PhD Dissertation

Contributions to Integral and Stochastic Geometry

Selected uniqueness problems for integral formulae and Minkowski tensors

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Contributions to Integral and Stochastic Geometry
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Preface

This dissertation presents the research results obtained through my PhD studies at the Department of Mathematics, Aarhus University, from August 2016 to July 2021. During this period I also obtained a master's degree in Mathematics. My studies were carried out under the supervision of Associate Professor Markus Kiderlen and were funded jointly by the Graduate School of Natural Sciences (GSNS formerly known as GSST) and the Centre for Stochastic Geometry and Advanced Bioimaging (CSGB) through a grant from the Villum Foundation. Four months of funding were also provided by the Department of Mathematics at Aarhus University. The dissertation consists of an introductory chapter and the following three self-contained papers:

Paper A Uniqueness of the Measurement Function in Crofton's Formula. Published in *Advances in Applied Mathematics*.

Paper B Reconstructing Planar Ellipses from Translation-Invariant Minkowski Tensors of Rank Two. Submitted to *Discrete and Computational Geometry*.

Paper C Mean Surface and Volume Particle Tensors under Restricted L-isotropy and Associated Ellipsoids. Submitted to *Advances in Geometry*.

Besides layout and minor adjustments, all papers correspond to their submitted versions. A draft of Paper A was included in my progress report used for the qualification examination in June 2019. This was later submitted and published in Advances in Applied Mathematics with only minor changes. Papers B and C are the result of the last two years of my studies and have thus not appeared in the progress report in any form. I have contributed extensively in both the research phase and the writing of all three papers.

The introductory chapter is meant to provide the reader with some basic theory and notation, which is beneficial for reading the papers of the dissertation. The chapter also gives an overview of the results of the papers and how they relate to the existing literature.

During my PhD studies I have been surrounded by a lot of amazing people, who each in their own way, have helped fill my everyday life with interesting challenges, laughs and exciting experiences. Here I first and foremost owe my gratitude to my supervisor Markus Kiderlen without whom this PhD dissertation would never have been possible. So thank you for the many hours of discussions and guidance which have been invaluable. My thousands upon thousands of questions must sometimes have taken a toll on your patience, but you somehow managed to keep a smile on your face even doing the most trivial of discussions for which I am grateful. I further want to thank Eva B. Vedel Jensen for allowing me to be a part of the amazing research

environment that was CSGB. A lot of my most treasured memories of my PhD study come from events with the people from this group.
I would also like to thank Professor Monika Ludwig from the University of Technology (TU) in Vienna for her hospitality, even though COVID-19 ended my visit in Vienna after only two weeks.
A huge thanks also goes out to all my colleagues at Aarhus University and CSGB for creating a great working environment. Without weekly meetings, daily lunches and weekly cake my PhD studies would have been a colorless affair. In particular, I want to thank Nick Larsen, Stine Hasselholt, Helene Svane, Jakob Thøstesen and Ragnhild Laursen for laughs, Friday beers and great discussions. A special thanks goes to Mads Stehr and Louis Jensen for being amazing officemates. Thank you for the many random conversations, much needed brain breaks and laughs. Helene Hauschultz also deserves a special thanks for being a great support both academically and socially these past eight years. Your support was particularly valuable doing the COVID-19 lockdowns, where our many walks might have prevented my research from coming to a complete standstill.
Finally, I owe my gratitude to my family and friends for their constant support. Here my boyfriend Jakob Krejberg Ørhøj deserves extraordinary attention. He has been my rock through my entire PhD study, always believing in me and listening attentively to every frustration and idea, that has risen over the past five years. Without him to pull me back up again, I might have been swallowed by one of the black holes of frustration that research contains, never to be seen again. His LaTeX skills have also prevented me from throwing my computer out of the window on multiple occasions for which both me and my computer are grateful.

Rikke Eriksen
Aarhus, July 2021

Summary

In convex geometry there exist a number of different integral geometric relations. One of these relations is the classical Crofton formula which states that the $(n-j)$th intrinsic volume of a compact convex set in $\mathbb{R}^n$ can be obtained as an invariant integral of the $(k-j)$th intrinsic volume of sections with k-planes. In the first part of the dissertation we consider the possibility of replacing the $(k-j)$th intrinsic volume in Crofton's formula by other functionals. Linearity of the integral implies that this is equivalent to investigating the functionals in the kernel of the invariant integral defined by Crofton's formula. The non-triviality of this kernel is shown by constructing examples of non-trivial local functionals in the kernel, where local refers to translation invariant, continuous and additive functionals, which only depend on a small neighbourhood around the location of interest. For $k = 1$, these examples turn out to be the only local functionals in the kernel and for $k = 2$ they are the only local even functionals in the kernel. Thus we provide a complete description of all local functionals in the kernel when $k = 1$ and all even local functionals in the kernel when $k = 2$. If the evenness assumption is omitted or $k > 2$ we prove that there exists additional examples of local functionals in the kernel.

The Minkowski tensors are the focus of the second part of the dissertation, more specifically, the volume and surface tensors. The volume and surface tensors are of great interest, as they contain information about the shape and orientation of the underlying convex body. In fact, it has previously been proven that the volume tensor of rank 2 uniquely determines centered full-dimensional ellipsoids. We investigate whether a similar result holds for the rank-2 surface tensor. We prove that this is in fact the case, when we consider ellipses in the two-dimensional Euclidean space. For the n-dimensional case, $n > 2$, we prove the result for all ellipsoids of revolution of dimension at least $n-1$, meaning all ellipsoids of dimension at least $n-1$, which are invariant under rotations fixing a one-dimensional plane pointwise. An algorithm for constructing the underlying ellipsoid given the surface tensor of rank 2 is also introduced.

The homeomorphism on the set of full-dimensional centered ellipsoids, given by the rank-2 volume tensor, has in the literature been used to define ellipsoidal set-valued summary statistics for stationary marked point processes with convex bodies as marks. Our newly proven homeomorphism, defined by the rank-2 surface tensor on ellipsoids of revolution, allows for an introduction of an alternative summary statistic when the stationary marked point process satisfies certain rotation invariance assumptions. When $n = 2$ the extra assumptions can be dropped.

Finally, we prove that, under these rotation invariance assumptions, the average volume and surface tensors of the typical particle (or mark) can be derived from lower dimensional sections.

Resumé

Crofton's formel giver, at det $(n-j)$'te indre volumen af en konveks og kompakt delmængde K af $\mathbb{R}^n$ kan bestemmes ved det invariante integral af det $(k-j)$'te indre volumen af snit af K med k-dimensionelle affine planer. I første del af afhandlingen undersøger vi om det $(k-j)$'te volumen er entydigt givet. Med andre ord om det er muligt at erstatte det $(k-j)$'te indre volumen med andre funktionaler. Vi viser, at det er muligt ved at konstruere eksempler på sådanne funktionaler. Da integralet er lineært kan en beskrivelse af alle funktionaler, som kan erstatte det $(k-j)$'te volumen, findes ved at kortlægge kernen af det invariante integral defineret ved Crofton's formel.

Vi er specielt interesseret i de lokale funktionaler, som er indeholdt i kernen, dvs. funktionaler som er translations invariante, kontinuerte og additive samt kun afhænger af små omegne omkring de områder som betragtes. Dette skyldes at denne egenskab ofte ses hos estimatorer, og at Crofton's formel anvendes indenfor f.eks. stereologi til at udlede sådanne estimatorer.

Vores undersøgelser viser, at når $k = 1$, kan alle lokale funktionaler beskrives ved den samme formel. Lignende gælder for alle lige, lokale funktionaler i kernen når $k = 2$. Hvis antagelsen om lige droppes eller $k > 2$ findes der lokale funktionaler som ikke kan beskrive på denne måde.

Anden del af afhandlingen omhandler Minkowski tensorer, mere præcist volumen- og overfladetensorer for kompakte legemer. Volumen- og overfladetensorer generaliserer velkendte geometriske begreber så som volumen og overfladearea og derudover indeholder de information om formen og orienteringen af det underliggende legeme. Et velkendt resultat fra litteraturen er at volumentensoren af rang 2 entydigt fastlægger fuld-dimensionelle centrerede ellipsoider. Vi viser, at lignende gælder for overfladetensoren af rang 2 i det to-dimensionelle Euklidiske rum. I det n-dimensionelle Euklidiske rum viser vi, at rang-2 overfladetensoren entydigt fastlægger omdrejningsellipsoider af dimension mindst $n-1$, dvs. ellipsoider som er invariante under rotationer som fastholder en linje punktvist.Vi introducerer også algoritmer til bestemmes af ellipsoiden givet dens rang-2 overfladetensor.

Volumentensoren af rang 2 definerer altså en homeomorfi mellem fuld-dimensionelle ellipsoider og symmetriske, positiv definite $n \times n$-matricer. Denne homeomorfi har i litteraturen været brugt til at definere estimatorer for den gennemsnitlige form og orientering af den typiske partikel (eller markering) af en stationær markeret punktproces med kompakte konvekse markeringer. Det nye resultat for overfladetensoren af rang 2 kan på lignende vis bruges til at introducere nye estimatorer, når bestemte rotationsinvariansantagelser er opfyldt. Under disse rotationsinvariansantagelser viser vi også at volumen- og overfladetensorerne kan bestemmes gennem lavere dimensionelle snit.

Introduction

This chapter serves as an introduction to the topics and main results of the dissertation. Seen as a whole, the dissertation lies within the area of convex, integral and stochastic geometry. As the subtitle indicates the dissertation contains considerations of uniqueness problems, more specifically two problems, the first being uniqueness in Crofton's formula where the research resulted in Paper *A* and secondly uniqueness of ellipsoids given their Minkowski tensors of rank 2 which resulted in Paper *B* and Paper *C*. Section 1 serves as an introduction to some notation and basic theory from convex geometry. This will be useful for the reading of all of the papers. Section 2 is an introduction to Paper *A*. It firstly contains a brief overview of some essential background material used in Paper *A*, such as Crofton's formula, local functionals and spherical harmonics. Afterwards, a short introduction to the main results and proofs of Paper *A* is given. Section 3 is an overview of papers *B* and *C*. An introduction to Minkowski tensors and hypergeometric functions is in this section, followed by a summary of the main results of the two papers.

1 Convex Geometry

The notation here widely follows [17]. Let $n \in \mathbb{N}$. We will generally work in the n-dimensional Euclidean vector space $\mathbb{R}^n$ equipped with the usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure. The n-dimensional Hausdorff measure of the unit ball B^n and the $(n-1)$-dimensional Hausdorff measure the unit sphere S^{n-1} are denoted $\kappa_n = \pi^{n/2}/\Gamma(1+n/2)$ and $\omega_n = n\kappa_n$ respectively.
A set $K \subset \mathbb{R}^n$ is called a convex body, if K is non-empty, convex and compact. The set of all convex bodies in $\mathbb{R}^n$ is denoted $\mathcal{K}^n$. In Paper *A* we will consider the set of all convex compact sets in a linear subspace X of $\mathbb{R}^n$ and denote it by $\mathcal{K}(X)$, hence here the empty set is included. Also the subfamily of all convex compact sets of dimension at most k, $0 \leq k \leq \dim X$, is denoted $\mathcal{K}_k(X)$.
The support function of a convex body K is

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle$$

for $u \in \mathbb{R}^n$. As an example the unit ball B^n has support function $\|u\|$ for $u \in \mathbb{R}^n$. The support function of an ellipsoid $K = AB^n$, with A being a symmetric positive semi-definite $n \times n$-matrix, can be given as

$$h_K(u) = \|\Lambda C^t u\| \tag{1.1}$$

where $C \in \mathrm{SO}(n)$ and the diagonal matrix Λ are given by the spectral decomposition, $A = C\Lambda C^t$.

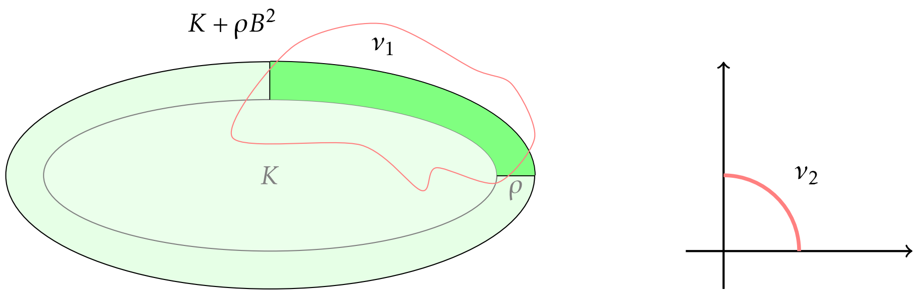

Figure 1: The figure shows a convex compact set K and the set $(K+\rho B^2)$ in light green. If $\nu = \nu_1 \times \nu_2$ then the local parallel set $M_\rho(K,\nu)$ is the set in green.

The distance between two convex bodies $K, M \in \mathcal{K}^n$ is defined by the Hausdorff metric,

$$d_H(K,M) = \min(\lambda \geq 0 : K \subset M + \lambda B^n, M \subset K + \lambda B^n).$$

This distance can also be expressed using support functions as (see [17, Lemma 1.8.14])

$$d_H(K,M) = \sup_{u \in S^{n-1}} |h_K(u) - h_M(u)|.$$

The *metric projection* of a convex body $K \in \mathcal{K}^n$ is the mapping $p(K,\cdot)$ such that for $x \in \mathbb{R}^n$, $p(K,x)$ is the closest point to x in K. The normalized vector, $u(K,x)$, pointing from $p(K,x)$ to x is the *outer unit normal* of K in direction x,

$$u(K,x) = \frac{x - p(K,x)}{d(K,x)},$$

where $d(K,x) = \|x - p(K,x)\|$. Pairs $(p(K,y), u(K,y))$ for $y \in \mathbb{R}^n \backslash K$ are *support elements* of K and the set of all these in $\Sigma = \mathbb{R}^n \times S^{n-1}$ is called the *normal bundle* of K, denoted $\operatorname{Nor} K$ (see [17, Chapter 2.6]). With this in mind we can define positive finite measures $\Lambda_0(K,\cdot), \dots, \Lambda_{n-1}(K,\cdot)$ on $\mathcal{B}(\Sigma)$ called the *support measures* by

$$\lambda_n(M_\rho(K,\nu)) = \sum_{m=0}^{n-1} \rho^{n-m} \kappa_{n-m} \Lambda_m(K,\nu) \tag{1.2}$$

for $\nu \in \mathcal{B}(\Sigma)$ and $\rho > 0$, where λ_n is the Lebesgue measure and

$$M_\rho(K,\nu) = \{x \in \mathbb{R}^n : 0 < d(K,x) \leq \rho \text{ and } (p(K,x), u(K,x)) \in \nu\}$$

is the *local parallel set* (see [17, Theorem 4.2.1]). The local parallel set consists of all $x \in \mathbb{R}^n \backslash K$ within a fixed distance ρ of K and where the corresponding support element is contain in the given subset $\nu \subset \Sigma$. An example of such a set is illustrated in Figure 1.
The support measures are concentrated on $\operatorname{Nor} K$ and they are closely connected to some of the fundamental measures in convex geometry, e.g. the total mass of the support measures of K are the intrinsic volumes, $V_j(K) = \Lambda_j(K,\Sigma)$. The measures $V_j(\cdot)$ are called intrinsic as they are independent of the ambient space. For a convex body K with $\dim K = m$, $V_m(K)$ is the volume of K, $2V_{m-1}(K)$ the surface area and $V_0(K)$ is

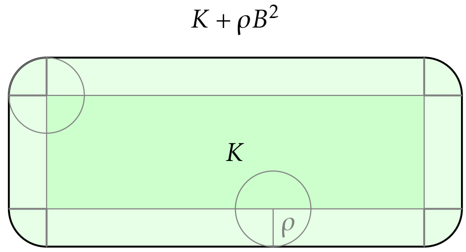

Figure 2: By the Steiner formula the volume of set $K + \rho B^2$ can be determined by decomposing it into the parts shown in the figure. Hence $V_2(K + \rho B^2) = V_2(K) + 2\rho V_1(K) + \pi\rho^2$.

the Euler characteristic.
A useful formula called the *classical Steiner formula* follows from (1.2),

$$V_n(K + \rho B^n) = \sum_{m=0}^{n} \rho^{n-m} \kappa_{n-m} V_m(K).$$

Thus the nth intrinsic volume of the expansion $K + \rho B^n$ of K can be determined by cutting the set into smaller pieces. An illustration of this can be found in Figure 2. The support measures also give rise to marginal measures called the *area measures*,

$$\mathrm{S}_j(K,\omega) = \frac{n\kappa_{n-j}}{\binom{n}{j}} \Lambda_j(K, \mathbb{R}^n \times \omega)$$

for $\omega \in \mathcal{B}(S^{n-1})$, $j = 1, \ldots, n-1$. These measures are locally determined, weakly continuous and additive. They further satisfy the following motion covariance. For a rigid motion g of $\mathbb{R}^n$ with corresponding rotation C_0

$$\mathrm{S}_j(gK, C_0\omega) = \mathrm{S}_j(K,\omega)$$

and for $\alpha > 0$,

$$\mathrm{S}_j(\alpha K, \omega) = \alpha^j \, \mathrm{S}_j(K,\omega).$$

The $(n-1)$th area measure is called the *surface area measure*. This is due to the following relation between this measure and the $(n-1)$-dimensional Hausdorff measure on the boundary of $K \in \mathcal{K}^n$, with $\dim K = n$,

$$\mathrm{S}_{n-1}(K,\omega) = \mathcal{H}^{n-1}(\tau(K,\omega)),$$

where $\tau(K,\omega)$ is called the reversed spherical image of K at ω and is the set of all boundary points of K at which there exists a normal vector of K belonging to ω [17, Chapter 2.2]. An illustration of this mapping can be found in Figure 3. If $\omega = S^{n-1}$ then the surface area measure is exactly the surface area of K and if further $K \in C^2_+$ then $S_{n-1}(K,\cdot)$ is closely connected to the Gauss-Kronecker curvature, H^{n-1}, as

$$S_{n-1}(K,\omega) = \int_\omega \frac{1}{H^{n-1}(x_K(u))} \mathcal{H}^{n-1}(du) \tag{1.3}$$

for $\omega \in \mathcal{B}(S^{n-1})$, where x_K is the reversed spherical image map. Further details can be found in [17, Chapter 2.5].

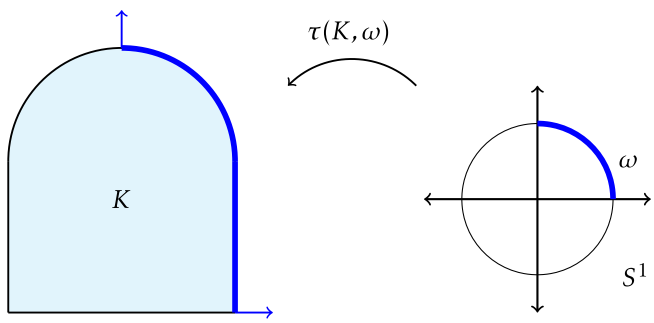

Figure 3: The figure shows a convex compact set K in blue. If ω is given as in the figure then the reverse spherical image at ω, $\tau(K,\omega)$, is the part of the boundary of K which is highlighted by dark blue.

2 Paper A

2.1 Crofton's formulae

In convex geometry a number of integral geometric relations exist. Here we will only mention Crofton's formula, as this formula is the focus of Paper A. As a reference we use [17] where Crofton's formula and multiple other integral geometric relations can be found.
Let $G(n,k)$ be the Grassmannian of k-dimensional linear subspaces and $A(n,k)$ be the group of affine k-dimensional subspaces in $\mathbb{R}^n$. On $G(n,k)$ an invariant probability measure ν_k and on $A(n,k)$ a rigid motion invariant measure μ_k can be defined using the normalized Haar measure on $\mathrm{SO}(n)$ (combined with the $(n-k)$-dimensional Lebesgue measure). See [17, Chapter 4.4] for further details.

Crofton's formula now states that the intrinsic volume of a convex body K can be determined by lower dimensional sections, i.e. for $k \in \{0,\ldots,n\}$ and $j \in \{0,\ldots,k\}$ then

$$\int_{A(n,k)} V_j(K \cap E)\, d\mu_k(E) = \alpha_{njk} V_{n+j-k}(K), \tag{2.1}$$

where

$$\alpha_{njk} = \frac{\binom{k}{j}\kappa_k \kappa_{n+j-k}}{\binom{n}{k-j}\kappa_j \kappa_n}.$$

Crofton's formula can be seen as a consequence of the famous Hadwiger characterization Theorem [6]. For this we observe that the left hand side of (2.1) is continuous, rigid motion invariant and additive. Due to Hadwiger's theorem it is a linear combination of intrinsic volumes. As the left hand side of (2.1) is further homogeneous of degree $n-k+j$, the equation follows. The constant α_{njk} can be determined by letting $K = B^n$.
Crofton's formula is particularly interesting in practical applications, as it can be used to extract different stereological estimators. Examples of such can be found in [13], where for instance an unbiased estimator of the surface area of a convex body K from one-dimensional linear sections is introduced or in [8] where Crofton's formula is used for calculating sampling probabilities associated with random grids.

2.2 Local functions

In Paper *A* we focus mainly on functionals satisfying similar properties as estimators known from stereology. Such estimators often only depend on small neighborhoods around the locations of interest. This property was formalized by Wolfgang Weil in his papers [19] (for functionals acting on the set of polytopes) and [20] (for functionals acting on convex bodies). He called the property *local* and the precise definition of this for functionals acting on $\mathcal{K}^n$ is the following.

Definition 2.1. *A functional $\varphi : \mathcal{K}^n \to \mathbb{R}$ is local if there is functional $\Phi : \mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$ which is measurable on $\mathcal{K}^n$ in the first variable, a finite signed Borel measure in the second and*

- $\varphi(K) = \Phi(K, \mathbb{R}^n)$ *for all* $K \in \mathcal{K}^n$
- Φ *is translation covariant, i.e.* $\Phi(K + x, A + x) = \Phi(K, A)$ *for all* $K \in \mathcal{K}^n$, $A \in \mathcal{B}(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$.
- Φ *is locally determined, i.e. for* $M, K \in \mathcal{K}^n$ *and* $A \in \mathcal{B}(\mathbb{R}^n)$ *then* $\Phi(M, A) = \Phi(K, A)$ *if there exists* $U \subset \mathbb{R}^n$ *open such that* $K \cap U = M \cap U$ *and* $A \subset U$.
- $K \mapsto \Phi(K, \cdot)$ *is weakly continuous (in the Hausdorff metric).*

By this definition, local functionals are clearly translation invariant and continuous. Note that the intrinsic volumes are clearly local with one possible local extension being the curvature measures.
Note further that a local functional is also local when restricted to the set of all polytopes in $\mathbb{R}^n$, $\mathcal{P}^n$, where local in $\mathcal{P}^n$ is in the sense of the definition from [19]. However the opposite does not necessarily hold, i.e. a functional acting on $\mathcal{K}^n$ which is local when restricted to $\mathcal{P}^n$, does not necessarily satisfy Definition 2.1.
Wolfgang Weil also proved in [20] an important theorem, which states that all local functionals on $\mathcal{K}^n$ admit a decomposition into local, homogeneous and additive parts. This implies in particular that all functionals satisfying Definition 2.1 are *standard* functionals (where standard is defined as in [18], i.e. continuous, translation invariant and additive).

2.3 Spherical harmonics

This section will give a brief introduction to the theory of spherical harmonics which is mainly important for that proofs in Paper *A*. However, the Laplace-Beltrami operator is also interesting in relation to Paper *B*.
We will consider functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and denote the restriction of f to the unit sphere S^{n-1} by $\hat{f}$ or $f\hat{}$. A function F acting on the unit sphere can be extended to $\mathbb{R}^n \backslash \{0\}$ by the radial extension

$$\check{F}(x) = F\left(\frac{x}{\|x\|}\right) \tag{2.2}$$

for $x \in \mathbb{R}^n \backslash \{0\}$. The Laplace operator Δ is important for the theory of spherical harmonic and acts on twice continuous differentiable functions, so for $f \in L^2(\mathbb{R}^n)$

$$\Delta f = \sum_{l=1}^{n} \frac{\partial^2 f}{(\partial x_l)^2}.$$

This operator gives rise through the radial extension to the Laplace-Beltrami operator (or the spherical Laplace operator) by

$$\Delta_S F = (\Delta \check{F})^{\wedge}$$

for $F \in L^2(S^{n-1})$. It is especially important for the proofs of Paper B, that the Laplace-Beltrami operator is self-adjoint with respect to the inner product

$$\langle F, G\rangle = \int_{S^{n-1}} F(u)G(u)\sigma_{n-1}(du),$$

where σ_{n-1} is the spherical Lebesgue measure on S^{n-1}. This can be proven using Greens formula (for further details see [5, section 1.2]). Hence

$$\langle F, \Delta_S G\rangle = \langle \Delta_S F, G\rangle$$

for all $F, G \in L^2(S^{n-1})$.
We can now define *harmonic* polynomials as the polynomials p which are homogeneous with $\Delta p = 0$. Examples of such polynomials are of course the constant polynomial and linear functions. Restrictions of harmonic polynomials on $\mathbb{R}^n$ to the unit sphere are called spherical harmonics of dimension n. The vector space of all harmonic polynomials of homogenuity degree k on $\mathbb{R}^n$ is denoted by $\mathcal{Q}_k^n$ and the restriction of these to S^{n-1} by $\mathcal{H}_k^n = \{\hat{p} : p \in \mathcal{Q}_k^n\}$.
The isomorphism $f \mapsto \hat{f}$ from $\mathcal{Q}_k^n$ to $\mathcal{H}_k^n$ implies that $\mathcal{H}_k^n \cap \mathcal{H}_m^n = \{0\}$ for $m \neq k$ (for further details see [5, Lemma 3.1.3]). So we can define the order of a non-zero n-dimensional harmonic H as the (unique) number k such that $H \in \mathcal{H}_k^n$. Thus H has order k, if it is the restriction to S^{n-1} of a k-homogeneous polynomial. The isomorphism further yields that $\mathcal{H}_k^n$ has finite dimension (see [5, Chapter 3.1]).
The Legendre polynomials, P_k^n, are the unique polynomials satisfying that for an orthonormal basis, $H_1, \ldots, H_N$, of $\mathcal{H}_k^n$,

$$\sum_{i=1}^{N} H_i(u)H_i(v) = \frac{N}{\omega_n}\mathrm{P}_k^n(\langle u, v\rangle). \tag{2.3}$$

(see [5, Theorem 3.3.3]). The degree of P_k^n is k and for fixed $v \in S^{n-1}$, $\mathrm{P}_k^n(\langle \cdot, v\rangle)$ is a spherical harmonic of order k. We adapt the notation $\mathrm{P}_{-1}^n(t) = 0$. By (2.3) it follows that the Legendre polynomials are even (odd) if k is even (odd). Also $\mathrm{P}_k^n(1) = 1$. For $n = 3$ we get the classical Legendre polynomials which are used in Paper A. These take the simpler form

$$\mathrm{P}_k^3(t) = \frac{1}{2^k k!}\frac{d^k}{dt^k}(t^2-1)^k \tag{2.4}$$

by the formula of Rodrigues (see [5, Prop. 3.3.7]) or alternatively,

$$\mathrm{P}_k^3(t) = \frac{1}{2^k}\sum_{j=0}^{k}(-1)^{k-j}\frac{(2j)!}{j!(2j-k)!(k-j)!}t^{2j-k}$$

where we understand $t^{2j-k} = 0$ if $2j - k < 0$. Paper A uses the above expression with $k = 5$, which is

$$\mathrm{P}_5^3(t) = \frac{1}{8}(63t^5 - 70t^3 + 15t).$$

2.4 Spherical projections and liftings

In Paper A we further use the theory of spherical projections and liftings. This will be a brief introduction to these operators based on the article [4]. For a compact set $W \subset \mathbb{R}^n$, $\mathcal{M}(W)$ will denote the dual space of W. This space is equivalent to the space of finite signed Borel measures on W. In this section we will assume that $k \in \{1, \ldots, n-1\}$, $m > -k$ is an integer and $L \in G(n,k)$. Let

$$H^{n-k}(L,v) = \{u \in S^{n-1} \backslash L^{\perp} | \, \mathrm{p}_L(u) = v\} \tag{2.5}$$

with

$$\mathrm{p}_L(u) = \frac{u|L}{\|u|L\|}$$

being the *spherical projection* of $S^{n-1} \backslash L^{\perp}$ onto $S^{k-1}(L) = S^{n-1} \cap L$, where $u|L$ is the orthogonal projection of u onto L. We let $\mathcal{M}_{m,L}(S^{n-1})$ be the subspace of $\mathcal{M}(S^{n-1})$ given by

$$\mathcal{M}_{m,L}(S^{n-1}) = \{\mu \in \mathcal{M}(S^{n-1}) : \int_{S^{n-1}} \|u|L\|^m |\mu|(du) < \infty\}.$$

We are now able to introduce mappings between $\mathcal{M}_{m,L}(S^{n-1})$ and $\mathcal{M}(S^{k-1}(L))$. The first being the *m-weighted spherical projection*, $\pi_{L,m}$, given by

$$\pi_{L,m}\mu = \int_{S^{n-1} \backslash L^{\perp}} \mathbb{1}_{(\cdot)}(\mathrm{p}_L(u)) \|u|L\|^m \mu(du)$$

for $\mu \in \mathcal{M}_{m,L}(S^{n-1})$. The other mapping is the *m-weighted spherical lifting* which disperses the measure $\mu \in \mathcal{M}(S^{k-1}(L))$ with a weight along the half-spheres orthogonal to L. In more detail, the m-weighted lifting is the function $\pi^*_{L,m} : \mathcal{M}(S^{k-1}(L)) \to \mathcal{M}(S^{n-1})$ such that

$$\pi^*_{L,m}\mu = \int_{S^{k-1}(L)} \int_{H^{n-k}(L,v) \cap (\cdot)} \langle v, w \rangle^{k+m-1} dw \mu(dv)$$

for $\mu \in \mathcal{M}(S^{k-1}(L))$. The weights are chosen such that $\pi^*_{L,0}\sigma^L_{k-1} = \sigma_{n-1}$, with σ^L_{k-1} being the spherical Lebesgue measure on $S^{k-1}(L)$. Both maps are weakly continuous and linear and maps positive measures to positive measures. A measurable function f, which is integrable with respect to σ_{n-1} respectively σ^L_{k-1}, can be identified with the measure $\int_{(\cdot)} f(u)du$, which in turn yields the spherical projections and the spherical liftings of functions. With this in mind $\pi^*_{L,m}$ and $\pi_{L,m}$ can be seen as the transpose of each other, (see [4, Section 5]).
The above gives rise to the *mean lifted projection*, $\pi^{(k)}_{m,j}$, $j < \infty$,

$$(\pi^{(k)}_{m,j}\mu)(A) = \int_{G(n,k)} (\pi^*_{L,m}\pi_{L,j}\mu)(A) dL$$

for $\mu \in \mathcal{M}(S^{n-1})$ and $A \in \mathcal{B}(S^{n-1})$. By the definitions of the spherical projection and the spherical lifting, the mean lifted projection acts on integrable functions f on S^{n-1} as

$$\int_{S^{n-1}} (\pi^{(k)}_{j,m} f) d\mu = \int_{S^{n-1}} f d(\pi^{(k)}_{m,j}\mu).$$

It further follows that $\pi^{(k)}_{m,j}$ is a self-adjoint and intertwining continuous linear operator on the set of continuous functions on S^{n-1}, $\mathcal{C}(S^{n-1})$, (for details see [4, Section 7 − 9]). So there are $a_{n,k,m,j,d} \in \mathbb{R}$ such that

$$a_{n,k,m,j,d} = \Big(\pi^{(k)}_{m,j}\,\mathrm{P}^n_d(\langle u,\cdot\rangle)\Big)(u),$$

see [4, Section 4]. The constants $a_{n,k,m,j,d}$ can be calculated using the Legendre polynomials and an explicit form is given in [4, Theorem 9.1]. We will only mention here that these formulas imply especially that $a_{3,2,1,2,5} = \frac{97}{1536}\pi$ and $a_{3,2,1,1,5} = \frac{344}{1575}$, which are the constants used in Paper *A*.

2.5 Paper A

In Section 2.1 we introduced the classical Crofton formula and using this we can define the *Crofton operator*, $\mathrm{C_k} : (\mathcal{K}_k(\mathbb{R}^n))^{\mathbb{R}} \to \mathbb{R}$, as

$$(\mathrm{C_k}\,\varphi)(K) = \int_{A(n,k)} \varphi(K \cap E)\mu_k(dE),$$

where $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is μ_k-integrable. With this in mind, we consider the question of whether there exist other measurable functionals than the jth intrinsic volume satisfying

$$(\mathrm{C_k}\,\varphi)(K) = \alpha_{njk} V_{n+j-k}(K).$$

By linearity of the Crofton operator, this question becomes equivalent to asking for a description of the kernel of $\mathrm{C_k}$, which is exactly the set-up in Paper *A*.
Paper *A* gives a partial answer to the question by proving that there indeed exist non-trivial functionals in the kernel of $\mathrm{C_k}$. However a full description of the kernel is not given.
The existence of functionals in $\ker \mathrm{C_k}$ is proven by construction. So we prove that for a non-trivial ν_k-integrable function $f : G(n,k) \to \mathbb{R}$, satisfying

$$\int_{G(n,k)} f(L)\nu_k(dL) = 0, \tag{2.6}$$

a non-trivial functional in the kernel of $\mathrm{C_k}$ can be given by

$$\varphi_f(K) = \begin{cases} V_k(K) f(\mathrm{lin}(K)) & \text{if } \dim K = k \\ 0 & \text{otherwise} \end{cases}, \tag{2.7}$$

see Proposition A.1.1. Now φ_f is translation invariant and additive. Also if f is continuous, then φ_f is continuous. Hence the functionals φ_f satisfy most of the natural geometric properties. Note that there are no non-trivial continuous additive rigid motion invariant functionals in the kernel of $\mathrm{C_k}$ due to Hadwiger's characterization theorem [6].
As described in Section 2.2 the interest in the kernel of the Crofton operator comes from the use of Crofton's formula for deriving estimators in fields such as stereology. Hence local functionals as defined in Section 2.2 are the main focus of Paper *A*.
In Paper *A* we extend the definition of local to include functionals acting on convex

compact sets of dimension at most $k \leq n$, by saying, that such a functional is local if it is translation invariant in the entire space and local in the sense of Definition 2.1 for each restriction to a k-dimensional linear subspace. As stated in Section 2.2 every local functional on $\mathcal{K}^n$ has a decomposition into homogeneous parts. We prove in Theorem A.1.2 that this also holds for local functionals on $\mathcal{K}_k(\mathbb{R}^n)$. A consequence of this theorem is that all local functionals in the kernel of C_1 can be written as φ_f for some measurable function f satisfying (2.6). When $k = 2$, a similar result holds for even, local functionals, i.e. all even local functionals in $\ker \mathrm{C}_2$ can be written as φ_f for some measurable function f satisfying (2.6). Thus Paper *A* contains a complete description of all local functionals in the kernel of C_1 and all even local functionals in $\ker \mathrm{C}_2$.

