(W)] - W[G(V)] &= iS(G(W) \cdot \omega \cdot \overline{G}(V)) - iS(G(V) \cdot \omega \cdot \overline{G}(W)) \\ &= -i \left(S(\overline{G}(V) \cdot \omega \cdot G(W)) + S(G(V) \cdot \omega \cdot \overline{G}(W)) \right) \\ &= -i\Theta(V,W). \end{aligned}$$

If we view G as an element in $\Omega^1(\mathcal{T}, S^2(T'M))$, we can rephrase it as

$$d_{\mathcal{T}} G = -i\Theta,$$

which is what we wanted.

The curvature of the canonical line bundle

The curvature of the canonical line bundle of a family of Kähler structures is given by the following proposition, see [5].

Proposition 2.7 ([5]). *The curvature of $\hat{\nabla}^K$ is given by*

$$F_{\hat{\nabla}^K}(X,Y) = i\rho(X,Y), \;\; F_{\hat{\nabla}^K}(V,X) = \frac{i}{2}\delta\tilde{G}(V) \cdot \omega \cdot X, \;\; F_{\hat{\nabla}^K}(V,W) = i\theta(V,W),$$

for any vector fields X, Y on M and V, W on $\mathcal{T}$

Proof. The curvature is a tensor, so we can assume all the vector fields to be commuting.

The first curvature comes from the fact that $\hat{\nabla}$ is an extension of the Levi-Civita connection in the direction of M.

First we calculate the curvature in mixed directions. Let $Z \in C^\infty(M, \hat{T}'M)$. We use the definition $\hat{\nabla}_V Z = \pi^{1,0}V[Z]$

$$\begin{aligned}
F_{\hat{\nabla}^K}(V,X)Z &= \hat{\nabla}_V\hat{\nabla}_X Z - \hat{\nabla}_X\hat{\nabla}_V Z \\
&= \pi^{1,0}V[\nabla_X Z] - \nabla_X \pi^{1,0}V[Z] \\
&= \pi^{1,0}V[\nabla_X Z] - \pi^{1,0}\nabla_X V[Z] \\
&= \pi^{1,0}V[\nabla]_X Z.
\end{aligned}$$

In the second equality, we use that $\hat{\nabla}$ is an extension of the Levi-Civita connection ∇. In the third equality we use that ∇_X preserves types. The last equality is the Leibniz rule.

Now fix $\sigma \in \mathcal{T}$, $p \in M$ and let $e_1, \dots, e_m$ be a basis for $\hat{T}'M_{\sigma,p} = T'_pM_\sigma$ which satisfies $g_\sigma(e_i, \overline{e}_j) = \delta_{ij}$. Then

$$\begin{aligned}
F_{\hat{\nabla}^K}(V,X) &= -\mathrm{tr}F_{\hat{\nabla}^{T'M}}(V,X) \\
&= -\mathrm{tr}(\pi^{1,0}V[\nabla]_X) = -\sum_j g(V[\nabla]_X e_j, \overline{e}_j).
\end{aligned}$$

We now use Equation (2.2) to get

$$\begin{aligned}
g(V[\nabla]_X e_j, \overline{e}_j) &= \frac{1}{2}(V[g])(e_j, \overline{e}_j) + \frac{1}{2}\nabla_{e_j}(V[g])(X, \overline{e}_j) - \frac{1}{2}\nabla_{\overline{e}_j}(V[g])(X, e_j) \\
&= 0 + \frac{1}{2}\nabla_{e_j}(\omega \cdot V[I])(X, \overline{e}_j) - \frac{1}{2}\nabla_{\overline{e}_j}(\omega \cdot V[J])(X, e_j) \\
&= \frac{1}{2}\omega(\nabla_{e_j}(V[I])X, \overline{e}_j) - \frac{1}{2}\omega(\nabla_{\overline{e}_j}(V[I])X, e_j) \\
&= -\frac{i}{2}g(\nabla_{e_j}(V[I])X, \overline{e}_j) - \frac{i}{2}g(\nabla_{\overline{e}_j}(V[I])X, e_j),
\end{aligned}$$

where we have used the $(1,1)$-part of $V[g]$ vanishes. Now we can finish the calculation

$$\begin{aligned}
F_{\hat{\nabla}^K}(V,X) &= -\sum_j \left(-\frac{i}{2}g(\nabla_{e_j}(V[I])X, \overline{e}_j) - \frac{i}{2}g(\nabla_{\overline{e}_j}(V[I])X, e_j) \right) \\
&= \frac{i}{2}\mathrm{tr}\nabla(V[I])X = \frac{i}{2}\delta(V[I])X = \frac{i}{2}\delta\tilde{G}(V) \cdot \omega \cdot X,
\end{aligned}$$

which is what we wanted.

Let us now take two vector fields on $\mathcal{T}$. Take an arbitrary $Z \in C^\infty(\mathcal{T} \times M, \hat{T}'M)$.

$$\begin{aligned}
\hat{\nabla_V} Z = \pi^{1,0}V[Z] &= V[\pi^{1,0}Z] - V[\pi^{1,0}]Z \\
&= V[Z] + \frac{i}{2}V[I]Z, \\
\hat{\nabla}_V\hat{\nabla}_W Z &= \hat{\nabla}_V\left(W[Z] + \frac{i}{2}W[I]Z\right) \\
&= VW[Z] + \frac{i}{2}VW[I]Z + \frac{i}{2}W[I]V[Z] + \frac{i}{2}V[I]W[Z] + \frac{i}{2}\frac{i}{2}V[I]W[I]Z.
\end{aligned}$$

We have assumed V and W to commute, so we know the curvature $F_{\hat{\nabla}^{T'M}}(V,W)Z$ is given by $\hat{\nabla}_V\hat{\nabla}_W Z - \hat{\nabla}_W\hat{\nabla}_V Z$. By using what we just calculated twice, and using V and W commute we get

$$\begin{aligned}F_{\hat{\nabla}^{T'M}}(V,W)Z &= \hat{\nabla}_V\hat{\nabla}_W Z - \hat{\nabla}_W\hat{\nabla}_V Z\\ &= \frac{1}{4}(-V[I]W[I]Z + W[I]V[I]Z)\\ &= -\frac{1}{4}[V[I],W[I]]Z.\end{aligned}$$

We have that $F_{\hat{\nabla}^K}$ is the trace of $F_{\hat{\nabla}^{T'M}}$ so we get

$$\begin{aligned}F_{\hat{\nabla}^K}(V,W) &= -\mathrm{tr} F_{\hat{\nabla}^{T'M}}(V,W)\\ &= \frac{1}{4}\mathrm{tr}\pi^{1,0}[V[I],W[I]]\\ &= i\theta(V,W),\end{aligned}$$

which is what we wanted.

We have now proven the theorem. □

Use of the Bianchi identity

We can use the Bianchi identity on the connection $\hat{\nabla}^K$ on different vector fields to get three useful results

Proposition 2.8 ([5]). *The two-form $\theta \in \Omega^2(\mathcal{T}, C^\infty(M))$ is closed.*

Proof. This follows directly when using the Bianchi identity on three vector fields on $\mathcal{T}$ □

Proposition 2.9 ([5]).

$$d(\theta(V,W)) = \frac{1}{2}W[\delta\tilde{G}(V)]\cdot\omega - \frac{1}{2}V[\delta\tilde{G}(W)]\cdot\omega$$

Proof. To prove this we use the Bianchi identity on $\hat{\nabla}^K$ on the vector fields V, W on M and X on $\mathcal{T}$.

$$\begin{aligned}0 &= F_{\hat{\nabla}^K}(V,W)X + F_{\hat{\nabla}^K}(W,X)V + F_{\hat{\nabla}^K}(X,V)W\\ &= i\theta(V,W)X + \left(\frac{i}{2}\delta\tilde{G}(W)\cdot\omega\cdot X\right)V - \left(\frac{i}{2}\delta\tilde{G}(V)\cdot\omega\cdot X\right)W\\ &= X[i\theta(V,W)] + V\left[\frac{i}{2}\delta\tilde{G}(W)\cdot\omega\cdot X\right] - W\left[\frac{i}{2}\delta\tilde{G}(V)\cdot\omega\cdot X\right]\\ &= iX[\theta(V,W)] + \frac{i}{2}V[\delta\tilde{G}(W)]\cdot\omega\cdot X - \frac{i}{2}W[\delta\tilde{G}(V)]\cdot\omega\cdot X.\end{aligned}$$

By isolating $X[\theta(V,W)] = d(\theta(V,W))(X)$ we get what we want

$$d(\theta(V,W)) = -\frac{1}{2}V[\delta\tilde{G}(W)]\cdot\omega + \frac{1}{2}W[\delta\tilde{G}(V)]\cdot\omega.$$

□

By using the Bianchi identity on one vector field V on M and two vector fields X and Y on $\mathcal{T}$ we get the last equality

Proposition 2.10 ([5]).

$$V[\rho] = \frac{1}{2} d(\delta\tilde{G}(V) \cdot \omega).$$

Proof. We compute this

$$\begin{aligned} 0 &= F_{\hat{\nabla}^K}(X,Y)V + F_{\hat{\nabla}^K}(Y,V)X + F_{\hat{\nabla}^K}(V,X)Y \\ &= iV[\rho(X,Y)] - X\left[\frac{i}{2}\delta\tilde{G}(V)\cdot\omega\cdot Y\right] + Y\left[\frac{i}{2}\delta\tilde{G}(V)\cdot\omega\cdot X\right] \\ &= iV[\rho(X,Y)] - \frac{i}{2} d(\delta\tilde{G}(V)\cdot\omega)(X,Y) \end{aligned}$$

By isolating $V[\rho(X,Y)]$ we get the identity we wanted

$$V[\rho(X,Y)] = \frac{1}{2} d(\delta\tilde{G}(V)\cdot\omega)(X,Y).$$

□

Holomorphic families of Kähler structures

If we further assume $\mathcal{T}$ to be a complex manifold, we can require I to be a holomorphic map.

Definition 2.11. Suppose $\mathcal{T}$ is a complex manifold and that I is a family of complex structures on M, parametrized by $\mathcal{T}$. Then I is holomorphic if and only if

$$V'[I] = V[I]' \quad \text{and} \quad V''[I] = V[I]''$$

for any vector field V on $\mathcal{T}$.

If we let J denote the almost complex structure on $\mathcal{T}$ induced by its complex structure, we get an almost complex structure $\hat{I}$ on $\mathcal{T}\times M$ defined by

$$\hat{I}(V\oplus X) = JV \oplus I_\sigma X,\ V\oplus X \in T_{(\sigma,p)}(\mathcal{T}\times M)$$

We can use this to give an alternative characterization of holomorphic families, given by the following proposition from [5]

Proposition 2.12 ([5]). *The family I is holomorphic if and only if $\hat{I}$ is integrable*

Proof. We have to show that I is holomorphic if and only if the Nijenhuis tensor for $\hat{I}$ vanishes. Since we know J is integrable, the Nijenhuis tensor will vanish when used on vector fields tangent to $\mathcal{T}$. Similarly since I is a family of integrable almost complex structures, the Nijenhuis tensor will vanish on vectors tangent to M. So we need to show it also vanishes in mixed directions.

Let X be a vector field on M and V a vector field on $\mathcal{T}$. We know that $[V, IX] = V[I]X$. We calculate the Nijenhuis tensor

$$\begin{aligned} N_{\hat{I}}(V',X) &= [JV', IX] - \hat{I}[JV', X] - \hat{I}[V', IX] - [V', X] \\ &= i[V', IX] - 0 - \hat{I}[V', IX] - 0 \\ &= iV'[I]X - \hat{I}V'[I]X \\ &= 2i\pi^{0,1}V'[I]X. \end{aligned}$$

Similarly we can show $N_{\hat{I}}(V'',X) = -2i\pi^{1,0}V''[I]X$. This means $N_{\hat{I}}$ vanishes if and only if

$$\pi^{0,1}V'[I]X = 0 \quad \text{and} \quad \pi^{1,0}V''[I]X = 0$$

which proves the proposition. □

Rigid families of Kähler structures

To construct a Hitchin connection, which we want to do in the next chapter, we need the notion of rigidity as introduced in [3].

Definition 2.13 (Rigid)**.** A family of Kähler structures is called *rigid* if

$$\nabla_{X''} G(V) = 0$$

for all vector fields V on $\mathcal{T}$ and X on M

Note that if we have a rigid family of Kähler structures, and V is real, we also get that

$$\nabla_{X'} \overline{G}(V) = 0$$

for all vector fields V on $\mathcal{T}$ and X on M. We have that $\tilde{G}(V) = G(V) + \overline{G}(V)$ is real, that means $\tilde{G}(V) = \overline{\tilde{G}(V)} = \overline{G(V)} + \overline{\overline{G}(V)}$, so we get $\overline{\overline{G}(V)} = G(V)$.

2.8 Construction of $H^{(k)}$ and $\mathcal{H}^{(k)}$

Define a family of $\overline{\partial}$-operators on $\mathcal{L}^k$ at $\sigma \in \mathcal{T}$ by

$$\nabla^{0,1}_\sigma = \frac{1}{2}(1 + iI_\sigma)\nabla.$$

For every $\sigma \in \mathcal{T}$ consider the subspace of $C^\infty(M, \mathcal{L}^k)$ given by

$$H^{(k)}_\sigma = H^0(M_\sigma, \mathcal{L}^k) = \left\{ s \in C^\infty(M, \mathcal{L}^k) \;\middle|\; \nabla^{0,1}_\sigma s = 0 \right\}.$$

We will assume these subspaces of holomorphic sections form a smooth finite rank sub-bundle $H^{(k)}$ of the trivial bundle $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$.

Let $\hat{\nabla}^t$ denote the trivial connection in the trivial bundle $\mathcal{H}^{(k)}$. Let $\mathcal{D}(M, \mathcal{L}^k)$ denote the vector space of differential operators acting on $C^\infty(M, \mathcal{L}^k)$. For any smooth one form u in $\mathcal{T}$ with values in $\mathcal{D}(M, \mathcal{L}^k)$ we have a connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$ given by

$$\hat{\nabla}_V = \hat{\nabla}^t_V - u(V)$$

for any vector field V on $\mathcal{T}$.

Chapter 3

Geometric quantization

The possible states of a quantum system are vectors in a Hilbert space called the state-space. Each observable is represented by a self-adjoint linear operator acting on the state space. Most quantum systems have a classical limit, and a way of relating the observables of the quantum mechanical system and the classical system. However often the construction is the other way around, one starts with a classical system, and wants to obtain a corresponding quantum mechanical system, by some way of *quantization*.

From physics we know canonical quantization, which for examples as the hydrogen atom matches observations. In mathematics, the general quest is to make a well defined quantization scheme, that can be used on any phase space, and that reproduce canonical quantization on $(\mathbb{R}^{2n}, \omega)$.

Geometric quantization is an attempt at such a quantization scheme, which in its most complete form involves metaplectic quantization. This quantization schemes however depends on the choice of a so-called *polarization*, which in the case we will consider will simply be a complex structure compatible with the given symplectic form. This quantization scheme, in its current state of development, fails to establish the independence of the polarization in general.

3.1 Prequantization

Definition 3.1 (Prequantum line bundle)**.** A *prequantum line bundle* $(\mathcal{L}, h, \nabla)$ over a symplectic manifold (M, ω) is a complex line bundle $\mathcal{L}$ with a Hermitian structure h and a compatible connection ∇ whose curvature satisfies

$$F_\nabla(X, Y) = -i\omega(X, Y).$$

A Hermitian structure h is called compatible with ∇ if for any vector field X and any two sections s_1, s_2 of $\mathcal{L}$ we have

$$X(s_1, s_2) = h(\nabla_X(s_1), s_2) + h(s_1, \nabla_X(s_2))$$

Definition 3.2 (Prequantizable)**.** We call a symplectic manifold (M, ω) *prequantizable* if there exists a prequantum line bundle over it.

Note that the curvature of a connection ∇ on $\mathcal{L} \to M$ is a 2-form $F_\nabla \in \Omega^2(\mathrm{End}(\mathcal{L}))$, with values in the Endomorphism bundle $\mathrm{End}(\mathcal{L}) = \mathcal{L} \otimes \mathcal{L}^*$. Since $\mathcal{L} \to M$ is a line bundle, we can create a global section of $\mathrm{End}(\mathcal{L}) \to M$ by choosing the identity, hence $\mathrm{End}(\mathcal{L})$ is trivial.

$$\mathrm{End}(\mathcal{L}) \simeq M \times \mathbb{C}$$

which means the curvature can be seen as a 2-form on M with values in $\mathbb{C}$.

A symplectic manifold (M, ω) is prequantizable if and only if

$$\left[\frac{\omega}{2\pi}\right] \in \mathrm{Im}(H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})),$$

see [48].

Hence for a symplectic manifold (M, ω), prequantization assigns a line bundle $\mathcal{L}$ with curvature ω. There is a prequantum operator $f \mapsto \hat{f}$ given by

$$\hat{f}\psi = (-i\hbar\nabla_{X_f} + f)s$$

for all $s \in C^\infty(M, \mathcal{L})$.

Example 3.3. Let us take a look at the example $\mathbb{R}^{2n}$ with coordinates (p^i, q^i), $i = 1, \dots, n$. Then $\omega = dp^i \wedge dq^i$ is a symplectic form, and $\omega = d(\sum_i p_i dq_i)$. Let $\alpha = \sum_i p_i dq_i$. We can take the trivial line bundle $\mathcal{L} = \mathbb{R}^{2n} \times \mathbb{C}$ with connection $\nabla_v = v + \frac{i}{\hbar} v \cdot \alpha$, as the prequantum line bundle. Then

$$X_{q_i} = \frac{\partial}{\partial p_i} \quad \text{and} \quad X_{p_i} = -\frac{\partial}{\partial q_i}.$$

We calculate $\nabla_{X_{q_i}}$ and $\nabla_{X_{p_i}}$

$$\nabla_{X_{q_i}} = \frac{\partial}{\partial p_i} + \frac{i}{\hbar}\frac{\partial}{\partial p_i} \cdot \sum_j p_j dq_j = \frac{\partial}{\partial p_i}$$

and

$$\nabla_{X_{p_i}} = -\frac{\partial}{\partial q_i} - \frac{i}{\hbar}\frac{\partial}{\partial q_i} \cdot \sum_i p_i dq_i = -\frac{\partial}{\partial q_i} - \frac{i}{\hbar} p_i.$$

Hence

$$\begin{aligned} \hat{q}_i s &= (-i\hbar\frac{\partial}{\partial p_i} + q_i)s \\ \hat{p}_i s &= (-i\hbar(-\frac{\partial}{\partial q_i} - \frac{i}{\hbar}p_i) + p_i)\psi = i\hbar\frac{\partial}{\partial q_i} s \end{aligned}$$

for all $s \in C^\infty(M, \mathcal{L})$.

Recall that in canonical quantization we consider wave functions ψ depending only on say the q_i-variables and the quantization of the coordinate functions are then given by $\hat{p}_i\psi = i\hbar\frac{\partial}{\partial q_i}\psi$ and $\hat{q}_i\psi = q_i\psi$. But notice that if we restrict to the subspace of s's in prequantization setup which are covariant constant along $\partial/\partial p_i$'s, we do actually perfectly recreate canonical quantization, since such sections will be determined by their restriction to say the subspace where p_i vanish and thus the resulting functions only depend on the p_i's and the prequantum operator reproduce the operators from canonical quantization perfectly. Requiring that the sections are covariant constant along the $\partial/\partial p_i$'s can be generalized to the notion of a polarization, which makes sense on any symplectic manifold. We will now briefly recall the definition of a general complex polarization, referring the reader to [48] for further details.

Complex Polarizations and geometric quantization

Let (M, ω) be a symplectic manifold. Assume that we have a prequantum line bundle $(\mathcal{L}, h, \nabla)$ on (M, ω).

A complex polarization of a symplectic manifold (M,ω) is a complex distribution P on M such that for each $m \in M$, $P_m \subset (T_mM)_{\mathbb{C}}$ is Lagrangian, the dimension of $D = P \cap \overline{P} \cap TM$ is constant and P is integrable.

Given a polarization P we can considers the vector space of $\overline{P}$-polarized sections

$$H_P^{(k)} = \left\{s \in C^\infty(M, \mathcal{L}^k) \mid \nabla_{\overline{Z}} s = 0, \quad \forall Z \in C^\infty(M,P)\right\}.$$

This is the quantum vector space which geometric quantization associated to (M,ω) equipped with the polarization P at *level* $k \in \mathbb{Z}$.

Example 3.4. For $\mathbb{R}^{2n}$ with the standard symplectic structure we can in the coordinates introduced in the previous section let $P = \operatorname{span}\{\partial/\partial p_1, \dots \partial/\partial p_n\}$. Then

$$H_P = \{s \in C^\infty(M,\mathcal{L}) \mid \nabla_{\frac{\partial}{\partial p_i}} s = 0\}$$

is precisely the subspace we considered to get canonical quantization.

As another example, let us first consider almost complex structures J which are compatible with ω, that is $g_J(X,Y) := \omega(X, JY)$ defines a Riemannian metric on M.

The almost complex structure J induces a splitting of the complexified tangent space

$$TM_{\mathbb{C}} = T'M_J \oplus T''M_J$$

into eigenspaces of J corresponding to the eigenvalues i and $-i$. Let

$$\begin{aligned}\pi_J^{1,0} &= \frac{1}{2}(\mathrm{Id} - iJ),\\ \pi_J^{0,1} &= \frac{1}{2}(\mathrm{Id} + iJ),\end{aligned}$$

denote the projections. Then $T'M_J = \operatorname{Im}(\pi_J^{1,0})$ and $T''M_J = \operatorname{Im}(\pi_J^{0,1})$.

Hence we can define $P = T'M_J$ and then P will be integrable if an only if J is integrable. We observe that the condition on the dimension of D is trivially true since $D = 0$ in this case. In fact, when $D = 0$ the sub-bundle P is always the i-eigenspace of some uniquely determined J.

However, geometric quantization does actually not reproduce the correct quantization of the harmonic oscillator, since the spectrum of the quantization of the Hamiltonian differs from the correct one by a shift. Metaplectic quantization is modification of geometric quantization, which does reproduce the canonical quantization of the harmonic oscillator.

3.2 Metaplectic quantization

Assume (M,ω) is prequantizable and we have a parallel almost complex structure J, which means we have a Kähler manifold. Fix a prequantum line bundle $(\mathcal{L}, h^{\mathcal{L}}, \nabla^{\mathcal{L}})$. Then $h^{\mathcal{L}}$ and h_J^δ induces a Hermitian structure h_J in Then $\mathcal{L}^k \otimes \delta_J$. $\nabla^{\mathcal{L}}$ and ∇_J^δ induces a h_J-compatible connection $\nabla_J = \nabla^{\mathcal{L}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla_J^\delta$ and ∇_J has curvature $-ik\omega + \frac{1}{2}\rho_J$, which is of type $(1,1)$ so $\nabla_J^{0,1} := \pi_J^{0,1}\nabla_J$ defines a $\overline{\partial}$ operator in $\mathcal{L}^k \otimes \delta_J$ making this a holomorphic line bundle over M_J. Consider the space $\mathcal{H}_{J,\delta}^{(k)} = C^\infty(M, \mathcal{L}^k \otimes \delta_J)$ of smooth sections. Then $\nabla_J^{0,1}$ gives rise to a subspace $H_{\delta,J}^{(k)}$ of holomorphic sections

$$H_{\delta,J}^{(k)} := H^0(M_J, \mathcal{L}^k \otimes \delta_J) = \left\{ s \in C^\infty(M_J, \mathcal{L} \otimes \delta_J) \,\middle|\, \nabla_J^{0,1} s = 0 \right\}.$$

If one insists on wanting a Hilbert space, one can consider the Hermitian inner product

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M h_J(s_1, s_2) \omega^m$$

and then restrict to the subspace of $H^{(k)}_{\delta,J}$, which consist of sections whose norm associated to this inner product is finite and one then actually obtains a Hilbert space.

It is however not clear to us that this Hermitian inner product is the correct one to consider. We are interested in constructing Hitchin connections, which in the first instance only preserves the space of holomorphic sections, but which then provides for an identification of the quantum vector spaces for the corresponding different polarizations induced from the family of J we consider. It would then be natural to ask that this identification also preserved the Hermitian inner products, which however in certain cases implies that this Hermitian inner product is not the right one.

CHAPTER 4

Construction of the general Hitchin connection

The main purpose of this chapter is to recall the construction of the Hitchin connection in geometric quantization, as it is done in [3] and then to recall the construction of a Hitchin connection in the setting of Metaplectic Quantization as it is done in [5].

4.1 The Hitchin connection in geometric quantization

Recall that we have a symplectic manifold (M, ω), which we assume is prequantizable with prequantum line bundle $(\mathcal{L}, h, \nabla)$ and that we further assume that we are given a smooth family

$$I : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$$

of complex structures on M such that $M_\sigma = (M, I_\sigma, \omega)$ is Kähler for all $\sigma \in \mathcal{T}$. We then consider the trivial $C^\infty(M, \mathcal{L}^k)$-bundle over $\mathcal{T}$, $\mathcal{H}^{(k)}$ and assume that the subspaces

$$H^0(M_\sigma, \mathcal{L}^k) \subset C^\infty(M, \mathcal{L}^k)$$

form a smooth subbundle $H^{(k)} \subset \mathcal{H}^{(k)}$.

Recall that a Hitchin connection is defined as follows.

Definition 4.1 (Hitchin connection)**.** A connection in $\mathcal{H}^{(k)}$ over $\mathcal{T}$ of the form

$$\hat{\nabla} = \hat{\nabla}^t - u \tag{4.1}$$

where $\hat{\nabla}^t$ is the trivial connection in $\mathcal{H}^{(k)}$ and u is a one form on $\mathcal{T}$ with values in differential operators acting on $C^\infty(M, \mathcal{L}^k)$ is called a *Hitchin connection* if it preserves the sub-bundle $H^{(k)}$.

Lemma 4.2 ([3])**.** *A connection $\hat{\nabla}$ of the form* (4.1) *in $\mathcal{H}^{(k)}$ induces a connection in $H^{(k)}$, i.e. $\hat{\nabla}$ is a Hitchin connection, if and only if*

$$\frac{i}{2} V[I] \nabla^{1,0} s + \nabla^{0,1} u(V) s = 0 \tag{4.2}$$

for all vector fields V on $\mathcal{T}$ and all smooth sections s of $H^{(k)}$.

Proof. Let s be a smooth section of $H^{(k)}$ over $\mathcal{T}$, then we know $\nabla^{0,1}_\sigma s = 0$ for all $\sigma \in \mathcal{T}$. Let V be a vector field over $\mathcal{T}$. Then $\hat{\nabla}_V s$ is a section of $\mathcal{H}^{(k)}$. Now

$$0 = \nabla^{0,1}_\sigma s = \frac{1}{2}(1 + iI_\sigma)\nabla s.$$

By taking the derivative of this in the direction of V, remembering that ∇ is independent of σ and using a Leibniz rule we get

$$\begin{aligned} 0 &= \frac{i}{2}(V[I]\nabla s)_\sigma + \frac{1}{2}(1 + iI_\sigma)\nabla(V[s]_\sigma) \\ &= \frac{i}{2}(V[I]\nabla s)_\sigma + \nabla^{0,1}_\sigma(V[s]_\sigma) \end{aligned}$$

and since $\nabla^{0,1}_\sigma s = 0$ we can write

$$\frac{i}{2}(V[I]\nabla^{1,0} s)_\sigma + \nabla^{0,1}_\sigma(V[s]_\sigma) = 0.$$

We can then compute

$$\begin{aligned} \nabla^{0,1}_\sigma((\hat{\nabla}_V(s))_\sigma) &= \nabla^{0,1}_\sigma((\hat{\nabla}^t_V(s) - u(V)s)_\sigma) \\ &= \nabla^{0,1}_\sigma(V[s]_\sigma) - \nabla^{0,1}_\sigma((u(V)s)_\sigma) \\ &= -\frac{i}{2}(V[I]\nabla^{1,0} s)_\sigma - \nabla^{0,1}_\sigma((u(V)s)_\sigma) \end{aligned}$$

hence $\hat{\nabla}$ preserves $H^{(k)}$ if and only if Equation (4.2) holds. □

By observing that $V''[I]\nabla^{1,0}s = 0$, we can write the equation as

$$\begin{aligned} 0 &= \frac{i}{2}V'[I]\nabla^{1,0}s + \nabla^{0,1}u(V)s \\ &= \frac{i}{2}\omega \cdot G(V) \cdot \nabla s + \nabla^{0,1}u(V)s \end{aligned}$$

instead.

Let us now recall the following Theorem from [3] together with its proof.

Theorem 4.3 ([3]). *Suppose there exists $l \in \mathbb{Q}$ such that the first Chern class of (M, ω) is $l[\omega] \in H^2(M, \mathbb{Z})$, $H^1(M, \mathbb{R}) = 0$ and M is compact. There exists a Hitchin connection $\hat{\nabla}$ in $\mathcal{H}^{(k)}$, which preserves the sub-bundle $H^{(k)}$. It is for all V smooth vector field on $\mathcal{T}$ given by*

$$\hat{\nabla}_V = \hat{\nabla}^t_V - u(V),$$

where $\hat{\nabla}^t_V$ is the trivial connection in $\mathcal{H}^{(k)}$, $u(V)$ is the second order differential operator given by

$$u(V) = \frac{1}{2k + l}\left(\frac{1}{2}\Delta_{G(V)}(s) + \nabla_{G(V)dF}(s) + 2kV'[F]s\right)$$

where $\Delta_{G(V)}$ is a certain second order operator depending linearly and smoothly on V defined by Equation (4.3). *Further V' denotes the $(1,0)$-part of V on $\mathcal{T}$ and $F\colon \mathcal{T} \to C^\infty_0(M)$ is determined by $F_\sigma \in C^\infty(M)$ being the Ricci potential for (M, I_σ) for all $\sigma \in \mathcal{T}$ with zero average.*

We remark that in the proof the compactness assumption on M is only used to obtained a smooth family of Ricci potentials F (using Hodge theory) and that this assumption therefore can be dropped if we are given a smooth family of Ricci potentials F by some other means.

Construction of Δ_G

We will now construct a u that solves equation (4.2) under certain conditions.

The manifold (M,ω) is a Kähler manifold, so we have the Kähler metric and the Levi-Civita connection ∇. Let $\rho_\sigma \in \Omega^{1,1}(M_\sigma)$ be the Ricci-form. By Hodge theory we have $\rho_\sigma = \rho_\sigma^H + d\alpha$, where ρ_σ^H is the harmonic part of the Ricci-form. Let the Ricci-potential be

$$F_\sigma \in C_0^\infty(M,\mathbb{R}) := \Big\{ f \in C^\infty(M,\mathbb{R}) \mid \int_M f\omega^m = 0 \Big\},$$

that satisfies $\rho_\sigma = \rho_\sigma^H + 2i\bar{\partial}_\sigma \partial_\sigma F_\sigma$. In this way we get a function $\sigma \mapsto F_\sigma$

$$F\colon \mathcal{T} \to C_0^\infty(M,\mathbb{R}).$$

By Hodge-theory F is smooth.

