PhD Thesis

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Graph complexes and cohomology of configuration spaces

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A page of the spectral sequence converging to the cohomology of the configuration space,
Paul Klee

Abstract

We study the generalised configuration space of points in a manifold depending on a graph, originally defined by *Eastwood* and *Huggett*. In particular, we examine its cohomology through graph complexes. One of those is the graph complex defined by *Baranowsky* and *Sazdanović*, denoted by $\mathcal{C}_{BS}$ that is the E_1 page of a spectral sequence converging to the homology of this type of configuration space. We compare $\mathcal{C}_{BS}$ with the graph complex GC defined by *Kontsevich* by defining a map between them.

In order to compute the rational homotopy type of the classical configuration space, *Kriz* and *Totaro* define a commutative differential graded algebra that serves as a rational model for it in the case the manifold is a complex projective variety. We generalise this commutative differential graded algebra by describing the complex $R(\Gamma, A)$, that depends on a graph Γ and on a commutative differential graded algebra A. We prove that the dual complex of $\mathcal{C}_{BS}$ is quasi equivalent to $R(\Gamma, A)$. In the case Γ is a complete graph and M is an even dimensional manifold, $R(\Gamma, A)$ is the commutative differential graded algebra that *Idrissi* proves to be a real model for the classical configuration space of points in M.

Finally, we compute the cohomology of the configuration space dependent on a graph of points in $\mathbb{R}^r$, $r \geq 0$. This is a generalization of the classical computation due to *Arnold* and *Cohen* that correspond to the case where the graph is complete. The cohomology of this graph configuration space is the cohomology of the *painted* little disks operad, that we define as a variation depending on a graph of the classical little disks operads.

Resumé

Vi studerer et generaliseret konfigurationsrum af punkter på en mangfoldighed, som oprindeligt blev defineret af *Eastwood* og *Huggett* og afhænger af en graf. Vi bruger grafkomplekser til undersøge dets kohomologi. Et af disse er grafkomplekset $\mathcal{C}_{BS}$, som blev defineret af *Baranowsky* og *Sazdanović*, og er den første side i en spektralfølge, der konvergerer til homologien for denne type konfigurationsrum. Vi sammenligner $\mathcal{C}_{BS}$ med grafkomplekset GC, defineret af *Kontsevich*, ved at beskrive et afbildning mellem dem.

For at beregne den rationelle homotopitype af det klassiske konfigurationsrum definerer *Kriz* og *Totaro* en differentieret algebra, der fungerer som en model for den rationelle homotopitype i tilfældet, hvor mangfoldigheden er en kompleks projektiv varietet. Vi konstruerer en ny algebra, $R(\Gamma, A)$, som er en generalisering af *Kriz* og *Totaro's* algebra, der ydderligere afhænger af en graf. Vi beviser at den er quasi-ækvivalent med det duale grafkompleks til $\mathcal{C}_{BS}$. Når Γ er en komplet graf og dimensionen af M er en lige, så er $R(\Gamma, A)$ en gradueret kommutativ, differential algebra, som *Idrissi* har bevist giver en model for den reelle homotopitype af det klassiske konfigurationsrum af punkter på M.

Endelig, beregner vi kohomologien af det generaliserede konfigurationsrum, som afhængigt af en graf, af punkter i $\mathbb{R}^r$, $r \geq 0$. Dette er en generalisering af en klassisk beregning af *Arnold* og *Cohen*, når grafen er komplet. Dette konfigurationsrums kohomologi er isomorft til kohomologien af *painted little disks operads*, som vi definerer til at være en variation, der afhænger af en graf, af de klassiske *little disks operads*.

Contents

Introduction

The present thesis is the result of my three year Ph.D. program at Aarhus University. The work focuses on the topological properties of a type of configuration space where some points are allowed to coincide. This can be encoded in a graph where the vertices correspond to the points in the configuration space and the edges record which of them are not allowed to overlap.

In the case where Γ is the complete graph this is the usual configuration space of n ordered points in a space X. We denote it by $\mathfrak{C}(X, n)$, where

$$\mathfrak{C}(X, n) = X^n \setminus \bigcup \Delta_{i,j}$$

$\Delta_{i,j} = \{(x_1, \ldots, x_n) \in X^n : x_i = x_j\}$. If the graph Γ is not necessary complete, the configuration space obtained has been defined by *Eastwood*and *Huggett*[12], in the study of the chromatic polynomial. Let M be a manifold and Γ a graph with set of vertices $V = \{v_1, \ldots, v_n\}$. Then the configuration space of n points in M, which we denote $\mathfrak{C}(M, \Gamma)$, is defined as

$$\mathfrak{C}(M, \Gamma) = M^n \setminus \bigcup_{e \in E(\Gamma)} \Delta_e,$$

where $E(\Gamma)$ is the set of edges in Γ, $\Delta_e = \{(m_1, \ldots, m_n) \in M^n; m_i = m_j\}$ and e is a directed edge from v_i to v_j. In 2012 *Baranovsky* and *Sazdanović* [2] define a graph cohomology inspired by the one defined by *Helme-Guizon* and *Rong* [16]. We will call this graph complex $\mathcal{C}_{BS}(\Gamma)$ and it is defined by

$$\mathcal{C}_{BS}(\Gamma) = \bigoplus_{S \subset E(\Gamma)} e_S \otimes A^{\otimes l(S)}$$

where A is a graded commutative algebra. This is related to the cohomology of configuration space. In [2] it is proven that there is a spectral sequence with E_1 page isomorphic to $\mathcal{C}_{BS}(\Gamma)$, converging to the homology of the *Eastwood* and *Huggett's* configuration space $\mathfrak{C}(M, \Gamma)$. The spectral sequence for the case where the graph is the complete graph was given in 1991 by *Bendersky* and *Gitler* [3]. Moreover, the complex $\mathcal{C}_{BS}(\Gamma)$ appears to be linked with *Hochschild homology* [33].

In 1992 *Kontsevich* [23][24] defined the graph homology complex (GC, d). This is given by

$$\mathrm{GC} = \bigoplus_k \mathrm{GC}_k$$

where GC_k is the $\mathbb{Q}$-vector space generated by the isomorphism classes of graphs with k vertices, for every $k \in \mathbb{N}$. The differential is

$$d(\Gamma) = \sum_{e \in E(\Gamma)} \Gamma/e,$$

here Γ/e is the graph obtained from Γ by contracting an edge e belonging to the set of edges $E(\Gamma)$. The classes of graphs are equipped with an orientation whose definition depends on the parity of a number n that represent the dimension of a manifold. Moreover, in the proof of formality of little disks operads, *Kontsevich* defines other graphs complexes. These latter, as well as the GC complex are involved in the computation of the real homotopy type of configuration spaces, as proved by *Willwacher* [8] and *Idrissi* [19], who define a rational model for the classical configuration space of points in a manifold, $\mathfrak{C}(X, n)$.

However, the study of the rational homotpopy type of $\mathfrak{C}(X, n)$ has a longer history. In 1994 *Fulton* and *MacPherson* [14] constructed a rational model for the configuration space when X is a non singular, compact, complex variety. This model depends on the cohomology ring $H^*(X, \mathbb{Q})$, the orientation and the Chern classes. The same year, *Kriz* in [26] described a differential graded algebra $E[n]$ that is a rational model for $\mathfrak{C}(X, n)$ and that is independent from the Chern classes. $E[n]$ is described in the same time in the work by *Totaro* [37] and it appeared to be isomorphic to the E_2 page of the Larey spectral sequence of the inclusion $\mathfrak{C}(X, n) \hookrightarrow X^n$. Later, *Lambecht* and *Stanley* studied the rational models for configurations spaces where X is a simply connected closed manifold. In 2004 they described the case $k = 2$, a configuration space of 2 points in a manifold [28]. In 2008 *Lambecht* and *Stanley* [30] presented a potential model, called G_A, for the general case such that the previous models, for example $E[n]$ are special cases of it. In 2019 *Idrissi* [19] proves that this is an actual model for the real homotopy type of configuration spaces. The proof makes use of the graph complexes introduced by *Kontsevich*.

In this thesis we compare the various graph complexes related to the homology or cohomology of configuration spaces and we extend some of the definitions to generalised configuration spaces depending on a graph. The work is divided in six chapter, of which the first three are background notions and previous results on which the final three chapters are based. These last present results that I have obtained under the supervision of *M. Bökstedt*. To summarize the content of this work, we briefly describe the various chapters, starting by the introductory ones.

The first chapter recalls the basic definitions of the theory of operads and describes *little disks operads*, the *partial operad of configuration spaces* and the construction of *spiders* that occurs in [10]. In the second chapter we present the aforementioned graph complexes. We first give the definition of *Kontsevich*'s graph complexes followed by the definition of $\mathcal{C}_{BS}$, together with results on the homology of configuration spaces. A brief introduction to homotopy theory and *Poincarè duality commutative differential graded algebras* is given in the third chapter. This serves as a background to the later section in the same chapter where the rational homotopy type of configuration spaces is discussed. Here we also give a short history of the study of rational homotopy types of configuration spaces. Moreover, we describe the *Kontsevich* graph complexes involved in the description of the real model for $\mathfrak{C}(X, n)$ carried out in [19] and [8].

The last three chapter are the core of this thesis. In *Chapter* 4 we compare the complex $\mathcal{C}_{BS}$, described above, with the graph complex GC defined by *Kontsevich*. In particular, we construct a variation of CG, called $\mathcal{K}(\Gamma)$. This chain complex differs from GC by the fact that it depends on a graph Γ and it is generated by graphs obtained from Γ by collapsing a subgraph, without considering isomorphism classes. The definition of $\mathcal{K}(\Gamma)$ varies, as in the case of GC according to the parity of a natural number. We therefore discuss the map between $\mathcal{C}_{BS}$ and $\mathcal{K}(\Gamma)$ in two cases: even and odd. We then build the complex $\mathcal{K}_\Sigma(\Gamma)$, generated by isomorphism classes of graphs obtained from Γ by contracting a subgraph. We build again a map for even and odd case from $\mathcal{K}(\Gamma)$ to $\mathcal{K}_\Sigma(\Gamma)$, and we discuss the relation between this last complex and GC.

The fifth chapter is based on the article by *Bökstedt* and *Minuz* [6], and it examines some results arising from the study of the rational homotopy type of configuration spaces. We compare the model $E[n]$, mentioned above, with the complex $\mathcal{C}_{BS}$. We generalise the definition of $E[n]$ by defining the differential graded algebra $R(\Gamma, A)$ depending on a graph Γ and with coefficients in any Frobenius algebra A. In the case where Γ is the complete graph, and $A = H^*(M)$ where M is a closed oriented compact manifold of even dimension, then $R(\Gamma, A) = G_A$ where G_A is the commutative differential graded algebra that *Idrissi* [19] proved to be a real model for $\mathfrak{C}(X, n)$. We define a map between the dual of $\mathcal{C}_{BS}(\Gamma)$ and $R(\Gamma, A)$, $F : \mathcal{C}_{BS}(\Gamma)^* \to R(\Gamma, A)$. The main theorem of the chapter is

Theorem. *The map*

$$F : \mathcal{C}_{BS}(\Gamma)^* \to R(\Gamma, A)$$

is a quasi equivalence.

The aim of the sixth and last chapter is to generalise the definition of *little disks operads* to the case of a configuration space depending on a graph. Little disks operads are operads given by embeddings of disjoint unions of disks. We describe in detail this construction in the first chapter, as well as the definition of the partial operad of configuration space. Little disks operads are formal and their cohomology is related to the cohomology of configuration spaces of points over $\mathbb{R}^r$, $r \geq 0$. We generalise the definition of the partial operad of configuration spaces and little disks operads to configuration spaces of the type $\mathfrak{C}(\mathbb{R}^r, \Gamma)$ over $\mathbb{R}^r$, that we call $\mathfrak{C}_r(\Gamma)$. We call this last operad *painted little disks operad*. Moreover, we compute the cohomology of this configuration space. In the case where Γ is complete, the computation is a classical result due to *Arnold* [1] and *Cohen* [9]. We adapt this result to $\mathfrak{C}_r(\Gamma)$ and we discuss two separate cases, when r is even and then when it is odd. It is a conjecture, that the construction of the *Fulton and McPherson-operad* can be extended to the generalised case. Moreover, if that is the case, one can ask if the *painted little disks operad* is formal.

Finally, one can inquire if the constructions presented in this thesis can be reproduced for a more general type of configuration space depending on a graph Γ and a function k on the set of vertices. This appears in two recent papers by *Bökstedt* and *Nuno Romão* [7] and *Bökstedt* [5] and it is related to the groups of generalised braids on Σ. The strands of the braids are colored, and some are allowed to pass through each other according to some rules. These are encoded in a graph Γ, where each vertex represents a color, and if there is an edge between two vertices, then the strands of these two colors are not allowed to intersect. On the other hand strands of the same color or of colors not connected by an edge can pass through each other. There can be multiple strands of the same color and this is represented by a function $\mathrm{k} : V(\Gamma) \to \mathbb{N}$ assigning a number of strands for every vertex. The equivalence classes of homotopic generalised braids on Σ with composition, denoted by $\mathrm{DB}_{\mathrm{k}}(\Sigma, \Gamma)$, is the fundamental group of a generalised kind of configuration space $\mathfrak{C}_{\mathrm{k}}(\Sigma, \Gamma)$, depending on Γ and $\mathrm{k} : V(\Gamma) \to \mathbb{N}$. Let Σ be a connected, oriented and closed surface, let $S^i(\Sigma)$ be the symmetric product of Σ. We define $S^{\mathrm{k}}\Sigma = \prod_{i \in V(\Gamma)} S^{\mathrm{k}(i)}\Sigma$ and

$$\mathfrak{C}_{\mathrm{k}}(\Sigma, \Gamma) = \{(x_1, ..., x_r) \in S^{\mathrm{k}}\Sigma : x_i \neq x_j \text{ if } \alpha_{i,j} \in E(\Gamma)\ \}$$

where $\alpha_{i,j}$ denote an edge between the vertices i and j, in the set of edges of Γ, $E(\Gamma)$. In this regard, this thesis represents a departure for future investigation.

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Chapter 1

Operads

1.1 Conventions and notation regarding graphs

We define a *graph* to be a 1-dimensional CW-complex, defined by the set of vertices, that we denote by $V(\Gamma)$, and the set of edges between them $E(\Gamma)$. We denote $\alpha_{a,b}$ an edge between the vertices a, b and $\alpha : a \to b$ when the edge is directed from a to b. In general we consider edges to be undirected, while we will specify when we add a direction on them. We call *loop* an edge between the same vertex $\alpha_{a,a}$, while *cycle* will denote a subset of edges of the form $\{\alpha_{1,2}, \alpha_{2,3}, \dots, \alpha_{k,1}\}$. A *tree* is then a non empty connected graph without loops or cycles and *forest* is the disjoint union of trees. Finally, if $e \in E(\Gamma)$,

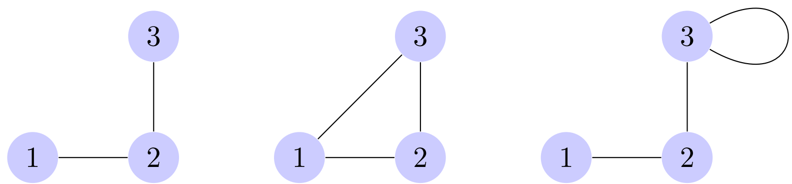

Figure 1.1: A tree, a graph with a cycle and a graph with a loop.

we denote by $\Gamma \smallsetminus e$ the graph obtained from Γ by deleting the edge e and Γ/e the graph obtained from Γ by contracting the edge e.

1.2 Operads

In this section we introduce the basics of the theory of *operads*. Operads are objects that encode the structure of an algebra. In particular, they describe its operations, and the word operad itself comes from the composition of *operation* and *monad*, as operad can be seen as a monad defining operations [38]. The operad was defined for the first time in 1972 in the article *The geometry* *of iterated loop spaces* by *May* [32]. However, the idea appeared earlier, for example in the works of *Boardman* and *Vogt*. Recently, the theory of operad has found applications in different branches of mathematics, from category theory to mathematical physics and it is in constant development.

We first give the definition of an operad. Then we focus on one type of operad, namely the *little disks operad*, that will be defined and characterized by some results that will be used later in this thesis. Finally, we discuss the partial operad of configuration spaces.

1.2.1 Definition of operad

We present operads over a *symmetric monoidal category*. Our definition corresponds to the one originally given by *May*, but in the language of category theory. We refer additionally to [31] and [13].

Definition 1.2.1 ([31], [22]). A *monoidal category* $(\mathcal{C}, \otimes, \mathbf{1})$ is a category $\mathcal{C}$ with a covariant functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and an object $\mathbf{1}$ in $\mathcal{C}$ called *unit object* that fulfill the following conditions.

- There is a commutative diagram

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \otimes"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] & \mathcal{C} \end{tikzcd}$$

- Let $\mathbf{1}$ be the category with only one element $\mathbf{1}$, and one morphism. There is a functor $\eta : \mathbf{1} \to \mathcal{C}$ such that the diagrams commute.

$$\begin{tikzcd} \mathbf{1} \times \mathcal{C} \arrow[r, "\eta \times \mathrm{id}"] \arrow[dr, "p"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\ & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C} \times \mathbf{1} \arrow[r, "\mathrm{id} \times \eta"] \arrow[dr, "p"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\ & \mathcal{C} \end{tikzcd}$$

  here p is the projection.

A monoidal category $\mathcal{C}$ is said to be *symmetric* if for every two objects X, Y in $\mathcal{C}$ there is a natural transformation, called *twist map*

$$\tau_{X,Y} : X \otimes Y \to Y \otimes X$$

such that

$$\tau_{Y,X}\tau_{X,Y} = \mathrm{id}_{X\otimes Y}$$

and the commutativity of the diagrams is satisfied.

$$\begin{tikzcd} X\otimes Y\otimes Z \arrow[r, "\tau_{X,Y}\otimes\mathrm{id}"] \arrow[dr, "\tau_{X,Y\otimes Z}"] & Y\otimes X\otimes Z \arrow[d, "\mathrm{id}\otimes\tau_{X,Z}"] \\ & Y\otimes Z\otimes X \end{tikzcd}$$

$$\begin{tikzcd} X\otimes Y\otimes Z \arrow[r, "\mathrm{id}\otimes\tau_{Y,Z}"] \arrow[dr, "\tau_{X\otimes Y,Z}"] & X\otimes Z\otimes Y \arrow[d, "\tau_{X,Z}\otimes\mathrm{id}"] \\ & Z\otimes X\otimes Y \end{tikzcd}$$

Remark 1.2.2. We denote the operation in a general monoidal category by $\otimes$, which is an abuse of notation, since generally $\otimes$ denotes the tensor product in the category of vector spaces, algebras or graded algebras. However, this choice is justified by the fact that this simplifies the notation, since we will mostly consider monoidal categories with multiplication given by the tensor product.

Example 1.2.3. Examples of symmetric monoidal categories are the category of vector spaces over a field $\mathbb{K}$ with tensor product $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$, the category of sets with the Cartesian product $(\mathbf{Set}, \times, \{*\})$, the category of Frobenius algebras over $\mathbb{K}$ with tensor product $(\mathbf{Frob}_{\mathbb{K}}, \otimes, \mathbb{K})$.

Definition 1.2.4. ([13]) An operad $\mathcal{O}$ on a symmetric monoidal category $\mathcal{C}$ is a sequence of objects $\mathcal{O}(n)$ in $\mathcal{C}$, $n \in \mathbb{N}$, such that there is

- an action of the symmetric group Σ_n on $\mathcal{O}(n)$
- a composition

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$$

  for all $k_1, \ldots, k_n \geq 1$ and we write

$$x(n) \circ x(k_1) \otimes \cdots \otimes x(k_n) \in \mathcal{O}(k_1 + \cdots + k_n)$$

  for every element $x(n)\otimes x(k_1)\otimes\cdots\otimes x(k_n) \in \mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\cdots\otimes\mathcal{O}(k_n)$

- an element $1 \in \mathcal{O}(1)$ called unit and a unit morphism

$$\mu : 1 \to \mathcal{O}(1)$$

Moreover the composition and the unit morphisms are required to satisfy some axioms expressed by the commutativity of the following diagrams

- Associativity:

$$\begin{tikzcd}
\begin{array}{c}(\mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\cdots\otimes\mathcal{O}(k_n))\\ \otimes\,\mathcal{O}(k_{1,1})\otimes\cdots\otimes\mathcal{O}(k_{n,k_n})\end{array} \arrow[r, "l"] \arrow[d, "\cong"] & \mathcal{O}(k_{1,1}+\cdots+k_{n,k_n}) \\
\begin{array}{c}\mathcal{O}(n)\otimes(\mathcal{O}(k_1)\otimes\cdots\otimes\mathcal{O}(k_n)\\ \otimes\,\mathcal{O}(k_{1,1})\otimes\cdots\otimes\mathcal{O}(k_{n,k_n}))\end{array} \arrow[ur, "h"] &
\end{tikzcd}$$

  where $l=\circ(\circ\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id})$ and $h=\circ(\mathrm{id}\otimes\circ\otimes\cdots\otimes\circ)$

- Equivariance: Let $\sigma\in\Sigma_n$ be a permutation, $\sigma_i\in\Sigma_{k_i}$ and we define $\sigma'\in\Sigma_{k_1+\cdots+k_n}$ to be composition of the permutations σ and $(\sigma_1,\ldots,\sigma_n)$. Finally, $\sigma^*:\mathcal{O}(k_1)\otimes\cdots\otimes\mathcal{O}(k_n)\to\mathcal{O}(k_{\sigma(1)})\otimes\cdots\otimes\mathcal{O}(k_{\sigma(n)})$ is the permutation of the factors of the tensor product by σ.

$$\begin{tikzcd}
\mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\cdots\otimes\mathcal{O}(k_n) \arrow[r, "\sigma\bigotimes_i\sigma_i"] \arrow[d, "\mathrm{id}\otimes\sigma^*"] & \mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\cdots\otimes\mathcal{O}(k_n) \arrow[d, "\circ"] \\
\mathcal{O}(n)\otimes\mathcal{O}(k_{\sigma(1)})\otimes\cdots\otimes\mathcal{O}(k_{\sigma(n)}) \arrow[d, "\circ"] & \mathcal{O}(k_1+\cdots+k_n) \\
\mathcal{O}(k_{\sigma(1)}+\cdots+k_{\sigma(n)}) \arrow[ur, "\sigma'"] &
\end{tikzcd}$$

- Unit:

$$\begin{tikzcd}
1\otimes\mathcal{O}(n) \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[dr, "\cong"] & \mathcal{O}(1)\otimes\mathcal{O}(n) \arrow[d, "\circ"] \\
& \mathcal{O}(n)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}(n)\otimes 1^{\otimes n} \arrow[r, "\mathrm{id}\otimes\mu^{\otimes n}"] \arrow[dr, "\cong"] & \mathcal{O}(n)\otimes\mathcal{O}(1)^{\otimes n} \arrow[d, "\circ"] \\
& \mathcal{O}(n)
\end{tikzcd}$$

We can think of an operad $\mathcal{O}(n)$ as an n-ary operation with n input and one output. The next picture illustrates the operadic composition.

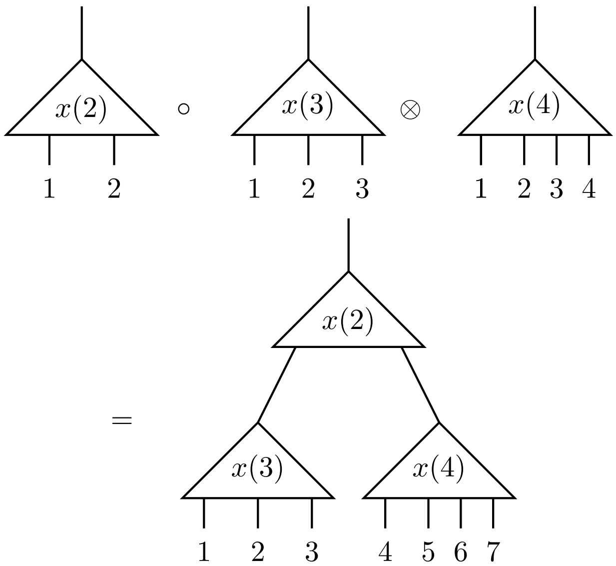

Figure 1.2: Operadic composition.

As mentioned, operads encode the structure of operations on an algebra. We call the algebra described by an operad *algebra over an operad*, and it is defined as follows:

Definition 1.2.5. ([13]) Let $\mathcal{O}$ be an operad over a symmetric monoidal category $\mathcal{C}$. An *algebra* $\mathcal{A}$ *over an operad* $\mathcal{O}$, or $\mathcal{O}$-algebra, is an object $\mathcal{A}$ in the category $\mathcal{C}$ with morphisms

$$\lambda : \mathcal{O}(n) \otimes \mathcal{A}^{\otimes n} \to \mathcal{A}$$

given for all $n \geq 0$, that respects the equivariance, associativity and unit relations. That is, the following diagrams commute for all $n \geq 0$:

- equivariance: Let $\sigma \in \Sigma_n$ be a permutation.

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \mathcal{A}^{\otimes n} \arrow[r, "\sigma\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\sigma^*"] & \mathcal{O}(n) \otimes \mathcal{A}^{\otimes n} \arrow[d, "\lambda"] \\
\mathcal{O}(n) \otimes \mathcal{A}^{\otimes n} \arrow[r, "\lambda"] & \mathcal{A}
\end{tikzcd}$$

- unit:

$$\begin{tikzcd}
1 \otimes \mathcal{A} \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[dr, "\cong"] & \mathcal{O}(1) \otimes \mathcal{A} \arrow[d, "\circ"] \\
& \mathcal{A}
\end{tikzcd}$$

- associativity: Let $X = (\mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n)) \otimes \mathcal{A}^{\otimes \sum k_i}$ and $Y = \mathcal{O}(n) \otimes (\mathcal{O}(k_1) \otimes \mathcal{A}^{k_1} \otimes \cdots \otimes \mathcal{O}(k_n) \otimes \mathcal{A}^{\otimes k_n})$.

$$\begin{tikzcd}
X \arrow[rr, "\cong"] \arrow[d, "\mu\otimes\mathrm{id}"] & & Y \arrow[d, "\mathrm{id}\otimes\lambda\otimes\cdots\otimes\lambda"] \\
\mathcal{O}(k_1+\cdots+k_n)\otimes\mathcal{A}^{\otimes\sum k_i} \arrow[dr, "\lambda"] & & \mathcal{O}(n)\otimes\mathcal{A}^{\otimes n} \arrow[dl, "\lambda"] \\
& \mathcal{A} &
\end{tikzcd}$$

Example 1.2.6. Examples of operads are the *commutative operads* and *associative operads* that encode the structure of commutative ad associative algebras. Another is the *little disks operads*, that we describe in the next paragraph.

In the next section we will consider *cyclic operads* in the category of real vector spaces with tensor product. We therefore, describe the condition for an operad to be cyclic.

Definition 1.2.7. ([31]) A *cyclic operad* is an operad $\mathcal{O}$ such that the action of Σ_n on $\mathcal{O}(n)$ extends to an action of Σ_{n+1} fulfilling the axioms given by the following commutative diagrams.

We denote by Σ_n^+ the group of automorphisms of the set $\{0, 1, \ldots, n\}$. Σ_n^+ is isomorphic to Σ_{n+1}, and Σ_n can be interpret as a subgroup of Σ_n^+ given by permutation σ such that $\sigma(0) = 0$. We denote by σ_n the permutation in Σ_n^+ given by $\sigma_n(0) = 1$, $\sigma_n(1) = 2, \ldots, \sigma_n(n) = 0$.

- Identity:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\mu"] \arrow[dr, "\mu"'] & \mathcal{O}(1) \arrow[d, "\sigma_1"] \\
& \mathcal{O}(1)
\end{tikzcd}$$

- The permutation s changes the position of $\mathcal{O}(n)$ and $\mathcal{O}(k)$ in the multiplication

$$\begin{tikzcd}
\mathcal{O}(n)\otimes\mathcal{O}(k)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} \arrow[r] \arrow[d, "\sigma_n\otimes\sigma_k"] & \mathcal{O}(n+k-1) \arrow[dd, "\sigma_{n+k-1}"] \\
\mathcal{O}(k)\otimes\mathcal{O}(n)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} \arrow[d, "s"] & \\
\mathcal{O}(k)\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\mathcal{O}(n) \arrow[r] & \mathcal{O}(n+k-1)
\end{tikzcd}$$

and

$$\begin{array}{ccc}
\mathcal{O}(n)\otimes \mathbf{1}\otimes\cdots\otimes \underbrace{\mathcal{O}(k)}_{i}\otimes\cdots\otimes\mathbf{1} & \longrightarrow & \mathcal{O}(n+k-1) \\
\downarrow{\scriptstyle \sigma_n\otimes\mathrm{id}} & & \downarrow{\scriptstyle \sigma_{n+k-1}} \\
\mathcal{O}(n)\otimes \mathbf{1}\otimes\cdots\otimes \underbrace{\mathcal{O}(k)}_{i-1}\otimes\cdots\otimes\mathbf{1} & \longrightarrow & \mathcal{O}(n+k-1)
\end{array}$$

We can think of a cyclic operad $\mathcal{O}(n)$ as obtained from an n-ary operation where the output is interpret as an input labeled by 0.

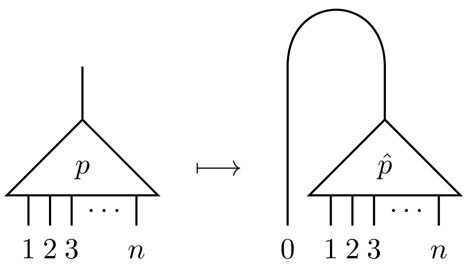

Figure 1.3: In a cyclic operad $\mathcal{O}(n)$ the action of the symmetric group Σ_n extend to an action of Σ_{n+1}[31].

1.2.2 Little disks operads

The *Little disks operad*, or LDO, is an operad such that for every n natural number the space $D_r(n)$ is the space of linear embeddings of n disks of dimension r in the r-dimensional unit disk. Little disks operads, were introduced in the 70s in the works by *Boardman* and *Vogt* [4] and *May* [32]. They have applications in topology, algebra and mathematical physics.

In this section we refer to [31] and [13] for the definitions.

Definition 1.2.8. ([13]) Let the standard r-disk be

$$\mathbb{D}_r = \{(x_1,\ldots,x_r)\in\mathbb{R}^r; x_1^2+\cdots+x_r^2\le 1\}\subset\mathbb{R}^r.$$

A linear embedding $c:\mathbb{D}_r\to\mathbb{D}_r$ is given for every $(x_1,\ldots,x_r)\in\mathbb{D}_r$ by

$$c(x_1,\ldots,x_r) = (a_1,\ldots,a_r)+t(x_1,\ldots,x_r),$$

where $(a_1,\ldots,a_r)\in\mathbb{D}_r$ and $t\in\mathbb{R}$ are such that if $c(x_1,\ldots,x_r)=(y_1\ldots y_r)$, then $\sum_i y_i^2(x_i)\le 1$. So a linear embedding of a disk is an embedding given by translations and shrinks.