The paper further provides a proof of the existence of additional local functionals in the kernel of the Crofton operator for $k \geq 2$. The proof of this is treated in two parts. Firstly when $k > 2$, it turns out that the Klain function of a local even $(k-1)$-homogeneous functional φ is in the kernel of the Radon transform if and only if

$$\int_{A(n,k)} \varphi(P \cap E)\mu_k(dE) = 0$$

for all convex polytopes $P \in \mathcal{K}_{n-1}(\mathbb{R}^n)$ (see Proposition A.2.5). As the kernel of the Radon transform is non-trivial when $k > 2$ we get the desired non-trivial $(k-1)$-homogeneous even local continuous functionals in the kernel of $\mathrm{C_k}$ (see [10] for details on Klain functions and [3] for detail on the Radon transform).

When $k = 2$ a construction of a non-trivial 1-homogeneous local functional in the kernel of C_2 is possible using spherical harmonics. Thus we construct a non-trivial function $\theta : \{(L,v) : L \in G(n,k), v \in S^{n-1} \cap L\} \to \mathbb{R}$ such that $\theta(L,\cdot)$ is continuous and centered on $S^{n-1} \cap L$ for each $L \in G(n,k)$. Letting

$$\varphi(K) = \int_{S^{n-1} \cap L} \theta(L,v)\mathrm{S}^L_{k-1}(K - x, dv)$$

for all compact convex sets K contained in $L + x$, $x \in L^{\perp}$, we get a local $(k-1)$-homogeneous functional. If

$$\int_{G(n,k)} \pi^*_{L,1}\theta(L,\cdot)\nu_k(dL) = 0, \tag{2.8}$$

then φ is in the kernel of C_2. Constructing this non-trivial continuous function θ satisfying (2.8) can be done using the theory of spherical liftings and projections introduced in Section 2.4. In Paper *A*, an example of such a construction is given as

$$\theta(L,v) = (\alpha\pi_{L,1} + \beta\pi_{L,2})\mathrm{P}^3_5(\langle \mathrm{p}_L(u_0),\cdot\rangle) - \langle x_L,\cdot\rangle,$$

where α, β chosen such that θ is non-trivial and satisfy (2.8).

The last part of Paper *A* is devoted to discussing the kernel of $\mathrm{C_k}$ for functionals now acting on $\mathcal{M}_k$, the set of all section profiles of the subset $\mathcal{M} \subset \mathcal{K}^n$ (i.e. $\mathrm{C_k}$ is now an operator acting on $\mathcal{M}_k^{\mathbb{R}}$). For $\mathcal{M}$ being the set of all rigid motions of a fixed $K_0 \in \mathcal{K}^n$, the problem of describing the kernel of C_0 is closely related to the *Pompeiu problem* (see for instance [15]), which is unsolved in the general case. However, Ramm [16] proved that the kernel is trivial, when K_0 has C^1-smooth boundary and $n = 3$. Paper

A fills a bit of this gap by proving, that, when $\mathcal{M}$ is the set of all n-dimensional balls and φ is a motion invariant functional, then φ is in the kernel of $\mathrm{C_k}$ if and only if $\varphi(K \cap E) = 0$ for μ_k-almost all $E \in A(n,k)$. In particular as all 1-dimensional convex compact sets are balls, this implies that for $k = 1$, there do not exist non-trivial motion invariant functionals in the kernel of the Crofton operator.

3 Papers B and C

3.1 Tensors

In papers B and C we work with surface and volume tensors. This section is therefore an introduction to the concept of tensors and some important properties of these. The notation widely follows [7].

An r-linear map from $(\mathbb{R}^n)^r$ into $\mathbb{R}$ is called a *rank-r tensor* and if it is invariant under permutations of the arguments it is *symmetric*. The real vector space of all rank-r symmetric tensors on $\mathbb{R}^n$ is denoted $\mathbb{T}^r$. The symmetric tensor product is for $a_i \in \mathbb{T}^{r_i}$, $i = 1,\dots,k$ given by

$$(a_1 \odot \cdots \odot a_k)(x_1,\dots,x_{s_k}) := \frac{1}{s_k!} \sum_{\sigma \in \mathcal{S}(s_k)} \prod_{i=1}^{k} a_i(x_{\sigma(s_{i-1}+1)},\dots,x_{\sigma(s_i)})$$

with $x_1,\dots,x_{s_k} \in \mathbb{R}^n$, where $s_0 = 0$, $s_i = r_1 + \cdots + r_i$ and $\mathcal{S}(m)$ denotes the group of permutations of $\{1,\dots,m\}$. Thus $a_1 \odot \cdots \odot a_k \in \mathbb{T}^{r_1+\cdots+r_k}$. To ease notation we abbreviate and write $a \odot b = ab$ and

$$a^r = \underbrace{a \odot a \cdots \odot a}_{r}.$$

With this notation $a^0 = 1$ and $a^r(x_1,\dots,x_r) = \langle a, x_1\rangle \cdots \langle a, x_r\rangle$ for $r \geq 1$ and $x_i \in \mathbb{R}^n$, $i = 1,\dots,r$. The scalar product of $\mathbb{R}^n$ given by

$$Q(x,y) = \langle x,y\rangle$$

for $x, y \in \mathbb{R}^n$, is a symmetric rank-2 tensor which we call the *metric tensor*.

Let $(e_1,\dots,e_n)$ be the standard basis of $\mathbb{R}^n$. A coordinate representation of $T \in \mathbb{T}^r$ is

$$T = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} t_{i_1\dots i_r} e_{i_1} \dots e_{i_r}$$

with

$$t_{i_1\dots i_r} = \binom{r}{m_1 \cdots m_n} T(e_{i_1},\dots,e_{i_r}),$$

where m_k count the number of times k appears among the indices $i_1,\dots,i_r$. Hence T can be represented as the array of elements

$$(T)_{i_1,\dots,i_n} = T(e_1,[i_1],\dots,e_n[i_n]) := T(\underbrace{e_1,\dots,e_1}_{i_1},\dots,\underbrace{e_n,\dots,e_n}_{i_n}).$$

This implies that rank-r tensors with $r = 0, 1, 2$ can be identified with numbers, vectors and symmetric $n \times n$-matrices, respectively.

A functional $\Gamma : \mathcal{K}^n \to \mathbb{T}^r$ is said to be *rotation covariant* if $\Gamma(CK) = C\Gamma(K)$ for all $C \in O(n)$ and to have *polynomial translation behaviour* if

$$\Gamma(K + t) = \sum_{j=0}^{r} \frac{1}{j!} \Gamma_{r-j}(K) t^j$$

for all $K \in \mathcal{K}^n$, $t \in \mathbb{R}^n$ where $\Gamma_{r-j}(K) \in \mathbb{T}^{r-j}$ are independent of t. If both rotation covariance and polynomial translation behaviour are satisfied, then Γ said to be

isometric covariant.
For $r, s \in \mathbb{N}_0$ and $k \in \{0, \dots, n-1\}$ the *Minkowski tensors* of a convex body $K \in \mathcal{K}^n$ are

$$\Phi_k^{r,s}(K) = \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma^n} x^r u^s \Lambda_k(K, d(x,u)).$$

If $k = n$ and $s = 0$, we define

$$\Phi_n^{r,0}(K) = \frac{1}{r!} \int_K x^r \, dx,$$

and if $k \notin \{0, \dots, n\}$ or $r \notin \mathbb{N}_0$ or $s \notin \mathbb{N}_0$ or $k = n$ and $s \neq 0$, we put $\Phi_k^{r,s} := 0$.
In Paper B and C we consider the surface tensors $\Phi_{n-1}^{0,s}$ and volume tensors $\Phi_n^{r,0}$. Note that the volume tensors are isometric covariant while the surface tensors are translation invariant, due to the translation covariance of the surface measure.
For $p \in \mathbb{N}_0$, Alesker's characterization theorem for tensors [1, Thm 2.5] states that, the space of all isometry covariant continuous valuations $\phi : \mathcal{K}^n \to \mathbb{T}^p$ is spanned by $Q^m \Phi_k^{r,s}$, where $m, r, s \in \mathbb{N}_0$, with $2m + r + s = p$ and $k \in \{0, \dots n\}$, and where $s = 0$ if $k = n$.
As the total mass of the jth support measure $\Lambda_j(K, \cdot)$ is the jth intrinsic volume, then

$$\Phi_k^{0,0}(K) = V_k(K).$$

Thus the Minkowski tensors are an extension of the set of intrinsic volumes. The intrinsic volumes satisfy the rather strong properties of continuity, additivity and motion invariance which limit their ability of describing the underlying set. E.g. the rotation invariance implies that orientation of the underlying set is not captured. The same is true for position in space due to the translation invariance. The Minkowski tensors however capture more information about the underlying set which makes them interesting for fields such as stereology and physics. It was for example proven in [12] and [11] that the surface tensors $\Phi_{n-1}^{0,s}(K)$, $s = 0, 1, \dots, q$ for large q or $q = \infty$ uniquely determines the convex body K.

3.2 Hypergeometric functions

Here we will give a brief introduction to the theory of hypergeometric functions. This is important for the proofs in both Paper *B* and *C*. We use the book [2] as main reference and proofs of results presented in this section can be found there. A hypergeometric series is a sum of the form

$$\sum_{m=0}^{\infty} c_m,$$

where $\frac{c_{m+1}}{c_m}$ is a rational function of m, so

$$\sum_{m=0}^{\infty} c_m = c_0 \cdot {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)$$

for non-negative integers (and not zero) b_i, where

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{x^m}{m!},$$

with $(a)_m$ being the Pochhammer symbol,

$$(a)_m = \begin{cases} 1 & \text{for } m = 0 \\ a(a+1)\dots(a+m-1) & \text{for } m > 0 \end{cases}.$$

The function ${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;x)$ converges absolutely for all x if $p \le q$, and for $\|x\| < 1$ if $p = q+1$ (see for instance [2, Theorem 2.1.1]). Some examples of functions which can be represented as a hypergeometric series are for instance $\log(1+x) = x \cdot {}_2F_1(1,1;2;-x)$ or $e^x = {}_0F_0(x)$. In Paper C we will work mainly with ${}_2F_1$ called the *hypergeometric function*, which is defined as

$${}_2F_1(a,b;c;x) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m m!} x^m \tag{3.1}$$

for $a,b,c \in \mathbb{R}$ and $\|x\| < 1$. This extend continuously to all other x. Easing notation we will from now on write ${}_2F_1 = \mathrm{F}$. Due to Euler we have the following integral representation of a hypergeometric function (see for instance [2, Theorem 2.2.1]),

$$\mathrm{B}(b,c-b)\,\mathrm{F}(a,b;c;x) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt, \tag{3.2}$$

where $\mathrm{B}(i,j)$ is the beta function. This integral representation, called *Euler's integral representation*, is useful for different rewritings of the hypergeometric functions. In particular it yields the following two transformations,

$$\mathrm{F}(a,b;c;x) = (1-x)^{-a}\,\mathrm{F}(a,c-b;c;\tfrac{x}{1-x})$$

called the *Pfaff transformation* and

$$\mathrm{F}(a,b;c;x) = (1-x)^{c-a-b}\,\mathrm{F}(c-a,c-b;c;x)$$

called the *Euler transformation*, see [2, Theorem 2.2.5]. Note that the Pfaff transformation extents the series representation of the hypergeometric functions to include $x < \frac{1}{2}$ and it can be used to prove known formulas for the trigeometric functions such as

$$\tan^{-1}(x) = x\,\mathrm{F}(\tfrac{1}{2},1;\tfrac{3}{2};-x^2) = \frac{x}{\sqrt{1+x^2}}\,\mathrm{F}(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};\tfrac{x^2}{1+x^2}) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

The integral representation (3.2) also implies that for $c-a-b > 0$ the radius of convergence of the series representation of the hypergeometric functions include $x = 1$, as

$$\mathrm{F}(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

If $c = a+b$ we get the convergence

$$\lim_{x\to 1^-} \frac{\mathrm{F}(a,b;a+b;x)}{-\log(1-x)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

Both convergences are important for the results of Paper C and can be found in [2, Theorem 2.1.3, 2.2.2].

It is easy to check that differentiating a hypergeometric function will again yield a hypergeometric function

$$\frac{d}{dx}\,\mathrm{F}(a,b;c;x) = \frac{ab}{c}\,\mathrm{F}(a+1,b+1;c+1;x).$$

This combined with the previously mentioned transformations can be used to yield six contiguous relations, meaning relations between two hypergemetric functions with the same power-series variable, two parameters which are pairwise equal and a third that only differs by one. We will only list the two which are used to make differential formulas for elliptic integrals in the next section, details on proofs can be found in [2, Section 2.5]. We oppress the variables and write $\mathrm{F} = \mathrm{F}(a,b;c;x)$ and $\mathrm{F}(a,b\pm 1;c;x) = \mathrm{F}(b\pm)$. So

$$x\frac{d\,\mathrm{F}}{dx} = b(\mathrm{F}(b+) - \mathrm{F}), \tag{3.3}$$

and

$$x(1-x)\frac{d\,\mathrm{F}}{dx} = (c-b)\,\mathrm{F}(b-) + (b-c+ax)\,\mathrm{F}. \tag{3.4}$$

An interesting application of these differential forms are the definition of the Jacobi polynomials of degree n,

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,\mathrm{F}(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1-x}{2})$$

which by putting $\alpha = \beta = 0$ yields the Rodrigues formula for the Legendre polynomials as described in (2.4), (see [2, Definition 2.5.1] for further details).

3.3 Elliptic integrals

For the proofs given in Paper *B* we use the complete elliptic integrals of first and second kind denoted K and E respectively. These are defined as

$$\mathrm{K}(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k\sin^2\theta}}\,d\theta,$$

$$\mathrm{E}(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k\sin^2\theta}\,d\theta,$$

for $k \in (0,1)$. They are closely related to the hypergeometric functions, as the binomial expansion of $(1-k\sin^2\theta)^{\pm\frac{1}{2}}$ implies

$$\mathrm{K}(k) = \frac{\pi}{2}\,\mathrm{F}(\tfrac{1}{2},\tfrac{1}{2};1;k),$$

$$\mathrm{E}(k) = \frac{\pi}{2}\,\mathrm{F}(\tfrac{1}{2},-\tfrac{1}{2};1;k).$$

The two mentioned contiguous relations for the hypergeometric functions (3.3), (3.4) yield the following differential formulas of the elliptic integrals

$$\frac{d}{dk}\,\mathrm{K}(k) = \frac{\mathrm{E}(k) - (1-k)\mathrm{K}(k)}{2k(1-k)}$$

and

$$\frac{d}{dk}\,\mathrm{E}(k) = \frac{\mathrm{E}(k) - \mathrm{K}(k)}{2k}.$$

3.4 Paper B

As described in Section 3.1 the Minkowski tensors play a crucial role, when considering tensor-valued continuous isometry covariant valuations on $\mathcal{K}^n$. Their ability to capture certain shape and orientation information of the underlying set is the focus of Paper *B*. In [9] it was proven that a full-dimensional centered ellipsoid is uniquely determined by its rank-2 volume tensor. This result follows as a full-dimensional centered ellipsoid E is an affine transformation of the unit ball, hence there is a symmetric positive definite $n \times n$-matrix A such that $E = AB^n$. Spectral decomposition of A yields the desired uniqueness and solving a system of equations gives an explicit reconstruction of E given $\Phi_n^{2,0}(E)$.
This uniqueness result found an application in stereology, where the Miles ellipsoid was defined in [21] for $n = 3$. Here the volume tensors up to rank 2 were used to associate an ellipsoid to a stationary marked point process giving an estimate of the average shape and orientation of the typical particle (or mark). The polynomial translation behaviour of the volume tensors implies that the volume tensors are dependent on both position and shape of the underlying set. Now, the translation covariance of the surface area measure implies that the surface tensors are translation invariant, and they therefore only depend on the shape of the underlying set. A natural continuation of the work on volume tensors is therefore to consider if similar results can be reached for the surface tensors, i.e. if the rank-2 surface tensor uniquely determines a centered ellipsoid. In Paper *B* we focus on the two-dimensional case and prove that in this setting the result holds. A sketch of the proof is as follows.
The spectral decomposition combined with motion covariance of the surface area measure allow us to restrict considerations to centered ellipses which are *axis-parallel*, meaning ellipses with principal axis along the standard basis of $\mathbb{R}^2$. The surface area measure of an ellipse is connected to the support function of E through the Laplace-Beltrami operator, as

$$\mathrm{S}_1(E, \cdot) = \mathrm{h}_E(\cdot) + \Delta_S \mathrm{h}_E(\cdot) := \square \mathrm{h}_E(\cdot).$$

As mentioned in Section 2.3, the Laplace-Beltrami operator is self-adjoint, and so the problem simplifies to proving that E is uniquely determined by the following rank-2 tensor

$$\int_{S^1} u^2 h_E(u) du. \tag{3.5}$$

Representing (3.5) as a diagonal matrix and using (1.1) imply that each diagonal entry in (3.5) can be expressed as a linear combination of the complete elliptic integrals of the first and second kind. Properties of these elliptic integrals finally yield that E is uniquely determined by $\Phi_{n-1}^{0,2}(E)$.
This result also implies that a centered ellipse is uniquely determined by its first three non-trivial Fourier coefficients. The proof of uniqueness further yields an algorithm for reconstructing the underlying centered ellipse given its rank-2 surface tensor. In Paper B, an algorithm allowing for an error of the inversion is also given, meaning an algorithm constructing an ellipse $\tilde{E}$ from $\Phi_{n-1}^{0,2}(E)$ such that the Hausdorff distance between E and $\tilde{E}$ is no larger than a fixed $\varepsilon > 0$.
The established uniqueness and reconstruction are then used in relation to stationary particle processes X with convex particles (or stationary marked point processes) in

$\mathbb{R}^2$. Here Paper *B* introduces a counterpart to the Miles ellipse by the *Blaschke ellipse* $\mathrm{e_B}(X)$, which is the associated centered ellipse with rank-2 surface tensor equal to the rank-2 average surface tensor of the typical particle, $\mathbf{K}_0$, i.e.

$$\Phi_{n-1}^{0,2}(\mathrm{e_B}(X)) = \Phi_{n-1}^{0,2}(\mathbb{E}\,\mathbf{K}_0). \tag{3.6}$$

The name "Blaschke" is chosen as the right hand side of (3.6) is the surface tensor of the Blaschke body of X (up to multiplication with the intensity of X) due to Minkowski's existence theorem (see for instance [18]).
Paper *B* continues by introducing an estimator of the right hand side of (3.6) which is ratio-unbiased.
Now the mean surface particle tensor of rank 2 contains information on both shape and orientation. However, this information is intertwined, which implies e.g. that a process consisting of the same particle K, rotated isotropically, will yield a ball as Blaschke ellipse. We therefore introduce another ellipse called the *mean shape ellipse* as an alternative. Here we treat the orientation and shape information separately by letting the length of each principal axis of this ellipse be the average of each principal axis lengths of the typical particle and the orientation be determined by the mean direction. Examples of the performance of the two ellipses can be found in Paper *B*.

3.5 Paper C

As in Paper *B* this paper considers the mean volume and surface particle tensors for stationary random collections of particles. However, in this paper we work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n > 2$. More specifically, we consider stationary marked point processes X with marks in $\mathcal{K}^n$ and the typical particle being the random convex body $\mathbf{K}_0$ with distribution equal to the mark distribution. We will restrict our considerations to processes where the distribution of the typical particle is invariant under rotations fixing a linear subspace L pointwise, which we call *L-restricted isotropy*.
For such a process X, with $L \in G(n,k)$, $k < n-1$, which satisfies some appropriate integrability assumptions, we prove that the mean volume particle tensor $\mathbb{E}\Phi_n^{r,0}(\mathbf{K}_0)$ can be derived from $(k+1)$-dimensional sections. This follows from the decomposition of $\mathbb{R}^n$ into L and $L^\perp$ and some basic integral transformations. The result is a generalization of the result given in [14], where the case $n = 3$ and $k = 1$ was treated.
Under the further assumption of X being non-degenerate, a similar result is proven for the mean surface particle tensor, $\mathbb{E}\Phi_n^{0,s}(\mathbf{K}_0)$. The proof of this is a bit more involved than the proof of the result for the mean volume particle tensors. In fact, we need a proposition saying that if a convex body K is invariant under rotations fixing L pointwise, then the surface area measure of K can be expressed as an integral relation dependent on the support measure in $(k+1)$-dimensional sections. We prove this using the Steiner formula for surface area measures combined with multiple integral transformations. This proposition now yields that the mean surface particle tensor can be derived from $(k+1)$-dimensional sections.
As previously mentioned the mean volume particle tensors up to rank 2 have been used in stereology to yield estimators of the average shape and orientation for the typical particle through e.g. the Miles ellipsoid. In Paper *B* we proved, that in the two-dimensional setting the mean surface particle tensor of rank 2 could also be used

to introduce ellipsoidal set-valued summary statistics, as it too uniquely determines centered ellipses. In Paper *C* we prove that the surface tensor of rank 2 is a homeomorphism from the set of all centered ellipsoids of revolution of dimension at least $n-1$, $n > 2$, to the set of all symmetric positive semi-definite $n \times n$-matrices with one eigenvalue of multiplicity $n-1$, and the other being positive, i.e. all centered ellipsoids of at least dimension $n-1$, which are invariant under rotations fixing a 1-dimensional linear subspace pointwise, are uniquely defined by their rank-2 surface tensor. This result relies heavily on the theory of hypergeometric functions (see Section 3.2), as the eigenvalues of the rank-2 surface tensor of a centered ellipsoid of revolution can be expressed in terms of these functions. This is due to the relation between the surface area measure and the Gauss-Kronecker curvature mentioned in Section 1.
As in the two dimensional setting an algorithm for constructing the centered ellipsoid of revolution given its rank-2 surface tensor can be introduced. Also, an algorithm allowing for a fixed error $\varepsilon > 0$ is introduced, meaning an algorithm where the Hausdorff distance between the constructed ellipsoid and the "real" ellipsoid does not exceed ε.
The last section of the paper concerns stationary marked point processes X, with marks in $\mathcal{K}^n$. Here we introduce associated ellipsoids based on the mean volume particle tensor up to rank 2 and the mean surface particle tensor of rank 2. We start by introducing the Miles ellipsoid, (known when $n = 3$ from [21]), and the inertia ellipsoid. These are both determined using the average volume tensors of the typical particle up to rank 2. Their difference lie in the way they center the typical particle before taking the average rank-2 volume tensor. The Miles ellipsoid centers the typical particle at

$$\bar{c} = \frac{\mathbb{E}\Phi_n^{1,0}(\mathbf{K}_0)}{\mathbb{E}V_n(\mathbf{K}_0)}.$$

If $\mathbf{K}_0$ is deterministic and full-dimensional, $\bar{c}$ is the center of mass of $\mathbf{K}_0$. However if $\mathbf{K}_0$ is random, then $\bar{c}$ might differ from the average center of mass of $\mathbf{K}_0$, $\mathbb{E}(\Phi_n^{1,0}(\mathbf{K}_0)/V_n(\mathbf{K}_0))$. Hence, in practice, if the reference points are systematically away from the centers of mass of the marks, the Miles ellipsoid might not be representative for the average shape of the typical particle. As the inertia ellipsoid is based on averaging the marks with center of mass at the origin, it does not suffer from this problem. However, in practice, estimation of the centers of mass of individual sampled particles can be subject to large variances.
When X is L-restricted isotropic with $L \in G(n,1)$, the previous mentioned homeomorhism defined by the rank-2 surface tensor allows for the introduction of a Blaschke ellipsoid $\mathrm{e_B}(X)$ as the centered ellipsoid of revolution with $\mathbb{E}\Phi_{n-1}^{0,2}(\mathbf{K}_0) = \Phi_{n-1}^{0,2}(\mathrm{e_B}(X))$, which corresponds to the definition given in Paper *B*. We conclude Paper *C* with a discussion of different estimation procedures for these three ellipsoids.

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PAPER A

Uniqueness of the Measurement Function in Crofton's Formula

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This paper is dedicated to the memory of Wolfgang Weil,
a kind colleague and teacher.

Abstract

Crofton's intersection formula states that the $(n-j)$th intrinsic volume of a compact convex set in $\mathbb{R}^n$ can be obtained as an invariant integral of the $(k-j)$th intrinsic volume of sections with k-planes. This paper discusses the question if the $(k-j)$th intrinsic volume can be replaced by other functionals, that is, if the measurement function in Crofton's formula is unique.

The answer is negative: we show that the sums of the $(k-j)$th intrinsic volume and certain translation invariant continuous valuations of homogeneity degree k yield counterexamples. If the measurement function is local, these turn out to be the only examples when $k = 1$ or when $k = 2$ and we restrict considerations to even measurement functions. Additional examples of local functionals can be constructed when $k \geq 2$.

MSC: 52A20; 52A22; 53C65

Keywords: Crofton's formula; Klain functional; Local functions; Spherical lifting; Uniqueness; Valuation

A.1 Introduction and main results

A.1.1 Uniqueness of local measurement functions in Crofton's formula

The classical Crofton formula [20] for compact convex sets K states that the invariantly integrated j-th intrinsic volume V_j of the intersection of K with a k-dimensional flat E is essentially an intrinsic volume of K:

$$\int_{A(n,k)} V_j(K \cap E)\mu_k(dE) = \alpha_{n,j,k} V_{n+j-k}(K). \tag{A.1.1}$$

Here μ_k is an (appropriately normalized) invariant measure on the space $A(n,k)$ of all k-flats (k-dimensional affine subspaces of $\mathbb{R}^n$), $\alpha_{n,j,k} > 0$ is a known constant and $0 \le j \le k \le n-1$.

We will make use of the following notation. For a linear topological space X of finite dimension, we will write $\mathcal{B}(X)$ for the Borel σ-algebra on X and denote the family of all compact convex subsets of X by $\mathcal{K}(X)$. We will write $\mathcal{K}_k(X)$ for the subfamily of all such sets of dimension at most k, $0 \le k \le \dim X$. Clearly, $\mathcal{K}_{\dim X}(X) = \mathcal{K}(X)$. In contrast to the standard literature (e.g. [20]) we include the empty set in these classes.

To simplify notation, we introduce *the Crofton operator* $\mathrm{C_k} : (\mathcal{K}_k(\mathbb{R}^n))^{\mathbb{R}} \to (\mathcal{K}(\mathbb{R}^n))^{\mathbb{R}}$ by

$$(\mathrm{C_k}\,\varphi)(K) = \int_{A(n,k)} \varphi(K \cap E)\mu_k(dE), \quad K \in \mathcal{K}(\mathbb{R}^n) \tag{A.1.2}$$

for a *measurement function* $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$. Here and in the rest of the paper we assume that $E \mapsto \varphi(K \cap E)$ is integrable for all $K \in \mathcal{K}(\mathbb{R}^n)$. Due to (A.1.1) there exists a measurement function φ solving $\mathrm{C_k}(\varphi) = V_j$ for any $j \in \{n-k, \dots, n\}$. The purpose of the present paper is to discuss uniqueness of such a solution, possibly under additional restrictions on φ. As $\mathrm{C_k}$ is linear, the equality $\mathrm{C_k}(\varphi) = V_j$ has at most one solution if and only if its kernel $\ker \mathrm{C_k}$ is trivial. We will therefore describe properties of the kernel of the Crofton operator.

Unless otherwise stated, we will assume that $n \in \{2, 3, \dots\}$ and $k \in \{1, \dots, n-1\}$, thereby excluding the trivial case $k = 0$. However, the case $k = 0$ will be discussed when measurement functions on smaller domains are considered; see Subsection A.1.2.

For general $k \in \{1, \dots, n-1\}$ the kernel of $\mathrm{C_k}$ is not trivial and we will give non-vanishing examples of measurement functions in $\ker \mathrm{C_k}$ later. We therefore impose additional assumption on φ, which are typically geometrically motivated. A set of rather strong assumptions would be the defining properties of the intrinsic volumes: continuity, motion invariance and additivity. However, due to Hadwiger's characterization theorem of the intrinsic volumes, applied in k-flats, such a measurement function must be a linear combination of $V_0, \dots, V_k$, and thus $\varphi \in \ker \mathrm{C_k}$ if and only if $\varphi \equiv 0$ by (A.1.1).

Can the assumptions imposed on φ be relaxed? The first result shows that there are non-trivial elements in $\ker \mathrm{C_k}$, if the motion invariance is weakened and replaced by translation invariance.

To state the result let $\operatorname{lin} K = \operatorname{aff} K - x$, where $\operatorname{aff} K$ is the affine hull of K, and x is an arbitrary element of $K \in \mathcal{K}(\mathbb{R}^n) \setminus \{\emptyset\}$. We will write ν_k for the invariant probability measure on the Grassmannian $G(n,k)$ of k-dimensional linear subspaces of $\mathbb{R}^n$.

Proposition A.1.1. *Let $f : G(n,k) \to \mathbb{R}$ be a ν_k-integrable function and define*

$$\varphi_f(K) = \begin{cases} V_k(K) f(\operatorname{lin} K), & \text{if } \dim K = k, \\ 0, & \text{otherwise,} \end{cases}$$

for $K \in \mathcal{K}_k(\mathbb{R}^n)$. Then

(i) *φ_f is translation invariant and additive.*

(ii) *If f is continuous, then φ_f is continuous.*

(iii) *If $f \not\equiv 0$ then $\varphi_f \not\equiv 0$.*

(iv) *We have*

$$\int_{G(n,k)} f(L) \nu_k(dL) = 0 \tag{A.1.3}$$

if and only if $\mathrm{C_k}(\varphi_f) \equiv 0$ *on* $\mathcal{K}(\mathbb{R}^n)$.

The aim of the following considerations is to introduce a natural geometric property and impose it to the measurement function φ. The Crofton formula and a number of other integral geometric relations are widely used in geometric sampling. Many of the stereological estimators obtained this way share a local property. Roughly speaking, this means that they can be seen as sums or integrals of contributions which only depend on an infinitesimal neighbourhood of the location considered. This is not only true for volume and surface area estimators under IUR sampling [3] but also under vertical and local designs. For instance the nucleator and the surfactor, defined as in [12], are of this type. Wolfgang Weil gave in [27] a formal definition of the local property (basing it on his work with polytopes in [26]), which we will recall below. He named functionals which have the local property and are in addition continuous and translation invariant, *local functionals* on $\mathcal{K}(\mathbb{R}^n)$. Among other things, he showed that every local functional φ is a *standard functional* in the sense of translation invariant valuation theory, that is, φ is continuous, translation invariant and additive on $\mathcal{K}(\mathbb{R}^n)$. It is an open problem if every standard functional is local, however the two notions are indeed equivalent for $n \in \{1,2\}$; see for instance Proposition A.2.3 combined with equation (A.1.8). As our main focus is Crofton formulae in $\mathbb{R}^3$, where only planes of dimension $k \in \{1,2\}$ are of practical interest, we thus could have developed our theory using standard functionals. An exact definition of local functionals on $\mathcal{K}(\mathbb{R}^n)$ is given in Definition A.2.1 in Section A.2.3, below. We already mention here that φ has the local property if it satisfies the following condition. For each $K \in \mathcal{K}(\mathbb{R}^n)$ there exists a finite signed Borel measure $\Phi(K,\cdot)$, such that $\varphi(K) = \Phi(K, \mathbb{R}^n)$, and $\Phi(K,\cdot)$ is local, meaning that the intersection of K with an open neighbourhood of a Borel set A already determines $\Phi(K,A)$. The transition kernel Φ is called a local extension of φ. For our considerations we need to extend this definition to include functionals only acting on compact convex sets of dimension at most k. A natural way of doing this is to consider the restrictions of φ to compact convex subsets of linear subspaces $L \in G(n,k)$ and to require them to be local in the sense of Definition A.2.1, when identifying L with $\mathbb{R}^k$. However, this would only give us translation invariance of the functional in each $L \in G(n,k)$ and not necessarily in all of $\mathbb{R}^n$. We therefore say that a functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is local if it is translation invariant and each restriction of φ to subsets of a k-dimensional linear subspace L are local in L in the sense of Wolfgang Weil; see Definitions A.2.1 and A.2.2 for details. Note that a local functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is continuous on $\mathcal{K}(L)$ for all $L \in G(n,k)$.

Our first main result is an extension of [27, Theorem 2.1] to local functionals on $\mathcal{K}_k(\mathbb{R}^n)$, $k \in \{1,\dots,n\}$ and it gives a decomposition of a local functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ into homogeneous, local functionals $\varphi^{(j)}$, $j = 0,\dots,k$.

Before stating this result we need to fix some notation. Let κ_k be the k-dimensional volume of the unit ball in $\mathbb{R}^k$. For each $L \in G(n,k)$, define the Euclidean unit ball in L to be $B_L = B^n \cap L$, where B^n is the unit ball in $\mathbb{R}^n$ and let $S^L_{k-1}(K,\cdot)$ be the $(k-1)$th surface measure of a compact convex set $K \in \mathcal{K}(L)$ with L as ambient space. Let $\tilde{\wp}^{\mathrm{k-1}}_{\mathrm{j-1}}(L)$ be the family of spherical polytopes of dimension at most $j-1$ in L, where the dimension of a spherical polytope is defined to be one smaller than the dimension j of its positive hull in L. For $k = n$, we simplify notation and write $\tilde{\wp}^{\mathrm{n-1}}_{\mathrm{j-1}} = \tilde{\wp}^{\mathrm{n-1}}_{\mathrm{j-1}}(\mathbb{R}^n)$. For a polytope P, let $\mathcal{F}_j(P)$ be the collection of j-faces of P, $j = 0,\dots,n$ and for $F \in \mathcal{F}_j(P)$ we denote by λ_F the restriction to F of the Lebesgue measure in the affine hull of F. We further let $N_L(P,F)$ be the normal cone of $P \subset L$ at F in the subspace L and let $n_L(P,F)$ denote the intersection of the unit sphere S^{n-1} with $N_L(P,F)$. For $k = n$, we write, $n(P,F) = n_{\mathbb{R}^n}(P,F)$. A function $\psi : S^{n-1} \cap L \to \mathbb{R}$ is called *centered* if

$$\int_{S^{n-1}\cap L} u\psi(u)\mathcal{H}^{k-1}(du) = 0,$$

where $\mathcal{H}^{k-1}$ is the $(k-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Theorem A.1.2. *Let $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ be a local functional with local extension $\Phi_L : \mathcal{K}(L) \times \mathcal{B}(L) \to \mathbb{R}$ for each $L \in G(n,k)$. Then φ has a unique representation*

$$\varphi(K) = \sum_{j=0}^{k} \varphi^{(j)}(K) \tag{A.1.4}$$

with j-homogeneous local functionals $\varphi^{(j)}$ on $\mathcal{K}_k(\mathbb{R}^n)$.

In addition, for each $L \in G(n,k)$ there is a decomposition

$$\Phi_L(K,\cdot) = \sum_{j=0}^{k} \Phi^{(j)}_L(K,\cdot), \tag{A.1.5}$$

$K \in \mathcal{K}(L)$, such that $\Phi^{(j)}_L$ is a local extension of $\varphi^{(j)}$ restricted to L, for $j = 0,\dots,k$.