Let $G \in C^\infty(M, S^2(T'M_\sigma))$. We get a linear bundle map $G\colon T'M_\sigma^* \to T'M_\sigma$. We can construct an operator $\Delta_G\colon C^\infty(M,\mathcal{L}^k) \to C^\infty(M,\mathcal{L}^k)$ in the following way

$$\begin{aligned}
\Delta_G\colon C^\infty(M,\mathcal{L}^k) &\xrightarrow{\nabla_\sigma^{1,0}} C^\infty(M, T'M_\sigma^* \otimes \mathcal{L}^k) \\
&\xrightarrow{G\otimes \mathrm{id}} C^\infty(M, T'M_\sigma \otimes \mathcal{L}^k) \\
&\xrightarrow{\nabla_\sigma^{1,0}\otimes \mathrm{id} + \mathrm{id}\otimes \nabla_\sigma^{1,0}} C^\infty(M, T'M_\sigma^* \otimes T'M_\sigma \otimes \mathcal{L}^k) \\
&\xrightarrow{\mathrm{Tr}} C^\infty(M,\mathcal{L}^k).
\end{aligned} \tag{4.3}$$

First we take the derivative in the $(1,0)$ direction, resulting in an element in $T'M_\sigma^* \otimes \mathcal{L}^k$. Then we contract with G on the $T'M_\sigma^*$ part and the identity on the other part, and end in $T'M_\sigma \otimes \mathcal{L}^k$. On this we use the tensor product connection (in the $(1,0)$ direction), and since $\nabla_\sigma^{1,0}$ preserves $T'M_\sigma$ we end in $T'M_\sigma^* \otimes T'M_\sigma \otimes \mathcal{L}^k$. Now we use trace and end back in $\mathcal{L}^k$.

Let f be a smooth function on M. We have the projection from $TM \simeq T'M_\sigma \oplus T''M_\sigma$ to $T'M_\sigma$, which takes df to $\partial_\sigma f$, so we can get a vector field $Gdf \in C^\infty(M, T_\sigma)$.

The existence of the Hitchin connection

We would now like to construct $u(V)$ that satisfies equation (4.2) using $\Delta_{G(V)}$. Assume the family of Kähler structures is rigid. We will do the calculations in the following steps. First we calculate $\nabla^{0,1}\Delta_{G(V)}$ and find

$$\nabla^{0,1}\Delta_{G(V)}s = -2ik\omega \cdot G(V)\cdot \nabla s - i\rho \cdot G(V) \cdot \nabla s - ik\omega \cdot \delta(G(V))s \tag{4.4}$$

for any (local) holomorphic section s of $\mathcal{L}^k$. So we see that $\Delta_{G(V)}$ almost satisfies equation (4.2), except for the last two terms.

Inspired by this, we let

$$u(V) = \frac{1}{4k+2n}\left(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]\right),$$

then we can show that $u(V)$ exactly satisfies equation (4.2). This is the last thing we need, because when this is done we know from Lemma 4.2 that $\hat{\nabla}$ preserves the sub-bundle $H^{(k)}$ under the stated conditions, and we have obtained Theorem 4.3.

We are doing the calculations step by step. First take the derivative of $\Delta_{G(V)}$

$$\begin{aligned}
\nabla^{0,1}\Delta_{G(V)}s &= \nabla_{a''}\nabla_{u'}G^{u'v'}\nabla_{v'}s \\
&= \nabla_{u'}\nabla_{a''}G^{u'v'}\nabla_{v'}s + [\nabla,\nabla]_{a''u'}G^{u'v'}\nabla_{v'}s \\
&= \nabla_{u'}G^{u'v'}\nabla_{a''}\nabla_{v'}s + \nabla_{u'}(\nabla_{a''}G^{u'v'})\nabla_{v'}s \\
&\quad + ([\nabla,\nabla]_{a''u'}G^{u'v'})\nabla_{v'}s + G^{u'v'}[\nabla,\nabla]_{a''u'}\nabla_{v'}s \\
&\quad + \text{four terms that cancel out} \\
&= \nabla_{u'}G^{u'v'}[\nabla,\nabla]_{a''v'}s + \nabla_{u'}G^{u'v'}\nabla_{v'}\nabla_{a''}s \\
&\quad + R^{w}_{awu}G^{uv}\nabla_v s - ik\omega_{au}G^{uv}\nabla_u s \\
&= \nabla_{u'}G^{u'v'}[\nabla,\nabla]_{a''v'}s - R^{w}_{wau}G^{uv}\nabla_v s - ik\omega_{au}G^{uv}\nabla_u s \\
&= -ik\nabla_{u'}G^{u'v}\omega_{av}\otimes s - r_{au}G^{uv}\nabla_v s - ik\omega_{au}G^{uv}\nabla_v s \\
&= -ik\nabla_{u'}G^{u'v}\omega_{av}\otimes s - J^x_a r_{xy}J^y_{u'}G^{u'v'}\nabla_{v'}s \\
&\quad - ik\omega_{au}G^{u'v'}\nabla_{v'}s \\
&= -ik(\nabla_{u'}G^{u'v})\omega_{av}\otimes s - ikG^{u'v}(\nabla_{u'}\omega_{av})\otimes s \\
&\quad - ikG^{u'v}\omega_{av}\nabla_{u'}s - iJ^x_a r_{xu'}G^{u'v'}\nabla_{v'}s - ik\omega_{au}G^{u'v'}\nabla_{v'}s \\
&= -ik(\nabla_{u'}G^{u'v})\omega_{av}\otimes s + 0 - ikG^{u'v}\omega_{av}\nabla_{u'}s \\
&\quad - iJ^x_a r_{xu'}G^{u'v'}\nabla_{v'}s - ik\omega_{au}G^{u'v'}\nabla_{v'}s \\
&= -ik\delta(G)^v\omega_{av}\otimes s - ik\omega_{av}G^{u'v}\nabla_{u'}s - i\rho_{au'}G^{u'v'}\nabla_{v'}s \\
&\quad - ik\omega_{au'}G^{u'v'}\nabla_{v'}s \\
&= -2ik\omega_{au}G^{uv}\nabla_v s - ik\omega_{av}\delta(G)^v\otimes s - i\rho_{au'}G^{u'v'}\nabla_{v'}s \\
&= -2ik\omega\cdot G\cdot\nabla s - ik\omega\cdot\delta(G)\otimes s - i\rho\cdot G\cdot\nabla s,
\end{aligned}$$

hence we have Equation (4.4). Next we want to prove that

$$\begin{aligned}
&\nabla^{0,1}(\Delta_{G(V)}s + 2\nabla_{G(V)\cdot dF}s) = \\
&\qquad -i(2k+\lambda)\omega\cdot G(V)\cdot\nabla s - ik\omega\delta(G(V))s - 2ik\omega\cdot G(V)\cdot dFs.
\end{aligned} \tag{4.5}$$

We know that $\rho = \lambda\omega + 2i\partial\overline{\partial}F$, so

$$\begin{aligned}
\nabla^{0,1}\Delta_{G(V)}s &= -i(2k+\lambda)\omega\cdot G(V)\cdot\nabla s - ik\omega\cdot\delta(G(V))s \\
&\quad + 2\partial\overline{\partial}F\cdot G(V)\cdot\nabla s.
\end{aligned}$$

So all we have to prove to get (4.5) is

$$\nabla^{0,1}(2\nabla_{G(V)\cdot dF}s) = -2ik\omega\cdot G(V)\cdot dFs - 2\partial\overline{\partial}F\cdot G(V)\cdot\nabla s.$$

We calculate

$$\begin{aligned}
\nabla^{0,1}\nabla_{G(V)\cdot dF}s &= \nabla^{0,1}((G(V)\cdot dF)^u\nabla_u s) \\
&= \nabla^{0,1}(G(V)\cdot dF)\cdot\nabla s + (G(V)\cdot dF)^u(\nabla^{0,1}\nabla_u s) \\
&= (G(V)\cdot\nabla^{0,1}dF)\cdot\nabla s - ik\omega\cdot G(V)\cdot dFs \\
&\quad + (G(V)\cdot dF)^u(\nabla_u\nabla^{0,1}s) \\
&= (G(V)\cdot\overline{\partial}\partial F)\cdot\nabla s - ik\omega\cdot G(V)\cdot dFs \\
&= -\partial\overline{\partial}F\cdot G(V)\cdot\nabla s - ik\omega\cdot G(V)\cdot dFs
\end{aligned}$$

using $\partial\overline{\partial} + \overline{\partial}\partial = 0$ and that $G(V)$ is symmetric.

Now we can use Lemma 2.10 to prove

$$4i\overline{\partial}V'[F] = \delta(G(V)) \cdot \omega + 2dF \cdot G(V) \cdot \omega \tag{4.6}$$

We are doing this by taking the derivative of $\rho = \lambda\omega + 2i\partial\overline{\partial}F = \lambda\omega + 2id\overline{\partial}F$ along V'.

$$\begin{aligned}
V'[\rho] &= \lambda V'[\omega] + 2idV'[\overline{\partial}]F + 2id\overline{\partial}V'[F] \\
&= 0 + 2idV'[\frac{1}{2} + i\frac{1}{2}iJ]dF + 2id\overline{\partial}V'[F] \\
&= -dV'[J] \cdot dF + 2id\overline{\partial}V'[F] \\
&= -dG(V) \cdot \omega \cdot dF + 2id\overline{\partial}V'[F] \\
&= d(dF \cdot \omega \cdot G(V) + 2i\overline{\partial}V'[F]).
\end{aligned}$$

Using the Lemma we conclude that the one form

$$-\delta G(V) \cdot \omega + 2dF \cdot \omega \cdot G(V) + 4i\overline{\partial}V'[F]$$

is closed. Since $H^1(M, \mathbb{R}) = 0$ this one form is exact. The form is of type $(0,1)$. Since M is compact there exists non- constant holomorphic functions, and therefore non constant anti-holomorphic functions. Since it is $\overline{\partial}$-exact, it must be 0. Hence we have Equation (4.6).

Putting together Equation (4.5) and Equation (4.6) we get

$$\begin{gathered}
\nabla^{0,1}(\Delta_{G(V)}s + 2\nabla_{G(V)\cdot dF}s) = \\
-i(2k+\lambda)\omega \cdot G(V)\nabla s - 4k\overline{\partial}V'[F] \otimes s
\end{gathered}$$

Now we are able to calculate $\nabla^{0,1}u(V)s$ for

$$u(V) = \frac{1}{4k+2\lambda}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]).$$

We see that

$$\begin{aligned}
\nabla^{0,1}u(V)s &= \frac{1}{4k+2\lambda}(-i(2k+\lambda)\omega \cdot G(V) \cdot \nabla s - 4k\overline{\partial}V'[F] \otimes s \\
&\quad + \nabla^{0,1}(4kV'[F]) \otimes s) \\
&= \frac{-i}{4k+2\lambda}(2k+\lambda)\omega \cdot G(V) \cdot \nabla s \\
&= -\frac{i}{2}\omega \cdot G(V) \cdot \nabla s = \frac{i}{2}V'[J] \cdot \nabla s \\
&= \frac{i}{2}V'[J] \cdot \nabla^{1,0}s
\end{aligned}$$

as we wanted. Which means we now have a Hitchin connection.

Andersen and Gammelgaard prove in [4] that the connection $\hat{\nabla}$ is projectively flat when there are no holomorphic sections.

Theorem 4.4 ([4]). *The connection $\hat{\nabla}$ defined in Theorem 4.3 is projectively flat, provided $H^0(M_\sigma, T_\sigma) = 0$ for all $\sigma \in \mathcal{T}$.*

4.2 The Hitchin connection in metaplectic quantization

Let (M, ω) be a symplectic manifold. Let $\mathcal{L}$ be a prequantum line bundle over the manifold. Let $I\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a smooth family of Kähler structures parametrized by a manifold $\mathcal{T}$, that is for every $\sigma \in \mathcal{T}$ I_σ defines a complex structure on M, turning this into a Kähler manifold M_σ. The Kähler metric is given by $g_\sigma(X, Y) = \omega(X, I_\sigma Y)$. As always I_σ induces a splitting of the complexified tangent bundle $TM_\mathbb{C} = T'M_\sigma \oplus T''M_\sigma$. For any vector field X on M we write $X'_\sigma = \pi^{1,0}_\sigma X$ and $X''_\sigma = \pi^{0,1}_\sigma X$. All of this is the same as in the case of geometric quantization. Instead of assuming $c_1(M)$ and $[\omega]$ are proportional, we assume that the first Chern class is even, such that we can pick a metaplectic structure δ on M. Then I can be viewed as a smooth map

$$I\colon \mathcal{T} \times M \to L^+M.$$

Take the pullback of the metaplectic structure on M by I. For any σ the restriction

$$\delta_\sigma = \delta|_{\{\sigma\}\times M} \to M$$

is a square root of K_σ in M_σ. The Hermitian structure h^δ_σ in δ_σ gives rise to a Hermitian structure h^δ on δ. Let

$$\pi_M\colon \mathcal{T} \times M \to M$$

denote the projection. Define

$$\hat{\mathcal{L}} = \pi_M^* \mathcal{L} = \mathcal{T} \times \mathcal{L}$$

with Hermitian metric $\hat{h}^{\mathcal{L}} = \pi_M^* h^{\mathcal{L}}$. Then $\hat{\mathcal{L}} \otimes \delta$ becomes a smooth line bundle over $\mathcal{T} \times M$ with Hermitian metric $\hat{h}$ induced by $\hat{h}^{\mathcal{L}}$ and h^δ.

Consider

$$\mathcal{H}^{(k)}_{\delta,\sigma} := C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta),$$

where we have the connection ∇_{I_σ} (denote this by ∇_σ). This connection gives rise to a subspace

$$H^{(k)}_{\delta,\sigma} := H^0(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) = \Big\{ s \in \mathcal{H}^{(k)}_{\delta,\sigma} \;\Big|\; \nabla^{0,1}_\sigma s = 0 \Big\}.$$

We shall assume the spaces $\mathcal{H}^{(k)}_{\delta,\sigma}$ form a smooth vector bundle $\mathcal{H}^{(k)}_\delta$ over $\mathcal{T}$. The aim of this chapter is to construct a connection in $\mathcal{H}^{(k)}_\delta$ preserving the spaces $H^{(k)}_{\delta,\sigma}$.

The reference connection

In standard geometric quantization we used the trivial connection as a reference point in the space of connections. Then we found an appropriate one form with values in differential operators, to add to the trivial connection, to construct a connection that preserves the subspace of holomorphic sections over M.

When doing metaplectic quantization we are working in $\mathcal{H}^{(k)}_\delta$, where we do not have a trivial connection available. So we must do something else to choose a good reference connection instead.

Define a connection $\hat{\nabla}^{\mathcal{L}}$ in $\hat{\mathcal{L}}$. For a vector field X on $\mathcal{T} \times M$ tangent to M and s a section of $\hat{\mathcal{L}}$ let

$$(\hat{\nabla}^{\mathcal{L}}_X s)_{(\sigma,p)} := (\nabla^{\mathcal{L}}_X s_\sigma)_p,$$

For a vector field V on $\mathcal{T} \times M$ tangent to $\mathcal{T}$ we have

$$(\hat{\nabla}^{\mathcal{L}}_V s)_{(\sigma,p)} = V[s_p]_\sigma,$$

where $V[s_p]_\sigma$ denotes the differentiation of s_p at σ along V. $\hat{\nabla}^{\mathcal{L}}$ can be seen to be compatible with the Hermitian structure $\hat{h}^{\mathcal{L}}$.

We recall the definition of a connection $\hat{\nabla}^T$ in $T \to \mathcal{T} \times M$ in the following way. For directions tangent to M let

$$(\hat{\nabla}^T_X Y)_{(\sigma,p)} := ((\nabla^T_\sigma)_X Y_\sigma)_p,$$

where ∇^T is the connection induced by the Levi-Civita connection, Y a section of T and $X \in T_pM$. For directions tangent to $\mathcal{T}$, let $V \in T_\sigma \mathcal{T}$ be any vector on $\mathcal{T}$, define

$$(\hat{\nabla}^T_V Y)_{(\sigma,p)} := \pi^{1,0}_\sigma V[Y_p]_\sigma,$$

where $V[Y_p]_\sigma$ is the differentiation of Y_p in the trivial bundle $\mathcal{T} \times T_pM_{\mathbb{C}}$ and $\pi^{1,0}_\sigma \colon \mathcal{T} \times TM_{\mathbb{C}} \to T_\sigma$ is the projection.

Then $\hat{\nabla}^T$ induces a connection $\hat{\nabla}^K$ in $K = \bigwedge^m T^*$, which induces a connection $\hat{\nabla}^\delta$ in δ. Which, with the help of $\hat{\nabla}^{\mathcal{L}}$, induces a connection $\hat{\nabla}^r$ in $\hat{\mathcal{L}}^k \otimes \delta$

Definition 4.5 (Reference connection)**.** The connection

$$\hat{\nabla}^r = (\hat{\nabla}^{\mathcal{L}})^{\otimes k} \otimes \mathrm{Id} + \mathrm{Id} \otimes \hat{\nabla}^\delta$$

in $\hat{\mathcal{L}}^k \otimes \delta \to \mathcal{T} \times M$ is called *the reference connection.*

The Hitchin connection

Let $\mathcal{D}(\mathcal{M}, \mathcal{L}^k \otimes \delta_\sigma)$ denote the space of differential operators on $\mathcal{L}^k \otimes \delta_\sigma$. These form a bundle $\hat{\mathcal{D}}(\mathcal{M}, \mathcal{L}^k \otimes \delta_\sigma)$ over $\mathcal{T}$. We seek a one form $u^\delta \in \Omega^1(\mathcal{T}, \hat{\mathcal{D}}(\mathcal{M}, \mathcal{L}^k \otimes \delta_\sigma))$ such that $\nabla^\delta = \hat{\nabla} + u^\delta$ preserves the subspaces $H^{(k)}_{\delta,\sigma}$ inside each fiber $\mathcal{H}^{(k)}_{\delta,\sigma}$. Such a connection is called a Hitchin connection.

Lemma 4.6 ([5] Lemma 5.1)**.** *The connection ∇^δ is a Hitchin connection if and only if the one form u^δ satisfies*

$$\nabla^{0,1} u^\delta(V)s + \frac{i}{2}\omega \cdot G(V) \cdot \nabla s + \frac{i}{4}\omega \cdot \delta(G(V))s = 0$$

for any V vector field on $\mathcal{T}$, any $\sigma \in \mathcal{T}$ and any $s \in H^{(k)}_\sigma$.

For the general case we need a second order operator Δ_G, which is defined similarly

$$\begin{aligned}\Delta_G \colon C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) &\xrightarrow{\nabla_\sigma} C^\infty(M_\sigma, TM^*_{\mathbb{C}} \otimes \mathcal{L}^k \otimes \delta_\sigma) \\ &\xrightarrow{G \otimes \mathrm{id} \otimes \mathrm{Id}} C^\infty(M_\sigma, T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma) \\ &\xrightarrow{\tilde{\nabla}_\sigma \otimes \mathrm{id} + \mathrm{id} \otimes \tilde{\nabla}_\sigma} C^\infty(M_\sigma, T^*_\sigma \otimes T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma) \\ &\xrightarrow{\mathrm{Tr}} C^\infty(M, \mathcal{L}^k \otimes \delta_\sigma),\end{aligned}$$

where $\tilde{\nabla}_\sigma$ is the Levi-Civita connection on M_σ induced by the metric on M_σ.

Again we have to assume the family of Kähler structures is rigid. Now using rigidity and much the same calculations as in proving Equation (4.4) one can prove the following Lemma

Lemma 4.7 ([5] Lemma 5.4)**.** *At every point $\sigma \in \mathcal{T}$ the operator $\Delta_{G(V)}$ satisfies*

$$\nabla^{0,1} \Delta_{G(V)} s = -2ik\omega \cdot G(V) \cdot \nabla s - ik\omega \cdot \delta(G(V))s + \frac{i}{2}\delta(\rho \cdot G(V))s$$

for all vector fields V on $\mathcal{T}$ and any local holomorphic sections s of the line bundle $\mathcal{L} \otimes \delta_\sigma \to M$.

This Lemma applies to all k, so also for $k = 0$. In this case the Lemma yields

$$\nabla^{0,1}\Delta_{G(V)}s = \frac{i}{2}\delta(\rho \cdot G(V))s.$$

Now apply the $\overline{\partial}$ operator on both sides, and we get

$$0 = \overline{\partial}(\delta(\rho \cdot G(V))),$$

thus if we assume $H^{0,1}(M) = 0$ we see that $\delta(\rho \cdot G(V))$ is exact with respect to the $\overline{\partial}$ operator on M, and we have proven the following

Corollary 4.8 ([5] Cor. 5.5)**.** *Provided that $H^{0,1}(M) = 0$, we have that $\delta(\rho \cdot G(V))$ is exact with respect to the $\overline{\partial}$ operator on M.*

For any compact Kähler manifold with $H^1(M, \mathbb{R}) = 0$, Hodge decomposition will give $H^{0,1}(M) = 0$. Now by Corollary 4.8 there exists a smooth one form $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$ such that

$$\overline{\partial}\beta(V) = -\frac{i}{2}\delta(\rho \cdot G(V))$$

for any vector field V on $\mathcal{T}$. We can now define $u(V)$ satisfying the wanted equation

$$u(V) = \frac{1}{4}(\Delta_{G(V)} + \beta(V))$$

Theorem 4.9 ([5] Theorem 1.2)**.** *Let (M, ω) be a prequantizable symplectic manifold with vanishing second Stiefel Whitney class. Let J be a rigid family of Kähler structures on M parametrized by a smooth manifold $\mathcal{T}$, all satisfying $H^{0,1}(M_\sigma) = 0$, $\sigma \in \mathcal{T}$. Then there exists a one form $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$ satisfying $\overline{\partial}\beta(V) = -\frac{i}{2}\delta(\rho \cdot G(V))$ and the connection*

$$\nabla^\delta_V = \hat{\nabla}_V + \frac{1}{4k}(\Delta_{G(V)} + \beta(V))$$

is a Hitchin connection on $\mathcal{H}^{(k)}_\delta$ over $\mathcal{T}$.

Note that the restrictions on M are a lot weaker than in the original setting with geometric quantization. We do not need that the first Chern class is proportional to $[\omega]$, now we just have to know that it is even, since the second Stiefel Whitney class is equal to the first Chern class modulo two.

It was proven by Gammelgaard in his thesis (Theorem 6.22), that

Theorem 4.10 ([25])**.** *The connection $\hat{\nabla}$ defined in Theorem 4.9 is projectively flat, provided $H^0(M_\sigma, T_\sigma) = 0$ for all $\sigma \in \mathcal{T}$.*

Chapter **5**

Review of the moduli space of flat connections on a compact surface

The main focus of this thesis is to construct a Hitchin connection in the case of a moduli space of a surface with marked points. We will construct the moduli space and a prequantum line bundle in that case. Before we do this, we will in this Chapter review the case of the closed surface, since many of the ideas are similar to how we do it in the case of the surface with marked points. The proofs will not be rigorous in this Chapter, since it is mainly here to serve as inspiration of how to do in the case of the surface with marked points.

5.1 The moduli space of flat connections

Let Σ be a compact surface, p a fixed base-point, $\pi_1(\Sigma, p)$ the fundamental group of Σ based at p and let G be a compact connected Lie-group.

Definition 5.1. The *representation variety of* Σ is the set

$$\mathcal{M}_G = \mathrm{Hom}(\pi_1(\Sigma, p), G)/G$$

of G-valued representations of $\pi_1(\Sigma, p)$ modulo conjugation in G.

Let us now recall the gauge theoretic description of $\mathcal{M}_G$ thus realizing it as the moduli space of flat G-connections on Σ.

Let $P \to \Sigma$ be a principal G-bundle. Let $\mathcal{A}_P$ denote the space of all connections in the principal G-bundle. Let $\mathcal{F}_P$ denote the subset of $\mathcal{A}_P$ consisting of all the flat connections. Define an equivalence relation on $\mathcal{F}_P$ by $A \sim A'$ if and only if they are gauge-equivalent. Then we can define the moduli space of flat connections as

$$\mathcal{M}_P = \mathcal{F}_P/_{\sim}.$$

Definition 5.2 (Moduli space of flat connections)**.** The *moduli space of flat connections* on a principal G-bundle $P \to M$ is the space

$$\mathcal{M}_P = \mathcal{F}_P/\mathcal{G}_P.$$

We will assume G to be simply connected, thus all principal G-bundles are trivializable, thus hence $\mathcal{M}_P$ is not dependent on P, and we will assume $P = \Sigma \times G$, .

Let us try to give a short explanation of the idea behind the proof of why these two definitions are the same; let α be a loop, $\alpha(0) = \alpha(1) = p$. Then for $p_0 \in \pi^{-1}(p)$, we know there exist a unique horizontal curve β, with starting point p_0 and $\pi \circ \beta = \alpha$. Since α is assumed to be a loop, $\beta(0)$ and $\beta(1)$ are both in the fibre P_p over p, so there exists a g such that $\beta(0) = \beta(1) \cdot g$. This g is called the holonomy of A along α with respect to p_0, denoted $\mathrm{hol}_{A,p_0}(\alpha)$. This induces a well-defined map

$$\mathrm{hol}\colon \mathcal{M}_P \to \mathrm{Hom}(\pi_1(\Sigma, p), G)/G = \mathcal{M}_G,$$

that sends $[A]$ to $[\mathrm{hol}_A]$.

Conversely if $\rho\colon \pi_1(\Sigma, p) \to G$ is a given homomorphism, we consider the trivial G bundle $\tilde{P} = \tilde{\Sigma} \times G$ over the universal covering space $\tilde{\Sigma}$ of Σ. We can get a right action of the fundamental group on $\tilde{P}$ by doing the following. Let $\gamma \in \pi_1(\Sigma, p)$ and $(y, g) \in \tilde{P}$. Define $(y, g) \cdot \gamma = (y \cdot \gamma, \rho(\gamma)^{-1} g)$, where $y \cdot \gamma$ denotes the natural action of $\pi_1(\Sigma, p)$ on the covering space. This action is free. We can also see that $P = \tilde{P}/\pi_1(\Sigma, p)$ is a principal G bundle over Σ. We have a free right action, so all we need is that $\tilde{P}$ is locally trivializable. Let π be the projection $\pi\colon P \to \Sigma$. We have to show that for all $x \in \Sigma$ there exists a neighborhood U such that $\pi\colon \pi^{-1}(U) \to U \times G$ is an equivariant diffeomorphism, which covers the identity on U. Let $q \in \Sigma$. Let $\tilde{\Sigma}$ denote the universal covering of Σ. Let U be an open neighborhood of q such that in $\tilde{\Sigma}$ the open neighborhoods of each pre-image of q are disjoint, this can be done since it is the universal covering. Let $\tilde{U}$ be one of these. $\tilde{P} = \tilde{\Sigma} \times G$ is locally trivializable, so $\tilde{P}|_{\tilde{U}} = \tilde{U} \times G \simeq U \times G$. Hence $\tilde{P}/\pi_1(\Sigma, p)$ is locally trivializable.

The trivial connection on $\tilde{P}$ (the pullback of the Maurer-Cartan form on G) is invariant under the action of $\pi_1(\Sigma, p)$, so it descends to a flat connection on P. Hence we have a well-defined map

$$\mathcal{M}_G \to \mathcal{M}_P.$$

5.2 Smooth structure

First we notice that in general the moduli space is not smooth, since it will have singular points. A way to see this is the following. We know

$$\pi_1(\Sigma) \simeq \langle \alpha_j, \beta_j \mid \prod_{j=1}^{g} [\alpha_j, \beta_j] = 1 \rangle.$$

Let $q\colon G^{2g} \to G$ denote the map

$$q(A_1, B_1, \ldots, A_g, B_g) = \prod_{j=1}^{g} [A_j, B_j].$$

Then we see that we can identify $\mathrm{Hom}(\pi_1(\Sigma), G)$ with $q^{-1}(e)$. This set is often singular, since in general $e \in G$ is not a regular point of q.

Goldman proved that this problem can be handled by only considering the irreducible representations. Define irreducible in the following way. A representation ρ is irreducible if the commutator $Z_\rho(\pi_1(\Sigma, p))$ is equal to the center of the group Z_G. In the case $g \geq 2$ it is well known that the space $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), \mathrm{SU}(n))$ of irreducible representations is a dense and open subset of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(n))$, and the quotient

$$\mathcal{M}' = \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), \mathrm{SU}(n))/\mathrm{SU}(n) \subseteq \mathcal{M}_{\mathrm{SU}(n)}$$

is a smooth manifold. Note that this is in no way an easy result, but has been proven by Goldman using explicit analysis of $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), \mathrm{SU}(n))/\mathrm{SU}(n)$.

Let $\mathcal{M}'_G$ denote the smooth part of $\mathcal{M}_G$.

5.3 Tangent space

The following two sections are about the tangent space and the symplectic structure. The statements and proofs in these sections are not going to be rigorous, since for instance the space $\mathcal{G}$ is infinite dimensional. To do it correctly, one should use Sobolev spaces (as we are doing in Chapter 6), but since this chapter purely serves as inspiration for how to do it in the case of the surface with marked points, we ignore in this chapter issues regarding infinite dimensional manifolds.

The following theorem states that the tangent space $T_{[A]}\mathcal{M}'_G$ at the gauge equivalence class $[A]$ of the moduli space of flat connections $\mathcal{M}'_G$ is identified with the first (de Rahm) cohomology group of Σ, with coefficients in the adjoint bundle $\operatorname{Ad} P$ and differential d_A induced by A.