Definition 1.2.9. ([13],[31]) The *little r-disks operad* $\mathcal{D}_r$ is given for every $n \in \mathbb{N}$ by the set of liner embeddings of the disjoint union of n little r-disks in $\mathbb{D}_r$

$$D_r(n) = \{(c_1, \dots, c_n) : \mathbb{D}_r^{\sqcup n} \to \mathbb{D}_r\}$$

such that $c_i : \mathbb{D}_r \to \mathbb{D}_r$ is a linear embedding, and if $\mathring{\mathbb{D}}_r$ denotes the interior of $\mathbb{D}_r$, $c_i(\mathring{\mathbb{D}}_r) \cap c_j(\mathring{\mathbb{D}}_r) = \emptyset$ for all $i \neq j$.

The operadic composition

$$\circ : D_r(n) \otimes D_r(k_1) \otimes \cdots \otimes D_r(k_n) \to D_r(k_1 + \cdots + k_n)$$

is given for $c \in D_r(n)$ and $c_i \in D_r(k_i)$ by

$$c \circ c_1 \otimes \cdots \otimes c_n : \coprod_{i=1}^{n} \coprod_{j=1}^{k_i} \mathbb{D}_r^{i,j} \to \mathbb{D}_r$$

such that the restriction to one little disk is given by the composition

$$(c \circ c_1 \otimes \cdots \otimes c_n)_{|\mathbb{D}_r^{i,j}} = c(c_i)$$

The symmetric group Σ_n acts on $D_r(n)$ by permuting the order of the n disks. More precisely, let $\sigma \in \Sigma_n$ and $c = (c_1, \dots, c_n) \in D_r(n)$ then

$$c * \sigma = (c_{\sigma(1)}, \dots, c_{\sigma(n)})$$

The unit element 1 is the identity $1 = \mathrm{id} : \mathbb{D}_r \to \mathbb{D}_r$.

Remark 1.2.10. The cohomology of little disks operads is known. In fact, the spaces $D_r(n)$ is homotopy equivalent to $\mathfrak{C}_r(n)$, where

$$\mathfrak{C}_r(n) = \{(x_1, \dots, x_n) \in \mathbb{R}^{rn}; x_i \neq x_j \text{ for } i \neq j, 0 < i, j \leq n\}$$

is the configuration space of n points in $\mathbb{R}^r$.

The cohomology of $\mathfrak{C}_r(n)$ has been computed by *Arnold* [1] in the case $r = 2$ and *Cohen* [9] for $r \geq 3$. The ring $H^*(\mathfrak{C}_r(n))$ is given by

$$H^*(\mathfrak{C}_r(n)) = \mathbb{Z}[e_{\alpha_{i,j}}]/\sim$$

where $0 < i, j \leq n$, $\mathbb{Z}[e_{\alpha_i}]$ is the free commutative graded algebra generated by $e_{\alpha_{i,j}}$ of degree $r - 1$ and $\sim$ are the relations

- $e_{\alpha_{i,j}} = (-1)^r e_{\alpha_{j,i}}$
- $e^2_{\alpha_{i,j}} = 0$ if r is odd
- $e_{\alpha_{a,b}} e_{\alpha_{b,c}} + e_{\alpha_{b,c}} e_{\alpha_{c,a}} + e_{\alpha_{c,a}} e_{\alpha_{a,b}} = 0$

This last relation is called in the literature *Arnold relation.*

We introduce the notion of *formality* for operads.

Definition 1.2.11. An operad $\mathcal{O}$ is called *formal* if there is a zig-zag of morphisms of operads which induce an isomorphism in homology

$$\mathcal{O} \leftarrow \cdots \rightarrow H(\mathcal{O}).$$

The following result holds for LDO.

Theorem 1.2.12 ([25], ([27])). *The little disks operads are formal.*

Remark 1.2.13. A brief history of this result appears in [25]. The first attempt on the proof is due to *Getzler* and *Jones*, and *Tamarkin*. Later, *Kontsevich* [25] gave a sketch of the proof, that was finally stated fully by *Lambrechts* and *Volić* in [27]. A brief description of this result will be discussed at the end of *Section* 2.1.

1.2.3 The partial operad of configuration spaces

Definition 1.2.14. ([31]) A *partial operad*, $\mathcal{O}$ in a based category $\mathcal{C}$ is a sequence of objects $\mathcal{O}(n)$ in $\mathcal{C}$, $n \in \mathbb{N}$, with a composition

$$\circ : \mathcal{O}(m) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_m) \rightarrow \mathcal{O}(n_1 + \cdots + n_m)$$

defined only for a subset of *composable elements* in $\mathcal{O}(m) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_m)$ that satisfy the associativity, unit and equivariance axioms of the definition of operad.

Definition 1.2.15. ([31]) Let $\mathfrak{C}_r(n)$ be the configuration space of n points over $\mathbb{R}^r$

$$\mathfrak{C}_r(n) = \{(x_1, \dots, x_n) \in \mathbb{R}^{rn}; x_i \neq x_j \text{ if } i \neq j\}$$

where $1 \leq i, j \leq n$.

We define $\mathfrak{C}_r$ to be the collection $\{\mathfrak{C}_r(n)\}_n$ for every $n \in \mathbb{N}$ with a composition

$$\circ : \mathfrak{C}_r(n) \otimes \mathfrak{C}_r(k_1) \otimes \cdots \otimes \mathfrak{C}_r(k_n) \rightarrow \mathfrak{C}_r(k_1 + \cdots + k_n)$$

given for every

$$(a, x_1, \dots, x_n) \in \mathfrak{C}_r(n) \otimes \mathfrak{C}_r(k_1) \otimes \cdots \otimes \mathfrak{C}_r(k_n)$$

where $a = (a_1, \dots, a_n)$ by

$$x \circ (x_1, \dots, x_n) = (\underbrace{(a_1, \dots, a_1)}_{k_1 \text{ times}} + x_1, \dots, \underbrace{(a_n, \dots, a_n)}_{k_n \text{ times}} + x_n)$$

The unit $1 \in \mathfrak{C}_r(1)$ is the set with one point and the unit morphism is

$$\mu : \{0\} \rightarrow \mathfrak{C}_r(1) = \mathbb{R}^r$$

.

Remark 1.2.16. ([31]) $\mathfrak{C}_r$ is a partial operad. The axioms of being an operad are satisfied, but the composition

$$x \circ (x_1, \ldots, x_n) = (\underbrace{(a_1, \ldots, a_1)}_{k_1 \text{ times}} + x_1, \ldots, \underbrace{(a_n, \ldots, a_n)}_{k_n \text{ times}} + x_n)$$

is not necessarily in $\mathfrak{C}_r(k_1 + \cdots + k_n)$ since $a_l + (x_l)_i$ is not necessarily different from $a_h + (x_h)_j$ for $i \neq j$ and $1 \leq h, l \leq n$.

Remark 1.2.17. The partiality in the definition of configuration space operads can be avoided by considering $FM_M(n)$, the *Fulton and McPherson compactification of configuration spaces* of n points over M [14]. This can be assembled to form an operad, called *Fulton and McPherson-operad*, that is weak equivalent to the little disks operad [15].

1.3 Spiders

This section introduces the mathematical notion of *spider*, a construction that will be used in the next chapters.

The objects are considered to be in the category of real vector spaces, therefore we specialize to *operads* in this category.

Definition 1.3.1. ([10]) An operad $\mathcal{O}$ in the category of real vector spaces, also called *linear operad*, is a collection of real vector spaces $\mathcal{O}(n)$, $n \geq 1$ together with a a right action of the symmetric group Σ_n over $\mathcal{O}(n)$ and composition law

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$$

that satisfy the associativity and equivariance axioms, and a unit $1_{\mathcal{O}} \in \mathcal{O}[1]$ satisfying the unit axioms.

Let an m-star $*_m$ be a tree with $m+1$ vertices and m edges such that there is a vertex v that is a common vertex for every edge. A *labeling* of the edges in a graph Γ, is a bijection

$$L : E(\Gamma) \to \{0, \ldots, |E(\Gamma)|\}$$

where $|E(\Gamma)|$ is the cardinality of the set of edges in Γ. Therefor a *labeled m-star* is an m-star together with a function L.

We can now define a *spider* as follows.

Definition 1.3.2. ([10]) Let $\mathcal{O}$ be a cyclic operad, $\mathcal{L}$ the set of labelings of an m-star, $m \geq 2$. $\mathcal{OS}[m]$, the space of *$\mathcal{O}$-spiders with m legs* is the space of coinvariants

$$\mathcal{OS}[m] = (\oplus_{\mathcal{L}} \mathcal{O}(m-1))_{\Sigma_m}$$

The symmetric group Σ_m acts by $\sigma(o_L) = \sigma(o)_{\sigma L}$ for some $\sigma \in \Sigma_m$ and $o \in \mathcal{O}(n-1)$.

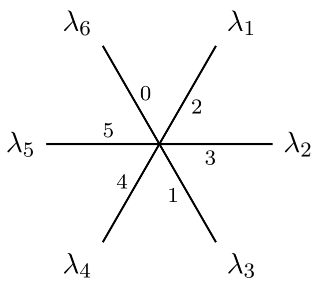

Figure 1.4: A labeled 6-star. In the picture λ_i are the edges, while the integers k, $k = 0, \ldots, 5$ the labelings of the edges [10].

We can think of a spider as superposition of an element of the cyclic operad $\mathcal{O}(m-1)$ and an m-star, where the labeling of the m-star correspond to the inputs and output of the operad. Dividing out by the action of Σ_m erases the labelings.

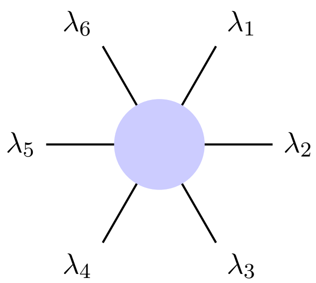

Figure 1.5: A spider with six legs. Dividing out by the action of Σ_6 erases the labelings [10].

The composition of operads induces a composition in the space of spiders, picturesquely called *mating law*.

Definition 1.3.3. ([10]) Let

$$\mathcal{OS} = \bigoplus_{m \geq 2} \mathcal{OS}[m]$$

be the space of $\mathcal{O}$-spiders.

Let S_1 and S_2 be two spiders with m and n legs respectively and the elements in the corresponding operads $o_1 \in \mathcal{O}(m)$ and $o_2 \in \mathcal{O}(n)$. The spider

$$(S_1, \lambda) \circ (\mu, S_2)$$

obtained by *mating* S_1 and S_2 along the legs λ and μ corresponds to the composition

$$o_2 \circ o_1 \otimes 1_{\mathcal{O}} \cdots \otimes 1_{\mathcal{O}}$$

obtained by composing o_1 and o_2. The composition is carried out by choosing a labeling $L_1(\lambda) = 0$ and $L_2(\mu) = 1$ so that λ is the output of o_1 and μ the first input of o_2. The edges in the underlying graph of the spiders that correspond to λ and μ are contracted, the remaining edges renamed and the inputs are relabeled so that the legs of S_1 are inserted in their order, into the ordered set of legs of S_2, at the former place of μ. The mated spider $S = (S_1, \lambda) \circ (\mu, S_2)$ is the equivalence class under the action of Σ_{m+n-2} of the result of this operation.

The following picture adapted from [10] illustrates the composition in the arachnid world.

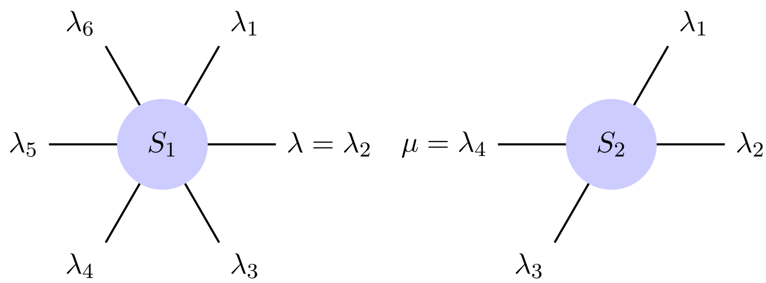

Figure 1.6: The spiders S_1 and S_2 to be mated along the legs λ and μ.[10]

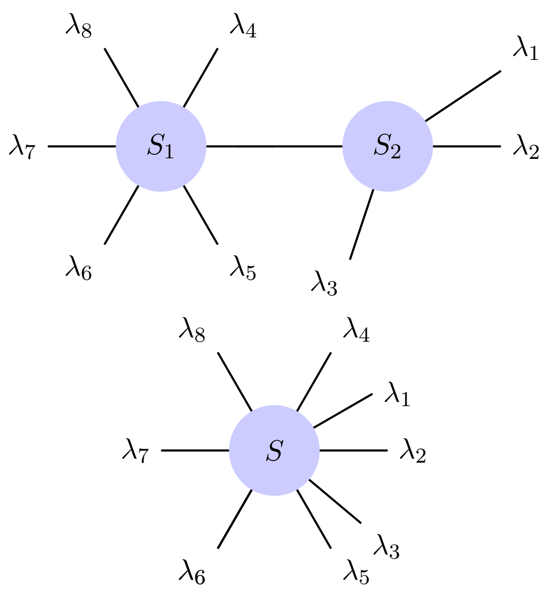

Figure 1.7: The spiders S_1 and S_2 mating along the legs λ and μ. In the second picture, the mated spider after the contraction of the edge.[10]

Chapter 2

Graph cohomologies

2.1 Kontsevich's graph cohomologies

The graph complex GC was introduced by *Kontsevich* [24][23] in the context of knots invariants and it was used to compute the homology of infinite dimensional Lie algebras. In the next paragraph we will describe two ways of defining GC and we give some results about its homology. In the proof of the formality of little disks operads *Kontsevich* [25] introduced other graph complexes, these will be constructed in the last paragraph of the section called *Formality of the little disks operads.*

2.1.1 The complex GC

In this section we present the graph complex defined by *Kontsevich* in [24]. We will refer for the definition and results to [24]. We call the graph complex GC, following the notation used by *Willwacher* in [40].

Definition 2.1.1. [24] Let GC_k be the abstract vector space over $\mathbb{Q}$ spanned by equivalence classes of pairs (Γ, o) where Γ is a connected non empty graph with k vertices, such that all vertices have valence ≥ 3 and o is the orientation of Γ.

- If the manifold M has odd dimension, o is the orientation of the real vector space $\mathbb{R}^{E(\Gamma)} \oplus H^1(\Gamma, \mathbb{R})$. The following relation holds

$$(\Gamma, -o) = -(\Gamma, o).$$

- If M is even dimensional, o is the orientation of $\mathbb{R}^{E(\Gamma)}$ and

$$(\Gamma, o) = \mathrm{Sgn}(\sigma)(\Gamma, o')$$

where o' differ from o by a permutation σ.

$$\mathrm{GC} = \bigoplus_k \mathrm{GC}_k$$

The differential $d : \mathrm{GC}_k \to \mathrm{GC}_{k-1}$ is given by

$$d(\Gamma, o) = \sum_{e \in E(\Gamma)} (\Gamma/e, o_{\Gamma/e})$$

where Γ/e is the graph obtained from Γ by contracting the edge e. $o_{\Gamma/e}$ is the induced orientation, defined to be the natural orientation on $\mathbb{R}^{E(\Gamma)\setminus e} \oplus H^1(\Gamma/e, \mathbb{R})$, in the odd case, while the induced orientation is the orientation on $\mathbb{R}^{E(\Gamma)\setminus e}$ in the even case.

Definition 2.1.2. ([40]) The *full graph complex* fGC is the graph complex given by linear combinations of equivalence classes of graphs, not necessary connected and without any valence condition, with differential and orientation defined as before. We have that GC is a subcomplex of fGC.

Remark 2.1.3. If follows from the definition that in the odd case changing the direction of one edge changes the sign of the orientation and

$$(\Gamma, -o) = -(\Gamma, o).$$

It follows that graphs with loops are zero.

In the even case the relation

$$(\Gamma, o) = \mathrm{Sgn}(\sigma)(\Gamma, o')$$

implies that graphs with double edges are zero.

Remark 2.1.4. The differential preserves the dimension of $H^1(\Gamma)$, so the complex GC decomposes into the direct sum of subcomplexes CG^{χ} given by the graphs with the same Euler Characteristic χ. In [23] *Kontsevich* denotes these complexes by G_*^m where $\chi = 1 - m$.

Remark 2.1.5. ([41]) The complex (GC, d^*), where d^* is the dual differential carries the structure of a dg Lie algebra. The differential d^* is defined by

$$d^*(\Gamma) = \sum_{v \in V(\Gamma)} \mathrm{split}(\Gamma, v).$$

Here $\mathrm{split}(\Gamma, v)$ is the operation that replaces the vertex v by two vertices connected by an edge and summing over all ways of reconnecting the edges incident at v to the two new vertices. Let n be an integer, one can define GC^n to be the differential graded $\mathbb{Q}$-vector space generated by isomorphism classes of graphs with an orientation. The orientation is defined as before, with the distinction between n being even or odd. The number n determines the cohomological degree of a graph. A graph $\gamma \in \mathrm{GC}^n$ has degree

$$|\gamma| = (|V(\gamma)| - 1)n - |E(\gamma)|(n-1)$$

here $|V(\gamma)|$ and $|E(\gamma)|$ are the cardinalities of the set of vertices and edges. The Lie bracket is defined by inserting one graph in the vertices of the other and vice versa

$$[\gamma, \nu] = \gamma \bullet \nu + (-1)^{|\gamma||\nu|} \nu \bullet \gamma$$

where $\gamma \bullet \nu = \sum_{v \in V(\gamma)}$ (inserting ν in v).

Remark 2.1.6. The homology of CG is notoriously hard to compute. Most of the work in this direction is due to *Kontsevich* and *Willwacxher* [40] and [20]. Some computations can be found in the article by *Barnatan* and *McKay* [11]. A feature characterizing the cohomology of GC_n is that it depends on the parity of n since

$$H^j(\mathrm{GC}^{\chi}_{n+2}) = H^{j+2k}(\mathrm{GC}^{\chi}_n)$$

where k is the number of cycles in the graphs with Euler characteristic χ [41]. A remarkable result due as well to *Willwacher* [40] is that

$$H^0(\mathrm{GC}_2) = \mathfrak{grt}_1$$

where $\mathfrak{grt}_1$ is the Grothendieck-Teichmüller Lie algebra, moreover $H^1(\mathrm{GC}_2) \cong \mathbb{K}$ and $H^{\leq 1}(\mathrm{GC}_2) = 0$.

2.1.2 The spiders definition

We present here the construction of GC given by *Conant* and *Vogtmann* in [10]. This definition corresponds to the one given by *Kontsevich* described above in the case where the manifold M is odd dimensional, with the difference that here the vertices in the graphs need to have valence ≥ 2.

Let Γ be a graph, for every edge $e \in E(\Gamma)$ we denote by e^+, e^- the half edges composing e.

Definition 2.1.7. ([10]) An *orientation* of a graph is determined by a choice of an ordering of the vertices and the half edges. Given a graph Γ with n vertices and m edges we then specify the orientation of Γ,

$$o(\Gamma) = \mathrm{Sgn}(v_1, \dots, v_n, e_1^+, e_1^-, \dots, e_m^+, e_m^-)$$

to be the equivalence classes of permutations up to sign of the word obtained by ordering the vertices and then the half edges of Γ.

Remark 2.1.8. As proved in [10], this definition of orientation is equivalent to the definition given by *Kontsevich* as orientation of the vector space $\mathbb{R}^{E(\Gamma)} \oplus H^1(\Gamma, \mathbb{R})$.

Definition 2.1.9. ([10]) An $\mathcal{O}$-*graph* $\mathcal{G}$ is an oriented graph without univalent vertices such that every vertex is colored by an $\mathcal{O}$-*spider* in a way that the half edges incident to the vertex are identified with the legs of the spider.

The graph cohomology chain complex is defined as follows:

Definition 2.1.10. ([10]) For all k in $\mathbb{N}$ the group $\mathcal{O}\mathcal{G}_k$ is the quotient of the vector space spanned by the $\mathcal{O}$-*graphs* with k vertices

$$\mathcal{O}\mathcal{G}_k = \mathbb{R}\{\mathcal{O}\text{-}\textit{graphs} \text{ with } k \text{ vertices}\}/\textit{relations}$$

where the relations are given by:

- (Orientation) $o(-\Gamma) = -o(\Gamma)$
- (Vertex linearity) If a vertex v in Γ is colored by an element $S_v = aF + bT$ $a, b \in \mathbb{R}$ and S, T in $\mathcal{O}\mathcal{S}[m]$ then $\Gamma = a\Gamma_S + b\Gamma_T$ where Γ_S and Γ_T are obtained by coloring v by S and T respectively.

Given an $\mathcal{O}$-*graphs* G and given an edge e in the underlying graph Γ we define a new $\mathcal{O}$-*graphs* G_e as follows. If e is a loop then G_e is zero. Otherwise, if e is not a loop, let Γ/e the graph obtained from Γ by collapsing the edge e. The induced orientation $o(\Gamma/e)$ is defined by:

Definition 2.1.11. ([10]) Let e be an edge in E that is not a loop, oriented so that v_i is the source and v_j the target, and let Γ have an orientation given by $o(\Gamma) = \mathrm{Sgn}(v_i, v_j, \ldots, v_n, e_1, \ldots, e_m)$. Then Γ/e has an induced orientation $o(\Gamma/e) = \mathrm{Sgn}(v_i, \hat{v_j}, \ldots, v_n, e_1, \ldots, \hat{e}, \ldots, e_m)$, obtained by removing the target vertex and the edge e.

Then the $\mathcal{O}$-*graphs* G_e has Γ/e as underlying graph, orientation $o(\Gamma/e)$ and the vertices are colored as follows. The vertices that are not in e are colored in the same way as in G, while suppose that v and w are the vertices in e and they are colored by the spiders S and T then the vertex formed by collapsing e is colored by the spider obtained by *mating* of the spiders S and T along the legs of the spiders corresponding to the half edges of e.

Definition 2.1.12. ([10]) The boundary operator $\delta : \mathcal{O}\mathcal{G}_k \to \mathcal{O}\mathcal{G}_{k-1}$ is defined by

$$\delta(G) = \sum_{e \in E(\Gamma)} G_e.$$

The graph chain complex is $(\mathcal{O}\mathcal{G}, \delta)$

$$\mathcal{O}\mathcal{G} = \bigoplus_k \mathcal{O}\mathcal{G}_k,$$

together with the differential.

Remark 2.1.13. If $\mathcal{O}$ is a commutative cyclic operad, the vector spaces $\mathcal{O}[n] = \mathbb{R}$ are 1-dimensional and the action of the symmetric group is trivial. The composition law is given by the multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. An $\mathcal{O}$-spider is given by an m-star weighted by a real number, and mating two spiders gives rise to a third spider weighted by the product of the weights of the first two. In this case $\mathcal{O}\mathcal{G}$ corresponds to the complex GC.

2.1.3 Formality of the little disks operads

In [25] *Kontsevich* proves that the little disks operad is formal. In the proof he introduces various graph complexes, of which we will give here a short description and we show how GC comes into the picture. We refer to [40], [8] and [19] for the construction of the complexes.

Definition 2.1.14. ([19]) Let $\mathrm{Gra}_n(N)$ be the commutative differential graded algebra generated by $e_{a,b}$, $a, b \in \{1, \ldots, N\}$ of degree $n-1$.

$$\mathrm{Gra}_n(N) = (\mathbb{Z}[e_{a,b}]/e_{a,b}^2 = 0, e_{a,b} = (-1)^n e_{b,a}; d = 0)$$

It can be interpreted as the commutative differential graded algebra spanned by graphs without loops or multiple edges, with N vertices, where the edges are the generators $e_{a,b}$ of degree $n-1$. The product $\mathrm{Gra}_n(N) \otimes \mathrm{Gra}_n(N) \to \mathrm{Gra}_n(N)$ consists of gluing two graphs together along their vertices. Moreover, $\mathrm{Gra}_n(N)$ for $N \in \mathbb{N}$ assembles to form a cooperad. The cooperadic structure is given by

$$\mathrm{Gra}_n(n) \to \mathrm{Gra}_n(m) \otimes \mathrm{Gra}_n(k_1) \otimes \cdots \otimes \mathrm{Gra}_n(k_m)$$

and it sends a graph Γ to $\sum \Gamma' \otimes \Gamma_1 \otimes \cdots \otimes \Gamma_k$ where the sum runs over all $(k+1)$-tuples of graphs such that when each graph Γ_i is inserted at the vertex i of Γ', there is a way of reconnecting the loose edges such that one obtains Γ [8].

The next complex is defined with the technique of *operadic twisting*. We will define it combinatorially in terms of graphs, without explaining the construction behind, since it would go out of the scope of this thesis. For the detail of twisting an operad we refer to *Appendix I* in [40].

Definition 2.1.15. ([19]) We define $\mathrm{TwGra}_n(N)$ to be the differential graded module spanned by graphs with two kind of vertices: N vertices called *external*, and some unlabeled *internal* vertices. The edges have degree $n-1$ and the internal vertices have degree $-n$. The differential is given by

$$d(\Gamma) = \sum_{e \in E(\Gamma)} \pm \Gamma/e$$

where e is an edge in Γ connecting an internal vertex to another vertex of either kind. The sign in the differential is given by the orientation of the graph defined as in the *Definition* 2.1.1, with distinction between n being odd or even. The product of two graphs glues them along their external vertices. Moreover, TwGra_n has the structure of a coopered.

Let $Graph_n(N) \subset \mathrm{TwGra}_n(N)$ be the subcomplex spanned by graphs with all internal vertices at least trivalent and and with no connected component consisting entirely of internal vertices.

This last result is due to *Kontsevich* [25], and *Lambrechts* and Volić [27].

Theorem 2.1.16. *Consider FM_n, the Fulton and McPherson compactification of the configuration space. There is a map of cooperads*

$$H^*(D_r(n)) \leftarrow Graphs_n \to \Omega^*_{PA}(FM_n)$$

that is a zigzag weak equivalence.

The result follows from the fact that the Fulton and McPherson operad FM is formal, so for all n $FM_n \cong \Omega^*_{PA}(FM_n)$, and that it is weakly equivalent as topological operad to the little disks operad D_r (*Propostion* 5.6 [27]).

Remark 2.1.17. The Lie algebra (GC, d^*), discussed in *Remark* 2.1.5 acts on $Graph_n$, as follows: let γ be an element in GC_n and Γ in $Graphs_n$, there is an element

$$\Gamma \circ \gamma \in Graphs_n$$

given by the contraction of subgraphs of shape γ in Γ (page 1255 in [41]).

2.2 Baranovsky-Sazdanović's graph cohomology

2.2.1 The complex $C_{BS}(\Gamma)$

This section presents the graph complex defined by *Baranovsky* and *Sazdanović* in [2]. Their definition is inspired by the work by *Helme-Guizon* and *Rong* [16], whose construction develops from the cohomology theory defined by *M. Khovanov* in [21]. There he associates to each link a family of cohomology groups whose Euler characteristic is the Jones polynomial of the link. *Helme-Guizon's* and *Rong's* graph cohomology expands the *Khovanov's* definition associating to each graph, graded cohomology groups whose Euler characteristic is the chromatic polynomial of the graph. *Baranovsky* and *Sazdanović* in [2] prove that there is a spectral sequence that relates the graph cohomology defined by *Helme-Guizon* and *Rong* with the cohomology of configuration spaces, verifying a conjecture posed by *Khovanov*.

We give here the definition of the graph cohomology complex. We refer to [2] for the definitions and the notation with the exception of the notation of the complex that we will call $\mathcal{C}_{BS}(\Gamma)$. Then, we state some results relating the complex to the homology of configuration spaces.

Let A be a graded commutative algebra over a commutative ring R, and assume that A is a projective R-module. Let Γ be a finite graph, $V = V(\Gamma)$ be the set of vertices and $E(\Gamma)$ the set of edges. We choose an order on the vertices. This gives an orientation on every edge α in $E(\Gamma)$, if α connects the vertices i and j and $i \leq j$, $\alpha : i \to j$. For any subset S of $E(\Gamma)$, we denote by $[\Gamma : S]$ the subgraph that has as vertices the same vertices of Γ and as edges the edges in S, we denote by $l(S)$ the number of connected components of $[\Gamma : S]$.

Definition 2.2.1 ([2])**.** Let Λ be an exterior algebra over R with generators e_α, $\alpha \in E(\Gamma)$, and e_S be the exterior product of e_α, $\alpha \in S$, ordered with the lexicographic order of the pair (i, j) where $\alpha : i \to j$.

The bigraded complex, that we will here denote by $\mathcal{C}_{BS}(\Gamma)$, is defined as

$$\mathcal{C}_{BS}(\Gamma) = \Lambda \otimes A^{\otimes n}/e_\alpha \otimes (a[i] - a[j]),$$

the algebra $\Lambda \otimes A^{\otimes n}$ quotient by the relation $e_\alpha \otimes (a[i] - a[j])$, where $a \in A$, $\alpha : i \to j \in E(\Gamma)$ and $a[i]$ denotes the element $1^{\otimes i-1} \otimes a \otimes 1^{\otimes n-i} \in A^{\otimes n}$. The complex has a bigrading given by the sum of the grading of the elements e_α of bidegree $(0, 1)$ and the elements $1 \otimes a_1 \otimes \cdots \otimes a_n$ with bidegree $(\sum_{i=1}^{n} \deg_A a_i, 0)$, so the degree of $e_S \otimes a_1 \otimes \cdots \otimes a_n$ in $\mathcal{C}_{BS}(\Gamma)$ is $(\sum_{i=1}^{n} \deg_A a_i, |S|)$.

The differential of degree $(0, 1)$ is given by the exterior product

$$\partial = \sum_{\alpha \in E(\Gamma)} e_\alpha$$

Remark 2.2.2. The assumption of A be a projective R-module is used in the proof of the convergence of the spectral sequence. We refer to [2] for the definition of the spectral sequence and the proof.