For a polytope $P \in \mathcal{K}(L)$, each $\Phi^{(j)}_L$ has the form

$$\Phi^{(j)}_L(P,\cdot) = \sum_{F\in\mathcal{F}_j(P)} g^{(j)}_L(n_L(P,F))\lambda_F, \tag{A.1.6}$$

where $g^{(j)}_L : \tilde{\wp}^{k-1}_{k-j-1}(L) \to \mathbb{R}$, $j = 0,\dots,k$, are (uniquely determined by $\Phi^{(j)}_L$) simple additive and continuous functions, the so-called associated functions of Φ_L.

Moreover we have

$$\varphi^{(k)}(K) = c^{(k)}_L V_k(K) \tag{A.1.7}$$

for all $K \in \mathcal{K}(L)$, where $c^{(k)}_L = \varphi^{(k)}(\kappa_k^{-1/k} B_L)$ and

$$\varphi^{(0)}(K) = c^{(0)} V_0(K) \tag{A.1.8}$$

for $K \in \mathcal{K}_k(\mathbb{R}^n)$, where $c^{(0)} = \varphi^{(0)}(\{0\})$. If further $x \in L^\perp$, then

$$\varphi^{(k-1)}(K) = \int_{S^{n-1}\cap L} \theta(L,v) S^L_{k-1}(K-x,dv) \tag{A.1.9}$$

for all $K \in \mathcal{K}(L+x)$, where $\theta(L,v) = g_L^{(k-1)}(\{v\})$, is continuous in v and centered for fixed L.

Note that (A.1.4)-(A.1.7) is an extension of [27, Theorem 2.1] to functionals acting on the subfamilies $\mathcal{K}_k(\mathbb{R}^n)$ of $\mathcal{K}(\mathbb{R}^n)$, $k < n$. However, continuity of $g_L^{(j)}$ appears to be a new result.

By linearity of the Crofton operator C_k, a local functional is in the kernel of C_k if and only if each homogeneous functional in its decomposition is in the kernel of C_k. By (A.1.7),(A.1.8) and (A.1.9) we have explicit descriptions of local functionals acting on compact convex sets of dimension at most 2. Using these descriptions our main result can be proven. It shows that the only local functionals in $\ker \mathrm{C}_1$ and the only even local functionals in $\ker \mathrm{C}_2$ are the examples given in Proposition A.1.1. This result cannot be improved as there are other examples in all remaining cases.

Theorem A.1.3. *Let $n \in \mathbb{N}$ and $k \le n-1$ be given.*

(1) *For $k = 1$ the local functionals in* $\ker \mathrm{C}_1$ *are precisely the functionals $\varphi = \varphi_f$ with some $f : G(n,k) \to \mathbb{R}$ satisfying* (A.1.3).

(2) *For $k = 2$ the* **even** *local functionals in* $\ker \mathrm{C}_2$ *are precisely the functionals $\varphi = \varphi_f$ with some $f : G(n,k) \to \mathbb{R}$ satisfying* (A.1.3).

(3) *For $k \ge 2$ there is a local functional φ of homogeneity degree $k-1$ in* $\ker \mathrm{C}_\mathrm{k}$, *which is not trivial, as there exists $K \in \mathcal{K}(\mathbb{R}^n)$ such that $\{E \in A(n,k) : \varphi(K \cap E) \neq 0\}$ is not a set of μ_k-measure zero.*

Thus we have a complete description of the kernel of the Crofton operator, when considering local and even functionals acting on intersections of compact convex sets in $\mathbb{R}^n$ with either 1- or 2-dimensional flats. The proof of the theorem makes use of the decomposition of φ into homogeneous parts (Theorem A.1.2), which reduces the problem to considering homogeneous local functionals in the kernel of the Crofton operator. The explicit expression for the 0-homogeneous functional $\varphi^{(0)}$ in (A.1.7) shows that $\varphi^{(0)} \in \ker \mathrm{C}_\mathrm{k}$ if and only if $\varphi^{(0)} = 0$. Together with (A.1.8) this gives the claim for $k = 1$. For $k = 2$ and φ being even rewriting the 1-homogeneous functional gives a connection to the kernel of the Radon transform on Grassmannians, which has well understood properties (see, for instance [7]). Using injectivity properties of this transform, the 1-homogeneous functional must vanish, yielding Theorem A.1.3(2).

The relation to the Radon transform also leads to the existence of a non-trivial $(k-1)$-homogeneous local functional in the kernel of the Crofton operator, when $k > 2$, hence yielding Theorem A.1.3(3) for $k > 2$. For $k = 2$ we start by explicitly constructing a 1-homogeneous local functional in the kernel of the Crofton operator when $n = 3$. Examples in higher dimensions are then constructed by averaging a three-dimensional counterexample over all three-dimensional subspaces containing a given $K \in \mathcal{K}_2(\mathbb{R}^n)$; for details see Section A.2.3. This yields Theorem A.1.3(3).

We conclude this section with some remarks connecting our results to the more algebraic oriented convex geometry literature. Alesker [1] defined a product on the space Val^∞ of *smooth* standard functionals (which is a dense subspace of the suitable normed space of *all* standard functionals). For any two smooth convex bodies K_1 and K_2 with positive Gauss curvature at every boundary point, the Alesker product of the smooth functionals

$$K \mapsto \varphi_i(K) = V_n(K + K_i), \tag{A.1.10}$$

$i = 1, 2$, is given by

$$\varphi_1 \cdot \varphi_2(K) = V_{2n}\Big(\Delta K + (K_1 \times K_2)\Big), \quad K \in \mathcal{K}(\mathbb{R}^n),$$

where $\Delta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, is the diagonal embedding. This property, combined with bilinearity and continuity, determines the Alesker product uniquely on Val^∞. The Crofton integral of a smooth standard functional φ can be expressed by means of Alesker's product:

$$\mathrm{C}_k\,\varphi = \alpha_{n,0,k} V_{n-k} \cdot \varphi; \tag{A.1.11}$$

with the constant $\alpha_{n,j,k}$ from (A.1.1). For a proof, see [4, Equations (2) and (16)]. Hence, a smooth standard functional φ is an element of $\ker \mathrm{C}_k$ if and only if $V_{n-k} \cdot \varphi = 0$. One might expect that our constructions, possibly slightly modified, also yield non-trivial smooth standard functionals satisfying this equation. We give one particular example showing that this is generally not the case. By the hard Lefschetz theorem (see [2, Theorem 3.12] and the references therein) the only smooth standard functional φ of homogeneity degree $j < n/2$ satisfying $V_1 \cdot \varphi = 0$ is trival. Hence, in contrast to Theorem A.1.3(3), there is no non-trivial 1-homogeneous smooth standard functional in $\ker \mathrm{C}_2$ in $\mathbb{R}^3$.

The tools of algebraic integral geometry appear not to be directly applicable for local functionals on $\mathcal{K}_k(\mathbb{R}^n)$ – note that such functionals need not even be continuous. However, at least some of our results are related or can even be proven using Alesker's product, and we illustrate this below by an example. Typically, as products of smooth standard functionals are difficult to evaluate, (A.1.11) is used to establish Crofton formulae by employing the Alesker–Fourier transform $\mathbb{F}$ on Val^∞. It satisfies the Plancherel-type formula $\mathbb{F}^2\varphi = \check{\varphi}$, where $\check{\varphi}(K) = \varphi(-K)$, $K \in \mathcal{K}(\mathbb{R}^n)$. In addition, we have

$$\mathbb{F}(\varphi * \varphi') = \mathbb{F}\varphi \cdot \mathbb{F}\varphi',$$

for all $\varphi, \varphi' \in \mathrm{Val}^\infty$. Here, the convolution product $*$ is uniquely determined by bilinearity, continuity and the requirement that

$$\varphi_1 * \varphi_2(K) = V_n\Big(K + K_1 + K_2\Big), \quad K \in \mathcal{K}(\mathbb{R}^n), \tag{A.1.12}$$

holds for any two standard functionals as defined in (A.1.10). Hence, if the convolution product ψ of $V_k = \mathbb{F}V_{n-k}$ and $\mathbb{F}\check{\varphi}$ can be determined, $\alpha_{n,0,k}\mathbb{F}\psi = \mathrm{C}_k\,\varphi$ is a Crofton integral for φ.

This can for instance be used to generalize our observation that the standard functional of the form (A.2.20) in the kernel of the Crofton operator must be trivial. We now sketch a proof showing that the smooth standard functional

$$\varphi_{j,\omega}(K) = \int_{S^{n-1}} \omega(u) S_j(K, du),$$

where ω is a smooth function on the sphere, can only be in the kernel of the Crofton operator C_k, $k > j$, when it is trivial. Such standard functionals are called *spherical valuations*; see [23] and [22]. It follows from [5, Theorem 1] that the Alesker–Fourier transform of the (non-vanishing) spherical valuation $\check{\varphi}_{j,\omega}$ is a (non-vanishing) spherical valuation $\varphi_{n-j,\tilde{\omega}}$, so we have to show that $V_k * \varphi_{n-j,\tilde{\omega}} = 0$ only has the trivial solution. In view of [20, Lemma 1.7.8], the function $\tilde{\omega}$ can be written as difference of support functions of smooth convex bodies M and $\tilde{M}$ with positive Gauss curvature at every boundary point. Hence $\varphi_{n-j,\tilde{\omega}}(K)$ is the difference of two mixed volumes; see [20, (5.19)]. The convolution product of mixed volumes is easy to determine by applying Steiner's formula and the characterizing property (A.1.12); see [4, Section 3.8]. Using this result, $V_k * \varphi_{n-j,\tilde{\omega}} = 0$ turns out to be equivalent to

$$V(K[k-j], M, B^n[n-k+j-1]) = V(K[k-j], \tilde{M}, B^n[n-k+j-1]);$$

see [20, p. 284] for notation. As this equality must hold for all $K \in \mathcal{K}(\mathbb{R}^n)$, the convex bodies M and $\tilde{M}$ must coincide up to translation by the multi-linearity of mixed volumes and [20, Theorem 7.6.2]. Thus, $\varphi_{n-j,\tilde{\omega}} = 0$, as claimed.

A.1.2 Variations: Measurement functions on smaller domains

The problem becomes more involved when considering functionals acting on specific subsets of $\mathcal{K}(\mathbb{R}^n)$. To discuss this more general setting, we fix $\mathcal{M} \subset \mathcal{K}(\mathbb{R}^n)$ and consider the collection

$$\mathcal{M}_k = \{K \cap E : K \in \mathcal{M}, E \in A(n,k)\}$$

of section profiles of sets in $\mathcal{M}$ with k-flats. The uniqueness of the measurement function $\varphi : \mathcal{M}_k \to \mathbb{R}$ in a suitable Crofton formula is again equivalent to a trivial kernel $\ker \mathrm{C_k}$, where the Crofton operator $\mathrm{C_k}$ is now a function from $\mathcal{M}_k^{\mathbb{R}}$ to $\mathcal{M}^{\mathbb{R}}$ defined by (A.1.2), but with $K \in \mathcal{M}$.

To appreciate the difficulty of the problem consider $k = 0$ and let $f(x) = \varphi(\{x\})$. The measurement function $\varphi : \mathcal{M}_k \to \mathbb{R}$ is an element of $\ker \mathrm{C}_0$ if and only if $\varphi(\emptyset) = 0$ and

$$\int_K f(x)dx = 0 \tag{A.1.13}$$

for all $K \in \mathcal{M}$. When $\mathcal{M}$ is 'large', for instance when it contains all axis-parallel cubes, we obviously have $f = 0$ almost everywhere by Dynkin's lemma (see, [6, Theorem 1.6.1]). However if

$$\mathcal{M} = \{gK_0 : g \text{ is a rigid motion in } \mathbb{R}^n\}$$

consists of all rigid motions of a fixed non-empty compact convex set K_0, the existence of non-vanishing functions f is not trivial at all. When K_0 is a Euclidean ball, a non-vanishing solution f of (A.1.13) can be given in terms of a Bessel function, and all solutions (within the Schwartz class of distributions) can be characterized. Whether there are other sets K_0 for which (A.1.13) has a non-vanishing solution f, is the *Pompeiu problem*; see, for instance [17]. This long-standing problem is still open in arbitrary dimension, but the case $n = 3$ has been settled by Ramm [18] even without convexity assumptions. His result implies that if K_0 has C^1-smooth boundary, but is not a Euclidean ball, and (A.1.13) holds for all $K \in \mathcal{M}$, then $f = 0$ almost everywhere.

We cannot solve this uniqueness problem in full generality, but state a result in the special case where $\mathcal{M}$ consist of all n-dimensional balls. We restrict attention to motion invariant measurement functions.

Theorem A.1.4. *Let $n \in \mathbb{N}$, $k \in \{0,\dots,n-1\}$ and $\mathcal{M}$ be the set of all n-dimensional balls and assume that $\varphi : \mathcal{M}_k \to \mathbb{R}$ is motion invariant. Then*

$$\int_{A(n,k)} \varphi(K \cap E)\mu_k(dE) = 0 \tag{A.1.14}$$

for all $K \in \mathcal{M}$ if and only if

$$\varphi(K \cap E) = 0$$

for μ_k-almost all $E \in A(n,k)$.

Remark that all compact convex sets of dimension 1 are balls and hence the above theorem states that the kernel of the Crofton operator is trivial when considering motion invariant functionals defined on all 1-dimensional compact, convex sets in $\mathbb{R}^n$.

The proof of this theorem makes use of the fact that a translation invariant functional of a lower dimensional ball does not depend on the center of the ball. Furthermore, due to rotation invariance each intersection $K \cap E$ can be replaced by a k-dimensional ball of equal radius within a fixed flat in $G(n,k)$. Hence φ only depends on the radius of the ball $E \cap K$. The proof of Theorem A.1.4 will be given in Section A.2.4. It exploits the fact that the left side of (A.1.14) can be written as a Riemann-Liouville integral whose injectivity properties are known.

A.1.3 Table of contents

The paper is structured as follows. In Section A.2.1 some preliminary definitions and notations are introduced. In Section A.2.2 the proof of Proposition A.1.1 is given by constructing non-zero functionals in the kernel of the Crofton operator. Section A.2.3 is devoted to considering local functionals. The definition of these is given and the first main result, Theorem A.1.2 is proven. The section ends with a proof of Theorem A.1.3, using the results of Theorem A.1.2. Finally, in Section A.2.4, we consider functionals on subsets of $\mathcal{K}_k(\mathbb{R}^n)$ and give a proof of Theorem A.1.4.

A.2 Proofs

A.2.1 Notation and preliminaries

Before giving the proofs of the above stated results we will introduce some further notation. Let $A \subset \mathbb{R}^n$, we will denote its boundary by $\operatorname{bd} A$, its interior by $\operatorname{int} A$ and its relative interior by $\operatorname{relint} A$. The orthogonal complement of A is given by $A^\perp$ and the convex hull by $\operatorname{conv}(A)$. If A is convex, its dimension is defined to be the dimension of its affine hull. The dual cone of A is given by

$$A^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 0 \,\forall y \in A\}.$$

Note that the dual of the convex cone $C = \{\alpha a : a \in A \text{ and } \alpha \ge 0\}$ of A satisfies $C^\circ = A^\circ$.

We have already introduced the notation $\tilde{\wp}^{n-1}_{j-1}$, the set of spherical polytopes in $\mathbb{R}^n$ of dimension at most $j-1$. We will make use of the subset $\wp^{n-1}_{j-1}$, consisting of all spherical polytopes of exact dimension $j-1$.

Throughout this paper we consider $G(n,k)$ and $A(n,k)$ endowed with the Fell topology, which induces Borel σ-algebras as in [21, p. 582]. Let $j,l \in \mathbb{N}$ with $j,l < n$ and consider $E \in A(n,l)$, ($E \in G(n,k)$). We define the space of all affine (linear) subspaces of dimension j incident to E by $A(E,j)$, ($G(E,j)$). As in [21, Section 13.2] we denote the appropriately normalized, invariant measures of these spaces by μ_j^E and ν_j^E, respectively. Further, for $L \in G(n,k)$, we let the set of all convex polytopes in L be denoted by $\mathcal{P}(L)$.

A.2.2 Construction of non-zero kernel functionals

Proof of Proposition A.1.1: Fix $f : G(n,k) \to \mathbb{R}$ and let φ_f be defined as in Proposition A.1.1. We show (i). By translation invariance of the intrinsic volume and the function $K \mapsto \operatorname{lin} K$, φ_f becomes translation invariant. To show additivity, we need to prove that

$$\varphi_f(K) + \varphi_f(M) - \varphi_f(K \cap M) = \varphi_f(K \cup M). \tag{A.2.1}$$

holds for all $K, M \in \mathcal{K}_k(\mathbb{R}^n)$ with $K \cup M \in \mathcal{K}_k(\mathbb{R}^n)$. Equation (A.2.1) is trivially true when $\dim(K \cup M) < k$, so we may assume $\dim(K \cup M) = k$. If $\dim K = \dim M = k$, then $\operatorname{lin} M = \operatorname{lin} K = \operatorname{lin}(K \cup M)$ and (A.2.1) follows by additivity of the intrinsic volume. This leaves us with the case where $\dim(K \cup M) = k$ and one of the sets has dimension strictly less than k. Without loss of generality we may assume that $j = \dim M < k$. As $\dim(K \cup M) = k$ then there exists $z \in K \setminus \operatorname{aff} M$. Since $K \cup M$ is convex we have

$$\{\alpha M + (1-\alpha)z : \alpha \in [0,1)\} \subset K \cup M \setminus \operatorname{aff} M.$$

As K is closed, we obtain $M \subset K$ and hence in this case (A.2.1) trivially follows. Thus φ_f is additive.

We now show (ii). Assume that f is continuous and let (K_m) be a sequence with $K_m \in \mathcal{K}_k(\mathbb{R}^n)$ converging to $K \in \mathcal{K}_k(\mathbb{R}^n)$. If $\dim K = k$ then $\operatorname{lin}(K_m)$ converges to $\operatorname{lin}(K)$ and so by continuity of the intrinsic volumes and f we get

$$\varphi_f(K_m) \to \varphi_f(K) \quad \text{for } m \to \infty.$$

If $\dim K < k$ then

$$|\varphi_f(K_m) - \varphi_f(K)| = |\varphi_f(K_m)| \leq \|f\|_\infty V_k(K_m) \to 0$$

as $m \to \infty$, where we used the facts that the continuous function f has a finite maximum norm on the compact set $G(n,k)$, and that V_k is continuous. Hence φ is continuous yielding (ii). As (iii) is obvious it remains to show (iv). For fixed $K \in \mathcal{K}(\mathbb{R}^n)$, $L \in G(n,k)$ and $x \in L^\perp$ remark that if $\dim(K \cap (L+x)) = k$ then $\operatorname{lin}(K \cap (L+x)) = L$ and otherwise $V_k(K \cap (L+x)) = 0$. Hence using Fubini's theorem

$$\int_{A(n,k)} |\varphi_f(K \cap E)| \mu_k(dE) = V_n(K) \int_{G(n,k)} |f(L)| \nu_k(dL).$$

Thus, the integrability of f implies the integrability of φ_f. The same arguments also show $[\mathrm{C_k}(\varphi_f)](K) = V_n(K) \int_{G(n,k)} f(L) \nu_k(dL)$, which clearly implies (iv). This finishes the proof of Proposition A.1.1. □

It should be remarked that the vector space of integrable real functions f on $G(n,k)$ satisfying (A.1.3) is infinite dimensional. Hence Proposition A.1.1 yields a large number of non-trivial functionals in $\ker \mathrm{C}_k$.

A.2.3 Local functionals

For the reader's convenience, we recall the definition of local functionals φ due to Wolfgang Weil in [27]. In contrast to [27] the empty set is an element of the domain of φ here.

Definition A.2.1. *A functional $\varphi : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ is called **local**, if it has a **local extension** $\Phi : \mathcal{K}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$, which is a measurable function on $\mathcal{K}(\mathbb{R}^n)$ in the first variable and a finite signed Borel measure on $\mathbb{R}^n$ in the second variable and such that Φ has the following properties:*

(i) *$\varphi(K) = \Phi(K, \mathbb{R}^n)$ for all $K \in \mathcal{K}(\mathbb{R}^n)$,*

(ii) *Φ is translation covariant, that is, $\Phi(K+x, A+x) = \Phi(K, A)$ for $K \in \mathcal{K}(\mathbb{R}^n)$, $A \in \mathcal{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

(iii) *Φ is locally determined, that is, $\Phi(K, A) = \Phi(M, A)$ for $K, M \in \mathcal{K}(\mathbb{R}^n)$, $A \in \mathcal{B}(\mathbb{R}^n)$, if there is an open set $U \subset \mathbb{R}^n$ with $K \cap U = M \cap U$ and $A \subset U$,*

(iv) *$K \mapsto \Phi(K, \cdot)$ is weakly continuous on $\mathcal{K}(\mathbb{R}^n)$ (w.r.t. the Hausdorff metric).*

If φ is local with local extension Φ, then $\Phi(\emptyset, \cdot) = 0$ due to (ii), and hence $\varphi(\emptyset) = 0$ by (i).

To prove Theorem A.1.2 we first need to extend the above definition to include functionals acting on compact convex subsets of $\mathbb{R}^n$ of dimension at most k, $k \in \{1, \ldots, n\}$.

Definition A.2.2. *A functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is called **local** if and only if φ is translation invariant and for all $L \in G(n,k)$, φ restricted to $\mathcal{K}(L)$ is local, i.e. $\varphi_L : \mathcal{K}(L) \to \mathbb{R}$, $K \mapsto \varphi(K)$ is local in the sense of Definition A.2.1 where L is identified with $\mathbb{R}^k$. The local extension of φ_L is denoted by $\Phi_L : \mathcal{K}(L) \times \mathcal{B}(L) \to \mathbb{R}$.*

More explicitly, choosing an orthonormal basis $u_1, \ldots, u_k$ of $L \in G(n,k)$, we can identify L with $\mathbb{R}^k$ using the isometry $\hat{}: \mathbb{R}^k \to L$, $a \mapsto \sum_{i=1}^k a_i u_i$. Then φ_L is local on L if and only if $\hat{\varphi}_L : \mathcal{K}(\mathbb{R}^k) \to \mathbb{R}$, $K \mapsto \varphi(\hat{K})$ is local on $\mathbb{R}^k$.

Now for $k = n$ Theorem A.1.2 was proven by Wolfgang Weil in [27] except for the continuity of the associated functions and equations (A.1.8) and (A.1.9), which both are consequences of this continuity property. We will therefore start out by proving the mentioned continuity in the case $k = n$. This will afterwards be used to prove Theorem A.1.2 for general $k \in \{1, \ldots, n\}$.

Proof of continuity of associated functions: Let $\varphi : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ be local with extension $\Phi : \mathcal{K}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$. The associated functions of Φ, $f^{(j)} : \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} \to \mathbb{R}$ are shown in [27] to vanish on $\tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} \setminus \wp^{\text{n-1}}_{\text{n-j-1}}$ for all $j \in \{0, \ldots, n\}$. Fix $j \in \{0, \ldots, n\}$ and consider the mapping $P : \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} \to \mathcal{P}(\mathbb{R}^n)$, $p \mapsto P(p) = p^\circ \cap Q$, where $Q = [-1/2, 1/2]^n$. This mapping is continuous which can be seen by decomposing it into the following three maps

$$p \mapsto \mathrm{conv}(p \cup \{0\}) \mapsto [\mathrm{conv}(p \cup \{0\})]^\circ = p^\circ \mapsto p^\circ \cap Q.$$

Continuity of the first and the last map is due to the continuity of the convex hull operator and the intersection operation of compact convex sets which cannot be separated by a hyperplane (see for instance [20, Theorem 1.8.10]), respectively. We note that the dual cone map on convex sets in the unit ball B^n is continuous in the Hausdorff metric by [24, Theorem 1]. The relationship between the Hausdorff metric and the Fell topology combined with the characterization of the Fell topology (see for instance [21, Theorem 12.2.2, 12.3.3]), yields continuity of the second map and therefore also continuity of P.

If $p \in \wp^{n-1}_{n-i-1}$ with $i \geq j$, then there is an i-face $F \in \mathcal{F}_i(P(p))$ with $0 \in \mathrm{relint}\, F$ and $n(P(p), F) = p$. Furthermore $F \cap \mathrm{int}\, Q \subset \mathrm{relint}\, F$ and all other i-faces of $P(p)$ do not hit $\mathrm{int}\, Q$. Now let (p_m) be a sequence in $\tilde{\wp}^{\text{n-1}}_{\text{n-j-1}}$ converging to $p \in \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}}$ and put $A = \epsilon B^n$ with $\epsilon < \frac{1}{2}$. The fact that $A \subset \mathrm{int}\, Q$ and the above observations imply

$$\begin{aligned} \mathbf{1}_{\{n-j-1\}}(\dim p_m) f^{(j)}(p_m) \epsilon^j \kappa_j &= \Phi^{(j)}(P(p_m), A) \\ &\to \Phi^{(j)}(P(p), A) = \mathbf{1}_{\{n-j-1\}}(\dim p) f^{(j)}(p) \epsilon^j \kappa_j \end{aligned} \tag{A.2.2}$$

as $m \to \infty$, where the weak continuity of $K \mapsto \Phi(K, \cdot)$ combined with the Portmanteau theorem was used, as $\Phi^{(j)}(P(p), \mathrm{bd}\, A) = 0$. As $f^{(j)}$ vanishes on $\tilde{\wp}^{\text{n-1}}_{\text{n-j-1}} \setminus \wp^{\text{n-1}}_{\text{n-j-1}}$, we have $\mathbf{1}_{\{n-j-1\}}(\dim q) f^{(j)}(q) = f^{(j)}(q)$ for all $q \in \tilde{\wp}^{\text{n-1}}_{\text{n-j-1}}$, so (A.2.2) implies the continuity of $f^{(j)}$. □

Proof of Theorem A.1.2: Let $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ be local in the sense of Definition A.2.2. Fix $L \in G(n,k)$ and $x \in L^\perp$. Let E denote the affine subspace $L + x$ and let $u_1, \dots, u_k$ denote an orthonormal basis of L. It follows that $\hat{\varphi}_L : \mathcal{K}(\mathbb{R}^k) \to \mathbb{R}$ is local by Definition A.2.2. Due to [27, Theorem 2.1], applied to $\mathbb{R}^k$, there are j-homogeneous local functionals $\hat{\varphi}^{(j)}_L$ on $\mathcal{K}(\mathbb{R}^k)$ such that $\hat{\varphi}_L = \sum_{j=0}^k \hat{\varphi}^{(j)}_L$. Furthermore, $\varphi^{(k)}_L = c^{(k)}_L V_k$ with some constant $c^{(k)}_L \in \mathbb{R}$, possibly depending on L. Let $\tilde{\ }: L \to \mathbb{R}^k$ be the inverse of $\hat{\ }: \mathbb{R}^k \to L$ and put $\varphi^{(j)}_L(K) = \hat{\varphi}_L(\widetilde{K - x})$ for $K \in \mathcal{K}(E)$ and $j = 0, \dots, k$. Then

$$\varphi(K) = \varphi_L(K - x) = \hat{\varphi}_L(\widetilde{K - x}) = \sum_{j=0}^{k-1} \varphi^{(j)}_L(K), \tag{A.2.3}$$

where $\varphi^{(j)}_L$ is local on L and j-homogeneous, and $\varphi^{(k)}_L = c^{(k)}_L V_k$.

For each $j \in \{0, \dots, k\}$ we make the above definition independent of L by defining the functionals $\varphi^{(j)}(K) = \varphi^{(j)}_L(K)$ for all $K \in \mathcal{K}_k(\mathbb{R}^n)$ contained in a translation of $L \in G(n,k)$. Note that this definition is independent of the choice of L. If $L, L' \in G(n,k)$ are such that K is contained in translates of L and L' then (A.2.3) and the homogeneity of $\varphi^{(j)}_L$ give

$$0 = \sum_{j=0}^{k} \alpha^j (\varphi^{(j)}_L(K) - \varphi^{(j)}_{L'}(K)),$$

for all $\alpha > 0$, implying $\varphi_L^{(j)}(K) = \varphi_{L'}^{(j)}(K)$. Hence (A.2.3) becomes (A.1.4). Representation (A.1.4) is unique, due to a standard homogeneity argument.

For the proof of the second part of the theorem, we let $L \in G(n,k)$. By [27, Theorem 2.1] there is a local extension, $\hat{\Phi}_L : \mathcal{K}(\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^k) \to \mathbb{R}$ of $\hat{\varphi}_L$ with a unique representation

$$\hat{\Phi}_L(K,\cdot) = \sum_{j=0}^{k} \hat{\Phi}_L^{(j)}(K,\cdot)$$

for $K \in \mathcal{K}(\mathbb{R}^k)$, such that $\hat{\Phi}_L^{(j)}$ is a local extension of $\hat{\varphi}_L^{(j)}$. Using the identification of L with $\mathbb{R}^k$ yields (A.1.5). Furthermore [27, Theorem 2.1] also gives that for a polytope $P \subset L$ and $A \in \mathcal{B}(L)$

$$\hat{\Phi}_L^{(j)}(\tilde{P},\tilde{A}) = \sum_{\tilde{F} \in \mathcal{F}_j(\tilde{P})} f_L^{(j)}(n(\tilde{P},\tilde{F}))\lambda_{\tilde{F}}(\tilde{A}).$$

We remark that $\tilde{F} \in \mathcal{F}_j(\tilde{P})$ if and only if $F \in \mathcal{F}_j(P)$ and by defining $g_L^{(j)} : \tilde{\wp}_{k-j-1}^{k-1}(L) \to \mathbb{R}$, $g_L^{(j)}(p) = f_L^{(j)}(\tilde{p})$, equation (A.1.6) follows from $n(\tilde{P},\tilde{F}) = \widetilde{n_L(P,F)}$ and

$$\Phi_L^{(j)}(P,A) = \hat{\Phi}_L^{(j)}(\tilde{P},\tilde{A}) = \sum_{F \in \mathcal{F}_j(P)} g_L^{(j)}(n_L(P,F))\lambda_F(A).$$

We have previously proved that $f^{(j)}$ is continuous for all j and so $g_L^{(j)}$ is continuous. Also by direct calculation it follows that $g_L^{(j)}$ inherits simple additivity from $f^{(j)}$ for all j and for any fixed local extension uniqueness follows by uniqueness of $f^{(j)}$ given in [27, Theorem 2.1]. This proves the asserted properties of the decomposition (A.1.6). We remark that

$$\varphi_L^{(j)}(P) = \hat{\Phi}_L^{(j)}(\tilde{P},\mathbb{R}^k) = \sum_{F \in \mathcal{F}_j(P)} g_L^{(j)}(n_L(P,F))V_j(F). \tag{A.2.4}$$

For $j = 0$, $V_0(\emptyset) = 0 = \varphi^{(0)}(\emptyset)$ and hence, for fixed $L \in G(n,k)$, we may consider a non-empty convex polytope $P \subset L$. Using (A.2.4) combined with the simple additivity of $g_L^{(0)}$ yields

$$\varphi_L^{(0)}(P) = \sum_{x \in \mathrm{vert}(P)} g_L^{(0)}(n_L(P,x)) = g_L^{(0)}(S^{n-1} \cap L),$$

where $\mathrm{vert}(P)$ denotes the set of vertices of the polytope P. Applying this twice, first with $P = \{0\}$, and then with an arbitrary $P \in \mathcal{P}(L)$, shows $\varphi_L^{(0)}(P) = \varphi^{(0)}(\{0\})V_0(P)$ for all $P \in \mathcal{P}(L)$. The functional $\varphi_L^{(0)}$ is local, so Definition A.2.1 (i) and (iv) imply that it is continuous on $\mathcal{K}(L)$. A standard approximation argument in L now shows

$$\varphi_L^{(0)} = \varphi^{(0)}(\{0\})V_0$$

on $\mathcal{K}(L)$, which yields (A.1.8).

Concerning the case $j = k$, we have already remarked after (A.2.3), that

$$\varphi^{(k)}(K) = \varphi_L^{(k)}(K) = c_L^{(k)} V_k(K)$$

for all $K \in \mathcal{K}(L)$ holds. □

For the proof of the case $j = k-1$ we will show the following more general result which essentially follows from Wolfgang Weil's paper [27] and McMullen's characterization of standard functionals of homogeneity degree $n-1$; see [16] and e.g. [2, Theorem 3.1(iii)]. Recall that a *standard* functional $\psi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is a continuous translation invariant valuation.

Proposition A.2.3. *Let $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ be given. The following statements are equivalent:*

(1) *φ is a local functional of homogeneity degree $k-1$.*

(2) *φ is translation invariant and homogeneous of degree $k-1$. For each $L \in G(n,k)$ the restriction of φ to L is a standard functional.*

(3) *There is a function*

$$\theta : \{(L,v) : L \in G(n,k),\, v \in S^{n-1} \cap L\} \to \mathbb{R}$$

such that $\theta(L,\cdot)$ is continuous and centered on $S^{n-1} \cap L$ for each $L \in G(n,k)$, and

$$\varphi(K) = \int_{S^{n-1}\cap L} \theta(L,v) S^L_{k-1}(K-x, dv) \tag{A.2.5}$$

for all $K \in \mathcal{K}(L+x)$, where $x \in L^\perp$ and $S^L_{k-1}(K-x,\cdot)$ is the $(k-1)$th surface measure of $K-x$ with L as ambient space.

The function θ in (3) *is uniquely determined by φ.*

Note that uniqueness of the function θ in (3) implies that φ is even if and only if $\theta(L,\cdot)$ is even for each $L \in G(n,k)$.

Another consequence of Proposition A.2.3 is the fact that every $(k-1)$-homogeneous local functional on $\mathcal{K}_k(\mathbb{R}^n)$ is a valuation. This can be seen as in the proof of Proposition A.1.1 taking into account the additivity in the first variable of the surface area measure (see, for instance [21, Theorem 14.2.2]).

Proof of Proposition A.2.3: Start by assuming that $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ is local and $(k-1)$-homogeneous. Identifying L with $\mathbb{R}^k$ it follows that $\hat{\varphi}_L : \mathcal{K}(\mathbb{R}^k) \to \mathbb{R}$ is local, hence it satisfies Definition A.2.1 and so by [27] it is a standard functional yielding the second statement.

Assuming (2) it follows by translation invariance of φ that it is enough to consider compact convex subsets of $L \in G(n,k)$. For each $L \in G(n,k)$ the restriction of φ to $\mathcal{K}(L)$ has a representation (A.2.5) with a function $\theta(L,\cdot)$ that is centered and continuous on $S^{n-1} \cap L$ by McMullen's characterization of standard functionals applied in L; see [2, Theorem 3.1(iii)]. This proves the implication (2) $\Rightarrow$ (3)

Assume now that (3) holds. Due to [2, Theorem 3.1(iii)] the restriction of φ to $\mathcal{K}(L)$ is a standard functional on $\mathcal{K}(L)$ of homogeneity degree $k-1$. Relation (A.2.5) implies $\varphi(K+x) = \varphi(K)$ for all $x \in (\operatorname{lin} K)^\perp$ and therefore φ is translation invariant and homogeneous of degree $k-1$. By [27, Theorem 3.1] it follows that each restriction of φ to $L \in G(n,k)$ is local and so φ is local of homogeneity degree $k-1$. □

We remark the following consequences of Theorem A.1.2 for the solution of our uniqueness problem.