Theorem 5.3. *Let A be a flat connection in a principal G-bundle $P \to \Sigma$. Then*

$$T_{[A]}\mathcal{M}'_G \simeq H^1(\Sigma, \operatorname{Ad} P; d_A).$$

We will give a sketch of the proof. First we will note that the tangent space of $\mathcal{F}$ is the set of co-cycles $Z^1(\Sigma, \operatorname{Ad} P; d_A)$. Note that $T_A\mathcal{A} \simeq \Omega^1(\Sigma, \operatorname{Ad} P)$. We want to find out when $a \in \Omega^1(\Sigma, \operatorname{Ad} P)$ is tangential to $\mathcal{F}$. Take the derivative of $\mathrm{F}_{A+ta} = 0$ with respect to t

$$\begin{aligned}
0 = \frac{d}{dt} F_{A+ta} &= \frac{d}{dt}(d(A+ta) + \frac{1}{2}[(A+ta) \wedge (A+ta)]) \\
&= \frac{d}{dt}(dA + tda + \frac{1}{2}[A \wedge A] + \frac{1}{2}t[A \wedge a] + \frac{1}{2}t[a \wedge A] + \frac{1}{2}t^2[a \wedge a]) \\
&= \frac{d}{dt}(F_A + tda + t[A \wedge a] + \frac{1}{2}t^2[a \wedge a]) \\
&= da + [A \wedge a] + t[a \wedge a],
\end{aligned}$$

evaluate at $t = 0$ to get

$$0 = da + [A \wedge a] = d_A a.$$

So we have the condition that a should be a closed one form with respect to d_A, to be tangential to $\mathcal{F}$ at A. That means

$$T_A\mathcal{F} = Z^1(\Sigma, \operatorname{Ad} P; d_A) \subseteq \Omega^1(\Sigma, \operatorname{Ad} P).$$

To show that $T_{[A]}\mathcal{M}_G \simeq H^1(\Sigma, \mathrm{Ad}P; d_A)$ we need to show that the subspace tangent to the action of the gauge group is

$$T_A(A\mathcal{G}) \simeq B^1(\Sigma, \operatorname{Ad} P; d_A),$$

which can be done in the following way. Let φ_t be a 1-parameter family of gauge transformations with $\varphi_0 = \mathrm{id}$. Let $g_t \colon P \to G$ be the associated family of G-equivariant maps ($g(ph) = h^{-1}g(p)h$) with $g_0 = e$. For a $p \in P$ $g_t(p)$ is a curve through $e \in G$, so $\frac{d}{dt}|_{t=0} g_t$ is a map $f \colon P \to \mathfrak{g}$. Since g_t is G-equivariant we see that f becomes G-equivariant.

$$\begin{aligned}
f(pg) &= \frac{d}{dt}\Big|_{t=0} g_t(pg) = \frac{d}{dt}\Big|_{t=0} g^{-1} g_t(p) g \\
&= \frac{d}{dt}\Big|_{t=0} (c(g^{-1}) \circ g_t)(p) = \operatorname{Ad}(g^{-1})(f(p)).
\end{aligned}$$

So we get $f \in \Omega^0(\Sigma, \mathrm{Ad}P)$.

It can be proven that

$$\varphi_t^*(A) = \operatorname{Ad}(g_t^{-1}) \circ A + g_t^*\theta.$$

We take the derivative of this and get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\varphi_t^*(A) &= \frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(g_t^{-1})\circ A + \frac{d}{dt}\Big|_{t=0} g_t^*\theta \\ &= \mathrm{ad}(-f)\circ A + \frac{d}{dt}\Big|_{t=0} g_t^*\theta.\end{aligned}$$

Let α be a curve in P with $\alpha(0) = p$ and $\alpha'(0) = X \in T_pP$.

$$\begin{aligned}g_t^*\theta(X) &= \theta_{g_t(p)}(D_p g_t(X)) = \theta(\frac{d}{ds}\Big|_{s=0} g_t(\alpha(s))) \\ &= D_{g_t(p)}L_{g_t(p)^{-1}}(\frac{d}{ds}\Big|_{s=0} g_t(\alpha(s))) \\ &= \frac{d}{ds}\Big|_{s=0}(g_t(p)^{-1}\cdot g_t(\alpha(s))),\end{aligned}$$

so by taking the derivative with respect to t we get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} g_t^*\theta &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}(g_t(p)^{-1}\cdot g_t(\alpha(s))) \\ &= \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}(g_t(p)^{-1}\cdot g_t(\alpha(s))) \\ &= \frac{d}{ds}\Big|_{s=0}(-f(p) + f(\alpha(s))) \\ &= df(X).\end{aligned}$$

Hence

$$\frac{d}{dt}\Big|_{t=0}\varphi_t^*(A) = [A\wedge f] + df = d_A f.$$

So a vector tangent to the gauge group action in A is an exact one form in $\Omega^1(\Sigma, \mathrm{Ad}P; d_A)$. The other way around. Given an G-equivariant map $f\colon P\to\mathfrak{g}$, then $g_t(p) = \exp(tf(p))$ will define a 1-parameter family of gauge transformations, which induces the tangent vector d_A in A. Hence we have now seen

$$T_A(A\mathcal{G}) = B^1(\Sigma, \mathrm{Ad}P; d_A).$$

All together we have

$$T_{[A]}\mathcal{M}_G \simeq H^1(\Sigma, \mathrm{Ad}P; d_A).$$

5.4 Symplectic structure

To define a symplectic structure, we assume the Lie algebra $\mathfrak{g}$ admits a non-degenerate, bilinear and symmetric form $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$, that is invariant under the adjoint action of G. For $G = \mathrm{SU}(n)$ we have such a form $B(X,Y) = \mathrm{tr}(X^*Y)$ for skew-hermitian and traceless matrices. Note that for every lie-algebra $\mathfrak{g}$ we have the Killing form which is bilinear and symmetric. If $\mathfrak{g}$ is semi-simple, then the Killing form is also non-degenerate.

Under this assumption Goldman proved that the smooth part of the moduli space over a closed surface admits a symplectic structure. B induces a bundle map $B_*\colon \mathrm{Ad}\,P\otimes\mathrm{Ad}\,P\to\Sigma\times\mathbb{R}$. Let $\varphi,\psi\in\Omega^1(\Sigma, \mathrm{Ad}\,P, d_A)$ be two d_A closed one forms representing tangent vectors at a point $[A]$ in $\mathcal{M}'_G$. Then $B_*(\varphi\wedge\psi)$ is a 2-form on Σ, so we define

$$\omega([\varphi],[\psi]) := \int_\Sigma B_*(\varphi\wedge\psi).$$

This can be proven to be a symplectic form on $\mathcal{M}'_G$.

5.5 The Chern-Simons line bundle

In the following we have our moduli space $\mathcal{M} = \mathcal{F}_P/\mathcal{G} \subset \mathcal{A}_P/\mathcal{G}$. We would like to construct a line bundle over $\mathcal{M}$. We consider the trivial bundle

$$\begin{tikzcd} \mathcal{F}_P \times \mathbb{C} \arrow[r, hook] \arrow[d, "\rho"] & \mathcal{A}_P \times \mathbb{C} \arrow[d] \\ \mathcal{F}_P \arrow[r, hook] \arrow[d] & \mathcal{A}_P \arrow[d] \\ \mathcal{M} \arrow[r, hook] & \mathcal{A}_P/\mathcal{G} \end{tikzcd}$$

$\mathcal{G}$ acts on $\mathcal{A}_P$, and we are going to lift that action to an action on $\mathcal{A}_P \times \mathbb{C}$, to get a line bundle

$$(\mathcal{F}_P \times \mathbb{C})/\mathcal{G}.$$

To lift the action we will construct a co-cycle $\Theta^k \colon \mathcal{A}_P \times \mathcal{G} \to U(1)$, and show that it satisfies the co-cycle condition. Then we can define the wanted action as

$$(\nabla_A, z) \cdot g = (\nabla_A^g, \Theta^k(\nabla_A, g) \cdot z).$$

Since Θ^k is a co-cycle, this will be an action of $\mathcal{G}$ on $\mathcal{A}_P \times \mathbb{C}$, and hence on $\mathcal{F}_P \times \mathbb{C}$, since $\mathcal{F}_P \subset \mathcal{A}_P$ and $\mathcal{G}$ preserves $\mathcal{F}_P$. The only thing left to check is that ρ is equivariant

$$\rho((\nabla_A, z) \cdot g) = \rho(\nabla_A^g, \Theta^k(\nabla_A, g) \cdot z) = \nabla_A^g = \rho(\nabla_A, z) \cdot g.$$

Hence we have a line bundle $(\mathcal{F}_P \times \mathbb{C})/\mathcal{G} \to \mathcal{M}$, if we can just construct the co-cycle Θ^k and check that stabilizers act trivially.

The Chern-Simons line bundle on a closed surface

We will construct the Chern-Simons line bundle, as it is done in [8].

The moment map is given by the curvature of a connection, hence the level set that we take the quotient of, consists exactly of the flat connections. We will like to lift the action of $\mathcal{G}$ to the trivial bundle $\mathcal{A}_P \times \mathbb{C}$. To do this define a co-cycle

$$\Theta^{(k)}(\nabla_A, g) := \exp(2\pi i k(\mathrm{CS}(\tilde{A}^{\tilde{g}}) - \mathrm{CS}(\tilde{A}))),$$

where $\tilde{A}$ and $\tilde{g}$ are any extensions of A and g to an arbitrary compact 3-manifold Y with boundary Σ, and

$$\mathrm{CS}(A) := \frac{1}{8\pi^2} \int_Y \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

The action of $\mathcal{G}$ on $\mathcal{A} \times \mathbb{C}$ is given by

$$(\nabla_A, z) \cdot g = (\nabla_A^g, \Theta^{(k)}(\nabla_A, g) \cdot z),$$

where $\nabla_A^g \colon = d + \mathrm{Ad}_{g^{-1}} A + g^*\theta$ denotes the gauge group action, with $\theta \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan form. Note that the Maurer-Cartan form is defined as $\theta(v) = (L_{g^{-1}})_* v$ for $v \in T_g G$.

A calculation shows that $\Theta^{(k)}$ satisfies the co-cycle condition

$$\Theta^{(k)}(\nabla_A, g)\Theta^{(k)}(\nabla_A^g, h) = \Theta^{(k)}(\nabla_A, gh),$$

and $\mathcal{G}$ preserves flat connections. So we obtain the induced Chern-Simons line bundle $\mathcal{L}^k_{\mathrm{CS}}$ over $\mathcal{M}$.

$\Theta^{(k)}$ satisfies the co-cycle condition

The proof of the co-cycle for $\Theta^{(k)}$ is almost the same in the closed and the punctured case. One can also see the calculations in [24, 8]. First one proves that $\Theta^{(k)}$ is not dependent on the extensions $\tilde{g}$ and $\tilde{A}$ of g and A. That this is true can be seen in the following way.

Let X and Y be 3-manifolds both with boundary Σ. Let $\tilde{g}_X$, $\tilde{g}_Y$, $\tilde{A}_X$ and $\tilde{A}_Y$ be extensions of g and A over X and Y. Let $X \cup -Y$ denote the closed 3-manifold that occurs when X and Y with switched orientation, are glued together along Σ. Together $\tilde{A}_X$ and $\tilde{A}_Y$ form a connection on $X \cup -Y$. Together $\tilde{g}_X$ and $\tilde{g}_Y$ form a gauge-transformation on $X \cup -Y$. Now

$$\begin{aligned}&(\mathrm{CS}(\tilde{A}_X^{\tilde{g}_X}) - \mathrm{CS}(\tilde{A}_X)) - (\mathrm{CS}(\tilde{A}_Y^{\tilde{g}_Y}) - \mathrm{CS}(\tilde{A}_Y))\\ &= (\mathrm{CS}((\tilde{A}_X \cup \tilde{A}_Y)^{\tilde{g}_X \cup \tilde{g}_Y}) - \mathrm{CS}(\tilde{A}_X \cup \tilde{A}_Y)) \in \mathbb{Z},\end{aligned}$$

this is exactly an element of $\mathbb{Z}$, which follows from the normalization of CS, when we are looking at a closed 3-manifold (for more details see [24]). This means $\Theta^{(k)}$ is independent of the extensions, which is what we wanted to prove. For more details see [8].

Knowing $\Theta^{(k)}$ is independent of choice of extension, we can show that it is indeed a co-cycle. We will prove it for $\Theta^{(k)}(\nabla_A, g) = \exp(-2\pi i k \mathrm{CS}_{[0,1]\times\Sigma}(\tilde{A}^{\tilde{g}}))$. Let $\tilde{h}_1 \colon [0,1] \times \Sigma \to G$ be an extension of h from Σ to $[0,1] \times \Sigma$, such that $\tilde{h}_1(0,\cdot) = h(\cdot)$ and $\tilde{h}_1(1,\cdot) = e(\cdot)$ (the identity gauge transformation). Define $\tilde{g}_1$ correspondingly. Define extensions $\tilde{h}_0$ and $\tilde{g}_0$ of h and g by

$$\tilde{h}_0(t,\cdot) = \begin{cases} \tilde{h}_1(2t,\cdot) & \text{for } t \leq \frac{1}{2} \\ \pi^* e(2t-1,\cdot) & \text{for } t \geq \frac{1}{2} \end{cases}$$

$$\tilde{g}_0(t,\cdot) = \begin{cases} \pi^* g(2t,\cdot) & \text{for } t \leq \frac{1}{2} \\ \tilde{g}_1(2t-1,\cdot) & \text{for } t \geq \frac{1}{2} \end{cases}.$$

By calculations we see that $\Theta^{(k)}$ satisfies the co-cycle condition

$$\begin{aligned}\Theta^{(k)}(\nabla_A, gh) &= \exp(-2\pi i k \mathrm{CS}_{[0,1]\times\Sigma}(\tilde{A}^{\tilde{gh}}))\\ &= \exp(-2\pi i k \mathrm{CS}_{[0,1]\times\Sigma}(\tilde{A}^{\tilde{g}_0\tilde{h}_0}))\\ &= \exp(-2\pi i k (\mathrm{CS}_{[0,\frac{1}{2}]\times\Sigma}(\tilde{A}^{\tilde{g}_0\tilde{h}_0}) + \mathrm{CS}_{[\frac{1}{2},1]\times\Sigma}(\tilde{A}^{\tilde{g}_0\tilde{h}_0})))\\ &= \exp(-2\pi i k (\mathrm{CS}_{[0,1]\times\Sigma}((\pi^* A)^g)^{\tilde{h}_1} + \mathrm{CS}_{[0,1]\times\Sigma}(\tilde{A}^{\tilde{g}_1})))\\ &= \Theta^{(k)}(\nabla_A^g, h) \cdot \Theta^{(k)}(\nabla_A, g).\end{aligned}$$

Remark 5.4. We can construct the co-cycle without requiring the existence of a bounding 3-manifold for Σ. Every gauge-transformation is homotopic to the identity, so we may extend g on Σ to $\tilde{g}$ one the cylinder $[0,1] \times \Sigma$ using a homotopy such that $\tilde{g}_0 = g$ and $\tilde{g}_1 = e$. For the natural projection $\pi \colon [0,1] \times \Sigma \to \Sigma$ extend ∇_A on Σ to $\tilde{\nabla}_A = \pi^* \nabla_A = d + \pi^* A$ on $[0,1] \times \Sigma$. Then $\tilde{\nabla}_A^{\tilde{g}}$ is an extension of ∇_A^g to $[0,1] \times \Sigma$. Choose the standard orientation on $[0,1] \times \Sigma$. Define

$$\Theta^{(k)}(\nabla_A, g) = \exp(-2\pi i k \mathrm{CS}_{[0,1]\times\Sigma}(\tilde{A}^{\tilde{g}})).$$

It can be shown that the two expressions agree. The latter is easier to generalize to a punctured surface, since it does not require the existence of a well-defined bounding 3-manifold.

Chapter **6**

The moduli spaces of parabolic bundles and flat connections

In this chapter we will construct the *moduli space of flat connections* over a surface with punctures. We will use the construction done by Andersen, Himpel, Jørgensen, Martens and McLellan in [8], which follows the work of Daskalapoulos and Wentworth in [22], only they do it for just one puncture, whereas [8] does it for any number of punctures.

This chapter is divided into six sections. We will define three moduli spaces over a surface with marked points. These moduli spaces are diffeomorphic on their irreducible loci. First, we define the moduli space of flat connections $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$, then in the second section, we define the moduli space of parabolic bundles $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$. We will however need another construction of the complex structure, which allows us to understand its variation with the complex structure on the surface better and further is in our favor when we later need to identify to the pre-quantum line bundle. To do this we use Sobolev completions of certain kinds of connections on $\tilde{\Sigma}$ to construct a moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)_\sigma$, whose irreducible locus is shown by Daskalapoulos and Wentworth to be smooth and naturally an almost complex manifold in [22]. In Section 3.3 in [22] they prove that the almost complex structure coincides with the complex structure from Mehta and Seshadri, Section 2.2, [35]. For small enough ϵ the moduli space is again homeomorphic (diffeomorphic on the irreducible locus) to $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})_\sigma$. In the fifth section, we review four lemmas proven by Andersen in [1], which proves that every element of the tangent space $T_{[A]}\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$ has a unique d_A-harmonic representation. In the last section we briefly introduce the moduli stacks, that we will be using in the rest of the Chapters in this thesis.

6.1 Definition of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$

Recall that we have a smooth surface Σ (of genus $g \geq 2$) with marked points $\mathcal{P} = \{p^{(1)}, \dots, p^{(b)}\}$ and that we further use the notation $\tilde{\Sigma} = \Sigma \backslash \mathcal{P}$. Each of the marked points $p^{(i)}$ are labelled by a $\lambda^{(i)} \in \Lambda$. We recall that $C^{(k)}_{\lambda^{(i)}}$ are the conjugacy classes of $G = SU(n)$ determined by $\lambda^{(i)}$, that is

$$\exp(i^{-1}(\lambda^{(i)})/k) \in C^{(k)}_{\lambda^{(i)}},$$

where we saw in the introduction that the matrix $\exp(i^{-1}(\lambda^{(i)})/k)$ was given by

$$\exp(\tfrac{1}{k}i^{-1}(\lambda^{(i)})) = \begin{pmatrix} e^{\frac{2\pi i}{k}(\lambda_1^{(i)}-\frac{1}{n}\sum_j \lambda_j^{(i)})} & 0 & \dots & 0 \\ 0 & e^{\frac{2\pi i}{k}(\lambda_2^{(i)}-\frac{1}{n}\sum_j \lambda_j^{(i)})} & \dots & 0 \\ \vdots & \dots & \dots & \dots \\ 0 & \dots & \dots & e^{\frac{2\pi i}{k}(\lambda_n^{(i)}-\frac{1}{n}\sum_j \lambda_j^{(i)})} \end{pmatrix}.$$

Suppose $\delta^{(i)}$ are the oriented boundary of small oriented embedded disjoint discs in Σ centered in $p^{(i)}$ as indicated in figure 6.1. Then we can define the following character variety description in complete analogy with the closed surface case

$$\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}) = \{\rho \in \mathrm{Hom}(\pi_1(\tilde{\Sigma}), G) \mid \rho(\delta^{(i)}) \in C^{(k)}_{\lambda^{(i)}}\}/G.$$

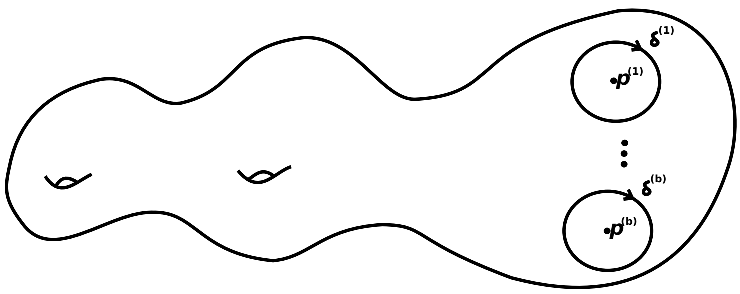


Figure 6.1: The surface Σ with small embedded discs around each puncture

We recall that the subset of irreducible such connections (or equivalently representations) is denoted $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ and it forms an open dense subset of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$. As it is proved in [17] the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ is smooth manifold of real dimension

$$\dim \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})' = 2(g-1)(n^2-1) + \sum_{i=1}^{b} \dim C^{(k)}_{\lambda^{(i)}}. \tag{6.1}$$

In Chapter 7 we will construct a symplectic form $\omega_{k,\overline{\lambda}}$ on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$.

Example: The moduli space of flat $SU(2)$-connections over $\tilde{\Sigma}$

The moduli space of flat connections $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ for $SU(2)$ is particular simple to understand. In the $SU(2)$ case we have that

$$\Lambda \cong \{0, \dots, k\}.$$

The conjugacy class associated to $\lambda \in \Lambda$ is determined by

$$\begin{pmatrix} e^{-\frac{i\pi}{k}\lambda} & 0 \\ 0 & e^{\frac{i\pi}{k}\lambda} \end{pmatrix} \in C^{(k)}_{\lambda}$$

We observe that $\dim C_\lambda^{(k)} = 2$ if $\lambda \notin \{0, k\}$ else $\dim C_\lambda^{(k)} = 0$. Let b_i be the number of points $p^{(i)}$ where $\lambda^{(i)} \notin \{0, k\}$. Thus we see that

$$\dim \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})' = 6(g-1) + 2b_i;$$

Suppose now $[\tilde{\rho}] \in \mathcal{M}(\tilde{\Sigma}, \overline{\lambda}) - \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$, which is the case if and only if there exists $\rho \in [\tilde{\rho}]$ such that $\rho(\pi_1(\tilde{\Sigma})) \subset \mathrm{U}(1) \subseteq \mathrm{SU}(2)$.

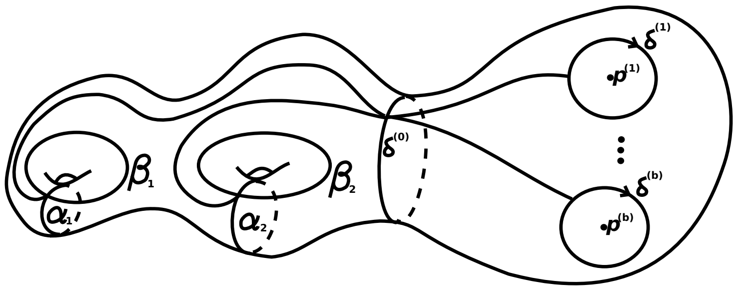


Figure 6.2: Σ with the curves $\delta^{(0)}$, $\alpha_1, \beta_1, \alpha_2, \beta_2$ and $\delta^{(1)}, \ldots, \delta^{(b)}$ in the case $g = 2$.

Let $\delta^{(0)}$ be the oriented boundary of an oriented embedded disc in Σ which contains all $p^{(i)}$ in its interior and let Σ' be the complement of this disc in Σ. Then we must have that $\rho(\delta^{(0)}) = 1$, since $\delta^{(0)}$ represents the conjugacy class of a product of commutators in $\pi_1(\tilde{\Sigma})$. Now let $s^{(i)} \in \{\pm 1\}$ be such that

$$\rho(\delta_i) = \begin{pmatrix} e^{-\frac{i\pi}{k} s^{(i)} \lambda^{(i)}} & 0 \\ 0 & e^{\frac{i\pi}{k} s^{(i)} \lambda^{(i)}} \end{pmatrix}.$$

Thus we see that $\rho(\delta^{(0)}) = 1$ if and only if

$$\sum_{i=1}^{b} s^{(i)} \lambda^{(i)} \in 2k\mathbb{Z}. \tag{6.2}$$

Thus we see that $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}) \neq \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ if and only if there exist $s^{(i)} \in \{\pm 1\}$ such that (6.2) is satisfied. Let $S_{\overline{\lambda}}$ be the set of solutions (if $\lambda^{(i)} \in \{0, k\}$ we don't distinguish $s^{(i)}$ from $-s^{(i)}$) to (6.2) and $S'_{\overline{\lambda}} = S_{\overline{\lambda}}/\{\pm 1\}$ where -1 acts by multiplying the $s^{(i)}$ by -1. If all $\lambda^{(i)} \in \{0, k\}$, then

$$\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}) = \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$$

if and only if the number of $p^{(i)}$ where $\lambda^{(i)} = k$ is odd. If under the other hand this number is even then $H^1(\Sigma', U(1))/\{\pm 1\}$ embeds in $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ as the strictly semi-stable locus by using the natural inclusion $U(1) \subset SU(2)$ combined with unique $SU(2)$ elements from $C^{(k)}_{\lambda^{(i)}}$

$$\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}) = H^1(\Sigma', U(1))/\{\pm 1\} \sqcup \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$$

If on the other hand there are $\lambda^{(i)} \notin \{0, k\}$, then we get a copy of $H^1(\Sigma', U(1))$, say $H^1(\Sigma', U(1))_{\overline{s}}$, in the strictly semi-stable locus of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ again by using the natural inclusion $U(1) \subset SU(2)$ combined with the above assignments on the generators δ_i depending on $\overline{s}$ and hence

$$\mathcal{M}(\tilde{\Sigma},\overline{\lambda}) = \bigsqcup_{\overline{s}\in S'_{\overline{\lambda}}} H^1(\Sigma', U(1))_{\overline{s}} \sqcup \mathcal{M}(\tilde{\Sigma},\overline{\lambda})'.$$

We stress that there are many cases where $S'_{\overline{\lambda}} = \emptyset$, in which case we get of course that $\mathcal{M}(\tilde{\Sigma},\overline{\lambda}) = \mathcal{M}(\tilde{\Sigma},\overline{\lambda})'$ as mentioned above. If we for example consider the particular case where $\lambda^{(i)} = \lambda$, $i = 1,\ldots,b$ for some $\lambda \in \{1,\ldots,k-1\}$, then if b is odd and $b\lambda < 2k$ then it is not possible to find $s^{(i)} \in \{\pm 1\}$ such that (6.2) is satisfied, so in this case $\mathcal{M}(\tilde{\Sigma},\overline{\lambda}) = \mathcal{M}(\tilde{\Sigma},\overline{\lambda})'$.

In general we see in this $SU(2)$ case that if $\mathcal{M}(\tilde{\Sigma},\overline{\lambda}) \neq \mathcal{M}(\tilde{\Sigma},\overline{\lambda})'$ then its co-dimension is equal to $4g-6+2b_i$ which is at least 4 unless $g=2$, $b_i=0$ and further the number of points $p^{(i)}$ labeled by k is even, in which case the co-dimension is $6-4=2$.

6.2 Lower bound on the co-dimension of the reducible locus of $\mathcal{M}(\tilde{\Sigma},\overline{\lambda})$

We now return to the general case of $SU(n)$. We start with the following observation. Suppose we have a compact connected semi-simple Lie group G. Then we have that the map

$$[\cdot,\cdot] : G\times G \to G$$

is surjective and there exists elements $a,b\in G$ such that the smallest Lie subgroup of G which contains a and b is G it self. Let us then consider the moduli space of flat G connection on Σ with holonomy contained in G-conjugacy classes $C^{(i)}$ around $p^{(i)}$. Let us denote this moduli space $M_G(\tilde{\Sigma},\overline{C})$. We claim that the irreducible locus $M_G(\tilde{\Sigma},\overline{C})'$ of this moduli space is non-empty. The argument is very simple. We consider standard generators (α_i,β_i), $i=1,\ldots g$ and $\delta^{(i)}$, $i=1,\ldots,b$ such that

$$\prod_{i=1}^{g}[\alpha_i,\beta_i]\delta^{(1)}\ldots\delta^{(b)} = 1.$$

Now we pick any $g^{(i)} \in^{(k)}_{\lambda^{(i)}}$ and consider $[a,b]g^{(1)}\ldots g^{(b)} \in G$. Then pick $a',b'\in G$ such that

$$[a',b'][a,b]g^{(1)}\ldots g^{(b)} = 1.$$

Now extend this to a G assignment on all the generators above, by assigning $1\in G$ to the $g-2$ remaining pairs (α_i,β_i) to get an irreducible representation of $\pi_1(\tilde{\Sigma})$ which is contained in the above moduli space

By similar considerations as one finds in [26] we see thus that

Proposition 6.1. *The dimension of $M_G(\tilde{\Sigma},\overline{C})'$ is*

$$\dim M_G(\tilde{\Sigma},\overline{C})' = 2(g-1)\dim G + \sum_{i=1}^{b}\dim C^{(i)}.$$

By the analysis of the moduli space of parabolic bundles in the following section we see that in order to estimate the co-dimension of the reducible locus of M we only need to consider the following subgroups $\tilde{G}\subset SU(n)$. Let $l_1,\ldots l_s$ be natural numbers such that their sum is n and consider

$$\tilde{G} = S(U(l_1)\times\ldots\times U(l_s)).$$

Since finite group quotients of each of the moduli spaces $M_{\tilde{G}}(\tilde{\Sigma}, \overline{C})$, where $\tilde{G}$ varies through the above groups and $C^{(i)} = \tilde{G} \cap C^{(k)}_{\lambda^{(i)}}$, maps injectively into the reducible locus of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$, and union of their images covers the whole reducible locus, we just need to understand that all the moduli spaces has dimension at most 4 less than that of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ to understand that the reducible locus in $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ has at least codimension 4.

We immediately observe that $\dim Z_{\tilde{G}} = s - 1$, since each $U(l_i)$ contributes a $U(1)$, but the determinant equal 1 condition reduced the dimension by exactly 1. Now we consider the Lie group $G = \tilde{G}/Z \cong PSU(l_1) \times \ldots \times PSU(l_s)$, which is clearly semi-simple. Let now $\overline{C}'$ be the string of conjugacy classes of G which $\overline{C}$ projects to. This allows us to see that modulo finite group quotients, the moduli space fibers over $M_G(\tilde{\Sigma}, \overline{C}')$ with fibers $H^1(\Sigma', Z_{\tilde{G}})$. Thus we conclude by Proposition 6.1 that

$$\dim M_{\tilde{G}}(\tilde{\Sigma}, \overline{C})' = 2(g-1)\dim G + 2\dim Z_G + \sum_{i=1}^{b} \dim C^{(i)}. \tag{6.3}$$

By an elementary counting argument we observe that the quantity $\dim G + \dim Z_G$ attains its maximum when we consider $G = S(U(n-1) \times U(1)) \cong U(n-1)$, thus we have that

$$\dim G + \dim Z_G \leq n^2 - 2n + 2.$$

Comparing the dimension formulae (6.1) and (6.3), we see that the difference $\dim \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})' - \dim M_G(\tilde{\Sigma}, \overline{C})'$ must be bounded from below by $2(n^2-1-(n^2-2n+2))+2(g-2)(n^2-(n-1)^2) = 4n - 6 + 2(g-1)(2n-1)$. Here we have used the simple estimate that $\dim C^{(i)} \leq \dim C^{(k)}_{\lambda^{(i)}}$. From this we see of course that if either $n > 2$ or if $g > 2$ then this is at least 4. If $(g, n) = (2, 2)$ our estimates are simply to crude to obtain our conclusion. However, the more refined analysis from the previous section (or an easy improvement on the argument in this section) gives the result we want.

Proposition 6.2. *The real codimension of the reducible locus in $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ is at least 4 unless we consider the very special case where $(g, n) = (2, 2)$ and $b_i = 0$ as detailed in the above $SU(2)$ example in which case we have that if the number of points $p^{(i)}$ which is labeled by k is even, then the real codimension of the reducible locus is 2.*

6.3 Definition of the moduli space of parabolic bundles

We will define the moduli space $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$ using the notion of parabolic bundles.

Recall from the introduction how we write $\lambda^{(i)}$ in a specific basis for $\mathfrak{h}$ and obtain a flag type $k_1^{(i)} < k_1^{(i)} + k_2^{(i)} \cdots < k_1^{(i)} + \cdots + k_r^{(i)} = n$

Definition 6.3 (Quasi-parabolic structure)**.** A quasi-parabolic structure on a holomorphic vector bundle $E \to \Sigma$ of rank n is a choice of a filtration of its fibers over each of the points in $\mathcal{P}$

$$E|_{p^{(i)}} = E_1^{(i)} \supsetneq E_2^{(i)} \supsetneq \cdots \supsetneq E_{r^{(i)}+1}^{(i)} = \{0\}.$$

Its multiplicities are $k_j^{(i)} = \dim(E_j^{(i)}/E_{j+1}^{(i)})$.