Remark 2.2.3. The complex $C_{BS}(\Gamma)$ is isomorphic to

$$\bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{l(S)}.$$

That is for every $n \in \mathbb{N}$

$$C^n_{BS}(\Gamma) = \bigoplus_{S \subseteq E(\Gamma), |S| = n} e_S \otimes A^{l(S)}.$$

For $S \subset E(\Gamma)$, each term a_i of the element $e_S \otimes a_1 \otimes \cdots \otimes a_{l(S)} \in A^{l(S)}$, corresponds to a component in $[\Gamma : S]$. In the case $S = \emptyset$, the components of $[\Gamma : \emptyset]$ are the vertices in Γ. We can construct a map

$$\phi : (\Lambda \otimes A^{\otimes n}/\sim) \to \bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{l(S)}$$

such that if α is an edge in $E(\Gamma)$, $\alpha : i \to j$, then

$$\phi(e_\alpha \otimes a_1 \otimes \cdots \otimes a_n) = e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^\tau a_i a_j \otimes \cdots \otimes a_{l(\alpha)}.$$

The terms a_i and a_j are multiplied with a sign that is the Kozul sign given by the permutation in the tensor product that brings a_j close to a_i, here $l(\alpha) = n-1$. The inverse is given by

$$\begin{aligned}&\phi^{-1}(e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^\tau a_i a_j \otimes \cdots \otimes a_{n-1}) \\ &\quad = e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^\tau a_i a_j \otimes \cdots \otimes 1 \otimes \cdots \otimes a_{n-1}\end{aligned}$$

where 1 is in the position j. Then, it is enough to notice that

$$e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^\tau a_i a_j \otimes \cdots \otimes 1 \otimes \cdots \otimes a_{n-1}$$

is in the same equivalence class of $e_\alpha \otimes a_1 \otimes \cdots \otimes a_n$, since for a,$b \in A$,

$$a \otimes b = (a \otimes 1)(1 \otimes b) \sim (1 \otimes a)(1 \otimes b) = (1 \otimes ab)$$

and

$$(a \otimes 1)(1 \otimes b) \sim (a \otimes 1)(b \otimes 1) = (ab \otimes 1).$$

Remark 2.2.4 ([2])**.** The differential $\partial : \mathcal{C}^k_{BS} \to \mathcal{C}^{k+1}_{BS}$ induced by Φ on

$$C^*_{BS}(\Gamma) = \bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{l(S)}$$

is

$$\begin{aligned}\partial(e_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) &= \sum_{\alpha \in E(\Gamma), l(S \cup \alpha) = l(S)} e_\alpha e_S \otimes a_1 \otimes \cdots \otimes a_{l(S)} \\ &+ \sum_{\alpha \in E(\Gamma), l(S \cup \alpha) = l(S)-1} (-1)^\tau e_\alpha e_S \otimes a_1 \otimes \cdots \otimes a_i \cdot a_j \otimes \cdots \otimes a_{l(S)}.\end{aligned}$$

The first term of the sum represents the case where the edge α connects two vertices of the edges in S that are in the same component. Therefore, the number of components of $[\Gamma : S]$ and $[\Gamma : S \cup \alpha]$ are the same, so $l(S) = l(S \cup \alpha)$. The second term of the sum refers to the case where the edge α connects two different components, so $l(S \cup \alpha) = l(S) - 1$. Suppose that $\alpha : i \to j$, and that a_h is the term corresponding to the component containing i, and a_k to the component containing j. Then τ is the Kozul sign given by the permutation in the tensor product that moves a_k to the position immediate to the right of a_h.

Example 2.2.5. Let Γ be K_3, the complete graph with 3 vertices. The order of the vertices induces an order on the edges given by the lexicographic order $E(K_3) = \{e_{0,1}, e_{0,2}, e_{1,2}\}$ and an orientation on the edges.

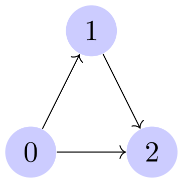

Figure 2.1: The graph K_3.

The chain $\mathcal{C}_{BS}(\Gamma)$ in this case is the following

$$A^{\otimes 3} \to A^{\otimes 2} \oplus A^{\otimes 2} \oplus A^{\otimes 2} \to A \oplus A \oplus A \to A$$

The chain groups are given by

$$\mathcal{C}_{BS}(\Gamma)^n = \bigoplus_{|S|=n} A^{\otimes l(S)}$$

where S is a subset of $E(\Gamma)$, and $l(S)$ is the number of components of $[\Gamma : S]$. The picture shows the components of $[\Gamma : S]$ with increasing cardinality of S and the differential that adds every time an edge in $[\Gamma : S]$, connecting its components.

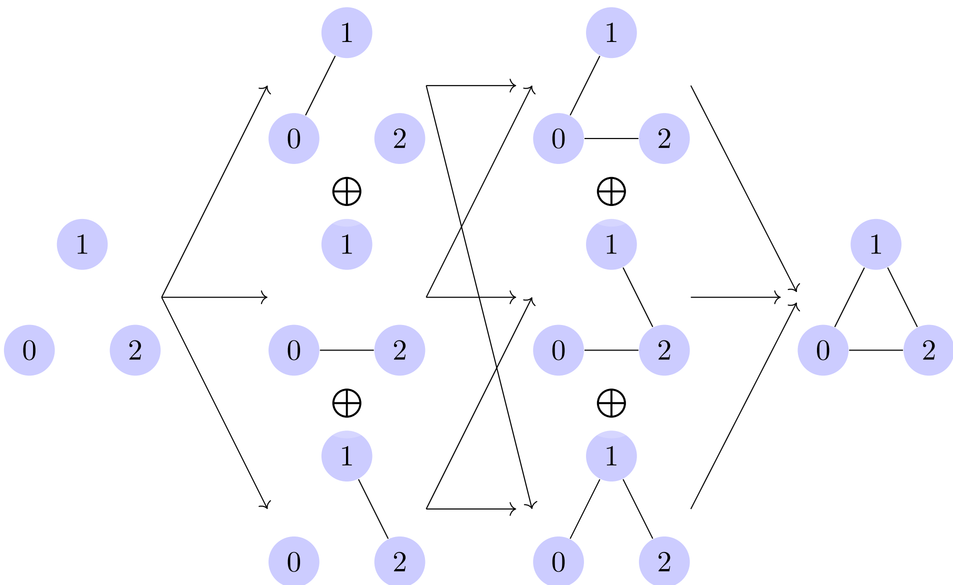

Figure 2.2: An example of $\mathcal{C}_{BS}(\Gamma)$ when $\Gamma = K_3$, the complete graph with three vertices.

Let $a_0 \otimes a_1 \otimes a_2 \in \mathcal{C}^0_{BS}$, then

$$\begin{aligned}&\partial(a_0 \otimes a_1 \otimes a_2)\\ &\quad= e_{0,1} \otimes a_0 a_1 \otimes a_2 + e_{1,2} \otimes a_0 \otimes a_1 a_2 + (-1)^{|a_1||a_2|} e_{0,2} \otimes a_0 a_2 \otimes a_1\end{aligned}$$

Notice that the fact that $\partial^2 = 0$ is provided by the sign coming from the graded commutativity of A. For example,

$$\partial_{e_{0,1}}((-1)^{|a_1||a_2|}e_{0,2} \otimes a_0a_2 \otimes a_1) \\ = (-1)^{|a_1||a_2|}e_{0,1}e_{0,2} \otimes a_0a_2a_1 = e_{0,1}e_{0,2} \otimes a_0a_1a_2$$

and

$$\partial_{e_{0,2}}(e_{0,1} \otimes a_0a_1 \otimes a_2) = e_{0,2}e_{0,1} \otimes a_0a_1a_2 = -e_{0,1}e_{0,2} \otimes a_0a_1a_2.$$

All the terms in ∂^2 given by adding an edge e_i and then e_j cancel with the terms given by adding the edges in opposite order, so $\partial^2 = 0$.

2.2.2 Results about the homology of configuration spaces

As anticipated, the complex $\mathcal{C}_{BS}$ is related to the homology of configuration spaces depending on a graph, as defined by *Eastwood* and *Hugget* [12].

Let M be simplicial complex and Γ a graph as defined in the first section. Let $\alpha : i \to j$ be an edge in $E(\Gamma)$, Z_α be the diagonal of the Cartesian product M^n corresponding to the edge α,

$$Z_\alpha = \{(m_1, \dots, m_n) \in M^n; m_i = m_j\}$$

and

$$Z_\Gamma = \bigcup_{\alpha \in E(\Gamma)} Z_\alpha.$$

We define the graph configuration space of M dependent on Γ to be

$$\mathfrak{C}(M, \Gamma) = M^n \backslash Z_\Gamma.$$

If M is a manifold the definition corresponds to the generalized configuration space depending on a graph studied by *Eastwood* and *Hugget* in [12].

Baranovsky and *Sazdanović* in [2] prove that $\mathcal{C}_{BS}$ is the E_1 page of a spectral sequence converging to the cohomology of such configuration space. This confirms a conjecture by *Khovanov* that there is a spectral sequence between the graph homology defined by *L.Helme-Guizon* and *Y. Rong* and the work by *Eastwood* and *Huggett*.

Theorem 2.2.6 ([2])**.** *Assume that the cohomology algebra $A = H^*(M, R)$ is a projective R-module and that Γ has no loops or multiple edges. There exist a spectral sequence with E_1 term isomorphic to $\mathcal{C}^*_{BS}$ which converges to the relative cohomology $H^*(M^n, Z_\Gamma; R)$.*

Remark 2.2.7 (*Remark* 4 [2])**.** When M is a compact R-oriented manifold of dimension m, the relative cohomology groups $H^*(M^n, Z_\Gamma; R)$ are isomorphic to the homology groups $H_{nm-*}(\mathfrak{C}(M, \Gamma); R)$ by Lefschetz duality.

Moreover in the case where M is a Kähler manifold the following result holds.

Remark 2.2.8 ([2]). If M is a compact Kähler manifold and the coefficient ring R is the rationals $\mathbb{Q}$ the spectral sequence degenerates at page E_2.

Chapter 3

Rational homotopy theory

3.1 Basic definitions and results

We recall here some basic definitions and results in homotopy theory that will be needed in the following chapters. We refer to [36], [18] and [29].

Two topological spaces X and Y have the same homotopy type if there are continuous maps f and g

$$f : X \rightleftarrows Y : g$$

such that the compositions are homotopic to the identity maps on X and Y, $fg \sim id_Y$ and $gf \sim \mathrm{id}_X$. Homotopy theory is the study of properties of topological spaces that depend on the homotopy type of the spaces, for example the homology and cohomology groups or the fundamental group.

Rational homotopy theory is a variation of homotopy theory where the properties of spaces are studied in their rationalization. For example the groups $H_i(X)$ and $\pi_i(X)$ can be rationalized in the vector spaces $H_i(X, \mathbb{Q})$ and $\pi_i(X) \otimes \mathbb{Q}$. This leads to a loss of information, because the torsion subgroups are ignored, but it creates a theory where computations are easier to carry out. We now describe the process in details.

Definition 3.1.1 ([18])**.** A simply connected space X is called *rational* if the equivalent conditions are satisfied:

- the collection of the homotopy groups $\pi_*(X)$ is a $\mathbb{Q}$-vector space
- the reduced homology $\widetilde{H}_*(X, \mathbb{Z})$ is a $\mathbb{Q}$-vector space
- the reduced homology of the loop space $\widetilde{H_*}(\Omega X, \mathbb{Z})$ is a $\mathbb{Q}$-vector space

The next definition and theorem show that a rational space X_0 can be associated to every simply connected topological space X.

Definition 3.1.2 ([18])**.** Let X be a simply connected space. A continuous map $l : X \to X_0$ is a rationalization of X if X_0 is simply connected and rational and

$$\pi_*(l) : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(X_0)$$

is an isomorphism.

Theorem 3.1.3 (1.5 [18])**.** *Let X be a simply connected topological space. There exists a relative CW complex (X_0, X), with no zero-cells and no one-cells such that the inclusion $i : X \to X_0$ is a rationalization. Moreover, such space is unique up to homotopy equivalence, relative to X.*

We then introduce the *rational homotopy type* of a simply connected topological space X. We recall the fact that two topological spaces have the same weak homotopy type if they are connected by a zig-zag of morphisms between them that induces an isomorphism in all homotopy groups.

Definition 3.1.4 ([18])**.** The rational homotopy type of a simply connected space X is the weak homotopy type of its rationalization X_0.

We can now redefine rational homotopy theory as the study of properties of topological spaces that depend on the rational homotopy type of the spaces.

3.1.1 Commutative differential graded algebra

Definition 3.1.5. A commutative differential graded algebra, or in short *CDGA*, is a graded algebra A over a ring R such that the multiplication

$$\cdot : A^n \times A^m \to A^{m+n}$$

is graded commutative, in the sense

$$a \cdot b = (-1)^{|a||b|} b \cdot a$$

here $| - |$ denotes the degree of the element $-$, and A^n denotes the set of elements of degree n. Moreover, A is equipped with a differential $d : A^n \to A^{n+1}$ that makes A into a cochain complex. The differential satisfies the graded Leibniz rule

$$d(ab) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$$

Definition 3.1.6 (2.1 [29])**.** An *oriented Poincarè duality algebra* of dimension n is an algebra A over a field $\mathbb{K}$ together with a linear map $\varepsilon : A^k \to \mathbb{K}$ such that the induced bilinear forms

$$A^k \otimes A^{n-k} \to \mathbb{K}$$

$$a \otimes b \mapsto \varepsilon(a \cdot b)$$

are non degenerate.

Definition 3.1.7 (2.2 [29])**.** An *oriented Poincarè duality CDGA* is a CDGA (A, d) such that the underlying algebra (A, ε) is an oriented Poincaré duality algebra and $\varepsilon(dA) = 0$.

Poincaré duality CDGAs have applications in computing the rational homotopy type of configuration spaces, as we will see in the next section. In [29] *Lambecht* and *Stanley* prove the following results.

Theorem 3.1.8 (1.1 [29])**.** *Let* $\mathbf{k}$ *be a field of any characteristic and let* (A, d) *be a CDGA over* $\mathbf{k}$ *such that* $H^*(A, d)$ *is a simply connected Poincaré duality algebra of dimension* n*. Then there exists a CDGA* (A', d') *weakly equivalent to* (A, d) *and such that* A' *is a simply-connected algebra satisfying Poincaré duality in dimension* n*.*

Definition 3.1.9 ([30], [28])**.** Let (A, ε) be an oriented Poincaré duality algebra. The *diagonal class* Δ is the element

$$\Delta = \sum_i (-1)^{|a_i|} a_i \otimes a_i^* \in A \otimes A$$

where $\{a_i\}_i$ is a basis for A and $\{a_i^*\}_i$ the Poincaré dual basis with respect to the orientation, that is

$$\varepsilon(a_i \cdot a_j^*) = \delta_{i,j}$$

where $\delta_{i,j}$ is the Kroneker delta.

Remark 3.1.10. Let M be a closed oriented manifold of dimension m. Then $H^*(M)$ is a Poincaré duality algebra and there is a preferred generator $[M] \in H_m(M)$. Choosing a basis $\{a_i\}_i$ for $H^*(M)$ there is a Poincaré dual basis $\{a_i^*\}_i$ characterized by the equation

$$\langle a_i \cup a_j^*, [M] \rangle = \delta_{i,j}$$

The diagonal class is the element of top degree in $H^*(M) \otimes H^*(M)$

$$\Delta = \sum_i (-1)^{|a_i|} a_i \otimes a_i^*.$$

3.1.2 Sullivan models

In the context of rational homotopy theory, we are interested in CDGA over the rationals. In 1977, *Sullivan* [34] constructed a functor from the category of topological spaces to the category of rational CDGA

$$A_{PL} : Top \to CDGA_{\mathbb{Q}}$$

such that if X is a simply connected space with rational homology of finite type, then the rational homotopy type of X is encoded in any CDGA weakly equivalent to $A_{PL}(X)$. $A_{PL}(X)$ is the CDGA of differential forms on X with coefficients in $\mathbb{Q}$.

Definition 3.1.11 ([36])**.** A *Sullivan algebra* is CDGA of the form $(\Lambda V, d)$, where

- $V = \{V^p\}_{p\geq 1}$ and ΛV denotes the free graded commutative algebra on V;
- $V = \bigcup_{k=0}^{\infty} V(k)$ where $V(0) \subset V(1) \subset \dots$ is an increasing sequence of graded subspaces such that $d = 0$ in $V(0)$ and $d : V(k) \to \Lambda V(k-1)$, $k \geq 1$.

Now we can define the *Sullivan model*.

Definition 3.1.12 ([36])**.** A *Sullivan model* for a CDGA (A, d) is a quasi isomorphism

$$m : (\Lambda V, d) \to (A, d)$$

from a Sullivan algebra $(\Lambda V, d)$. If X is a path connected topological space then a Sullivan model for A_{PL}

$$m : (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$$

is called *Sullivan model for* X. A *minimal Sullivan model* is a Sullivan model such that $d(V) \subset \Lambda^{\geq 2} V$.

Sullivan models encode the rational homotopy information of the space, as the following results shows.

Remark 3.1.13. If $(\Lambda V, d)$ be a Sullivan model for X then

$$H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}).$$

If X and Y are simply connected topological spaces with the same rational homotopy type then $A_{PL}(X)$ and $A_{PL}(Y)$ are weak equivalent. Moreover, X and Y have the same rational homotopy type if and only if their Sullivan minimal models are isomorphic. In fact, there is a bijection between the sets

$$\left\{ \begin{array}{c} \text{rational} \\ \text{homotopy types} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{minimal Sullivan algebras} \end{array} \right\}.$$

Definition 3.1.14. We will call a *rational model* for X any CDGA (A, d) weakly equivalent to $A_{PL}(X)$, and so

$$H^*(A, d) \cong H^*(X, \mathbb{Q}).$$

3.2 Rational and real homotopy type of configuration spaces

Let X be a topological space. The ordered configuration space of n points in X that we denote by $\mathfrak{C}(X,n)$ is defined as

$$\mathfrak{C}(X,n) = X^n \setminus \bigcup \Delta_{i,j}$$

where $\Delta_{i,j} = \{(x_1,\dots,x_n) \in X^n : x_i = x_j\}$

We recall here the most important results about the rational homotopy type of this space. We give first a general overview and then analyze more in detail some constructions that will be needed later in this thesis.

3.2.1 Historical notes

The study of the rational homotopy type of configuration spaces $\mathfrak{C}(X,n)$ dates back to 1994 when *Fulton* and *MacPherson* [14] constructed a rational model $\mathcal{A}^{\cdot}(X,n)$ for $\mathfrak{C}(X,n)$, where X is a non singular, compact, complex variety. This model depends on the cohomology ring $H^*(X,\mathbb{Q})$, the orientation and the Chern classes.

The same year, *Kriz* in [26] described a differential graded algebra $E[n]$ that is a rational model for $\mathfrak{C}(X,n)$ and that is independent from the Chern classes. He defined it as

$$E[n] = H^*(X^n,\mathbb{Q})[G_{a,b}]/\sim$$

where $G_{a,b}$ are generators of degree $2m-1$, a, $b \in \{1,\dots,n\}$, $a \neq b$ and $\sim$ are the relations

- $G_{a,b} = G_{b,a}$
- $p_a^*(x)G_{a,b} = p_b^*(x)G_{a,b}$, $x \in H^*(X)$
- $G_{a,b}G_{b,c} + G_{b,c}G_{c,a} + G_{c,a}G_{a,b} = 0$,

and p_a^* is the pullback of the projection $p_a : X^n \to X$. The differential is given by

$$d(G_{a,b}) = p_{a,b}^*\Delta$$

where Δ is the class of the diagonal.

$E[n]$ was described in the same time in the work by *Totaro* [37] and it appeared to be isomorphic to the E_2 page of the Larey spectral sequence of the inclusion $\mathfrak{C}(X,n) \hookrightarrow X^n$. The algebra $E[n]$ will be here discussed in details in *Theorem* 5.2.1.

Later, *Lambecht* and *Stanley* studied the rational models for configurations spaces where X is a simply connected closed manifold. They showed that a simply connected closed manifold always admits a Poincaré duality model A

[29]. In 2004 they described the case $k = 2$, a configuration space of 2 points in a manifold [28] and they defined a model for its rational homotopy type, $G_A(2)$, that is

$$G_A(2) = A^{\otimes 2}/\Delta,$$

where Δ is the diagonal class in A. In 2008 they presented a potential model for the general case [30]. They conjectured that if X is a simply connected m-manifold, $G_A(n)$ is a rational model for $\mathfrak{C}(X, n)$, where $G_A(n)$ is defined as follows.

$$G_A(n) = A^{\otimes n}[G_{a,b}]/\sim$$

where $G_{a,b}$ are generators of degree $m-1$, a, $b \in \{1, \dots, n\}$, $a \neq b$, and $\sim$ are the relations

- $G_{a,b} = (-1)^m G_{b,a}$
- $p_a^*(x)G_{a,b} = p_b^*(x)G_{a,b}$, $x \in A$
- $G_{a,b}G_{b,c} + G_{b,c}G_{c,a} + G_{c,a}G_{a,b} = 0$,

here p_a^* is the pullback of the projection $p_a : X^n \to X$. The differential is given by

$$d(G_{a,b}) = p_{a,b}^*\Delta$$

where Δ is the class of the diagonal.

In 2019 *Idrissi* [19] proved the conjecture true for the real homotopy type and manifolds of dimension at least 4. *Campos* and *Willwacher* few years before [8] constructed a real model for the configuration space of points in a manifold.

3.2.2 A real model for configuration spaces

In [30] *Lambecht* and *Stanley* conjectured that G_A is a rational model for configuration spaces of points $\mathfrak{C}(X, n)$ for any simply connected manifold. In [19] *Idrissi* gives a positive answer to the problem in the case of real homotopy type. We will give a brief description of the construction leading to the proof since we will refer to it later in this thesis.

The proof is inspired by the one of formality of little disks operad that we described at the end of *Section* 2.1 and it involves variations of the graph complexes defined there.

Let M be a manifold of dimension n, A a Poincarè duality CDGA with its linear map ε, and $\Omega^*_{PA}(M)$ the CDGA of piecewise algebraic differential forms on M.

Theorem 3.2.1 ([30]). *There exists a zigzag of weak equivalences of CDGAs*

$$A \xleftarrow{\rho} R \xrightarrow{\sigma} \Omega^*_{PA}(M)$$

such that:

- *A is a Poincaré duality CDGA of dimension n;*
- *R is a quasi-free CDGA generated in degrees ≥ 2;*
- *for all $x \in R$, $\varepsilon(\rho(x)) = \int_M \sigma(x)$*

Definition 3.2.2 ([19])**.** Let $\mathrm{Gra}_R^{\circlearrowleft}(N)$ be the differential graded commutative algebra spanned by graphs with loops with N vertices such that every vertex is colored by an element in R.

$$\mathrm{Gra}_R^{\circlearrowleft}(N) = (R^{\otimes N} \otimes \mathbb{Z}[e_{a,b}]/e_{a,b} = (-1)^n e_{b,a}; d(e_{a,b}) = p_{a,b}^*(\Delta))$$

where $e_{a,b}$ are edges in the graphs and their degree is $n-1$. One can define Δ to be the inverse image under $\rho \otimes \rho$ of the class of the diagonal in A, as described in *Proposition* 3.3 in [19]. For $N \in \mathbb{N}$ $Gra_R^{\circlearrowleft}(N)$ assemble to form Hopf right $\mathrm{Gra}_n^{\circlearrowleft}$-comodule. Here $\mathrm{Gra}_n^{\circlearrowleft}$ is the cooperad Gra_n where loops are allowed.

The second step is to construct the twisted operad $\mathrm{TwGra}_R^{\circlearrowleft}$, analogously to the construction of TwGra_n in *Section* 2.1.

Definition 3.2.3 ([19])**.** $\mathrm{TwGra}_R^{\circlearrowleft}(N)$ is the dg module spanned by graph with two kind of vertices: N external vertices and unlabeled internal vertices. All vertices are labeled by element in R. The edges have degree $n-1$ while the internal vertices have degree $-n$. The differential is given by

$$d = d_1 + d_2 + d_R$$

where d_R is the differential coming from R, $d_2(e_{a,b}) = p_{a,b}^*(\Delta)$ is the differential coming from $\mathrm{Gra}_R(N)$ and d_1 is defined as

$$d_1(\Gamma) = \sum_{e \in E(\Gamma)} \pm \Gamma/e$$

and e is an edge in Γ connecting an internal vertex to another vertex of either kind, multiplying the labels. As $Gra_R^{\circlearrowleft}(N)$, $\mathrm{TwGra}_R^{\circlearrowleft}$ assemble to form Hopf right $\mathrm{TwGra}_n^{\circlearrowleft}$-comodule. We denote by $\mathrm{TwGra}_R \subset \mathrm{TwGra}_R^{\circlearrowleft}$ the sub CDGA spanned by graphs with no loops.

The last step of the proof is to define a restriction of TwGra_R. We need first to define the *partition function*.

Definition 3.2.4. The partition function $\mathbb{Z}_\phi : \mathrm{fGC}_R \to \mathbb{Z}$ is the restriction of $w : \mathrm{TwGra}_R \to \Omega^*_{PA}(FM_M)$ to $\mathrm{fGC}_R = \mathrm{TwGra}_R(0)$. The map w is defined as follows:

$$w(\Gamma) = \int_{p_N : FM(N+I) \to FM(N)} w'(\Gamma) = (p_N)_*(w'(\Gamma))$$

where $FM_M(n)$ is the *Fulton-McPherson operad*, p_N the projection and I the number of internal vertices.

Definition 3.2.5 ([19])**.** Let

$$\mathrm{fGC}_R = \mathrm{TwGra}_R(0)$$

and let $\mathbb{R}_\phi$ be the fGC_R -module of dimension one induced by $Z_\phi : \mathrm{fGC}_R \to \mathbb{R}$. We define $Graphs_R^\phi$ to be

$$Graphs_R^\phi = \mathbb{R}_\phi \otimes_{\mathrm{fGC}_R} \mathrm{TwGra}_R(N)$$

$Graphs_R^\phi$ is spanned by graphs in TwGra_R with no component containing only internal vertices. The orientation on the graphs is given as in the *Definition* 2.1.1, with distinction between odd and even case.

The main result of the paper is the following

Theorem 3.2.6 (Theorem 4.14 [19])**.** *Let $FM_M(N)$ be the Fulton and McPherson compactification of the configuration space of N points over M. For all $N \in \mathbb{N}$ there is a zigzag sequence of quasi-isomorphisms of CDGA*

$$G_A(N) \leftarrow \cdots \to Graphs_R^\phi \to \Omega^*_{PA}(FM_M(N)).$$

Since $FM_M(n)$, the Fulton and McPherson compactification of the configuration space is homotopy equivalent to $\mathfrak{C}(M,n)$ [14] then if follows that $\Omega^*_{PA}(FM_M(n)) \cong \Omega^*_{PA}(\mathfrak{C}(M,n))$ and so we can conclude that G_A is a real model for $\mathfrak{C}(X,n)$.

Chapter 4

Relation between graph complexes

4.1 Introduction

In this chapter we build a chain map between the complex defined by *Baranovsky* and *Sazdanović* in [2], described in *Section* 2.2, and the *Kontsevich*'s graph complex fGC. In the first sections we define a chain complex that is a variation of fGC. This complex, differently from fGC, depends on a graph Γ and has ordered vertices colored by elements in a graded algebra, that is the cohomology of a manifold M. However, similarly to fGC, it has two different definitions according to the parity of an integer n, that stands for the dimension of the manifold M. We denote this intermediate complex by $\mathcal{K}(\Gamma)$. In *Section* 4.3 we build a map between $\mathcal{C}_{BS}(\Gamma)$ and $\mathcal{K}(\Gamma)$ for the even and odd case. Then, in Section 4.4 we introduce the complex $\mathcal{K}/_{\Sigma}(\Gamma)$ that differs from $\mathcal{K}(\Gamma)$ by having unordered vertices, and we build a map between $\mathcal{K}(\Gamma)$ and $\mathcal{K}/_{\Sigma}(\Gamma)$. In the last section we describe the relation between this complex and *Kontsevich's* graph complex fGC.

4.2 A variation of fGC: the chain $\mathcal{K}(\Gamma)$

We define here the chain complex $\mathcal{K}(\Gamma)$. As anticipated, it is a variation of Kontsevich's complex fGC, depending on a graph Γ with ordered vertices colored by elements in a graded algebra. $\mathcal{K}^k(\Gamma)$ is the vector space generated by graphs Γ/S with k vertices obtained from Γ by contracting S, a subset of $E(\Gamma)$. The definition depends on an integer n and varies according to the parity of this number.

Definition 4.2.1. Let M be a compact manifold of dimension n and A a $\mathbf{k}$-CDG algebra that is a model for the cohomology of M with coefficient in $\mathbf{k}$, where $\mathbf{k}$ is a field of characteristic 0. Let Γ be a graph without loops or multiple edges with an order on the vertices, that gives an orientation on every edge α in $E(\Gamma)$ in a way that, if α connects the vertices i and j and $i \leq j$, then $\alpha : i \to j$. Let S be a subset of $E(\Gamma)$ such that S is a forest. We define the chain $\mathcal{K}(\Gamma)$ to be

$$\mathcal{K}(\Gamma) = \bigoplus_{S \subset E} \Gamma/S \otimes A^{\otimes l(S)}$$

where Γ/S stands for the $\mathbf{k}$ vector space generated by Γ/S, $l(S)$ is the number of components of $[\Gamma : S]$. For every $k \in \mathbb{N}$ $\mathcal{K}^k(\Gamma)$ we have

$$\mathcal{K}^k(\Gamma) = \bigoplus_{S \subset E, |S|=k} \Gamma/S \otimes A^{\otimes l(S)}$$

We have the following two cases:

- If n is odd, we assign to the graph Γ the orientation o as defined in *Definition* 2.1.7, that is given by a choice of an order on the vertices and half edges. Let e be an edge in Γ, to the graph Γ/e is assign an induced orientation as defined in *Definition* 2.1.11. In the odd case the following relation holds
$$o(-\Gamma) = -o(\Gamma)$$

- If n is even we assign to Γ an orientation given by a choice of an order on the set of edges and an induced orientation on Γ/e as in *Definition* 4.2.2. The following relation holds
$$\sigma(o)(\Gamma) = \mathrm{Sgn}(\sigma)o(\Gamma)$$
where σ is a permutation of the set edges.

The differential $\delta : \mathcal{K}(\Gamma)^n \to \mathcal{K}(\Gamma)^{n+1}$ is defined by

$$d = \sum_{e \in E(\Gamma)} d_e,$$

and for $e \in \Gamma/S$ such that $e : i \to j$

$$\begin{aligned} &d_e(\Gamma/S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) \\ &\quad = o(\Gamma/S/e)(-1)^{\nu}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_i a_j \otimes \ldots a_{l(S \cup e)} \end{aligned}$$

where $(-1)^{\nu}$ is the Kozul sign given by moving a_j to the immediate right of a_i in the tensor product.