Lemma A.2.4. *For $k \in \{1,\dots,n-1\}$, a local functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ with decomposition* (A.1.4) *satisfies*

(1) $\varphi \in \ker \mathrm{C_k}$ *if and only if* $\varphi^{(0)},\dots,\varphi^{(k)} \in \ker \mathrm{C_k}$.

(2) $\varphi^{(0)} \in \ker \mathrm{C_k}$ *if and only if* $\varphi^{(0)} = 0$.

(3) $\varphi^{(k)} \in \ker \mathrm{C_k}$ *if and only if* $\varphi^{(k)} = \varphi_f$ *with* $f : G(n,k) \to \mathbb{R}$ *satisfying* (A.1.3).

Proof. Let $k \in \{1,\dots,n-1\}$ and φ be given as in the lemma. By decomposition (A.1.4) we have

$$\begin{aligned}
&\int_{A(n,k)} \varphi(\alpha K \cap E)\mu_k(dE)\\
&= \sum_{j=0}^{k} \alpha^{n-k+j} \int_{G(n,k)} \int_{L^\perp} \varphi^{(j)}(K \cap (L+t))\lambda_{L^\perp}(dt)\nu_k(dL)\\
&= \sum_{j=0}^{k} \alpha^{n-k+j} \int_{A(n,k)} \varphi^{(j)}(K \cap E)\mu_k(dE)
\end{aligned}$$

for all $\alpha > 0$, which yields (1) by comparing coefficients. Due to (A.1.8), combined with Crofton's formula (A.1.1), we have $\varphi^{(0)} \in \ker \mathrm{C_k}$ if and only if

$$0 = \int_{A(n,k)} \varphi^{(0)}(\{0\})V_0(K \cap E)\mu_k(dE) = \varphi^{(0)}(\{0\})\alpha_{n,0,k}V_{n-k}(K)$$

for all $K \in \mathcal{K}(\mathbb{R}^n)$, where the constant $\alpha_{n,0,k}$ is positive. Thus 2. holds.

In view of (A.1.7) we have $\varphi^{(k)} = \varphi_f$ with $f(L) = c_L^{(k)}$ for all $L \in G(n,k)$, and the last claim follows from Proposition A.1.1(iv). □

This lemma implies Theorem A.1.3(1). For A.1.3(2) we need to treat the $(k-1)$-homogeneous part. Let therefore φ be a local functional of homogeneity degree $k-1$, and assume that φ is even. In particular, due to Theorem A.1.2 and the remark given below the Theorem, φ is a translation invariant even valuation. Its *Klain function* $\mathbf{K}_\varphi : G(n,k-1) \to \mathbb{R}$ is defined by

$$\mathbf{K}_\varphi(M) = \frac{1}{\kappa_{k-1}}\varphi\Big(B^n \cap M\Big),$$

for $M \in G(n,k-1)$. Strictly speaking, this is a slight extension of Klain's [13] original definition (he calls $\mathbf{K}_\varphi$ the *generating function* of φ), which was only formulated for *continuous* translation invariant even valuations. In the present context φ need not be continuous, but (A.2.5) and the fact that $\theta(L,\cdot)$ in this formula must be even, imply

$$\mathbf{K}_\varphi(M) = 2\theta\Big(\operatorname{span}(M \cup \{u\}), u\Big) \tag{A.2.6}$$

for all unit vectors $u \in M^\perp$, and thus, using (A.2.5) again,

$$\varphi(K) = \mathbf{K}_\varphi(M)V_{k-1}(K) \tag{A.2.7}$$

for all $K \in \mathcal{K}(M)$. Although $\mathbf{K}_\varphi$ need not be continuous on $G(n,k-1)$, it determines φ like in the classical case. In fact, (A.2.5) and (A.2.6) yield the explicit inversion formula

$$\varphi(K) = \frac{1}{2}\int_{S^{n-1}\cap L} \mathbf{K}_\varphi(L\cap v^\perp) S_{k-1}^L(K,dv) \tag{A.2.8}$$

for all $K \in \mathcal{K}(L)$ and $L \in G(n,k)$.

For $i,j \in \{1,\dots,n-1\}$ *the Radon transform on Grassmannians* $\mathrm{R}_{i,j} : \mathrm{L}^1(G(n,i)) \to \mathrm{L}^1(G(n,j))$ is defined by

$$(\mathrm{R}_{i,j} f)(L) = \int_{G(L,i)} f(M)\nu_i^L(dM)$$

for $L \in G(n,j)$ and an integrable function $f \in L^1(G(n,i))$. We remark that $\mathrm{R}_{i,j}$ is well defined since for $f \in \mathrm{L}^1(G(n,i))$ we have

$$\int_{G(n,j)} |(\mathrm{R}_{i,j} f)(L)|\nu_j(dL) \le \int_{G(n,i)} |f(M)|\nu_i(dM).$$

This also implies that the Radon transform is Lipschitz continuous.

Proposition A.2.5. *Let $k \in \{1,\dots,n-1\}$. Assume that the even $(k-1)$-homogeneous local functional $\varphi : \mathcal{K}_k(\mathbb{R}^n) \to \mathbb{R}$ has Klain function $\boldsymbol{K}_\varphi$.*

Then

$$\int_{A(n,k)} \varphi(P\cap E)\mu_k(dE) = 0 \tag{A.2.9}$$

for all convex polytopes $P \in \mathcal{K}_{n-1}(\mathbb{R}^n)$ if and only if $R_{k-1,n-1}\Big(\boldsymbol{K}_\varphi\Big) = 0$.

Proof. Let $P \subset v^\perp$ be a convex polytope of dimension $n-1$ with $v \in S^{n-1}$. For any fixed $L \in G(n,k)$ (A.2.7) implies

$$\int_{L^\perp} \varphi(P\cap(L+x))\lambda_{L^\perp}(dx) = \mathbf{K}_\varphi(L\cap v^\perp)\int_{L^\perp} V_{k-1}(P\cap(L+x))\lambda_{L^\perp}(dx).$$

The translative integral on the right is proportional to the mixed volume

$$V\Big(P[n-1], B_L[1]\Big) = \frac{2}{n}\|v|L\| V_{n-1}(P);$$

see, e.g. [25, p. 177] and [20, Section 5.1]. Hence, (A.2.9) holds for all convex polytopes of dimension $n-1$ if and only if

$$\int_{G(n,n-k)} \mathbf{K}_\varphi(L^\perp\cap v^\perp)\|v|L^\perp\|\nu_{n-k}(dL) = 0$$

for all $v \in S^{n-1}$. Here we also replaced the integration with respect to ν_k by an integration with respect to ν_{n-k} by taking orthogonal complements. Using a Blaschke-Petkantschin formula (see, for instance [21, Theorem 7.2.4]) this can be shown to be equivalent to

$$\int_{G(\mathrm{span}\{v\},n-k+1)} \mathbf{K}_\varphi(L^\perp)\, h(L,v)\, \nu_{n-k+1}^{\mathrm{span}\{v\}}(dL) = 0,$$

where
$$h(L,v) = \int_{G(L,n-k)} \|v|M^{\perp}\| [M, \operatorname{span}\{v\}]^{k-1} \nu_{n-k}^{L}(dM)$$
depends on the subspace determinant $[M, \operatorname{span}\{v\}]$, which is here the sine of the angle between v and M. The function $h(L,v)$ is clearly positive and independent of $L \in G(\operatorname{span}\{v\}, n-k+1)$, as any plane in this space can be rotated to any other plane by a rotation fixing v.

Putting things together, we see that (A.2.9) holds for all convex polytopes of dimension $n-1$ if and only if
$$\int_{G(v^{\perp},k-1)} \mathbf{K}_{\varphi}(L) \nu_{k-1}^{v^{\perp}}(dL) = 0$$
for all unit vectors v, where we again took orthogonal complements. This is the assertion. □

By [7], the Radon transform, $\mathrm{R}_{i,j}$, $i < j$, on the set of all square integrable functions $\mathrm{L}^2(G(n,i))$ is injective if and only if $i + j \leq n$. Hence $\mathrm{R}_{k-1,n-1}$ is injective when acting on $\mathrm{L}^2(G(n,i))$ if and only if $k \in \{1,2\}$. We note that the kernel of $\mathrm{R}_{i,j}$ when $\mathrm{R}_{i,j}$ acts on $\mathrm{L}^2(G(n,i))$ is trivial if and only if its kernel is trivial when it acts on $\mathrm{L}^1(G(n,i))$. This can be proven using similar arguments as given in the proof of the theorem below.

Theorem A.2.6. *There is no non-trivial even local functional on $\mathcal{K}_2(\mathbb{R}^n)$ of homogeneity degree 1 in the kernel of the Crofton operator with 2-flats.*

Let $2 < k < n$. There are non-trivial even local functionals on $\mathcal{K}_k(\mathbb{R}^n)$ of homogeneity degree $k-1$ in the kernel of the Crofton operator with k-flats.

Proof. For $k = 2$ the kernel of $\mathrm{R}_{k-1,n-1}$ is trivial and for any $\varphi \in \ker \mathrm{C_k}$, Proposition A.2.5 gives that its Klain function $\mathbf{K}_{\varphi}$ is in the kernel of $\mathrm{R}_{k-1,n-1}$, which implies that $\mathbf{K}_{\varphi} \equiv 0$ and hence φ is trivial. This proves that there are no non-trivial even local functionals on $\mathcal{K}_2(\mathbb{R}^n)$ of homogeneity degree 1 in the kernel of the Crofton operator with 2-flats.

On the other hand if $2 < k < n$ then there exists a non-trivial function f in $\ker \mathrm{R}_{k-1,n-1}$. Using convolution on the compact Lie group $SO(n)$ of all proper rotations and the subgroup $SO(n)/SO(k \times (n-k))$, which can be identified with $G(n,k)$ (see [10]) we can approximate f with continuous functions in the kernel of $\mathrm{R}_{k-1,n-1}$. This implies that there exist non-trivial continuous functions in $\ker \mathrm{R}_{k-1,n-1}$. Letting $\mathbf{K}_{\varphi}$ be one such function we can construct φ by (A.2.8) yielding a continuous even local $(k-1)$-homogeneous function. By Proposition A.2.5 we have $(\mathrm{C_k}\,\varphi)(P) = 0$ for all polytopes $P \in \mathcal{K}_{n-1}(\mathbb{R}^n)$. Using approximation of compact convex sets by polytopes from the outside implies that $\varphi \in \ker \mathrm{C_k}$ and so the last statement of the theorem follows. □

Proof of Theorem A.1.3(1) **and A.1.3**(2)**:** Note that by Lemma A.2.4 a local functional φ with decomposition (A.1.4) is in $\ker \mathrm{C_k}$ if and only if $\varphi^{(0)} = 0$, the functionals $\varphi^{(1)}, \ldots, \varphi^{(k-1)}$ are in $\ker \mathrm{C_k}$ and $\varphi^{(k)} = \varphi_f$ for some f satisfying (A.1.3). If $k = 1$ the latter condition is equivalent to $\varphi = \varphi_f$ proving Theorem A.1.3(1). Assuming that $k = 2$ and that φ is in addition even, $\varphi^{(1)} = 0$ by Theorem A.2.6 and hence again, $\varphi \in \ker \mathrm{C_k}$ is equivalent to $\varphi = \varphi_f$. □

For the proof of Theorem A.1.3(3) we note that Theorem A.2.6 states that if $k > 2$ then there exists non-trivial $(k-1)$-homogeneous even local functionals $\varphi \in \ker \mathrm{C_k}$ and hence we only need to construct examples of non-trivial 1-homogeneous local functionals in $\ker \mathrm{C}_2$. To explicitly make such a construction, we need to consider local functionals which are not necessarily even. For this we will make use of the translational Crofton formula for the surface area measures; see [8, Theorem 3.1]: For $K \in \mathcal{K}(\mathbb{R}^n)$ and $L \in G(n,k)$, we have

$$\int_{L^\perp} S^L_{k-1}((K-x)\cap L, \cdot)\lambda_{L^\perp}(dx) = \pi_{L,1} S_{n-1}(K,\cdot). \tag{A.2.10}$$

This relation makes use of the operator $\pi_{L,m}$ with $m = 1$, which is described in the following. For $m \in \mathbb{Z}$, $m > -k$, the *m-weighted spherical projection* $\pi_{L,m}$ maps any finite signed Borel measure μ on S^{n-1} into the space of finite signed Borel measures on $S^{n-1} \cap L$. The measure $\pi_{L,m}\mu$ is defined as the image of the measure μ_m, given by

$$\mu_m(A) = \int_A \|u|L\|^m \, \mu(du),$$

(where $A \subset S^{n-1}$ is a Borel set) under the *spherical projection*

$$p_L : S^{n-1} \setminus L^\perp \to S^{n-1} \cap L, v \mapsto \frac{v|L}{\|v|L\|}.$$

For $m \leq 0$ one must restrict considerations to a subclass of measures to assure that $\pi_{L,m}\mu$ is a well-defined finite signed measure.

If f is a continuous function on the sphere, its m-weighted spherical projection on L is the density of the m-weighted spherical projection of the measure $\int_{(\cdot)} f(v)\,dv$. More explicitly, this is the function given by

$$[\pi_{L,m} f](u) = \int_{p_L^{-1}(\{u\})} f(v)\langle u, v\rangle^{k+m-1}\, dv, \tag{A.2.11}$$

for $u \in S^{n-1} \cap L$. For more details on $\pi_{L,m}$ and the *m-weighted spherical lifting* $\pi^*_{L,m}$, see [9]. For later use, we remark that

$$[\pi^*_{L,m} f(\cdot)](u) = \|u|L\|^m f(p_L(u)) \tag{A.2.12}$$

for all integrable functions $f : S^{n-1} \cap L \to \mathbb{R}$, $u \in S^{n-1} \setminus L^\perp$ and $L \in G(n,k)$. By [9, (5.4) and (5.5)], $\pi_{L,m}$ and $\pi^*_{L,m}$ can be considered as transpose operators, as

$$\int_{S^{n-1}\cap L} f\, d(\pi_{L,m}\mu) = \int_{S^{n-1}} (\pi^*_{L,m} f)\, d\mu \tag{A.2.13}$$

for all integrable functions f on $S^{n-1} \cap L$ and all finite signed measures μ, and

$$\int_{S^{n-1}\cap L} (\pi_{L,m} f)\, d\mu = \int_{S^{n-1}} f\, d(\pi^*_{L,m}\mu) \tag{A.2.14}$$

for all integrable functions f on S^{n-1} and all finite signed measures μ on $S^{n-1} \cap L$.

The following proposition is a counterpart to Proposition A.2.5 for $(k-1)$-homogeneous local functionals, but without the evenness assumption. We recall that, according to (A.1.9) for any $(k-1)$-homogeneous local functional φ there is an associated function θ on the compact domain $D = \{(L,v) \in G(n,k) \times S^{n-1} : v \in L\}$ such that

$$\varphi(K) = \int_{S^{n-1}\cap L} \theta(L,v) S^L_{k-1}(K-x, dv) \tag{A.2.15}$$

for all compact convex sets K contained in $x+L$, where $L \in G(n,k)$ and $x \in L^\perp$. To avoid technicalities, we consider only the case where the associated function θ is continuous on D.

Theorem A.2.7. *Let φ be a $(k-1)$-homogeneous local functional on $\mathcal{K}_k(\mathbb{R}^n)$ such that its associated function $\theta(L,v)$ given by* (A.2.15) *is continuous.*

Then $\varphi \in \ker \mathrm{C_k}$ if and only if

$$\int_{G(n,k)} \Big[\pi^*_{L,1}\theta\big(L,\cdot\big)\Big]\, \nu_k(dL) = 0 \tag{A.2.16}$$

on S^{n-1}.

Proof. Let φ be a $(k-1)$-homogeneous local functional on $\mathcal{K}_k(\mathbb{R}^n)$ and let $\theta(L,v)$ given by (A.2.15). Due to (A.2.15) and (A.2.10), we have

$$\begin{aligned}(\mathrm{C_k}\,\varphi)(K) &= \int_{G(n,k)} \int_{S^{n-1}\cap L} \theta\big(L,v\big)\Big[\pi_{L,1} S_{n-1}(K,\cdot)\Big](dv)\, \nu_k(dL)\\ &= \int_{G(n,k)} \int_{S^{n-1}} \Big[\pi^*_{L,1}\theta\big(L,\cdot\big)\Big](v)\, S_{n-1}(K,dv)\, \nu_k(dL)\end{aligned}$$

for all $K \in \mathcal{K}(\mathbb{R}^n)$, where we used (A.2.13) for the last equality. In view of (A.2.12) and the assumption that θ is continuous, $(L,v) \mapsto [\pi^*_{L,1}\theta\big(L,\cdot\big)](v)$ is continuous and hence bounded, so an application of Fubini's theorem implies

$$(\mathrm{C_k}\,\varphi)(K) = \int_{S^{n-1}} h(v)\, S_{n-1}(K,dv), \tag{A.2.17}$$

where

$$h(v) = \int_{G(n,k)} \Big[\pi^*_{L,1}\theta\big(L,\cdot\big)\Big](v)\, \nu_k(dL)$$

defines a continuous function on S^{n-1}. For instance by decomposing the Hausdorff measure on the sphere like in [9, (3.3)], one can show that the function h is centered.

Concluding, we see that $\varphi \in \ker \mathrm{C_k}$ if and only if the right hand side of (A.2.17) is zero for all $K \in \mathcal{K}(\mathbb{R}^n)$. [16, Theorem 3] now shows that this is equivalent to (A.2.16). □

We use the notation in [9] to give an example of a function θ satisfying (A.2.16) by writing it as linear combination of spherical projections of spherical harmonics.

Let ω be a continuous function on S^{n-1}. If we put $\theta(L,\cdot) = \pi_{L,m}\omega$, then (A.2.16) is equivalent to $\pi^{(k)}_{1,m}\omega = 0$ on S^{n-1}, where

$$\pi^{(k)}_{1,m}\omega = \int_{G(n,k)} [\pi^*_{L,1}\pi_{L,m}\omega]\, \nu_k(dL). \tag{A.2.18}$$

The operators $\pi_{1,m}^{(k)}$ act as multiples of the identity on any space of spherical harmonics of order $r \in \mathbb{N}_0$: if ω_r is a spherical harmonic of order r, then

$$\pi_{1,m}^{(k)}\omega_r = a_{n,k,1,m,r}\,\omega_r, \tag{A.2.19}$$

with the multiplier $a_{n,k,1,m,r}$ given as a rather complicated finite sum in [9, Theorem 9.1]. This suggests finding m, r such that the multipliers vanishes. However, the choice of m is non-trivial. If we for instance choose $m = 1-k$, which can be shown to correspond to

$$\varphi(K) = \binom{n-1}{k-1}\int_{S^{n-1}} \omega(u) S_{k-1}(K, du), \tag{A.2.20}$$

$K \in \mathcal{K}(\mathbb{R}^n)$, by [9, Theorem 6.2], then all multipliers in (A.2.19) are non-vanishing by [9, p. 41]. Hence there are only trivial maps φ of type (A.2.20) in $\ker \mathrm{C_k}$. For many other choices of m explicit formulae for $a_{n,k,1,m,r}$ are not available.

We therefore construct counterexamples as follows: Fix $r \in \mathbb{N}_0 \setminus \{1\}$ and choose different integers m and m'. As two numbers are always linearly dependent, there are $\alpha, \beta \in \mathbb{R}$ with $(\alpha, \beta) \neq 0$ such that

$$\alpha a_{n,k,1,m,r} + \beta a_{n,k,1,m',r} = 0. \tag{A.2.21}$$

Let ω_r be an n-dimensional spherical harmonic of order r and set

$$\theta(L,\cdot) = [\alpha\pi_{L,m} + \beta\pi_{L,m'}]\omega_r - \langle x_L, \cdot\rangle, \tag{A.2.22}$$

where $x_L \in L$ is chosen such that $\theta(L,\cdot)$ is centered. This defines a continuous function θ. Due to (A.2.19) and (A.2.21), equation (A.2.16) holds on S^{n-1}, and Theorem A.2.7 thus implies that the $(k-1)$-homogeneous local functional φ with this associated function θ is an element of $\ker \mathrm{C}_k$.

Proposition A.2.8. *There exist non-trivial local functionals φ on $\mathcal{K}_2(\mathbb{R}^n)$ of homogeneity degree 1 in $\ker \mathrm{C}_2$.*

Before giving the proof we will introduce some notation. Let $j, l \in \mathbb{N}$ with $j < l \leq n$ and $L \in G(n,l)$. The spherical projection on the sphere in $L' \in G(L, j)$ in the ambient space L is

$$p_{L'}^L : (S^{n-1}\setminus(L')^\perp) \cap L \to S^{n-1} \cap L', v \mapsto \frac{v|L'}{\|v|L'\|}.$$

If $L = \mathbb{R}^n$ then $p_{L'}^L$ coincides with the already defined spherical projection.

Proof. Let $n \in \mathbb{N}$, $n \geq 3$ and fix $u_0 \in S^{n-1}$. For each $L \in G(n,3)$ define $\psi^L : \mathcal{K}_2(L) \to \mathbb{R}$ as follows. For $L' \in G(L,2)$ put

$$\psi^L(K') = \int_{S^{n-1}\cap L'} \theta^L(L', v) S_1^{L'}(K', dv),$$

$K' \in \mathcal{K}_2(L)$ with $K' \subset L'$ and $L \not\subseteq u_0^\perp$. When $L \subseteq u_0^\perp$ put $\psi^L(K') = 0$. Here the function $\theta^L : \{(L', v) \in G(L,2) \times S^{n-1} : v \in L'\} \to \mathbb{R}$ is given as in (A.2.22) when identifying L with $\mathbb{R}^3$ and choosing ω_r to be a certain spherical harmonic of order $r = 5$. More explicitly, we set

$$\theta^L(L',\cdot) = (\alpha\pi_{L',1}^L + \beta\pi_{L',2}^L)P_5^3(\langle p_L(u_0), \cdot\rangle) - \langle x_{L'}, \cdot\rangle. \tag{A.2.23}$$

with

$$(\pi^L_{L',m} f)(v) = \int_{(p^L_{L'})^{-1}(\{v\})} f(u)\langle v, u\rangle^{m+1}\,du$$

and $P^3_5(t) = \frac{1}{8}(63t^5 - 70t^3 + 15t)$ being the fifth order Legendre polynomial of dimension 3; see for instance [11, p. 85]. $P^3_5(\langle p_L(u_0), \cdot\rangle)$ is a 3-dimensional spherical harmonic of order 5 on $S^{n-1} \cap L$ for any $L \in G(n,3)$ with $u_0 \in S^{n-1} \cap L$. As P^3_5 is odd also $\theta^L(L', \cdot)$ is an odd function. Furthermore, we define $\psi^{L+x}(K) = \psi^L(K - x)$ for all $x \in \mathbb{R}^n$ and $K \in \mathcal{K}_2(L + x)$. Using this we define $\varphi : \mathcal{K}_2(\mathbb{R}^n) \to \mathbb{R}$ as

$$\varphi(K') = \int_{A(\operatorname{aff} K', 3)} \psi^F(K')\mu_3^{\operatorname{aff} K'}(dF) \tag{A.2.24}$$

for $K' \in \mathcal{K}_2(\mathbb{R}^n)$. Note that by definition of ψ^F, the map φ is translation invariant. As ψ^F is 1-homogeneous and local for each $F \in A(n,3)$ the map φ is 1-homogeneous and local. If θ^L satisfies (A.2.16) for $k = 3$ then by Theorem A.2.7, $\psi^F \in \ker \mathrm{C}_2$ for all $F \in A(n,3)$ and using [21, Theorem 7.1.2] we get

$$\int_{A(n,2)} \varphi(K \cap E)\mu_2(dE) = \int_{A(n,3)} \int_{A(F,2)} \psi^F(K \cap E)\mu_2^F(dE)\mu_3(dF) = 0$$

for $K \in \mathcal{K}(\mathbb{R}^n)$. Hence we need to construct functionals θ^L satisfying (A.2.16) for all $L \in G(n,3)$ such that φ is non-trivial. In view of Lemma A.2.9, below, we need to find $L'_0 \in G(n,2)$, $L_0 \in G(n,3)$ with $u_0 \in L'_0 \subset L_0$ and a set $K' \in \mathcal{K}(L'_0)$ such that $\psi^{L_0}(K') > 0$ holds. From now on we fix $L'_0 \in G(n,2)$ and $L_0 \in G(n,3)$ with $u_0 \in L'_0 \subset L_0$ (implying $p_{L_0}(u_0) = u_0$) and show the existence of $K' \subset L'_0$ with the required properties. We identify these spaces with $\mathbb{R}^3$ and a two-dimensional subspace L of $\mathbb{R}^3$, respectively. With this identification in mind, the map $K' \mapsto \psi(K')$ given by the right hand side of (A.2.15) corresponds to ψ^{L_0}. Let $\theta^{\mathbb{R}^3} = \theta$ be given as in (A.2.22) with $m = 1$, $m' = 2$ and $r = 5$. Explicit calculation gives

$$a_{3,2,1,2,5} = \frac{97}{1536}\pi \quad \text{and} \quad a_{3,2,1,1,5} = \frac{344}{1575}.$$

We therefore put

$$\alpha = -176128 \quad \text{and} \quad \beta = 50925\pi. \tag{A.2.25}$$

This implies that the function θ satisfies (A.2.16) when $n = 3$ and $k = 2$. Let u_1 be a unit vector in L orthogonal to u_0. For abbreviation, we put $f_m(\cdot) = \pi_{L,m} P^3_5(\langle p_L(u_0), \cdot\rangle)$. Note that $\theta(L, \cdot)$ is the sum of the function $f = \alpha f_1 + \beta f_2$ and the linear function $\langle x_L, \cdot\rangle$. In view of its definition, x_L is invariant under all rotations $\rho \in O(3)$ such that $\rho u_0 = u_0$ and $\rho L = L$. Hence x_L is a multiple of u_0 and it is therefore enough to show that $s \mapsto f(su_0 + \sqrt{1-s^2}u_1)$ is not linear. Now, for $m \in \mathbb{N}$ and $s \in (0,1)$ we have

$$\begin{aligned}
g_m(s) &= \frac{\partial^4}{\partial s^4} f_m(su_0 + \sqrt{1-s^2}u_1) \\
&= \frac{\partial^4}{\partial s^4} \int_{p_L^{-1}(\{su_0 + \sqrt{1-s^2}u_1\})} P^3_5(\langle u_0, v\rangle)\langle su_0 + \sqrt{1-s^2}u_1, v\rangle^{m+1}\,dv \\
&= 2\int_0^1 \frac{\partial^4}{\partial s^4} P^3_5\big(st\big) t^{m+1}(1-t^2)^{-1/2}\,dt \\
&= 1890s \int_0^1 t^{m+6}(1-t^2)^{-1/2}\,dt.
\end{aligned}$$

Now
$$\int_0^1 t^\gamma (1-t^2)^{-1/2} dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma+2}{2}\right)}, \quad \gamma > -1,$$
yields
$$g_m(s) = s\frac{1890\sqrt{\pi}}{2}\frac{\Gamma(\frac{m+7}{2})}{\Gamma(\frac{m+8}{2})},$$
so $g_2(s) = \frac{33075\pi}{128}s$ and $g_1(s) = 864s$. Due to (A.2.25), we obtain
$$\frac{\partial^4}{\partial s^4} f(su_0 + \sqrt{1-s^2}u_1) = \alpha g_1(s) + \beta g_2(s) = -1512000\pi s \neq 0,$$
for $s \neq 0$. So $\theta(L,\cdot)$ does not vanish. As $\theta(L,\cdot)$ is centered and non-vanishing, the associated local functional ψ given by (A.2.15) cannot be vanishing by Proposition A.2.3. Therefore, there must be a set $K' \in \mathcal{K}(L)$ with $\psi(K') \neq 0$. Possibly changing the signs of α and β, we can assure $\psi(K') > 0$ and the proposition is shown. □

In the above proof Lemma A.2.9 was used. In the proof of this lemma we will work with Lipschitz functions and therefore introduce metrics that induce the natural topologies. Identifying rotations with their matrix representations with respect to the standard basis, we can for instance use the Frobenius norm on $SO(n)$. On $G(n,k)$ we can work with the metric d_k given by
$$d_k(L,\tilde{L}) = d(B^n \cap L, B^n \cap \tilde{L}), \quad L,\tilde{L} \in G(n,k),$$
where d is the Hausdorff metric on $\mathcal{K}(\mathbb{R}^n)$. It is now clear that for $L_0 \in G(n,k)$, $\vartheta \mapsto \vartheta L_0'$ is non-expansive, that is, this mapping is Lipschitz with constant at most 1.

Lemma A.2.9. *Let $n \geq 3$, $u_0 \in S^{n-1}$, $L_0' \in G(n,2)$ and $L_0 \in G(n,3)$ with $u_0 \in L_0' \subset L_0$ be given. Furthermore, let φ be defined as in* (A.2.24).

If there is a compact convex set $K' \in \mathcal{K}_2(L_0')$ with $\psi^{L_0}(K') > 0$, then there is a compact convex set K_0 in $\mathbb{R}^n$ such that
$$G = \{E \in G(n,2) : \varphi(K_0 \cap E) > 0\} \tag{A.2.26}$$
contains an open neighbourhood of L_0' and therefore has positive μ_2-measure.

Proof. Let L_0, L_0', K' and u_0 as assumed in the Lemma. The assumption on K' implies
$$\psi^L(K') = \psi^{L_0}(K') > 0 \tag{A.2.27}$$
for all $L \in G(L_0', 3)$. In fact, if ϑ is a rotation fixing L_0' pointwise and satisfying $\vartheta L = L_0$, then
$$[\pi^{L_0}_{L_0',m} P_5^3(\langle u_0, \cdot\rangle)](v) = [\pi^{\vartheta L}_{\vartheta L_0',m} P_5^3(\langle \vartheta u_0, \cdot\rangle)](\vartheta v) = [\pi^{L}_{L_0',m} P_5^3(\langle u_0, \cdot\rangle)](v),$$
$v \in S^{n-1} \cap L_0'$, implying $\theta^L(L_0',\cdot) = \theta^{L_0}(L_0',\cdot)$ and hence $\psi^L(K') = \psi^{L_0}(K')$.

As $\theta^{L_0}(L_0',\cdot)$ is an odd function, $K' \in \mathcal{K}_2(L_0')$ must be two-dimensional by (A.2.27) and without loss of generality, we may assume that 0 is a relative interior point of K'.

Let $R_0 > 0$ be such that K' is contained in the interior of R_0B^n, and put $K_0 = (K' + (L_0')^\perp) \cap RB^n$. Then L_0' hits the interior of K_0 and $K_0 \cap L_0' = K'$. More generally, for all $z \in (L_0')^\perp$ in a bounded neighbourhood U of 0 the plane L_0' hits the interior of $K_0 - z$ and $(K_0 - z) \cap L_0' = K'$. All the sets $K_0 - z$ with $z \in U$ are contained in a ball of radius $R > 0$, say. In addition, we may assume that all these sets contain the ball rB^n for some $r > 0$. We will later require that all translations $K_0 - x$ with $x \in \mathbb{R}^n$ and $\|x\| \le \sup_{z\in U} \|z\|$ are contained in a given neighbourhood W of K_0. This can be achieved by shrinking U even further, if necessary.

For all $L \in G(L_0', 3)$ we have $u_0|L = u_0$, so $\|u_0|\vartheta L_0\| \ge \frac{1}{2}$ in a neighbourhood of the identity id in $SO(n)$. It is therefore easy to show that $\vartheta \mapsto p_{\vartheta L}(u_0)$ is Lipschitz in this neighbourhood with a Lipschitz constant that is independent of L. As P_5^3 is Lipschitz on $[-1, 1]$, the function

$$\vartheta \mapsto P_5^3\Big(\langle \vartheta^{-1} p_{\vartheta L}(u_0), v\rangle\Big)$$

is Lipschitz in a neighbourhood of id in $SO(n)$ with a Lipschitz constant that can be chosen independent of $v \in S^{n-1}$ and $L \in G(L_0', 3)$. It follows that

$$\begin{aligned}\vartheta \mapsto &\pi^{\vartheta L}_{\vartheta L_0', m} P_5^3\Big(\langle p_{\vartheta L}(u_0), \cdot\rangle\Big)(\vartheta w)\\ &= \int_{\left(p^L_{L_0'}\right)^{-1}(\{w\})} P_5^3\Big(\langle \vartheta^{-1} p_{\vartheta L}(u_0), v\rangle\Big)\langle v, w\rangle^{m+1}\,dv\end{aligned}$$

and hence $\vartheta \mapsto \theta^{\vartheta L}(\vartheta L_0', \vartheta w)$ is Lipschitz in a neighbourhood of id with a Lipschitz constant that can be chosen independent of $w \in S^{n-1} \cap L_0'$ and $L \in G(L_0', 3)$. Now let K be a convex body in RB^n containing the ball rB^n and observe that

$$\psi^{\vartheta L}(K \cap \vartheta L_0') = \int_{S^{n-1}\cap L_0'} \theta^{\vartheta L}(\vartheta L_0', \vartheta w) S_1^{L_0'}\Big((\vartheta^{-1}K) \cap L_0', dw\Big).$$

Hence,

$$\begin{aligned}&|\psi^{\vartheta L}(K \cap \vartheta L_0') - \psi^L(K_0 \cap L_0')| \qquad\qquad \text{(A.2.28)}\\ &\le \int_{S^{n-1}\cap L_0'} \sup_{w\in S^{n-1}\cap L_0'} |\theta^{\vartheta L}(\vartheta L_0', \vartheta w) - \theta^L(L_0', w)| S_1^{L_0'}\Big((\vartheta^{-1}K) \cap L_0', dw\Big)\\ &\quad + \left|\int_{S^{n-1}\cap L_0'} \theta^L(L_0', w)[S_1^{L_0'}\Big((\vartheta^{-1}K) \cap L_0', \cdot\Big) - S_1^{L_0'}\Big(K_0 \cap L_0', \cdot\Big)](dw)\right|\end{aligned}$$

Due to the monotonicity and motion invariance of the first intrinsic volume V_1, the total mass of $S_1^{L_0'}\Big((\vartheta^{-1}K) \cap L_0', \cdot\Big)$ is

$$2V_1((\vartheta^{-1}K) \cap L_0') \le 2V_1(\vartheta^{-1}K) \le 2V_1(RB^n),$$

so the first expression in (A.2.28) is Lipschitz in ϑ in a neighbourhood of id with a Lipschitz constant that does not depend on K or L. The expression of θ in (A.2.23) and continuity of P_5^3 implies that the integrand in the second term of (A.2.28) is bounded by a constant M, which does not depend on L, and thus this second term is bounded by

$$M\left|V_1\Big(K \cap \vartheta L_0'\Big) - V_1\Big(K_0 \cap L_0'\Big)\right|.$$

Now $(\vartheta, K) \mapsto V_1(K \cap \vartheta L_0')$ is a Lipschitz function when we restrict considerations to compact convex sets contained in RB^n and containing rB^n, with a Lipschitz constant that depends on r and R only. In fact, that $L \mapsto K \cap L$ is Lipschitz with a constant that depends on r and R only, follows for instance from [14, Lemma 2.2] and the explicit form of the Lipschitz constant in [15]. Due to its interpretation as multiple of the mean width (see for instance [20, pages 50, 297 and 231]) the first intrinsic volume is Lipschitz with constant $n\kappa_n/\kappa_{n-1}$.