The tuple $(k_1^{(i)}, \ldots, k_{r^{(i)}}^{(i)})$ is said to be the *flag type* at $p^{(i)}$. If all multiplicities are 1, or equivalently $r^{(i)} = N$, we say the flag at $p^{(i)}$ is full.

Definition 6.4 (Parabolic bundle)**.** A *parabolic bundle* is a vector bundle $E \to \Sigma$ with a quasi-parabolic structure, that is further equipped with *parabolic weights* $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(n)})$ for all flags. This is a choice of real numbers

$$\alpha^{(i)} = (\alpha_1^{(i)}, \ldots, \alpha_{r(i)}^{(i)}),\ 0 \leq \alpha_1^{(i)} < \cdots < \alpha_{r^{(i)}}^{(i)} < 1$$

Notice that given a weight $\lambda \in \Lambda = \{(0, \lambda_2, \ldots \lambda_n) \in \mathbb{Z}^n \mid 0 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq k\}$, we get a matrix

$$\exp(\tfrac{1}{k} i^{-1}(\lambda)) = \begin{pmatrix} e^{\frac{2\pi i}{k}(-\frac{1}{n}\sum_i \lambda_i)} & 0 & \ldots & 0 \\ 0 & e^{\frac{2\pi i}{k}(\lambda_2 - \frac{1}{n}\sum_i \lambda_i)} & \ldots & 0 \\ \vdots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & e^{\frac{2\pi i}{k}(\lambda_n - \frac{1}{n}\sum_i \lambda_i)} \end{pmatrix}$$

The numbers $\beta_j := \frac{1}{k}(\lambda_j - \frac{1}{n}\sum_i \lambda_i)$ are contained in an interval of length one 1. We do not change the matrix when we add $m \in \mathbb{Z}$ to the β_j's. Hence we can make sure $\beta_1 \geq 0$. Now for some l we might have $1 \leq \beta_i \leq 2$ for $i \geq l$. Hence we define $\alpha_i = \beta_{l+i} + m - 1 = \frac{1}{k}(\lambda_{l+i} - \frac{1}{n}\sum_j \lambda_j) + m - 1$ and $\alpha_{n-l+i} = \beta_i + m = \frac{1}{k}(\lambda_{n-l+i} - \frac{1}{n}\sum_j \lambda_j) + m$. In this way we can go back and forth between the α's and the λ's.

Recall that for any such weight $\lambda^{(i)}$ there is a unique standard parabolic subgroup $P^{(i)}$.

Definition 6.5 (Parabolic degree and slope)**.** The *parabolic degree* of E is $\mathrm{pdeg}(E) := \deg(E) + \sum_{i,j} \lambda_j^{(i)} k_j^{(i)}$, and its *slope* is

$$\mu(E) := \frac{\mathrm{pdeg}(E)}{\mathrm{rk}(E)}.$$

For E a parabolic bundle, any sub-bundle $F \leq E$ will inherit a canonical structure of a parabolic bundle. The same is true for quotient bundles. Therefore it makes sense to state the following definition

Definition 6.6 ((Semi-)stable)**.** We say a parabolic bundle E is *(semi-)stable* if for every sub-bundle F of E we have that

$$\mu(F) \underset{(=)}{<} \mu(E).$$

For E a semi-stable (but not stable) parabolic bundle with degree 0. Let E_1 denote the maximal stable subbundle of E, whose isomorphism class is uniquely determined by E. Then E/E_1 is a semi-stable bundle of lower rank, but same slope. Let E_2 be the maximal stable subbundle of E/E_1, and so on. Then define $\mathrm{gr}(E) = \oplus_i E_i$. Define E and E' to be S-equivalent if $\mathrm{gr}(E)$ is isomorphic to $\mathrm{gr}(E')$.

Hence given a weight $\lambda^{(i)}$ we get a flag-type, and can define the moduli space of S-equivalence classes of semi-stable parabolic bundles with trivial determinant, which we will denote by $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$, this is exactly the moduli space $\mathcal{M}(\Sigma_\sigma, \overline{\alpha})$ defined by [35]. By [35] there is a homeomorphism between $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$ and $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$, which is a diffeomorphism between $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})'$ and $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$.

6.4 Sobolev spaces

In each marked point there should be a chosen direction $v^{(i)}$

$$v^{(i)} \in \mathbb{P}(T_{p^{(i)}}\Sigma) := (T_{p^{(i)}}\Sigma \backslash \{0\})/\mathbb{R}_+.$$

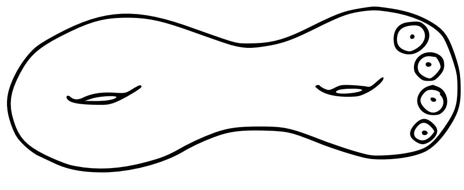


Figure 6.3: Surface with marked point and small embedded discs around each puncture, along which we compute the holonomy around each puncture in the direction induced from the orientation of the surface.

Recall that the punctured surface is denoted by $\tilde{\Sigma}$. As before Σ_σ denotes Σ equipped with a complex stucture. For each i we let $z^{(i)} \colon U^{(i)} \to D$ denote a complex analytic isomorphism with $D \subset \mathbb{C}$ the unit disk, such that $p^{(i)} = (z^{(i)})^{-1}(0)$ and $Dz^{(i)}(v^{(i)}) \in \mathbb{R}_+(\frac{d}{dx}) \subseteq \mathbb{P}(T_0 D)$. We may define new coordinates on $(U^{(i)})^* = (U^{(i)}) \backslash \{p^{(i)}\}$. Set $w^{(i)} := -\log z^{(i)}$, then $w^{(i)}$ maps $(U^{(i)})^*$ analytically to the semi-infinite cylinder $C^{(i)} = \{(\tau, \theta) \mid \tau \geq 0, 0 \leq \theta \leq 2\pi\}/(\tau, 0) \sim (\tau, 2\pi)$. Around each puncture, we find a neighborhood, conformally equivalent to the standard semi-infinite straight cylinder $S^1 \times [0, \infty)$, see Figure 6.4, and the direction $v^{(i)}$ corresponds to the line $(0,0) \times [0, \infty)$ on the cylinder in polar coordinates.

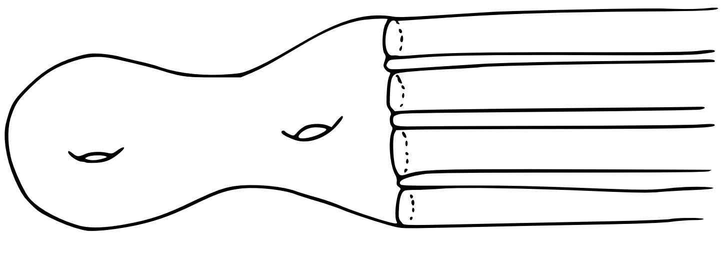


Figure 6.4: Surface with semi-infinite cylinders

Fix a metric h on $\tilde{\Sigma}$ compatible with the complex structure Σ_σ, such that it restricts to the standard flat metric on the semi-infinite ends of Σ, $h|_{U^{(i)} \backslash \{p^{(i)}\}} = d(\tau^{(i)})^2 + d(\theta^{(i)})^2$.

We are going to construct a further moduli space denoted by $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)_\sigma$. To do this we need the notion of Sobolev spaces, which we will introduce in the following section. We will follow the construction of Andersen in [1] closely.

Recall that we have our surface Σ with punctures $\mathcal{P} = \{p^{(1)}, \ldots, p^{(b)}\}$. Around each puncture we have an open neighborhood $U^{(i)}$ that we can map analytically to a semi-infinite cylinder $C^{(i)} \simeq S^1 \times [1, \infty)$.

First we will take a look at the semi-infinite ends of the surface, $C^{(i)} = S^1 \times [0, \infty)$. Over $C^{(i)}$ consider the trivial principal G-bundle $Q^{(i)}$ with connection $\nabla_0 = d + A^{(i)}$, where $A^{(i)}$ is a constant 1-form with values in $\mathfrak{g}$. in $Q^{(i)}$ over $C^{(i)}$. Let $d_{A^{(i)}}$ denote the covariant derivative in the associated adjoint bundle of $Q^{(i)}$, $\mathrm{Ad}Q^{(i)}$.

Define the $*$ operator such that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{Vol}$$

If we let $(\theta^{(i)}, r^{(i)})$ be the coordinates on the cylinder, then we have $\mathrm{Vol} = d\theta^{(i)} \wedge dr^{(i)}$. By defining

$$*d\theta^{(i)} = dr^{(i)} \,,\; *d\theta^{(i)} = -d\theta^{(i)}$$

we see that

$$d\theta^{(i)} \wedge dr^{(i)} = d\theta^{(i)} \wedge *d\theta^{(i)} = \langle d\theta^{(i)}, d\theta^{(i)} \rangle \mathrm{Vol} = \mathrm{Vol}.$$

We can calculate $*(d\theta^{(i)} \wedge dr^{(i)})$ by looking at $\alpha = \beta = d\theta^{(i)} \wedge dr^{(i)}$.

$$(d\theta^{(i)} \wedge dr^{(i)}) \wedge *(d\theta^{(i)} \wedge dr^{(i)}) = |d\theta^{(i)} \wedge dr^{(i)}|^2 d\theta^{(i)} \wedge dr^{(i)}$$
$$*(d\theta^{(i)} \wedge dr^{(i)}) = 1.$$

We can now define $d^*_{A^{(i)}} := - * d_{A^{(i)}} *$ and we get a commutative diagram

$$\begin{tikzcd}
\Omega^2(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \arrow[r, "*"] \arrow[d, "d^*_{A^{(i)}}"] & \Omega^0(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \arrow[d, "d_{A^{(i)}}"] \\
\Omega^1(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) & \Omega^1(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \arrow[l, "-*"']
\end{tikzcd} .$$

We can combine the operators $d_{A^{(i)}}$ and $d^*_{A^{(i)}}$ to get an operator

$$\tilde{\delta}_{A^{(i)}} : \Omega^0(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \oplus \Omega^2(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \to \Omega^1(C^{(i)}, \mathrm{Ad}\, Q^{(i)}).$$

Weighted Sobolev spaces on $C^{(i)}$

We are now going to construct some weighted Sobolev spaces on the cylinder $C^{(i)}$. The ideas used here will be the same, when we construct Sobolev spaces on the whole surface.

Let $\epsilon \in \mathbb{R}$ and $\psi \in \Omega^j(C^{(i)}, \mathrm{Ad} Q^{(i)})$. We define a norm

$$|\psi|^2_{\epsilon,k} := \int_{C^{(i)}} \sum_{0 \leq l \leq k} |\nabla^l (e^{\epsilon u} \psi(u, \theta))|^2 du d\theta.$$

Let $\Omega^j_{\epsilon,k}(C^{(i)}, \mathrm{Ad}\, Q^{(i)})$ denote the completion of $\Omega^j(C^{(i)}, \mathrm{Ad}\, Q^{(i)})$ in the norm $|\cdot|_{\epsilon,k}$.

Let $N^{(i)}_{A^{(i)}}$ denote the kernel of $\frac{\partial}{\partial \theta^{(i)}} + [A^{(i)}, \cdot]$

$$N^{(i)}_{A^{(i)}} = \left\{ \varphi \in \Omega^0(S^1, \mathrm{Ad}\, Q^{(i)}) \;\middle|\; \frac{\partial \varphi}{\partial \theta^{(i)}} + [A^{(i)}, \varphi] = 0 \right\}.$$

Fix a function $\rho^{(i)}$ with support in $(0, \infty)$ and constant 1 near ∞. For $j = 0, 2$ define the subspaces $\Omega^j_{N^{(i)}_{A^{(i)}}}(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \subseteq \Omega^j(C^{(i)}, \mathrm{Ad}\, Q^{(i)})$ given by

$$\Omega^j_{N^{(i)}_{A^{(i)}}}(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) := \left\{ \varphi \in \Omega^j(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \;\middle|\; \exists \varphi_\infty \in N^{(i)}_{A^{(i)}} \text{ s.t. } \varphi - \rho^{(i)} \varphi_\infty \in \Omega^j_C(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \right\}.$$

In the same way we define $\Omega^1_{N^{(i)}_{A^{(i)}}}(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \subseteq \Omega^1(C^{(i)}, \mathrm{Ad}\, Q^{(i)})$ as

$$\Omega^1_{N^{(i)}_{A^{(i)}}}(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) := \{ \varphi \in \Omega^1(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \mid \exists \varphi_\infty \in N^{(i)}_{A^{(i)}} \oplus N^{(i)}_{A^{(i)}}$$
$$\text{s.t. } \varphi - \rho^{(i)} \varphi_\infty \in \Omega^j_C(C^{(i)}, \mathrm{Ad}\, Q^{(i)}) \}.$$

On these spaces we can consider the norm

$$|\varphi|^2_{\epsilon,k,\infty} := \int_{C^{(i)}} \sum_{0\le l\le k} |\nabla^l(e^{\epsilon u}(\varphi(u,\theta) - \rho\varphi_\infty(\theta)))|^2 dud\theta + \int_{S^1} |\varphi_\infty(\theta)|^2 d\theta.$$

Let $\Omega^j_{\epsilon,k,\infty}(C^{(i)}, \operatorname{Ad} Q^{(i)})$ denote the completion of $\Omega^j_{N^{(i)}_{A^{(i)}}}(C^{(i)}, \operatorname{Ad} Q^{(i)})$ in the norm $|\cdot|_{\epsilon,k,\infty}$.

Weighted Sobolev spaces on $\tilde\Sigma$

We will now define Sobolev spaces on $\tilde\Sigma$, which locally on the ends of $\tilde\Sigma$ are equivalent to the ones we have just defined on the semi-infinite cylinders $C^{(i)}$.

Let $Q = \tilde\Sigma \times G$ be the trivial bundle. As in the beginning of Section 6 we have disk neighborhoods $U^{(i)}$ of each puncture $p^{(i)}$, $i = 1,\dots,b$. Let A' be a flat connection in Q over $\tilde\Sigma$, $[A'] \in \mathcal{M}(\tilde\Sigma, \overline{\lambda})$. By Lemma 2.7 in [22] for each puncture $p^{(i)}$ there exists a gauge-equivalent connection $A^{(i)}$ via a gauge-transformation $g^{(i)}$, such that in the chosen trivialization $A^{(i)}$ is given by $\xi^{(i)} d\theta^{(i)}$, where $\xi^{(i)} \in \mathfrak{g}$. Note that $\xi^{(i)}$ does not depend on $r^{(i)}$. Let $\tilde S^{(i)}$ be a circle around $p^{(i)}$ inside $U^{(i)}$. Let $D^{(i)}$ denote the disk with boundary $\tilde S^{(i)}$. On $\tilde S^{(i)}$ we have $g^{(i)} \colon S^1 \to G$. We have assumed that G is simply connected, so we can use a homotopy from $g^{(i)}$ to the identity, to expand $g^{(i)}$ to a gauge transformation on the cylinder, that is $g^{(i)}$ in one end and the identity in the other. Now for each i we have a gauge transformation $\tilde g^{(i)}$, that is the identity on all of $\Sigma \backslash U^{(i)}$ and $g^{(i)}$ on $D^{(i)}$. Since the $U^{(i)}$'s are disjoint, we can compose all the $\tilde g^{(i)}$'s to get a gauge transformation g. Let A be the connection gauge equivalent to A' via g. Let $S^{(i)}$ be a circle around $p^{(i)}$ inside $D^{(i)}$.

Let $A^{(i)} := \xi^{(i)} d\theta$ and

$$N_{A^{(i)}} := \left\{ \varphi \in \Omega^0(S^1, \operatorname{Ad} Q^{(i)}) \,\middle|\, \frac{\partial\varphi}{\partial\theta^{(i)}} + [A^{(i)}, \varphi] = 0 \right\}.$$

Definition of the Sobolev spaces

We want to construct the Sobolev spaces as before, but since our surface is now more complicated than a simple cylinder, we have to do it in a little more complicated way. Earlier we had u that ran from 0 to ∞ along the cylinder. Pick a Riemannian metric $\tilde g$ on $\tilde\Sigma$, such that it is equal to the standard metric on the cylinders. Now we pick a point $q \in \tilde\Sigma$, and let d be the function that measures the distance from any point $p \in \tilde\Sigma$ to q. This d will now play the role that u did before. Fix a function ρ on $\tilde\Sigma$ with support on the cylinders and such that ρ is 1 near the ends of the cylinders. Consider the spaces

$$\Omega^j_{\epsilon,k,\infty}(\tilde\Sigma, \operatorname{Ad} Q) := \Big\{ f \in \Omega^j_{L^2_{\mathrm{loc}}}(\tilde\Sigma, \operatorname{Ad} Q) | \\ \exists f_\infty \in N_{A^{(1)}} \times \cdots \times N_{A^{(b)}} \colon \sum_{0\le l\le k} \int_{\tilde\Sigma} |\nabla^l(e^{\epsilon d}(f - \rho f_\infty))|^2 + \int_{\bigcup_i S^{(i)}} |f_\infty|^2 < \infty \Big\}$$

for $j = 0, 2$. Similarly let

$$\Omega^1_{\epsilon,k,\infty}(\tilde\Sigma, \operatorname{Ad} Q) := \Big\{ f \in \Omega^1_{L^2_{\mathrm{loc}}}(\tilde\Sigma, \operatorname{Ad} Q) | \exists f^1_\infty, f^2_\infty \in N_{A_1} \times \cdots \times N_{A_n} \colon \\ \sum_{0\le l\le k} \int_{\tilde\Sigma} \left| \nabla^l(e^{\epsilon d}(f - \rho(f^1_\infty \oplus f^2_\infty))) \right|^2 + \int_{\bigcup_i S^{(i)}} |f^1_\infty \oplus f^2_\infty|^2 < \infty \Big\},$$

and

$$\Omega^j_{\epsilon,k}(\tilde{\Sigma}, \operatorname{Ad} Q) := \Big\{\varphi \in \Omega^j_{L^2_{\mathrm{loc}}}(\tilde{\Sigma}, \operatorname{Ad} Q) \,\Big|\, \sum_{0\le l\le k} \int_{\tilde{\Sigma}} |\nabla^l(e^{\epsilon d}\varphi)|^2 < \infty\Big\}.$$

Note that $\Omega^j_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \subseteq \Omega^j_{\epsilon,k}(\tilde{\Sigma}, \operatorname{Ad} Q)$, since we can choose $f_\infty = 0$.

The operator δ_A

The flat connection A in Q gives us a covariant derivative in $\operatorname{Ad} Q$ over $\tilde{\Sigma}$, and we get a complex

$$0 \to \Omega^0_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \Omega^2_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \to 0$$

with d_A the boundary map. We will denote the first cohomology group of this complex by

$$H^1_{\epsilon,k}(\tilde{\Sigma}, d_A) = \frac{\ker d_A}{\operatorname{Im} d_A}.$$

By using the Hodge-star operator associated to the metric $\tilde{g}$ on Σ we can consider $d^*_A = - * d_A *$ on $\Omega^i(\tilde{\Sigma}, \operatorname{Ad} Q)$. As before we consider

$$\tilde{\delta}_A \colon \Omega^0_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \oplus \Omega^2_{\epsilon,k+1,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{Ad} Q).$$

Similarly we can look at $d_A \colon \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q)$, and d^*_A as an operator $d^*_A \colon \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q)$. Denote the sum of these two by δ_A as in [1],

$$\delta_A \colon \Omega^1_{\epsilon,k,\infty}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q).$$

It is shown in [1] that δ_A and $\tilde{\delta}_A$ are both Fredholm for ϵ positive and sufficiently small. For ϵ sufficiently small and positive, the $\Omega^1_{\epsilon,k}$-kernel and the L_2-kernel of δ_A are the same, see [1] .

6.5 Construction of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)_\sigma$

To construct the final moduli space, we are going to use the Sobolev spaces just constructed above, see [22] for more details. Let ∇_0 denote the trivial connection in $\operatorname{Ad} Q \simeq \tilde{\Sigma} \times \mathfrak{g}$. Define a space of connections modeled on the Sobolev spaces.

$$\mathcal{A}_\epsilon := \{\nabla_0 + A \,|\, A \in \Omega^1_{\epsilon,1}(\tilde{\Sigma}, \operatorname{Ad} Q)\}.$$

Let $\mathcal{A}_{F,\epsilon}$ denote the subspace of $\mathcal{A}_\epsilon$ of flat connections, and $\mathcal{A}_{\mathrm{irr},F,\epsilon}$ the subspace of irreducible flat connections.

Define

$$\mathcal{D} := \{\varphi \in L^2_{2,\mathrm{loc.}}(\tilde{\Sigma}, \operatorname{End}(\operatorname{Ad} Q)) \,|\, \|\nabla_0\varphi\|^2_{1,\delta} < \infty\}.$$

We define the map $r \colon \mathcal{D} \to \prod_i l^{(i)}$ by $r(\varphi) = (r^{(1)}(\varphi), \ldots, r^{(b)}(\varphi))$, where $l^{(i)}$ is the space of parallel sections with respect to ∇_0 restricted to the circle $S^{(i)}$ around $p^{(i)}$ in $D^{(i)}$, see [8], and $r^{(i)}(\varphi)(\theta) = \lim_{\tau\to\infty}\varphi((w^{(i)})^{-1}(\tau, \theta))$.

We can now give the definitions

$$\mathcal{G}_\epsilon := \{\varphi \in \mathcal{D} \,|\, \varphi\varphi^* = I, \det\varphi = 1\} \quad \text{and} \quad \mathcal{G}_{0,\epsilon} := \{\varphi \in \mathcal{G}_\epsilon \,|\, r(\varphi) = I\}.$$

The two groups act on $\mathcal{A}_\epsilon$, and they both preserve $A_{F,\epsilon}$ and $\mathcal{A}_{\mathrm{irr},F,\epsilon}$. Now denote

$$\mathcal{F}(\tilde{\Sigma},\overline{\lambda},\epsilon) := \mathcal{A}_{F,\epsilon}/\mathcal{G}_{0,\epsilon} \quad \text{and} \quad \mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)_\sigma := \mathcal{A}_{F,\epsilon}/\mathcal{G}_\epsilon$$
$$\mathcal{F}(\tilde{\Sigma},\overline{\lambda},\epsilon)' := \mathcal{A}_{\mathrm{irr},F,\epsilon}/\mathcal{G}_{0,\epsilon} \quad \text{and} \quad \mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma := \mathcal{A}_{\mathrm{irr},F,\epsilon}/\mathcal{G}_\epsilon.$$

The moduli space $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma$ is the called irreducible locus of the moduli space $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)_\sigma$. It is proven by Daskalapoulos and Wentworth in [22] that the Gauge group $\mathcal{G}_{0,\epsilon}$ is connected and path connected.

Daskalapoulos and Wentworth prove in [22] that $\mathcal{F}(\tilde{\Sigma},\overline{\lambda},\epsilon)'$ and $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma$ are both smooth manifolds, and show that $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma$ naturally has a structure of an almost complex manifold.

Let $N^{(i)}$ denote the centralizer of $e^{\lambda^{(i)}}$ in $\mathrm{SU}(n)$, then $N^{(i)} = \mathrm{S}(\mathrm{U}(k_1^{(i)}) \times \cdots \times \mathrm{U}(k_{r(i)}^{(i)}))$. Hence let $d^{(i)}$ be $d^{(i)} := \dim(\mathrm{SU}(n)/N^{(i)}) = \dim C^{(k)}_{\lambda^{(i)}}$. Then we can state the theorem from [22]:

Theorem 6.7 (Thm. 3.7 [22])**.** *$\mathcal{F}(\tilde{\Sigma},\overline{\lambda},\epsilon)'$ and $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma$ are both smooth manifolds of dimensions $(2(g-1)+b)(n^2-1)$ and $2(g-1)(n^2-1)+\sum_i d^{(i)}$ respectively. $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma$ has naturally the structure of an almost complex manifold.*

In Section 3.3 in [22] Daskalapoulos and Wentworth prove that the almost complex structure coincides with the complex structure from Mehta and Seshadri, Section 2.2, [35]. For small enough ϵ the moduli space is again homeomorphic (diffeomorphic on the irreducible locus) to $\mathcal{M}(\tilde{\Sigma},\overline{\lambda})$.

6.6 Hodge theory

The following lemmas, all proven by Andersen in [1] allows us to use Hodge-theory to get a model for the tangent space of $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma$. The lemmas were proved in the setting with a closed surface that is cut along a number of disjoint closed curves which are leaves of a Strebel foliation. But none of the Lemmas or proofs use the original surface, they just use the surface $\tilde{\Sigma}$.

Lemma 6.8 ([1])**.** *Any element of $H^1_{\epsilon,k}(\tilde{\Sigma}, d_A)$ can be represented by a d_A-closed form on Σ with compact support.*

Lemma 6.9 ([1])**.** *Any element of* $\operatorname{coker}(\delta_A)$ *can be represented by an element of*

$$\Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q)$$

with compact support.

With these two lemmas we can establish the wanted Hodge-theory

Lemma 6.10 (Lemma 5.3 in [1])**.** *We have a natural isomorphism*

$$H^1_{\epsilon,k}(\tilde{\Sigma}, d_A) \simeq \ker\left(\delta_A \colon \Omega^1_{\epsilon,k}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \cdot\right)$$

Lemma 6.11 (Lemma 5.4 in [1])**.** *We have a natural isomorphism*

$$\operatorname{coker}(\delta_A) \simeq \ker\left(\tilde{\delta}_A \colon \Omega^0_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q) \oplus \Omega^2_{\epsilon,k-1}(\tilde{\Sigma}, \operatorname{Ad} Q) \to \cdot\right)$$

With these lemmas established, we get the structure of a manifold on the set $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$ and the following cohomology description of the tangent space

$$T_{[A]}\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma \simeq H^1_{\epsilon,k}(\tilde{\Sigma}, d_A),$$

where each of the elements has a unique d_A-harmonic representative.

6.7 The moduli stack of parabolic bundles $\mathfrak{B}_{\Sigma_\sigma, \overline{P}}$

As just described we are interested in looking at the moduli space of flat connections on the marked surface Σ with prescribed holonomy around each marked point, which can be identified with the moduli space of semi-stable parabolic bundles. Occasionally we choose to use moduli stacks to solve some of the problems, and then afterwards translate the solution to a solution for the moduli space. This is a simplifying tool, since when looking at the moduli stack of parabolic bundles, we can construct a map to the stack of bundles, that just forgets the parabolic structure. The same thing is not possible when looking at the moduli spaces. If attempt to constructed a map from $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$ that forgets the parabolic structure, then this map will typically only be well define on a Zariski open if we insist the map goes to the moduli space with one less marked point.

The language of stacks is big and involved, and we will not go into the details of the precise definition of a stack here. See Appendix B for a short note defining stacks.

Recall that we have a weight $\lambda^{(i)}$ defining a flag type, for each marked point $p^{(i)}$ on our smooth surface Σ. Let $\overline{P} = (P^{(1)}, \ldots, P^{(b)})$ denote the parabolic subgroups corresponding to $\overline{\lambda}$, as defined earlier. A parabolic bundle on Σ with marked points $\{p^{(1)}, \ldots, p^{(b)}\}$ is a bundle $\mathcal{E}$ with parabolic structures at $p^{(i)}$, meaning a marking $\lambda^{(i)}$ and a reduction of structure $\varphi_i \in \mathcal{E}_{p^{(i)}}/P^{(i)}$.

There is a morphism from the sub-stack of semi-stable bundles $\mathfrak{B}^{ss}_{\Sigma_\sigma, \overline{P}}$ to the moduli space of semi-stable parabolic bundles $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$, which induces an injection on the level of Picard groups [37]. Furthermore Pauly analyses in [37] precise which line bundles on the stack descends to the moduli space, something will be will recall in Chapter 9, where we also explicitly recall the structure of the Picard group for the stacks $\mathfrak{B}_{\Sigma_\sigma, \overline{P}}$ and compute its canonical bundle.

Chapter **7**

The line bundle, the mapping class group and the Teichmüller space

7.1 The Chern-Simons line bundle

To construct a line bundler over $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_{\sigma}$ we construct a co-cycle in terms of the Chern-Simons action on the cylinder $[0,1] \times \tilde{\Sigma}$. Recall $\mathcal{G}_{0,\epsilon}$ is the gauge group, $\mathcal{A}_{\epsilon}$ the space of connections and $\mathcal{A}_{F,\epsilon}$ the subspace of flat connections, as defined in Chapter 6. Let

$$\tilde{\mathcal{G}}_{0,\epsilon} := \big\{\tilde{g} \colon [0,1] \times \tilde{\Sigma} \to G \,|\, \tilde{g}(t,\cdot) \in \mathcal{G}_{0,\epsilon}, \forall t \in [0,1], \\ \text{and is continuous and piecewise smooth in } t\big\}.$$

By Proposition 3.3 in [22] $\mathcal{G}_{0,\epsilon}$ is connected and path-connected, so every gauge transformation $g \in \mathcal{G}_{0,\epsilon}$ is smoothly homotopic to the identity. Because of this we can extend $g \in \mathcal{G}_{0,\epsilon}$ to $\tilde{g} \in \tilde{\mathcal{G}}_{0,\epsilon}$ so that $\tilde{g}_0 = \tilde{g}(0,\cdot) = g$ and $\tilde{g}_1 = \tilde{g}(1,\cdot) = e$. Similarly we can use the natural projection $\pi \colon [0,1] \times \tilde{\Sigma} \to \tilde{\Sigma}$ to extend $\nabla_A = \nabla_0 + A$ on Σ^0 to $\tilde{\nabla}_A = \pi^* \nabla_A = d + \widetilde{A + A_0}$, where $\widetilde{A + A_0} = \pi^*(A + A_0)$. Then $\tilde{\nabla}_A^{\tilde{g}} \in \tilde{\mathcal{A}}_{\epsilon}$ is an extension of ∇_A^g to $[0,1] \times \tilde{\Sigma}$ and we can define a co-cycle

$$\Theta^k(\nabla_A, g) := \exp\left(-2\pi i k \mathrm{CS}_{[0,1]\times\tilde{\Sigma}}\left(\widetilde{A + A_0}^{\tilde{g}}\right)\right).$$

Daskalapoulos and Wentworth define a different co-cycle $\tilde{\Theta}^k \colon L^2_{1,\epsilon}(T^*\tilde{\Sigma} \otimes \mathfrak{g}_P) \times \mathcal{G}_{0,\epsilon} \to \mathrm{U}(1)$,

$$\tilde{\Theta}^k(\nabla_A, g) := \exp\left(\frac{ik}{4\pi}\int_{\tilde{\Sigma}} \mathrm{tr}\left(\mathrm{Ad}_{g^{-1}}(A + A_0) \wedge g^{-1}dg\right) - \frac{ik}{12\pi}\int_{[0,1]\times\tilde{\Sigma}} \mathrm{tr}(\tilde{g}^{-1}\tilde{d}\tilde{g})^3\right),$$

where $\tilde{d} = d + \frac{d}{dt}$. The two co-cycles are equal, see [8], Lemma 3.4. The later definition of the co-cycle is independent of choice of path in $\mathcal{G}_{0,\epsilon}$, see [22], Lemma 5.2. The action of $\mathcal{G}_{0,\epsilon}$ on $\mathcal{A}_{\epsilon} \times \mathbb{C}$ is given by

$$(\nabla_A, z) \cdot g := (\nabla_A^g, \Theta^k(\nabla_A, g) \cdot z). \tag{7.1}$$

Θ^k satisfies the co-cycle condition, see Lemma 3.5 [8]. So since Θ^k satisfies the co-cycle condition and $\mathcal{G}_{0,\epsilon}$ preserves flat connections, we obtain the induced Chern-Simons line bundle over $\mathcal{F}_{\epsilon}$. To get the line bundle over $\mathcal{M}_{\epsilon}$ we need some restrictions on the weights. The full result is Theorem 3.6 in [8].