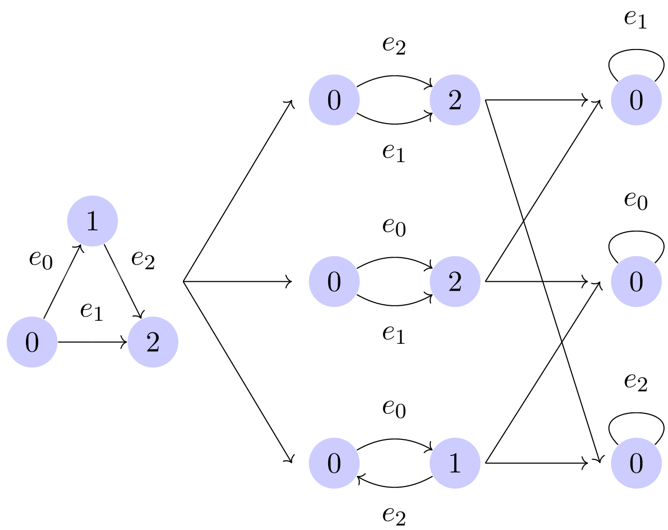

Figure 4.1: An example of $\mathcal{K}(\Gamma)$ where $\Gamma = K_3$. According to the case being odd or even the loops or the double edges are zero.

We can define the induced orientation in the even case in the definition of GC as follows.

Definition 4.2.2. Let Γ be a graph and e an edge in $E(\Gamma)$ that is not a loop. If Γ has an orientation given by the order of the edges $o(\Gamma) = \mathrm{Sgn}(e_1, \dots, e_{|E(\Gamma)|})$, then Γ/e has an induced orientation $o(\Gamma/e) = \mathrm{Sgn}(\hat{e}, e_1, \dots, e_{|E(\Gamma)|})$, obtained by moving the edge e in the first place and then removing it.

Theorem 4.2.3. *The differential* $d : \mathcal{K}^k(\Gamma) \to \mathcal{K}^{k+1}(\Gamma)$ *makes* $\mathcal{K}(\Gamma)$ *into a chain complex, that is* $d^2 = 0$.

Proof. The fact that $d^2 = 0$ is a consequence of the definition of orientation of a graph. We therefore divide the proof in the odd and even case.

Suppose first that the complex is defined in the odd case. Let $o(\Gamma/S)\Gamma/S \otimes a_1 \otimes \cdots \otimes a_{l(S)}$ be an element in $\mathcal{K}^k$, let e and f be edges in Γ/S. Suppose that they are not loops, we shall show that the orientation obtained by removing from Γ/S first e and then f is the opposite of the orientation obtained by removing first f and then e. Suppose without loss of generality that e is the edge $e : v_i \to v_j$ and $f : v_k \to v_l$ and we give an orientation to the graph Γ/S in a way that the edges v_i, v_j, v_k, v_l comes first in this order and the others follow. We first remove e and then f, the removing of e implies the removing of the vertex v_j and $\mathrm{Sgn}(v_i, v_j, v_k, v_l \dots) = \mathrm{Sgn}(v_i, v_k, v_l \dots) = \mathrm{Sgn}(v_k, v_l, v_i \dots)$ and removing the edge f gives $\mathrm{Sgn}(v_k, v_i \dots) = -\mathrm{Sgn}(v_i, v_k \dots)$. On the other hand if we first remove f we obtain that $\mathrm{Sgn}(v_i, v_j, v_k, v_l \dots) = \mathrm{Sgn}(v_k, v_l, v_i, v_j, \dots)$

and deleting the vertex v_l we have $\mathrm{Sgn}(v_k, v_i, v_j, \dots) = \mathrm{Sgn}(v_i, v_j, v_k, \dots) = \mathrm{Sgn}(v_i, v_k, \dots)$. Suppose now $\mathcal{K}(\Gamma)$ to be defined in the even case, then the orientation of a graph is given by an ordering of the set of edges $E(\Gamma)$. Suppose that e and f are edges in Γ/S that are not loops. Let $o(\Gamma/S)$ be the orientation given by ordering the edges in increasing order, then $o((\Gamma/S)/e) = \mathrm{Sgn}(e, e_1, \dots, e_k) = (-1)^\tau o(\Gamma/S)$, where τ is the number of edges before e. Removing then the edge f gives a sign $(-1)^\nu$, where ν is the number of edges before f in Γ/S if f comes before e, otherwise we get a sign $(-1)^{\nu-1}$. Now, removing first f and then e gives signs $(-1)^\nu$ and $(-1)^{\tau-1}$ if f comes before e, while $(-1)^\nu$ and $(-1)^\tau$ otherwise. So, if f comes before e then $o((\Gamma/S)/e/f) = (-1)^{\tau+\nu}$ and $o((\Gamma/S)/f/e) = (-1)^{\tau+\nu-1}$. If f comes after e then $o((\Gamma/S)/e/f) = (-1)^{\tau+\nu-1}$ and $o((\Gamma/S)/f/e) = (-1)^{\tau+\nu}$.

This shows that terms in d^2 given by removing first e and then f cancel out with terms obtained by removing first f and then e. We can conclude that $d^2 = 0$. □

4.3 From $\mathcal{C}_{BS}(\Gamma)$ to $\mathcal{K}(\Gamma)$

We will first discuss the case where M is an odd dimensional manifold and then the case where it has even dimension.

Let Γ be a graph as before, M an odd dimensional manifold and A a CDGA as before. We consider the chain complex $\mathcal{C}_{BS}(\Gamma)$ as in *Definition* 2.2,

$$\mathcal{C}_{BS}(\Gamma) = \bigoplus_{S \subset E(\Gamma)} e_S \otimes A^{\otimes l(S)}$$

where e_S denotes the product $e_S = e_{i_1} \cdot \ldots \cdot e_{i_{|S|}}$ in increasing order of generators e_α corresponding to edges α in S, and $l(S)$ is the number of components of $[\Gamma : S]$.

Definition 4.3.1. Let M be an odd dimensional manifold. We define the map $\psi : \mathcal{C}_{BS}(\Gamma) \to \mathcal{K}(\Gamma)$ for all $n \in \mathbb{N}$ as the map of $\mathbf{k}$-modules,

$$\psi^n : \mathcal{C}^n_{BS}(\Gamma) \to \mathcal{K}(\Gamma)^n$$
$$e_S \otimes a \mapsto \begin{cases} 0 & \text{if } S \text{ is not a forest} \\ \eta_{\Gamma/S} \Gamma/S \otimes a & \text{otherwise} \end{cases}$$

where $\eta_{\Gamma/S}$ is the sign given by the induced orientation obtained by contracting from Γ the edges in S in decreasing order. That is, if $S = \{e_1, \dots, e_k\}$, $\eta_{\Gamma/S}$ is the product of the induced orientations obtained by removing the edges in S, $\eta_{\Gamma/S} = o(\Gamma/e_k) \cdot o((\Gamma/e_k)/e_{k-1}) \cdot \ldots \cdot o((\Gamma/e_k/e_{k-1}/ \cdots /e_2)/e_1)$.

Theorem 4.3.2. *$\psi : \mathcal{C}_{BS}(\Gamma) \to \mathcal{K}(\Gamma)$ is a chain map.*

Proof. Let $S \subseteq E(\Gamma)$, $|S| = n$ and $l(S) = k$, where $l(S)$ is the number of components of the graph $[\Gamma : S]$. If e is an edge in E such that $S \cup e$ is a forest then

$$\begin{aligned} &d_e(\eta_{\Gamma/S}\Gamma/S \otimes a_1 \otimes \cdots \otimes a_k) \\ &\qquad = o((\Gamma/S)/e)\eta_{\Gamma/S}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_{k-1} \end{aligned}$$

where $o((\Gamma/S)/e) \in \{1, -1\}$ is the sign given by the orientation of $(\Gamma/S)/e$ induced from Γ/S. By the relation $o((\Gamma/e)/f) = -o((\Gamma/f)/e)$, we have that

$$o((\Gamma/S)/e)\eta_{\Gamma/S} = (-1)^{\tau}\eta_{\Gamma/S\cup e} \tag{4.1}$$

where τ is the number of edges smaller than e. The notation $o((\Gamma/S)/e)\eta_{\Gamma/S}$ indicates the sign of the graph $\Gamma/e_n/e_{n-1}/\cdots/e_1/e$ obtained by contracting first the edges $e_i \in S$ in decreasing order and then the edge e in $E(\Gamma)$ not belonging to S, while $\eta_{\Gamma/S\cup e}$ is the sign of the graph $\Gamma/e_n/e_{n-1}/\cdots/e/\cdots/e_1$ obtained from Γ by contracting the edges in $S \cup e$ in decreasing order. Therefore,

$$\begin{aligned} &o(\Gamma/S/e)\eta_{\Gamma/S}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_{k-1} \\ &\qquad = (-1)^{\tau}\eta_{\Gamma/S\cup e}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_{k-1}. \end{aligned}$$

If e is an edge in Γ such that $S \cup e$ is not a forest then

$$d_e(\eta_{\Gamma/S}\Gamma/S \otimes a_1 \otimes \cdots \otimes a_k) = 0.$$

On the other hand, the sign given by applying the differential of the chain $\mathcal{C}_{BS}(\Gamma)$ to an element $e_S \otimes a_1 \otimes \cdots \otimes a_k \in \mathcal{C}^n_{BS}(\Gamma)$ is

$$\begin{aligned} &\partial_e(e_S \otimes a_1 \otimes \cdots \otimes a_k) \\ &\qquad = e \cdot e_S \otimes a_1 \otimes \cdots \otimes a_{k-1} = (-1)^{\tau} e_{S\cup e} \otimes a_1 \otimes \cdots \otimes a_{k-1}. \end{aligned}$$

Again, τ is the number of edges smaller than e and the sign is given by permuting the factor in the multiplication in the exterior algebra,

$$e \cdot e_S = e \cdot e_1 \cdot \ldots \cdot e_n = (-1)^{\tau} e_1 \cdot \ldots \cdot e \cdot \ldots \cdot e_n = e_{S\cup e}.$$

Suppose that $S \subset E(\Gamma)$ is a forest and e and edge that is not in S, we want to prove the commutativity of the diagram

$$\begin{tikzcd} \mathcal{C}_{BS}(\Gamma)^n \arrow[r, "\psi"] \arrow[d, "\partial_e"'] & \mathcal{K}(\Gamma)^n \arrow[d, "d_e"] \\ \mathcal{C}_{BS}(\Gamma)^{n+1} \arrow[r, "\psi"] & \mathcal{K}(\Gamma)^{n+1} \end{tikzcd}$$

$$\begin{tikzcd} e_S a_1 \otimes \cdots \otimes a_k \arrow[r, "\psi"] \arrow[d, "\partial_e"'] & \eta_{\Gamma/S}\Gamma/S \otimes a_1 \otimes \cdots \otimes a_k \arrow[d, "d_e"] \\ e_{S\cup e} a_1 \otimes \cdots \otimes a_{k-1} \arrow[r, "\psi"] & \eta_{\Gamma/S\cup e}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_{k-1} \end{tikzcd}$$

Let first $e : i \to j$ be an edge such that $S \cup e$ is a forest

$$\begin{aligned}
&d_e \circ \psi(e_S \otimes a_1 \otimes \cdots \otimes a_k) \\
&\quad = d_e(\eta_{\Gamma/S} a_1 \otimes \cdots \otimes a_k) \\
&\quad = o((\Gamma/S)/e)\eta_{\Gamma/S}(-1)^{\nu} a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1} \\
&\quad = (-1)^{\tau}\eta_{\Gamma/S\cup e}(-1)^{\nu} a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1},
\end{aligned}$$

where the last equality is due to 4.1. $(-1)^{\nu}$ is the Kozul sign given by moving a_j in the tensor product. On the other hand,

$$\begin{aligned}
&\psi \circ \partial_e(e_S \otimes a_1 \otimes \cdots \otimes a_k) \\
&\quad = \psi((-1)^{\tau} e_{S\cup e}(-1)^{\nu} \otimes a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1}) \\
&\quad = (-1)^{\tau}\eta_{\Gamma/S\cup e}(-1)^{\nu} a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1}
\end{aligned}$$

If S is a graph such that Γ/S has a loop then for every edge e in $E(\Gamma) \setminus S$ that is not a loop, $\Gamma/S \cup e$ also has a loop. Then

$$(d_e \circ \psi)((e_S \otimes a_1 \otimes \cdots \otimes a_k) = 0$$

and

$$(\psi \circ \partial_e)((e_S \otimes a_1 \otimes \cdots \otimes a_k) = 0$$

since $\Gamma/S = 0$. The same equation holds in the case where e is a loop. □

We now describe the even dimensional case.

Definition 4.3.3. Let M be an even dimensional manifold. The map $\phi : \mathcal{C}_{BS}(\Gamma) \to \mathcal{K}(\Gamma)$ is defined for all $n \in \mathbb{N}$ as the map of **k**-modules,

$$\phi^n : \mathcal{C}(\Gamma)^n_{BS} \to \mathcal{K}(\Gamma)^n$$

$$e_S \otimes a \mapsto \begin{cases} 0 & \text{if } S \text{ is not a forest or } \Gamma/S \text{ contains a loop} \\ \mu_{\Gamma/S}\Gamma/S \otimes a & \text{otherwise} \end{cases}$$

where $\mu_{\Gamma/S}$ is the sign of the induced orientation obtained by contracting from Γ the edges in S in decreasing order.

Theorem 4.3.4. *ϕ is a chain map.*

Proof. Suppose that S is a subset of the set of edges $E(\Gamma)$ that is a forest and e an edge that is not in S, we want to prove the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{C}(\Gamma)^n_{BS} \arrow[r, "\phi"] \arrow[d, "\partial_e"'] & \mathcal{K}(\Gamma)^n \arrow[d, "d_e"] \\
\mathcal{C}(\Gamma)^{n+1}_{BS} \arrow[r, "\phi"] & \mathcal{K}(\Gamma)^{n+1}
\end{tikzcd}$$

$$
\begin{tikzcd}
e_S a_1 \otimes \cdots \otimes a_k \arrow[r, "\phi"] \arrow[d, "\partial_e"'] & \mu_{\Gamma/S}\Gamma/S \otimes a_1 \otimes \cdots \otimes a_k \arrow[d, "d_e"] \\
e_{S\cup e} a_1 \otimes \cdots \otimes a_{k-1} \arrow[r, "\phi"] & \mu_{\Gamma/S\cup e}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_{k-1}
\end{tikzcd}
$$

Note that, if $S = \{e_1, \ldots, e_s\}$ the induced orientation $\mu(\Gamma/S) = (-1)^{l_S}$, where $l_S = \sum_{i=1}^{s} n_{e_i}$ and n_{e_i} are the number of edges in Γ before e_i. The differential ∂_e adds a sign that is $(-1)^{n_{S,e}}$, where $n_{S,e}$ is the number of edges before e in S. The differential d_e gives a sign $(-1)^{n_{\Gamma/S,e}}$ where $n_{\Gamma/S,e}$ is the number of edges before e in Γ/S. Since the number of edges before e in Γ is the sum of the number of edges before e in Γ/S and the number of edges before e in S, that is $n_e = n_{\Gamma/S,e} + n_{S,e}$. We have that

$$(-1)^{l_{S\cup e}}(-1)^{n_{S,e}} = (-1)^{l_S}(-1)^{n_{\Gamma/S,e}}(-1)^{2n_{S,e}},$$

this implies

$$(-1)^{l_{S\cup e}}(-1)^{n_{S,e}} = (-1)^{l_S}(-1)^{n_{\Gamma/S,e}} \tag{4.2}$$

Suppose that $S \cup e$ is a forest, then

$$
\begin{aligned}
d_e \circ \phi(e_S \otimes a_1 \otimes \cdots \otimes a_k) &= d_e(\mu_{\Gamma/S}\Gamma/S \otimes a_1 \otimes \cdots \otimes a_k) \\
&= o((\Gamma/S)/e)\mu_{\Gamma/S}(-1)^{\nu}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1} \\
&= (-1)^{n_{\Gamma/S,e}+l_S}(-1)^{\nu}\Gamma/S \cup e \otimes a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1}
\end{aligned}
$$

where $(-1)^{\nu}$ is the Kozul sign given by moving a_j in the tensor product. On the other hand,

$$
\begin{aligned}
&\phi \circ \partial_e(e_S \otimes a_1 \otimes \cdots \otimes a_k) \\
&\quad = \phi((-1)^{n_{S,e}}(-1)^{\nu} e_{S\cup e} \otimes a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1}) \\
&\quad = (-1)^{n_{S,e}}\eta_{\Gamma/S\cup e}(-1)^{\nu} a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1} \\
&\quad = (-1)^{n_{S,e}}(-1)^{l_{S\cup e}}(-1)^{\nu} a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1}
\end{aligned}
$$

By the argument above 4.2, the last expression equals

$$(-1)^{l_S + n_{\Gamma/S,e}}(-1)^{\nu} a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{k-1}.$$

Let S be a graph such that Γ/S has a loop or a double edge. Then for every edge e in $E(\Gamma) \setminus S$ that is not a loop

$$(d_e \circ \psi)(e_S \otimes a_1 \otimes \cdots \otimes a_k) = 0$$

and

$$(\psi \circ \partial_e)(e_S \otimes a_1 \otimes \cdots \otimes a_k) = 0$$

since $\Gamma/S \cup e$ is a graph with a loop or a double edge. □

4.4 From $\mathcal{K}(\Gamma)$ to $\mathcal{K}_\Sigma(\Gamma)$

We define $\mathcal{K}_\Sigma(\Gamma)$ to be the complex $\mathcal{K}(\Gamma)$ such that the vertices are unordered in the following sense:

Definition 4.4.1. Let Γ be a graph without loops or multiple edges with an order on the vertices that gives an orientation on every edge α in $E(\Gamma)$ in a way that if α connects the vertices i and j and $i \leq j$ then $\alpha : i \to j$. We define $\mathcal{K}_\Sigma(\Gamma)$ to be the complex given for every $n \geq 0$ by the set of coinvariants

$$\mathcal{K}^n_\Sigma(\Gamma) = \Big(\bigoplus_{S \subset E(\Gamma), |S|=n} \Gamma/S \otimes A^{l(S)} \Big)_{\Sigma_{l(S)}}$$

where $\Sigma_{l(S)}$ acts on the vertices of Γ/S and to $A^{l(S)}$.

The differential is given by $d_\Sigma : \mathcal{K}_\Sigma(\Gamma)^n \to \mathcal{K}_\Sigma(\Gamma)^{n+1}$ such that

$$d_\Sigma = \sum_{e \in E(\Gamma)} d_{\Sigma,e}$$

and for $e \in \Gamma/S$ such that $e : i \to j$

$$\begin{aligned} &d_{\Sigma,e}([\Gamma/S]_\Sigma \otimes a_1 \otimes \cdots \otimes a_{l(S)}) \\ &\qquad = o((\Gamma/S)/e)(-1)^\nu [\Gamma/S \cup e]_\Sigma \otimes a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{l(S \cup e)} \end{aligned}$$

here $(-1)^\nu$ is the Kozul sign given by moving a_j to the immediate right of a_i in the tensor product. In particular, for $x \in \mathcal{K}^k(\Gamma)$

$$d_{\Sigma,e}([x]_\Sigma) = [d_e(x)]_\Sigma.$$

Theorem 4.4.2. *The differential $d_\Sigma : \mathcal{K}^k_\Sigma(\Gamma) \to \mathcal{K}^{k+1}_\Sigma(\Gamma)$ makes $\mathcal{K}_\Sigma(\Gamma)$ into a chain complex, that is $d^2 = 0$.*

Proof. The fact that $d^2 = 0$ is a consequence of the definition of orientation of a graph. This is defined as in the complex $\mathcal{K}(\Gamma)$, so the proof follows the same argument as in *Theorem* 4.2.3. □

Remark 4.4.3. Note that Γ and Γ' are graphs that differ for a permutation of the vertices $\sigma \in \Sigma$ or an inversion of the direction of the edges, then $\Gamma' = (-1)^{|\sigma|}(-1)^m \Gamma$ where $|\sigma|$ is the sign of the permutation and m in the number of flipped edges. Therefor, $[\Gamma']_\Sigma = (-1)^{|\sigma|}(-1)^m [\Gamma]_\Sigma$

Remark 4.4.4. As for $\mathcal{K}$, the definition of orientation of the graph varies according to the parity of an integer n, that is the dimension of he manifold M, such that A is a CGD model for M.

Moreover, we have a map of $\mathbf{k}$-module for every $n \in \mathbb{N}$

$$\phi^n : \mathcal{K}^n(\Gamma) \to \mathcal{K}^n_\Sigma(\Gamma)$$

$$x \mapsto [x]_\Sigma$$

Theorem 4.4.5. *The map* $\phi : \mathcal{K}(\Gamma) \to \mathcal{K}_\Sigma(\Gamma)$ *is a chain map.*

Proof. We want to prove the commutativity of the following diagram

$$\begin{tikzcd} \mathcal{K}(\Gamma)^n \arrow[r, "\phi"] \arrow[d, "d_e"'] & \mathcal{K}(\Gamma)^n_\Sigma \arrow[d, "d_{\Sigma,e}"] \\ \mathcal{K}(\Gamma)^{n+1} \arrow[r, "\phi"] & \mathcal{K}(\Gamma)^{n+1}_\Sigma \end{tikzcd}$$

Let S be a subset of $E(\Gamma)$, $x = \Gamma/S \otimes a_1 \otimes \cdots \otimes a_{l(S)}$ an element in $\mathcal{K}^n$ and let $[x]_\Sigma$ be a representative of the class of x with a specific orientation o.

Now, for an edge e in Γ/S,

$$d_{\Sigma,e} \circ \phi(x) = d_{\Sigma,e}([x]_\Sigma) = [d_e(x)]_\Sigma,$$

the last equality is a direct consequence of the definition of the differential in $\mathcal{K}/_\Sigma$. On the other hand

$$\phi \circ d_e(x) = [o(x/e)d_e(x)]_\Sigma = [d_e(x)]_\Sigma.$$

$\square$

4.5 The full graph complex fGC_R

We describe in details the definition of fGC_R, briefly mentioned in 3.2.5. The full graph complex fGC_R was defined by *Idrissi* in [19], and in a particular case by *Campos* and *Willawacher* [8].

Let R be the CDGA introduced in *Theorem* 3.2.1. Then, the *full graph complex* fGC_R is the CDGA $\mathrm{TwGra}_R(0)$. It is the free algebra generated by unlabeled connected graphs whose vertices are colored by elements in R. The product is given by the disjoint union of graphs. More precisely

$$\mathrm{fGC}_R = \bigoplus \mathbb{Z}\left[\left[\gamma \otimes A^{\otimes V(\gamma)}\right]_{\Sigma_{V(\gamma)}}\right]$$

here $\Sigma_{V(\gamma)}$ is the the symmetric group on $V(\gamma)$. As before, the definition depends on a natural number n, equivalence classes of graphs are equipped with orientation as defined in *Definition* 2.1.1, with the distinction between n being even or odd. The edges have degree $n-1$ while the vertices have degree $-n$.

The differential is given by $d : \mathrm{fGC}^k_R \to \mathrm{fGC}^{k+1}_R$ is defined by

$$d = \sum_e d_e + d_{\text{split}}$$

such that e is an edge between i and j that is not a loop

$$d_e(\gamma \otimes a_1 \otimes \cdots \otimes a_l) = o(\gamma/e)\gamma/e \otimes a_1 \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_{l-1}$$

and

$$d_{\mathrm{split}}(\gamma \otimes a_1 \otimes \cdots \otimes a_l) = p_{i,j}^*(\Delta)\gamma \setminus e \otimes a_1 \otimes \cdots \otimes a_l$$

where Δ is the class of the diagonal as defined in *Proposition* 3.3. in [19], and $p_{i,j}^*$ is the pull back of the projection.

Remark 4.5.1. The complex $\mathcal{K}_\Sigma(\Gamma)$ can be interpret as a version the *full graph complex* fGC_R but relative to a graph and with differential given only by the term $\sum_e d_e$. Note that fGC_R depends on a natural number n, in addition edges have degree $n-1$ and vertices have degree $-n$. On the other hand, $\mathcal{K}_\Sigma(\Gamma)$ depends only on the parity of this number, and edges have degree -1.

Chapter 5

Graph cohomologies and rational homotopy type of configuration spaces

5.1 Introduction

In the present chapter we compare the graph cohomology complex $\mathcal{C}_{BS}(\Gamma)$ defined by *Baranovsky* and *Sazdanović* in [2] and described in *Section* 2.2, with the model for the rational homotopy type given by *Kriz* and *Totaro* denoted by $E[n]$. We first describe this commutative differential graded algebra, that was briefly introduced in *Section* 3.2. Then we define Frobenius algebras and give some technical results that will be later used. In *Section* 5.4 we define the dual of the complex $(\mathcal{C}_{BS}(\Gamma), \partial)$ that we will call $(\mathcal{C}_{BS}(\Gamma)^*, \delta)$. The complex $(\mathcal{C}_{BS}(\Gamma), \partial)$ is the E_1 page of a spectral sequence converging to the relative cohomology $H^*(M^n, Z_\Gamma, R)$, and by *Remark* 4 in [2], if the space M is a compact oriented manifold of dimension m, the cohomology is isomorphic to the homology $H_{mn-*}(\mathfrak{C}(M, \Gamma), R)$. In this case the dual complex $(\mathcal{C}^*_{BS}(\Gamma), \delta)$ converges to the cohomology of the configuration space $H^{mn-*}(\mathfrak{C}(M, \Gamma), R)$. On the other hand, the cohomology of the complex $E[n]$ is the cohomology of the configuration space. By *Remark* 2.2.8, if M is a compact Kähler manifold and the coefficient ring is $R = \mathbb{Q}$, the spectral sequence degenerates at page E_2. In this case the two complexes are quasi equivalent. In the following sections we prove that there is in general a quasi equivalence between $C_{BS}(\Gamma)^*$ and a generalized version of $E[n]$, called $R(\Gamma, A)$. In the definition of this generalised complex, a ring $\Delta[G_{a,b}]/\sim$ is involved. This is the exterior algebra over the generator corresponding to the edges in the graph Γ quotient by a relation, that we call the *generalised Arnold relation.* We denote this ring by $R(\Gamma)$. *Section* 5.5 describes it for a complete graph K_n, and in this case the relation is the usual Arnold relation. We will call it $R(K_n)$. In the following section we define $R(\Gamma, A)$ for an even dimensional manifold. It depends on a graph Γ not necessarily complete, and a Frobenius algebra A over any field. In the case of an even dimensional manifold and a complete graph Γ, $R(\Gamma, A)$ coincide with the CDGA that *Idrissi* [19] proves to be a real model for $\mathfrak{C}(M, n)$. *Section* 5.6.1 contains the main theorem of the chapter.

Theorem. *Let $S \subseteq \Gamma$. The map*

$$F : \mathcal{C}_{BS}(\Gamma)^* \to R(\Gamma, A)$$
$$F(G_S \otimes x) = [G_S \otimes x]$$

is a quasi equivalence.

In [35] *Thomas* and *Felix* prove that the mn suspension of the E_2 term of the *Bendersky-Gitler* spectral sequence is isomorphic to the E_2 term of of the *Cohen* and *Taylor* spectral sequence of which the *Kriz*'s model is a special case. Our theorem presents an alternative proof and generalization of this result. In the last section we discuss the chain complex $\mathcal{C}_{BS}(\Gamma)^*/I(\Gamma)$, where $I(\Gamma)$ is the ideal generated by the generalised Arnold relation and we show that it is isomorphic to $R(\Gamma, A)$.

All the sections in this chapter refer to the article by *Bökstedt* and *Minuz* [6].

5.2 The Kriz model

In this section we describe the rational model for the configuration space of points in a complex projective variety defined by *Kriz* in [26] and introduced in *Section* 3.2.

Let X be a smooth projective variety over $\mathbb{C}$ and $\mathfrak{C}(X, n)$ be the ordered configuration space of n points in a space X,

$$\mathfrak{C}(X, n) = X^n \setminus \bigcup_{i \neq j} \Delta_{i,j}$$

$\Delta_{i,j} = \{(x_1, \ldots, x_n) \in X^n; x_i = x_j\}$. For a, $b \in \{1, \ldots, n\}$, $a \neq b$, let $p_a^* : H^*(X) \to H^*(X^n)$ the pullback of the projection $p_a : X^n \to X$ to the a-th coordinate and let $p_{a,b}^* : H^*(X^2) \to H^*(X^n)$ the pullback of the projection $p_{a,b} : X^n \to X^2$. Let $\Delta \in H^*(X^2)$ be the class of the diagonal.

Theorem 5.2.1 ([26])**.** *Let X be a complex projective variety of complex dimension m. Then the space $\mathfrak{C}(X, n)$ has a model $E(n)$ that is isomorphic to*

$$H^*(X^n, \mathbb{Q})[G_{a,b}]$$

where $G_{a,b}$ are generators of degree $2m - 1$, a, $b \in \{1, \ldots, n\}$, $a \neq b$ modulo the relations

- $G_{a,b} = G_{b,a}$
- $p_a^*(x)G_{a,b} = p_b^*(x)G_{a,b},\ x \in H^*(X)$
- $G_{a,b}G_{b,c} + G_{b,c}G_{c,a} + G_{c,a}G_{a,b} = 0$

The differential is given by $d(G_{a,b}) = p_{a,b}^*\Delta$.

Remark 5.2.2. The third relation $G_{a,b}G_{b,c}+G_{b,c}G_{c,a}+G_{c,a}G_{a,b} = 0$ is known in the literature as *Arnold relation*.

The definition of this graded algebra presents some similarities with the graded complex defined in *Section* 2.2: the structure of the exterior algebra with generators $G_{a,b}$ and the first two relations. However, the differential in $\mathcal{C}_{BS}(\Gamma)$ "adds edges" while the one in $E[n]$ "removes edges". Therefore, we would like to relate the dual of the graded complex $\mathcal{C}_{BS}(\Gamma)$ with the DGA $E(n)$.

Moreover, the complex $\mathcal{C}_{BS}(\Gamma)$ makes perfect sense in positive characteristic, so that we will also consider the following situation. Let $\mathbf{k}$ be a ground ring, which could typically be $\mathbb{Z}$, $\mathbb{Q}$, or a prime field $\mathbb{F}_p$. Assume that $A = H^*(X, \mathbf{k})$ is an algebra over $\mathbf{k}$ which is free as a $\mathbf{k}$-module. We extend the definition given by Kriz to this case by defining $E[n]$ as $A[G_{a,b}]/\sim$, where the relations are given by exactly the same three formulas as in the theorem above. It will be convenient to extend the definition further to the case where A is a Frobenius algebra. To do this, we have to give a definition of Δ in this case, we do that in the next section.