Concluding, as (A.2.27) implies that $2\epsilon = \psi^L(K_0 \cap L_0')$ is positive, the bound (A.2.28) yields

$$\psi^{\vartheta L}(K \cap \vartheta L_0') > \epsilon > 0 \tag{A.2.29}$$

for all ϑ in a sufficiently small open neighbourhood V of id, all $L \in G(L_0', 3)$, and all K in a sufficiently small neighbourhood W of K_0. Hence, if $z \in U$ and $\vartheta \in V$, we get

$$\varphi(K_0 \cap \vartheta(L_0' + z)) = \int_{G(L_0',3)} \psi^{\vartheta L}((K_0 - \vartheta z) \cap \vartheta L_0')\nu_2^{L_0'}(dL) > \epsilon,$$

where we used $K = K_0 - \vartheta z$ in (A.2.29). It follows that G in (A.2.26) contains an open neighbourhood of L_0' and the assertion is shown. □

A.2.4 Motion invariant functionals

The proof of Theorem A.1.4 makes use of the Riemann-Liouville integral

$$(I^\alpha g)(x) = \Gamma(\alpha)^{-1} \int_0^x g(t)(x-t)^{\alpha-1}\,\lambda(dt)$$

of locally integrable functions $g : [0, \infty) \to \mathbb{R}$, where $\alpha > 0$ is a parameter. For arbitrary $\alpha, \beta > 0$ we have

$$I^{\alpha+\beta} g = I^\alpha I^\beta g \tag{A.2.30}$$

and

$$\frac{d}{dx}(I^{\alpha+1} g) = I^\alpha g; \tag{A.2.31}$$

see [19] for details.

Lemma A.2.10. *Let $\alpha > 0$ and a locally integrable function $g : [0, \infty) \to \mathbb{R}$ be given. If $I^\alpha g(x) \equiv 0$ on $[0, \infty)$ then $g \equiv 0$ almost everywhere on $[0, \infty)$.*

Proof. Let $m > \alpha$ be an integer and set $\beta = m - \alpha > 0$. The assumption $I^\alpha g \equiv 0$ and (A.2.30) yield $I^m g \equiv 0$ on $[0, \infty)$, and applying (A.2.31) $(m-1)$ times gives

$$0 = I^1 g(x) = \int_0^x g(t)dt$$

for all $x \geq 0$. As the Radon-Nikodym derivative of a measure is uniquely determined almost everywhere, the claim follows. □

This can now be used to prove Theorem A.1.4.

Proof of Theorem A.1.4: Let $L \in G(n,k)$ and let $\varphi : \mathcal{M}_k \to \mathbb{R}$ be a motion invariant map. Let $h : (0,\infty) \to \mathbb{R}$ be given by $h(r) = \varphi(r^{1/2}(B^n \cap L))$. By the motion invariance of φ, h is independent of $L \in G(n,k)$. Since $rB^n \cap (L+x)$ is either empty, a point or a k-dimensional ball of radius $\sqrt{r^2 - \|x\|^2}$ for $x \in L^\perp$ we get that

$$\begin{aligned}\int_{A(n,k)} \varphi(rB^n \cap E)\mu_k(dE) &= \int_{G(n,k)} \int_{L^\perp \cap rB^n} h(r^2 - \|x\|^2)\lambda_{L^\perp}(dx)\nu_k(dL) \\ &= (n-k)\kappa_{n-k} \int_0^r h(r^2 - s^2)s^{n-k-1}\,ds\end{aligned}$$

for all $rB^n \in \mathcal{M}$ with $r > 0$. The last equality follows by identifying $L^\perp$ with $\mathbb{R}^{n-k}$ and introducing spherical coordinates (see, for instance [12]). As ν_k is a probability measure and $(n-k)\kappa_{n-k} \neq 0$, a substitution shows that the left hand side of the last displayed formula is zero if and only if

$$0 = \int_0^{r^2} h(t)(r^2 - t)^{\alpha - 1}(dt) = \Gamma(\alpha)(I^\alpha h)(r^2),$$

where $\alpha = \frac{n-k}{2} > 0$. Concluding, (A.1.14) is equivalent to $(I^\alpha h)(t) = 0$ for all $t > 0$, and Lemma A.2.10 proves the claim. □

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PAPER B

Reconstructing Planar Ellipses from Translation- Invariant Minkowski Tensors of Rank Two

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Abstract

Minkowski tensors contain information about shape and orientation of the underlying convex body. We make this precise by showing that reconstructing a centered ellipse in two-dimensional Euclidean space from its rank-2 surface tensor is a well-posed inverse problem. It turns out that this result can be restated equivalently with other geometric tomography data derived from the support function of the ellipse, such as the first three non-trivial Fourier coefficients. We present explicit reconstruction algorithms for all three types of input. The relevance of these findings is illustrated in an application to stationary particle processes. We define and discuss two shape ellipses, each containing information about the mean shape and orientation of the typical particle.

MSC: 52A10; 52A22; 52A38; 60G55

Keywords: Blaschke ellipse; Geometric tomography; Minkowski tensor; Particle process; Support function

B.1 Introduction

Hadwiger's famous characterization theorem [8] states that any real-valued continuous, rigid motion invariant and additive functional on the family $\mathcal{K}^n$ of convex bodies (non-empty compact convex subsets of $\mathbb{R}^n$) is a linear combination of the intrinsic volumes. In the seminal work [1], Alesker extended this result to tensor-valued functionals on $\mathcal{K}^n$, where the role of the intrinsic volumes is now assumed by the so-called *Minkowski tensors*

$$\Phi_j^{r,s}(K) = c_{n,j}^{r,s} \int_{\mathbb{R}^n \times S^{n-1}} x^r u^s \Lambda_j\Big(K, d(x,u)\Big),$$

$j = 0, \ldots, n$ and $r, s \in \mathbb{N}_0$. Here $x^r u^s$ is to be understood as a symmetric tensor product of rank $r+s$, $\Lambda_j(K,\cdot)$ is the jth support measure of K, S^{n-1} is the unit sphere in $\mathbb{R}^n$, and $c_{n,j}^{r,s}$ is a known constant.

In view of their crucial role in pure convex geometry, it is not surprising that Minkowski tensors (of low rank) have been used as summary statistics in mathematical physics [21] for instance as anisotropy indices for galaxies [2], and for the analysis of ice grains in arctic drill cores [20]. A biological application is given in [25].

In contrast to the motion invariant intrinsic volumes, Minkowski tensors capture certain orientation and shape information of the underlying set. This can be illustrated by restricting attention to the *volume tensors* $\Phi_n^{r,0}(K)$, $r = 0, 1, 2, 3, \ldots$; see for instance [10] for an overview of uniqueness, reconstruction and stability results when volume tensors up to a certain (usually large) rank are known. We only name here that if p is a polynomial of degree at most $d \geq 2$ such that $K = \{x \in \mathbb{R}^n : p(x) \geq 0\}$ is a full-dimensional convex body, then K is determined by $\{\Phi_n^{r,0}(K) : r = 0, 1, \ldots, d\}$; see [10] and [13]. In particular, an ellipsoid in $\mathbb{R}^n$ is determined by its volume tensors up to rank 2 and an explicit reconstruction of this ellipsoid can be given; cf. [9, p. 266].

Uniqueness, stability and reconstruction algorithms based on *surface tensors* $\Phi_{n-1}^{0,s}(K)$, $s = 0, 1, 2, 3, \ldots, q$ for large q or $q = \infty$ can be found in [12] for $n = 2$ and [11] for $n \geq 3$. In contrast to the volume tensors, the surface tensors are translation invariant, so uniqueness can only hold up to translation. Strikingly though, it appears to be unknown if an ellipsoid is uniquely determined up to translation by surface tensors up to rank 2 and if so, how to reconstruct the ellipsoid from the tensors.

The present paper fills this gap in the planar case $n = 2$, showing in Theorem B.2.1 that a centered ellipse in $\mathbb{R}^2$ is uniquely determined by its rank-2 surface tensor. This theorem also implies a weak stability result: if the rank-2 surface tensors of two centered ellipses are close to one another, the two ellipse must be close in the Hausdorff metric. This stability statement will be slightly strengthened in Proposition B.2.4. In Section B.2 we state equivalent formulations of this result in terms of data derived from the support function (Theorem B.2.2) and its Fourier coefficients (Theorem B.2.3). Our paper is therefore also a contribution to the field of geometric tomography [3]. After proving these results in Section B.3, we will give explicit reconstruction algorithms in Section B.4 for all three kinds of input data. Finally, in Section B.5, these findings are applied to stationary particle processes by inferring certain shape and orientation information using two 'mean particle ellipses', which will be called the 'Blaschke ellipse' and the 'mean shape ellipse'.

B.2 Main uniqueness and continuity results

Let $\mathcal{E}$ be the family of (possible degenerate) centered ellipses in $\mathbb{R}^2$. Here, an ellipse is called *centered* if its center of symmetry coincides with the origin. We will work in $\mathbb{R}^2$ and focus on the surface tensor $\Phi_1^{0,2}(\cdot)$ of rank $s = 2$, which simplifies to

$$\Phi(E) = \Phi_1^{0,2}(K) = \frac{1}{8\pi} \int_{S^1} u^2 \, \mathrm{S}_1(E, du). \tag{B.2.1}$$

Here, $\mathrm{S}_1(E, \cdot)$ denotes the first order surface area measure of E. If E has interior points the latter is the spherical image of the one-dimensional Hausdorff-measure restricted to the boundary of E. For this and other notions from convex geometry, we refer the reader to the monograph [18]. Continuity statements on $\mathcal{K}^2$ will be understood with respect to the Hausdorff metric. Choosing a suitable basis, we will always identify the occurring rank-two tensors with symmetric 2×2-matrices. For instance, the entries $\Phi(E)_{ij}$ of the matrix $\Phi(E)$ are $\int_{S^1} u_i u_j \, \mathrm{S}_1(E, du)$, $i, j = 1, 2$, so $\Phi(E)$ is an element of the class $\mathcal{U}$ consisting of all positive semi-definite symmetric real 2×2-matrices. Our first result implies that reconstructing $E \in \mathcal{E}$ from $\Psi(E)$ is a well-posed inverse problem in the sense of Hadamard.

Theorem B.2.1. *The function $\Phi : \mathcal{E} \to \mathcal{U}$ given by* (B.2.1) *is a homeomorphism.*

To formulate this result in a slightly different setting, we will make use of the *support function* $\mathrm{h}_E(\cdot) = \max_{x \in E} \langle x, \cdot \rangle$ of E. (Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.) It is well-known that $\mathrm{S}_1(E, \cdot)$ can be represented by the support function $\mathrm{h}_E(\cdot)$: in fact, there is a self-adjoint second order differential operator $\square$ on the unit circle S^1 such that $\mathrm{S}_1(E, \cdot) = \square \mathrm{h}_E(\cdot)$; see for instance [18]. This implies that Φ is closely related to the mapping $\Psi : \mathcal{E} \to \mathcal{U}$ given by

$$\Psi(E) = \int_{S^1} u^2 \mathrm{h}_E(u) du, \tag{B.2.2}$$

$E \in \mathcal{E}$. We will see that no eigenvalue of $\Psi(E)$ can be larger than twice the other, so $\Psi(E) \in \mathcal{V}$ where

$$\mathcal{V} = \{A \in \mathcal{U} : \text{ eigenvalues } \mu_1, \mu_2 \text{ of } A \text{ satisfy } \tfrac{1}{2}\mu_1 \le \mu_2 \le 2\mu_1\}.$$

The connection between $\mathrm{S}_1(E, \cdot)$ and h_E implies that Theorem B.2.1 is equivalent to the following statement.

Theorem B.2.2. *The function $\Psi : \mathcal{E} \to \mathcal{V}$ given by* (B.2.2) *is a homeomorphism.*

The equivalence of Theorem B.2.1 and B.2.2 will be shown in Section B.3 and a proof of Theorem B.2.2 will also be given there.

Using the standard parametrization $\varphi \mapsto u_\varphi = (\cos\varphi, \sin\varphi)^t$ of the unit circle, the support function of $E \in \mathcal{E}$ can be identified with the continuous 2π-periodic function $\varphi \mapsto \mathrm{h}_E(u_\varphi)$, $\varphi \in \mathbb{R}$, with Fourier series representation

$$\mathrm{h}_E(u_\varphi) = \frac{a_0}{2} + \sum_{n=2}^{\infty} \Big(a_n \cos(n\varphi) + b_n \sin(n\varphi) \Big), \tag{B.2.3}$$

where we used the well-known fact that $a_1 = b_1 = 0$ as E is centered; see Section B.3. Clearly, *all* Fourier coefficients determine the support function h_E, and thus the ellipse E. The following equivalent reformulation of Theorem B.2.2 however shows that the first three non-trivial Fourier coefficients actually are sufficient for this purpose. To see this, define $\Upsilon : \mathcal{E} \to \mathbb{R}^3$ by

$$\Upsilon(E) = (a_0, a_2, b_2), \tag{B.2.4}$$

where a_0, a_2, b_2 are the first three non-trivial Fourier coefficients of $E \in \mathcal{E}$ in (B.2.3). An easy calculation using trigonometric identities shows that the injective map

$$S : \mathcal{V} \to \mathbb{R}^3, A = (A_{ij}) \mapsto \tfrac{1}{\pi}(A_{11} + A_{22}, A_{11} - A_{22}, A_{12} - A_{21}) \tag{B.2.5}$$

satisfies $S \circ \Psi = \Upsilon$. Thus, if $\mathcal{W} = \Upsilon(\mathcal{E})$ denotes the range of Υ, it can be concluded that Theorem B.2.2 is equivalent to the following reformulation; see Section B.3.2 for details, where we will also show

$$\mathcal{W} = \left\{\alpha(1, y, z) : \alpha \geq 0, y^2 + z^2 \leq \left(\tfrac{1}{3}\right)^2\right\}. \tag{B.2.6}$$

Theorem B.2.3. *The function $\Upsilon : \mathcal{E} \to \mathcal{W}$ given by* (B.2.4) *is a homeomorphism.*

As mentioned above, Theorem B.2.3 implies in particular that any $E \in \mathcal{E}$ is uniquely determined by its Fourier-coefficients (a_0, a_2, b_2). In view of the geometric interpretation of a_1 and b_1 described in Section B.3.2, this also shows that any (not necessarily centered) ellipse in $\mathbb{R}^2$ is determined by the five coefficients $(a_0, a_1, b_1, a_2, b_2)$.

A (weak) stability result can be stated in terms of the Hausdorff metric d_H, which can be defined by

$$d_H(K, K') = \max_{u \in S^1} |h_K(u) - h_{K'}(u)|. \tag{B.2.7}$$

for $K, K' \in \mathcal{K}^n$. We will use the *Frobenius norm* $\|\cdot\|_F$ on $\mathcal{V}$.

We can show a weak stability result for Ψ, Φ and Υ, but we only state it explicitly for the former. An ellipse $E \in \mathcal{E}$ has its image under Ψ in the interior $\operatorname{int}\mathcal{V}$ of $\mathcal{V}$ if and only if it has interior points (and is thus not a line-segment or a singleton).

Proposition B.2.4. *For any $A \in \operatorname{int}\mathcal{V}$ there is a $c = c(A) > 0$ such that*

$$d_H\left(\Psi^{-1}(A), \Psi^{-1}(A')\right) \leq c\|A - A'\|_F,$$

for all A' in a neighborhood of A in $\mathcal{V}$.

Another consequence of the above Theorems can be seen by considering an ellipse E and letting X be a uniform stochastic variable on the boundary ∂E of E. In other words, the distribution of X is the normalized one-dimensional Hausdorff measure, restricted to ∂E. As E is smooth, there is a unique outer unit normal U_E to E at X. By definition, the distribution of U_E coincides up to normalization with $S_1(E, \cdot)$ so Theorem B.2.1 implies the following stochastic result.

Corollary B.2.5. *Assume that $E, E' \in \mathcal{E}$ are not singletons. Then* $\operatorname{Var} U_E = \operatorname{Var} U_{E'}$ *if and only if E and E' are dilatations of one another, i.e. there exists $\lambda > 0$ such that $E' = \lambda E$.*

B.3 Proofs

B.3.1 Preliminaries on elliptic integrals

We will repeatedly make use of elliptic integrals of the first and second kind, given by

$$\mathrm{K}(z) = \int_0^{\frac{\pi}{2}} (1 - z\sin^2\varphi)^{-1/2} d\varphi,$$

for $z \in [0,1)$, and

$$\mathrm{E}(z) = \int_0^{\frac{\pi}{2}} (1 - z\sin^2\varphi)^{1/2} d\varphi,$$

for $z \in [0,1]$, respectively. These functions are described for instance in [6], but in a slightly modified notation replacing the parameter z used here by z^2 on the right sides of the above definitions. We will also need the following differentiation formulas (see [6, p. 863, formula 8.123]):

$$\frac{d}{dz}\mathrm{K}(z) = \frac{\mathrm{E}(z) - (1-z)\mathrm{K}(z)}{2(1-z)z}, \tag{B.3.1}$$

and

$$\frac{d}{dz}\mathrm{E}(z) = \frac{\mathrm{E}(z) - \mathrm{K}(z)}{2z}, \tag{B.3.2}$$

for $z \in (0,1)$. The functions f and g in the following lemma will play a prominent role later.

Lemma B.3.1. *The following statements hold.*

(i) *The function* $f : [0,1] \to \left[-1, -\frac{1}{2}\right]$, *given by*

$$f(z) = \begin{cases} \frac{(1-z)\mathrm{K}(z) - \mathrm{E}(z)}{z\,\mathrm{E}(z)} & \text{for } z \in (0,1], \\ -\frac{1}{2} & \text{for } z = 0, \end{cases} \tag{B.3.3}$$

is continuous, strictly decreasing and surjective. Its inverse is Lipschitz continuous with Lipschitz constant 8.

(ii) *The function* $g : [0,1] \to \left[0, \frac{1}{2}\right]$, *given by*

$$g(z) = \begin{cases} \frac{z}{(1+z)E(z) + (1-z)K(z)}, & \text{for } z \in [0,1), \\ \frac{1}{2} & \text{for } z = 1, \end{cases} \tag{B.3.4}$$

is continuous, strictly increasing and surjective. In addition, for any $0 < \overline{z} < 1$, *the function* g *is Lipschitz on* $[0, \overline{z}]$ *with Lipschitz constant* $\mathrm{K}(\overline{z})/2$.

Proof. To show (i) we note that $\lim_{z\to 0+} f(z) = -1/2$ and $f(1) = -1$, so f is continuous and its image contains $[-1, -1/2]$. We will see that f is strictly decreasing, implying that this interval is the range of f.

We want to bound f'. For $z \in (0,1)$, the relations (B.3.1), and (B.3.2) yield

$$f'(z) = \frac{1}{2(z\,\mathrm{E}(z))^2} f_1(z)$$

where $f_1(z) = (1-z)\,\mathrm{K}(z)^2 + 3\,\mathrm{E}(z)^2 - 2(2-z)\,\mathrm{K}(z)\,\mathrm{E}(z)$.

We define

$$h(z) = (z-1)\,\mathrm{K}(z)^2 + \mathrm{E}(z)^2.$$

Now $h'(z) = \frac{1}{z}(\mathrm{E}(z) - \mathrm{K}(z))^2 > 0$ for all $z \in (0,1)$. As $h(0) = 0$ this implies $h(z) \geq 0$ for all $z \in [0,1)$. Hence, $(1-z)\,\mathrm{K}(z)^2 \leq \mathrm{E}(z)^2$ and so

$$f_1(z) \leq 2\,\mathrm{E}(z) f_2(z),$$

where $f_2(z) = 2\,\mathrm{E}(z) - (2-z)\,\mathrm{K}(z)$. Differentiating this function gives

$$f_2'(z) = -\frac{1}{4(1-z)} f_3(z) \tag{B.3.5}$$

where $f_3(z) = 2\,\mathrm{E}(z) - 2(1-z)\,\mathrm{K}(z)$ and $f_3'(z) = \mathrm{K}(z) \geq \pi/2$ for all $z \in (0,1)$. As $f_3(0) = 0$ this implies $f_3(z) \geq (\pi/2)z$ for all $z \in [0,1)$. We now work our way up through the functions again. From (B.3.5) get $f_2'(z) \leq -(\pi/8)z$, and $f_2(0) = 0$ yields $f_2(z) \leq -(\pi/16)z^2$. Hence, as $\mathrm{E}(z) \leq \pi/2$ for $z \in [0,1]$, we get

$$f'(z) \leq \frac{1}{z^2\,\mathrm{E}(z)} f_2(z) \leq -\frac{\pi}{16}\frac{1}{\mathrm{E}(z)} \leq -\frac{1}{8},$$

for $z \in (0,1)$. In conclusion, this shows that f is strictly decreasing on $(0,1)$.

Hence, the inverse f^{-1} of f exists, and $|(f^{-1})'| = 1/|f' \circ f^{-1}| \leq 8$ on $(-1,-1/2)$. This implies the Lipschitz property of f^{-1} and (i) is shown.

We turn to the proof of (ii). The continuity of g follows from $\lim_{z\to 1-} g(z) = 1/2$. The function $w(z) = 1/g(z)$ has derivative

$$w'(z) = \frac{1}{2z^2}(z\,\mathrm{E}(z) - 4\,\mathrm{K}(z)) \leq \frac{1}{2z^2}\left(\frac{\pi}{2} - 4\frac{\pi}{2}\right) \leq -\frac{3}{4}\pi < 0.$$

Hence, w is strictly decreasing and g is strictly increasing and surjective. Similar arguments show that $g(z)/z$ is strictly increasing, so $g(z)/z \leq g(1) = 1/2$.

For $0 < z < \overline{z} < 1$ thus get

$$\begin{aligned} |g'(z)| &= \left|-\frac{w'(z)}{w^2(z)}\right| = \left|\frac{g(z)}{z}\right|^2 \left|z^2 w'(z)\right| \\ &\leq \frac{1}{8}(4\,\mathrm{K}(z) - z\,\mathrm{E}(z)) \leq \frac{\mathrm{K}(\overline{z})}{2}. \end{aligned}$$

This implies the asserted Lipschitz property of g and the proof is complete. □

B.3.2 Proof of the equivalence of Theorems B.2.1, B.2.2 and B.2.3

Proposition B.3.2. *Theorem B.2.1 implies Theorem B.2.2 and vice versa.*

Proof. Let Δ_S denote the *Laplace-Beltrami operator* on the circle S^1 and $\square = 1 + \Delta_S$, so we have $S_1(E,\cdot) = \square h_E(\cdot)$ for any $E \in \mathcal{E}$. If E is degenerate, this equality holds in the sense of generalized functions; see e.g. [5] for details. We thus obtain

$$\Phi(E) = \frac{1}{8\pi}\int_{S^1} u^2\Big(\square h_E(u)\Big)du = \frac{1}{8\pi}\int_{S^1} (\square u^2) h_E(u)du,$$

where we used the self-adjointness of Δ_S (see for instance [7, Chapter 1]) for the second equality, and where $\Box u^2$ is understood component-wise. Applying the last displayed formula in [18, p. 119] component-wise to the 2-homogeneous function $x \mapsto x^2$, $x \in \mathbb{R}^2$, we see that $\Box u^2 = 2I - 3u^2$, where $I \in \mathcal{U}$ is the identity matrix.

In view of the definition (B.2.2) of Ψ, and the fact that $\operatorname{tr}\Psi = \int_{S^1} h_E(u)du$ we arrive at

$$\Phi(E) = \frac{1}{8\pi}\Big(2(\operatorname{tr}\Psi(E))I - 3\Psi(E)\Big). \tag{B.3.6}$$

If μ_1, μ_2 are the two (possibly coinciding) eigenvalues of $\Psi(E)$, the matrix in (B.3.6) has eigenvalues $(2\mu_1 - \mu_2)/(8\pi)$ and $(2\mu_2 - \mu_1)/(8\pi)$. As this matrix is positive semi-definite, we conclude $\Psi(E) \in \mathcal{V}$. Hence, (B.3.6) can be written more concisely as

$$\Phi = \frac{1}{8\pi} T \circ \Psi, \tag{B.3.7}$$

with $T(A) = 2(\operatorname{tr} A)I - 3A$, $A \in \mathcal{V}$. It is not difficult to see that the range of T is $\mathcal{U}$, and that $T^{-1} : \mathcal{U} \to \mathcal{V}$ is given by

$$T^{-1}(A) = \tfrac{1}{3}\Big(2(\operatorname{tr} A)I - A\Big). \tag{B.3.8}$$

Hence, $T : \mathcal{V} \to \mathcal{U}$ is a homeomorphism and (B.3.7) shows that Theorem B.2.1 is a consequence of Theorem B.2.2, and vice versa. □

For a centered ellipse $E \in \mathcal{E}$ there exist a symmetric positive semi-definite matrix $A \in \mathcal{U}$ such that $E = AB^2$, where B^2 is the Euclidean unit disk in $\mathbb{R}^2$. Now by eigendecomposition we can write $E = C\Lambda B^2$, where C is an element of the special orthogonal group SO(2) with columns which are eigenvectors of A. The diagonal matrix Λ has the corresponding eigenvalues on its diagonal. The support function of E now takes the simple form

$$\mathrm{h}_E(u) = \max_{x \in E}\langle x, u\rangle = \|\Lambda C^t u\|, \tag{B.3.9}$$

for $u \in S^1$. The Fourier coefficients of $\varphi \mapsto \mathrm{h}_E(u_\varphi)$ for $E = C\Lambda B^2$, are given by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \|\Lambda C^t u_\varphi\| \cos(n\varphi)d\varphi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} \|\Lambda C^t u_\varphi\| \sin(n\varphi)d\varphi,$$

for $n \geq 0$ due to (B.3.9). For reference, the monograph [7] can be consulted. By definition [18, Equation (1.30)], the *mean width* $w(E) = \frac{1}{\pi}\int_{S^1} \mathrm{h}_E\, du$ of E coincides with a_0. By [7, Theorem 4.2.1] the vector $(a_1, b_1)^t$ is the Steiner point of E. As the Steiner point coincides with the center of symmetry of E, and E is centered, $a_1 = b_1 = 0$. The Fourier coefficients for $n \geq 2$ do not have similarly simple geometric interpretations.

Proposition B.3.3. *Theorem B.2.2 implies Theorem B.2.3 and vice versa.*

Proof. We first remark that the mapping S in (B.2.5) is injective on $\mathcal{V}$ and that its inverse is continuous on the range of S. The relation $S \circ \Psi = \Upsilon$ follows from the definitions and the trigonometric identities $\cos^2\varphi + \sin^2\varphi = 1$, $\cos^2\varphi - \sin^2\varphi = \cos(2\varphi)$, and $\cos\varphi\sin\varphi + \sin\varphi\cos\varphi = \sin(2\varphi)$.

Now if Theorem B.2.2 holds $\Psi : \mathcal{E} \to \mathcal{V}$ is surjective, so $S \circ \Psi = \Upsilon$ implies $S(\mathcal{V}) = S(\Psi(\mathcal{E})) = \mathcal{W}$, and Theorem B.2.3 follows. On the other hand, if Theorem B.2.3 holds, $S : \mathcal{V} \to \mathcal{W}$ is surjective, so $\Psi = S^{-1} \circ \Upsilon$ is a homeomorphism, implying Theorem B.2.2. □

The explicit representation of $\mathcal{W}$ in (B.2.6) can easily be derived from the parametrization $A_{\varphi,\mu_1,\mu_2} = C_\varphi \Lambda C_\varphi^t$ of all elements of $\mathcal{V}$, where C_φ is a rotation about 0 with angle $\varphi \in [0, 2\pi)$ and Λ is a diagonal matrix with non-negative eigenvalues μ_1 and μ_2, with $\mu_1/2 \leq \mu_2 \leq 2\mu_1$. In fact, in the nontrivial case $\mu_1 + \mu_2 > 0$, we have

$$S\Big(A_{\varphi,\mu_1,\mu_2}\Big) = (\mu_1 + \mu_2)\Big(1, \beta\cos 2\varphi, \beta\sin 2\varphi\Big)$$

with $\beta = \frac{\mu_1-\mu_2}{\mu_1+\mu_2} \in \Big[-\frac{1}{3}, \frac{1}{3}\Big]$.

B.3.3 Proofs of Theorem B.2.2, Proposition B.2.4, and Corollary B.2.5

This section is dedicated to the proof of Theorem B.2.2 (implying, as we have seen, Theorems B.2.1 and B.2.3), the weak stability result in Proposition B.2.4, and Corollary B.2.5. For later use, we note that the definitions of $\Psi(E)$ and h_E imply

$$C\Psi(E)C^t = \Psi(CE) \tag{B.3.10}$$

for all C in the orthogonal group $O(2)$.

We say that an ellipse $E \in \mathcal{E}$ is *axis-parallel* if its principal axes directions are parallel to the standard coordinate axes. By convention, if E is a disk, any direction is a principal axis direction, so any centered disk is axis-parallel. The next two propositions show that $\Psi(E)$ can be used to find a rotation C of $E \in \mathcal{E}$ such that CE is axis-parallel with the largest principal axis parallel to the y-axis.

Proposition B.3.4. *The following statements hold.*

(i) *If the axis-parallel ellipse*

$$E = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} B^2 \in \mathcal{E} \tag{B.3.11}$$

with $x, y \geq 0$ is given, then

$$\Psi(E) = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \tag{B.3.12}$$

with $\mu_1, \mu_2 \geq 0$.

(ii) *In (i) we have $x \leq y$ if and only if $\mu_1 \leq \mu_2$. Furthermore, $x = y$ if and only if $\mu_1 = \mu_2$.*

(iii) *An arbitrary $E \in \mathcal{E}$ is a disk if and only if $\Psi(E)$ is a multiple of the identity matrix.*

Proof. Let E be given by (B.3.11) in (i). Direct calculations using (B.3.9) and the standard parametrization of S^1 show that the off-diagonal elements of $\Psi(E)$ are zero and that

$$\mu_1 = \int_{S^1} u_1^2 h_E(u)du = 4\int_0^{\pi/2} \sin^2\varphi \sqrt{x^2\sin^2\varphi + y^2\cos^2\varphi}\, d\varphi, \tag{B.3.13}$$

and

$$\mu_2 = 4\int_0^{\pi/2} \cos^2\varphi \sqrt{x^2\sin^2\varphi + y^2\cos^2\varphi}\, d\varphi. \tag{B.3.14}$$

This shows (i).

Trigonometric identities allow a split of these integrals at $\varphi = \pi/4$ to obtain

$$\mu_2 - \mu_1 = 4\int_0^{\pi/4} (2\cos^2\varphi - 1)\times \left(\sqrt{x^2\sin^2\varphi + y^2\cos^2\varphi} - \sqrt{x^2\cos^2\varphi + y^2\sin^2\varphi}\right) d\varphi.$$

Without loss of generality, we may assume $(x, y) \neq (0, 0)$. Putting

$$a = \sqrt{x^2\sin^2\varphi + y^2\cos^2\varphi} \text{ and } b = \sqrt{x^2\cos^2\varphi + y^2\sin^2\varphi},$$

we see from the binomial formula $(a-b)(a+b) = a^2 - b^2$ that

$$\mu_2 - \mu_1 = 4(y^2 - x^2)\int_0^{\pi/4} \frac{(2\cos^2\varphi - 1)^2}{a+b} d\varphi.$$

The integrand is positive on $(0, \pi/4)$, so the difference $\mu_2 - \mu_1$ has the same sign as $y - x$, and vanishes iff $y - x$ vanishes. This proves (ii).

In (ii) we have shown $x = y$ iff $\mu_1 = \mu_2$, which directly implies that $\Psi(E)$ of a disk E is a multiple of I. For the converse assume that $\Psi(E)$ is a multiple of I. Then $\Psi(CE) = \Psi(E)$ for all $C \in SO(2)$ due to (B.3.10), so we may apply the above to C_0E, where $C_0 \in SO(2)$ is chosen so that C_0E is axes-parallel. We infer that C_0E, and hence E, must be a disk. □

Proposition B.3.5. *Fix $E \in \mathcal{E}$. If $C \in \mathrm{SO}(2)$ is such that $C\Psi(E)C^t$ has diagonal form, then CE is an axis-parallel ellipse.*

Proof. As $\Psi(E)$ is symmetric, there is a rotation $C \in SO(2)$ such that $C\Psi(E)C^t$ has diagonal form. By (B.3.10) we see that $\Psi(E')$ with $E' = CE$ is a diagonal matrix. Due to Proposition B.3.4(iii), E' is a disk (and hence axis parallel) if $\Psi(E')$ is a multiple of the identity matrix I. We may therefore assume that $\Psi(E')$ has two different eigenvalues $\mu_1 \neq \mu_2$.

Now let $v \in S^1$ be a principal axis direction of E' and let $C_0 = 2vv^t - I \in S(2)$ be the matrix of the reflection across the line spanned by v. As $C_0E' = E'$, equation (B.3.10) implies $C_0\Psi(E')C_0^t = \Psi(E')$, that is, $\Psi(E')$ and C_0 commute. This, and the fact that e_1 is an eigenvector of $\Psi(E')$ with eigenvalue μ_1 shows that C_0e_1 is also an eigenvector of $\Psi(E')$ with the same eigenvalue. As $\mu_1 \neq \mu_2$ this implies $C_0e_1 = \pm e_1$, so v must be parallel to one of the coordinate axes. This shows that $E' = CE$ is axes-parallel and concludes the proof. □

Proposition B.3.6. *The mapping $\Psi : \mathcal{E} \to \mathcal{V}$ is bijective.*

Proof. To show the injectivity, let $E \in \mathcal{E}$. In view of Propositions B.3.4 and B.3.5, we may assume an axis-parallel ellipse (B.3.11) with $x \leq y$ and have to show that x and y can be recovered from the two numbers μ_1 and μ_2 in (B.3.12).

Clearly, $y = 0$ occurs if and only if $\mu_1 = \mu_2 = 0$ due to (B.3.13) and (B.3.14), so we may assume $y > 0$ and $\mu_2 > 0$ from now on. It is convenient to work with the squared eccentricity $z = 1 - x^2/y^2 \in [0,1]$ of E, as

$$\begin{aligned}\frac{z}{4}\mu_1 &= yz\int_0^{\pi/2} \sin^2\varphi\sqrt{1 - z\sin^2\varphi}\,d\varphi \\ &= \frac{y}{3}\Big((2z-1)E(z) + (1-z)K(z)\Big),\end{aligned} \tag{B.3.15}$$

where the first equality follows from (B.3.13) and the second equality is a special case of [6, p. 186, formula 2.583(4.)]. Similarly,

$$\frac{z}{4}\mu_2 = \frac{y}{3}\Big((1+z)E(z) - (1-z)K(z)\Big), \tag{B.3.16}$$

which can be derived from (B.3.15) and (B.3.14) observing that $z(\mu_1 + \mu_2)/4 = yzE(z)$. The function $f(z) = 3\mu_1/(\mu_1 + \mu_2) - 2$ can be derived from $\Psi(E)$ and is independent of y when $z \in [0,1]$ is given. In fact, when $z > 0$, Equations (B.3.15) and (B.3.16) show that

$$f(z) = \frac{(1-z)K(z) - E(z)}{zE(z)}.$$

When $z = 0$, we must have $x = y$ and thus $\mu_1 = \mu_2$ by Proposition B.3.4(ii), hence $f(0) = -1/2$. By Lemma B.3.1, the function f is strictly decreasing on $[0,1]$, implying that z is uniquely determined by μ_1 and μ_2. Solving (B.3.15) with the z-value just obtained determines y and we must have $x = y\sqrt{1-z}$. Concluding, we have shown that $\Psi(E)$ determines E uniquely.