Theorem 7.1 ([8]). *Suppose $\lambda^{(i)}$ is contained in the interior of $\mathcal{C}^+$ and that $k\sum_i \lambda^{(i)}$ is in the co-root lattice of* $\mathrm{SU}(n)$. *Then the line bundle on $\mathcal{F}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)$ constructed above descends to $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$. It comes naturally equipped with a connection, whose curvature is $\frac{1}{2\pi i}\omega$.*

This Theorem gives us a symplectic form, which we will denote by $\omega_{k,\overline{\lambda}}$. Remark that as the notion implies this symplectic form does not depend on σ, see [8]. Also note that this means the Chern-Simons line bundle just constructed, is a prequantum line bundle, see Definition 3.1. Denote the line bundle by $\mathcal{L}_{k,\overline{\lambda}}$.

By the co-dimension estimates in Section 6.2 and since the reducible locus corresponds to the strictly semi-stable locus, which is a complex sub-variety of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})_\sigma$, we see that the singularities of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})_\sigma$ is of at least complex co-dimension 2.

7.2 The mapping class group

On the moduli space over a closed surface, we define the mapping class group as $\Gamma(\Sigma) = \mathrm{Diff}(\Sigma)/\mathrm{Diff}_0(\Sigma)$, where $\mathrm{Diff}(\Sigma)$ is the group of orientation-preserving diffeomorphisms on Σ and $\mathrm{Diff}_0(\Sigma)$ is the subgroup of diffeomorphisms isotopic to the identity.

In the case of the punctured surface, the diffeomorphisms of the mapping class group should also preserve some information about the marked points. Let $\mathrm{Diff}_+(\Sigma, \mathcal{P}, \overline{V}, \overline{\lambda})$ be the diffeomorphisms of Σ that preserve the $(\mathcal{P}, \overline{V}, \overline{\lambda})$. Thus we allow in particular diffeomorphism which permute the marked points, but only if they do so in a way which preserved the labelling by $\overline{\lambda}$. We define the mapping class group as

$$\Gamma_{(\Sigma,\mathcal{P},V)} := \mathrm{Diff}_+(\Sigma, \mathcal{P}, \overline{V}, \overline{\lambda})/\mathrm{Diff}_0(\Sigma, \mathcal{P}, \overline{V}),$$

the group $\mathrm{Diff}_0(\Sigma, \mathcal{P}, \overline{V})$ contains diffeomorphism which preserves each $(p^{(i)}, v^{(i)})$ and which is isotopic to the identity among such.

By [8] this mapping class group is isomorphic to a mapping class group, where a neighborhood around each marked point is asked to be preserved. This is mapping class group is defined in the following way.

Let $\mathrm{Diff}_+(\tilde{\Sigma}, \overline{z}, \overline{\lambda})$ be the diffeomorphisms of $\tilde{\Sigma}$ that preserve the chosen local coordinates around each puncture, only permuting those with equal weights, so $z^{(j)} = f \circ z^{(i)}$ if $f(p^{(i)}) = p^{(j)}$. In this way the direction $v^{(i)}$ in a marked point $p^{(i)}$ is preserved.

Note that $\mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda})$ acts by pullback on $\mathcal{A}_\epsilon$, since by construction the weights used in the Sobolev norms are preserved. We lift this action to the trivial line bundle $\mathcal{A}_\epsilon \times \mathbb{C}$ by

$$f^*(\nabla_A, z) := (f^*\nabla_A, z) \tag{7.2}$$

for $f \in \mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda})$. Define a morphism $\Psi\colon \mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda}) \to \mathrm{Aut}(\mathcal{G}_{0,\epsilon})$ by $\Psi(f)(g) := g \circ f$. By Lemma 3.7 in [8] the two lifts (7.1) and (7.2) combine to an action of $\mathcal{G}_{0,\epsilon} \rtimes_\Psi \mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda})$ on $\mathcal{A}_\epsilon \times \mathbb{C}$. This implies that the action of $\mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda})$ on $\mathcal{A}_\epsilon \times \mathbb{C}$ descends to an action on the Chern-Simons line bundle over $\mathcal{F}_\epsilon$.

Let $\mathrm{Diff}_0(\Sigma, \overline{z}, \overline{\lambda})$ be the diffeomorphisms in $\mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda})$ that are isotopic to the identity, through an isotopy that preserves the neighborhoods $D^{(i)}$. Then the mapping class group defined as

$$\Gamma_{(\Sigma,\mathcal{P},V)} \cong \mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda})/\mathrm{Diff}_0(\Sigma, \overline{z}, \overline{\lambda}).$$

is isomorphic to $\Gamma_{(\Sigma,\mathcal{P},V)}$, see [8].

Suppose we have an isotopy f_t given with $f_0 = \mathrm{id}$ and $f_1 = f$. The action of f_t on $\nabla_A \in \mathcal{A}_\epsilon$ can be understood as a gauge transformation, if we let $g_t(p)$ be the holonomy of ∇_A along the path $s \mapsto f_{(1-t)(1-s)}(p)$. Since $f \in \mathrm{Diff}_0(\Sigma, \overline{z}, \overline{\lambda})$, it will preserve local coordinates around the punctures, so g_t will be constant near punctures, hence $g_t \in \mathcal{G}_{0,\epsilon}$ for all t. This shows that $\mathrm{Diff}_0(\Sigma, \overline{z}, \overline{\lambda})$ acts trivially on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$, which means we get an action of $\mathrm{Diff}_+(\Sigma, \overline{z}, \overline{\lambda})/\mathrm{Diff}_0(\Sigma, \overline{z}, \overline{\lambda})$ on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$. It is proven in [8] that for ∇_A and g_t as above, we have $\Theta^k(\nabla_A, g_0) = 1$, which means $\mathrm{Diff}_0(\Sigma, \overline{z}, \overline{\lambda})$ acts trivially on the Chern-Simons line bundle, and hence the action of the mapping class group lifts to an action on the Chern-Simons line bundle:

Proposition 7.2 ([8])**.** *We have an induced action of the mapping class group $\Gamma_{(\Sigma,\mathcal{P},V)}$ on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$ with a lift to $\mathcal{L}_{k,\overline{\lambda}}$.*

7.3 Kähler structure parametrized by the Teichmüller space

For a closed surface Σ the Teichmüller space is defined as $\mathcal{T}(\Sigma) := \mathcal{C}(\Sigma)/\mathrm{Diff}_0(\Sigma)$, where $\mathcal{C}(\Sigma)$ is the space of conformal structures on Σ.

In general we define the Teichmüller space for $(\Sigma, \mathcal{P}, \overline{V})$ as follows.

Definition 7.3 (Teichmüller space)**.** The Teichmüller space is defined as

$$\mathcal{T}_{(\Sigma,\mathcal{P},V)} := \mathcal{C}(\Sigma)/\mathrm{Diff}_0(\Sigma, \mathcal{P}, \overline{V}).$$

By the above discussion we see that the Teichmüller space parametrizes Kähler structures on the moduli space.

Lemma 7.4 ([8])**.** *The Teichmüller space $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$ parametrizes Kähler structures on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ in a $\Gamma_{(\Sigma,\mathcal{P},V)}$-equivariant way.*

Proof. Suppose $\tilde{g}$ is a metric on $\tilde{\Sigma}$ with the properties specified in Section 6.4 which further represents a point σ in $\mathcal{C}(\Sigma)$. By the construction of the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$ in Section 6.5 and the Hodge-theory in Section 6.6 we get that the tangent space of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$ can be identified with $\ker(d_A^* + d_A)$. Harmonicity is preserved by $*$, and $*^2 = -1$, so we get the wanted almost complex structure on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$. That this almost complex structure is integrable comes from using Theorem 3.8 and Theorem 3.13 in [22], that says there is a diffeomorphism between $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$ and $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})'$, and that the almost complex structures are equivalent, hence by Mehta-Seshadri, the almost complex structure is integrable. Using further the identification

$$\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})' = \mathcal{M}(\tilde{\Sigma}, \overline{\lambda})',$$

we get a complex structure I_σ on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$ and by using Hodge-theory we see that the complex structure I_σ is compatible with the symplectic structure ω on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$. Hence we get a map

$$I \colon \mathcal{C}(\Sigma) \to C^\infty(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', \mathrm{End}(T\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'))$$

such that (M, I_σ, ω) is Kähler for all $\sigma \in \mathcal{C}(\Sigma)$. The group of diffeomorphisms $\mathrm{Diff}_+(\Sigma, \mathcal{P}, \overline{V}, \overline{\lambda})$ acts on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma$ via pullback, this induces an action on $C^\infty(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma, \mathrm{End}(T\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma))$. The map I is equivariant with respect to this action. The group $\mathrm{Diff}_0(\Sigma, \mathcal{P}, \overline{V})$ acts trivially on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'$, so we obtain a map

$$\mathcal{T}_{(\Sigma,\mathcal{P},V)} \to C^\infty(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', \mathrm{End}(T\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})')),$$

such that (M, I_σ, ω) is Kähler for all $\sigma \in \mathcal{T}_{(\Sigma,\mathcal{P},V)}$, hence we finally see that $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$ parametrizes Kähler structures on $(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', \omega)$ in a $\Gamma_{(\Sigma,\mathcal{P},V)}$-equivariant way. □

Chapter **8**

The first Chern class of the moduli stack of bundles

Let Σ be a smooth surface. Assume the genus of Σ is $g \geq 2$. Let G be a simple and simply connected Lie group. Let $\mathfrak{M}_{\Sigma_\sigma}$ be the moduli stack of $G^{\mathbb{C}}$-bundles over Σ_σ. The aim of this chapter is to calculate the first Chern class of $K_{\mathfrak{M}_{\Sigma_\sigma}}$. We know that $c_1(K_{\mathfrak{M}_{\Sigma_\sigma}}) = -c_1(T\mathfrak{M}_{\Sigma_\sigma})$, so the focus will be on calculating $c_1(T\mathfrak{M}_{\Sigma_\sigma})$. To do this we rely on the work of Teleman and Woodward in [41].

Let L denote the determinant bundle over $\mathfrak{M}_{\Sigma_\sigma}$. Let E' be a universal $G^{\mathbb{C}}$-bundle over $\mathfrak{M}_{\Sigma_\sigma} \times \Sigma$. E' has the property that for any $m \in \mathfrak{M}_{\Sigma_\sigma}$ the bundle $E'_m \to \Sigma \times \{m\} \simeq \Sigma$ satisfies $[E'_m] = m$. Now note that for any line bundle $F \to \mathfrak{M}_{\Sigma_\sigma}$ the bundle $E' \otimes \pi^* F \to \Sigma \times \mathfrak{M}_{\Sigma_\sigma}$ will have the same universal property. The first Chern class of E' is an element in $H^2(\Sigma \times \mathfrak{M}_{\Sigma_\sigma})$, and via the Künneth Formula we can see it as an element $\in H^0(\Sigma) \otimes H^2(\mathfrak{M}_{\Sigma_\sigma}) \oplus H^2(\Sigma) \otimes H^0(\mathfrak{M}_{\Sigma_\sigma})$. Let c''_1 denote the part in $H^0(\Sigma) \otimes H^2(\mathfrak{M}_{\Sigma_\sigma})$ and c'_1 the part in $H^2(\Sigma) \otimes H^0(\mathfrak{M}_{\Sigma_\sigma})$. Let F be the dual line bundle to the line bundle with first Chern class c''_1. Then $c''_1(E' \otimes \pi^* F) = 0$. Let $E := E' \times F$.

For a $G^{\mathbb{C}}$-representation V we denote the associated bundle $E(V)$.

We are going to calculate the first Chern class of $T\mathfrak{M}_{\Sigma_\sigma}$, using the Grothendieck-Riemann-Roch Theorem, which we will now state.

Theorem 8.1 (The Grothendieck-Riemann-Roch theorem)**.** *Let X be a smooth quasi projective scheme over the complex numbers. Let $K_0(X)$ denote the Grothendieck group of X. We can consider the Chern character as a functorial transformation*

$$\mathrm{ch}\colon K_0(X) \to H^*(X, \mathbb{Q})$$

Let $f\colon X \to Y$ be a proper morphism. Then the Grothendieck-Riemann-Roch theorem relates the push-forward map

$$f_! = \sum_i (-1)^i R^i f_* \colon K_0(X) \to K_0(Y)$$

with the push-forward

$$f_* \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}),$$

by

$$\mathrm{ch}(f_! \mathcal{F}^\bullet) = f_*(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{Td}(T_f))$$

where T_f is the relative tangent sheaf.

Teleman and Woodward (see [41]) use this version of the Grothendieck-Riemann-Roch Theorem on stacks, hence for a $f\colon \mathfrak{M} \to \mathfrak{N}$ between two stacks, the Theorem relates a push-forward map $f_!$, taking a vector bundle over $\mathfrak{M}$ to a vector bundle over $\mathfrak{N}$, with the push-forward $f_*\colon H^\bullet(\mathfrak{M}) \to H^\bullet(\mathfrak{N})$, by

$$\mathrm{ch}(f_!\mathcal{F}^\bullet) = f_*(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{Td}(T_f)),$$

where T_f is the relative tangent sheaf.

Using this we will prove the following:

Lemma 8.2 ([6]). *The first Chern class $c_1(T\mathfrak{M}_{\Sigma_\sigma})$ of the tangent bundle $T\mathfrak{M}_{\Sigma_\sigma}$ of the stack of bundles is*

$$c_1(T\mathfrak{M}_{\Sigma_\sigma}) = 2\check{h}\chi,$$

where $\chi := c_2(E) \cap [\Sigma]$ is a generator of $H^2(\mathfrak{M}_{\Sigma_\sigma})$ and $\check{h}$ is the dual Coxeter number.

Note that this was also proven in [32], but the proof is very different.

Proof. Define the K-theory class

$$\Omega := R(\pi_{\mathfrak{M}_{\Sigma_\sigma}})_*(E(\mathfrak{g}) \otimes K).$$

On the level of K-theory $\Omega[1]$ is the cotangent bundle to $\mathfrak{M}_{\Sigma_\sigma}$, see Andersen, Gukov and Pei [6]. Using this we will calculate the first Chern class of Ω, and hence find the first Chern class of $T\mathfrak{M}_{\Sigma_\sigma}$.

Let $T_{\pi_{\mathfrak{M}_{\Sigma_\sigma}}}$ be the relative tangent sheaf along the projection $\pi_{\mathfrak{M}_{\Sigma_\sigma}}: \mathfrak{M}_{\Sigma_\sigma} \times \Sigma \to \mathfrak{M}_{\Sigma_\sigma}$. Let x^* be the generator of $H^2(\Sigma)$ Poincare dual to a point $x \in \Sigma$, so $x^* \cap [\Sigma] = 1$. To calculate the Chern character of Ω we start by using the Grothendiek-Riemann-Roch theorem

$$\begin{aligned}
\mathrm{ch}(\Omega) &= \pi_{\mathfrak{M}_{\Sigma_\sigma}*}\Big(\mathrm{ch}(E(\mathfrak{g}) \otimes K) \cup \mathrm{Td}(T_{\pi_{\mathfrak{M}_{\Sigma_\sigma}}})\Big) \\
&= \mathrm{ch}(E(\mathfrak{g}) \otimes K) \cup \mathrm{Td}(T_{\pi_{\mathfrak{M}_{\Sigma_\sigma}}}) \cap [\Sigma] \\
&= \mathrm{ch}(E(\mathfrak{g})) \cup \mathrm{ch}(K) \cup \mathrm{Td}(T_{\pi_{\mathfrak{M}_{\Sigma_\sigma}}}) \cap [\Sigma]
\end{aligned}$$

We now use that for a line bundle $\mathrm{ch}(L) = \sum_n \frac{c_1(L)^n}{n!}$ and for a vector bundle the Todd class is $\mathrm{Td}(L) = 1 + \frac{1}{2}c_1(L) + \dots$

$$= \mathrm{ch}(E(\mathfrak{g})) \cup (1 + c_1(K)) \cup (1 + \frac{1}{2}c_1(T_{\pi_{\mathfrak{M}_{\Sigma_\sigma}}})) \cap [\Sigma]$$

Using that $c_1(T_{\pi_{\mathfrak{M}_{\Sigma_\sigma}}}) = (2-2g)x^*$ we see

$$= \mathrm{ch}(E(\mathfrak{g})) \cup \Big(1 + c_1(K)\Big) \cup \Big(1 + \frac{1}{2}(2-2g)x^*\Big) \cap [\Sigma]$$

and then we use that $c_1(K) = (2-2g)x^*$ hence

$$\begin{aligned}
&= \mathrm{ch}(E(\mathfrak{g})) \cup \Big(1 + (2-2g)x^*\Big) \cup \Big(1 + \frac{1}{2}(2-2g)x^*\Big) \cap [\Sigma] \\
&= \mathrm{ch}(E(\mathfrak{g})) \cup \Big(1 + \frac{1}{2}(2-2g)x^* + (2-2g)x^* + \frac{1}{2}(2-2g)^2(x^*)^2\Big) \cap [\Sigma] \\
&= \mathrm{ch}(E(\mathfrak{g})) \cup (1 + 3(1-g)x^*) \cap [\Sigma]
\end{aligned}$$

Hence

$$\mathrm{ch}(T\mathfrak{M}_{\Sigma_\sigma}) = -\mathrm{ch}(\Omega) = \mathrm{ch}(E(\mathfrak{g})) \cup (-1 - 3(1-g)x^*) \cap [\Sigma]$$

To find $c_1(T\mathfrak{M}_{\Sigma_\sigma})$ we write

$$\mathrm{ch}(E(\mathfrak{g})) = \mathrm{rk}(E(\mathfrak{g})) + c_1(E(\mathfrak{g})) + \frac{1}{2}(c_1(E(\mathfrak{g}))^2 - 2c_2(E(\mathfrak{g}))) + \dots$$

To see what part of this product is in $H^2(\mathfrak{M}_{\Sigma_\sigma})$ we write $H^n(\Sigma \times \mathfrak{M}_{\Sigma_\sigma})$ as a sum using the Künneth formula, then remember that $H^n(\Sigma) = 0$ for $n \geq 3$, $\cup x^* \colon H^n(\Sigma) \to H^{n+2}(\Sigma)$ and $\cap[\Sigma] \colon H^n(\Sigma) \to H^{n-2}(\Sigma)$. We have $c_1(E(\mathfrak{g})) = c_1' + c_1''$ where $c_1' \in H^2(\Sigma) \otimes H^0(\mathfrak{M}_{\Sigma_\sigma})$ and $c_1'' \in H^0(\Sigma) \otimes H^2(\mathfrak{M}_{\Sigma_\sigma})$. We note that all $G^{\mathbb{C}}$-bundles over Σ are trivial since $G^{\mathbb{C}}$ is connected and simply connected, which means $c_1' = 0$. Hence $c_1(E(\mathfrak{g})) \in H^0(\Sigma) \otimes H^2(\mathfrak{M}_{\Sigma_\sigma})$. When constructing E we also made sure that $c_1'' = 0$, so $c_1(E(\mathfrak{g})) \cup x^* \cap [\Sigma] = 0$. Hence the only non-zero part of $\mathrm{ch}(T\mathfrak{M}_{\Sigma_\sigma})$ in $H^2(\mathfrak{M}_{\Sigma_\sigma})$ is

$$c_1(T\mathfrak{M}_{\Sigma_\sigma}) = c_2(E(\mathfrak{g})) \cap [\Sigma] \in H^2(\mathfrak{M}_{\Sigma_\sigma})$$

From [14] we know that $c_2(E) \cap [\Sigma]$ generates $H^2(\mathfrak{M}_{\Sigma_\sigma})$. From Equation (8.3) in [7] we get

$$c_2(E(\mathfrak{g})) = 2\check{h}c_2(E),$$

where $\check{h}$ is the dual Coxeter number. So we conclude

$$c_1(T\mathfrak{M}_{\Sigma_\sigma}) = 2\check{h}c_2(E) \cap [\Sigma] = 2\check{h}\chi.$$

□

Chapter 9

The canonical bundle of the moduli stack of parabolic bundles

Recall that Σ is a surface with genus $g \geq 2$. It has marked points $\mathcal{P} = \{p^{(1)}, \ldots, p^{(b)}\}$ each with a weight $\lambda^{(i)} \in \Lambda$. Let $\mathfrak{M}_{\Sigma_\sigma}$ denote the moduli stack of bundles, let $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ denote the moduli stack of parabolic bundles with reduction of structure group to $P^{(i)}$, where $P^{(i)}$ is the parabolic subgroup corresponding to $\lambda^{(i)}$. We wish to describe the canonical bundle of $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$.

Let G be a semi-simple and simply connected compact group. In this chapter we will only use the complexification $G^{\mathbb{C}}$. Let M be a complex manifold. In Section 9.2 we will find the canonical bundle of Q/P, $Q \to M$ a holomorphic principal $G^{\mathbb{C}}$-bundle, and $P \subset G^{\mathbb{C}}$ a parabolic subgroup of $G^{\mathbb{C}}$. We will prove that

$$K_{Q/P} \simeq \mathcal{L} \otimes \pi^* K_M,$$

where $\mathcal{L}$ is a line bundle that depends on the parabolic subgroup P. We will use this to prove that the canonical bundle of $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ is the tensor product of a line bundle for each marked point and the canonical bundle of $\mathfrak{M}_{\Sigma_\sigma}$ pull back to $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$. The idea is to see $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ as the fibered product, and then remove the parabolic structure one marked point at a time.

To do this we need to prove Equation 9, which is done in Section 9.2, and to do so we need some basic knowledge about Lie algebras and Lie groups, this can be found in Section 9.1. In Section 9.3, we go through how to use Equation 9 in the case of the moduli stack of parabolic bundles. We then use this to conclude when the canonical bundle of the stack of parabolic bundles has a square root. Finally we restrict to the case $\mathrm{SL}(n, \mathbb{C})$ in Section 9.4.

9.1 Lie algebras and Lie groups

Let $G^{\mathbb{C}}$ be a Lie group and $\mathfrak{g}$ its Lie algebra. A torus in a complex Lie group $G^{\mathbb{C}}$ is a compact, connected, abelian Lie subgroup of $G^{\mathbb{C}}$. We will let T denote a maximal torus of $G^{\mathbb{C}}$. Let $\mathcal{X}(T)$ denote the character group of T. The Lie group $G^{\mathbb{C}}$ acts on its Lie algebra $\mathfrak{g}$, hence we can write $\mathfrak{g}$ as a sum

$$\mathfrak{g} = \oplus_{\lambda \in \mathcal{X}(T)} \mathfrak{g}_\lambda,$$

where

$$\mathfrak{g}_\lambda := \{x \in \mathfrak{g} \mid tx = \lambda(t)x \text{ for all } t \in T\}.$$

The character λ is called a root of $G^{\mathbb{C}}$ if $\lambda \neq 0$ and $g_\lambda \neq 0$. The set of roots is denoted by R.

Given a set of roots, one can always choose a set of positive roots, which we will denote by R^+. A set of positive roots R^+ is a subset of R such that for each $\alpha \in R$ exactly one of α and $-\alpha$ is in the subset $R^+ \subseteq R$, and if the sum of two elements α and β in R^+ is a root, then $\alpha + \beta$ is also in R^+. Note that this can be chosen in many different ways. The other roots are the negative roots, and will be denoted R^-. A positive root that cannot be described as the sum of two positive roots, is called a simple root. We will denote the simple roots by

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}.$$

For each root we can define a corresponding co-root as

$$\check{\alpha} := \frac{2}{\langle \alpha, \alpha \rangle} \alpha.$$

The co-roots also form a root system, which we will call the co-root system and denote by $\check{R}$. If $\alpha_1, \ldots, \alpha_n$ are the simple roots for R, then $\check{\alpha}_1, \ldots, \check{\alpha}_n$ are the simple roots for $\check{R}$.

Fundamental weights

We define the weight lattice of the Lie group $G^{\mathbb{C}}$ to be

$$\mathcal{X} = \mathrm{Hom}_{\text{alg.gp.}}(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$$

One can see that this is exactly the character group of the maximal torus T. Hence

$$\mathcal{X} = \mathcal{X}(T).$$

Let $Y := \mathrm{Hom}_{\text{alg.gp.}}(\mathbb{C}^*, T)$. When we assume $G^{\mathbb{C}}$ is simply connected, the $\mathbb{Z}$-span $\mathbb{Z}R$ of the root system R is a proper subset of $\mathcal{X}$. The $\mathbb{Z}$-span $\mathbb{Z}\check{R}$ of the co-root system $\check{R}$ is equal to Y. This means there exists $\omega_1, \ldots, \omega_n \in \mathcal{X}$ such that

$$\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$$

We call $\omega_1, \ldots, \omega_n$ the fundamental weights. Obviously these span $\mathcal{X}$.

We can now write the Lie algebra as the sum

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} g_\alpha$$

where $\mathfrak{h}$ is the called the Cartan sub-algebra, which is equal to $\mathfrak{g}_0$.

The roots of parabolic subgroups

We are interested in looking at parabolic subgroups. The smallest parabolic subgroup of a Lie group $G^{\mathbb{C}}$ is called a Borel subgroup. It is defined as follows

Definition 9.1 (Borel subgroup)**.** $B \subset G^{\mathbb{C}}$ is called a *Borel subgroup* if it is a maximal closed and connected solvable subgroup.

A parabolic subgroup can then be defined as

Definition 9.2 (Parabolic subgroup)**.** A subgroup $P \subset G^{\mathbb{C}}$ containing B is called *parabolic*. Hence the Borel subgroup is the smallest parabolic subgroup.

For P a parabolic subgroup of G, P can be decomposed as

$$P = U \rtimes L$$

where U is unipotent and L is the Levi factor, see [30]. L is reductive, and hence we can calculate the roots of L by making a decomposition

$$\mathfrak{l} = \oplus_{\lambda \in \mathcal{X}(T)} \mathfrak{l}_\lambda,$$

where we note that the maximal torus of L is the same as the maximal torus of $G^{\mathbb{C}}$. The roots of L are also roots of $G^{\mathbb{C}}$, and therefore spanned by a number of the simple roots of G, $I \subset \{\alpha_1, \ldots, \alpha_n\}$. We define the root system of P to be the roots of L, so

$$R(P) = R(L) = \mathbb{Z}I \cap R.$$

Sometimes we will write P_I to emphasize the simple roots determining P.

The Lie algebras of parabolic subgroups

For a root $\alpha \in R$ we have a map

$$X_\alpha \colon \mathbb{C} \to G^{\mathbb{C}}$$

defined by

$$tX_\alpha(a)t^{-1} = X_\alpha(\alpha(t)a) \text{ for all } t \in T.$$

It is well known that we can see the Borel group as a span of these maps X_α for all the negative roots $\alpha \in R^-$ and the maximal torus T. Then

$$\mathfrak{b} = \mathfrak{h} \oplus \oplus_{\alpha \in R^-} \mathfrak{g}_\alpha.$$

Every parabolic subgroup P_I is then spanned by B and the maps X_α for $\alpha \in R(P)$, hence

$$\mathfrak{p} = \mathfrak{h} \oplus \oplus_{\alpha \in R^- \cup R(P)} \mathfrak{g}_\alpha.$$

The roots of $G^{\mathbb{C}}$ that are not roots of P is the set $R^+ \backslash R(P)$, we will denote this by $R(\mathfrak{g}/\mathfrak{p})$. Hence we can write $\mathfrak{g}/\mathfrak{p}$ as the sum of $\mathfrak{g}_\alpha$'s of all the αs in $R(\mathfrak{g}/\mathfrak{p})$,

$$\mathfrak{g}/\mathfrak{p} \simeq \oplus_{\alpha \in R(\mathfrak{g}/\mathfrak{p})} \mathfrak{g}_\alpha.$$

We now know almost all we need about Lie groups and Lie algebras, before proving the theorem about the canonical bundle of $Q/P \to M$. But we need a simple result about the fiber-wise tangent bundle. When we have a principal $G^{\mathbb{C}}$ bundle $Q \to M$ we have a short exact sequence,

$$0 \to T_{Q/M} \to TQ \to \pi^* TM \to 0,$$

where $T_{Q/M}$ is the fiber-wise tangent bundle. Equivalently one can look at the bundle $Q/P \to M$, and find the fiber-wise tangent bundle here

$$0 \to T_{(Q/P)/M} \to T(Q/P) \to \pi^* TM \to 0.$$

Lemma 9.3. *Let $Q \to M$ be a principal $G^{\mathbb{C}}$-bundle for G a Lie group. Let P be a parabolic subgroup of $G^{\mathbb{C}}$. Then the fiber-wise tangent bundle of $Q/P \to M$ is*

$$T_{(Q/P)/M} \simeq (Q \times \mathfrak{g}/\mathfrak{p})/P.$$

Proof. We know that for a Lie group $G^{\mathbb{C}}$ the tangent bundle is $TG^{\mathbb{C}} \simeq G^{\mathbb{C}} \times \mathfrak{g}$. Equivalently we see that for a principal $G^{\mathbb{C}}$ bundle $Q \to M$, the fiber-wise tangent bundle $T_{Q/M}$ is $T_{Q/M} \simeq Q \times \mathfrak{g}$, by the map $\mathfrak{g} \to \mathfrak{X}(Q)$, $x \mapsto \xi_x(q) = \frac{d}{dt}\big|_{t=0} q g_t$.