5.3 Structures of tensor powers of Frobenius algebras

We will consider a graded version of Frobenius algebras. To be precise about how we understand that term in this chapter:

Definition 5.3.1. A graded commutative Frobenius algebra A over a commutative ground ring $\mathbf{k}$ is a graded commutative ring, free and finite over $\mathbf{k} = A^0$ as a module, together with a perfect, graded symmetrical pairing

$$\langle -, - \rangle : A \otimes A \to \mathbf{k},$$

such that

$$\langle ab, c \rangle = \langle a, bc \rangle.$$

Remark 5.3.2. Main example: Let X be a compact, connected $\mathbf{k}$-orientated manifold such that each cohomology group $H^i(X;\mathbf{k})$ is a free $\mathbf{k}$-module. The cohomology ring $H^*(X, \mathbf{k})$ is a graded commutative Frobenius algebra over $\mathbf{k}$. In this case, the pairing has degree $-\dim(X)$.

If A is a graded Frobenius algebra, so is $A\otimes A$. The multiplication is given by the usual tensor product of DGAs, involving the Koszul sign

$$(a\otimes b)(c\otimes d) = (-1)^{|b|\cdot|c|} ac\otimes bd,$$

and the pairing is given by

$$\langle a\otimes b, c\otimes d\rangle_2 = (-1)^{|b|\cdot|c|}\langle a,c\rangle\langle b,d\rangle$$

where $|-|$ stands for the degree of the an element in the graded algebra.

We can construct the dual $A^* = \hom(A, \mathbf{k})$ and we have an isomorphism of vector spaces given by

$$k : A \cong A^*$$

$$a \mapsto k(a)(-) = \langle -, a\rangle$$

A is equipped with a multiplication $m : A\otimes A \to A$ and A^* a dual map given by $m^* : A^* \to (A\otimes A)^* \cong A^*\otimes A^*$. Therefore we have a map $\mu^* : A \to A\otimes A$ defined by composing the map k that gives the isomorphism with the dual:

$$\mu^* : A \xrightarrow{k} A^* \xrightarrow{m^*} A^*\otimes A^* \xrightarrow{k^{-1}\otimes k^{-1}} A\otimes A.$$

Alternatively, μ^* is defined by that

$$\langle x\otimes y, \mu^*(a)\rangle_2 = \langle xy, a\rangle$$

We see from this definition that $\mu^* : A \to A\otimes A$ is an $A\otimes A^{op}$ module map, since

$$\begin{aligned}
&\langle x\otimes y, (a\otimes 1)\mu^*(b)(1\otimes c)\rangle_2\\
&\qquad = (-1)^{|a|\cdot|xy|+|b|\cdot|c|}\langle (a\otimes 1)(x\otimes y)(1\otimes c), \mu^*(b)\rangle_2\\
&\qquad = (-1)^{|a|\cdot|xy|+|b|\cdot|c|}\langle axyc, b\rangle\\
&\qquad = \langle xy, abc\rangle.
\end{aligned}$$

We define $\Delta \in A\otimes A$ by the property

$$\langle a\otimes b, \Delta\rangle_2 = \langle ab, 1\rangle.$$

Remark 5.3.3. In the case $A = H^*(M, \mathbf{k})$ as considered above, $A\otimes A \cong H^*(M\times M, \mathbf{k})$, and Δ corresponds under this isomorphism to the Poincaré dual of the homology class of the diagonal $M \subset M\times M$.

Lemma 5.3.4. *The class Δ satisfies that $(a\otimes b)\Delta = \mu^*(ab)$. In particular, $\mu^* : A \to A\otimes A$ is given by $\mu^*(a) = (a\otimes 1)\Delta$.*

Proof. Because the paring $\langle -, - \rangle_2$ is perfect, it suffices to prove that for any $x, y \in A$, we have that $\langle x \otimes y, \mu^*(ab) \rangle_2 = \langle x \otimes y, (a \otimes b)\Delta \rangle_2$. We do the computation

$$\begin{aligned} \langle x \otimes y, (a \otimes b)\Delta \rangle_2 &= \langle (x \otimes y)(a \otimes b), \Delta \rangle_2 \\ &= \langle xyab, 1 \rangle \\ &= \langle xy, ab \rangle \\ &= \langle x \otimes y, \mu^*(ab) \rangle_2 . \end{aligned}$$

□

Remark 5.3.5. Δ has the property that $(1 \otimes a)\Delta = (a \otimes 1)\Delta$, $a \in A$.

We introduce some notation. Let S be a subset of the set of edges $E(\Gamma)$. Each S determines a partition of the set of vertices so we have a map

$$\Phi : \mathcal{E}(\Gamma) \to \mathcal{P}(\Gamma)$$

where $\mathcal{P}(\Gamma)$ is the set of all partitions of $V(\Gamma)$ and $\mathcal{E}(E)$ the set of subsets of $E(\Gamma)$. The sets $\mathcal{E}(\Gamma)$ form a partially ordered set by inclusion, and $\mathcal{P}(\Gamma)$ form partially ordered sets by refinement. The map Φ is order preserving.

Note that the number $l(S)$ introduced in *Definition* 2.2 corresponds to the number of sets in the partition $P = \Phi(S)$, that is the cardinality of $\phi(S)$. We denote also by $|P|$ the number $l(S)$.

There is a contravariant functor Ψ from $\mathcal{P}(\Gamma)$ to graded algebras given by

$$\Psi(P) = A^{\otimes |P|}.$$

The dual of the canonical surjective map $\Psi(P \to V(\Gamma)) : A^{\otimes n} \to A^{\otimes |P|}$ is a canonical injective map

$$A^{\otimes |P|} \cong (A^{\otimes |P|})^* \hookrightarrow (A^{\otimes n})^* \cong A^{\otimes n}.$$

For any partition P we consider the image $\Delta_P \in A^{\otimes n}$ of $1 \in A^{\otimes |P|}$. Using lemma 5.3.4 inductively, we see that multiplying with this element is dual to the multiplication map in the sense that the following diagram commutes:

$$\begin{tikzcd} A^{\otimes |P|} \arrow[r, "\Delta_P \cdot"] \arrow[d, "k^{\otimes |P|}"] & A^{\otimes n} \arrow[d, "k^{\otimes n}"] \\ (A^{\otimes |P|})^* \arrow[r, "\Psi(P \to V(\Gamma))^*"] & (A^{\otimes n})^* \end{tikzcd}$$

This element is invariant under any permutation in S_n preserving P. If Q is a refinement of P, there is similarly a relative element $\Delta_{Q,P} \in A^{\otimes |Q|}$ such that the following diagram commutes:

$$\begin{tikzcd} A^{\otimes |Q|} \arrow[rr, "\Delta_{Q,P}"] \arrow[dr, "\Delta_Q"'] & & A^{\otimes |P|} \arrow[dl, "\Delta_P"] \\ & A^{\otimes n} & \end{tikzcd}$$

Each algebra $A^{\otimes|P|}$ is a module over $A^{\otimes n}$, and multiplication by $\Delta_{P,Q}$ is a map of $A^{\otimes n}$-modules.

5.4 The dual graded complex

Using the notation of the previous section, we can re-write $\mathcal{C}_{BS}(\Gamma)$ as the graded chain complex

$$\mathcal{C}_{BS}(\Gamma) = \bigoplus_{S\subseteq E(\Gamma)} e_S \otimes A^{\otimes l(S)} = \bigoplus_{P\in\mathcal{P}} (\bigoplus_{S,\phi(S)=P} e_S \otimes A^{\otimes|P|}).$$

The differential is given by

$$\partial = \sum_{e\in E(\Gamma)} \partial_e$$
$$\partial_e(e_S \otimes x) = (-1)^\tau e_{S\cup\{e\}} \otimes \Psi(S \cup \{e\}).$$

The sign $(-1)^\tau$ is determined by the number τ of edges in S that precede e in the chosen ordering of the edges.

We note that as a graded vector space

$$\mathcal{C}_{BS}(\Gamma) = \Lambda(e_\alpha) \otimes (A^{\otimes n})/\sim$$

where $\sim$ indicates the relation $e_\alpha \otimes (a[i] - a[j])$, $a \in A$, $\alpha : i \to j \in E(\Gamma)$ and $a[i]$ denotes the element $1^{\otimes i-1} \otimes a \otimes 1^{\otimes n-i} \in A^{\otimes n}$, described in the first section in *Definition* 2.2.1. This relation corresponds to the second relation of the definition of the DGA defined by *Kriz*, since $p_a(x) = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1 \in A^{\otimes n}$ where x is the a-th component of the tensor product.

We want to describe the dual graded chain complex

$$(\mathcal{C}_{BS}(\Gamma))^* = \Big(\bigoplus_{S\subseteq E(\Gamma)} e_S \otimes A^{\otimes l(S)} \Big)^*.$$

We will denote by G_α the dual of the element e_α, for the edge $\alpha : i \to j$. We can so write the dual graded chain complex

$$\begin{aligned}(\mathcal{C}_{BS}(\Gamma))^* &= \Big(\bigoplus_{S\subseteq E(\Gamma)} e_S \otimes A^{\otimes l(S)} \Big)^* \\ &= \bigoplus_{S\subseteq E(\Gamma)} (e_S)^* \otimes (A^*)^{\otimes l(S)} = \bigoplus_{S\subseteq E(\Gamma)} G_S \otimes A^{\otimes l(S)}.\end{aligned}$$

Equivalently

$$(\mathcal{C}_{BS}(\Gamma))^* = \Lambda(G_\alpha) \otimes A^{\otimes n}/\sim .$$

The dual of the differential ∂, that we denote by δ, acts by removing edges in the graph and therefore increasing the number of components. Let G_S be the product of all the G_{ij} where $\alpha : i \to j$ is an edge in S,

$$\delta(G_S) = \sum_{i<j} (-1)^\nu \delta_{i,j}(G_S) = \sum_{i<j} (-1)^\nu G_{S \smallsetminus \alpha_{i,j}}$$

where ν is the number corresponding to the position of the edge $\alpha_{i,j}$ in the ascending order. We have

$$\delta(G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) = \sum_{i<j} (-1)^\nu \delta_{i,j}(G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)})$$

and

$$\begin{aligned} &\delta_{i,j}(G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) \\ &\quad = (-1)^\tau G_{S \smallsetminus \alpha} \otimes (\Delta_{S,S \smallsetminus \alpha} \cdot a_1 \otimes \cdots \otimes a_{l(S)}) \end{aligned}$$

in the case $\alpha : i \to j$ is an edge belonging to S and $l(S \smallsetminus \alpha) = l(S) - 1$, and τ is the Kozul sign given by moving the factor in $\mu(a)$ in the j-th position. While

$$\delta_{i,j}(G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) = G_{S \smallsetminus \alpha} \otimes a_1 \otimes a_i \otimes \cdots \otimes a_{l(S)}$$

in the case $\alpha : i \to j$ is an edge belonging to S and $l(S \smallsetminus \alpha) = l(S)$. Finally,

$$\delta_{ij}(G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) = 0$$

if α does not belong to S.

Remark 5.4.1. We discuss here the grading of the dual complex. Let $S \subseteq E(\Gamma)$. We assign to an element $G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}$ in $\mathcal{C}_{BS}(\Gamma)^*$ the grading

$$(m-1)r_{\text{ext}} - r_{\text{int}} + \sum_i |a_i|,$$

where m is the dimension of the manifold. r_{ext} is the number of external edges, that are the edges that, if removed, disconnect components, r_{int} the number of internal edges, that are the edges that do not disconnect components if removed and $|a_i|$ is the degree of the element a_i in A. The differential has degree 1 since

$$\begin{aligned} &\delta(G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) \\ &= \begin{cases} 0 & \text{if } \alpha \notin S \\ \sum_{\alpha \in E(\Gamma)} G_{S \smallsetminus \alpha} \otimes \Delta_{S,S \smallsetminus \alpha} \cdot a_1 \otimes \cdots \otimes a_{l(S)}, & \text{if } \alpha \text{ disconnects } S \\ \sum_{\alpha \in E(\Gamma)} G_{S \smallsetminus \alpha} \otimes a_1 \otimes \cdots \otimes a_{l(S)}, & \text{if } \alpha \text{ non disconnects } S \end{cases} \end{aligned}$$

and $\Delta_{S,S \smallsetminus \alpha}$ has degree m. If S is a forest, the grading of $\mathcal{C}_{BS}(\Gamma)^*$ and the DGA $R(\Gamma, A)$ that we define later, coincide.

5.5 The ring $R(K_n)$

In this section we want to study the ring defined by the exterior algebra $\Lambda[G_{a,b}]$, where $G_{a,b}$ are edges in a complete graph with n vertices K_n, quotient by the relations introduced by Kriz in *Theorem* 5.2.1.

Let K_n be a complete graph with n vertices and $\Lambda[G_{a,b}]$ be the exterior algebra with generators $G_{a,b}$ given corresponding to the edges in K_n. We define

$$R(K_n) = \Lambda[G_{a,b}]/\sim$$

where $\sim$ is the Arnold relation $G_{i,j}G_{j,k} + G_{j,k}G_{k,i} + G_{k,i}G_{i,j}$. We call $I(K_n)$ the ideal generated by this relation, in order to simplify the notation this will be denoted also by I.

The following lemmas characterize the ideal I. We denote by G_Γ the product of the generators corresponding to edges in Γ.

Lemma 5.5.1. *Let $v = (v_1, \dots, v_k)$, $k \geqslant 3$ a set of vertices in K_n and denote by $s(v)$ the product $s(v) = G_{v_1,v_2} \cdot \dots \cdot G_{v_i,v_{i+1}} \cdot G_{v_k,v_1}$ where $G_{v_i,v_{i+1}}$ is the generator in the exterior algebra corresponding to the edge $\alpha : v_i \to v_{i+1}$, so $s(v)$ is the product of the generators corresponding to edges of a cycle of length k. Let J be the ideal generated by the elements $s(v)$ with $k \geqslant 3$. Let I be the ideal generated by $\delta(s(v))$ with $k \geqslant 3$. Then J is contained in I and I is generated by $\delta(s(v_1, v_2, v_3)) = G_{v_1,v_2}G_{v_2,v_3} + G_{v_2,v_3}G_{v_3,v_1} + G_{v_3,v_1}G_{v_1,v_2}$.*

Proof. We first show that J is contained in I.

$$\begin{aligned}
&G_{v_1,v_2} \cdot \delta(s(v)) \\
&\quad = G_{v_1,v_2} \cdot (G_{v_2,v_3} \cdot \dots \cdot G_{v_i,v_{i+1}} \cdot G_{v_{i+1},v_{i+2}} \cdot \dots \cdot G_{v_k,v_1}) \\
&\qquad - G_{v_1,v_2} \cdot (G_{v_1,v_2} \cdot G_{v_3,v_4} \cdot \dots \cdot G_{v_i,v_{i+1}} \cdot G_{v_{i+1},v_{i+2}} \cdot \dots \cdot G_{v_k,v_1}) \\
&\qquad + \cdots \\
&\quad = G_{v_1,v_2} \cdot G_{v_2,v_3} \cdot \dots \cdot G_{v_i,v_{i+1}} \cdot G_{v_{i+1},v_{i+2}} \cdot \dots \cdot G_{v_k,v_1} = s(v).
\end{aligned} \tag{5.1}$$

Now we want to show that I is generated by $\delta(s(v_1, v_2, v_3)) = G_{v_1,v_2}G_{v_2,v_3} + G_{v_2,v_3}G_{v_3,v_1} + G_{v_3,v_1}G_{v_1,v_2}$. Let I_k the ideal generated by $\delta(s(v))$ where $v = (v_1, \dots, v_l)$, $l \leqslant k$. $I = \cup I_k$, we want to show by induction that $I_k = I_3$, $k \geqslant 3$ where I_3 is generated by $\delta(s(v_1, v_2, v_3))$. It is obviously true for $k = 3$. Suppose it true for $k-1$, we want to show that for every $s(v_1, \dots, v_k)$, $\delta(s(v_1, \dots, v_k)) \in I_{k-1}$. Consider $X = \delta(s(v_k, v_1, v_2))\delta(G_{v_2,v_3} \cdot \dots \cdot G_{v_{k-1},v_k}) \in I_3 = I_{k-1}$, we can expand the expression

$$\begin{aligned}
X = {} & (G_{v_1,v_2} \cdot G_{v_2,v_k} - G_{v_k,v_1} \cdot G_{v_2,v_k} + G_{v_k,v_1} \cdot G_{v_1,v_2}) \cdot \\
& \cdot \Big(\sum_{2 \le j \le k-1} (-1)^j G_{v_2,v_3} \cdot \dots \cdot \widehat{G}_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k} \Big)
\end{aligned}$$

$$
\begin{aligned}
&= (G_{v_1,v_2}) \cdot \Big(\sum_{2 \le j \le k-1} (-1)^j G_{v_k,v_2} \cdot G_{v_2,v_3} \cdot \dots \cdot \widehat{G}_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k} \Big) \\
&\quad - (G_{v_k,v_1}) \cdot \Big(\sum_{2 \le j \le k-1} (-1)^j G_{v_k,v_2} \cdot G_{v_2,v_3} \cdot \dots \cdot \widehat{G}_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k} \Big) \\
&\quad + \sum_{2 \le j \le k-1} (-1)^j G_{v_k,v_1} \cdot G_{v_1,v_2} \cdot G_{v_2,v_3} \cdot \dots \cdot \widehat{G}_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k} \\
&= (G_{v_1,v_2}) \cdot \big(-\delta(s(v_k, v_2, \dots, v_{k-1})) \\
&\qquad\qquad + G_{v_2,v_3} \cdot \dots \cdot G_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k} \big) \\
&\quad - (G_{v_k,v_1}) \big(-\delta(s(v_k, v_2, \dots, v_{k-1})) \\
&\qquad\qquad + G_{v_2,v_3} \cdot \dots \cdot G_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k} \big) \\
&\quad + \delta(s(v_k, v_1, \dots, v_{k-1})) - G_{v_1,v_2} \cdot G_{v_2,v_3} \cdot \dots \cdot G_{v_j,v_{j+1}} \\
&\quad + G_{v_k,v_1} \cdot G_{v_2,v_3} \cdot \dots \cdot G_{v_j,v_{j+1}} \\
&= -G_{v_1,v_2} \cdot \delta(s(v_k, v_2, \dots, v_{k-1})) \\
&\quad + G_{v_k,v_1} \delta(s(v_k, v_2, \dots, v_{k-1})) + \delta(s(v_k, v_1, \dots, v_{k-1})).
\end{aligned}
$$

Note that the third equality comes from the fact that

$$
\begin{aligned}
&\delta(s(v_k, v_2, \dots, v_{k-1})) \\
&\qquad = \sum_{2 \le j \le k} (-1)^{j-1} G_{v_k,v_2} \cdot \dots \cdot \widehat{G}_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k}
\end{aligned}
$$

that equals the first term in the sum in the expression apart from the missing term $G_{v_2,v_3} \cdot \dots \cdot G_{v_j,v_{j+1}} \cdot \dots \cdot G_{v_{k-1},v_k}$.

Now, $\delta(s(v_k, v_1, \dots, v_{k-1})) = (-1)^k \delta(s(v_1, v_2, \dots, v_k))$, so we can write

$$
\begin{aligned}
&\delta(s(v_1, v_2, \dots, v_k)) \\
&\qquad = -G_{v_1,v_2} \cdot \delta(s(v_2, v_3, \dots, v_k)) + G_{v_k,v_1} \delta(s(v_2, v_3, \dots, v_k)) - X.
\end{aligned}
$$

We can conclude that $\delta(s(v_1, v_2, \dots, v_k) \in I_{k-1} = I_3$. This end the proof by induction, so $I_k = I_3$ for all k and so $I = I_3$. □

Corollary 5.5.2. *If the graph K_n contains a cycle, then $G_\Gamma \in I$.*

We conclude that every element in $R_n(\Gamma)$ where $\Gamma = K_n$ can be written as a linear combination of the classes $G_{\Gamma'}$ where Γ' are graphs which do not contain any cycles. Such a graph is a disjoint union of trees, that is, it is a forest. However, these classes are not linearly independent in $R_n(\Gamma)$. We can rewrite the complex $R_n(\Gamma)$ as

$$
R_n(\Gamma) = \Lambda[G_{a,b}]/\sim = \bigoplus \bigotimes \mathbb{Z}[T]/\sim
$$

where $\mathbb{Z}[T]$ is the free group generated by the trees. We have from a result by Vassilev [39] that $\mathbb{Z}[T] = \mathbb{Z}^{(n-1)!}$.

5.6 The generalised DGA

We want to extend the definition of $E[n]$ to any graph Γ and to a Frobenius algebra over any ring. In order to do so, we need to modify the ideal $I(K_n)$ and introduce the following definition:

Definition 5.6.1. Let Γ be a graph and $(v_1, \ldots, v_k)$, $k \geqslant 3$ a set of vertices in Γ, we call a cycle w a subset of the set of edges of Γ of the form $\{(v_1, v_2), \ldots, (v_i, v_{i+1}), \ldots, (v_k, v_1)\}$. Let $\Lambda[G_{a,b}]$ be the exterior algebra with generators $G_{a,b}$ given corresponding to the edges in Γ. We denote by G_w the product $G_w = G_{v_1,v_2} \cdot \ldots \cdot G_{v_i,v_{i+1}} \cdot \ldots \cdot G_{v_k,v_1}$. We define

$$R(\Gamma) = \Lambda[G_{a,b}]/\sim$$

where $\sim$ is the relation

$$\delta(G_{w_j}) = \sum_i (-1)^i G_{v_{1,j},v_{2,j}} \cdot \ldots \cdot \hat{G}_{v_{i,j},v_{i+1,j}} \cdot \ldots \cdot G_{v_{k,j},v_{1,j}} = 0$$

for all j, where w_j is a cycle in Γ. We call *generalised Arnold relations* these last relations and $I(\Gamma)$ the ideal generated by them.

Remark 5.6.2. Note that by the results in the previous section, if $\Gamma = K_n$ then $I(\Gamma) = I(K_n)$.

Lemma 5.6.3. *If v is a cycle in Γ then $G_v \in I(\Gamma)$.*

Proof. Let v be the cycle with edges $\{(v_1, v_2), \ldots, (v_i, v_{i+1}), \ldots, (v_k, v_1)\}$

$$\begin{aligned}
&G_{v_1,v_2} \cdot d(G_v) \\
&\quad = G_{v_1,v_2} \cdot (G_{v_2,v_3} \cdot \ldots \cdot G_{v_i,v_{i+1}} \cdot G_{v_{i+1},v_{i+2}} \cdot \ldots \cdot G_{v_k,v_1}) \\
&\qquad - G_{v_1,v_2} \cdot (G_{v_1,v_2} \cdot G_{v_3,v_4} \cdot \ldots \cdot G_{v_i,v_{i+1}} \cdot G_{v_{i+1},v_{i+2}} \cdot \ldots \cdot G_{v_k,v_1}) \\
&\qquad + \cdots \\
&\quad = G_{v_1,v_2} \cdot G_{v_2,v_3} \cdot \ldots \cdot G_{v_i,v_{i+1}} \cdot G_{v_{i+1},v_{i+2}} \cdot \ldots \cdot G_{v_k,v_1} = G_v.
\end{aligned}$$

□

Corollary 5.6.4. *If Γ contains a cycle then $G_\Gamma \in I(\Gamma)$.*

We can conclude that the elements in $R_n(\Gamma)$ are linear combinations of forests. We now define the generalized complex.

Definition 5.6.5. Let M a compact, connected **k**-orientated manifold of even dimension m, $A = H^*(M, \mathbf{k})$ be a Frobenius algebra, where **k** is the ground ring. Let Γ be a graph with n edges and k cycles w_j $j = 0, \ldots, k$. We define the complex

$$R(\Gamma, A) = \Lambda[G_{a,b}] \otimes A^{\otimes n} / \sim$$

where $G_{a,b}$ are generators of degree $m - 1$, $(a, b) \in E(\Gamma)$, and $\sim$ are the relations

- $G_{a,b} = G_{b,a}$
- $p_a^*(x)G_{a,b} = p_b^*(x)G_{a,b}$, $x \in H^*(X)$
- $\delta(G_{w_j}) = \sum_i (-1)^i G_{v_{1,j},v_{2,j}} \cdot \ldots \cdot \hat{G}_{v_{i,j},v_{i+1,j}} \cdot \ldots \cdot G_{v_{h,j},v_{1,j}} = 0$, for all $j = 0, \ldots, k$

The differential is given by

$$d(G_{a,b}) = p_{a,b}^* \Delta$$

here Δ is the class of the diagonal as described in *Section* 5.3.

5.7 A quasi equivalence

Let M be an even dimensional, compact, connected **k**-orientated manifold of dimension m, $A = H^*(M, \mathbf{k})$ a graded commutative Frobenius algebra and Γ be any graph. Let $(\mathcal{C}_{BS}(\Gamma)^*, \delta)$ be the dual complex defined in *Section* 5.4 as $\mathcal{C}_{BS}(\Gamma)^* = \bigoplus_{S \subseteq E(\Gamma)} G_S \otimes A^{\otimes l(S)}$. We consider the generalized complex given in *Definition* 5.6.5

$$R(\Gamma, A) = \Lambda[G_{a,b}] \otimes A^{\otimes n}/\sim$$

where $\sim$ are the relations introduced in *Definition* 5.6.5 and $\Lambda[G_{a,b}]$ is the exterior algebra with generators given by the edges in Γ. We want to show that there is a quasi equivalence between $(\mathcal{C}_{BS}(\Gamma)^*, \delta)$ and $(R(A, \Gamma), d)$.

Remark 5.7.1. The differential in $\mathcal{C}_{BS}(\Gamma)^*$ can be written as

$$\delta = \delta_{\text{int}} + \delta_{\text{ext}}$$

where δ_{int} is the differential that removes internal edges, meaning edges such that if removed they don't disconnect components, and δ_{ext} is the differential that removes external edges, that are the edges that if removed they disconnect components. By *Lemma* 5.6.3 we have that $R(\Gamma)$ is given by linear combination of forests and therefore $d = \delta_{\text{ext}}$.

Definition 5.7.2. Let $S \subseteq E(\Gamma)$. We define the following map of graded groups:

$$F : \mathcal{C}_{BS}(\Gamma)^* \to R(\Gamma, A)$$
$$F(G_S \otimes x) = [G_S \otimes x]_\sim.$$

In order to simplify the notation we will write $G_S \otimes x$ instead of $[G_S \otimes x]_\sim$.

Lemma 5.7.3. *The map F is compatible with the differential.*

Proof. Let $G_{\Gamma'} \otimes x \in \mathcal{C}_{BS}(\Gamma)^*$, where Γ' is a subgraph of Γ. We want to check the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{C}_{BS}(\Gamma)^* \arrow[r, "F"] \arrow[d, "\delta"'] & R(A,\Gamma) \arrow[d, "d"] \\ \mathcal{C}_{BS}(\Gamma)^* \arrow[r, "F"] & R(A,\Gamma) \end{tikzcd}$$

We consider first the case where Γ' does not contain any cycle. If Γ' does not contain any cycle

$$d \circ F(G_{\Gamma'} \otimes x) = d(G_{\Gamma'} \otimes x).$$

On the other hand by *Remark* 5.7.1

$$F \circ \delta(G_{\Gamma'} \otimes x) = F \circ \delta_{\text{ext}}(G_{\Gamma'} \otimes x) = d(G_{\Gamma'} \otimes x).$$

Now, suppose that Γ' contains a cycle, that we denote by S, then

$$d \circ F(G_{\Gamma'} \otimes x) = d(0) = 0,$$

by definition of F. To prove the commutativity of the diagram we want to show that $F \circ \delta(G_{\Gamma'} \otimes x) = 0$. By the previous remark,

$$\delta(G_{\Gamma'} \otimes x) = \delta_{\text{int}} + \delta_{\text{ext}}(G_{\Gamma'} \otimes x),$$

the first summand is given by $\delta_{\text{int}}(G_{\Gamma'} \otimes x) = \delta(G_{S \cup S'})G_{\Gamma'/S \cup S'} \otimes x$, where S' is the graph given by the internal edges in Γ' that are not in S. The differential δ_{int} doesn't change the number of components and so it doesn't act on $x \in A^{\otimes l(\Gamma)}$. Now,

$$\delta(G_{S \cup S'}) = \delta(G_S)G_{S'} + G_S\delta(G_{S'}) \in I(\Gamma)$$

by *Lemma* 5.6.3 because G_S and $\delta(G_S)$ belongs to $I(\Gamma)$, so $F \circ \delta_{\text{int}}(G_{\Gamma'} \otimes x) = 0$. The second summand is

$$\delta_{\text{ext}}(G_{\Gamma'} \otimes x) = \delta_{\text{ext}}(G_{\Gamma'/S \cup S'})G_S G_{S'} \otimes x'$$

where $x' \in A^{\otimes l(\Gamma')-1}$. The term belongs to the ideal $I(\Gamma)$ since $G_S \in I(\Gamma)$ and so

$$F \circ \delta_{\text{ext}}(G_{\Gamma'} \otimes x) = 0.$$

$\square$

Theorem 5.7.4. *The map F is a quasi equivalence.*

Proof. We want to introduce two filtrations on $\mathcal{C}_{BS}(\Gamma)^*$ and on $R(A,\Gamma)$, and prove that F is compatible with them and that it induces a quasi equivalence on the filtrations' quotients.

Let Γ be a graph with n vertices, S be a subset of the set of edges $E(\Gamma)$. S determines a partition of the set of vertices, so we have a map $\Phi : \mathcal{E} \to \mathcal{P}$ where $\mathcal{P}$ is the set of all partitions of $V(\Gamma)$ and $\mathcal{E}$ the set of subsets of $E(\Gamma)$. As noted in *Section* 5.4 we can rewrite the complex $\mathcal{C}_{BS}(\Gamma)^*$ as

$$\mathcal{C}_{BS}(\Gamma)^* = \bigoplus_{P\in\mathcal{P}} (\bigoplus_{S,\phi(S)=P} G_S \otimes A^{\otimes|P|})$$

$|P|$ is the number of classes in the partition P.
There is a filtration of $\mathcal{C}_{BS}(\Gamma)(A)^*$ given by

$$\mathcal{F}_k = \bigoplus_{P\in\mathcal{P},|P|\geq k} \Big(\bigoplus_{S,\phi(S)=P} G_S \otimes A^{\otimes|P|}\Big).$$

$\mathcal{F}_k$ is a subcomplex of $\mathcal{C}_{BS}(\Gamma)^*$ since the differential δ acts by removing edges and so increasing the number of components. So

$$\mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_k \subseteq \mathcal{F}_{k-1} \subseteq \cdots \subseteq \mathcal{C}_{BS}(\Gamma)^*$$

and $|V(\Gamma)| = n$.