To show surjectivity, let $A \in \mathcal{V}$ be given. Applying a suitable rotation C, we may assume that e_1 and e_2 are eigenvectors of C^tAC, where the corresponding eigenvalues μ_1 and μ_2 satisfy $\mu_1 \le \mu_2 \le 2\mu_1$. It follows that the number $b = 3\mu_1/(\mu_1 + \mu_2) - 2$ is an element of $[-1,-1/2]$, so there is a $z_0 \in [0,1]$ such that $f(z_0)$ coincides with b due to Lemma B.3.1. Inserting z_0 into (B.3.15) yields a value $y_0 = y$ and we may put $x_0 = y_0\sqrt{1-z_0}$. Then the ellipse E_0 of the form (B.3.11) with $x = x_0$ and $y = y_0$ satisfies $\Psi(E_0) = C^tAC$. In view of (B.3.10), we thus arrive at $\Psi(CE_0) = C\Psi(E_0)C^t = A$, showing surjectivity. This concludes the proof. □

Proof of Theorem B.2.2: Endow the space of continuous functions on S^1 with the maximum norm. In view of our definition (B.2.7) of the Hausdorff metric, $E \mapsto h_E$ is an isometry. This clearly implies the Lipschitz-continuity of Ψ.

Due to Proposition B.3.6, the inverse of Ψ exists. It thus remains to show that the inverse is continuous. For a given $r > 0$ we consider the set

$$\mathcal{V}(r) = \{A \in \mathcal{V} : \text{all eigenvalues of } A \text{ are in } [0,r]\}.$$

As it is enough to show the continuity of Ψ^{-1} on $\mathcal{V}(r)$ for all $r > 0$, we now fix $r > 0$. We will show below that there is an $R > 0$ such that

$$\Psi^{-1}\big(\mathcal{V}(r)\big) \subset \mathcal{E}(R) = \{E \in \mathcal{E} : E \subset RB^2\}. \tag{B.3.17}$$

As $\mathcal{E}$ is closed in the family $\mathcal{K}^2$ of convex bodies, the set $\mathcal{E}(R)$ is compact due to the Blaschke selection theorem (see, e.g. [18, Theorem 1.8.7]), so the continuity of Ψ^{-1} on $\Psi(\mathcal{E}(R)) \supset \mathcal{V}(r)$ follows from topological standard arguments for continuous maps on compact spaces; see for instance [23, Proposition 5.2.5].

It remains to show (B.3.17). If $E \in \mathcal{E}$ is such that $\Psi(E) \in \mathcal{V}(r)$, the same holds true for any rotation of E due to (B.3.10), so we may assume that E is axis-parallel with half-axes lengths $0 \le x \le y$. By (B.3.13), the eigenvalue μ_1 of $\Psi(E)$ satisfies

$$\begin{aligned} \mu_1 &= 4\int_0^{\frac{\pi}{2}} \sin^2\varphi\sqrt{x^2\sin^2\varphi + y^2\cos^2\varphi}\,d\varphi \\ &\ge 4\int_0^{\frac{\pi}{2}} \sin^2\varphi\sqrt{y^2\cos^2\varphi}\,d\varphi = \frac{4y}{3}. \end{aligned} \tag{B.3.18}$$

As $\mu_1 \le r$, (B.3.17) holds with $R = 3r/4$. This completes the proof of Theorem B.2.2. □

Tracking the continuity properties of all involved mappings in the proof above, the stability result in Proposition B.2.4 can be shown.

Proof of Proposition B.2.4: Let $A \in \operatorname{int}\mathcal{V}$ be given, and let $C \in SO(2)$ be such that CAC^t is the diagonal matrix with diagonal entries $\mu_1 \le \mu_2$. Perturbation theory (see, e.g. [22, Lemma 4.3]) yields the existence of a neighborhood $\mathcal{N}$ of A and a constant $c_1 = c_1(A) > 0$ such that the following holds: for any A' in $\mathcal{N}$ there is a $C' \in SO(2)$ such that $C'A'(C')^t$ is a diagonal matrix with diagonal entries $\mu_1' \le \mu_2'$, and

$$\|C - C'\|_F \le c_1\|A - A'\|_F. \tag{B.3.19}$$

We may additionally assume that $\mathcal{N}$ is a compact subset of $\operatorname{int}\mathcal{V}$.

The properties of the Frobenius norm imply $|\mu_i - \mu_i'| \le \|A - A'\|_F$ for $i = 1, 2$. Hence, the eigenvalues of $A' \in \mathcal{N}$ are uniformly bounded away from 0 and from above; and μ_1'/μ_2' is uniformly bounded away from 1/2. Setting $b = 3\mu_1/(\mu_1 + \mu_2) - 2$, $z = f^{-1}(b)$, and $b' = 3\mu_1'/(\mu_1' + \mu_2') - 2$, $z' = f^{-1}(b')$, using this boundedness and the Lipschitz property of f^{-1} in Lemma B.3.1 gives the existence of a constant $c_2 = c_2(A) > 0$

$$|z - z'| \le c_2\|A - A'\|_F.$$

We may in addition conclude that b' is uniformly bounded from below by a constant $\underline{b} = \underline{b}(A) > -1$. As f is strictly decreasing,

$$z' = f^{-1}(b') \le f^{-1}(\underline{b}) = \overline{z} < 1.$$

Solving (B.3.16) for y suggests the definition $y = \frac{3}{4}\mu_2 g(z)$ and $y' = \frac{3}{4}\mu_2' g(z')$. As both, the restriction of g to $[0, \overline{z}]$ (by Lemma B.3.1), and the eigenvalue μ_2' are bounded and Lipschitz, there is a constant $c_3 = c_3(A) > 0$ such that

$$|y - y'| \le c_3\|A - A'\|_F. \tag{B.3.20}$$

Similar arguments for $x = y\sqrt{1-z}$ and $x' = y'\sqrt{1-z'}$ yield the existence of a constant $c_4 = c_4(A) > 0$ such that

$$|x - x'| \le c_4\|A - A'\|_F. \tag{B.3.21}$$

Due to (B.3.10), the ellipses $E = C^t\Lambda B^2$ and $E' = (C')^t\Lambda' B^2$ with the diagonal matrices $\Lambda = \operatorname{diag}(x, y)$ and $\Lambda' = \operatorname{diag}(x', y')$ satisfy $\Psi(E) = A$ and $\Psi(E') = A'$. Equations

(B.2.7) and (B.3.9), imply

$$\begin{aligned} d_H(\psi^{-1}(A), \psi^{-1}(A')) &= d_H(E, E') \\ &= \max_{u \in S^1} \Big| \|\Lambda C u\| - \|\Lambda' C' u\| \Big| \\ &\le \max_{u \in S^1} \Big\| \big(\Lambda C - \Lambda' C'\big) u \Big\| \\ &\le \|\Lambda C - \Lambda' C'\|_F \\ &\le \|\Lambda\|_F \|C - C'\|_F + \|\Lambda - \Lambda'\|_F \|C'\|_F, \end{aligned}$$

as the Frobenius norm is subordinated and sub-multiplicative. As $\|C'\|_F = \sqrt{2}$, the assertion now follows from (B.3.19), (B.3.20), and (B.3.21). □

Corollary B.2.5 is a simple consequence of Theorem B.2.1.

Proof of Corollary B.2.5: Firstly observe that

$$\mathrm{Var}(U_E) = \mathbb{E}(U_E^2) = \alpha \Phi(E) = \Phi(\alpha E),$$

with $\alpha^{-1} = \mathrm{S}_1(E, S^1)/(8\pi) > 0$. Similarly we get $\mathrm{Var}(U_{E'}) = \Phi(\alpha' E')$ with $(\alpha')^{-1} = \mathrm{S}_1(E', S^1)/(8\pi)$. Hence, if $\mathrm{Var}(U_E) = \mathrm{Var}(U_{E'})$, Theorem B.2.1 implies $E' = \frac{\alpha}{\alpha'} E$. If, on the other hand, there exist $\lambda > 0$ such that $E' = \lambda E$ then $\alpha' E' = \alpha E$, which proves the corollary. □

B.4 Reconstruction algorithms

The proof of Theorem B.2.2 is constructive and gives rise to the following reconstruction algorithm based on the functions f and g given in (B.3.3) and (B.3.4), respectively.

Algorithm SupportTensorData

Input: Let $A \in \mathcal{V}$ be given.
Task: Find $E \in \mathcal{E}$ such that $\Psi(E) = A$.
Action: Find $C \in \mathrm{SO}(2)$ such that $C^t A C = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $\mu_1 \le \mu_2$.
(a) Find the squared eccentricity $z \in [0, 1]$ by solving $f(z) = 3\frac{\mu_1}{\mu_1 + \mu_2} - 2$.
(b) Put $y = \frac{3}{4}\mu_2 g(z)$ and $x = y\sqrt{1 - z}$.
Output: The ellipse $E = C \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} B^2$.

To find f^{-1}, a numerical procedure is required. The following variant of the above algorithm takes into account that this inversion comes with an error.

Algorithm SupportTensorData*

Input:	Let $A \in \mathcal{V}$ and $\varepsilon > 0$ be given.
Task:	Find $E \in \mathcal{E}$ such that $d_H(E, \Psi^{-1}(A)) \le \varepsilon$.
Action:	Find $C \in \mathrm{SO}(2)$ such that $C^t A C = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $\mu_1 \le \mu_2$. If $b = 3\frac{\mu_1}{\mu_1+\mu_2} - 2$ is equal to -1, let $x = 0$ and $y = \frac{3}{4}\mu_1$. Otherwise, (a) Find $\underline{z}$ and $\overline{z}$ in $[0,1)$ with $f(\overline{z}) \le b \le f(\underline{z})$ and $\overline{z} - \underline{z} \le \delta_1$, where $\delta_1 = \delta_1(\overline{z}, \varepsilon) = \varepsilon \frac{4}{3} \min\{\frac{1}{\mu_2 \mathrm{K}(\overline{z})}, \frac{\sqrt{1-\overline{z}}}{\mu_1}\}$. (b) Put $z = \frac{\underline{z}+\overline{z}}{2}$, $y = \frac{3}{4}\mu_2 g(z)$ and $x = y\sqrt{1-z}$.
Output:	The ellipse $E = C\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} B^2$.

The two numbers $\underline{z}$ and $\overline{z}$ in step (a) can be found iteratively by the method of nested intervals that are halved in each iteration, starting with $\underline{z}_0 = 0$ and $\overline{z}_0 = 1$. One first iterates ignoring the requirement $\overline{z}_n - \underline{z}_n \le \delta_1$. Eventually we will have $\overline{z}_n < 1$ as $b > -1$, and the corresponding δ_1 will be positive. In each consecutive step, δ_1 will remain unchanged or become larger, and the condition in step (a) can be reached in finitely many steps.

That **Algorithm SupportTensorData*** yields an ellipse within the desired precision when $b > -1$ can be seen as follows. Let $0 \le x_0 \le y_0$ be the half-axes lengths of the ellipse $E_0 = \Psi^{-1}(A)$ and let z_0 be its squared eccentricity. If x, y and z are the corresponding quantities determined in the algorithm, we have $\underline{z} \le z, z_0 \le \overline{z}$, so $|z - z_0| \le \overline{z} - \underline{z} \le \delta_1$. Hence,

$$|y - y_0| \le \tfrac{3}{4}\mu_2 |g(z) - g(z_0)| \le \tfrac{3}{8}\mu_2 \mathrm{K}(\overline{z})\delta_1 \le \frac{\varepsilon}{2}, \tag{B.4.1}$$

by Lemma B.3.1(ii) and the definition of δ_1. In addition, using the inequality (B.3.18) and the bound $|z - z_0| \le \delta_1$, and (B.4.1), we see that

$$\begin{aligned} |x - x_0| &= \left| y\sqrt{1-z} - y_0\sqrt{1-z_0} \right| \\ &\le \left|\sqrt{1-z}\right| \left|y - y_0\right| + y_0 \left|\sqrt{1-z} - \sqrt{1-z_0}\right| \\ &\le \left|y - y_0\right| + \tfrac{3}{4}\mu_1 \frac{1}{2\sqrt{1-\overline{z}}} \left|z - z_0\right| \\ &\le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As E_0 and E are obtained from rotating axis-parallel ellipses with the same rotation, it is easy to see that their Hausdorff distance is not exceeding ε.

Algorithm SurfaceTensorData

Input:	Let $A \in \mathcal{U}$ be given.
Task:	Find $E \in \mathcal{E}$ such that $\Phi(E) = A$.
Action:	Apply **Algorithm SupportTensorData** to $T^{-1}(8\pi A)$, given by (B.3.8), with output E.
Output:	The ellipse E.

The reconstruction from Fourier coefficients makes use of the inverse

$$S^{-1}(a_0, a_2, b_2) = \frac{\pi}{2}\begin{pmatrix} a_0 + a_2 & b_2 \\ b_2 & a_0 - a_2 \end{pmatrix},$$

$(a_0, a_2, b_2) \in \mathcal{W}$, of the mapping S in (B.2.5).

Algorithm FourierData

Input:	Let $(a_0, a_2, b_2) \in \mathcal{W}$ be given.
Task:	Find $E \in \mathcal{E}$ such that $\Upsilon(E) = A$.
Action:	Apply **Algorithm SupportTensorData** to $S^{-1}(a_0, a_2, b_2)$ with output E.
Output:	The ellipse E.

Naturally, the last two algorithms also can be defined in 'starred' versions by applying **Algorithm SupportTensorData*** instead of **Algorithm SupportTensorData** in either of them.

B.5 An application to particle processes

In this section we will consider a number of set-valued summary statistics for stationary particle processes making use of the results in the previous sections. Let X be a stationary particle process in $\mathbb{R}^2$ with convex particles, intensity $\gamma > 0$ and grain distribution $\mathbb{Q}$. For details on the theory of particle processes we recommend [19, Sections 4.1 and 4.2]. The grain distribution is concentrated on

$$\mathcal{K}_0^2 = \{K \in \mathcal{K}^2 : z(K) = 0\},$$

where $z : \mathcal{K}^2 \to \mathbb{R}^2$ is a *center function*, that is, it is measurable and satisfies $z(K + x) = z(K) + x$ for all $K \in \mathcal{K}^2$ and $x \in \mathbb{R}^2$. We will use the canonical choice for $z(K)$, which is the circumcenter of K, unless stated otherwise.

Let $\mathbf{K}_0$ be a random convex body with distribution $\mathbb{Q}$. It can be considered as the *typical particle* of X in the Palm sense. Here and in the following, we restrict considerations to convex particles, although extensions to polyconvex particles, or even larger set classes that allow for well-defined support measures, are possible.

By definition, the intensity measure of X is locally finite. This property is by [19, Theorem 4.1.2] equivalent to the $\mathbb{Q}$-integrability of the perimeter $2V_1(\mathbf{K}_0)$ and the area $V_2(\mathbf{K}_0)$ of $\mathbf{K}_0$. Hence, $\mathbb{E}\mathrm{S}_1(\mathbf{K}_0, \cdot)$ is a finite measure, as $\mathrm{S}_1(K, \cdot) \le 2V_2(K)$, and the smallest disk containing $\mathbf{K}_0$ has an integrable radius.

We now give a short overview of common set-valued summary statistics of X and their interrelation. The *Aumann expectation* $\mathbb{E}\mathbf{K}_0$ is also called *selection expectation*, as it is defined as the closure of the set containing all means of integrable selections of $\mathbf{K}_0$. Equivalently, by the convexity of $\mathbf{K}_0$, the expectation $\mathbb{E}\mathbf{K}_0$ is the unique convex body satisfying

$$\mathrm{h}_{\mathbb{E}\mathbf{K}_0}(u) = \mathbb{E}\mathrm{h}_{\mathbf{K}_0}(u),$$

$u \in S^1$; see [17, Thorem 2.1.35]. As Minkowski-addition (corresponding to the addition of support functions) and Blaschke-addition (corresponding to the addition of surface area measures) coincide in $\mathbb{R}^2$, we also have

$$\mathrm{S}_1(\mathbb{E}\mathbf{K}_0, \omega) = \mathbb{E}\mathrm{S}_1(\mathbf{K}_0, \omega), \tag{B.5.1}$$

for any measurable $\omega \subset S^1$.

Due to Minkowski's existence theorem, there exists a unique set $B(X) \in \mathcal{K}_0^2$ such that

$$\mathrm{S}_1(B(X), \omega) = \gamma \mathbb{E}\,\mathrm{S}_1(\mathbf{K}_0, \omega), \tag{B.5.2}$$

for any measurable $\omega \subset S^1$, and this set is called the Blaschke body of X. In contrast to higher dimensions this construction also works for degenerate particle processes in the planar case. It follows that the usual *mean body* $M(X)$ and the *Blaschke body* $B(X)$ of X are closely related to $\mathbb{E}\mathbf{K}_0$, as a comparison of (B.5.1) and (B.5.2) shows that $\mathbb{E}\mathbf{K}_0$ is a translate of the *Blaschke shape* $\gamma^{-1}B(X)$. In a similar way, it can be seen that $\mathbb{E}\mathbf{K}_0$ is a translate of $\gamma^{-1}M(X)$, as the latter is defined as a support function mean (though with the Steiner point map as center function).

Hence, the mean $\mathbb{E}\mathbf{K}_0$, and the sets $\gamma^{-1}B(X)$ and $\gamma^{-1}M(X)$ contain essentially the same mean information about the typical particle. Direct estimation of these sets is not practical in many applications, as they cannot be described by a finite-dimensional parameter vector. We therefore suggest to pool this information even further by considering the surface tensor of $\mathbb{E}\mathbf{K}_0$ and representing it as an ellipse. This simplified set-valued summary statistic still captures certain orientation and shape information. To implement this program, the results of Section B.2 are needed.

For any convex body $K \in \mathcal{K}^2$ the tensor $\Phi(K)$ is clearly in $\mathcal{U}$, so Theorem B.2.1 yields the existence of a unique centered ellipse $\mathrm{e}(K) \in \mathcal{E}$ with

$$\Phi\Big(\mathrm{e}(K)\Big) = \Phi(K).$$

Clearly, $\mathrm{e}(K)$ remains unchanged when K is translated.

Definition B.5.1. *The* Blaschke ellipse *of a stationary particle process X of convex particles is the unique centered ellipse* $\mathrm{e}_{\mathrm{B}}(X) \in \mathcal{E}$ *with*

$$\Phi(\mathrm{e}_{\mathrm{B}}(X)) = \Phi(\mathbb{E}\boldsymbol{K}_0). \tag{B.5.3}$$

Equivalently, $\mathrm{e}_{\mathrm{B}}(X) = \mathrm{e}(\mathbb{E}\boldsymbol{K}_0)$.

We decided to define the Blaschke ellipse without the factor γ (present in the definitions of $M(X)$ and $B(X)$) to ease comparison with the Miles ellipsoid in [25], which is based on an average of rank-2 volume tensors of $\mathbf{K}_0$. In view of the above,

$$\mathrm{e}_{\mathrm{B}}(X) = \gamma^{-1}\,\mathrm{e}\Big(M(X)\Big) = \gamma^{-1}\,\mathrm{e}\Big(B(X)\Big), \tag{B.5.4}$$

and the Blaschke ellipse can be derived from any of the above set-valued means. But it is not necessary to determine these set-valued means to derive $\mathrm{e}_{\mathrm{B}}(X)$, as the standard proof of measure theory shows $\mathbb{E}\Phi(\mathbf{K}_0) = \Phi(\mathbb{E}\mathbf{K}_0)$, so

$$\Phi(\mathrm{e}_{\mathrm{B}}(X)) = \mathbb{E}\Phi(\mathbf{K}_0), \tag{B.5.5}$$

by (B.5.3), and $\mathrm{e}_{\mathrm{B}}(X) \in \mathcal{E}$ is uniquely determined by (B.5.5). In view of Theorems B.2.2 and B.2.3, the Blaschke ellipse could equivalently be defined as the ellipse in $\mathcal{E}$ with rank-2 support tensor or first non-trivial three Fourier coefficients equal to the corresponding means of $\mathbf{K}_0$. If $\mathbf{K}_0$ coincides almost surely with a deterministic convex body K, then $\mathrm{e_B}(X) = \mathrm{e}(K)$. If $\mathbf{K}_0$ is isotropic, $\mathrm{e_B}(X)$ is a disk, but the converse is false.

We now turn attention to the estimation of $\mathrm{e_B}(X)$ based on observations of X in a compact window $W \in \mathcal{K}^2$. The idea is to reconstruct $\mathrm{e_B}(X)$ from an estimator of $\Phi(\mathrm{e_B}(X)) = \gamma^{-1}\Phi(B(X))$, which can be based on known estimators for $S_1(B(X),\cdot)$ in the literature. However, many such estimators, like those that can be naturally derived from [24, Theorem 4.4] are only *asymptotically* unbiased when the window grows in an appropriate manner. If W is the unit square, an *unbiased* procedure can be derived from [24, Cor. 4.5] suggesting to estimate $\Phi(B(X))$ by

$$\tilde{\Phi}_W(X) = \sum_{K\in X}[\Phi(K\cap W) - \Phi(K \cap \partial^+ W)], \tag{B.5.6}$$

where $\partial^+ W = \{(x_1, x_2) \in \mathbb{R}^2 : \max(x_1, x_2) = 1\}$ is the upper right boundary of W, and Φ is additively extended. However, we base our later analysis on the estimator

$$\hat{\Phi}_W(X) = \frac{\sum_{K\in X}\Phi_W(K)}{V_2(W)}, \tag{B.5.7}$$

where

$$\Phi_W(K) = \frac{1}{2\pi}\int_{\operatorname{int} W\times S^1} u^2 \Lambda_1(K\cap W, d(x,u))$$

only takes into account the part of the boundary of the particle K which is contained in the interior $\operatorname{int} W$ of W. Firstly, $\hat{\Phi}_W(X)$ is more flexible and not restricted to rectangular windows, and secondly, for $W = [0,1]^2$ its variance is smaller than the one of $\tilde{\Phi}_W(X)$, at least when X is isotropic. The latter statement is made precise in Proposition B.5.2(ii), where a rank-2 tensor Φ is identified with the vector $(\Phi_{11}, \Phi_{22}, \Phi_{12})$ of its essential entries, so the covariance matrices are in $\mathbb{R}^{3\times 3}$. This Proposition also collects other basic properties of $\hat{\Phi}_W(X)$.

Proposition B.5.2. *For any $W \in \mathcal{K}^2$ with positive area we have:*

(i) *The estimator $\hat{\Phi}_W(X)$ in* (B.5.7) *is unbiased for* $\Phi(\mathrm{B}(X))$.

(ii) *If $W = [0,1]^2$ and X is isotropic, the covariance matrix of $\hat{\Phi}_W(X)$ is larger or equal to the covariance matrix of $\tilde{\Phi}_W(X)$ in* (B.5.6) *in the sense of Loewner partial order.*

Proof. To show (i) we use the fact [4, p. 103] that the support measure is locally determined, implying that

$$\Lambda_1(K\cap W, \operatorname{int} W \times \omega) = \Lambda_1(K, \operatorname{int} W\times\omega)$$

holds for all measurable $\omega \subset S^1$. Combining this with Campbell's theorem [19, Theorem 3.1.2], and the motion covariance [19, Theorem 14.2.2] of $\Lambda_1(K,\cdot)$ yields the asserted unbiasedness.

To show (ii) assume $W = [0,1]^2$ and note that the definition of the support measure implies

$$\Phi(K\cap W) - \Phi(K\cap\partial^+ W) = \Phi_W(K) + \frac{1}{2}\left[\Phi(K\cap\partial^- W) - \Phi(K\cap\partial^+ W)\right],$$

where $\partial^- W$ consists of all boundary points of W which are not in $\partial^+ W$. Hence, $\tilde{\Phi}_W(X) = \hat{\Phi}_W(X) + \Gamma_W(X)$ with

$$\Gamma_W(X) = \frac{1}{2} \sum_{K \in X} [\Phi(K \cap \partial^- W) - \Phi(K \cap \partial^+ W)].$$

Let X' be the rotation of X about the midpoint of W with angle π. Stationarity and isotropy of X imply that X and X' have the same distribution. But $\hat{\Phi}_W(X') = \hat{\Phi}_W(X)$ and $\Gamma_W(X') = -\Gamma_W(X)$, so $(\hat{\Phi}_W(X), \Gamma_W(X))$ has the same distribution as $(\hat{\Phi}_W(X), -\Gamma_W(X))$. We conclude

$$\mathrm{Cov}\left(\hat{\Phi}_W(X), \Gamma_W(X)\right) = -\mathrm{Cov}\left(\hat{\Phi}_W(X), \Gamma_W(X)\right).$$

Hence, the two stochastic vectors are uncorrelated, and

$$\mathrm{Var}\left(\tilde{\Phi}_W(X)\right) = \mathrm{Var}\left(\hat{\Phi}_W(X)\right) + \mathrm{Var}\left(\Gamma_W(X)\right).$$

We see that $\mathrm{Var}\left(\tilde{\Phi}_W(X)\right) - \mathrm{Var}\left(\hat{\Phi}_W(X)\right)$ is positive semi-definite, so the former variance is larger or equal than the latter by definition of Loewner's partial order. This concludes the proof. □

Let $\hat{\gamma}$ be an unbiased estimator of the intensity γ. Canonically, $\hat{\gamma}$ is chosen as the mean number of particles with center function in W, divided by the area $V_2(W)$. Then

$$\frac{\hat{\Phi}_W(X)}{\hat{\gamma}} \tag{B.5.8}$$

is a ratio-unbiased estimator for $\gamma^{-1}\Phi(\mathrm{B}(X))$. In view of (B.5.4), an estimator $\widehat{\mathrm{e}_\mathrm{B}}(X)$ for $\mathrm{e}_\mathrm{B}(X)$ can now be obtained using **Algorithm SurfaceTensorData*** with (B.5.8) as input. In the examples given in Figure B.1 we employed this method with the *lower tangent point* [19, p. 110] as center function for intensity estimation, and with precision $\varepsilon = 5 \cdot 10^{-5}$ in **Algorithm SurfaceTensorData***. We considered several stationary Poisson particle processes and sampled them in the compact unit square $W = [0,1]^2$. In all examples the intensity $\gamma = 300$ was chosen, but the distribution $\mathbb{Q}$ varied. Each row in Figure B.1 corresponds to a different process model. The first column shows realizations of X in W, where in (*a*) we chose $\mathbb{Q}(\{K\}) = 1$ for a fixed triangle $K \in \mathcal{K}^2$, in (*d*) $\mathbb{Q}$ is the image of a uniform distribution of $\varphi \in [0, 2\pi)$ under the mapping

$$\varphi \mapsto C_\varphi K, \tag{B.5.9}$$

where C_φ is the rotation about 0 with angle φ, turning X into an isotropic process, and in (*g*) the opening angle of the random isosceles triangle $\mathbf{K}_0$ is distributed uniformly in $[\pi/100, \pi/7]$. The second column depicts the corresponding $\widehat{\mathrm{e}_\mathrm{B}}(X)$ in black. The gray-shaded shape ellipse in the second column and the directional histogram in the last column will be explained after Definition B.5.3. Both estimators of the Blaschke ellipse in Figures B.1(*b*) and B.1(*h*) capture the intuitive 'orientation' and summarize the peakedness of the triangles. However, in the isotropic case (*e*) the Blaschke ellipse is a disk and contains no particle shape information (other than their average perimeter).

As noted before, isotropic processes are not the only cases where the Blaschke ellipse $\mathrm{e}_\mathrm{B}(X)$ is a disk. In fact, due to Proposition B.3.4, $\mathrm{e}_\mathrm{B}(X)$ is a disk if and only if

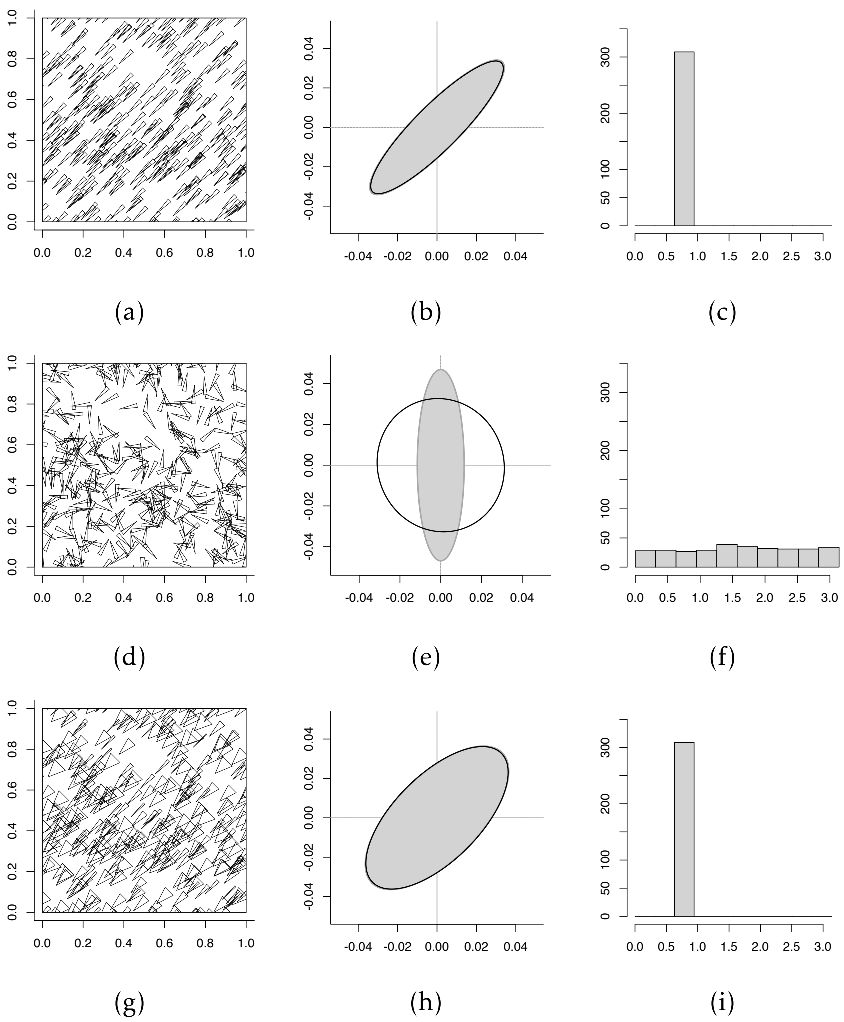

Figure B.1: Simulation of three different Poisson particle processes and their associated ellipses and directional information. The first column shows one realization of each process with intensity $\gamma = 300$. In (a) the typical particle $\mathbf{K}_0$ is a fixed triangle, in (d) $\mathbf{K}_0$ is the isotropic rotation of this triangle, and in (g) $\mathbf{K}_0$ can attain triangles with different shapes but the same 'orientation'. In the second column the corresponding estimated Blaschke ellipses (black) and the mean shape ellipses (shaded gray) are shown. The last column shows the histograms of the directions of $\mathrm{e}(K)$ for each sampled particle K with reference point in W.

the mean surface tensor is a multiple of the identity matrix. A possible example is a modification of the process in Figure B.1(a), where every other triangle is rotated by $\frac{\pi}{2}$ (not shown here). Another example of such an anisotropic process with circular $\mathrm{e}_{\mathrm{B}}(X)$ can be seen in Figure B.2(d) where $\mathbb{Q}$ is the image under (B.5.9) of a uniform angle $\varphi \in [0, \pi)$, and K is a fixed triangle.

In a way, the Blaschke ellipse mixes information about orientation and shape. This undesirable behaviour of the Blaschke ellipse is due to the averaging over all observed particles in its definition. If individual particles can be analyzed, we therefore suggest

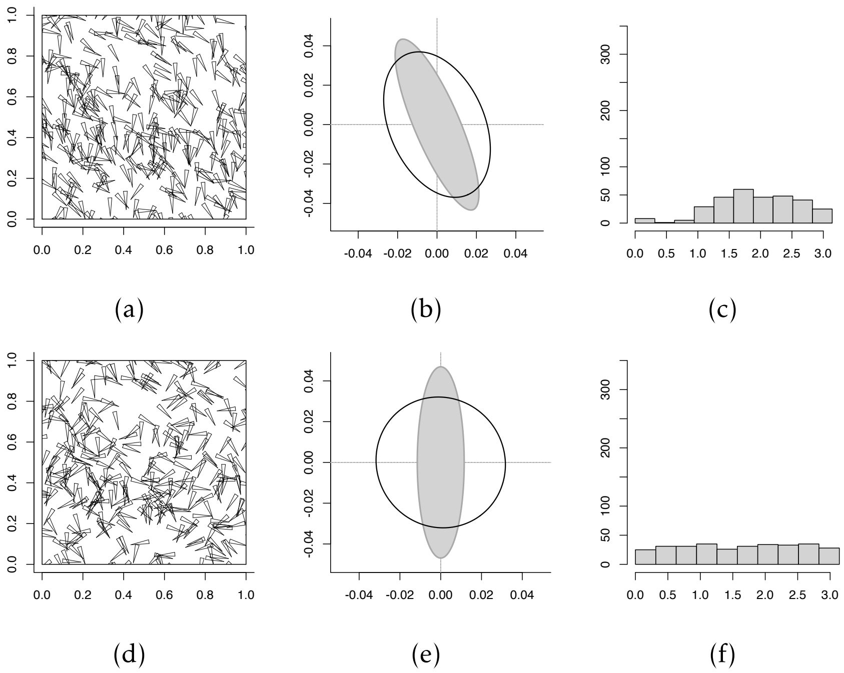

Figure B.2: Simulation of two different Poisson particle processes and their associated ellipses and directional information. The intensity was $\gamma = 300$ in both cases. The first row depicts a process with one deterministic triangle, rotated by a scaled beta-distributed angle. In (b), the estimated mean shape ellipse (gray shaded) clearly has a larger eccentricity than the estimated Blaschke ellipse (black), and the directional histogram in (c) captures the scaled beta-distribution. The second row depicts a process with one deterministic triangle, rotated by a uniform angle in $[0,\pi)$. In the corresponding (e), the estimated mean shape ellipse (gray shaded) captures (an ellipsoidal approximation of) the triangle shape, whereas $\widehat{e_B}(X)$ (black) is a disk. But in this case, the directional histogram in (f) is close to a uniform distribution, although the process is not isotropic.

to separate orientation and shape for each of them. This is made precise in the following definition of the *mean shape ellipse*. For $K \in \mathcal{K}^2$, we will write $\lambda_1(K) \le \lambda_2(K)$, $u(K) \in S^1_+ = \{u_\varphi : \varphi \in [0,\pi)\}$, and $\varphi(K) \in [0,\pi)$ for the smallest eigenvalue, the largest eigenvalue, the direction of the longest principal axis of $\mathrm{e}(K)$ and the angle of this direction with the x-axis, respectively.