We will construct a map from $Q \times \mathfrak{g}/\mathfrak{p}$ to $T(Q/P)$, $(q, A) \mapsto \xi(q, A)$ such that $\pi_*(\xi(q, A))$ is zero. And then prove that $(q, A) \cdot p = (q \cdot p, \mathrm{Ad}(p^{-1})A)$ is mapped to the same element of $T(Q/P)$. Afterwards we prove that the map from the quotient is injective. Hence we prove that we have a short exact sequence

$$0 \to (Q \times \mathfrak{g}/\mathfrak{p})/P \to T(Q/P) \to \pi^* TM \to 0.$$

Which means that by definition of the fiber-wise tangent bundle, we have

$$T_{(Q/P)/M} \simeq (Q \times \mathfrak{g}/\mathfrak{p})/P.$$

Define the map $\xi\colon Q \times \mathfrak{g}/\mathfrak{p} \to T(Q/P)$ to be $\xi(q, A) = \frac{d}{dt}\big|_{t=0}[q g_t]$, where g_t is a map in Q such that $g_0 = e$ and $\frac{d}{dt}\big|_{t=0} g_t = A$, and $[\cdot]$ means we have an equivalence class in Q/P.

The element $(q, A) \cdot p$ is mapped to $\xi((q, A) \cdot p) = \frac{d}{dt}\big|_{t=0}[q \cdot p\tilde{g}_t]$, where we can use $p^{-1} g_t p$ as $\tilde{g}_t$ since $\frac{d}{dt}\big|_{t=0} p^{-1} g_t p = \mathrm{Ad}(p^{-1})A$. Hence

$$\xi((q, A) \cdot p) = \frac{d}{dt}\Big|_{t=0}[q \cdot pp^{-1} g_t p] = \frac{d}{dt}\Big|_{t=0}[q \cdot g_t] = \xi(p, A).$$

We see that we have a well-defined map $(Q \times \mathfrak{g}/\mathfrak{p})/P \to T(Q/P)$. Now we want to prove that this map is injective. Let $(q, A), (\tilde{q}, \tilde{A}) \in Q \times \mathfrak{g}/\mathfrak{p}$ and assume $\xi(q, A) = \xi(\tilde{q}, \tilde{A})$. Then $T_{[q]}Q/P = T_{[\tilde{q}]}Q/P$ so $[q] = [\tilde{q}]$. Hence there exists a $p \in P$ such that $q = \tilde{q} \cdot p$.

$$\frac{d}{dt}\Big|_{t=0}[\tilde{q} \cdot \tilde{g}_t] = \frac{d}{dt}\Big|_{t=0}[q \cdot p^{-1}\tilde{g}_t] = \frac{d}{dt}\Big|_{t=0}[q \cdot p^{-1}\tilde{g}_t p],$$

hence

$$A = \frac{d}{dt}\Big|_{t=0}[g_t] = \frac{d}{dt}\Big|_{t=0}[p^{-1}\tilde{g}_t p] = \mathrm{Ad}(p^{-1})\tilde{A}.$$

Which means the map $(Q \times \mathfrak{g}/\mathfrak{p})/P \to T(Q/P)$ is injective. Hence we have proven the Lemma. ☐

9.2 Canonical bundle of $Q/P \to M$

We will prove the following theorem about the canonical bundle of B.

Theorem 9.4. *Let $G^{\mathbb{C}}$ be a semi simple and simply connected Lie group. Let R denote the set of roots of $G^{\mathbb{C}}$. Let $P_I \subset G^{\mathbb{C}}$ be a parabolic subgroup corresponding to the simple roots $I \subset R$. Let $Q \to M$ be a principal $G^{\mathbb{C}}$-bundle. Let $B = Q/P$. Then*

$$K_B \simeq \mathfrak{L}_\kappa \otimes \pi^* K_M,$$

where $\mathfrak{L}_\kappa$ is the element of the Picard group $\mathrm{Pic}(G^{\mathbb{C}}/P)$ *with $\kappa = -\sum_{\alpha \in R(\mathfrak{g}/\mathfrak{p})} \alpha$.*

Proof. From the principal $G^{\mathbb{C}}$-bundle $Q \to M$ we get a short exact sequence of bundles over Q, where $T_{Q/M}$ denotes the fiber-wise tangent bundle

$$0 \to T_{Q/M} \to TQ \to \pi^* TM \to 0.$$

Modding out by P reveals a short exact sequence

$$0 \to T_{(Q/P)/M} \to TB \to \pi^* TM \to 0.$$

It follows that the canonical bundle of B is

$$K_B = \bigwedge^{\text{top}} T^* B = \bigwedge^{\text{top}} T^*_{(Q/P)/M} \otimes \pi^* K_M.$$

We saw in Lemma 9.3 that

$$T_{(Q/P)/M} \simeq (Q \times \mathfrak{g}/\mathfrak{p})/P.$$

From Equation 9.1 we know that

$$\mathfrak{g}/\mathfrak{p} \simeq \oplus_{\alpha \in R(\mathfrak{g}/\mathfrak{p})} \mathfrak{g}_\alpha.$$

Let $\pi_{Q/P}$ denote the projection map $\pi_{Q/P} \colon Q \to Q/P$. Then

$$Q \times \mathfrak{g}/\mathfrak{p} \simeq_P \pi^*_{Q/P} T_{(Q/P)/M}$$

Taking the top exterior power we get

$$Q \times \bigwedge^{\text{top}} (\oplus_{\alpha \in R(\mathfrak{g}/\mathfrak{p})} R_\alpha) \simeq_P \pi^*_{Q/P} \bigwedge^{\text{top}} T_{(Q/P)/M}.$$

We recognize this as the line bundle $\mathfrak{L}_\kappa \to Q/P$ where $\kappa = -\sum_{\alpha \in R(\mathfrak{g}/\mathfrak{p})} \alpha \in \mathcal{X}(P)$ is an element of the character group of P. Hence we have proven the theorem. □

Remark 9.5. Note the Picard group of $G^{\mathbb{C}}/P$ is isomorphic to the character group of P, see [31] Corollary 3.3.

$$\operatorname{Pic}(G^{\mathbb{C}}/P) \simeq \mathcal{X}(P).$$

We want to know when the line bundle has a square root. This is fairly simple, using results which can be found in [30]. We prove the following Lemma

Lemma 9.6. *Let $\mathfrak{L}_\kappa$ be an element of $\operatorname{Pic}(G^{\mathbb{C}}/P)$. It corresponds to an element κ in the weight lattice $\mathcal{X}$. Then $\mathfrak{L}_\kappa$ has a square root in $\operatorname{Pic}(G^{\mathbb{C}}/P)$ when κ is divisible by two in the character group of P, denoted by $\mathcal{X}(P_I)$.*

Note that this means $\langle \kappa, \check{\alpha} \rangle$ has to be even for all simple roots, and has to be zero for the simple roots in I.

Proof. To prove the Lemma we need the remark and that the character group is

$$\mathcal{X}(P_I) \simeq \{\lambda \in \mathcal{X}(T) \mid \langle \lambda, \check{\alpha} \rangle = 0 \,\forall\, \alpha \in I\},$$

where T is the maximal torus in P, but since $B \subset P$ this is the same as the maximal torus of $G^{\mathbb{C}}$, see [30] on page 169. In conclusion we have

$$\operatorname{Pic}(G^{\mathbb{C}}/P) \simeq \{\lambda \in \mathcal{X}(T) \mid \langle \lambda, \check{\alpha} \rangle = 0 \,\forall\, \alpha \in I\}.$$

The character group of P, where κ lives, is a subset of the weight lattice $\mathcal{X}$, which is spanned by the fundamental weights $\omega_1, \dots, \omega_n$, hence there exists $a_1, \dots, a_n$ such that

$$\kappa = a_1 \omega_1 + \cdots + a_n \omega_n.$$

Then κ is even in $\mathcal{X}$ when all the a_is are even. If κ is even in $\mathcal{X}$, one wants to know wether $\frac{1}{2}\kappa$ which we will denote by $\nu := \frac{a_1}{2}\omega_1 + \cdots + \frac{a_n}{2}\omega_n$ is still an element of $\mathcal{X}(P)$. If this is true, then κ is even in the character group $\mathcal{X}(P)$ and hence the line bundle $\mathcal{L}_\kappa$ has a square root in the Picard group $\operatorname{Pic}(G^{\mathbb{C}}/P)$. In conclusion κ is even in $\mathcal{X}(P)$ if $\langle \kappa, \check{\alpha} \rangle$ is even for all simple roots α and zero for all simple roots in I. Hence we have proven the Lemma. □

As an example we can look at the case, where P is the smallest parabolic subgroup, the Borel subgroup.

Example 9.7 (The Borel subgroup). When looking at the Borel subgroup $\mathrm{Pic}(G^{\mathbb{C}}/B) \simeq \mathcal{X}(B)$ and $\kappa = \sum_{\alpha \in R^+} \alpha$. The Weyl weight ρ is defined as the sum of the fundamental weights, so it is an element of the weight lattice $\mathcal{X}(B) \simeq \mathrm{Pic}(G/B)$. It can be proven that the Weyl vector equals half the sum of the positive roots $\frac{1}{2} \sum_{\alpha \in R^+} \alpha$. So we can conclude that $\sum_{\alpha \in R^+} \alpha$ is even in the weight lattice, and hence the line bundle $\mathfrak{L}_\kappa$, where $\kappa = \sum_{\alpha \in R^+} \alpha$ has a square root, namely the line bundle $\mathfrak{L}_\rho$.

Since this is very useful, we write the result as a corollary.

Corollary 9.8. *Let $G^{\mathbb{C}}$ be a semi-simple and simply connected Lie group. Let R denote the set of roots of $G^{\mathbb{C}}$. Let $B \subset G^{\mathbb{C}}$ be the Borel subgroup. Let $Q \to M$ be a principal $G^{\mathbb{C}}$-bundle. Let $Y = Q/B$. Assume K_M has a square root. Let $\kappa = -2\rho$, where ρ is the Weyl vector. Then $K_Y \simeq \mathfrak{L}_\kappa \otimes \pi^* K_M$ and furthermore K_Y has a square root.*

As another example we can look at the case where P is the maximal parabolic subgroup in type A.

Example 9.9 (Maximal parabolic subgroup in type A). We get a maximal parabolic subgroup, when we just take away the last root. So $I = \{\alpha_1, \dots, \alpha_{n-1}\}$ and $R(P_I) = \mathbb{Z}\{\alpha_1, \dots, \alpha_{n-1}\} \cap R$. We see that $R(\mathfrak{g}/\mathfrak{p}) = \{\alpha_1 + \cdots + \alpha_n, \alpha_2 + \cdots + \alpha_n, \dots, \alpha_n\}$. Then $\kappa = -\sum_{1 \leq i \leq n} \alpha_i + \cdots + \alpha_n = -2\rho + \sum_{1 \leq i \leq j < n} \alpha_i + \cdots + \alpha_j$. To check whether this is even, we have to calculate $\langle \kappa, \check{\alpha}_k \rangle$ and see if this is even for all j. Since

$$\begin{aligned} \langle \alpha_i, \check{\alpha}_i \rangle &= 2, \\ \langle \alpha_i, \check{\alpha}_{i-1} \rangle &= -1, \\ \langle \alpha_i, \check{\alpha}_{i+1} \rangle &= -1, \text{ and} \\ \langle \alpha_i, \check{\alpha}_j \rangle &= 0 \text{ in all other cases,} \end{aligned}$$

we get

$$\begin{aligned} \langle 2\rho, \check{\alpha}_k \rangle &= 2 \,\forall\, k \\ \langle \sum_{1 \leq i \leq j < n} \alpha_i + \cdots + \alpha_j, \check{\alpha}_k \rangle &= 2 \text{ for } k < n \\ \langle \sum_{1 \leq i \leq j < n} \alpha_i + \cdots + \alpha_j, \check{\alpha}_n \rangle &= \sum_{1 \leq i < n} -1 = -(n-1), \end{aligned}$$

so $\kappa = -(n-1)\omega_n$. For n odd this is even, so now we just have to check that $\frac{n-1}{2}\omega_n \in \{\mu \in X(T) \mid \langle \mu, \check{\alpha} \rangle = 0 \,\forall\, \alpha \in I\}$ for n odd. This is true, since $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$ and $I = \{\alpha_1, \dots, \alpha_{n-1}\}$. We see that λ is even in $X(P)$ if and only if n is odd.

9.3 The canonical bundle of the moduli stack of parabolic bundles

Recall that Σ is a surface with marked points $p^{(1)}, \dots, p^{(b)}$ each with a weight $\lambda^{(i)} \in \Lambda$. Let $\mathfrak{M}_{\Sigma_\sigma}$ denote the moduli stack of bundles, let $\mathfrak{B}_{\Sigma_\sigma, \overline{P}}$ denote the moduli stack of parabolic bundles, where $\overline{P}$ denote the

We will prove the following

Theorem 9.10. *The canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ has the form*

$$K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} = \otimes_{i=1}^b \mathcal{L}_{\kappa^{(i)}} \otimes \pi^* K_{\mathfrak{M}_{\Sigma_\sigma}}.$$

where $\kappa^{(i)} \in \mathcal{X}(P^{(i)})$ is the element $\kappa^{(i)} = -\sum_{\alpha \in R(\mathfrak{g}/\mathfrak{p}^{(i)})} \alpha$.

Proof. Let

$$E \to \Sigma \times \mathfrak{M}_{\Sigma_\sigma}$$

denote the universal bundle. Let $E^{(i)} = E|_{\{p^{(i)}\}\times \mathfrak{M}_{\Sigma_\sigma}} \to \mathfrak{M}_{\Sigma_\sigma}$ denote E restricted to $\{p^{(i)}\} \times \mathfrak{M}_{\Sigma_\sigma} \subset \Sigma \times \mathfrak{M}_{\Sigma_\sigma}$. Then we have a fibered product presentation of $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$

$$\mathfrak{B}_{\Sigma_\sigma,\overline{P}} = E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b)}/P^{(b)}.$$

We have proven Theorem 9.4 that states that for $G^{\mathbb{C}}$ a semi simple and simply connected Lie group, $Q \to M$ a principal $G^{\mathbb{C}}$-bundle, and $P \subset G^{\mathbb{C}}$ a parabolic subgroup of $G^{\mathbb{C}}$, the canonical bundle of Q/P is

$$K_{Q/P} \simeq \mathcal{L}_\kappa \otimes \pi^* K_M,$$

where $\mathfrak{L}_\kappa$ is the element of the Picard group $\mathrm{Pic}(G^{\mathbb{C}}/P)$ with $\kappa = -\sum_{\alpha\in R(\mathfrak{g}/\mathfrak{p})} \alpha$. We will use this to prove that the canonical bundle of $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ is the tensor product of a line bundle for each marked point and the canonical bundle of $\mathfrak{M}_{\Sigma_\sigma}$ pull back to $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$. The idea is to see $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ as the fibered product, and then remove the parabolic structure one marked point at a time. To see how this is done we write the pull back diagram

$$\begin{tikzcd}
E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-1)}/P^{(b-1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b)} \arrow[r] \arrow[d] & E^{(b)} \arrow[d, "\pi_b"] \\
E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-1)}/P^{(b-1)} \arrow[r] & \mathfrak{M}_{\Sigma_\sigma}
\end{tikzcd}$$

and see that

$$E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-1)}/P^{(b-1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b)} \simeq \pi_b^*(E^{(b)})$$

hence

$$\mathfrak{B}_{\Sigma_\sigma,\overline{P}} = E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b)}/P^{(b)} \simeq \pi_b^*(E^{(b)})/P^{(b)}.$$

Now we have the setup we wanted, since $\pi_b^*(E^{(b)}) \to E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-1)}/P^{(b-1)}$ is a principal $G^{\mathbb{C}}$ bundle, that we want to mod out by a parabolic subgroup $P^{(b)}$. Hence we can use Theorem 9.4 to get the canonical bundle of $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ as the product of a line bundle and the canonical bundle of $E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-1)}/P^{(b-1)}$:

$$K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} \simeq \mathcal{L}_{\kappa^{(b)}} \otimes \pi_b^*(K_{E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-1)}/P^{(b-1)}}).$$

Doing exactly the same, we can remove the parabolic structure from $p^{(b-1)}$ to get

$$K_{E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-1)}/P^{(b-1)}} \simeq \mathcal{L}_{\kappa^{(b-1)}} \otimes \pi_{b-1}^*(K_{E^{(1)}/P^{(1)} \times_{\mathfrak{M}_{\Sigma_\sigma}} \cdots \times_{\mathfrak{M}_{\Sigma_\sigma}} E^{(b-2)}/P^{(b-2)}}).$$

Hence starting from b and removing the parabolic structure one point at a time, each time getting a line bundle, we get that

$$\begin{aligned} K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} &\simeq \mathcal{L}_{\kappa^{(b)}} \otimes \pi_b^*(K_{E^{(1)}/P^{(1)}\times_{\mathfrak{M}_{\Sigma_\sigma}}\cdots\times_{\mathfrak{M}_{\Sigma_\sigma}}E^{(b-1)}/P^{(b-1)}}) \\ &\simeq \mathcal{L}_{\kappa^{(b)}} \otimes \pi_b^*(\mathcal{L}_{\kappa^{(b-1)}} \otimes \pi_{b-1}^*(K_{E^{(1)}/P^{(1)}\times_{\mathfrak{M}_{\Sigma_\sigma}}\cdots\times_{\mathfrak{M}_{\Sigma_\sigma}}E^{(b-2)}/P^{(b_2)}})) \\ &\simeq \mathcal{L}_{\kappa^{(b)}} \otimes \pi_b^*(\mathcal{L}_{\kappa^{(b-1)}} \otimes \pi_{b-1}^*(\mathcal{L}_{\kappa^{(b-2)}} \otimes \pi_{b-2}^*(\ldots K_{\mathfrak{M}_{\Sigma_\sigma}}))), \end{aligned}$$

and so forth. By abuse of notation, seeing all the line bundles as bundles over $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ we get

$$K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} = \otimes_{i=1}^b \mathcal{L}_{\kappa^{(i)}} \otimes \pi^* K_{\mathfrak{M}_{\Sigma_\sigma}}.$$

□

The Picard group of the stack $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ is $\mathbb{Z}$ direct sum the character group of each $P^{(i)}$, see [8].

Theorem 9.11. *The Picard group of* $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ *is*

$$\mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma,\overline{P}}) = \mathbb{Z} \oplus \bigoplus_{i=1}^b \mathcal{X}(P^{(i)}).$$

where

$$\mathcal{X}(P^{(i)}) = \{\mu \in \mathcal{W} \mid \langle \mu, \alpha \rangle = 0 \,\forall\, \alpha \in I^{(i)}\}.$$

and the generator of the $\mathbb{Z}$ *factor of* $\mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma,\overline{P}})$ *is* $\pi^*\chi$*, where* $\chi \in \mathrm{Pic}(\mathfrak{M}_{\Sigma_\sigma})$*.*

Proof. From [8] we have that the Picard group of the moduli stack of quasi parabolic bundles $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ is

$$\mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma,\overline{P}}) = \mathbb{Z} \oplus \bigoplus_i \mathcal{X}(P^{(i)}),$$

where $X(P^{(i)})$ denotes the character group of $P^{(i)}$. From [30] we know that the character group of $P^{(i)}$ is

$$\mathcal{X}(P^{(i)}) = \{\mu \in \mathcal{W} \mid \langle \mu, \alpha \rangle = 0 \,\forall\, \alpha \in I^{(i)}\}.$$

where I_i are the simple roots determining the parabolic subgroup $P^{(i)}$. □

As remarked in Chapter 6, there is a morphism from the sub-stack of semi-stable bundles $\mathfrak{B}^{ss}_{\Sigma_\sigma,\overline{P}}$ to the moduli space of semi-stable parabolic bundles $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$, which induces an injection on the level of Picard groups. As it is argued in [37] we get the following proposition

Proposition 9.12 ([37])**.** *The line bundle* $\mathcal{L}_{k,\overline{\mu}}$ *associated to* $(k, \overline{\mu}) \in \mathbb{Z} \oplus \bigoplus_{i=1}^b \mathcal{X}(P^{(i)})$ *descends to* $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$ *if and only if* $\exp(\sum_i \mu^{(i)})$ *acts trivial on the center of* $SU(n)$

E.g. the line bundle descends to $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$ if $\sum_i \mu^{(i)} \in \mathcal{R}$. Which means that when we write each $\mu^{(i)}$ in the $\mathbb{Z}$-basis L_i of $\mathcal{W}$, then the total sum of all coefficients must be divisible by n.

Using this description for the Picard group, and that we have found the first Chern class of $K_{\mathfrak{M}_{\Sigma_\sigma}}$ in Chapter 8 Lemma 8.2, we get the following corollary to Theorem 9.10.

Theorem 9.13. *The canonical bundle* $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ *correspond to the element*

$$K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} \simeq (-2\check{h}, \kappa^{(1)}, \ldots, \kappa^{(b)}) \in \mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma,\overline{P}})$$

We get an immediate use of Theorem 9.10 by looking at Example 9.7. We can us this Example to conclude that if all the parabolic groups were the Borel subgroup, then the line bundles $\mathcal{L}_{\kappa^{(i)}}$ each have a square root. And since $K_{\mathfrak{M}_{\Sigma_\sigma}}$ always has a square root, this will mean that $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ has a square root, when all the parabolic subgroups are the Borel subgroup.

Corollary 9.14. *Assume the weights $\lambda^{(i)}$ in each marked point are such that the parabolic subgroup $P^{(i)} \subseteq G$ associated to the marked point $p^{(i)}$ is the Borel subgroup of G. Then the canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\mathcal{P},B}}$ of $\mathfrak{B}_{\Sigma_\sigma,\mathcal{P},B}$ has the first Chern class*

$$c_1(K_{\mathfrak{B}_{\Sigma_\sigma,\mathcal{P},B}}) = (-2\check{h}, -2\rho, -2\rho, \dots, -2\rho),$$

where ρ is the Weyl vector. Hence $K_{\mathfrak{B}_{\Sigma_\sigma,\mathcal{P},B}}$ has a square root.

9.4 Restricting to SU(n)

We will restrict to the case of SU(n) bundles, hence the parabolic subgroups will be subgroups of the complexification SL($n, \mathbb{C}$).

The Lie algebra $\mathfrak{sl}(n)$

The Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ consists of all the traceless $n \times n$ complex matrices. The Cartan algebra $\mathfrak{h}$ is the subspace of diagonal traceless matrices, each of which we identify with an n-tuple $(a_1, \dots, a_n)$ with $\sum_i a_i = 0$. Let $L_i \colon \mathfrak{h} \to \mathbb{C}$ be defined by $L_i(a_1, \dots, a_n) = a_i$, then the dual to the Cartan algebra is

$$\mathfrak{h}^* = \mathbb{C}\langle L_1, \dots, L_n\rangle / \langle L_1 + \cdots + L_n\rangle.$$

Define for $i \neq j$, $E_{ij} \in \mathfrak{sl}(n, \mathbb{C})$ to be the matrix which has a 1 in the (i,j)-entry and zeros otherwise. Then E_{ij} is an eigenvector for $\mathfrak{h}$ under the adjoint action with eigenvalue $L_i - L_j$. The weight lattice is $\mathcal{W} = \mathbb{Z}\langle L_1, \dots, L_n\rangle / \langle L_1 + \cdots + L_n\rangle$ and the root lattice is $\mathcal{R} = \operatorname{span}_{\mathbb{Z}}\{L_i - L_j \mid i < j\}$. Note that we have an isomorphism $\mathcal{W}/\mathcal{R} \cong \mathbb{Z}/n\mathbb{Z}$ given by $\sum \alpha_i L_i \mapsto \sum \alpha_i \in \mathbb{Z}/n\mathbb{Z}$. We can define a set of positive roots by $R^+ = \{L_i - L_j \mid i > j\}$. Then the simple roots are $\Pi = \{L_{i+1} - L_i \mid i = 1, \dots, n-1\}$. The positive Weyl chamber is $\mathcal{C}^+ = \{\sum_i a_i L_i \mid a_1 \leq \cdots \leq a_n\}$. We define the positive weights to be $\mathcal{W}^+ = \mathcal{W} \cap \mathcal{C}^+$. In general, when we have a positive weight $\lambda \in \mathcal{W}^+$, we get an n-tuple $[\lambda_1, \dots, \lambda_n]$ of integers such that $\lambda_1 \leq \cdots \leq \lambda_n$. We observe that $\theta = L_n - L_1$, thus we see that

$$\Lambda = \{(0, \lambda_2, \dots \lambda_n) \in \mathbb{Z}^n \mid 0 \leq \lambda_2 \leq \dots \leq \lambda_n \leq k\}.$$

Some of the entries in this n-vector might be equal, hence we get a reduced vector consisting of r different numbers, which we will denote $\tilde{\lambda}$,

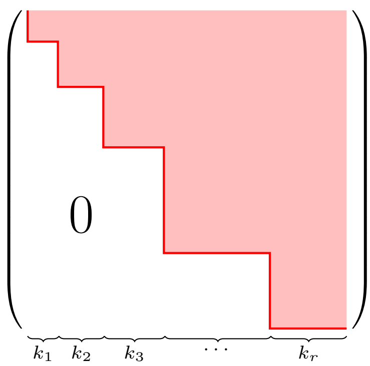
$$\begin{aligned} \lambda_1 &= \dots = \lambda_{k_1} \\ \lambda_{k_1+1} &= \dots = \lambda_{k_1+k_2} \\ &\dots \\ \lambda_{k_1+\cdots+k_{r-1}+1} &= \dots = \lambda_n. \end{aligned}$$

Let $\tilde{\lambda} = [\lambda_{k_1}, \dots, \lambda_{k_{r-1}}, k]$. This specifies a flag-type $(k_1, \dots, k_r)$, where $\sum_i k_i = n$.

The sum of the roots of $\mathrm{SL}(n,\mathbb{C})$ that are not roots of P

We will rewrite Theorem 9.13 assuming $G^{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C})$. Hence we need to find the sum of the roots of $\mathrm{SL}(n,\mathbb{C})$ that are not roots of the parabolic subgroup P, since the first Chern class of the canonical line bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ has the form $(-2\check{h}, \kappa^{(1)}, \ldots, \kappa^{(b)})$, where $\kappa^{(i)} = -\sum_{\alpha\in R(\mathfrak{g}/\mathfrak{p}_i)} \alpha$. Hence we are interested in finding all the roots of $\mathfrak{g} = \mathfrak{sl}(n)$ that are not roots of $\mathfrak{p}$.

We know that elements of P have the matrix form



hence the roots of P that are not roots of $\mathfrak{sl}(n)$ are the elements $E_i - E_j$, where $P_{ij} = 0$. So to take this sum, we start by calculating the sum over the first column, then we can calculate it over the first block, and then take a sum of all the blocks.

The first column has zeros from $i = k_1 + 1$ to n, hence

$$\sum_{i=k_I+1}^{n} (-E_1 + E_i) = -(n-k_1)E_1 + \sum_{i=k_1+1}^{n} E_i.$$

Then first block is then P_{ij} for $k_1 < i \leq n$ and $1 \leq j \leq k_1$. Hence when we sum over these entries we get

$$\begin{aligned}
&-(n-k_1)E_1 + \sum_{i=k_1+1}^{n} E_i \\
&-(n-k_1)E_2 + \sum_{i=k_1+1}^{n} E_i \\
&\ldots \\
&= -(n-k_1)\sum_{i=1}^{k_1} E_i + k_1 \sum_{i=k_1+1}^{n} E_i
\end{aligned}$$

When we look at all the blocks we get

$$
\begin{aligned}
&-(n-k_1)\sum_{i=1}^{k_1} E_i + k_1 \sum_{i=k_1+1}^{n} E_i \\
&-(n-(k_1+k_2))\sum_{i=k_1+1}^{k_1+k_2} E_i + k_2 \sum_{i=k_1+k_2+1}^{n} E_i \\
&+\dots \\
&-(n-(k_1+\dots+k_{r-1}))\sum_{i=k_1+\dots+k_{r-2}+1}^{k_1+\dots+k_{r-1}} E_i - k_{r-1}\sum_{i=k_1+\dots+k_{r-1}+1}^{n} E_i \\
=&-(n-k_1)\sum_{i=1}^{k_1} E_i \\
&-(n-(2k_1+k_2))\sum_{i=k_1+1}^{k_1+k_2} E_i \\
&\ \dots \\
&-(n-(2k_1+\dots+2k_{r-2}+k_{r-1}))\sum_{i=k_1+\dots+k_{r-2}+1}^{k_1+\dots+k_{r-1}} E_i \\
&+(k_1+\dots+k_{r-1})\sum_{i=k_1+\dots+k_{r-1}+1}^{n} E_i.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\kappa &= [\kappa_1,\dots,\kappa_{r(i)}] \\
&= [n-k_1, n-(2k_1+k_2), n-(2k_1+2k_2+k_3),\dots, \\
&\quad n-(2k_1+\dots+2k_{r-2}+k_{r-1}), -(k_1+\dots+k_{r-1})],
\end{aligned}
$$

up to adding the same integer to all entries.

Using this we can rewrite Theorem 9.10 to the following.

Corollary 9.15. *The canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ correspond to the element*

$$K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}} \simeq (-2\check{h}, \kappa^{(i)},\dots,\kappa^{(b)}) \in \mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma,\overline{P}})$$

where $\check{h}$ is the dual Coxeter number, which for $SU(n)$ is n and for $G=SU(n)$ and the above specified parabolic sub-groups $\overline{P}$, we have that

$$\kappa^{(i)} = [(n-k_1^{(i)}),(n-(2k_1^{(i)}+k_2^{(i)})),\dots,(n-(2k_1^{(i)}+\dots+2k_{r-2}^{(i)}+k_{r-1}^{(i)})),-(k_1^{(i)}+\dots+k_{r-1}^{(i)})]$$

When will $K_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda})'}$ have a square root

We already know that $c_1(K_{\mathfrak{B}_{\Sigma_\sigma,\mathcal{P},B}})$ is even when all the parabolic subgroups are the Borel subgroup. This can easily be verified now. For the Borel subgroup $k_i=1$ and $r=n$. Hence we get

$$\kappa^{(i)} = [n-1,n-3,n-5,\dots,-n+1] = [n,n-2,\dots,-n+2]$$

so this can always be even, hence the canonical bundle of the moduli stack of parabolic bundles always has a square root, when the parabolic subgroups are the Borel subgroup.

We know the canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ has a square root when the numbers

$$n - k_1^{(i)}, n - (2k_1^{(i)} + k_2^{(i)}), \dots, n - (2k_1^{(i)} + \dots + 2k_{r-2}^{(i)} + k_{r-1}^{(i)}), -(k_1^{(i)} + \dots + k_{r-1}^{(i)})$$

have the same parity for each i. Let us assume this and then in this case we let $b_o(\overline{P})$ be the number of points $p^{(i)}$ where these numbers are odd. Then we can prove when this line bundle descends to a line bundle over $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$.

Proposition 9.16. *In the case where the $\kappa_j^{(i)}$'s have the same parity for each i, $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$ has a unique square root of its canonical bundle if and only if $b_o(\overline{P})$ is even.*

Proof. Assume for $i = 1, \dots, b$ that

$$\begin{aligned}
\kappa^{(i)} &= [\kappa_1^{(i)}, \dots, \kappa_{r(i)}^{(i)}] \\
&= [(n - k_1^{(i)}), (n - (2k_1^{(i)} + k_2^{(i)})), (n - (2k_1^{(i)} + 2k_2^{(i)} + k_3^{(i)})), \dots, \\
&\quad (n - (2k_1^{(i)} + \dots + 2k_{r-2}^{(i)} + k_{r-1}^{(i)})), -(k_1^{(i)} + \dots + k_{r-1}^{(i)})]
\end{aligned}$$

all have the same parity for each i.