Similarly we have a filtration on $R(A,\Gamma)$ in terms of partitions. Since

$$R(\Gamma) = \bigoplus_{P\in\mathcal{P}} \bigoplus_{S,\phi(S)=P} G_S \Big/ \bigoplus_{S,\phi(S)=P} G_S \cap I(\Gamma)$$

we can define

$$\mathcal{F'}_k = \bigoplus_{P\in\mathcal{P},|P|\geq k} \Big(\bigoplus_{S,\phi(S)=P} G_S \Big/ \bigoplus_{S,\phi(S)=P} G_S \cap I(\Gamma)\Big) \otimes A^{\otimes|P|}$$

as before $\mathcal{F'}_k$ is a subcomplex of $R(A,\Gamma)$ since the differential d acts by removing edges and so increasing the number of components. So

$$\mathcal{F'}_n \subseteq \cdots \subseteq \mathcal{F'}_k \subseteq \mathcal{F'}_{k-1} \subseteq \cdots \subseteq R(A,\Gamma)$$

and $|V(\Gamma)| = n$. We want to show that F is compatible with the filtrations, that is $F(\mathcal{F}_k) \subseteq \mathcal{F'}_k$. This is clearly true since $F(G_\Gamma \otimes x) = G_\Gamma \otimes x$ is Γ if a forest and 0 otherwise.

There are two short exact sequences given by inclusion and quotient map

$$\mathcal{F}_{k-1} \longrightarrow \mathcal{F}_k \longrightarrow \mathcal{F}_{k-1}/\mathcal{F}_k$$

and

$$\mathcal{F'}_{k-1} \longrightarrow \mathcal{F'}_k \longrightarrow \mathcal{F'}_{k-1}/\mathcal{F'}_k.$$

The last step of the proof consists in showing that for every k,

$$F : \mathcal{F}_{k-1}/\mathcal{F}_k \longrightarrow \mathcal{F}'_{k-1}/\mathcal{F}'_k$$

is a quasi equivalence and then use the long exact sequences in homology induced from the short exact sequences to prove the result. Now,

$$\mathcal{F}_{k-1}/\mathcal{F}_k = \bigoplus_{P\in\mathcal{P},|P|=k-1} \Big(\bigoplus_{S,\phi(S)=P} G_S \otimes A^{\otimes|P|} \Big)$$

is determined by the partitions with exactly $k-1$ classes.

Let S be a subset of $E(\Gamma)$ that determines a partition of the set of vertices $P = \{P_1, \dots, P_l\}$ and let Γ_i^S, $0 \leq i \leq l$, be the connected subgraph of S corresponding to the element P_i in the partition. By *Remark* 5.7.5 we can rewrite $\mathcal{F}_{k-1}/\mathcal{F}_k$ as

$$\mathcal{F}_{k-1}/\mathcal{F}_k = \bigoplus_{P\in\mathcal{P},|P|=k-1} \bigoplus_{S,\phi(S)=P} \Big(\bigotimes_{1\leq i\leq k-1} C_{\text{con}}(\Gamma_i^S) \Big) \otimes A^{\otimes|P|}.$$

We define the chain complex $C_{\text{con}}(\Gamma)$ for connected graphs with h vertices, and for every $0 \leq i \leq h$, $C_{\text{con}}(\Gamma)^i$ is the free group generated by all connected subgraphs of Γ with i edges. Let S be a connected subgraph of Γ, the differential is given by

$$d_{\text{con}}(S) = \sum_{e\in E(\Gamma)} (-1)^{\nu} (S \smallsetminus e)$$

where ν is the position of edge $e \in E(\Gamma)$ in ascending order. If $S \smallsetminus e$ is not connected $d_{\text{con}}(S) = 0$.
$\mathcal{F}'_{k-1}/\mathcal{F}'_k$ can be written as

$$\mathcal{F}'_{k-1}/\mathcal{F}'_k = \bigoplus_{P\in\mathcal{P},|P|=k-1} \bigoplus_{S,\phi(S)=P} \Big(\bigotimes_{1\leq i\leq k-1} C_{\text{con}}(\Gamma'_i) \Big) \otimes A^{\otimes|P|}$$

where now Γ'_i are spanning trees, in particular $C_{\text{con}}(\Gamma') = C_{\text{con}}(\Gamma)/I(\Gamma)$. By the *Künneth formula*, the problem reduces to checking if

$$q : C_{\text{con}}(\Gamma) \to C_{\text{con}}(\Gamma)/I(\Gamma)$$

is a quasi equivalence. Here by $I(\Gamma)$ we mean the subgroup given by $\alpha(I(\Gamma))$ and α is the isomorphism defined in *Lemma* 5.7.5. By *Lemma* 5.7.8 the homology of $C_{\text{con}}(\Gamma)$ is concentrated in degree $n-1$. Now, $C_{\text{con}}(\Gamma)/I(\Gamma)$ is a complex concentrated in dimension $n-1$ by *Remark* 5.7.6, that is the chain group generated by the trees and

$$(C_{\text{con}}(\Gamma)/I(\Gamma))^{n-1} = C_{\text{con}}(\Gamma)^{n-1}/d_{\text{con}}(C_{\text{con}}(\Gamma)^n)$$

because by *Lemma* 5.7.7 $d_{\text{con}}(C_{\text{con}}(\Gamma)^n) = I(\Gamma)$. Since $C_{\text{con}}^{n-2} = 0$, we have

$$H_{n-1}(C_{\text{con}}(\Gamma)) = C_{\text{con}}(\Gamma)^{n-1}/I(\Gamma) = H_{n-1}(C_{\text{con}}(\Gamma)/I(\Gamma))$$

and

$$H_i(C_{\mathrm{con}}(\Gamma)) = H_i(C_{\mathrm{con}}(\Gamma)/I(\Gamma)) = 0$$

for $i \neq n-1$. Then

$$H_*(C_{\mathrm{con}}(\Gamma)) = H_*(C_{\mathrm{con}}(\Gamma)/I(\Gamma)),$$

so $q : C_{\mathrm{con}}(\Gamma) \to C_{\mathrm{con}}(\Gamma)/I(\Gamma)$ is a quasi equivalence.

Finally we consider the long exact sequences in homology

$$\begin{tikzcd}
H_{i+1}(\mathcal{F}_{k-1}/\mathcal{F}_k) \arrow[r] \arrow[d, "\simeq"] & H_i(\mathcal{F}_k) \arrow[r] \arrow[d, "F_*"'] & H_i(\mathcal{F}_{k-1}) \arrow[r] \arrow[d, "F_*"] & H_i(\mathcal{F}_{k-1}/\mathcal{F}_k) \arrow[d, "\simeq"] \\
H_{i+1}(\mathcal{F'}_{k-1}/\mathcal{F'}_k) \arrow[r] & H_i(\mathcal{F'}_k) \arrow[r] & H_i(\mathcal{F'}_{k-1}) \arrow[r] & H_i(\mathcal{F'}_{k-1}/\mathcal{F'}_k)
\end{tikzcd}$$

Since $\mathcal{F}_n = \mathcal{F'}_n$, we have that $H_i(\mathcal{F}_n) = H_i(\mathcal{F'}_n)$ for every $i \geq 0$. We can then use the *Five Lemma* and induction on k with initial step given by $H_i(\mathcal{F}_n) = H_i(\mathcal{F'}_n)$.

$$\begin{tikzcd}
H_i(\mathcal{F}_{n-1}/\mathcal{F}_n) \arrow[r] \arrow[d, "\simeq"] & H_{i-1}(\mathcal{F}_n) \arrow[r] \arrow[d, "="'] & H_{i-1}(\mathcal{F}_{n-1}) \arrow[r] \arrow[d, "q"] & H_{i-1}(\mathcal{F}_{n-1}/\mathcal{F}_k) \arrow[r] \arrow[d, "\simeq"] & H_{i-2}(\mathcal{F}_n) \arrow[d, "="'] \\
H_i(\mathcal{F'}_{n-1}/\mathcal{F'}_n) \arrow[r] & H_{i-1}(\mathcal{F'}_n) \arrow[r] & H_{i-1}(\mathcal{F'}_{n-1}) \arrow[r] & H_{i-1}(\mathcal{F'}_{n-1}/\mathcal{F'}_n) \arrow[r] & H_{i-2}(\mathcal{F'}_n)
\end{tikzcd}$$

Therefore $H_i(\mathcal{F}_k) \simeq H_i(\mathcal{F'}_k)$ for every k and i. This concludes the proof that F is a quasi equivalence.

□

Lemma 5.7.5. *Let*

$$\mathcal{F}_{k-1}/\mathcal{F}_k = \bigoplus_{P\in\mathcal{P},|P|=k-1} \Big(\bigoplus_{S,\phi(S)=P} G_S \otimes A^{\otimes|P|} \Big).$$

The map

$$\alpha : \bigoplus_{P\in\mathcal{P},|P|=k-1} \bigoplus_{S,\phi(S)=P} G_S \otimes A^{\otimes|P|}$$
$$\longrightarrow \bigoplus_{P\in\mathcal{P},|P|=k-1} \bigoplus_{S,\phi(S)=P} \Big(\bigotimes_{1\leq i\leq k-1} C_{\mathrm{con}}(\Gamma_i) \otimes A^{\otimes|P|} \Big)$$

is a group isomorphism.

Proof. Let P be a partition with $k-1$ classes and consider $S \subseteq E(\Gamma)$, such that $\Phi(S) = P$. Consider a class P_i corresponding to a connected subgraph of S, Γ_i. Since the tensor product $A^{\otimes|P|}$ is not affected by the differential, we can reduce to building a map

$$\alpha : \{G_{S'} \in \Lambda[G_S]; \phi(S') = P\} \to \bigotimes_{1\leq i\leq k-1} C_{\mathrm{con}}(\Gamma_i)$$

where $\Lambda[G_S]$ is the exterior algebra with generators given by the edges in S. Let S' be a subgraph of S. It can be written as $S' = S'_1 \cup \cdots \cup S'_{k-1}$, where S'_i are connected subgraphs of Γ_i. Then

$$\begin{aligned}\alpha(G_{S'}) &= \alpha(G_{S'_1} \cdot \ldots \cdot G_{S'_{k-1}}) \\ &= S'_1 \otimes \cdots \otimes (-1)^{\xi_i} S'_i \otimes \cdots \otimes (-1)^{\xi_i} S'_{k-1},\end{aligned}$$

ξ_i is the number of edges smaller (in S') than the smallest edge in S'_i plus the sum of the edges in $S'_1 \cup \cdots \cup S'_{i-1}$. We can suppose that the edges is S_i are ordered in increasing order and the edges in S_i are smaller then the edges in S_j is $i > j$. In this case the map α reduces to

$$\alpha(G_{S'}) = S'_1 \otimes \cdots \otimes S'_i \otimes \cdots \otimes S'_{k-1}.$$

We show that α commutes with the differentials δ and $d^{\otimes}_{\text{con}}$ that is the differential of the tensor product of complexes,

$$\begin{aligned}d^{\otimes}_{\text{con}}(\alpha(G_{S'})) &= d^{\otimes}_{\text{con}}(S'_1 \otimes \cdots \otimes S'_i \otimes \cdots \otimes S'_{k-1}) \\ &= \sum (-1)^{\nu_{1,j}} S'_1 \setminus e_{1,j} \otimes \cdots \otimes S'_i \otimes \cdots \otimes S'_{k-1} \\ &\quad + \cdots + S'_1 \otimes \cdots \otimes S'_i \otimes \cdots \otimes \sum (-1)^{\nu_{k-1,j}} (-1)^{\tau_{k-1,j}} S'_{k-1} \setminus e_{k-1,j}\end{aligned}$$

where $\tau_{k-1,j}$ are the number of edges in S' smaller than $e_{k-1,j}$. Moreover,

$$\begin{aligned}\alpha(\delta(G_S)) &= \alpha\Big(\sum (-1)^{\nu_i} S \setminus e_i\Big) \\ &= \sum S'_1 \otimes \cdots \otimes (-1)^{\nu_i} S_j \setminus e_i \otimes \cdots \otimes S_{k-1},\end{aligned}$$

and $(-1)^{\nu_i} = (-1)^{\nu_{i,j} + \tau_{i,j}}$. It is a group isomorphism, since there is a bijection between elements of the base of $\{G_{S'} \in \Lambda[G_S]; \phi(S') = P\}$ and elements $S_1 \otimes \cdots \otimes (-1)^{\nu_i} S_i \otimes (-1)^{\nu_i} S_l$ of the base of $\bigotimes_{1 \leq i \leq k-1} C_{\text{con}}(\Gamma_i)$. □

Remark 5.7.6. As a consequence of *Lemma* 5.6.3 we have that $\alpha(I(\Gamma)) \cap C_{\text{con}}$ contains the connected graphs with cycles, so the connected graphs that are not spanning trees. Therefore, $C^i_{\text{con}}/I(\Gamma)$ is trivial for all $i \neq n-1$.

Lemma 5.7.7. *Let $(C_{\text{con}}(\Gamma), d_{\text{con}})$ be the chain complex defined in the proof, Γ a connected graph with n vertices, then $d_{\text{con}}(C^n_{\text{con}}) = I(\Gamma) \cap C^{n-1}_{\text{con}}$.*

Proof. C^n_{con} is the free group generated by connected subgraph S of Γ with n edges. That is the image under the map α of the algebra

$$\bigoplus_{S \subseteq E(\Gamma), |S| = n} G_S.$$

Since Γ has n vertices, S must contain a cycle. We call C the cycle, and the exterior algebra G_S is given by the product of the edges in C and the product of the rest of the edges, that we call S'. Now $d_{\text{con}}(S) = \sum_e d_{\text{con},e}(C)S'$, since removing one edge in S' will give a non connected graph. By *Lemma* 5.7.7 the ideal generated by $d_{\text{con}}(C^n)$ is $\alpha(I(\Gamma)) \cap C^{n-1}_{\text{con}}$. □

Lemma 5.7.8. *Let Γ be a graph with n vertices. The homology of the complex $C_{\text{con}}(\Gamma)$ is concentrated in degree $n-1$.*

Proof. Let Γ be a connected graph with n vertices and let e be an edge in $E(\Gamma)$. We order the edges in Γ so that e is the last edge. We denote by $\Gamma \smallsetminus e$ the graph obtained from Γ by deleting the edge e and Γ/e the graph obtained from Γ by contracting the edge e. We have a short exact sequence

$$0 \to C_{\text{con}}(\Gamma \smallsetminus e) \xrightarrow{\alpha} C_{\text{con}}(\Gamma) \xrightarrow{\beta} C_{\text{con}}(\Gamma/e) \to 0.$$

The map α is the inclusion of subgraphs and is injective since $\ker(\alpha)$ is given by graphs that are mapped to 0, but these are the disconnected graph that are 0 also in $C^i_{\text{con}}(\Gamma \smallsetminus e)$, so $\ker(\alpha) = 0$. β is the contraction of the edge e and it is surjective since every element in $C^i_{\text{con}}(\Gamma/e)$ is the image of an element in $C^i_{\text{con}}(\Gamma)$, and if a graph is disconnected in $C^i_{\text{con}}(\Gamma/e)$ so it is in $C^i_{\text{con}}(\Gamma)$. Now we want to show that α and β are chain maps, so that the squares in the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C^{i+1}_{\text{con}}(\Gamma \smallsetminus e) \arrow[r, "\alpha"] \arrow[d, "d_{\text{con}}"] & C^{i+1}_{\text{con}}(\Gamma) \arrow[r, "\beta"] \arrow[d, "d_{\text{con}}"] & C^{i}_{\text{con}}(\Gamma/e) \arrow[r] \arrow[d, "d_{\text{con}}"] & 0 \arrow[d] \\
0 \arrow[r] & C^{i}_{\text{con}}(\Gamma \smallsetminus e) \arrow[r, "\alpha"] & C^{i}_{\text{con}}(\Gamma) \arrow[r, "\beta"] & C^{i-1}_{\text{con}}(\Gamma/e) \arrow[r] & 0
\end{tikzcd}$$

The right and left square are clearly commutative. Consider the second square, let $S \in C^i_{\text{con}}(\Gamma \smallsetminus e)$ then

$$\alpha(d_{\text{con}}(S)) = \alpha\Big(\sum_{l\in E(S)} (-1)^{\nu_l} S \smallsetminus l\Big) = \sum_{l\in E(S)} (-1)^{\nu_l} S \smallsetminus l$$

and

$$d_{\text{con}}(\alpha(S)) = d_{\text{con}}(S) = \sum_{l\in E(S)} (-1)^{\nu_l} S \smallsetminus l.$$

Consider now the third square, let $S \in C^i_{\text{con}}(\Gamma)$,

$$\beta(d_{\text{con}}(S)) = \beta\Big(\sum_{l\in E(S)} (-1)^{\nu_l} S \smallsetminus l\Big) = \sum_{l\in E(S \smallsetminus e)} (-1)^{\nu_l} (S \smallsetminus l)/e$$

and

$$d_{\text{con}}(\beta(S)) = d_{\text{con}}(S/e) = \sum_{l\in E(S \smallsetminus e)} (-1)^{\nu_l} S/e \smallsetminus l,$$

since we ordered the edges in Γ so that e is the last edge. Reordering the edges of Γ commutes with the differential so considering the chain where the edge

e is the last edge in Γ doesn't affect the computation of the homology. There are long exact sequences in homology and we proceed by induction.

$$\begin{aligned} H_{i-1}(C_{\text{con}}(\Gamma/e)) &\longrightarrow H_{i-1}(C_{\text{con}}(\Gamma \setminus e)) \longrightarrow H_{i-1}(C_{\text{con}}(\Gamma)) \\ &\longrightarrow H_{i-2}(C_{\text{con}}(\Gamma/e)) \longrightarrow H_{i-2}(C_{\text{con}}(\Gamma \setminus e)) \end{aligned}$$

From the long exact sequence follows that if the homology of $C_{\text{con}}(\Gamma \setminus e)$ is concentrated in degree $n-1$ and the homology of $C_{\text{con}}(\Gamma/e)$ is concentrated in degree $n-2$, that are the degrees represented by trees, then the homology of $C_{\text{con}}(\Gamma)$ is concentrated in degree $n-1$. We prove by induction on the number of edges in Γ that the homology of $C_{\text{con}}(\Gamma)$ is concentrated in degree $n-1$, where n is the number of vertices in Γ. Let Γ be a connected graph with one edge e and two vertices, then $\Gamma \setminus e$ is a disconnected graph and so $C_{\text{con}}(\Gamma \setminus e)$ is trivial. Γ/e is a graph with one vertex and no edges, the complex $C_{\text{con}}(\Gamma/e)$ is concentrated in degree 0. $H_*(C_{\text{con}}(\Gamma/e))$ is concentrated in degree 0 that is $n-2$ and so $H_*(C_{\text{con}}(\Gamma))$ is concentrated in degree $1 = n-1$. Now suppose by induction that the statement is true for any graph Γ with $|E(\Gamma)| < k$. We want to prove it for $|E(\Gamma)| = k$. Then $\Gamma \setminus e$ is either disconnected or it is a connected graph with $k-1$ edges and n vertices. In the first case the homology is trivial, in the second case the homology $H_*(C_{\text{con}}(\Gamma \setminus e))$ is concentrated in degree $n-1$ by inductive hypothesis. Γ/e is a graph with $k-1$ edges and $n-1$ vertices, it can have a loop or a multiple edge, by *Lemma* 5.7.10 the homology $H_*(C_{\text{con}}(\Gamma/e))$ is either trivial or concentrated in degree $n-2$. □

Remark 5.7.9. This lemma provides an alternative proof of the result given by *Vassilev* in [39] regarding the homology of the complex of connected subgraphs of the complete graph. The proof could be already present in the literature but we are not aware of any references.

Lemma 5.7.10. *Let Γ be a graph. If Γ contains a loop, then the homology groups $H_i(C_{\text{con}}(\Gamma))$ are trivial, and replacing a multiple edge by a single edge doesn't change the homology.*

Proof. Let Γ be a graph with a loop l, then $\Gamma \setminus l$ and Γ/l are the same graph and so $H(C_{\text{con}}(\Gamma \setminus l)) = H(C_{\text{con}}(\Gamma/l))$. By the long exact sequence we have that $H(C_{\text{con}}(\Gamma)) = 0$. Let now Γ have a multiple edge e. Then $\Gamma \setminus e$ is a graph with single edges and Γ/l is a graph with a loop, and so $H(C_{\text{con}}(\Gamma/e)) = 0$. By the long exact sequence we have that $H(C_{\text{con}}(\Gamma \setminus e)) = H(C_{\text{con}}(\Gamma))$. □

Remark 5.7.11. The idea in the proof of *Lemma* 5.7.10 is the same as the proof of the similar statement in *Corollary* 3.2 in [17].

5.8 The chain $\mathcal{C}^*_{BS}(\Gamma)/I(\Gamma)$

In this chapter we want to discuss the complex $\mathcal{C}^*_{BS}(\Gamma)/I(\Gamma)$ and show that it is isomorphic to $R(A, \Gamma)$.

The complex $R(A,\Gamma)$ is defined as $\Lambda[G_{a,b}]\otimes A^{\otimes n}$ quotient by the relations

- $G_{a,b}=G_{b,a}$
- $p_a^*(x)G_{a,b}=p_b^*(x)G_{a,b}$, $x\in H^*(X)$
- $\delta(w_i)=0$

where w_i is a cycle in Γ, and with differential

$$d(G_{a,b})=p^*_{a,b}\Delta.$$

Let $\widetilde{R}(A,\Gamma)$ be $\Lambda[G_{a,b}]\otimes A^{\otimes n}$ quotient only by the first two relations, with the same differential d and let $(\mathcal{C}^*_{BS}(\Gamma),\delta)$ be the complex defined in *Section* 5.4.

Remark 5.8.1. The complexes $(\mathcal{C}^*_{BS}(\Gamma),\delta)$ and $\widetilde{R}(A,\Gamma)$ are the same as vector spaces and they differ only for the differentials. The differential of the first complex is defined as

$$\delta(G_S\otimes a_1\otimes\cdots\otimes a_{l(S)})$$
$$=\begin{cases}0 & \text{if } \alpha\notin S\\ \sum_{\alpha\in E(\Gamma)}G_{S\smallsetminus\alpha}p_{i,j}(\Delta)\otimes a_1\otimes\cdots\otimes a_n, & \text{if } \alpha \text{ disconnects } S\\ \sum_{\alpha\in E(\Gamma)}G_{S\smallsetminus\alpha}\otimes a_1\otimes\cdots\otimes a_n, & \text{if } \alpha \text{ non disconnects } S\end{cases}$$

Here $\alpha: i\to j$ and $p_{i,j}(\Delta)$ is the pullback of the projection. Note that $G_{S\smallsetminus\alpha}p_{i,j}(\Delta)\otimes a_1\otimes\cdots\otimes a_n=G_{S\smallsetminus\alpha}\Delta_{S,S\smallsetminus\alpha}\otimes a_1\otimes\cdots\otimes a_{l(S)}$, due to the relation in the definition of $\mathcal{C}_{BS}(\Gamma)$. The differential multiplies by the diagonal class only when removing the edge disconnects components. On the other hand, the differential in $E[n]$, given by

$$d(G_{a,b})=p^*_{a,b}\Delta,$$

always multiplies by the diagonal class. The differentials δ and d can be therefore written as $\delta=\delta_{\text{int}}+\delta_{\text{ext}}$ and $d=d_{\text{int}}+d_{\text{ext}}$ (see *Remark* 5.7.1), and $d_{\text{ext}}=\delta_{\text{ext}}$. Removing an edge in a cycle leaves the number of components unchanged, while removing edges in a forest disconnects components, so from the point where S is a spanning forest the two complexes are the same.

We now prove that there is an isomorphism of chain complexes between $\mathcal{C}^*_{BS}(\Gamma)/I(\Gamma)$ and $R(A,\Gamma)$.

Theorem 5.8.2. *There is an isomorphism of chain complexes,*

$$\mathrm{id}:(\mathcal{C}^*_{BS}(\Gamma),\delta)/I(\Gamma)\to(R(A,\Gamma),d)$$

Proof. Consider the identity map id : $\mathcal{C}^*_{BS}(\Gamma)/I(\Gamma) \to R(A,\Gamma)$. By *Remark* 5.8.1, the complexes $\mathcal{C}^*_{BS}(\Gamma)$ and $\widetilde{R}_n(A,\Gamma)$ differ only from the internal differential since the two differentials agree in the case where the subgraph S of the set of vertices $E(\Gamma)$ is a forest. By *Lemma* 5.6.3 the third relation assures that elements in the complexes corresponding to graphs S with a component containing cycles are zero. Therefore $\mathcal{C}^*_{BS}(\Gamma)/I(\Gamma)$ and $R(A,\Gamma)$ agree as vector spaces, as the two differentials $\delta = d$, since $d_{\text{int}} = \delta_{\text{int}} = 0$. $\square$

Chapter 6

Generalised configuration spaces

6.1 Introduction

As described in the first chapter, *little disks operads* are operads such that for every natural number n, $D_r(n)$ is the space of linear embeddings of n disks of dimension r into the r-dimensional unit disk,

$$D_r(n) = Emb(\mathbb{D}_r^{\sqcup n}, \mathbb{D}_r).$$

Moreover, we have an homotopy equivalence $D_r(n) \cong \mathfrak{C}_r(n)$, where

$$\mathfrak{C}_r(n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{rn}; x_i \neq x_j \text{ for } i \neq j, 0 < i, j \leq n\}$$

is the configuration space of n points in $\mathbb{R}^r$.

The operadic composition

$$D_r(n) \circ D_r(k_1) \circ \cdots \circ D_r(k_n)$$

is given by inserting the disk $D_r(k_i)$ in the i-th disk of $D_r(n)$.

Little disks operads have been introduced in the $70's$ with the works by *Boardman* and *Vogt* [4] and *May* [32], and they had been applied in topology, algebra and mathematical physics. The cohomology of LDO is known, and it is related to the cohomology of the configuration space of n points in $\mathbb{R}^d$, $\mathfrak{C}_r(n)$. This has been computed by Arnold [1] in the case $r = 2$ and Cohen [9] for $r \geq 3$. The ring $H^*(\mathfrak{C}_r(n))$ is given by

$$\mathbb{Z}[e_{\alpha_{i,j}}]/\sim$$

where $0 < i, j \leq n$, $\mathbb{Z}[e_{\alpha_i}]$ is the free commutative graded algebra generated by $e_{\alpha_{i,j}}$ of degree $r - 1$ and $\sim$ are the relations

- $e_{\alpha_{i,j}} = (-1)^r e_{\alpha_{j,i}}$

- $e^2_{\alpha_{i,j}} = 0$ if r is odd
- $e_{\alpha_{a,b}} e_{\alpha_{b,c}} + e_{\alpha_{b,c}} e_{\alpha_{c,a}} + e_{\alpha_{c,a}} e_{\alpha_{a,b}} = 0$

In the present chapter we generalise the definition of little disks operads to the configuration space depending on a graph defined by *Heaswood* and *Huggett* [12], and described in *Section* 2.2. In particular, we consider the configuration spaces of points in $\mathbb{R}^r$ depending on a graph Γ, and we denote it by $\mathfrak{C}_r(\Gamma)$. We define a partial operad of this generalised configuration space over $\mathbb{R}$, and a generalised version of the little disks operads, that we call *painted little disks operads*. Moreover, we compute the cohomology of $\mathfrak{C}_r(\Gamma)$. In the first section we give the definition of painted little disks operads. Then, in the second section we define $\mathfrak{C}_r(\Gamma)$ and a candidate for its cohomology, denoted by $R^r(\Gamma)$. We then have two cases according to r being even or odd. We first prove that in the even case the candidate ring is isomorphic to the cohomology of $\mathfrak{C}_r(\Gamma)$. Then we reproduce the proof for the odd case.

6.2 Painted little disks operads

We extend the idea of configuration space partial operad and little disks operads, defined in *Section* 1.2, to configuration spaces over $\mathbb{R}^r$ depending on a graph.

Definition 6.2.1. Let Γ be a graph and $V(\Gamma)$ its set of vertices. Let, by abuse of notation, denote by $V(\Gamma)$ the cardinality of the set $V(\Gamma)$. The generalised configuration space over $\mathbb{R}^r$ $r \geq 0$, is

$$\mathfrak{C}_r(\Gamma) = \{(x_1, \dots, x_{V(\Gamma)}) \in \mathbb{R}^{rV(\Gamma)}; x_i \neq x_j \text{ if } e_{i,j} \in E(\Gamma)\}$$

where $e_{i,j}$ is an edge in Γ.

6.2.1 The partial operad of generalised configuration spaces

We now define an operad indexed by natural numbers such that for every $n \in \mathbb{N}$, we associate the space of disjoint unions of configuration spaces dependent on a graph with n vertices.

Definition 6.2.2. Let Γ be a graph and $V(\Gamma)$ its set of vertices. We define the *partial operad of generalised configuration spaces* to be a collection of spaces

$$\widetilde{\mathfrak{C}}_r(n) = \coprod_{\Gamma; V(\Gamma)=n} \mathfrak{C}_r(\Gamma)$$

for every $n \in \mathbb{N}$, with composition

$$\circ : \widetilde{\mathfrak{C}}_r(n) \otimes \widetilde{\mathfrak{C}}_r(k_1) \otimes \cdots \otimes \widetilde{\mathfrak{C}}_r(k_n) \to \widetilde{\mathfrak{C}}_r(k_1 + \cdots + k_n)$$

given for every

$$(a, x_1, \dots, x_n) \in \mathfrak{C}_r(n) \otimes \mathfrak{C}_r(k_1) \otimes \cdots \otimes \mathfrak{C}_r(k_n)$$

where $a = (a_1, \dots, a_n)$, by

$$x \circ (x_1, \dots, x_n) = (\underbrace{(a_1, \dots, a_1)}_{k_1 \text{ times}} + x_1, \dots, \underbrace{(a_n, \dots, a_n)}_{k_n \text{ times}} + x_n).$$

The composition is defined for the elements $(a, x_1, \dots, x_n)$ such that

$$x \circ (x_1, \dots, x_n) \in \mathfrak{C}_r(\Gamma')$$

where Γ' is a graph with $k_1 + \cdots + k_n$ vertices obtained by inserting in the i-th vertex of Γ the graph Γ_i, and connecting the vertices of Γ_i with a vertex v of Γ if there was an edge in Γ between the i-th vertex and v.

Let $\widetilde{\mathfrak{C}}_r(1)$ be the configuration space $\mathfrak{C}_r(\Gamma)$ where $\Gamma = *$ the graph with only one vertex. The unit $1 \in \widetilde{\mathfrak{C}}_r(1)$ is the set with one point and the unit morphism is

$$\mu : \{0\} \to \mathfrak{C}_r(*) = \mathbb{R}^r.$$

An example of composition is shown in the following picture.