Definition B.5.3. *The* mean shape ellipse *of a stationary particle process X of convex particles is the unique centered ellipse* $\mathrm{e_S}(X) \in \mathcal{E}$ *with lengths of principal axis*

$$\overline{\lambda}_1 = \mathbb{E}\lambda_1(\boldsymbol{K}_0), \textit{ and } \overline{\lambda}_2 = \mathbb{E}\lambda_2(\boldsymbol{K}_0),$$

and the longer principal axis direction U, where U is the mean direction *of $u(\boldsymbol{K}_0)$.*

The mean direction of $u(\mathbf{K}_0)$ must be understood as an appropriate modification of the standard notion in directional statistics [14, Section 2.2.1], taking into account that the direction is an element of S^1, but with antipodal points identified. Thus, $U = u_\varphi$ with $\varphi \in [0,\pi)$ must satisfy

$$u_{2\varphi} = r\mathbb{E}u_{2\varphi(\mathbf{K}_0)} \tag{B.5.10}$$

with some $r > 0$. If the mean on the right is zero, U is undefined and we set $U = (0,1)^t$ in this case. In simulations, we estimated U replacing the expectation in (B.5.10) with an empirical mean. From a theoretical point of view, one could have chosen to define U as the Fréchet mean of $u(\mathbf{K}_0)$ on the manifold S^1_+ with endpoints identified, if we insisted on an intrinsic mean. However, the Fréchet mean is computationally more involved. Furthermore, the analysis of the performance of these two estimators for a variety of circular distributions in [15] indicates that none of them is generally worse than the other.

The estimation of $\mathrm{e_S}(X)$ from observations in a sampling window W requires a careful treatment of edge effects. To avoid censoring by the window, we employed Miles' *associated point rule* [16] where all particles with their center function (lower tangent point) in W are assumed to be observable, even though they may not be completely contained in W. The numbers $\overline{\lambda}_1$ and $\overline{\lambda}_2$ where then estimated using empirical averages. As the number of observations contributing to these averages is random, the resulting estimators are only ratio-unbiased.

Returning to Figure B.1, we see that the estimated mean shape ellipse (gray shaded) coincides with $\widehat{\mathrm{e_B}}(X)$ in (b) and (h). The corresponding empirical histograms of $u(\mathbf{K}_0)$ in (c) and (i) reveal why this is the case: the individual ellipses associated to the observed particles all have the same principal axis direction. This is different in the isotropic case: in (f) the empirical histogram is close to a uniform distribution, but the estimated $\mathrm{e_S}(X)$ captures an elliptical approximation of a 'mean shape', which clearly indicates elongated particles. These examples illustrate how $\mathrm{e_S}(X)$ separates certain 'shape' and 'direction' information of the particles, which is mixed when $\mathrm{e_B}(X)$ is considered. This can also be observed in the first row of Figure B.2 for a process where $\mathbb{Q}$ is the image under (B.5.9), where $\varphi = B\pi$ and B follows a beta distribution with shape parameters $\alpha = 2$ and $\beta = 3$.

We emphasize the value of the histograms in the last columns of Figures B.1 and B.2. Clearly, if this histogram is close to uniform or is multimodal, the value of the direction U is debatable. In such cases, it might be better to report $\mathrm{e_S}(X)$ in an axis-parallel version to avoid misinterpretations.

As the histogram is close to uniform when the process is isotropic, one may employ a statistical test with the null hypotheses of isotropy. In the present context, where the Poisson assumption implies independence of the observed particles, a Kolmogorov-Smirnov test can be used for this purpose. Examples where the hypothesis is not rejected can be seen in Figures B.1(f) and B.2(f). Here they each have a p-values above 0.05, but only the first process is actually isotropic. False positives like in Figure B.2(f) occur as the distribution of $u(\mathbf{K}_0)$ (like any other summary statistic in a sufficiently complex setting) captures certain but not all features of the underlying process. In practical applications the observed particles may stochastically depend on one another and thus more elaborate inference tools must be used instead of a Kolmogorov-Smirnov test.

We summarize this applied section. We introduced two new ellipsoidal summary statistics for stationary particle processes. The first one is the Blaschke ellipse, which does not depend on the choice of the center function in contrast to the volume-tensor-based Miles ellipse in [25]. The second one is the mean shape ellipse, which allows to make inference on an average shape even if the Aumann expectation and thus a number of other set-valued means are disks.

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PAPER C

Mean Surface and Volume Particle Tensors under Restricted L-isotropy and Associated Ellipsoids

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Abstract

The convex-geometric Minkowski tensors contain information about shape and orientation of the underlying convex body. They therefore yield valuable summary statistics for stationary marked point processes with marks in the family of convex bodies, or, slightly more specialised, for stationary particle processes. We show here that if the distribution of the typical particle is invariant under rotations about a fixed k-plane, then the average volume tensors of the typical particle can be derived from $k+1$-dimensional sections. This finding extends the well-known three-dimensional special case to higher dimensions. A corresponding result for the surface tensors is also proven.

In the last part of the paper we show how Minkowksi tensors can be used to define three ellipsoidal set-valued summary statistics, discuss their estimation and illustrate their construction and use in a simulation example. Two of these, the so-called Miles ellipsoid and the inertia ellipsoid are based on mean volume tensors of ranks up to 2. The third, based on the mean surface tensor of rank-2 will be called Blaschke ellipsoid and is only defined when the typical particle has a rotationally symmetric distribution about an axis, as we then can use uniqueness and reconstruction results for centered ellipsoids of revolution from their rank-2 surface tensor. The latter are also established here.

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Keywords: Blaschke ellipsoid; Blaschke shape; Miles ellipsoid; Minkowski tensor; stationary marked point process; stationary particle process; restricted isotropy

C.1 Introduction

Any real-valued continuous, rigid motion invariant and additive functional on the family $\mathcal{K}^n$ of convex bodies (non-empty compact convex subsets of $\mathbb{R}^n$) is a linear combination of the intrinsic volumes due to Hadwiger's famous characterization theorem [5]. Alesker [1] extended this result to the space of certain tensor-valued functionals on $\mathcal{K}^n$, which are in turn spanned by the *Minkowski tensors*

$$\Phi_j^{r,s}(K) = c_{n,j}^{r,s} \int_{\mathbb{R}^n \times S^{n-1}} x^r u^s \Lambda_j\Big(K, d(x,u)\Big), \tag{C.1.1}$$

with $r,s \in \mathbb{N}_0$ if $j = 0,\dots,n-1$, and $r \in \mathbb{N}_0, s = 0$ for $j = n$. Here $x^r u^s$ is a symmetric tensor of rank $r+s$, $\Lambda_j(K,\cdot)$ is the jth support measure of K, S^{n-1} is the unit sphere in $\mathbb{R}^n$, and $c_{n,j}^{r,s}$ is a known constant. We call $\Phi^s(K) = \Phi_{n-1}^{0,s}(K)$ the *surface tensors* of K and $\Psi^r(K) = \Phi_n^{r,0}(K)$ the *volume tensors* of K.

Alesker's characterization theorem gives the Minkowski tensors a crucial role in convex geometry. As they also capture certain orientation and shape information of the underlying set, they have been used as summary statistics, e.g. as anisotropy indices for galaxies [3], or for the analysis of ice grains in arctic drill cores [17]. They also found applications in biology, where cells in tissue were modelled as particles with their nucleus or nucleolus as reference points, (see for instance [19]). We will follow [19] and model stochastic collections of particles with associated reference points as stationary marked point processes with convex particles as marks. For abbreviation, a mean Minkowski tensor of the typical particle (typical mark) of such a process will be called *mean particle tensor*. The present paper gives in Section C.2 a simplification of the mean volume and surface particle tensors under the assumption that the distribution of the typical particle is invariant under rotations about a fixed linear plane L. The latter property will be called *L-restricted isotropy* as it is a weak form of the common isotropy assumption. We will show how the average *volume* tensor of the typical particle can be derived from the intersection of this particle with a linear subspace of dimension $\dim L+1$ containing L. This generalizes the main result in [11], where the three-dimensional case has been treated. A corresponding result for the *surface* tensor is also discussed. This is of particular interest in local stereology where estimators such as the nucleator are made by intersecting the particles with linear subspaces, see for instance [6].

To illustrate the orientation and shape information carried by the Minkowski tensors consider a polynomial p of degree at most $d \geq 2$ such that $K = \{x \in \mathbb{R}^n : p(x) \geq 0\}$ is a convex body. Then K is determined among all Borel sets up to a set of Lebesgue measure zero by $\{\Psi^r(K) : r = 0,1,\dots,d\}$; see [8] and [10]. In particular, an ellipsoid in $\mathbb{R}^n$ is determined by its volume tensors up to rank 2 among all convex bodies and an explicit reconstruction of this ellipsoid can be given; cf. [7, p. 266]. This uniqueness has been used in stereology to construct ellipsoids that capture the shape and orientation of the typical particle of stationary particle processes. The Miles ellipsoid, a set-valued summary statistic based on this construction was introduced in [19] in the three-dimensional case – its extension to general dimension is straightforward and will be given in Section C.4. As shown in Example C.4.5, the Miles ellipsoid as a test statistic for isotropy loses power as it mixes shape and position information of the individual particles. We therefore introduce the so-called inertia ellipsoid, which is based on the same volume tensors but centers the particles more appropriately before averaging their tensors.

An alternative, which also eliminates the influence of particle position are ellipsoids based on surface tensors. In contrast to the volume tensors, the surface tensors are translation invariant, so a set K can at the best be determined up to translation by finitely many surface tensors. In Paper B it was proven that for $n = 2$ the rank-2 surface tensor of a centered ellipse determines this ellipse uniquely among all planar convex bodies. The corresponding result in higher dimensions is still open. We will in Section C.3 narrow this gap by showing uniqueness in the case where the ellipsoids are rotation invariant about a fixed line. This allows us to introduce the so-called Blaschke ellipsoid of processes that are L-restricted isotropic with L being a line. The Blaschke ellipsoid is based on the mean rank-2 surface tensor of the typical particle and turns out to be the unique centered ellipsoid that has the same rank-2 surface tensor as the Blaschke shape of the process.

The paper is structured as follows. Section C.1.1 introduces basic definitions and notations including some facts about hypergeometric functions, which will be useful later on. In Section C.2 we consider a stationary marked point process in $\mathbb{R}^n$ with convex bodies as marks which is L-restricted isotropic, and we show how the mean volume and the mean surface particle tensors can be derived from the intersection with a linear $(\dim L + 1)$-dimensional subspace containing L. Section C.3 is devoted to proving that the rank-2 surface tensor uniquely determines centered ellipsoids of revolution, i.e. centered ellipsoids which are invariant under rotations about a fixed axis. It also contains an algorithm to reconstruct the ellipsoid given its surface tensor. This is then used in Section C.4 to introduce the Blaschke ellipsoid in the case of L-restricted isotropy, where L is a line. By definition, its rank-2 surface tensor coincides with the mean surface particle tensor. Section C.4 also contains definitions and an introduction to the inertia and the Miles ellipsoids both of which are associated ellipsoids constructed using the mean volume particle tensors of ranks up to 2. A discussion of the strengths and weaknesses of these ellipsoids can also be found in Section C.4.

C.1.1 Preliminaries and notation

In this section we introduce basic concepts and results relevant for the rest of the paper. The notation follows widely [13], which also is an excellent reference for convex geometric notions. We will write $e_1, \ldots, e_n$ for the standard basis in $\mathbb{R}^n$, $n \in \mathbb{N}$. For $k, r \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{0, \ldots, r\}$ with $i_1 + \cdots + i_k = r$ we use the notation

$$(z_1[i_1], \ldots, z_k[i_k]) = (\underbrace{z_1, \ldots, z_1}_{i_1}, \ldots, \underbrace{z_k, \ldots, z_k}_{i_k}) \in \mathbb{R}^{nr},$$

$z_1 \ldots, z_r \in \mathbb{R}^n$. If one of these vectors only occurs once, we sometimes omit the corresponding parenthesis as in $(z_1[r-1], z_2) = (z_1[r-1], z_2[1])$.
For $u \in \mathbb{R}^n, u \neq 0$, we define the closed half space

$$u^+ = \{x \in \mathbb{R}^n : \langle x, u \rangle \geq 0\},$$

where $\langle x, u \rangle$ is the usual inner product of x and u.
The m-dimensional Hausdorff measure is $\mathcal{H}^m(\cdot)$, and $\omega_n = \mathcal{H}^{n-1}(S^{n-1}) = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$

and $\kappa_n = \mathcal{H}^n(\mathbb{R}^n) = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$ denote the surface area of the unit sphere S^{n-1} and the volume of the Euclidean unit ball B^n in $\mathbb{R}^n$, respectively.

C.1.2 Volume- and surface tensors

The family of non-empty compact convex sets in $\mathbb{R}^n$ will be denoted by $\mathcal{K}^n$. Its elements are called convex bodies. Two convex bodies $K, L \in \mathcal{K}^n$ have Hausdorff distance $d_H(K,L)$. For $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n \backslash K$ there is a unique point nearest point $p(K,x) \in K$ to x. The normalized vector pointing from $p(K,x)$ to x will be denoted by $u(K,x)$, and is given by

$$u(K,x) = \frac{x - p(K,x)}{d(K,x)},$$

where $d(K,x) = \|x - p(K,x)\|$. The projection onto a linear subspace $M \subset \mathbb{R}^n$ will be denoted by $\mathrm{p}_M(\cdot)$. The *support function*

$$h_K(u) = \max_{x \in K} \langle u, x \rangle,$$

$u \in S^{n-1}$, determines $K \in \mathcal{K}^n$ uniquely.

For $k \in \{1,\ldots,n\}$, $G(n,k)$ is the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$. Given $M \in G(n,k)$, the support set of a convex body K in M with ambient space M, is denoted by $\mathrm{Nor}_M(K)$ and the j'th support measure in M by $\Lambda_j^M(K,\cdot)$. If $k = n$, and hence the ambient space is $\mathbb{R}^n$, we shorten the notation and write $\mathrm{Nor}(K)$ and $\Lambda_j(K,\cdot)$ respectively. The total mass of the j'th support measure

$$V_j(K) = \Lambda_j\left(K, \mathbb{R}^n \times S^{n-1}\right)$$

is called the j'th intrinsic volume.

The j'th surface measure is up to normalization the marginal measure of the support measure defined on the set of normal vectors:

$$S_j(K,\omega) = n\kappa_{n-j}\binom{n}{j}^{-1}\Lambda_j(K, \mathbb{R}^n \times \omega),$$

$\omega \subset S^{n-1}$ measurable.

We will work with *symmetric tensors* on $\mathbb{R}^n$, which are r-linear symmetric functionals on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (r copies) for some $r \in \mathbb{N}_0$. The number r is called the *rank* of the corresponding tensor. For $x \in \mathbb{R}^n$, $z \mapsto \langle x, z \rangle$ is a tensor of rank 1, and

$$x^r = \underbrace{x \odot x \cdots \odot x}_{r}$$

is a tensor of rank r, where $\odot$ is the symmetrized tensor product. We will also write $x^r u^s = x^r \odot u^s$, where $u \in \mathbb{R}^n$. Using the standard basis of $\mathbb{R}^n$, a tensor T of rank r can be identified with a symmetric array, as

$$T(z_1,\ldots,z_r) = \sum_{j_1=1}^{n} \sum_{j_r=1}^{n} T(e_{j_1},\ldots,e_{j_r}) z_{1,j_1} \cdots z_{r,j_r},$$

$z_1, \dots, z_r \in \mathbb{R}^n$. With this identification, tensors of rank $0, 1, 2$ are identified with real numbers, vectors and symmetric $n \times n$ matrices, respectively. Due to the symmetry of this array, T can equivalently be described by the numbers

$$T_{i_1,\dots,i_n} = T\Big(e_1[i_1], \dots, e_n[i_n]\Big), \tag{C.1.2}$$

where $i_1, \dots, i_n \in \mathbb{N}_0$ with $\sum_{j=1}^n i_j = r$. For instance, $(x^r)_{i_1,\dots,i_n} = x_1^{i_1} \cdots x_n^{i_n}$. Adopting the usual notation for the *metric tensor* Q with $Q(z_1, z_2) = \langle z_1, z_2 \rangle$, the definition of the symmetric tensor product implies

$$(Q^r)_{2i_1,\dots,2i_n} = \binom{r}{i_1, \dots, i_n} \binom{2r}{2i_1, \dots, 2i_n}^{-1} \tag{C.1.3}$$

for $\sum_{j=1}^n i_j = r$.

The *Minkowski tensors* $\Phi_j^{r,s}(K)$, $K \in \mathcal{K}^n$, are defined by (C.1.1) with the constant

$$c_{n,j}^{r,s} = \frac{1}{r!s!} \frac{\omega_{n-j}}{\omega_{n-j+s}}.$$

Hence we get the following expressions for the *volume tensors*

$$\Psi^r(K) = \frac{1}{r!} \int_K x^r \, dx.$$

$r \in \mathbb{N}_0$, and the *surface tensors*,

$$\Phi^s(K) = \frac{1}{s!\omega_{s+1}} \int_{S^{n-1}} u^s \, \mathrm{S}_{n-1}(K, du),$$

$s \in \mathbb{N}_0$. When $K \in \mathcal{K}^n$ is full-dimensional, $c(K) = \Psi^1(K)/\Psi^0(K)$ (identified with a vector) is simply the physical *center of mass* of the rigid body K when assumed to have constant density. In contrast, $\Phi^1(K) = \mathbf{0}$ does not carry any information on K.

Section C.3 will rely heavily on the theory of *hypergeometric functions*. We will therefore introduce some of the basic theory of these functions in the next subsection.

C.1.3 Preliminaries on hypergeometric functions

The Gaussian hypergeometric functions (see e.g. [2, Chapter 2]) can be represented by the following power series for $a, b \in \mathbb{R}$, $c > 0$

$$\mathrm{F}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{C.1.4}$$

$|z| < 1$, where $(q)_n$ is the Pochhammer symbol.

For $c > a + b$ the value $\mathrm{F}(a, b; c; 1)$ is well-defined as (C.1.4) converges in this case at $z = 1$. In any case, there exists an analytic continuation of this function on the cut complex plane $\mathbb{C} \setminus [1, \infty)$, but we will only work with real arguments here.

The derivative of a hypergeometric function is

$$\mathrm{F}'(a, b; c; z) = \frac{ab}{c} \mathrm{F}(a + 1, b + 1; c + 1; z), \tag{C.1.5}$$

see [2, Section 2.5]. For $c > b > 0$ and $z < 1$, Euler's integral representation is given by

$$B(b, c-b)\,\mathrm{F}(a,b;c;z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx, \tag{C.1.6}$$

Pfaff's transformation by

$$\mathrm{F}(a,b;c;z) = (1-z)^{-b}\,\mathrm{F}(c-a,b;c;\tfrac{z}{z-1}) \tag{C.1.7}$$

and Euler's transformation by

$$\mathrm{F}(a,b;c;z) = (1-z)^{c-a-b}\,\mathrm{F}(c-a,c-b;c;z). \tag{C.1.8}$$

Further details can be found in [2, Section 2.2]. For $z = 1$ we have [2, Thm 2.2.2]

$$\mathrm{F}(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{C.1.9}$$

if $c > a+b$, and if $c = a+b$ then [2, Thm 2.1.3] states

$$\lim_{z\to 1-} \frac{\mathrm{F}(a,b;a+b;z)}{-\ln(1-z)} = \frac{\Gamma(a+b)}{\Gamma(b)\Gamma(a)}. \tag{C.1.10}$$

The following Lemma will be applied in Subsection C.3.1 with the function $f_n = f_{\frac{1}{2},\frac{n+1}{2},\frac{n}{2}+1}$.

Lemma C.1.1. *Let $a,b,c \in \mathbb{R}$ with $0 < a \le c$. The function $f_{a,b,c} : (-\infty,1] \to [0,\infty)$ given by*

$$f_{a,b,c}(z) = \begin{cases} \frac{\mathrm{F}(a,b;c;z)}{\mathrm{F}(a+1,b;c;z)}, & \text{if } z < 1,\\ 0, & \text{if } z = 1,\end{cases}$$

is a homeomorphism.

Proof. We start by proving that $f = f_{a,b,c}$ is a strictly decreasing function.

Note that if $z < 0$, Pfaff's transformation (C.1.7) yields

$$f(z) = \frac{(1-z)^{-b}\,\mathrm{F}(c-a,b;c;\frac{z}{z-1})}{(1-z)^{-b}\,\mathrm{F}(c-(a+1),b;c;\frac{z}{z-1})} = \left(\frac{\mathrm{F}(c-a-1,b;c;\frac{z}{z-1})}{\mathrm{F}(c-a,b;c;\frac{z}{z-1})}\right)^{-1}.$$

As $z \mapsto \frac{z}{z-1}$ is monotonically decreasing when $z < 0$ it is enough to prove the monotonicity for $0 < z < 1$. We abbreviate $\mathrm{F}(z) = \mathrm{F}\big(a,b;c;z\big)$ and $\mathrm{F}_+(z) = \mathrm{F}\big(a+1,b;c;z\big)$. Now

$$f'(z) = \frac{f_1(z)}{\mathrm{F}_+(z)^2} \tag{C.1.11}$$

with

$$f_1(z) = (\mathrm{F}_+(z)\,\mathrm{F}'(z) - \mathrm{F}_+'(z)\,\mathrm{F}(z)).$$

The differentiation formula (C.1.5) gives

$$\begin{aligned} f_1(z) = \frac{b}{c}\Big(&a\,\mathrm{F}(a+1,b+1;c+1;z)\,\mathrm{F}(a+1,b;c;z)\\ &-(a+1)\,\mathrm{F}(a,b;c;z)\,\mathrm{F}(a+2,b+1;c+1;z)\Big).\end{aligned}$$

Putting $y_n = \frac{(a)_n(b)_n}{(c)_n n!}$, the series representation (C.1.4) combined with $(x+1)_n = \frac{x+n}{x}(x)_n$ yield

$$\mathrm{F}(a+1,b+1;c+1;z) = \frac{c}{ab}\sum_{n=0}^{\infty} y_n \frac{(a+n)(b+n)}{c+n} z^n,$$

$$\mathrm{F}(a+1,b;c;z) = \frac{1}{a}\sum_{n=0}^{\infty} y_n (a+n) z^n,$$

$$\mathrm{F}(a,b;c;z) = \sum_{n=0}^{\infty} y_n z^n,$$

$$\mathrm{F}(a+2,b+1;c+1;z) = \frac{c}{ab(a+1)}\sum_{n=0}^{\infty} y_n \frac{(a+n)(a+n+1)(b+n)}{c+n} z^n.$$

Hence,

$$f_1(z) = \frac{1}{a}\sum_{n=0}^{\infty} (c_n - \tilde{c}_n) z^n, \tag{C.1.12}$$

where Cauchy's product formula implies

$$c_n = \sum_{k=0}^{n} y_k y_{n-k} \frac{(a+k)(b+k)}{c+k}(a+n-k),$$

$$\tilde{c}_n = \sum_{k=0}^{n} y_k y_{n-k} \frac{(a+k)(b+k)}{c+k}(a+k+1).$$

Hence

$$c_n - \tilde{c}_n = \sum_{k=0}^{n} y_k y_{n-k} s_k = \frac{1}{2}\sum_{k=0}^{n} y_k y_{n-k}[s_k + s_{n-k}], \tag{C.1.13}$$

with

$$s_k = \frac{(a+k)(b+k)}{c+k}(n-2k-1).$$

We claim that s_k+s_{n-k} is negative for all $k = 0,\dots,n$. By symmetry, and the fact that $s_{n/2}$ is negative for even n, it is enough to consider $k < n/2$. For these k we have $k < n-k$. As $0 < a \le c$ the monotonicity of $x \mapsto (a+x)/(c+x)$ and of $x \mapsto (b+x)$, gives

$$s_k + s_{n-k} \le \frac{(a+n-k)(b+n-k)}{c+n-k}[(n-2k-1)+(n-2(n-k)-1)] < 0.$$

Thus, in view of (C.1.13), all coefficients in the expansion (C.1.12) are negative, implying $f_1(z) < 0$ for all $0 < z < 1$, and the continuity of f now yields that f is strictly decreasing. This implies injectivity and surjectivity of f. The continuity of its inverse is a consequence of the fact that f is strictly decreasing. □

We will also need the following lemma, which is a direct consequence of (C.1.5) and the definition of hypergeometric functions. To ease notation we define functions $F(z) = \mathrm{F}(\frac{1}{2}, \frac{n-1}{2}; \frac{n}{2}+1; z)$ and $F_+(z) = \mathrm{F}(\frac{3}{2}, \frac{n+1}{2}; \frac{n}{2}+2; z)$.

Lemma C.1.2. *For $0 \le a < b < 1$ the function*

$$l_n(z) = F(z)^{-\frac{1}{n-1}}$$

is strictly decreasing and Lipschitz on $[a,b]$ with constant $\frac{1}{2(n+2)}F_+(b)F(a)^{-\frac{n}{n-1}}$.

C.2 L-restricted isotropy

In this section we will consider a stationary random collection of particles X and show that the mean tensors of the typical particle can be derived from $(k+1)$-dimensional sections when X is L-restricted isotropic, $L \in G(n,k)$.

In view of applications, we model $X = \{(x_i, K_i) : i = 0, 1, 2, \ldots\}$ as a stationary marked point process with marks K_i in the family $\mathcal{K}^n$ of convex bodies, see [16, Chapter 3.5] for details. Its *intensity* $\gamma > 0$ is the mean number of points per unit volume, and its *mark distribution* $\mathbb{Q}$ can be understood as the distribution of the mark of a typical point in the Palm sense. The *typical particle* $\mathbf{K}_0$ is the random convex body with distribution $\mathbb{Q}$.
We denote the associated process consisting of the translated marks by

$$\tilde{X} = \{x_i + K_i | (x_i, K_i) \in X\}.$$

The point x_i will be called the *reference point* of the particle $x_i + K_i$. For $\tilde{X}$ to become a stationary particle process, $\mathbb{Q}$-integrability of all intrinsic volumes

$$\mathbb{E} V_j(\mathbf{K}_0) < \infty, \quad i = 0, \ldots, n, \tag{C.2.1}$$

is needed. We therefore assume from now on that this integrability condition is met.

Modelling collections of particles via the stationary marked point process model X is slightly more general then modelling them as a classical stationary particle process. In fact, any particle process is a germ-grain process and thus can be written as $\tilde{X}$ with X being a marked point process with a centered typical particle satisfying (C.2.1). To do so, one chooses $x_i = z(\tilde{K}_i)$ and $K_i = \tilde{K}_i - x_i$ for particle $\tilde{K}_i$, where $z(\cdot)$ is a translation covariant center function, see [16, Section 4.2]. But the representation as marked point process is more general, as it can be thought of as collection of particles, each of which carrying a individually selectable reference point. This is relevant for instance in biological applications, where cells are modelled as particles with their nucleus or nucleolus as reference points.

We say X is degenerate if there is a hyperplane such that the orthogonal projection of $\mathbf{K}_0$ on this hyperplane has a.s. vanishing $\mathcal{H}^{n-1}$-content, see [16, Chapter 4.6]. If, for instance $V_n(\mathbf{K}_0) > 0$ with positive probability, then X is not degenerate. For non-degenerate processes Minkowski's theorem (see for instance [16, Theorem 14.3.1]) yields the existence of a full dimensional convex body $\mathrm{B_s}(X)$ such that

$$S_{n-1}(\mathrm{B_s}(X), \cdot) = \mathbb{E} S_{n-1}(\mathbf{K}_0, \cdot). \tag{C.2.2}$$

Adopting the usual convention that the circumcenter of $\mathrm{B_s}(X)$ coincides with the origin, this determines $\mathrm{B_s}(X)$ uniquely. The body $\mathrm{B_s}(X)$ is called *Blaschke shape* in [18]. Note that [16, Section 4.6] works with a scaled version of this mean set, the so-called *Blaschke body* $B(X) = \gamma^{\frac{1}{n-1}} \mathrm{B_s}(X)$.

We will work under the assumption of the typical particle being invariant under rotations fixing a lower dimensional subspace $L \in G(n,k)$ pointwise. The group of such rotations will be denoted by $\mathrm{SO}_n(L^\perp) = \{C \in \mathrm{SO}(n) | Cy = y \ \forall y \in L\}$. This property of X is called *L-restricted isotropy* and is defined as follows.

Definition C.2.1. *Let $k \in \{1,\dots,n-1\}$ and $L \in G(n,k)$.*
*For $k < n-1$, X is called L-**restricted isotropic** if its mark distribution $\mathbb{Q}$ is invariant under rotations fixing the linear subspace L pointwise, that is, if the distributions of $\boldsymbol{K}_0$ and $C\boldsymbol{K}_0$ coincide for all $C \in \mathrm{SO}_n(L^\perp)$.*

*If $k = n-1$, X is L-**restricted isotropic** if the distribution, $\mathbb{Q}$, is invariant under the reflection in L.*

In the case $n = 3$, $k = 1$ the L-restricted isotropy of X was called *restricted isotropy* in [11]. Adopting usual parlance, the property in Definition C.2.1 should be called *weak L-restricted isotropy*, in analogy to *weakly stationary* processes X, which need not be stationary, but have the property that their intensity measure is translation invariant. However, we will omit the word *weakly* throughout for brevity.

Under such isotropy assumptions we can simplify the general expression of the mean volume tensor $\mathbb{E}\Psi^r(\mathbf{K}_0)$ and the mean surface particle tensor $\mathbb{E}\Phi^s(\mathbf{K}_0)$ to tensors only depending on the section profiles of the typical particle with $L \cup \mathrm{span}(w)$, where $w \in L^\perp$, $\|w\| = 1$, is fixed. To assure that $\mathbb{E}\Psi^r(\mathbf{K}_0)$ is finite, we will assume

$$\mathbb{E}\int_{\mathbf{K}_0} \|x\|^r \mathcal{H}^n(dx) < \infty. \tag{C.2.3}$$

Note that this assumption is necessary when $r = 2$ and does not follow from the general assumption (C.2.1). One may for instance consider the case $n = 1$ and the typical particle $\mathbf{K}_0 = [-T, T]$, where T is a positive random variable with finite mean and infinite third moment. An even stronger condition than (C.2.3) for processes in $\mathbb{R}^n$, which is easier to check, is the condition that

$$R(\mathbf{K}_0) = \max\{\|x\| : x \in \mathbf{K}_0\}$$

has finite moments up to order $n + r$.

Theorem C.2.2. *Let $r \in \mathbb{N}_0$, $k \in \{1,\dots,n-2\}$ and $L \in G(n,k)$ be given. If X is an L-restricted isotropic marked point process satisfying* (C.2.3) *then*

$$\begin{aligned}\mathbb{E}\Psi^r(\boldsymbol{K}_0) = &\frac{2}{r!}\sum_{s=0}^{\lfloor \frac{r}{2}\rfloor}\binom{r}{2s}\frac{\omega_{2s+n-k}}{\omega_{2s+1}}\,\mathrm{Q}^s_{L^\perp}\\ &\int_L\int_0^\infty \mathbb{P}((\alpha w + y) \in \boldsymbol{K}_0)\alpha^{n-k-1+2s}\,d\alpha\, y^{r-2s}\mathcal{H}^k(dy),\end{aligned} \tag{C.2.4}$$

for any $w \in L^\perp \cap S^{n-1}$. Here, $\mathrm{Q}_{L^\perp}(z_1, z_2) = \langle \mathrm{p}_{L^\perp}(z_1), \mathrm{p}_{L^\perp}(z_2)\rangle$ is the metric tensor in $L^\perp$.

Theorem C.2.2 generalizes the main theoretical result in [11], where the case $n = 3$ and $k = 1$ was treated, see its reformulation in Corollary C.2.5, below.
Note that if L is a hyperplane, and thus $k = n-1$, the simpler expression

$$\mathbb{E}\Psi^r(\mathbf{K}_0) = 2\mathbb{E}\Psi^r(\mathbf{K}_0 \cap M_+)$$

can be obtained using a decomposition of the Lebesgue measure on $L^\perp$ and L. Here M_+ is one of the closed half planes in $\mathbb{R}^n$ with L as its boundary.

Although this theorem is stated in terms of marked point processes, it has a deterministic counterpart, which is obtained in a straightforward manner assuming that $\mathbf{K}_0$ coincides with a fixed convex body K almost surely. The statement involves the *radial function*

$$\rho_K(x) = \sup\{\alpha \geq 0 : \alpha x \in K\}, \quad x \in \mathbb{R}^n,$$

of K.

Corollary C.2.3. *Let $k \in \{1, \dots, n-2\}$ and $L \in G(n,k)$ be given. If $K \in \mathcal{K}^n$ is invariant under all rotations in $SO_n(L^\perp)$, then*

$$\Psi^r(K) = \frac{2}{r!} \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2s} \frac{\kappa_{2s+n-k}}{\omega_{2s+1}} \mathrm{Q}^s_{L^\perp} \int_L \rho_{K-y}^{2s+n-k}(w) y^{r-2s} \mathcal{H}^k(dy),$$

for any $w \in L^\perp \cap S^{n-1}$ and $r \in \mathbb{N}_0$.

A similar result can be obtained for the mean surface particle tensor. In contrast to the case of volume tensors, no additional integrability assumption is needed, as (C.2.1) with $j = n-1$ implies the existence of $\mathbb{E}\Phi^s(\mathbf{K}_0)$ for all $s \in \mathbb{N}_0$. Fubini's theorem implies that

$$\mathbb{E}\Phi^s(\mathbf{K}_0) = \mathbb{E}\Phi^s(\mathrm{B_s}(X)) \tag{C.2.5}$$

if X is non-degenerate. If X is L-restricted isotropic, the rotation covariance of the surface measure (see for instance [16, Theorem 14.2.2]) implies that the Blaschke shape $\mathrm{B_s}(X)$ must be rotation invariant under all rotations fixing L pointwise.

Theorem C.2.4. *Let $s \in \mathbb{N}_0$, $k \in \{0, \dots, n-2\}$, $L \in G(n,k)$, $M \in G(n,k+1)$ with $L \subset M$ and $w \in M \cap L^\perp \cap S^{n-1}$ be given. If X is a non-degenerate L-restricted isotropic point process then*

$$\begin{aligned}\mathbb{E}\Phi^s(\boldsymbol{K}_0) = \sum_{t=0}^{\lfloor \frac{s}{2} \rfloor} b_{n,k,s}(t) \mathrm{Q}^t_{L^\perp} \int_{\mathrm{Nor}_M(\mathrm{B_s}(X)\cap M_+)} &\langle u, w\rangle^{2t} \langle x, w\rangle^{n-k-1} \\ &\times \mathrm{p}_L(u)^{s-2t} \Lambda_k^M\Big(\mathrm{B_s}(X) \cap M_+, d(x,u)\Big),\end{aligned} \tag{C.2.6}$$

for $s \in \mathbb{N}_0$, where $M_+ = M \cap w^+$ and

$$b_{n,k,s}(t) = \binom{s}{2t} \frac{4\omega_{2t+n-k}}{s!\,\omega_{s+1}\,\omega_{2t+1}}.$$

As before, this theorem has a deterministic counterpart when $\mathbf{K}_0 = K$ is a deterministic convex body that is invariant under rotations in $\mathrm{SO}(L^\perp)$. In this case, $\mathrm{B_s}(X) = K$ can be inserted into (C.2.6) yielding an expression for $\Phi^s(K)$ only depending on $K \cap M_+$.