Assume $\kappa_1^{(i)}, \dots, \kappa_{r(i)}^{(i)}$ are all even. Then we have the sum

$$\begin{aligned}
\sum_j^r k_j \tfrac{1}{2}\kappa_j &= k_1 \tfrac{1}{2}(n - k_1) + k_2 \tfrac{1}{2}(n - (2k_1 + k_2)) + \dots + k_{r-1} \tfrac{1}{2}(n - (2k_1 + \dots + 2k_{r-2} + k_{r-1}) \\
&\quad - k_r \tfrac{1}{2}(k_1 + \dots + k_{r-1}) \\
&= \tfrac{1}{2}\Big(n(k_1 + \dots + k_r) - nk_r - \sum_{j,m}^{r-1} k_j k_m - k_r(n - k_r)\Big) \\
&= \tfrac{1}{2}\Big(n^2 - \sum_{j,m}^{r-1} k_j k_m - 2k_r n - k_r^2\Big) \\
&= \tfrac{1}{2}\Big(n^2 - \sum_{j,m}^{r} k_j k_m\Big) = 0.
\end{aligned}$$

Assume $\kappa_1^{(i)}, \dots, \kappa_{r(i)}^{(i)}$ are all odd. Then since $\lambda \in \mathcal{W}_+$ we can add 1 to each entry, and still represent the same λ. Now all entries are even. Hence we have the sum

$$\begin{aligned}
\sum_j^r k_j \tfrac{1}{2}(\kappa_j + 1) &= k_1 \tfrac{1}{2}(n - k_1 + 1) + k_2 \tfrac{1}{2}(n - (2k_1 + k_2) + 1) + \dots \\
&\quad + k_{r-1} \tfrac{1}{2}(n - (2k_1 + \dots + 2k_{r-2} + k_{r-1} + 1) - k_r \tfrac{1}{2}(k_1 + \dots + k_{r-1} + 1) \\
&= \sum_j^r k_j \tfrac{1}{2}\kappa_j + \tfrac{1}{2}n \\
&= \tfrac{1}{2}n
\end{aligned}$$

When we add up over all points we get $\frac{1}{2}n$ for each marked point, where the $\kappa_j^{(i)}$'s were initially odd. Hence the line bundle $K^{1/2}_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ will descend to a line bundle over $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$ when we have an even amount of marked points where the $\kappa_j^{(i)}$'s are odd. If we have a canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ that has a square root, then $b_o(\overline{P})$ of the marked points will have $\kappa_j^{(i)}$ odd. Hence we end up with the sum $\frac{1}{2}b_o(\overline{P})n$, which is only 0 modulo n, if $b_o(\overline{P})$ is even. □

From this we immediately get

Corollary 9.17. *In the case where $k_j^{(i)} = 1$ for all $j = 1, \ldots, r^{(i)}$, $i = 1, \ldots b$, the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$ has a unique square root of its canonical bundle if n is even or if n is odd, we need b even.*

As described in the introduction, this is enough for us. Since we can always arrange that the number of marked points, b, is even, by propagation of vacua as described in [42, 11].

9.5 Example: Surface with genus 0 and b punctures

We have now calculated the first chern class of the moduli space for a surface with genus greater than or equal to 2. We have also calculated the first Chern class for genus 0 in the case of SU(2).

The surface we are going to study in this example is the surface with genus 0 and b punctures, denoted by $\tilde{\Sigma}$. The weights on the punctures will be the representations $\lambda^{(1)}, \ldots, \lambda^{(b)}$. We want to find the first Chern class of the moduli space of flat SU(2)-connections $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ and the class of symplectic form $\omega_{\mathrm{SU}(2)}$ of that moduli space. These both live in the second cohomology group $H^2(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}), \mathbb{Z})$, so first and foremost this is the group we will try to understand.

Each weight $\lambda^{(i)}$ is an element of the dual of the Cartan algebra, and hence and element of $\mathfrak{su}(2)^*$. Let $\mathcal{O}_\lambda = \mathrm{SU}(2) \cdot \lambda$ denote the coadjoint orbit, which is a Kähler manifold $(\mathbb{C}P^1, k \cdot \omega_{FS})$, where k is such that λ becomes the $k+1$ dimensional irreducible representation of SU(2). We have the Fubini-study symplectic form $\omega_{FS} = \frac{i}{2}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$, which corresponds to $1 \in H^2(\mathbb{C}P^1, \mathbb{Z})$.

When we take the product of all the $\mathcal{O}_{\lambda^{(i)}}$s we get a new Kähler manifold, which we will denote by X,

$$X := \mathcal{O}_{\lambda^{(1)}} \times \cdots \times \mathcal{O}_{\lambda^{(b)}}.$$

There is a SU(2) action on each, which is just rotation on $\mathbb{C}P^1$. Let μ be the map

$$\begin{aligned} \mu \colon X &\to \mathfrak{su}(2)^* \\ (x^{(1)}, \ldots, x^{(b)}) &\mapsto \sum_i x^{(i)}. \end{aligned}$$

We will denote the Kähler quotient by $X_{\mathrm{SU}(2)}$, then

$$X_{\mathrm{SU}(2)} := X//\mathrm{SU}(2) = \mu^{-1}(0)/\mathrm{SU}(2).$$

Let $X_0 := \mu^{-1}(0)$. We have a projection $p \colon X_0 \to X_{\mathrm{SU}(2)}$, which makes this a SU(2)-bundle. $X_{\mathrm{SU}(2)}$ is a symplectic manifold, denote the symplectic form by ω_b. Let ω denote the symplectic form on X. Then

$$p^*(\omega_b) = i^*(\omega).$$

From Jeffrey we know that

$$\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}) \simeq X_{\mathrm{SU}(2)}.$$

From Kirwan we get that taking the GIT quotient is the same as taking the Kähler quotient, hence

$$X//^{\mathrm{GIT}}\mathrm{SL}_2(\mathbb{C}) \simeq X_{\mathrm{SU}(2)},$$

and $q\colon X \to X//^{\mathrm{GIT}}\mathrm{SL}_2(\mathbb{C}) \simeq X_{\mathrm{SU}(2)}$ is a $\mathrm{SL}_2(\mathbb{C})$-bundle. We write the first page of the Serre spectral sequence for $\mathrm{SL}_2(\mathbb{C}) \to X \to X_{\mathrm{SU}(2)}$, where we use that $H^*(\mathrm{SL}_2(\mathbb{C})) = H^*(S^3)$

$$\begin{array}{c|ccc}
3 & \mathbb{Z} & H^1(X_{\mathrm{SU}(2)}) & H^2(X_{\mathrm{SU}(2)}) \\
2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & \mathbb{Z} & H^1(X_{\mathrm{SU}(2)}) & H^2(X_{\mathrm{SU}(2)}) \\
\hline
 & 0 & 1 & 2
\end{array}$$

We see that it converges, hence

$$H^2(X_{\mathrm{SU}(2)}, \mathbb{Z}) \to H^2(X, \mathbb{Z}).$$

We know the cohomology of X

$$H^2(X, \mathbb{Z}) = \oplus_{i=1}^{b} \mathbb{Z},$$

so we can use this as a coordinate system on $H^2(X_{\mathrm{SU}(2)}, \mathbb{Z})$.

Since X is a fibre bundle, the tangent bundle will be

$$TX = q^*(TX_{\mathrm{SU}(2)}) \oplus TF,$$

where $TF \simeq X \times \mathfrak{sl}(2, \mathbb{C})$ is trivial.

By a few calculations we see that the first Chern class of TX is the push forward of the first Chern class of $TX_{\mathrm{SU}(2)}$, which is what we are interested in

$$\begin{aligned}
c_1(TX) &= c_1(q^*(TX_{\mathrm{SU}(2)})) \cdot c_0(TF) + c_0(q^*(TX_{\mathrm{SU}(2)})) \cdot c_1(TF) \\
&= c_1(q^*(TX_{\mathrm{SU}(2)})) + c_1(TF) \\
&= q^* c_1(TX_{\mathrm{SU}(2)}) + 0.
\end{aligned}$$

Since $c_1(\mathbb{C}P^1) = 2$ we get that $c_1(TX) = (2, \dots, 2)$. We saw that q^* is an isomorphism, so we have found the Chern class we were looking for:

$$q^* c_1(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})) = (2, \dots, 2).$$

To calculate the class of the symplectic form $\omega_{\mathrm{SU}(n)}$ on $X_{\mathrm{SU}(2)}$, we use the fact that we know the class of the symplectic form ω on X, since this is $[\omega] = (k^{(1)}, \dots, k^{(b)})$. By the same argument as before and the fact that $q^*([\omega_{\mathrm{SU}(2)}]) = [\omega]$, we see that the symplectic form is

$$q^*[\omega_{\mathrm{SU}(n)}] = (k^{(1)}, \dots, k^{(b)}).$$

This all means, that if all the weights are the same, $k^{(1)} = \dots = k^{(b)}$, then the first Chern class will be $l[\omega]$ for some $l \in \mathbb{Q}$, which is what we wanted. Note also that if the weights are not the same, the first Chern class can never be l times the class of the symplectic form.

Chapter **10**

The Hitchin connection on the moduli space of parabolic bundles when we have a Fano type condition

We would now like to use Andersen's general construction of the Hitchin connection, Theorem 4.3, to construct a Hitchin connection in $\mathcal{V}^{(k)}_{\overline{\lambda}}$. Recall that we assume the genus of the marked surface Σ is $g \geq 2$.

Theorem 10.1. *Assume there exists $l \in \mathbb{Q}$ such that the first Chern class of the canonical bundle of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$ is $l[\omega_{k,\overline{\lambda}}]$ and that the weights $\overline{\lambda}$ are contained in the interior of the Weyl Chamber. Then we can use Theorem 4.3 to construct a Hitchin connection in the bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$.*

Proof. From Theorem 7.1 we know that the Chern-Simons line bundle is a prequantum line bundle since the weights $\overline{\lambda}$ are contained in the interior of the Weyl Chamber. From [8] we know that the moduli space is simply connected. To prove $G(V)$ is holomorphic, and hence the family of Kähler structures is rigid, one can follow exactly what Hitchin did in [29] his proof of Lemma 2.13 and the remark following. The only things used is that

$$T_{[A]}\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}, \epsilon)'_\sigma \simeq H^1_{\epsilon,k}(\tilde{\Sigma}, d_A),$$

where each of the elements has a unique d_A-harmonic representative, which we know from Section 6.6. In the construction of the Hitchin connection we need a Ricci potential, e.g. a map $F\colon \mathcal{T}_{(\Sigma,\mathcal{P},V)} \to C^\infty(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})', \mathbb{R})$ that satisfies the equation

$$\rho_\sigma = 2n\omega_{k,\overline{\lambda}} + 2id\overline{\partial}F_\sigma,$$

Zograf and Takhtajan has constructed such an F in [40], so we use this Ricci potential F.

Now we have seen that all the required assumptions of Theorem 4.3 are met, except that the first Chern class is $l[\omega_{k,\overline{\lambda}}]$. When we assume that the first Chern class is in fact on the form $l[\omega_{k,\overline{\lambda}}]$, then we can construct the wanted Hitchin connection. □

We remember from Section 6.7 that for the stack $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ of parabolic $\mathrm{SL}(n,\mathbb{C})$-bundles, there is a morphism from the sub-stack of semi-stable parabolic bundles $\mathfrak{B}^{ss}_{\Sigma_\sigma,\overline{P}}$ to the moduli space of semi-stable parabolic bundles $\mathcal{M}_{\mathrm{par}}(\Sigma_\sigma, \overline{\lambda})$, which induces an injection on the level of the Picard groups.

We just concluded that we can use Andersen's general construction of the Hitchin connection, Theorem 4.3 ([3]), and the projective flatness proved by Andersen and Gammelgaard ([4]) to construct a projectively flat Hitchin connection in the bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$ when the first Chern class of the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})$ is $l[\omega_{k,\overline{\lambda}}]$ for some $l \in \mathbb{Q}$. We know from [8] that the class of $\omega_{k,\overline{\lambda}}$ is given by

$$(k, \lambda^{(1)}, \ldots, \lambda^{(b)}) \in \mathbb{Z} \oplus \bigoplus_i X(P^{(i)}) \simeq \mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma, \overline{P}}),$$

From Corollary 9.15 we know the first Chern class of $K_{\mathfrak{B}_{\Sigma_\sigma, \overline{P}}}$ as

$$K_{\mathfrak{B}_{\Sigma_\sigma, \overline{P}}} \simeq (-2\check{h}, \kappa^{(i)}, \ldots, \kappa^{(b)}) \in \mathrm{Pic}(\mathfrak{B}_{\Sigma_\sigma, \overline{P}})$$

where

$$\begin{aligned}\kappa &= [\kappa^{(1)}, \ldots, \kappa^{(r)}] \\ &= [(n - k_1), (n - (2k_1 + k_2)), (n - (2k_1 + 2k_2 + k_3)), \ldots, \\ &\quad (n - (2k_1 + \cdots + 2k_{r-2} + k_{r-1})), -(k_1 + \cdots + k_{r-1})].\end{aligned}$$

Hence we can work out when the first Chern class of the canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma, \overline{P}}}$ and the class of the symplectic form $\omega_{k,\overline{\lambda}}$ are proportional for the stack of parabolic bundles. When they are indeed proportional, they will also be so the for moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$. We conclude

Theorem 10.2. *If for all $i = 1, \ldots, b$ there exists an $l \in \mathbb{Q}$ such that*

$$\begin{aligned}-2\check{h} \cdot l &= k \\ \lambda^{(i)} \cdot l &= \kappa_1^{(i)} = -(n - k_1^{(i)}) \\ &\cdots \\ \lambda^{(i)} \cdot l &= \kappa_r^{(i)} = k_1^{(i)} + \cdots + k_{r-1}^{(i)},\end{aligned}$$

up to adding an integer $m^{(i)} \in \mathbb{Z}$ to each equation. Then we can apply Theorem 4.3 and Theorem 4.4 to construct a Hitchin connection in $\mathcal{V}^{(k)}_{\overline{\lambda}}$, which is unique up to projective equivalence, $\Gamma_{(\Sigma, \mathcal{P}, V)}$-invariant and projectively flat.

Chapter 11

The Hitchin connection on the moduli space of parabolic bundles

In this final chapter we will prove the main result of this thesis. The aim is to construct a quantization of the moduli space defined in Chapter 6, and find a Hitchin connection in this setting, without having any assumptions on the weights, like we had in Theorem 10.2. This is done using metaplectic quantization and the Hitchin connection constructed in this setting from [5] by Andersen, Gammelgaard and Roed. We wish to construct a Hitchin connection in the bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$ over $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$. We can use Theorem 4.9 to construct such a Hitchin connection, when the canonical bundle $K_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}}}$ has a square root, and there exists a pre-quantum line bundle $\tilde{L}_{\overline{\lambda}}$ such that $K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda})'} \otimes \tilde{L}_{\overline{\lambda},\sigma} \simeq \mathcal{L}_{k,\overline{\lambda}}$ as holomorphic line bundles for all $\sigma \in \mathcal{T}_{(\Sigma,\mathcal{P},V)}$.

Recall Σ is a closed oriented 2-manifold with marked points $\mathcal{P} = \{p^{(1)}, \dots, p^{(b)}\}$, each with a direction $v^{(i)} \in \mathbb{P}(T_{p_i}\Sigma)$ and a weight $\lambda^{(i)} \in \Lambda$. As in earlier chapters $\tilde{\Sigma} = \Sigma - \mathcal{P}$.

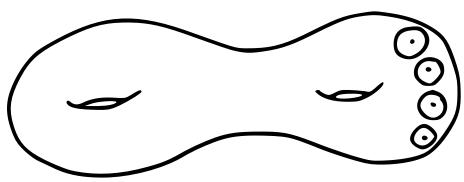


Figure 11.1: Punctured surface

In Chapter 6 we constructed the moduli space $\mathcal{M}(\tilde{\Sigma},\overline{\lambda})$, which is the moduli space of flat connections whose holonomy around $p^{(i)}$ lie in $C_{\lambda^{(i)}}$. By a result of Metha and Seshadri [35], this moduli space is homeomorphic (diffeomorphic on the smooth locus) to the moduli space $\mathcal{M}_{\text{par}}(\Sigma_\sigma,\overline{\lambda})$ of semi-stable parabolic bundles with trivial determinant and weights determined by the $\lambda^{(i)}$'s. Using Sobolev spaces we constructed a moduli space $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)_\sigma$ in Section 6.5. The smooth part $\mathcal{M}(\tilde{\Sigma},\overline{\lambda},\epsilon)'_\sigma$ was shown by Daskalapoulos and Wentworth to be naturally an almost complex manifold in [22]. For small enough ϵ the moduli space is again diffeomorphic to $\mathcal{M}(\tilde{\Sigma},\overline{\lambda})'$.

In Section 4.2 we recall how the Hitchin connection can be constructed, when using metaplectic correction. It is Theorem 4.9 that we wish to use to construct a Hitchin connection in this case. In this theorem we do not need the first Chern class to be $l[\omega]$, but we do still

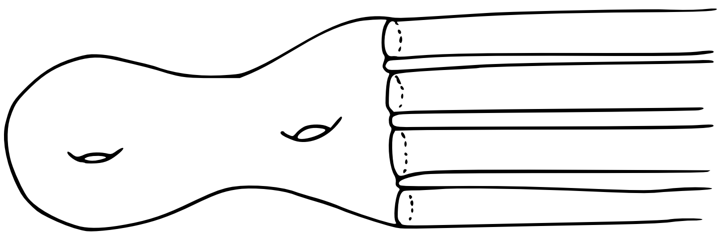


Figure 11.2: Surface with semi-infinite cylinders

need it to be even. When looking at the moduli space, we do not know in general that the first Chern class is even (for example see Example 9.9), so we need another way of using metaplectic quantization. The idea is to look at a bigger moduli space, where the first Chern is in fact even, and then use this to construct a quantization of the original moduli space via a projection from the bigger moduli space to the original moduli space.

Let ρ denote the Weyl vector and $\overline{\rho} = (\rho, \dots, \rho)$. Given the weight $\overline{\lambda}$ we define a new weight $\overline{\lambda}' = \overline{\lambda} + \overline{\rho}$. Observe that the weights $\overline{\lambda}'$ are in the interior of the positive Weyl chamber. Then $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')$ and $\mathcal{L}_{k+\check{h}, \overline{\lambda}'}$ are the moduli space and the prequantum line bundle associated to $\overline{\lambda}'$ as described in Chapter 6 and Chapter 7. The following two results about the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'$ are needed in order to prove the main Theorem, that the bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$ supports a projectively flat Hitchin connection.

Proposition 11.1.

$$H^{0,1}(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma) = 0.$$

Proof. Using Serre spectral sequences for the projection $\pi \colon \mathfrak{B}_{\Sigma_\sigma, \overline{P}'} \to \mathfrak{M}_{\Sigma_\sigma}$ we see that $H^1(\mathfrak{B}_{\Sigma_\sigma, \overline{P}'}, \mathcal{O}) = 0$ (see Appendix A). Using that the complex co-dimension of the reducible locus of $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma$ is ≥ 2 argument, to go from $\mathfrak{B}_{\Sigma_\sigma, \overline{P}'}$ to $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma$, we get

$$H^{0,1}(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma) \simeq H^1(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma, \mathcal{O}) \simeq H^1(\mathfrak{B}_{\Sigma_\sigma, \overline{P}'}, \mathcal{O}).$$

Hence $H^{0,1}(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma) = 0$. □

Proposition 11.2. *The moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma$ has no holomorphic vector fields.*

Proof. The proof of this theorem is completely parallel to Hitchin's original proof of the same fact for the moduli spaces of semi-stable bundles, using his integrable system on the corresponding Higgs bundle moduli space. Let $\mathcal{M}_{\mathrm{Higgs}, \overline{\lambda}'}(\Sigma_\sigma)$ denote the moduli space of $\overline{\lambda}'$ semi-stable parabolic Higgs bundles of rank n and with trivial determinant. On this moduli space we an integrable system by the means of the Hitchin map. Given a vector bundle E on Σ_σ, any invariant homogeneous degree i polynomial naturally defines a map

$$H^0(\Sigma_\sigma, \mathrm{End}_0(E) \otimes K(\mathcal{P})) \to H^0(\Sigma_\sigma, K(\mathcal{P})^i).$$

Take the elementary symmetric polynomials (of degree at least 2) as a homogeneous basis of polynomials on $\mathfrak{sl}(n)$ invariant under the adjoint action of $\mathrm{SL}(n)$, then the corresponding maps a_i combine to give the Hitchin map

$$h_{\overline{\lambda}'} : \mathcal{M}_{\mathrm{Higgs},\overline{\lambda}'}(\Sigma_\sigma) \to \mathbb{H},$$

where

$$\mathbb{H} = H^0(\Sigma_\sigma, K(\mathcal{P})^2) \oplus H^0(\Sigma_\sigma, K(\mathcal{P})^3) \oplus \cdots \oplus H^0(\Sigma_\sigma, K(\mathcal{P})^n).$$

The components of h_α are defined as follows. For any parabolic Higgs bundle (E, Φ) and any $x \in \Sigma_\sigma$, let $k \in K(\mathcal{P})|_x$. Then we have

$$\det(k.\mathrm{Id}_{E|_x} - \Phi|_x) = k^n + a_2(\Phi)(x)k^{n-2} + \cdots + a_{n-1}(\Phi)(x)k + a_n(\Phi)(x),$$

and $h_\alpha(E, \Phi)$ is given by $(a_2(\Phi), \ldots, a_n(\Phi))$. Since the Hitchin map is proper it follows that the holomorphic functions on $\mathcal{M}_{\mathrm{Higgs},\overline{\lambda}'}(\Sigma_\sigma)$ must all be pull backs of holomorphic functions on the Hitchin base $\mathbb{H}$. But now we recall that $T^*\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$ embeds in $\mathcal{M}_{\mathrm{Higgs},\overline{\lambda}'}(\Sigma_\sigma)$, simply because cotangent vectors to $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$ are precisely the Higgs fields

$$T^*_{[E]}\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma \cong H^0(\Sigma_\sigma, \mathrm{End}_0(E) \otimes K(\mathcal{P})).$$

Thus if we assume that we have a holomorphic tangent field on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$, then this will dually induce a holomorphic function on $T^*\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma$, which by Hartog's theorem, again using complex co-dimension at least 2 will extend to a holomorphic function on $\mathcal{M}_{\mathrm{Higgs},\overline{\lambda}'}(\Sigma_\sigma)$. Now this function will be homogenous of degree 1 with respect to the $\mathbb{C}^*$ action there is on $\mathcal{M}_{\mathrm{Higgs},\overline{\lambda}'}(\Sigma_\sigma)$ induced by multiplication of scalars on the Higgs field. This however contradicts the above description of the space of holomorphic functions on this Higgs bundle moduli space, since they all have degree at least two with respect to the $\mathbb{C}^*$ action. □

Theorem 11.3. *The bundle $\mathcal{V}^{(k)}_{\overline{\lambda}}$ supports a projectively flat Hitchin connection which is mapping class group invariant and unique up to projective equivalence.*

Proof. First of all, note that the new weights $\overline{\lambda}'$ are in the interior of the positive Weyl Chamber, and hence the line bundle $\mathcal{L}_{k+\check{h},\overline{\lambda}'}$ is a prequantum line bundle on $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'$. Let $\overline{P}' = (P^{(1)'}, \ldots, P^{(b)'})$ be the parabolic subgroups associated to the weight $\overline{\lambda}'$. Note that for $\overline{\lambda}'$ we have all $k_j^{(i)}$'s equal to one, since all components of $\overline{\lambda}'$ are in the interior of the Weyl chamber, and thus $\overline{P}'$ will be b copies of the Borel subgroup. Hence by Corollary 9.17 the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'$ has a square root of its canonical bundle $K_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'}$.

Let $\tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'}$ to be the bundle over $\mathcal{T}_{(\Sigma,\mathcal{P},V)}$ whose fiber at $\sigma \in \mathcal{T}_{(\Sigma,\mathcal{P},V)}$ is

$$\tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}',\sigma} = H^0(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma, \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'_\sigma}).$$

We observe that pulling back over the natural fibration

$$\pi' : \mathfrak{B}_{\Sigma_\sigma,\overline{P}'} \to \mathfrak{B}_{\Sigma_\sigma,\overline{P}}$$

gives us

$$(\pi')^*\mathcal{L}_{k,\overline{\lambda}} = \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'},$$

and induces a natural isomorphism of bundles

$$(\pi')^* : \tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'} \to \mathcal{V}^{(k)}_{\overline{\lambda}}. \tag{11.1}$$

This follows since $H^0(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma, \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'_\sigma})$ is the same as taking H^0 of the corresponding bundle over the sub-stack of stable parabolic bundles. As we saw in Chapter 6, the complex co-dimension of the strictly semi-stable locus is at least two (for $g > 2$), hence by Hartog's extension Theorem we have

$$H^0(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma, \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathcal{M}(\tilde{\Sigma},\overline{\lambda}')'_\sigma}) \simeq H^0(\mathfrak{B}_{\Sigma_\sigma,\overline{P}'}, \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}'}}).$$

Since π' is a proper morphism, we have

$$(\pi')^* \colon H^0(\mathfrak{B}_{\Sigma_\sigma,\overline{P}}, \mathcal{L}_{k,\overline{\lambda}}) \to^{\simeq} H^0(\mathfrak{B}_{\Sigma_\sigma,\overline{P}'}, \mathcal{L}_{k+\check{h},\overline{\lambda}'} \otimes K^{1/2}_{\mathfrak{B}_{\Sigma_\sigma,\overline{P}'}}).$$

Using the same co-dimension ≥ 2 argument for $\mathfrak{B}_{\Sigma_\sigma,\overline{P}}$ we get

$$H^0(\mathfrak{B}_{\Sigma_\sigma,\overline{P}}, \mathcal{L}_{k,\overline{\lambda}}) \simeq H^0(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda})'_\sigma, \mathcal{L}_{k,\overline{\lambda}}),$$

hence $(\pi')^* : \tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'} \to \mathcal{V}^{(k)}_{\overline{\lambda}}$ is an isomorphism.

Since we know that $H^{0,1}(\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma) = 0$ by Proposition 11.1, we conclude that Theorem 4.9 applies to provide a Hitchin connection on $\tilde{\mathcal{V}}^{(k)}_{\overline{\lambda}'}$.

From Proposition 11.2 we know that the moduli space $\mathcal{M}(\tilde{\Sigma}, \overline{\lambda}')'_\sigma$ has no holomorphic vector fields, hence we can apply Theorem 4.10 (proven in [25]) to conclude this projection is projectively flat.

Now combine this with the isomorphism (11.1) to reach the same conclusion for $\mathcal{V}^{(k)}_{\overline{\lambda}}$.

Uniqueness follows since any other second order Hitchin connection needs to have the same symbol as the one constructed here. But then the difference is of first order, and since there is no holomorphic vector fields of degree 1, it is actually of order zero. The invariance of this connection under the action of $\Gamma_{(\Sigma,\mathcal{P},V)}$ now follows by this uniqueness or directly by the naturality of the construction.

Thus we have proven the theorem. □

This allows us to now give the gauge theory definition of the quantum representations of the mapping class group at least projectively, simply as the action of the mapping class group on the space of projectively covariant constant sections. The next step would then we to understand that the Pauly isomorphism is a projectively flat isomorphism (in analogy with Laszlo's result in the case of no marked points [33]) between $\mathcal{V}^{(k)}_{\overline{\lambda}}$ with the projectively flat connection constructed in this thesis and then the bundle of sheaf of vacua for the weights $\overline{\lambda}$ together with the TUY-connection in this bundle constructed in [42]. Once this has been done one would by combining with the work of Andersen and Ueno [10, 11, 12, 13] have the gauge theory construction of the WRT-modular function discussed in the beginning of this introduction. The flatness of Pauly's isomorphism however goes beyond the scope of this work.

Appendix **A**

Cohomology of a fibration

Let G be a simple and simply connected compact Lie group. Let Σ be a smooth algebraic curve over the complex numbers. Let $\mathcal{M}$ be the moduli space of semi-stable holomorphic $G^{\mathbb{C}}$-bundles on Σ. Let $\mathfrak{M}$ be the corresponding stack. Let $\{x_1, \ldots, x_m\}$ be marked points on Σ. Fix the divisor $D = \sum_i x_i$. Let $B \to \Sigma$ be a parabolic $G^{\mathbb{C}}$-bundle with reduction of structure group to a parabolic subgroup P_i over each parabolic point x_i. Let $G_i = P_i \cap G$. Assign integral dominant weights $\overline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ such that G_i preserves λ_i. Let $\mathrm{ad}_S(B)$ be the adjoint bundle of B with reduction of structure group corresponding to P_i over x_i.

Let (B, Φ) be a Higgs bundle, with B as before and

$$\Phi \in H^0(\Sigma, \mathrm{ad}_s(B) \otimes K(D))$$

Let $\mathcal{B}$ ($\mathcal{B}_H$) denote the moduli space of semi-stable parabolic $G^{\mathbb{C}}$-bundles (Higgs bundles). Let $\mathfrak{B}$ ($\mathfrak{B}_H$) denote the corresponding stacks. We are interested in calculating

$$c_1(T\mathfrak{B}) \in H^2(\mathfrak{B}).$$

For each marked point x_i the parabolic structures correspond to $G^{\mathbb{C}}/P_i$. Since we have this for each marked point we get a fibration

$$G^{\mathbb{C}}/P_1 \times \cdots \times G^{\mathbb{C}}/P_n \to \mathfrak{B} \to \mathfrak{M}.$$

To ease notation we will denote $G^{\mathbb{C}} \times \cdots \times G^{\mathbb{C}}$ by F and $P_1 \times \cdots \times P_n$ by P. Then the fibration will look like

$$F/P \to \mathfrak{B} \to \mathfrak{M}$$

We have that $\mathfrak{M}$ is the stack of holomorphic $G^{\mathbb{C}}$-bundles, so every q in $\mathfrak{M}$ is an isomorphism class of principal $G^{\mathbb{C}}$-bundles. There is a principal $G^{\mathbb{C}}$-bundle $E \to \mathfrak{M} \times \Sigma$ such that $E_q \to \{q\} \times \Sigma$ is a principal bundle over a copy of Σ which is isomorphic to q. Now take this bundle E and restrict it to $\mathfrak{M} \times \{p_i\}$ for each marked point $p_i \in \Sigma$. We can take the fiberwise product of these to get a bundle over $\mathfrak{M}$.

$$G^{\mathbb{C}} \times \cdots \times G^{\mathbb{C}} \to E_1 \times_{\mathfrak{M}} \cdots \times_{\mathfrak{M}} E_n \to \mathfrak{M}.$$

Again to make notation easier we will denote $E_1 \times_{\mathfrak{M}} \cdots \times_{\mathfrak{M}} E_n$ simply by $\tilde{E}$. Then we have the fibration

$$F \to \tilde{E} \to \mathfrak{M}.$$

Lemma A.1. *The induced map $H^i(\mathfrak{M}) \to H^i(\tilde{E})$ is injective*

Proof. We write the spectral sequence for $F \to \tilde{E} \to \mathfrak{M}$, using the assumption that $\tilde{H}^i(F) = 0$ for $i \leq 2$.

$$
\begin{array}{c|cccc}
3 & H^3(F) & & & \\
2 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & H^1(M) & H^2(M) & H^3(M) \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

We see that no d can hit $H^3(\mathfrak{M})$, so the map is injective. $\square$

We have $P = P_1 \times \cdots \times P_n \subset G^{\mathbb{C}} \times \cdots \times G^{\mathbb{C}}$, if we mod out by this we get a fibration

$$F/P \to E_1/P_1 \times_{\mathfrak{M}} \cdots \times_{\mathfrak{M}} E_n/P_n \to \mathfrak{M}.$$

We have that $\mathfrak{B} \simeq E_1/P_1 \times_{\mathfrak{M}} \cdots \times_{\mathfrak{M}} E_n/P_n$.