Example 6.2.3. We give an example of composition of generalised configuration space operad. Let Γ be the graph given by two vertices 1 and 2, and an edge $e_{1,2}$ between them. Let Γ_1 be the graph given by two vertices and no edges, and Γ_2 the graph with vertices 1, 2, 3 and edges $e_{1,2}$ and $e_{2,3}$. Now, $\mathfrak{C}_r(\Gamma)$ and $\mathfrak{C}_r(\Gamma_1)$ are contained in $\widetilde{\mathfrak{C}}_r(2)$ and $\mathfrak{C}_r(\Gamma_2)$ in $\widetilde{\mathfrak{C}}_r(3)$. The composition

$$\circ : \widetilde{\mathfrak{C}}_r(2) \otimes \widetilde{\mathfrak{C}}_r(2) \otimes \widetilde{\mathfrak{C}}_r(3) \to \widetilde{\mathfrak{C}}_r(5)$$

can be represented as in *Figure* 6.1.

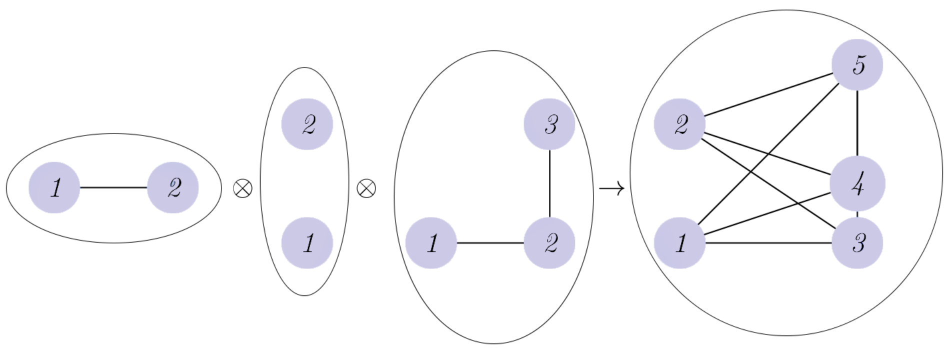

Figure 6.1: The composition law in generalised configuration space operads and in painted little disks operads.

So if

$$(x, x_1, x_2) \in \mathfrak{C}_r(\Gamma) \otimes \mathfrak{C}_r(\Gamma_1) \otimes \mathfrak{C}_r(\Gamma_2)$$

then

$$x \circ (x_1, x_2) \in \mathfrak{C}_r(\Gamma_5)$$

where Γ_5 is the graph with five vertices on the right.

Proposition 6.2.4. *The generalised configuration space operad is a partial operad.*

Proof. We want to check that it satisfies the properties defining operads. To simplify the notation we denote by $\widetilde{d_r}(n)$ an element in $\widetilde{\mathfrak{C}}_r(n)$.

Associativity: We want to prove that for every $\widetilde{\mathfrak{C}}_r(n)$, n operads $\widetilde{\mathfrak{C}}_r(k_1)$,...,$\widetilde{\mathfrak{C}}_r(k_n)$ and $k_1+\cdots+k_n$ operads the associativity property holds

$$\begin{aligned}&\widetilde{d_r}(n) \circ (\widetilde{d_r}(k_1) \circ (\widetilde{d_r}(k_{1,1}), \dots, \widetilde{d_r}(k_{1,l_1})),\\ &\quad \dots, \widetilde{d_r}(k_n) \circ (\widetilde{d_r}(k_{n,1}), \dots, \widetilde{d_r}(k_{n,l_n})))\\ &\quad\quad = (\widetilde{d_r}(n) \circ (\widetilde{d_r}(k_1), \dots, \widetilde{d_r}(k_n))) \circ (\widetilde{d_r}(k_{1,1}), \dots, \widetilde{d_r}(k_{n,l_n})).\end{aligned}$$

This property holds for the standard partial operad of configuration spaces, that is the case where all graphs are complete. On the other hand, if two vertices for example v_1 and v_2 in the graph defining $\mathfrak{C}_r(\Gamma) \ni \widetilde{d_r}(n)$ are not connected by any edge, then the graphs inserted in v_1 in the composition $\widetilde{d_r}(k_1) \circ (\widetilde{d_r}(k_{1,1}), \dots, \widetilde{d_r}(k_{1,l_1}))$ and in v_2 in $\widetilde{d_r}(k_2) \circ (\widetilde{d_r}(k_{2,1}), \dots, \widetilde{d_r}(k_{2,l_2}))$ will not be connected by edges. Similarly, in $\widetilde{d_r}(n) \circ (\widetilde{d_r}(k_1), \dots, \widetilde{d_r}(k_n))$ the graph inserted in the first two vertices will not be connected by edges.

Equivariance: Let $\sigma \in \Sigma_n$ be a permutation and let $\sigma' \in \Sigma_{k_1+\cdots+k_n}$ the function that permutes by t the n blocks given by k_i terms. Then

$$\begin{aligned}&(\widetilde{d_r}(n) * \sigma) \circ (\widetilde{d_r}(k_{\sigma(1)}), \dots, \widetilde{d_r}(k_{\sigma(n)}))\\ &\quad = \widetilde{d_r}(n) \circ (\widetilde{d_r}(k_1), \dots, \widetilde{d_r}(k_n)) * \sigma'\end{aligned}$$

since σ permutes the vertices in Γ and σ' permutes in the same way the block of vertices in the graphs inserted in the vertices of Γ. Let $\sigma_i \in \Sigma_{k_i}$ then in a similar way

$$\begin{aligned}&\widetilde{d_r}(n) \circ (\widetilde{d_r}(k_1) * \sigma_1, \dots, \widetilde{d_r}(k_n) * \sigma_n)\\ &\quad = \widetilde{d_r}(n) \circ (\widetilde{d_r}(k_1), \dots, \widetilde{d_r}(k_n)) * (\sigma_1, \dots, \sigma_n)\end{aligned}$$

Unit: Let $\widetilde{d_r}(n) \in \mathfrak{C}_r(\Gamma)$, and $\widetilde{d_r}(n) \circ (\widetilde{d_r}(1), \dots, \widetilde{d_r}(1))$ be contained in a configuration space dependent on a graph Γ' obtained from Γ by inserting in each vertex a graph given by only one vertex. Then Γ' clearly is Γ itself and we have diagrams

$$\begin{tikzcd} 1 \otimes \widetilde{\mathfrak{C}}_r(n) \arrow[r, "\mu\otimes id"] \arrow[dr, "\cong"] & \widetilde{\mathfrak{C}}_r(1) \otimes \widetilde{\mathfrak{C}}_r(n) \arrow[d, "\circ"] \\ & \widetilde{\mathfrak{C}}_r(n) \end{tikzcd}$$

$$\begin{tikzcd} \widetilde{\mathfrak{C}}_r(n) \otimes 1^{\otimes n} \arrow[r, "id\otimes\mu^{\otimes n}"] \arrow[dr, "\cong"] & \widetilde{\mathfrak{C}}_r(n) \otimes \widetilde{\mathfrak{C}}_r(1)^{\otimes n} \arrow[d, "\circ"] \\ & \widetilde{\mathfrak{C}}_r(n) \end{tikzcd}$$

These diagrams commute since

$$\widetilde{d}_r(n) \circ (1, \dots, 1) = \widetilde{d}_r(n) = 1 \circ \widetilde{d}_r(n)$$

where 1 is the unit that is $(0, \dots, 0) \in \mathbb{R}^r$.

Since the composition is defined for a subset of $\widetilde{\mathfrak{C}}_r(n) \otimes \widetilde{\mathfrak{C}}_r(k_1) \otimes \cdots \otimes \widetilde{\mathfrak{C}}_r(k_n)$ then $\widetilde{\mathfrak{C}}_r$ is a partial operad. □

Remark 6.2.5. We conjecture that, as in the case of LDO and the partial operad of configuration spaces, one can construct a generalised version of the Fulton and McPherson-operad.

6.2.2 Painted little disks operads

Analogously to the definition of LDO we can define a generalised version of them, where we consider linear embeddings of union of disks of which some are disjoint.

Definition 6.2.6. Let Γ be a graph. The *painted little r-disks operad* $\widetilde{D}_r$ is defined for every $n \in \mathbb{N}$ to be a disjoint union of sets of linear embeddings of the union of n little r-disks in $\mathbb{D}_r$. That is

$$\widetilde{D}_r(n) = \coprod_{\Gamma, |V(\Gamma)|=n} \mathrm{Emb}(\Gamma)$$

where

$$\mathrm{Emb}(\Gamma) = \{(c_1, \dots, c_n) : \bigcup_{i=1}^{n} \mathbb{D}_r^i \to \mathbb{D}_r\}$$

and $\mathbb{D}_r^i \cup \mathbb{D}_r^j$ is a disjoint union if there is an edge α in Γ with vertices i and j. Here, $c_i : \mathbb{D}_r \to \mathbb{D}_r$ is a linear embedding such that, if $\mathring{\mathbb{D}}_r$ denotes the interior of $\mathbb{D}_r$, $c_i(\mathring{\mathbb{D}}_r) \cap c_j(\mathring{\mathbb{D}}_r) = \emptyset$ for all α, edge between i and j.

The operadic composition

$$\circ : \widetilde{D}_r(n) \otimes \widetilde{D}_r(k_1) \otimes \cdots \otimes \widetilde{D}_r(k_n) \to \widetilde{D}_r(k_1 + \cdots + k_n)$$

is given for $c \in \mathrm{Emb}(\Gamma) \in \widetilde{D}_r(n)$ and $c_i \in \mathrm{Emb}(\Gamma_i) \in \widetilde{D}_r(k_i)$ by

$$c \circ c_1 \otimes \cdots \otimes c_n : \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \mathbb{D}_r^{i,j} \to \mathbb{D}_r$$

where the unions $\mathbb{D}_r^{i,j} \cup \mathbb{D}_r^{h,k}$ are disjoint if there is an edge α between i and h in Γ, and/or when $i = h$ there is an edge β between j and k in Γ_i. The operadic composition is defined by the composition of embeddings

$$(c \circ c_1 \otimes \cdots \otimes c_n)_{|\mathbb{D}_r^{i,j}} = c(c_i)$$

The symmetric group Σ_n acts on $\widetilde{D}_n$ by permuting the order of the n disks. More precisely, if $\sigma \in \Sigma_n$ and $c = (c_1, \ldots, c_n) \in \widetilde{D}_r(n)$, then

$$c * \sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)}).$$

The unit element 1 is the identity $1 = \mathrm{id} : \mathbb{D}_r \to \mathbb{D}_r$.

Remark 6.2.7. The composition of painted little disks operads can be depicted as in *Figure* 6.1, where the edges represent the disks that cannot intersect.

Lemma 6.2.8. *The painted little disks operads satisfy the axioms of operad.*

Proof. This follows from the fact that LDO is an operad and it goes analogously to the proof for the generalised configuration space partial operad. Let $\widetilde{d}(n)$ be an element in $\widetilde{D}(n)$.
Associativity: This property holds for standard LDO. On the other hand, if there is no edge between two vertices so the two corresponding disks are allowed to overlap, then they will overlap also after the insertion. This is because the inserted disks belong to $\bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \mathbb{D}_r^{i,j}$ where the unions $\mathbb{D}_r^{i,j} \cup \mathbb{D}_r^{h,k}$ are disjoint if there is an edge α between i and h in Γ, and/or when $i = h$ there is an edge β between j and k in Γ_i.
Equivariance: The proof works analogously to the one for the partial operad of generalised configuration spaces.
Unit: The following equation holds, since $1 = \mathrm{id}$.

$$\widetilde{d}_r(n) \circ (1, \ldots, 1) = \widetilde{d}_r(n) = 1 \circ \widetilde{d}_r(n).$$

□

6.3 The cohomology of the generalised configuration space over $\mathbb{R}^r$

In this section we define a candidate for the cohomology of $\mathfrak{C}_r(\Gamma)$, that later we show being actually isomorphic to it. We introduce the graded commutative ring $R^r(\Gamma)$. We give first a general definition, and in the next sections we describe the cases of r odd and even. We refer to the literature (*Getzler* and *Jones* [15], *Kontsevich* [24], *Willwacher* [40]) for the choice of denoting the even (respct. odd) case in relation to the dimension of the space on which the configuration space is defined. So with this convention r even (respect. odd) corresponds to odd (respect. even) generators.

Definition 6.3.1. Let Γ be a graph, with an orientation on the edges. Let r be a natural number and $\mathbb{Z}[e_{\alpha_i}]$ the free commutative graded algebra generated by $e_{\alpha_{i,j}}$ of degree $r-1$, where $\alpha_{i,j}$ is an edge in Γ between the vertices i and j oriented from i to j. Let w be a cycle in Γ and we denote by $e_w = e_{v_{1,2}} \cdot e_{v_{2,3}} \cdot \ldots \cdot e_{v_{s,1}}$ the product of the generators corresponding to the edges in the cycle w. We define the graded commutative ring

$$R^r(\Gamma) = \mathbb{Z}[e_{\alpha_{i,j}}]/\sim$$

where $\sim$ are the relations

- $e_{\alpha_{i,j}} = (-1)^r e_{\alpha_{j,i}}$
- $e^2_{\alpha_{i,j}} = 0$ if r is odd
- $d(e_{w_j}) = \sum_i (-1)^{(r-1)i} e_{v_{1,2}} \cdots \hat{e}_{v_{i,j}} \cdots e_{v_{s_j,1}} = 0$ for every cycle w_j in Γ.

We call these last relations the *generalised Arnold relations*, and we denote by $I(\Gamma)$ the ideal generated by them.

6.4 Algebraic description of $R^r(\Gamma)$, the even case

In this section we describe the case where r is an even natural number. We suppose r even for the rest of the section.

Definition 6.4.1. Let Γ be a graph, not oriented and let $E(\Gamma)$ be its set of edges. Let r be an even natural number and $\Lambda[e_{\alpha_i}]$ be the exterior algebra over the generators e_{α_i} of degree $r-1$ where $\alpha_i \in E(\Gamma)$. Let w be a cycle in Γ and we denote by $e_w = e_{v_{1,2}} \cdot e_{v_{2,3}} \cdot \ldots \cdot e_{v_{s,1}}$ the product of the generators corresponding to the edges in the cycle w. We define the graded commutative ring

$$R^r(\Gamma) = \Lambda[e_\alpha]/\sim$$

where $\sim$ are the relations

- $e_{\alpha_{i,j}} = e_{\alpha_{j,i}}$
- $d(e_{w_j}) = \sum_i (-1)^i e_{v_{1,2}} \cdots \hat{e}_{v_{i,j}} \cdots e_{v_{s_j,1}} = 0$, for every cycle w_j in Γ.

Notation. We will sometimes denote $\Lambda[e_\alpha]$, the exterior algebra generated by the edges $\alpha \in E(\Gamma)$ by $\Lambda[E(\Gamma)]$ in order to simplify the notation.

We can suppose that Γ has no multiple edges, since the following lemma holds.

Lemma 6.4.2. *Let r be an even natural number. Let Γ be a graph with a double edge, that is two edges e and e' incident to the same vertices. Let $\Gamma' = \Gamma - e'$. Then, the rings $R^r(\Gamma)$ and $R^r(\Gamma')$ are isomorphic.*

Proof. In $R^r(\Gamma)$ we have the generalised Arnold relation relation $d(ee') = 0$. This is $d(ee') = e' - e = 0$ that implies $e' = e$. □

6.4.1 Deletion-contraction short exact sequence for $R^r(\Gamma)$

Let Γ be a graph without multiple edges and $\alpha \in E(\Gamma)$. Let $\Lambda[e_{\alpha_i}]$ be the exterior algebra over the generators e_{α_i}. Then there is a diagram of ring maps

$$\begin{tikzcd}
\Lambda(\Gamma \setminus \alpha) \arrow[r, "f_\Lambda"] \arrow[d, "\iota_\Lambda"] & \Lambda(\Gamma/\alpha) \arrow[d, "i_\Lambda"] \\
\Lambda(\Gamma) \arrow[r, "l_\Lambda"] & \Lambda(\Gamma/\alpha) \otimes \Lambda[e_\alpha]
\end{tikzcd}$$

where i_Λ is the inclusion map, the map $f_\Lambda : \Lambda[\Gamma \setminus \alpha] \to \Lambda[\Gamma/\alpha]$ is defined for every edge $\eta \in E(\Gamma)$, $f(e_\eta) = e_\eta$. The map $l_\Lambda : \Lambda[\Gamma] \to \Lambda(\Gamma/\alpha) \otimes \Lambda[e_\alpha]$ is defined by $l_\Lambda(e_\eta) = e_\eta \otimes 1$ if $\eta \neq \alpha$ and $l_\Lambda(e_\alpha) = 1 \otimes e_\alpha$.

Lemma 6.4.3. *The diagram above is commutative and the maps are compatible with the generalised Arnold relation.*

Proof. We first show that the image under these maps of an element in the ideal generated by the generalised Arnold relations is still in the ideal. If w is a cycle in $\Gamma \setminus \alpha$ then it is a cycle in Γ/α and in Γ. If $d(e_w) \in I(\Gamma \setminus \alpha)$ then $f_\Lambda(d(e_w)) \in I(\Gamma/\alpha)$ and $\iota_\Lambda(d(e_w)) \in I(\Gamma)$. i_Λ is the inclusion so it sends element in the ideal generated by the Arnold relation to themselves. Suppose that w is a cycle in Γ, if w contains α then it is also a cycle in Γ/α (we can suppose Γ without double edges by *lemma* 6.4.2) and so $l_\Lambda(d(e_w)) \in I(\Gamma/\alpha) \otimes \Lambda[e_\alpha]$. If w does not contain α then $l_\Lambda(d(e_w)) = d(e_w) \otimes 1 \in I(\Gamma/\alpha) \otimes \Lambda[e_\alpha]$. The diagram is commutative since for a subgraph S in $\Gamma \setminus \alpha$,

$$l_\Lambda(\iota_\Lambda(e_S)) = e_S \otimes 1 = i_\Lambda(f_\Lambda(e_S)).$$

□

Theorem 6.4.4. *The diagram*

$$\begin{tikzcd} R^r(\Gamma \setminus \alpha) \arrow[r, "f_R"] \arrow[d, "\iota_R"] & R^r(\Gamma/\alpha) \arrow[d, "i_R"] \\ R^r(\Gamma) \arrow[r, "l_R"] & R^r(\Gamma/\alpha) \otimes \Lambda[\alpha] \end{tikzcd}$$

is a pullback diagram, in particular the map

$$\iota_R : R^r(\Gamma \setminus \alpha) \to R^r(\Gamma)$$

is injective.

As consequences of the previous theorem we have

Corollary 6.4.5. *Suppose that Γ' is a subgraph of Γ such that $V(\Gamma') = V(\Gamma)$, then $\iota : R^r(\Gamma') \to R^r(\Gamma)$ is injective.*

Corollary 6.4.6. *For every $\alpha \in E(\Gamma)$ there is a short exact sequence*

$$0 \to R^r(\Gamma \setminus \alpha)_k \to R^r(\Gamma)_k \to R^r(\Gamma/\alpha)_{k-r+1} \to 0$$

where the indices k and $k-r+1$ denote the grading in the ring.

6.4.2 Proof of *Theorem* 6.4.4

Let $I(\Gamma)$ denote the ideal of $\Lambda[E(\Gamma)]$ generated by the generalised Arnold relations. Let r be an even natural number. We notice that

$$\Lambda[E(\Gamma)] = \Lambda[E(\Gamma/\alpha)] \otimes \Lambda[\alpha],$$

and so

$$\Lambda[E(\Gamma/\alpha)] \otimes e_\alpha = (\Lambda[E(\Gamma/\alpha)] \otimes \Lambda[\alpha])/(\Lambda[E(\Gamma/\alpha)] \otimes 1).$$

We define $g_\Lambda : \Lambda[E(\Gamma)] \to \Lambda[E(\Gamma/\alpha)] \otimes e_\alpha$ by $g(e_S) = 0$ if S does not contain α, and $g(e_S) = e_1 \cdots e_s \otimes e_\alpha$ if S contains α and S is the subgraph of Γ with edges $1, \ldots, s$, α. There is a diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & I(\Gamma \setminus \alpha) \arrow[r] \arrow[d] & \Lambda[E(\Gamma \setminus \alpha)] \arrow[r] \arrow[d] & R^r(\Gamma \setminus \alpha) \arrow[r] \arrow[d, "\iota_R"] & 0 \\ 0 \arrow[r] & I(\Gamma) \arrow[r] \arrow[d, "g_I"] & \Lambda[E(\Gamma)] \arrow[r] \arrow[d, "g_\Lambda"] & R^r(\Gamma) \arrow[r] \arrow[d, "g_R"] & 0 \\ 0 \arrow[r] & I(\Gamma/\alpha) \otimes e_\alpha \arrow[r] \arrow[d] & \Lambda[E(\Gamma/\alpha)] \otimes e_\alpha \arrow[r] \arrow[d] & R^r(\Gamma/\alpha) \otimes e_\alpha \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd} \tag{6.1}$$

Lemma 6.4.7. *There is a map $\iota_R : R_r(\Gamma \setminus \alpha) \to R_r(\Gamma)$ such that the diagram 6.1 is commutative. Moreover, the three rows and the middle column are exact.*

Proof. The map ι_R is well defined: if w is a cycle in $\Gamma \setminus \alpha$ then it is a cycle in Γ, and elements in the ideal generated by $d(w)$ are sent to the same element in the ideal in $R^r(\Gamma)$. Let w' be a cycle in Γ, if α is not in w' then it is a cycle in $\Gamma \setminus \alpha$. If α is an edge in w' then $w' \setminus \alpha$ is not a cycle in $\Gamma \setminus \alpha$.

Consider first the left upper square. If w is a cycle in $\Gamma \setminus \alpha$ then it is a cycle in Γ. The diagram commutes since all the maps in the square are inclusions. We move now to the lower square on the left. Suppose that w is a cycle in Γ, if w contains α then its image in Γ/α is also a cycle and

$$\iota(g_{\widetilde{I}}(d(w))) = d(w) \otimes \alpha = g_\Lambda(\iota(d(w))).$$

If w does not contain α then $\iota(g_{\widetilde{I}}(d(w))) = 0 = g_\Lambda(\iota(d(w)))$. Now the upper square on the right is commutative since the maps are compatible with the generalised Arnold relations. The lower right square also commutes since the maps are compatible with the relations, and if w contains α then the image in Γ/α is also a cycle, so

$$q(g_\Lambda(d(w))) = 0 = g_R q((d(w))).$$

If w does not contain α then $q(g_\Lambda(d(w))) = 0 = g_R q((d(w)))$. If the set S in Γ does not contain any cycle, clearly

$$q(g_\Lambda(S)) = g_R(q(S)).$$

The three rows are exact by definition, since the maps are left and right row maps are given by inclusion and quotient respectively. The middle column is also exact since the upper map $\iota : \Lambda[E(\Gamma \setminus \alpha)] \to \Lambda[E(\Gamma)]$ is injective, g_Λ is surjective and $\mathrm{im}(\iota) = \Lambda[E(\Gamma \setminus \alpha)] \otimes 1 = \Lambda[E(\Gamma/\alpha)] \otimes 1 = \ker(g_\Lambda)$. □

Lemma 6.4.8. *The last column is exact.*

Proof. The map g_R is surjective since g_Λ is surjective and if w is a cycle in Γ not containing α then it is mapped to zero, while if w contains α then w/α is a cycle in Γ/α. We need to check that $\mathrm{im}(\iota_R) = \ker(g_R)$. Consider the long exact sequence in cohomology. Since the second and third rows are exact then $H^*(\Lambda[E(\Gamma)]) = 0$ and $H^*(\Gamma/\alpha \otimes e_\alpha) = 0$, that implies

$$H^*(R(\Gamma)) \cong H^*(I(\Gamma/\alpha) \otimes e_\alpha).$$

So $\mathrm{im}(\iota_R) = \ker(g_R)$ if and only if g_I is surjective. We want to check this last condition, that is we want to show that every generator in $I(\Gamma/\alpha)$ is the image of an element in $I(\Gamma)$. Let v_1 and v_2 be the two vertices incident to α and their common image in $V(\Gamma/\alpha)$ be v. Now, let $e_w = e_1 \cdots e_s$ be the product of the generators corresponding to edges in a cycle w in Γ/α. If the vertex v does not occur in w then w is a cycle in Γ. There is then an element $y = d(e_w)e_\alpha \in I(\Gamma)$ such that $g_I(y) = d(w) \otimes \alpha$. If v is a vertex in w incident to exactly two edges in Γ/α, suppose them e_1 and e_s, then there is a cycle $w' = \alpha e_1 \cdots e_s$ in Γ. So $d(e_{w'})$ is the corresponding generator in $I(\Gamma)$ and its image under the map g_I is

$$g_I(d(\alpha e_1 \cdots e_s)) = g(e_1 \cdots e_s) - g(\alpha d(e_1 \cdots e_s)) = -d(w) \otimes \alpha$$

Finally, if v is incident to more then two edges in w then we can decompose w in a product of cycles where v is incident to only two vertices and deduce the result from the single factors using that d is a derivation. □

We conclude the proof of *Theorem* 6.4.4 with an inductive argument. We proceed with a double induction on the number of vertices m and edges n of a graph $\Gamma(m, n)$. If $n = 0$, $\Gamma(m, 0)$ is the graph with m vertices and no edges. In this case the theorem holds since $R^r(m, 0) = \Lambda[\Gamma(m, 0)]$. The case where $m = 0$ corresponds to the empty graph. As inductive step we want to prove that if all graphs $\Gamma(m, n-1)$ and $\Gamma(m-1, n-1)$ satisfy *Theorem* 6.4.4, then it also holds for $\Gamma(m, n)$. In order to prove the inductive step we need that $\Gamma(m, n)$ satisfies the following property.

Property 1. *For $e \in E(\Gamma)$, let $\ker(g_R^e)$ be the kernel of the map $g_R^e : R^r(\Gamma) \to R^r(\Gamma/e) \otimes e$. Then $\cap_e \ker(g_R^e) = \mathbb{Z}$.*

We prove the property by induction. The following lemma provides the inductive step.

Lemma 6.4.9. *Assume that all graphs $\Gamma(m, n-1)$ and $\Gamma(m-1, n-1)$ satisfy* Theorem *6.4.4 and that $\Gamma(m, n-1)$ and $\Gamma(m-1, n-1)$ satisfy **Property 1**. Then $\Gamma(m, n)$ satisfy **Property 1**.*

Proof. Let $\varepsilon : R^r(\Gamma) \to \mathbb{Z}$ be the canonical map which sends all edges to 0. There is a split short exact sequence

$$0 \to \ker(\varepsilon) \to R^r(\Gamma) \to \mathbb{Z} \to 0.$$

The sequence splits by the inclusion $i : \mathbb{Z} \to \Lambda[E(\Gamma)]$, since the composition $\varepsilon \circ i = \mathrm{id}_{\mathbb{Z}}$. The image of i is contained in $\cap_e \ker(g_R^e)$ since $\ker(g_R^e) = \{e_S\} \in R^r(\Gamma)$ and S is a subset of Γ that does not contain the edge $e \in E(\Gamma)$. In order to prove the lemma it's enough to check

$$\bigcap_e \ker(g_R^e) \cap \ker(\varepsilon) = \{0\}.$$

Let $x \in \bigcap_e \ker(g_R^e) \cap \ker(\varepsilon)$. We have to prove that $x = 0$. Consider the following commutative diagram

$$
\begin{tikzcd}
R^r(\Gamma \setminus e) \arrow[rr, "\prod_{f\in E(\Gamma\setminus e)} g_R^f"] \arrow[d, "\iota_R"] & & \prod_{f\in E(\Gamma\setminus e)} R^r(\Gamma \setminus e/f) \otimes f \arrow[dd, "\prod_{f\in E(\Gamma\setminus e)} \iota_R"] \\
R^r(\Gamma) \arrow[r, "\prod_{f\in E(\Gamma)} g_R^f"] \arrow[d, "g_R^e"] & \prod_{f\in E(\Gamma)} R^r(\Gamma/f) \otimes f \arrow[dr, "P"] & \\
R^r(\Gamma/e) \otimes e_e & & \prod_{f\in E(\Gamma\setminus e)} R^r(\Gamma/f) \otimes f
\end{tikzcd}
$$

where P is the projection map. Now, from *Lemma* 6.4.8 we have that there is some class $y \in R^r(\Gamma \setminus e)$ such that $i_R(y) = x$. Since $x \in \ker(\varepsilon) \in R^r(\Gamma)$ then it comes from a class in $\ker(\varepsilon) \in R^r(\Gamma \setminus e)$. By hypothesis $\Gamma(m, n-1)$ and $\Gamma(m-1, n-1)$ satisfy *Theorem* 6.4.4 so $\Gamma \setminus e$ and Γ/f do. Then the right vertical map is injective. By using the commutativity of the diagram we see that $\prod_{f\in E(\Gamma\setminus e)} g_R^f(y) = 0$. Since $\Gamma \setminus e$ satisfies **Property 1** and we have that x belongs to $\cap_{f\in(\Gamma\setminus e)} \ker(g_R^f)$, it follows that $y = 0$, so that $x = 0$. □

We can conclude the proof of *Theorem* 6.4.4 by proving that if it holds for $\Gamma(m, n-1)$ and $\Gamma(m-1, n-1)$ so it does for $\Gamma(m, n)$. We want to prove that

$$\iota_R : R^r(\Gamma \setminus e) \to R^r(\Gamma)$$

is injective, so that if $\iota_R(x) = 0$, $x \in R^r(\Gamma \setminus e)$ then $x = 0$. Consider the commutative diagram in *Lemma* 6.4.9, we have that $\iota_R g_R^f(x) = g_R^f \prod_{f\in E(\Gamma\setminus e)} \iota_R$ so if $\iota_R g_R^f(x) = 0$, then $x = 0$ since the right vertical map is injective and the upper horizontal map is injective on $\ker(\varepsilon)$. The injectivity of this last map is due to the fact that by inductive hypothesis $\bigcap_f \ker(g_R^f) = \mathbb{Z}$ and that $\bigcap_f \ker(g_R^f) \cap \ker(\varepsilon) = \{0\}$ in $R^r(\Gamma \setminus e)$.

6.5 Deletion-contraction in the space $\mathfrak{C}_r(\Gamma)$

In this section we describe a type of deletion-contraction exact sequence that occurs for configuration spaces. The main result in this section, provided by *Corollary* 6.5.3, is the existence of a short exact sequence of the form

$$0 \to H^*(\mathfrak{C}_r(\Gamma \setminus e)) \to H^*(\mathfrak{C}_r(\Gamma)) \to H^{*-r+1}(\mathfrak{C}_r(\Gamma/e)) \to 0.$$

We start by proving the following theorem.

Theorem 6.5.1. *There is a long exact sequence in cohomology*

$$
\begin{aligned}
\cdots \longrightarrow H^*(\mathfrak{C}_r(\Gamma \setminus e)) &\longrightarrow H^*(\mathfrak{C}_r(\Gamma)) \\
&\longrightarrow H^{*-r+1}(\mathfrak{C}_r(\Gamma/e)) \longrightarrow H^{*+1}(\mathfrak{C}_r(\Gamma \setminus e)) \longrightarrow \cdots
\end{aligned}
$$

Proof. Let e be an edge in Γ between the vertices a and b. The space $\mathfrak{C}_r(\Gamma)$ is an open subspace of $\mathfrak{C}_r(\Gamma \setminus e)$. The complement

$$A_e(\Gamma) = \mathfrak{C}_r(\Gamma \setminus e) - \mathfrak{C}_r(\Gamma)$$

is a closed subspace in $\mathfrak{C}_r(\Gamma \setminus e)$ and

$$A_e(\Gamma) = \{(x_1, \dots, x_n) \in \mathbb{R}^{rn}; x_i \neq x_j \text{ if } \alpha_{i,j} \in E(\Gamma) \setminus \{e\} \text{ while } x_a = x_b\}.$$

There is a canonical homeomorphism between $A_e(\Gamma)$ and $\mathfrak{C}_r(\Gamma/e)$ sending $(x_1, \dots, x_n)$ to $(x_1, \dots, x_a, \dots, \widehat{x_b}, \dots, x_{n-1})$. We define an open neighborhood $V_e(\Gamma)$ of $\mathfrak{C}_r(\Gamma \setminus e)$ for a fix $\epsilon > 0$ in the following way

$$V_e(\Gamma) = \{(x_1, \dots, x_n) \in \mathfrak{C}_r(\Gamma \setminus e); |x_a - x_b| < \epsilon\}$$

We then have two open subspaces $\mathfrak{C}_r(\Gamma)$ and $V_e(\Gamma)$ of $\mathfrak{C}_r(\Gamma \setminus e)$ such that $\mathfrak{C}_r(\Gamma) \cup V_e(\Gamma) = \mathfrak{C}_r(\Gamma \setminus e)$. There is a pushout diagram

$$\begin{tikzcd}
V_e(\Gamma) \cap \mathfrak{C}_r(\Gamma) \arrow[r] \arrow[d] & V_e(\Gamma) \arrow[d] \\
\mathfrak{C}_r(\Gamma) \arrow[r] & \mathfrak{C}_r(\Gamma \setminus e)
\end{tikzcd}$$

We obtain a Mayer-Vietoris long exact sequence in cohomology

$$\begin{aligned}
\cdots \longrightarrow H^*(\mathfrak{C}_r(\Gamma \setminus e)) &\xrightarrow{\phi^*} H^*(\mathfrak{C}_r(\Gamma)) \oplus H^*(V_e(\Gamma)) \\
&\xrightarrow{\psi^*} H^*(\mathfrak{C}_r(\Gamma) \cap V_e(\Gamma)) \xrightarrow{\delta^*} H^{*+1}(\mathfrak{C}_r(\Gamma \setminus e)) \longrightarrow \cdots
\end{aligned}$$

where ϕ is the map assigning to each cohomology class x its restrictions $(x|_{\mathfrak{C}_r(\Gamma)}, x|_{V_e(\Gamma)})$ and $\psi(x, y) = x - y$.

Finally, we notice that V_e is homotopy equivalent to $\mathfrak{C}_r(\Gamma/e)$ and by *Lemma* 6.5.2 below $\mathfrak{C}_r(\Gamma) \cap V_e$ is homotopy equivalent to $\mathbb{S}^{r-1} \times \mathfrak{C}_r(\Gamma/e)$. Let $[\mu]$ denote the fundamental class of $\mathbb{S}^{r-1}$. By the Kunneth formula, we can rewrite the long exact sequence as

$$\begin{aligned}
\cdots \longrightarrow H^*(\mathfrak{C}_r(\Gamma \setminus e)) &\longrightarrow H^*(\mathfrak{C}_r(\Gamma)) \oplus H^*(\mathfrak{C}_r(\Gamma/e)) \\
&\longrightarrow \bigoplus_{k+l=*} H^k(\mathbb{S}^{r-1}) \otimes H^l(\mathfrak{C}_r(\Gamma/e)) \longrightarrow H^{*+1}(\mathfrak{C}_r(\Gamma \setminus e) \longrightarrow \cdots
\end{aligned}$$

This implies the existence of the long exact sequence

$$\begin{aligned}
\cdots \longrightarrow H^*(\mathfrak{C}_r(\Gamma \setminus e)) &\longrightarrow H^*(\mathfrak{C}_r(\Gamma)) \\
&\longrightarrow [\mu] H^*(\mathfrak{C}_r(\Gamma/e)) \longrightarrow H^{*+1}(\mathfrak{C}_r(\Gamma \setminus e)) \longrightarrow \cdots
\end{aligned}$$

Finally, replacing $[\mu]H^{*-r+1}(\mathfrak{C}_r(\Gamma/e) \cong H^{*-r+1}(\mathfrak{C}_r(\Gamma/e)$ we have the deletion-contraction long exact sequence for generalised configuration spaces:

$$\begin{aligned} \cdots \longrightarrow H^*(\mathfrak{C}_r(\Gamma \setminus e)) \longrightarrow H^*(\mathfrak{C}_r(\Gamma)) \\ \longrightarrow H^{*-r+1}(\mathfrak{C}_r(\Gamma/e)) \longrightarrow H^{*+1}(\mathfrak{C}_r(\Gamma \setminus e)) \longrightarrow \cdots \end{aligned}$$

□

Lemma 6.5.2. *$\mathfrak{C}_r(\Gamma) \cap V_e$ is homotopy equivalent to $\mathbb{S}^{r-1} \times \mathfrak{C}_r(\Gamma/e)$.*

Proof. $\mathfrak{C}_r(\Gamma) \cap V_e$ is the space

$$\{(x_1, \ldots, x_n) \in \mathfrak{C}_r(\Gamma) : 0 < |x_a - x_b| < \epsilon\}$$

We define the maps

$$f : \mathfrak{C}_r(\Gamma) \cap V_e \to \mathbb{S}^{r-1} \times \mathfrak{C}_r(\Gamma/e)$$

by

$$f((x_1, \ldots, x_n)) = \left(\frac{x_a - x_b}{|x_a - x_b|}, (x_1, \ldots, x_a, \ldots, \widehat{x_b}, \ldots, x_n) \right)$$

and

$$g : \mathbb{S}^{r-1} \times \mathfrak{C}_r(\Gamma/e) \to \mathfrak{C}_r(\Gamma) \cap V_e$$

by

$$g(y, (x_1, \ldots, x_n)) = \left(x_1, \ldots, x_a, \ldots, x_a + \tfrac{\epsilon}{2} y, \ldots, x_n\right)$$

Now gf is clearly homotopic to the identity and fg equals the identity. □

Corollary 6.5.3. *There is a short exact sequence*

$$0 \to H^*(\mathfrak{C}({}_r\Gamma \setminus e)) \to H^*(\mathfrak{C}_r(\Gamma)) \to H^{*-r+1}(\mathfrak{C}_r(\Gamma/e)) \to 0$$

Proof. Consider the long exact sequence

$$\cdots \to H^*(\mathfrak{C}_r(\Gamma \setminus e)) \xrightarrow{\phi^*} H^*(\mathfrak{C}_r(\Gamma)) \xrightarrow{\psi^*} H^{*-r+1}(\mathfrak{C}_r(\Gamma/e)) \to \cdots$$

The map ϕ^* is injective since it is defined to be the restriction to $H^*(\mathfrak{C}_r(\Gamma))$. The proof that ψ^* is surjective is contained in *Lemma* 6.6.2. □

6.6 The isomorphism, the even case

Let r be an even natural number. For any edge $e = e(v, w) \in E(\Gamma)$, there is a map

$$p_e : \mathfrak{C}_r(\Gamma) \to S^{r-1}$$

defined by

$$p_e(x) \mapsto \frac{x_i - x_j}{|x_i - x_j|} \in S^{r-1} \subset \mathbb{R}^r \setminus \{0\}.$$

Combining them, we obtain a map

$$p(\Gamma) : \mathfrak{C}_r(\Gamma) \to (S^{r-1})^{E(\Gamma)}.$$

We identify $H^*((S^{r-1})^{E(\Gamma)})$ with the ring $\Lambda[E(\Gamma)]$ and we choose a standard generator $[S^{r-1}] \in H^{r-1}(S^{r-1})$. The following definition will depend on the above choice.

Definition 6.6.1. Let r be an even number, the maps $p_r(\Gamma)$ induce ring homeomorphisms

$$p_r(\Gamma)^* : \Lambda[E(\Gamma)] \to H^*(\mathfrak{C}_r(\Gamma)); \quad p_r(\Gamma)(e) = p_e^*([S^{r-1}]).$$

Lemma 6.6.2. *The map $p_r(\Gamma)^*$ is surjective.*

Proof. We prove it by induction on the number of edges in Γ. The lemma is true if Γ has one edge. Now we suppose the result true for graphs with $n-1$ edges. We have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda[E(\Gamma \setminus e)] \arrow[r] \arrow[d, "p_r(\Gamma - e)"'] & \Lambda[E(\Gamma)] \arrow[r, "g"] \arrow[d, "pr_(\Gamma)"'] & \Lambda[E(\Gamma/e)] \arrow[r] \arrow[d, "p_r(\Gamma/e)"] & 0 \\
\cdots \arrow[r] & H^*(\mathfrak{C}_r(\Gamma \setminus e)) \arrow[r, "\phi^*"'] & H^*(\mathfrak{C}_r(\Gamma)) \arrow[r, "\psi^*"'] & H^*(\mathfrak{C}_r(\Gamma/e)) \arrow[r] & \cdots
\end{tikzcd}$$

The first and last vertical maps are surjective by hypothesis and g is also surjective. By the commutativity of the diagram $p_r(\Gamma/e) \circ g = \psi^* \circ p_r(\Gamma)$. Moreover $p_r(\Gamma/e) \circ g$ is surjective since it is the composition of surjective maps. It follows that ψ^* is surjective. The map ϕ^* is injective since it is defined to be the restriction to $H^*(\mathfrak{C}_r(\Gamma))$. Therefore we obtain the short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Lambda[E(\Gamma \setminus e)] \arrow[r] \arrow[d] & \Lambda[E(\Gamma)] \arrow[r, "g"] \arrow[d] & \Lambda[E(\Gamma/e)] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^*(\mathfrak{C}_r(\Gamma \setminus e)) \arrow[r, "\phi^*"'] & H^*(\mathfrak{C}_r(\Gamma)) \arrow[r, "\psi^*"'] & H^*(\mathfrak{C}_r(\Gamma/e)) \arrow[r] & 0
\end{tikzcd}$$

By the five lemma the middle map is also surjective. □

Lemma 6.6.3. *The map $p_r(\Gamma)^*$ maps elements in the ideal generated by the generalised Arnold relations to* 0.

Proof. Let w_n be a cycle of length n in Γ and $I(\Gamma)$ the ideal in $R^r(\Gamma)$ generated by the generalised Arnold relation. We want to prove by induction that the image of $I(\Gamma)$ under the map $p_r(\Gamma)^*$ is trivial. It is sufficient to show it for the generators $d(w_i)$, so we can prove the theorem for $\Lambda[w_n]$ for all n. We proceed by induction.

Let $n = 3$. In this case the generalised Arnold relation corresponds to the standard Arnold relation and the theorem holds. Suppose now that the theorem is true for w_n, we want to show the result for w_{n+1}. Let e be an edge in w_{n+1}, by *Corollary* 6.5.3 we have a short exact sequence

$$0 \longrightarrow H^*(\mathfrak{C}_r(w_{n+1} \setminus e)) \longrightarrow H^*(\mathfrak{C}_r(w_{n+1})) \longrightarrow H^*(\mathfrak{C}_r(w_{n+1}/e)) \longrightarrow 0$$

Now $w_{n+1} \setminus e$ is a tree with n edges and w_{n+1}/e is a cycle with n edges. Let T be a tree with k vertices such that every vertex has valence smaller or equal then 2, then

$$\mathfrak{C}_r(T) = \{(x_1, \ldots, x_k) \in \mathbb{R}^{kr} : x_1 \neq x_2, \ldots, x_{k-1} \neq x_k\}$$

that is

$$\begin{aligned} \mathfrak{C}_r(T) &= \{(x_1, \ldots, x_k) \in \mathbb{R}^{kr} : x_1 - x_2 \neq 0, \ldots, x_{k-1} - x_k \neq 0\} \\ &\subset (\mathbb{R}^r \setminus \{0\})^k. \end{aligned}$$

Since r is even then $r-1$ is odd and so $R^r(T) = \Lambda[E(T)] = H^*(\mathbb{S}^{k(r-1)})$. It follows that the two rings are isomorphic $H^*(\mathfrak{C}_r(T)) \cong R^r(T)$.

We have the following two short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \Lambda[E(w_{n+1} \setminus e)] \arrow[r, "f"] \arrow[d, "\cong"'] & \Lambda[E(w_{n+1})] \arrow[r, "g"] \arrow[d, "p_r(w_{n+1})"'] & \Lambda[E(w_{n+1}/e)] \arrow[r] \arrow[d, "p_r(w_{n+1}/e)"] & 0 \\ 0 \arrow[r] & H^*(\mathfrak{C}_r(w_{n+1} \setminus e)) \arrow[r, "\phi^*"'] & H^*(\mathfrak{C}_r(w_{n+1})) \arrow[r, "\psi^*"'] & H^*(\mathfrak{C}_r(w_{n+1}/e)) \arrow[r] & 0 \end{tikzcd}$$

Let $x \in I(w_{n+1})$ be $d(w_{n+1})$ the generator of the generalised Arnold identity. Then $g(x) \in I(w_{n+1}/e)$ and by inductive hypothesis $p_r(w_{n+1}/e) \circ g(x) = 0$. Since ψ^* is surjective and the diagram is commutative, $p_r(w_{n+1}/e) \circ g(x)$ is the image of an element $y = p_r(w_{n+1})(x)$. The sequences are exact, so y in in the image of ϕ^*, $y = \phi^*(z)$ for some $z \in H^*(\mathfrak{C}_r(\Gamma \setminus e))$. The map $p_r(w_{n+1} \setminus e)$ is an isomorphism so there is an element $k = z$ such that $\phi^* \circ p_r(w_{n+1} \setminus e)(k) = y = p_r(w_{n+1}) \circ f(k)$. Now suppose that $y \neq 0$, since f is injective then $z \neq 0$ and $f(z) \neq x$ since x is not in the kernel of g. Now consider the element $x - f(z)$. We have that $p_r(\Gamma)(x - f(z)) = 0$. So $x - f(z)$ is in the kernel of $p_r(\Gamma)$. Since the symmetric group acts on $\Lambda[E(\Gamma)]$ and on $H^*(\mathfrak{C}_r(\Gamma))$ and they are invariant under cyclic permutation of the edges, then the kernel is invariant under cyclic permutation of the edges. But the elements in $\Lambda[w_{n+1} \setminus e]$ are not invariant, so also $x - f(z)$ is not. This is a contradiction. That means that $y = 0$, and proves the theorem. $\square$

Corollary 6.6.4. *The map $p_r(\Gamma)^* : \Lambda[E(\Gamma)] \to H^*(\mathfrak{C}_r(\Gamma))$ factors uniquely over maps $\lambda_r : R^r(\Gamma) \to H^*(\mathfrak{C}_r(\Gamma))$.*

Theorem 6.6.5. *There is a isomorphism of graded commutative rings*

$$\lambda_r : R^r(\Gamma) \to H^*(\mathfrak{C}_r(\Gamma)).$$

Proof. We prove the theorem by double induction on the number of edges and vertices in the graph. Let $\Gamma(m,n)$ be a graph with m vertices and n edges. The theorem clearly holds for every m for the graphs $\Gamma(m,0)$. Now, suppose that it is true for $\Gamma(m,n-1)$ and $\Gamma(m-1,n-1)$, we want to prove that it is true also for $\Gamma(m,n)$.Let e be an edge in $\Gamma = \Gamma(m,n)$, by the previous sections we have the short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & R^r(\Gamma \smallsetminus e) \arrow[r] \arrow[d, "\cong"'] & R^r(\Gamma) \arrow[r] \arrow[d] & R^r(\Gamma/e) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H^*(\mathfrak{C}_r(\Gamma \smallsetminus e)) \arrow[r, "\phi^*"'] & H^*(\mathfrak{C}_r(\Gamma)) \arrow[r, "\psi^*"'] & H^*(\mathfrak{C}_r(\Gamma/e)) \arrow[r] & 0
\end{tikzcd}$$

By the five lemma if follows that the middle map is also an isomorphism. □

6.7 The odd case

In this section we want to adapt the arguments used to compute the cohomology of the generalised configuration space $\mathfrak{C}_r(\Gamma)$, to the case when when r is odd.

6.7.1 Algebraic description of the ring $P^r(\Gamma)$

Let r be an odd natural number and let Γ be a graph. In this case the ring $R^r(\Gamma)$ is defined as follows:

Definition 6.7.1. Let Γ be a graph with an orientation on the edges and let r be an odd natural number. Let $\mathbb{Z}[e_{\alpha_i}]$ the free commutative graded algebra generated by $e_{\alpha_{i,j}}$ of degree $r-1$ where $\alpha_{i,j}$ is an edge in Γ between the vertices i and j, oriented from i to j. Let w be a cycle in Γ, and we denote by $e_w = e_{v_{1,2}} \cdot e_{v_{2,3}} \dots \cdot e_{v_{s,1}}$ the product of the generator corresponding to the edges in the cycle w. We define the graded commutative ring

$$P^r(\Gamma) = \mathbb{Z}[e_{\alpha_{i,j}}]/\sim$$

where $\sim$ are the relations

- $e_{\alpha_{i,j}} = -e_{\alpha_{j,i}}$
- $e^2_{\alpha_{i,j}} = 0$
- $d(e_{w_j}) = \sum e_{v_{1,2}} \cdots \hat{e}_{v_{i,j}} \cdots e_{v_{s_j,1}} = 0$ for every cycle w_j in Γ.

$P^r(\Gamma)$ is the symmetric algebra $\mathbb{Z}[E(\Gamma)]/e_{i,j}^2$ quotient by the generalised Arnold relations.

Remark 6.7.2. The first relation implies that edges that are loops are zero in $P^r(\Gamma)$.

6.7.2 Deletion-contraction short exact sequence for P^r

There is a diagram of ring maps

$$\begin{tikzcd}
\mathbb{Z}[\Gamma \setminus \alpha]/e_{i,j}^2 \arrow[r, "f_S"] \arrow[d, "\iota_S"] & \mathbb{Z}[\Gamma/\alpha]/e_{i,j}^2 \arrow[d, "i_S"] \\
\mathbb{Z}[\Gamma]/e_{i,j}^2 \arrow[r, "l_S"] & \mathbb{Z}[\Gamma/\alpha]/e_{i,j}^2 \otimes \mathbb{Z}[\alpha]/e_{i,j}^2
\end{tikzcd}$$

where i_S is the inclusion map, the map $f_S : \mathbb{Z}[\Gamma \setminus \alpha]/e_{i,j}^2 \to \mathbb{Z}[\Gamma/\alpha]/e_{i,j}^2$ is defined for every edge $\eta \in E(\Gamma)$, $f_S(e_\eta) = e_\eta$. The map $l_S : \mathbb{Z}[\Gamma]/e_{i,j}^2 \to \mathbb{Z}[\Gamma/\alpha]/e_{i,j}^2 \otimes \mathbb{Z}[\alpha]/e_{i,j}^2$ is defined by $l_S(e_\eta) = e_\eta \otimes 1$ if $\eta \neq \alpha$ and $l_S(e_\alpha) = 1 \otimes e_\alpha$.

We prove the respective of *Theorem* 6.4.4 for the odd case.

Theorem 6.7.3. *The diagram*

$$\begin{tikzcd}
P^r(\Gamma \setminus \alpha) \arrow[r, "f_P"] \arrow[d, "\iota_P"] & P^r(\Gamma/\alpha) \arrow[d, "i_P"] \\
P^r(\Gamma) \arrow[r, "l_P"] & P^r(\Gamma/\alpha) \otimes \mathbb{Z}[\alpha]/\alpha^2
\end{tikzcd}$$

is a pullback diagram. In particular the map $\iota_P : P^r(\Gamma \setminus \alpha) \to P^r(\Gamma)$ is injective and we have a short exact sequence

$$0 \to P^r(\Gamma \setminus \alpha) \to P^r(\Gamma) \to P^r(\Gamma/\alpha) \to 0.$$

Proof. Let $I(\Gamma)$ the ideal generated by the Generalised Arnold relations for r odd. Following the definitions and notation introduced in *Section* 6.4.2 we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & I(\Gamma \setminus \alpha) \arrow[r] \arrow[d] & \mathbb{Z}[E(\Gamma \setminus \alpha)]/e_{i,j}^2 \arrow[r] \arrow[d] & P^r(\Gamma \setminus \alpha) \arrow[r] \arrow[d, "\iota_P"] & 0 \\
0 \arrow[r] & I(\Gamma) \arrow[r] \arrow[d, "g_I"] & \mathbb{Z}[E(\Gamma)]/e_{i,j}^2 \arrow[r] \arrow[d, "g_\Lambda"] & P^r(\Gamma) \arrow[r] \arrow[d, "g_P"] & 0 \\
0 \arrow[r] & I(\Gamma/\alpha) \otimes e_\alpha \arrow[r] \arrow[d] & \mathbb{Z}[E(\Gamma/\alpha)]/e_{i,j}^2 \otimes e_\alpha \arrow[r] \arrow[d] & P^r(\Gamma/\alpha) \otimes e_\alpha \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the rows and the middle column are exact. We want to show that the last column is also exact.

The map g_P is surjective. We need to check that $\mathrm{im}(\iota_P) = \ker(g_P)$. Consider the long exact sequence in cohomology. Since the second row and third rows are exact then $H^*(\mathbb{Z}[E(\Gamma)]/e_{i,j}^2) = 0 = H^*(\mathbb{Z}[E(\Gamma)]/e_{i,j}^2 \otimes e_\alpha)$, that implies

$$H^*(P(\Gamma)) = H^*(I(\Gamma/\alpha) \otimes \alpha).$$

So $im(\iota_P) = \ker(g_P)$ if and only if g_I is surjective. We want to check this last condition, that is we want to show that every generator in $I(\Gamma/\alpha)$ is the image of an element in $I(\Gamma)$. Let v_1 and v_2 be the two vertices incident to α and their common image in $V(\Gamma/\alpha)$ be v. Now, let $e_w = e_1 \cdots e_s$ be the product of the generators corresponding to edges in a cycle w in Γ/α. If the vertex v does not occur in w, then w is a cycle in Γ. There is then an element $y = d(e_w)e_\alpha \in I(\Gamma)$ such that $g_I(y) = d(w) \otimes \alpha$. If v is a vertex in w incident to exactly two edges in Γ/α, suppose them e_1 and e_s then there is a cycle $w' = \alpha e_1 \cdots e_s$ in Γ. So $d(e_{w'})$ is the corresponding generator in $I(\Gamma)$ and its image under the map g_I is

$$g_{\tilde{I}}(d(\alpha e_1 \cdots e_s)) = g(e_1 \cdots e_s) + g(\alpha d(e_1 \cdots e_s)) = d(w) \otimes \alpha.$$

Finally, if v is incident to more then two edges in w then we can decompose w in product of cycles where v is incident to only two vertices and deduce the result from the single factors using that d is a derivation.

We conclude the proof by double induction on m, n that are respectively the vertices and the edges of a graph $\Gamma(m, n)$. The proof proceeds analogously to the even case. The following property can be proved by induction in the same way as the even case since *Lemma* 6.4.9 holds also for P^r. Therefore, we have the following

Property 1. *For $e \in E(\Gamma)$, let $\ker(g_P^e)$ be the kernel of the map $g_P^e : P^r(\Gamma) \to P^r(\Gamma/e) \otimes e$. Then $\bigcap_e \ker(g_P^e) = \mathbb{Z}$.*

Lemma 6.7.4. *Assume that $\Gamma(m, n-1)$ and $\Gamma(m-1, n-1)$ satisfy* Theorem *6.7.3 and that $\Gamma(m, n-1)$ and $\Gamma(m-1, n-1)$ satisfy the property above. For $e \in E(\Gamma)$, let $\ker(g_P^e)$ be the kernel of the map $g_P^e : P^r(\Gamma) \to P^r(\Gamma/e) \otimes e$. Then $\bigcap_e \ker(g_P^e) = \mathbb{Z}$.*

As in the even case, the inductive step of the proof of *Theorem* 6.7.3 uses the property above. □

6.7.3 The isomorphism, the odd case

For any edge $e = e(v, w) \in E(\Gamma)$, consider the map

$$p_e : \mathfrak{C}_r(\Gamma) \to S^{r-1}$$

defined by

$$p_e(x) \mapsto \frac{x_i - x_j}{|x_i - x_j|} \in S^{r-1} \subset \mathbb{R}^r \setminus \{0\}$$

as before. Combining them, we obtain a map

$$p(\Gamma) : \mathfrak{C}_r(\Gamma) \to (S^{r-1})^{E(\Gamma)}.$$

Since r is odd we identify $H^*((S^{r-1})^{E(\Gamma)})$ with the ring $\mathbb{Z}[E(\Gamma)]/\{e_{i,j}^2\}$. We choose a standard generator $[S^{r-1}] \in H^{r-1}(S^{r-1})$, for each edge in $E(\Gamma)$ we chose one of the two orientations of this edge and denote it by $e \in \tilde{E}(\Gamma)$, so that every oriented edge can be written uniquely as either e or $\bar{e}$. We define the following map.

Definition 6.7.5. Let r be an odd number, the maps $p_r(\Gamma)$ induce ring homeomorphisms

$$\begin{aligned} p_r(\Gamma)^* : \mathbb{Z}[E(\Gamma)]/\{e_{i,j}^2\} &\to H^*(\mathfrak{C}_r(\Gamma)) \\ p_r(\Gamma)(e) &= p_e^*([S^{r-1}]) \\ p_r(\Gamma)(\bar{e}) &= -p_e^*([S^{r-1}]) \end{aligned}$$

Lemma 6.7.6. *The map $p_r(\Gamma)^*$ is surjective.*

Proof. As in the even case, we prove the lemma by induction on the number of edges in Γ. The lemma is true if Γ has one edge. Now we suppose the result true for graph with less then n edges. By *Corollary* 6.5.3 we have the two short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}[E(\Gamma \setminus e)]/\{e_{i,j}^2\} \arrow[r] \arrow[d] & \mathbb{Z}[E(\Gamma)]/\{e_{i,j}^2\} \arrow[r] \arrow[d] & \mathbb{Z}[E(\Gamma/e)]/\{e_{i,j}^2\} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^*(\mathfrak{C}_r(\Gamma \setminus e)) \arrow[r, "\phi^*"'] & H^*(\mathfrak{C}_r(\Gamma)) \arrow[r, "\psi^*"'] & H^*(\mathfrak{C}_r(\Gamma/e)) \arrow[r] & 0 \end{tikzcd}$$

The first and last map are surjective by hypothesis. By the five lemma the middle map is also surjective. □

Lemma 6.7.7. *The map $p_r(\Gamma)^*$ maps elements in the ideal generated by the generalised Arnold relations to* 0*.*

Proof. We can repeat the same argument of *Lemma* 6.6.3 for the odd case. □

Finally, we have the isomorphism.

Theorem 6.7.8. *There is a isomorphism of graded commutative rings*

$$\lambda_r : P^r(\Gamma) \to H^*(\mathfrak{C}_r(\Gamma)).$$

Proof. We prove the theorem by double induction on the number of edges and vertices in the graph. Let $\Gamma(m,n)$ be a graph with m vertices and n edges. The theorem clearly holds for every m for the graphs $\Gamma(m,0)$. Now, suppose that it is true for $\Gamma(m,n-1)$ and $\Gamma(m-1,n-1)$, we want to prove that it is true also for $\Gamma(m,n)$. Let e be an edge in $\Gamma = \Gamma(m,n)$, we have the short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & P^r(\Gamma \setminus e) \arrow[r] \arrow[d, "\cong"'] & P^r(\Gamma) \arrow[r] \arrow[d] & P^r(\Gamma/e) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H^*(\mathfrak{C}_r(\Gamma \setminus e)) \arrow[r, "\phi^*"'] & H^*(\mathfrak{C}_r(\Gamma)) \arrow[r, "\psi^*"'] & H^*(\mathfrak{C}_r(\Gamma/e)) \arrow[r] & 0
\end{tikzcd}$$

By the five lemma if follows that the middle map is also an isomorphism. □

List of symbols

$\mathfrak{C}(X,n)$	*Classical configuration space of points on X*, Definition 3.2.
$\mathfrak{C}(X,\Gamma)$	*Generalised configuration space*, Definition 2.2.2.
$\mathfrak{C}_r(\Gamma)$	*Generalised configuration space* when $X=\mathbb{R}^r$, Definition 6.2.1.
$\mathcal{C}_{BS}(\Gamma)$	*Baranovsky-Sazdanović graph complex*, Definition 2.2.
GC	*Kontsevich graph complex*, Definition 2.1.1.
fGC	*Full graph complex*, Definition 2.1.2
fGC_R	*Full graph complex*, Definition 3.2.5 and Section 4.5
$E[n]$	*Kriz*'s rational model for configuration spaces on complex projective varieties, Definition 5.2.1.
$\mathcal{K}(\Gamma)$	Variation of GC, Definition 4.2.1.
$\mathcal{K}_\Sigma(\Gamma)$	Variation of GC, Definition 4.4.1.
$R(\Gamma)$	Definition 5.6.1.
$R(\Gamma,A)$	Definition 5.6.5.
$I(\Gamma)$	Ideal generated by the *generalised Arnold relations*, Definition 5.6.1.
$\widetilde{\mathfrak{C}}_r$	*Generalised configuration space operad*, Definition 6.2.2.
$\widetilde{D}_r$	*Painted little disks operad*, Definition 6.2.6.
$R^r(\Gamma)$	*Cohomology ring for $\mathfrak{C}_r(\Gamma)$ when r is even*, Definition 6.4.1.
$P^r(\Gamma)$	*Cohomology ring for $\mathfrak{C}_r(\Gamma)$ when r is odd*, Definition 6.7.1.

List of Figures

Corrections to the submitted version

Page 28, *Definition* 3.1.14:" We will call a *rational model* for X any CDGA (A, d) weakly equivalent to $A_{PL}(X)$".

Page 30, definition of $G_A(n)$: "$G_{a,b}$ are generators of degree $m-1$", in the first relation "$G_{a,b} = (-1)^m G_{b,a}$".

Page 69 definition of R^r, *Definition* 6.4.1 and page 79 *Definition* 6.7.1: "$e_w = e_{v_{1,2}} \cdot e_{v_{2,3}} \cdot \ldots \cdot e_{v_{s,1}}$" in the product of the generators corresponding to the edges in the cycle "w", "$\sum e_{v_{1,2}} \cdots \hat{e}_{v_{i,j}} \cdots e_{v_{s_j,1}}$" in the third relation in the same definitions.

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