We also have a fibration

$$F/P \to (F/P) \times \tilde{E} \to \tilde{E}$$

If we mod out by F we get maps

$$p_1 \colon (F/P) \times (\tilde{E}) \to (F/P) \times (\tilde{E})/F \simeq \tilde{E}/P \simeq \mathfrak{B}$$

and

$$p_2 \colon \tilde{E} \to \tilde{E}/F \simeq \mathfrak{M}.$$

For $\mathbb{C}P^\infty$ we know that for any $\alpha \in H^2(X)$ there exists a map $X \to \mathbb{C}P^\infty$ such that α is the pullback of $c_1(\mathbb{C}P^\infty) \in H^2(\mathbb{C}P^\infty)$. We do this for $\mathfrak{M}$. We have that $c_2(E) \cap [\Sigma]$ generates $H^2(\mathfrak{M})$, so let $f \colon \mathfrak{M} \to \mathbb{C}P^\infty$ be the map such that $c_2(E) \cap [\Sigma] \in H^2(\mathfrak{M})$ is the pullback $c_1(\mathbb{C}P^\infty)$. We have now described all of the following maps, such that we get a commutative diagram of fibrations

$$
\begin{tikzcd}
F/P \arrow[r] \arrow[d] & F/P \times \tilde{E} \arrow[r] \arrow[d, "p_1"] & \tilde{E} \arrow[d, "p_2"] \\
F/P \arrow[r] \arrow[d] & \mathfrak{B} \arrow[r] \arrow[d] & \mathfrak{M} \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathbb{C}P^\infty \arrow[r, "\mathrm{id}"] & \mathbb{C}P^\infty
\end{tikzcd}
.
$$

We wish to prove the following by the use of spectral sequences

Theorem A.2. *For a fibration*

$$F/P \to \mathfrak{B} \to \mathfrak{M}$$

with $\pi_1(\mathfrak{M}) = 1$, $H^2(\mathfrak{M}) = \mathbb{Z}$, $H^n(F/P) = 0$ *for* n *odd and* F/P *and* $\mathfrak{M}$ *connected, the sequence*

$$0 \to H^2(\mathfrak{M}) \to H^2(\mathfrak{B}) \to H^2(F/P) \to 0$$

is exact.

Proof. We begin to write the spectral sequence for the fibration $F/P \to \mathfrak{B} \to \mathfrak{M}$.

$$E^2_{p,q} = H^p(\mathfrak{M}; H^q(F/P;\mathbb{Z})).$$

For q odd we notice that

$$E^2_{p,q} = H^p(\mathfrak{M}; 0) = 0.$$

And for $p = 1$ we similarly have

$$E^2_{1,q} = 0.$$

To calculate $E^2_{2,q} = H^2(\mathfrak{M}; H^q(F/P))$ we use universal coefficient theorem twice. We have $H^2(\mathfrak{M}) = \mathbb{Z}$, so if there is no torsion in $H_2(\mathfrak{M})$ we get $H_2(\mathfrak{M}) = \mathbb{Z}$ by UCT. Again using UCT we get the exact sequence

$$0 \to \operatorname{Ext}(H_1(\mathfrak{M}), H^q(F/P)) \to H^2(\mathfrak{M}; H^q(F/P)) \to \operatorname{Hom}(H_2(\mathfrak{M}), H^q(F/P)) \to 0.$$

Using what we have just found we get

$$0 \to 0 \to H^2(\mathfrak{M}; H^q(F/P)) \to \operatorname{Hom}(\mathbb{Z}, H^q(F/P)) \to 0,$$

which tells us that $H^2(\mathfrak{M}; H^q(F/P)) \simeq H^q(F/P)$.

Since we have $\mathfrak{M}$ is connected a similar argument reveals $E^2_{0,q} = H^0(\mathfrak{M}; H^q(F/P)) \simeq H^q(F/P)$. (using Poincare duality)

The lower corner of the page E^2 will look like the following

$$\begin{array}{c|cccc}
2 & H^2(F/P) & 0 & H^2(F/P) & \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & \mathbb{Z} & H^3(\mathfrak{M}) \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

We notice that $d_2 = 0$, so the page E^3 will be the same. Using the main theorem for spectral sequences, we can get an exact sequence containing $H^2(\mathfrak{B})$. The theorem states that there exists subgroups

$$0 \subset F^2_2 \subset F^2_1 \subset F^2_0 = H^2(\mathfrak{B})$$

such that

$$E_\infty^{2,0} \simeq F_2^2\,,\; E_\infty^{1,1} \simeq F_1^2/F_2^2 \text{ and } E_\infty^{0,2} \simeq F_0^2/F_1^2.$$

So we have the following

$$\begin{aligned} \mathbb{Z} = E_\infty^{2,0} &\simeq F_2^2 \\ 0 = E_\infty^{1,1} &\simeq F_1^2/F_2^2 = F_1^2/\mathbb{Z} \\ \ker(d_3) = E_\infty^{0,2} &\simeq F_0^2/F_1^2 = H^2(\mathfrak{B})/F_1^2. \end{aligned}$$

From (2.21) we get the short exact sequence

$$0 \to \mathbb{Z} \to F_1^2 \to 0 \to 0,$$

so we conclude $F_1^2 = \mathbb{Z}$. From (2.22) we get the short exact sequence

$$0 \to F_1^2 \to H^2(\mathfrak{B}) \to \ker(d_3) \to 0,$$

which then becomes

$$0 \to \mathbb{Z} \to H^2(\mathfrak{B}) \to \ker(d_3) \to 0.$$

So we get an exact sequence

$$0 \to \mathbb{Z} \to H^2(\mathfrak{B}) \to H^2(F/P) \to^{d_3} H^3(\mathfrak{M}).$$

Since we have a commuting diagram

$$\begin{tikzcd} F/P \arrow[r] \arrow[d] & F/P \times \tilde{E} \arrow[r] \arrow[d, "p_1"] & \tilde{E} \arrow[d, "p_2"] \\ F/P \arrow[r] \arrow[d] & \mathfrak{B} \arrow[r] \arrow[d] & \mathfrak{M} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbb{C}P^\infty \arrow[r, "\mathrm{id}"] & \mathbb{C}P^\infty \end{tikzcd}.$$

we can calculate the spectral sequence of $F/P \to F/P \times \tilde{E} \to \tilde{E}$ and prove that $d_3' = 0$. We can then use this to conclude $d_3 = 0$ in our original spectral sequence.

We have the map $f\colon \mathfrak{M} \to \mathbb{C}P^\infty$ such that the generator $c_2(E) \cap [\Sigma] \in H^2(\mathfrak{M})$ is the pullback of $c_1(\mathbb{C}P^\infty) \in H^2(\mathbb{C}P^\infty)$. Since $H^2(\mathfrak{M}) \simeq \mathbb{Z}$ and $H^2(\mathbb{C}P^\infty) \simeq \mathbb{Z}$ we have that f induces an isomorphism

$$H^2(\mathbb{C}P^\infty) \to^{\simeq} H^2(\mathfrak{M}).$$

By universal coefficient theorem $H^2(\mathfrak{M}) \simeq H_2(\mathfrak{M})$ and $H^2(\mathbb{C}P^\infty) \simeq H_2(\mathbb{C}P^\infty)$. By Hurewicz Theorem we get isomorphisms $\pi_2(\mathfrak{M}) \simeq H_2(\mathfrak{M})$ and $\pi_2(\mathbb{C}P^\infty) \simeq H_2(\mathbb{C}P^\infty)$, since $\mathfrak{M}$ and $\mathbb{C}P^\infty$ are simply connected. So by putting all of this together we see that f induces an isomorphism

$$\pi_2(\mathfrak{M}) \to \pi_2(\mathbb{C}P^\infty).$$

We can then write the long exact sequence for homotopy

$$\cdots \to \pi_2(\tilde{E}) \to \pi_2(\mathfrak{M}) \to^{\simeq} \pi_2(\mathbb{C}P^\infty) \to \pi_1(\tilde{E}) \to 0.$$

So we can find $\pi_1(\tilde{E})$

$$\begin{aligned}\pi_1(\tilde{E}) &= \pi_2(\mathbb{C}P^\infty)/\ker(\pi_2(\mathbb{C}P^\infty) \to \pi_1(\tilde{E}))\\ &= \pi_2(\mathbb{C}P^\infty)/\mathrm{Im}(\pi_2(B) \to \pi_2(\mathbb{C}P^\infty))\\ &= \pi_2(\mathbb{C}P^\infty)/\pi_2(\mathbb{C}P^\infty) = 0.\end{aligned}$$

We write E_2':

$$\begin{array}{c|cccc} 2 & H^2(F/P) & 0 & & \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{Z} & 0 & H^2(\tilde{E}) & H^3(\tilde{E}) \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

As before we can get an exact sequence from the spectral sequence

$$0 \to H^2(\tilde{E}) \to H^2(\tilde{E} \times F/P) \to H^2(F/P) \to^{d_3'} H^3(\tilde{E}).$$

From Kunneth formula we get

$$\begin{aligned}H^2(\tilde{E} \times F/P) &= H^0(\tilde{E}) \otimes H^2(F/P) \oplus H^1(\tilde{E}) \otimes H^1(F/P) \oplus H^2(\tilde{E}) \otimes H^0(F/P)\\ &= H^2(F/P) \oplus H^2(\tilde{E}).\end{aligned}$$

So

$$\ker(d_3 \colon H^2(F/P) \to H^3(\tilde{E})) = \mathrm{Im}(H^2(F/P) \oplus H^2(\tilde{E}) \to H^2(F/P)) = H^2(F/P).$$

And then $d_3' = 0$.

We now use the maps between the fibrations

$$\begin{tikzcd} {} \arrow[r] & H^2(F/P) \arrow[r, "d_3"] \arrow[d, "="] & H^3(\mathfrak{M}) \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & H^2(F/P) \arrow[r, "0"] & H^3(\tilde{E}) \arrow[r] & {} \end{tikzcd}.$$

Since we proved in Lemma that $H^3(\mathfrak{M}) \to H^3(\tilde{E})$ is injective, $d_3 = 0$. Then we get the exact sequence

$$0 \to \mathbb{Z} \to H^2(\mathfrak{B}) \to H^2(F/P) \to 0.$$

$\square$

Remark A.3. We have

$$F/P \to^i \mathfrak{B} \to^j \mathfrak{M}.$$

Take the tangentbundles and we get a commuting diagram

$$\begin{tikzcd}
T(F/P) \arrow[r, "i_*"] \arrow[d] & T\mathfrak{B} \arrow[r, "j_*"] \arrow[d, "\pi_1"] & T\mathfrak{M} \arrow[d, "\pi_2"] \\
F/P \arrow[r] & \mathfrak{B} \arrow[r] & \mathfrak{M}.
\end{tikzcd}$$

Construct a map $f\colon T\mathfrak{B} \to j^*T\mathfrak{M} = \{(v,b)|\pi_2 v = j(b)\} \subseteq T\mathfrak{M} \times \mathfrak{B}$ by $f(w) := (j_*(w), \pi_1(w)) \in j^*T\mathfrak{M}$, where $f(w) \in j^*T\mathfrak{M}$ since $\pi_2(j_*(w)) = j(\pi_1(w))$ since the diagram commutes. Using this we get

$$\ker(f) \to T\mathfrak{B} \to j^*T\mathfrak{M}.$$

Remark A.4. Note that we can use the same arguments as in the proof of Theorem A.2 to prove the sheaf cohomology group $H^1(\mathfrak{B}_{\Sigma_\sigma,\overline{P}}, \mathcal{O}) = 0$.

Appendix **B**

A brief introduction to stacks

In 1959 Grothendieck introduced the notion of stacks, when he saw that one cannot construct a good moduli space when automorphisms exist. Often when this is the problem, one can still define a moduli stack. Stacks were defined by Giraud in 1966 and was given the name stack by Deligne and Mumford in 1969.

The theory of stacks is big and complicated, even the definition is hard to understand. But using stacks, many moduli problems become easier to write, understand and solve, therefore stacks seems to be the natural language to solve these questions in.

So in this chapter, we are trying to give an overview of the definition and some useful theorems concerning stacks. It will not be concise or precise, but should give the idea of what stacks are and how to use them when working with moduli problems. This chapter is written after reading [28] and [23], where the proofs of the mentioned lemmas can be found.

Before defining a stack we need the definition of a scheme and a sheaf. Loosely defined a scheme is a structure that enlarges the notion of an algebraic variety. A *Scheme* is a topological space together with commutative rings for all of its open subsets, which arises from gluing together spaces of prime ideals of commutative rings along their open subsets. So it is a locally ringed space, which is locally a spectrum of a commutative ring.

Definition B.1 (Affine scheme)**.** An *affine scheme* is a locally ringed space isomorphic to the spectrum $\mathrm{Spec}(A)$ of a commutative ring A.

On can think of a scheme as being covered by coordinate charts of affine schemes

Definition B.2 (Scheme)**.** A *scheme* is a locally ringed space X admitting a covering by open sets U_i, such that the restriction of the structure sheaf $\mathcal{O}_X$ to each U_i is an affine scheme.

Definition B.3 (Presheaf)**.** Let X be a topological space, let C be a category. A *presheaf* F on X is a contravariant functor $\mathrm{Open}(X)$ to C, where $\mathrm{Open}(X)$ has the objects open subsets of X and morphisms inclusions.

Definition B.4 (Sheaf)**.** A *sheaf* is a presheaf with values in the category of sets that satisfies

Locality If (U_i) is an open covering of an open set U, and if $s, t \in F(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each set U_i of the covering, then $s = t$

Gluing If (U_i) is an open covering of an open set U, and for each i we have a section $s_i \in F(U_i)$ such that for each pair U_i, U_j of the covering sets, the restrictions of s_i and s_j agree on $U_i \cap U_j$, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i.

As a reason to inventing stacks we can look at the following problem. It would be nice to describe a classifying space for vector bundles of rank n. So we want a space $B\mathrm{GL}_n$ such that for any scheme T

$$\mathrm{Mor}(T, B\mathrm{GL}_n) = \mathrm{Cat}(\text{Vector bundles of rank } n \text{ on } T)/\text{isomorphisms},$$

but such a space cannot exist, since every vector bundle on $\mathcal{E}$ on T is locally trivial, so the map $T \to B\mathrm{GL}_n$ corresponding to $\mathcal{E}$ is locally constant, so $\mathcal{E}$ will be globally trivial. So we cannot make this definition, even though it would have been nice, since in topology we do have a classifying space $B\mathrm{GL}_n$ such that for any space T, the homotopy classes of maps $f\colon T \to B\mathrm{GL}_n$ correspond to isomorphism classes of vector bundles on T. The problem is that we do not have a good algebraic replacement for homotopy classes of maps. To solve this problem we can choose not to pass to isomorphism classes, and this is what is done when defining *stacks*.

Remark B.5. Any scheme X is determined by its functor of points. So X is determined by the functor

$$\mathrm{Mor}(\cdot, X)\colon \text{Schemes} \to \text{Sets}$$

sending a scheme T to the set $\mathrm{Mor}(T, X)$. This functor is a sheaf, in the sense that a morphism $T \to$ can be obtained from glueing morphisms on a covering of T.

To define a stack we use this idea. We define a stack to be given by its functor of points. For example for $B\mathrm{GL}_n$ we define $B\mathrm{GL}_n(T)$ for a scheme T to be the category of vector bundles of rank n on T.

Definition B.6 (Stack)**.** A *stack* is a sheaf of groupoids

$$\mathcal{M}\colon \text{Schemes} \to \text{Groupoids} \subseteq \text{Categories}.$$

By this we mean that a stack is the following assignment

- For any scheme T we assign a category $\mathcal{M}(T)$ in which all morphisms are isomorphisms
- For any morphism $f\colon T \to S$ between schemes we assign a functor $f^*\colon \mathcal{M}(S) \to \mathcal{M}(T)$
- For any pair of composable morphisms $R \to^f S \to^g T$ we assign a natural transformation $\varphi_{f,g}\colon f^* \circ g^* \to (g \circ f)^*$. These transformations should be associative for composition. If f or g is the identity, then $\varphi_{f,g}$ should be the identity.

The assignment should satisfy that objects glue and morphisms glue. In other words

- By objects gluing we mean, given a covering $U_i \to T$ of a scheme T, objects $\mathcal{E}_i \in \mathcal{M}(U_i)$ and isomorphisms $\varphi_{ij}\colon \mathcal{E}_i|_{U_i \cap U_j} \to \mathcal{E}_j|_{U_i \cap U_j}$ that satisfy a cocycle condition on 3-fold intersections, there exists an object $\mathcal{E} \in \mathcal{M}(T)$, which is unique up to isomorphisms, and isomorphisms $\psi_i\colon \mathcal{E}_i|_{U_i} \to \mathcal{E}_i$ such that $\varphi_{ij} = \psi_j \circ \psi_i^{-1}$.
- By morphisms glueing we mean, given a covering $U_i \to T$, object $\mathcal{E}, \mathcal{F} \in \mathcal{M}(T)$ and morphisms $\varphi_i\colon \mathcal{E}|_{U_i} \to \mathcal{F}|_{U_i}$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there is a unique morphism $\varphi\colon \mathcal{E} \to \mathcal{F}$ such that $\varphi|_{U_i} = \varphi_i$.

We can now give some simple examples of stacks

Example B.7 ([28]). [Bun_n] Let C be a smooth projective curve. Let Bun_n be the stack given by

$$\mathrm{Bun}_n(T) := \mathrm{Cat}(\text{Vector bundles of rank } n \text{ on } C \times T).$$

The morphisms in the category are isomorphisms of vector bundles and the functors f^* are given by the pull back of bundles. The gluing condition are satisfied by descent for vector bundles.

Example B.8 ([28]). For schemes S and T let $\mathrm{Mor}(T, S)$ denote the category of morphisms from T to S, in which the only morphisms of the category, are the identities. Let the pull back functors f^* for $f: S \to T$ be given by composition with f. Then given scheme S, $\underline{S}(T) := \mathrm{Mor}(T, S)$ defines a stack.

Lemma B.9 ([28])**.** *Let $\mathcal{M}$ be a stack. Then for any scheme T there is a natural equivalence of categories*

$$\mathrm{Mor}_{stacks}(\underline{T}, \mathcal{M}) \simeq \mathcal{M}(T).$$

So using this lemma we see that one can choose to write T instead of $\underline{T}$.

Definition B.10 (Algebraic Stack)**.** A stack $\mathcal{M}$ is called algebraic if

1. For all schemes $X \to \mathcal{M}$ and $Y \to \mathcal{M}$ the fibre product $X \times_{\mathcal{M}} Y$ is representable. Note a functor $\mathcal{F}$ is called representable if there exists an object such that $\mathcal{F}$ is the functor of points for this object.

2. There exists a scheme $u: U \to \mathcal{M}$ such that for all schemes $X \to \mathcal{M}$ the projection $X \times_{\mathcal{M}} U \to X$ is a smooth projection.

3. The forgetfull map $\mathrm{Isom}(u, v) = U \times_{\mathcal{M}} U \to U \times U$ is quasi compact and separated.

An algebraic stack $\mathcal{M}$ is called smooth (respectively normal/ locally noetherian) if there exists an atlas $u: U \to \mathcal{M}$ with U being smooth (respectively normal/ locally noetherian).

Example B.11 ([28] , Bun_n). Bun_n is a smooth stack. In [28] Example 1.14 they construct an atlas for Bun_n and argue that this can be shown to be smooth.

Lemma B.12 ([28] Cor. 3.4)**.** *Let $\mathcal{M}$ be a smooth, noetherian algebraic stack and $\mathcal{U} \subset \mathcal{M}$ an open substack. Let $\mathcal{L}_{\mathcal{U}}$ be a line bundle on $\mathcal{U}$, then there exists a line bundle $\mathcal{L}$ on $\mathcal{M}$ such that $\mathcal{L}|_{\mathcal{U}} \simeq \mathcal{L}_{\mathcal{U}}$.*

Example B.13 ([28] , Bun_n). There is a universal vector bundle $\mathcal{E}_{\mathrm{univ}}$ on $C \times \mathrm{Bun}_n$ because any morphism $T \to \mathrm{Bun}_n$ defines a bundle on $C \times T$.

To give a line bundle on Bun_n is the same as a functorial assignment of a line bundle to any family of vector bundles.

Definition B.14 (Coarse moduli space)**.** Let $\mathcal{M}$ be a algebraic stack. An algebraic space M together with a map $p: \mathcal{M} \to M$ is called a *coarse moduli space* for $\mathcal{M}$ if

- For all schemes T and morphisms $q: \mathcal{M} \to T$ there exists a unique morphism $M \to T$ making the diagram commutative

$$\begin{tikzcd} \mathcal{M} \arrow[rr] \arrow[dr] & & T \\ & M \arrow[ur] & \end{tikzcd}$$

- For all algebraicaly closed fields $\overline{K}$ we have $\mathcal{M}(\overline{K})/\text{isomorphism} = M(\overline{K})$.

Example B.15 ([28]). In [28] they argue that if we let $\mathrm{Bun}_n^{\mathrm{stable}}$ be the moduli stack for stable bundles on a curve. Then the coarse moduli space of stable bundles M^{stable} constructed by geometric invariant theory is a coarse moduli space for $\mathrm{Bun}_n^{\mathrm{stable}}$.

A Poincare family is a vector bundle on $C \times M_n^{d,\mathrm{stable}}$ such that the fibre over every point of $M_n^{d,\mathrm{stable}}$ lies in the isomorphism class of bundles defined by this point. So such a bundle is the same as a section of the map $\mathrm{Bun}_n^{d,\mathrm{stable}} \to M_n^{d,\mathrm{stable}}$.

Theorem B.16 ([28] Cor. 3.12)**.** *Let C be a curve with genus bigger than* 1*.*

- *If $(n, d) = 1$ then there ecists a Poincare family on the coarse moduli space $M_n^{d,stable}$ of stable vector bundles on C*
- *If $(n, d) \neq 1$ then there is no open subset $U \subset M_n^{d,stable}$ ($U \neq \emptyset$) such that there exists a Poincare family on $C \times U$.*

Appendix C

Fibre bundles

Of course the notion of fibre bundles is well known, but in calculations throughout this thesis, we need some basic results about fibre bundles, for example how to get a line bundle from a cocycle or when a principal G-bundle is trivializable. In this appendix these things are written out.

A fibre bundle is a space, that locally is a product, but might look different globally.

Definition C.1 (Fibre bundle)**.** A *fibre bundle* is a (E, B, F, π) where E, B and F are topological spaces and $\pi\colon E \to B$ is a continuous surjection such that for all $x \in B$ there exists a neighborhood U_x of x and a homeomorphism $\varphi_x\colon \pi^{-1}(U_x) \to U_x \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_x) \arrow[r] \arrow[dr, "\pi"'] & U_x \times F \arrow[d, "\mathrm{pr}_1"] \\ & U_x \end{tikzcd}$$

We call B the base space, E the total space, F the fibre and π the projection. The set of all $\{(U_x, \varphi_x) \mid x \in B\}$ is called a trivialization.

When we have a fibre bundle $E \to B$, we can look at the set of trivializations, and from them define the notion of a transition function.

$$h_{\alpha\beta} := \varphi_\alpha \varphi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F.$$

For pairs U_α, U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, and triplets $U_\alpha, U_\beta, U_\gamma$ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, these map satisfies

$$h_{\alpha\alpha}(x) = \mathrm{Id}_F \ , \ h_{\alpha\beta}(x) = h^{-1}_{\beta\alpha}(x) \text{ and } h_{\alpha\beta} \circ h_{\beta\gamma}(x) = h_{\alpha\gamma}(x).$$

The last equation is called the cocycle condition.

Definition C.2 (Section)**.** A *section* of a fibre bundle is a continuous map $s\colon B \to E$ such that $\pi \circ s = \mathrm{id}_X$. Note that not all fibre bundles have globally defined sections.

Important examples of fibre bundles are the trivial bundle and line bundles.

Definition C.3 (Trivial bundle)**.** Let $E = B \times F$, let π be the projection onto the first factor. Then $E \to B$ is a fibre bundle called the *trivial bundle*.

A special case of fibre bundles are vector bundles, which are those bundles whose fibers are vector spaces.

Definition C.4 (Line bundle)**.** A line bundle is a vector bundle, where all fibers have dimension 1.

Sometimes it is useful to know, that if we have a cocycle, we can construct a line bundle, by seeing the cocycle as the transition functions. The construction is the following. Let $(U_\alpha)_{\alpha\in A}$ be a covering of a space X. Let $g_{\alpha,\beta}\colon U_{\alpha\beta} \to \mathrm{GL}(1)$ be the cocycle, that is $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. Then we get a line bundle by setting

$$\amalg_{\alpha\in A} U_\alpha \times \mathbb{R}/(x,t) \sim (x, g_{\alpha\beta}(x)t) \text{ for } x \in U_{\alpha\beta}$$

Example C.5 (Endomorphism bundle). From a vector bundle $E \to X$ we can construct a new vector bundles called the *Endomorphism bundle* $\mathrm{End}(E) \to X$, where the total space now consists of all the endomorphisms $E \to E$.

The next lemma gives us a way to check if a vector bundle is trivial

Lemma C.6. *Let $E \to X$ be a rank n vector bundle. Then E is trivial if and only if π admits n sections $s_1,\dots,s_n$ such that $s_1(x),\dots,s_n(x)$ form a basis of E_x for any $x \in X$.*

Proof. To see this we assume E is trivial. Then $E \simeq^\varphi X \times \mathbb{R}^n$. Define $s_i(x) = \varphi^{-1} \circ e_i$, where $e_i\colon X \to X \times \mathbb{R}^n$ sends x to $(x, 0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the i'th place. Conversely if E admits n such sections, define a map $E \to X \times \mathbb{R}^n$ by $s_i(x) \to (x, 0, \dots, 0, 1, 0, \dots, 0)$, again with 1 on the i'th place and extend linearly. □

Definition C.7 (Principal G-bundle)**.** Let M be a manifold and G a Lie-group. A *Principal G-bundle over M* consists of a smooth manifold P that satisfies

- There is a free right action of G on P, such that M is the quotient space under this action and the projection $\pi\colon P \to P/G = M$ is smooth.
- P is locally trivializable, that is for every point in M there exists a neighborhood U with an equivariant diffeomorphism $\pi^{-1}(U) \to U \times G$ covering the identity on M.

Several times we are going to need, that if G is a simply connected group, and Σ is a surface, then any principal G-bundle $P \to \Sigma$ is trivializable, i.e. $P \simeq \Sigma \times G$.

Lemma C.8. *Let G be a simply connected Lie-group, and Σ a surface. Then any principal G-bundle P is trivializable*

Proof. Let G be a simply connected Lie-group and Σ a oriented surface. Let $P \to \Sigma$ be a principal G-bundle. As in [36] define

$$EG := \Big\{(t_1g_1,\dots,t_kg_k,\dots) \in ([0,1]\times G)^{\mathbb{N}} \;\Big|\; \sum_i t_i = 1 \text{ and } t_i = 0 \text{ for all but finitely many } i\Big\}.$$

Let $BG = EG/G$. Then it is proven in [36] that there exists $f\colon \Sigma \to BG$ such that $f^*(BG) \simeq P$. So if we can prove f is homotopic to the identity, then $P \simeq \Sigma \times G$, as we wanted.

BG is a connected CW-complex, so $\pi_0(BG) = 0$. From [36] we have that $\pi_i(BG) \simeq \pi_{i-1}(G)$. Since G is assumed to be simply connected we have $\pi_0(G) = \pi_1(G) = 0$, so $\pi_1(BG) = \pi_2(BG) = 0$. This means there exists a CW-complex homotopy equivalent to BG consisting of cells of dimension greater than or equal to 3. Since Σ is a surface, the cells of this CW-complex will be of dimension less than or equal to 2. Hence f has to be null-homotopic. Which means we have proven our lemma. □

Definition C.9 (Adjoint bundle)**.** Let P be a principal G-bundle, for G a Lie-group with Lie-algebra $\mathfrak{g}$.

Let $\Psi\colon G \to \mathrm{Aut}(G)$ be the map that sends $g \in G$ to Ψ_g, where $\Psi_g(h) = ghg^{-1}$. Let $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ be the map that sends $g \in G$ to Ad_g, where $\mathrm{Ad}_g = d(\Psi_g)_e\colon \mathfrak{g} \to \mathfrak{g}$.

Then we can define the adjoint bundle as

$$\mathrm{Ad}(P) := P \times \mathfrak{g}/[p \cdot g, x] \sim [p, \mathrm{Ad}_{g^{-1}(x)}] = P \times_{\mathrm{Ad}} \mathfrak{g}$$

Definition C.10. A *connection* on a principal G-bundle $P \to M$ is a $\mathfrak{g}$-valued one form A on P such that

- $A(X^*) = X$ for all $X \in \mathfrak{g}$. By X^* we mean the vector field on P associated to X by differentiating the G action on P.
- A is G-equivariant in the sense that for all $g \in G$: $r_g^*(A) = \mathrm{Ad}_{g^{-1}} A$, where $r_g\colon P \to P$ denote the map $r_g(p) = p \cdot g$.

Definition C.11. A *principal bundle homomorphism* between two principal G-bundles P and P' is a G-equivariant bundle homomorphism. If $P = P'$ it is called a *gauge transformation*. Denote by $\mathcal{G}_P$ the group of all gauge transformations $P \to P$.

Remark C.12. Let $u\colon P \to G$ be a G-equivariant map, $p \mapsto u_p$. We can associate a gauge transformation $\Phi\colon P \to P$ by letting $\Phi(p) = p \cdot u_p$.

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