C.2.1 Proof of Theorem C.2.2

Proof of Theorem C.2.2: Let k, r, w, L and X be given as in the theorem. In view of (C.2.3), Fubini's theorem implies

$$\begin{aligned}\mathbb{E}\Psi^r(\mathbf{K}_0) &= \frac{1}{r!}\mathbb{E}\int_{\mathbf{K}_0} x^r\,dx \\ &= \frac{1}{r!}\int_L \int_{L^\perp} (x+y)^r \mathbb{P}(x+y \in \mathbf{K}_0) \mathcal{H}^{n-k}(dx) \mathcal{H}^k(dy).\end{aligned}$$

For each $x \in L^\perp$ there is a rotation $C \in SO_n(L^\perp)$ such that $Cx = \|x\|w$ and, obviously, $Cy = y$ for all $y \in L$. Thus, the L-restricted isotropy assumption, the use of spherical coordinates in $L^\perp$, and Fubini's theorem give

$$\mathbb{E}\Psi^r(\mathbf{K}_0) = \frac{1}{r!}\int_L \int_0^\infty \alpha^{n-k-1} f_r(\alpha, y)\mathbb{P}(\alpha w + y \in \mathbf{K}_0)\, d\alpha\, \mathcal{H}^k(dy),$$

where

$$f_r(\alpha, y) = \int_{S^{n-1}\cap L^\perp} (\alpha u + y)^r \mathcal{H}^{n-k-1}(du), \tag{C.2.7}$$

$\alpha \in [0,\infty)$ and $y \in L$. Evaluating the integral defining f_r (using the binomial formula for tensors and, for instance, [14, Equation (24)]) yields

$$f_r(\alpha, y) = 2\sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2s} \frac{\omega_{2s+n-k}}{\omega_{2s+1}} \alpha^{2s} y^{r-2s}\, \mathrm{Q}_{L^\perp}^s, \tag{C.2.8}$$

which implies the assertion. □

Identifying tensors with finitely many numbers, as outlined in Subsection C.1.2, Equation (C.1.2), allows us to restate (C.2.4) in a form which emphasizes that the mean volume particle tensor only depends on $\mathbf{K}_0 \cap M$ where M is a $(k+1)$-dimensional subspace containing L. We state this only for axes parallel subspaces and use the notation $\Psi_M^r(K)$ for the relative volume tensor of a convex body K contained in a linear subspace M.

Corollary C.2.5. *Let $r \in \mathbb{N}_0$, $k \in \{1,\dots,n-2\}$, and put*

$$L = \operatorname{span}\{e_{n-k+1},\dots,e_n\},\ M = \operatorname{span}\{e_{n-k},\dots,e_n\}, M_+ = M \cap e_{n-k}^+. \tag{C.2.9}$$

Assume that X is L-restricted isotropic and satisfies (C.2.3). *For $i_1,\dots,i_n \in \{0,\dots,r\}$ with $\sum_{j=1}^n i_j = r$ the mean volume particle tensor satisfies*

$$\begin{aligned}\mathbb{E}\Psi^r(\boldsymbol{K}_0)_{i_1,\dots,i_n} = &\tfrac{(r+n-k-1)!}{r!} c_{i_1,\dots,i_{n-k}} \\ &\times \mathbb{E}\Psi_M^{r+n-k-1}(\boldsymbol{K}_0 \cap M_+)_{i,i_{n-k+1},\dots,i_n},\end{aligned}$$

where $i = n-k-1+\sum_{j=1}^{n-k} i_j$ and

$$c_{i_1,\dots,i_{n-k}} = \begin{cases} \frac{2}{\Gamma(\frac{i+1}{2})}\prod_{j=1}^{n-k}\Gamma\left(\frac{i_j+1}{2}\right) & \text{for } i_1,\dots,i_{n-k} \text{ even,} \\ 0 & \text{otherwise.}\end{cases} \tag{C.2.10}$$

Proof. Fix $i_1\dots,i_n \in \{1,\dots,r\}$ such that $r = \sum_{j=1}^n i_j$ and abbreviate $r' = \sum_{j=1}^{n-k} i_j$. Using the same notation for $f_r(\alpha,y)$ as in the proof of Theorem C.2.2, Equation (C.2.8) gives

$$\begin{aligned} f_r(\alpha,y)_{i_1,\dots,i_n} &= 2\binom{r}{r'}\frac{\omega_{r'+n-k}}{\omega_{r'+1}}\alpha^{r'}\Big[y^{r-r'}\,\mathrm{Q}_{L^\perp}^{\frac{r'}{2}}\Big](e_1[i_1],\dots,e_n[i_n]) \\ &= 2\frac{\omega_{r'+n-k}}{\omega_{r'+1}}\alpha^{r'}\Big[\mathrm{Q}_{L^\perp}^{\frac{r'}{2}}(e_1[i_1],\dots,e_{n-k}[i_{n-k}])\Big] \\ &\quad\times\Big[y^{r-r'}(e_{n-k+1}[i_{n-k+1}],\dots,e_n[i_n])\Big],\end{aligned}$$

as the metric tensor in $L^\perp$ and y^{r-2s} are zero when applied to at least one vector in L and $L^\perp$, respectively. If at least one of the indices $i_1,\dots,i_{n-k}$ is odd, this expression is zero. Otherwise Equation (C.1.3) ($L^\perp$ can be identified with $\mathbb{R}^{n-k}$) yields

$$\begin{aligned} f_r(\alpha,y)_{i_1,\dots,i_n} &= 2\frac{\omega_{r'+n-k}}{\omega_{r'+1}}\alpha^{r'} y_{n-k+1}^{i_{n-k+1}}\cdots y_n^{i_n}\frac{\binom{r'/2}{i_1/2,\dots,i_{n-k}/2}}{\binom{r'}{i_1,\dots,i_{n-k}}} \\ &= \alpha^{r'} y_{n-k+1}^{i_{n-k+1}}\cdots y_n^{i_n} c_{i_1\dots,i_{n-k}}, \end{aligned} \tag{C.2.11}$$

where Legendre's duplication formula was used in the last step. This implies

$$\begin{aligned} &r!\mathbb{E}\Psi^r(\mathbf{K}_0)_{i_1,\dots,i_n} \\ &= c_{i_1,\dots,i_{n-k}}\int_L\int_0^\infty \mathbb{P}((\alpha e_1+(y_1,\dots,y_k))\in\mathbf{K}_0)\alpha^{n-k-1+r'} \\ &\qquad\times y_{n-k+1}^{i_{n-k+1}}\cdots y_n^{i_n}\,d\alpha\,\mathcal{H}^k(dy) \\ &= (r+n-k-1)!c_{i_1,\dots,i_{n-k}}\mathbb{E}\Psi_M^{r+n-k-1}(\mathbf{K}_0\cap M_+)_{n-k-1+r',i_{n-k+1},\dots,i_n}, \end{aligned}$$

as $M=\operatorname{span}(L\cup\{e_{n-k}\})$, and proves the Corollary. □

C.2.2 Proof of Theorem C.2.4

Before we can prove Theorem C.2.4 we need a decomposition of the surface area measure which is given in the following proposition. It makes use of the unique invariant probability measure $\nu_{L^\perp}$ on $\mathrm{SO}_n(L^\perp)$; see e.g. [16].

Proposition C.2.6. *Let $k\in\{0,\dots,n-2\}$, $L\in G(n,k)$ and $M\in G(n,k+1)$ with $L\subset M$. Fix $w\in M\cap L^\perp\cap S^{n-1}$ and set $M_+=M\cap w^+$.*
If $K\in\mathcal{K}^n$ satisfies $CK=K$ for all rotations $C\in\mathrm{SO}_n(L^\perp)$ then

$$\begin{aligned} \mathrm{S}_{n-1}(K,\beta) = 2\omega_{n-k}&\int_{\mathrm{SO}_n(L^\perp)}\int_{\mathrm{Nor}_M(K\cap M)}\mathbb{1}_{C^{-1}\beta}(v)\|p_{\mathrm{span}(w)}(x)\|^{n-k-1} \\ &\times\Lambda_k^M(K\cap M_+,d(x,v))\nu_{L^\perp}(dC), \end{aligned} \tag{C.2.12}$$

for all $\beta\in\mathcal{B}(S^{n-1})$.

Proof. Let $\varepsilon>0$ be given and consider $B_\varepsilon(K,\beta)=\{x\in(K+\varepsilon B^n)\backslash K\,|\,u(K,x)\in\beta\}$. By a version of the Steiner formula (see for instance [13, p. 215])

$$\frac{1}{\varepsilon}\mathcal{H}^n(B_\varepsilon(K,\beta)) = \frac{1}{n}\sum_{m=0}^{n-2}\varepsilon^{n-m-1}\binom{n}{m}\mathrm{S}_m(K,\beta)+\mathrm{S}_{n-1}(K,\beta). \tag{C.2.13}$$

Hence for $\varepsilon\to 0$ the expression $\mathrm{S}_{n-1}(K,\beta)$ is the limit of the left hand side of (C.2.13). By a decomposition of the Lebesgue measure as in the proof of Theorem C.2.2, and using the rotational invariance of K, the latter can be written as

$$\begin{aligned} \frac{1}{\varepsilon}\mathcal{H}^n(B_\varepsilon(K,\beta)) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n}\mathbb{1}_{(K+\varepsilon B^n)\backslash K}(x)\mathbb{1}_{u(K,x)\in\beta}\,dx \\ &= \frac{\omega_{n-k}}{\varepsilon}\int_{\mathrm{SO}_n(L^\perp)}\int_L\int_0^\infty \mathbb{1}_{(K+\varepsilon B^n)\backslash K}(\alpha w+y) \\ &\qquad\times\mathbb{1}_{u(K,C(\alpha w+y))\in\beta}\,\alpha^{n-k-1}\,d\alpha\,\mathcal{H}^k(dy)\,\nu_{L^\perp}(dC) \end{aligned}$$

The innermost two integrals can be combined and one obtains

$$\begin{aligned}\frac{1}{\varepsilon}\,\mathcal{H}^n(B_\varepsilon(K,\beta)) = \frac{\omega_{n-k}}{\varepsilon}\int_{\mathrm{SO}_n(L^\perp)}\int_{M_+} &\mathbb{1}_{K+\varepsilon B^n\setminus K}(z)\\ &\times \mathbb{1}_{u(K,Cz)\in\beta}\|p_{\mathrm{span}(w)}(z)\|^{n-k-1}\,dz\,\nu_{L^\perp}(dC).\end{aligned}$$

The rotational invariance property of K implies $p(K,Cz) = Cp(C^{-1}K,z) = Cp(K,z)$ and so $u(K,Cz) = Cu(K,z)$. We also have $p(K,z) = p(K\cap M_+,z)\in M_+$ as $z\in M_+$ due to the Pythagorean theorem implying in particular $u(K,z) = u(K\cap M_+,z)$ and

$$(K+\varepsilon B^n)\cap M_+ = [(K\cap M_+)+\varepsilon(B^n\cap M)]\cap M_+.$$

Combining these facts yields

$$\begin{aligned}\frac{1}{\varepsilon}\,\mathcal{H}^n(B_\varepsilon(K,\beta)) = \frac{\omega_{n-k}}{\varepsilon}\int_{\mathrm{SO}_n(L^\perp)}\int_{M_+\setminus(K\cap M_+)} &\mathbb{1}_{(K\cap M_+)+\varepsilon(B^n\cap M)}(z)\\ &\times \mathbb{1}_{u(K\cap M_+,z)\in C^{-1}\beta}\|p_{\mathrm{span}(w)}(z)\|^{n-k-1}\,dz\,\nu_{L^\perp}(dC).\end{aligned}$$

The isometric identification of M with $\mathbb{R}^{k+1}$ which sends w to the first vector of the standard basis of $\mathbb{R}^{k+1}$ and [13, Theorem 4.2.8] imply

$$\begin{aligned}\frac{1}{\varepsilon}\,\mathcal{H}^n(B_\varepsilon(K,\beta)) = \omega_{n-k}\sum_{j=0}^{k}\frac{\omega_{k-j+1}}{\varepsilon}\int_{\mathrm{SO}_{k+1}(e_1^\perp)}\int_0^\varepsilon \rho^{k-j}\int_{\mathrm{Nor}(\tilde K)} &\mathbb{1}_{C^{-1}\beta}(v)\\ \times\mathbb{1}_{e_1^+}(x+\rho v)\|p_{\mathrm{span}(e_1)}(x+\rho v)\|^{n-k-1}\Lambda_j(\tilde K, d(x,v))\,d\rho\,\nu_{e_1^\perp}(dC),\end{aligned}\tag{C.2.14}$$

where the convex body $\tilde K\subset\mathbb{R}^{k+1}$ is the image of $K\cap M_+$ under this identification.

There exists $R>0$ such that $\tilde K\subset RB^{k+1}$. If $0<\varepsilon<1$ the contraction property of the orthogonal projection implies that the absolute jth summand in (C.2.14) is bounded by

$$\frac{\omega_{k-j+1}}{\varepsilon}\int_0^\varepsilon \rho^{k-j}\,d\rho\,(R+1)^{n-k-1}V_j(\tilde K) = c\varepsilon^{k-j}\to 0,$$

as $\varepsilon\to 0$ whenever $j<k$.

The function

$$g(\rho) = \mathbb{1}_{e_1^+}(x+\rho v)\|p_{\mathrm{span}(e_1)}(x+\rho v)\|^{n-k-1}$$

is continuous at $\rho=0$. In fact, the indicator can only be discontinuous at zero when $x\in e_1^\perp$, but in this case $\lim_{\rho\to 0} g(\rho) = 0 = g(0)$ as $n-k-1>0$. Hence the summand in (C.2.14) with $j=k$ converges for $\varepsilon\to 0$ to

$$2\int_{\mathrm{SO}_{k+1}(e_1^\perp)}\int_{\mathrm{Nor}(\tilde K)}\mathbb{1}_{C^{-1}\beta}(v)\|p_{\mathrm{span}(e_1)}(x)\|^{n-k-1}\Lambda_k(\tilde K, d(x,v))\,\nu_{e_1^\perp}(dC)$$

by dominated convergence. We thus may take the limit $\varepsilon\to 0$ in (C.2.13) and arrive at

$$\begin{aligned}S_{n-1}(K,\beta) = 2\omega_{n-k}\int_{\mathrm{SO}_{k+1}(e_1^\perp)}\int_{\mathrm{Nor}(\tilde K)} &\mathbb{1}_{C^{-1}\beta}(v)\\ &\times\|p_{\mathrm{span}(e_1)}(x)\|^{n-k-1}\Lambda_k(\tilde K, d(x,v))\,\nu_{e_1^\perp}(dC).\end{aligned}$$

This is the claim (C.2.12) if we again use the identification of $\mathbb{R}^{k+1}$ with M. □

Proof of Theorem C.2.4: By (C.2.5) it is enough to consider the surface tensor of the Blaschke shape of X. The L-restricted isotropy and Proposition C.2.6 yield

$$\Phi^s(\mathrm{B_s}(X)) = \frac{2\omega_{n-k}}{s!\omega_{s+1}} \int_{\mathrm{Nor}_M(\mathrm{B_s}(X)\cap M_+)} \int_{\mathrm{SO}_n(L^\perp)} (Cu)^s \nu_{L^\perp}(dC)$$
$$\times \|p_{\mathrm{span}(w)}(x)\|^{n-k-1} \Lambda_k^M(\mathrm{B_s}(X)\cap M_+, d(x,u)).$$

Note that $Cu = \langle u, w\rangle(Cw) + \mathrm{p}_L(u)$, and that the image of the measure $\omega_{n-k}\nu_{L^\perp}$ under $C \mapsto Cw$ is the restriction of $\mathcal{H}^{n-k-1}$ on $S^{n-1}\cap L^\perp$. Hence,

$$\Phi^s(\mathrm{B_s}(X)) = \frac{2}{s!\omega_{s+1}} \int_{\mathrm{Nor}_M(\mathrm{B_s}(X)\cap M_+)} f_s\big(\langle u, w\rangle, \mathrm{p}_L(u)\big)$$
$$\times \langle x, w\rangle^{n-k-1} \Lambda_k^M(\mathrm{B_s}(X)\cap M_+, d(x,u)). \tag{C.2.15}$$

with the function f_s given by (C.2.7) with $r = s$. Inserting the representation (C.2.8) for f_s into (C.2.15) now yields the assertion (C.2.6). □

As in Corollary C.2.5 we can represent the mean surface particle tensor as an array of elements.

Corollary C.2.7. *Under the assumptions of Theorem C.2.4 let in addition* $i_1,\dots,i_n \in \{0,\dots,s\}$ *with* $\sum_{j=1}^n i_j = s$ *and* (C.2.9) *be given. The mean surface particle tensor can be represented as*

$$\mathbb{E}\Phi^s(\boldsymbol{K}_0)_{i_1,\dots,i_n} = \frac{2}{s!\omega_{s+1}} c_{i_1,\dots,i_{n-k}} \int_{\mathrm{Nor}_M(\mathrm{B_s}(X)\cap M_+)} u_{n-k}^{s'} u_{n-k+1}^{i_{n-k+1}} \cdots u_n^{i_n}$$
$$\times x_{n-k}^{n-k-1} \Lambda_k^M(\mathrm{B_s}(X)\cap M_+, d(x,u)), \tag{C.2.16}$$

with the constants $s' = \sum_{j=1}^{n-k} i_j$ *and* $c_{i_1,\dots,i_{n-k}}$ *from Corollary C.2.5.*

Proof. Putting $w = e_{n-k}$ and inserting the representation (C.2.11) into (C.2.15), the assertion follows. □

Corollary C.2.5 (or Theorem C.2.2 after applying Tonelli's theorem to (C.2.4)) implies that the mean volume particle tensor can be written as average of tensors of lower-dimensional sections of $\mathbf{K}_0$. This is relevant for applications, since the average can be replaced by an empirical mean in an inference procedure. Equation (C.2.16) in Corollary C.2.7 (and thus also (C.2.6) in Theorem C.2.4) is qualitatively different, as the right hand side involves $\mathrm{B_s}(X)\cap M^+$, which is a full-dimensional set-valued average intersected with a lower-dimensional set. We note that the integral w.r.t. the support measure in (C.2.16) cannot be written as an *average integral* of a relative support measure of a section of $\mathbf{K}_0$. Indeed, we claim that

$$\mathbb{E}\Phi^s(\mathbf{K}_0)_{i_1,\dots,i_n} = \mathbb{E}\varphi(\mathbf{K}_0 \cap M_+) \tag{C.2.17}$$

cannot hold for any $j \in \{0,\dots,k\}$ and any real-valued function φ on the set of convex bodies in M_+. For $n = 3$, $s = 2$ and $k = 1$ it is enough to consider a stationary marked point process for which the typical particle is a uniform random rotation about the e_3-axis of the fixed triangle K with vertices $-e_3, e_3, te_1$, $t > 0$. Then the right hand side of (C.2.17) is independent of t as $\mathbf{K}_0 \cap M_+$ is almost surely the line segment from $-e_3$ to e_3, but the left hand side depends on t when $i_1 = 2$.

C.3 Rank-2 surface tensors and ellipsoids

Let X be a stationary process as in Section C.2. The mean rank-2 volume tensor of its typical particle is the volume tensor of a unique centered ellipsoid. This follows from [7] where it was proven that the rank-2 volume tensor Ψ^2 is a continuous bijective map from the space of all centered full-dimensional ellipsoids in $\mathbb{R}^n$ into the space of symmetric, positive definite $n \times n$-matrices. The inversion formula given in [7, p. 266] and standard topological arguments imply that Ψ^2 is in fact a homeomorphism between these spaces.

A corresponding result for the rank-2 surface tensor in $\mathbb{R}^2$ was proven in Paper *B*. However it is still an unsolved problem if the statement also hold for general $n > 2$. In this section we will consider ellipsoids of revolution, that is, ellipsoids which are invariant under rotations fixing a one-dimensional linear subspace pointwise. We will prove that the rank-2 surface tensor uniquely determines these.

If we consider a stationary marked point process X which is non-degenerate and L-restricted isotropic with $L \in G(n,1)$, then the Blaschke shape of X is rotation invariant under rotations fixing L pointwise and so Theorem C.3.1 below yields the existence of a unique ellipsoid of revolution with its rank-2 surface tensor coinciding with the mean rank-2 surface particle tensor of the typical particle of X. This will be used in Section C.4 to define an ellipsoidal summary statistic for X based on surface tensors.

Let $n \geq 3$ and $\mathcal{E}$ be the space of all centered ellipsoids of revolution of dimension at least $n-1$. As we only will work with surface tensors of rank two in this section, we will write $\Phi(K)$ instead of $\Phi^2(K)$ for abbreviation. We will later see that $\Phi(E)$ for $E \in \mathcal{E}$ is an element of the space $\mathcal{A}$ of all $n \times n$ positive semidefinite symmetric matrices such that one eigenvalue has multiplicity $n-1$ and the other eigenvalue is positive. The following theorem is actually stronger than the mentioned uniqueness result.

Theorem C.3.1. *The function $\Phi : \mathcal{E} \to \mathcal{A}$ is a homeomorphism.*

C.3.1 Proof of Theorem C.3.1

For a centered ellipsoid $E \in \mathcal{E}$ of revolution there exists a symmetric positive semidefinite matrix M such that $E = MB^n$ with one positive eigenvalue of multiplicity $n-1$. Spectral decomposition allows us to find a rotation $C \in \mathrm{SO}(n)$ and a diagonal matrix with $a(E) > 0$ on the first $n-1$ diagonal entries and $b(E)$ on the remaining, i.e. $\Lambda = \mathrm{diag}(a(E)[n-1], b(E))$ such that $M = C\Lambda C^t$. Let $z(E) = 1 - (\frac{b(E)}{a(E)})^2$ be the squared *eccentricity* of E. Clearly, E is uniquely determined by $a(E), z(E)$ and the rotation $C \in \mathrm{SO}(n)$. For later use, we note that the definition of $\Phi(E)$ implies

$$\tilde{C}\Phi(E)\tilde{C}^t = \Phi(\tilde{C}E) \tag{C.3.1}$$

for all $\tilde{C}$ in the orthogonal group $O(n)$.

Proposition C.3.2. *Let $E \in \mathcal{E}$ with rotation axis direction e_n, principal axes lengths $a = a(E)$ and $b = b(E)$ and squared eccentricity $z = z(E)$ be given.*

Then $\Phi(E) = \mathrm{diag}\big(\mu_1[n-1], \mu_2\big)$ with

$$\mu_1 = \frac{\kappa_n}{8\pi} a^{n-1}(1-z)\mathrm{F}(\tfrac{1}{2}, \tfrac{n+1}{2}; \tfrac{n}{2}+1; z) \tag{C.3.2}$$

(continuously extended with value 0 *at* $z = 1$*) and*

$$\mu_2 = \frac{\kappa_n}{8\pi} a^{n-1}(1-z)\,\mathrm{F}(\tfrac{3}{2}, \tfrac{n+1}{2}; \tfrac{n}{2}+1; z), \tag{C.3.3}$$

(continuously extended with value $a^{n-1}\kappa_{n-1}/(4\pi)$ *at* $z = 1$*).*
As a consequence,

$$\frac{\mu_1}{\mu_2} = f_n(z), \tag{C.3.4}$$

where $f_n = f_{\frac{1}{2},\frac{n+1}{2},\frac{n}{2}+1}$ *is given in Lemma C.1.1.*

For later use, we remark that (C.3.3) can be rewritten as

$$\mu_2 = \frac{\kappa_n}{8\pi} a^{n-1}\,\mathrm{F}(\tfrac{1}{2}, \tfrac{n-1}{2}; \tfrac{n}{2}+1; z) \tag{C.3.5}$$

using Euler's transformation (C.1.8) and the symmetry of the hypergeometric function in its first two arguments.

Proof. Consider first the case where $z < 1$, which is equivalent to $\dim E = n$. As $E = \Lambda B^n$ with $\Lambda = \mathrm{diag}(a[n-1], b)$ the support function of E is differentiable and takes the simple form

$$\mathrm{h}_E(u) = \|\Lambda u\|$$

By [13, Cor 1.7.3 and p. 115] the reverse spherical image map is given by

$$x_E(u) = \frac{1}{\mathrm{h}_E(u)} \Lambda^2 u$$

for $u \in S^{n-1}$. The $(n-1)$th normalized elementary symmetric function of the principal radii of curvature of E is equal to the inverse of the Gauss-Kronecker curvature, composed with the reverse spherical image map (see [13, p 117]), hence the standard proof of measure theory and [13, p. 217] yield

$$\Phi(E) = \frac{a^{2(n-1)}b^2}{8\pi} \int_{S^{n-1}} u^2 \frac{1}{\mathrm{h}_E(u)^{n+1}}\, du. \tag{C.3.6}$$

as the Gauss-Kronecker curvature of E at x in the boundary of E is given by

$$\mathrm{K}(p) = \frac{1}{a^{2(n-1)}b^2 \|\Lambda^{-2}p\|^{n+1}}$$

where we used $a, b > 0$, see [12, Prop. 3.1]. Note that $\Phi(E)$ has diagonal form. Using (C.3.6), cylindrical coordinates, $\int_{S^{n-1}\cap e_n^\perp} v_i^2 \mathcal{H}^{n-2}(dv) = \kappa_{n-1}$, Euler's integral representation (C.1.6) and $F(a, b; c; z) = F(b, a; c; z)$ we get

$$\begin{aligned} \mu_1 &= \frac{a^{2n}(1-z)}{8\pi} \int_{S^{n-1}} u_i^2\, \mathrm{h}_E(u)^{-(n+1)} du \\ &= \frac{\kappa_n}{8\pi} a^{n-1}(1-z)\,\mathrm{F}(\tfrac{1}{2}, \tfrac{n+1}{2}; \tfrac{n}{2}+1; z), \end{aligned}$$

$i = 1, \ldots, n-1$, and, in a similar way,

$$\mu_2 = \frac{a^{2n}(1-z)}{8\pi} \int_{S^{n-1}} u_n^2\, \mathrm{h}_E(u)^{-(n+1)} du \tag{C.3.7}$$

$$= \frac{\kappa_n}{8\pi} a^{n-1}(1-z)\,\mathrm{F}(\tfrac{3}{2}, \tfrac{n+1}{2}; \tfrac{n}{2}+1; z). \tag{C.3.8}$$

Hence we have $\Phi(E) = \operatorname{diag}\big(\mu_1[n-1], \mu_2\big)$ with its eigenvalues given by (C.3.2) and (C.3.3) whenever $z < 1$.

We now consider the case $z = 1$. The ellipsoid E has dimension $n-1$, and its surface area measure is concentrated on the two points $\pm e_n$ with mass $a^{n-1}\kappa_{n-1}$ at each, so

$$\Phi(E) = \frac{a^{n-1}\kappa_{n-1}}{4\pi} e_n^2,$$

proving that $\Phi(E)$ is a diagonal matrix with eigenvalues $\mu_1 = 0$ and $\mu_2 = a^{n-1}\kappa_{n-1}/(4\pi)$. In view of (C.1.10), Equation (C.3.2) thus also holds for $z = 1$ if the right hand side is continuously extended. Similarly, (C.1.9) and (C.3.5) show that the continuous extension of the right side of (C.3.3) at $z = 1$ attains the correct value $a^{n-1}\kappa_{n-1}/(4\pi)$. This shows the assertions. □

Proposition C.3.3. *Any $E \in \mathcal{E}$ with $\Phi(E) = \operatorname{diag}\big(\mu_1[n-1], \mu_2\big)$ for some $\mu_1 \geq 0$ and $\mu_2 > 0$ has rotation axis direction e_n (and possibly others).*

Proof. The set $E \in \mathcal{E}$ is rotationally symmetric about an axis with direction $u \in S^{n-1}$, say. We have $CE = E$ for all rotations C fixing u. From (C.3.1), we deduce that $\Phi(E)$ commutes with all such rotations. If $u \neq \pm e_n$ the matrix $\Phi(E)$ must thus be a multiple of the identity matrix and $\mu_1 = \mu_2$ as $n \geq 3$. Due to (C.3.1), we thus have $\Phi(CE) = \Phi(E)$ for all rotations C, and in particular for a rotation C_0 which maps u to e_n. As C_0E is rotationally symmetric about the axis with direction e_n, Proposition C.3.2 can be applied to it. By (C.3.4) and Lemma C.1.1 its eccentricity is zero, so C_0E, and hence E, is a ball. This shows that u can always be chosen equal to e_n. □

Using these two propositions we can now prove the main theorem of this section.

Proof of Theorem C.3.1: As the surface area measure $S_{n-1}(E, \cdot)$ is weakly continuous as a function of E (see for instance [13, page 215]), Φ is continuous.

To show surjectivity let $A \in \mathcal{A}$ be given. There is a rotation $C \in \mathrm{SO}(n)$ such that $A = C\operatorname{diag}(\mu_1[n-1], \mu_2)C^t$, where $\mu_1 \geq 0$ and $\mu_2 > 0$. In view of (C.3.1) it is enough to find an ellipsoid $E \in \mathcal{E}$ such that $\Phi(E) = \operatorname{diag}(\mu_1[n-1], \mu_2)$. Lemma C.1.1 yields the existence of $z_0 \in (-\infty, 1]$ such that

$$\frac{\mu_1}{\mu_2} = f_n\big(z_0\big).$$

Define $a_0 = a$ as the solution of (C.3.3) when $z = z_0$. By construction, the ellipsoid E with rotation axis e_n, squared eccentricity z_0 and axes lengths a_0 in all directions in $e_n^\perp$ satisfies $\Phi(E) = \operatorname{diag}(\mu_1[n-1], \mu_2)$ by Proposition C.3.2, proving surjectivity.

To show injectivity assume that E and $E' \in \mathcal{E}$ have the same rank-2 surface tensor. In view of (C.3.1) and Proposition C.3.2 we may assume that E and E' have been mapped by the same rotation in such a way that E has rotation axis e_n and thus $\Phi(E) = \Phi(E') = \operatorname{diag}(\mu_1[n-1], \mu_2)$. By Proposition C.3.3 also E' has rotation axis e_n, so Proposition C.3.2 can be applied to both ellipsoids. Due to (C.3.4) and Lemma C.1.1 they have the same eccentricity, and (C.3.3) shows $a(E) = a(E')$. As the centered ellipsoids E and E' have the same orientation, this gives $E = E'$. The injectivity assertion is shown.

To show the continuity of the inverse of Φ let $\Phi(E_k) \to \Phi(E)$ for $k \to \infty$ in $\mathcal{A}$ be given. Assume first that all ellipsoids E_k have rotation axis direction e_n. By Proposition C.3.2, the matrix $\Phi(E_k)$ has diagonal form $\mathrm{diag}(\mu_1^{(k)}[n-1], \mu_2^{(k)})$ with non-negative diagonal elements, $\mu_2^{(k)} > 0$ and $z(E_k) = f_n^{-1}(\mu_1^{(k)}/\mu_2^{(k)})$. The continuity of eigenvalues shows $\Phi(E) = \mathrm{diag}(\mu_1[n-1], \mu_2)$, where $\mu_i = \lim_{k\to\infty} \mu_i^{(k)}$, $i = 1, 2$. Note that E has rotation axis direction e_n by Proposition C.3.3. As $\Phi(E) \in \mathcal{A}$ we must have $\mu_2 > 0$. Due to Lemma C.1.1, the function f_n^{-1} is continuous, so $z(E_k) \to z(E)$, $k \to \infty$ by (C.3.4). From (C.3.3) we now deduce $a(E_k) \to a(E)$ and thus also $b(E_k) \to b(E)$ as $k \to \infty$. As all ellipsoids have the same orientation, we arrive at $E_k \to E$, $k \to \infty$.
If not all ellipsoids E_k have rotation axis direction e_n, suppose for contradiction that there is $\varepsilon > 0$ and a subsequence $(E_{k'})$ of (E_k) such that $d_H(E_{k'}, E) > \varepsilon$. Let $(C_{k'})$ be a sequence of rotations such that $C_{k'}E_{k'}$ has rotation axis direction e_n. By compactness of $\mathrm{SO}(n)$ there is a subsequence $(C_{k''})$ converging to some rotation C_0. Applying the above to the rotated ellipsoids gives $C_{k''}E_{k''} \to C_0 E$, so

$$\begin{aligned}\varepsilon &\le d_H(E_{k''}, E)\\ &= d_H(C_{k''}E_{k''}, C_{k''}E)\\ &\le d_H(C_{k''}E_{k''}, C_0E) + d_H(C_0E, C_{k''}E) \to 0,\end{aligned}$$

which is a contradiction. Hence, $E_k \to E$ as $k \to \infty$, so the inverse of Φ is continuous.

□

C.3.2 Reconstruction algorithm

Theorem C.3.1 implies that given a positive semidefinite, symmetric $n \times n$-matrix A such that one eigenvalue has multiplicity $n-1$ and the other eigenvalue is positive, then there exists a unique centered ellipsoid of revolution E with surface tensor $\Phi(E) = A$. As the proof of Theorem C.3.1 is constructive, it gives rise to the following reconstruction algorithm. To simplify notation we introduce the function g_n on $(-\infty; 1]$ given by

$$g_n(z) = \left(\frac{8\pi\mu_2}{\kappa_n \mathrm{F}(z)}\right)^{\frac{1}{n-1}}.$$

which is (C.3.5) solved for a where the abbreviation $\mathrm{F}(z) = \mathrm{F}(\frac{1}{2}, \frac{n-1}{2}; \frac{n}{2}+1; z)$ is used. Recall also that $\mathrm{F}_+(z) = \mathrm{F}(\frac{3}{2}, \frac{n+1}{2}; \frac{n}{2}+2; z)$ and that $f_n = f_{\frac{1}{2}, \frac{n+1}{2}, \frac{n}{2}+1}$ is given in Lemma C.1.1.

Algorithm TensorData

Input: Let $A \in \mathcal{A}$.
Task: Find $E \in \mathcal{E}$ such that $\Phi(E) = A$.
Action: Find $C \in \mathrm{SO}(n)$ such that $C^t A C = \mathrm{diag}(\mu_1[n-1], \mu_2)$.
(a) Find the squared eccentricity $z \in (-\infty, 1]$ by solving $f_n(z) = \frac{\mu_1}{\mu_2}$.
(b) Put $a = g_n(z)$ and $b = a\sqrt{1-z}$.
Output: The ellipsoid $E = C\,\mathrm{diag}(a[n-1], b)B^n$.

As finding f_n^{-1} requires a numerical procedure the following variant of **Algorithm TensorData** is introduced to take the error of a numerical inversion into account. It makes use of the constants $c_1 = c_1(\varepsilon)$ for some $\varepsilon > 0$ satisfying $0 < c_1 < 1$ and

$$|g_n(1) - g_n(1 - c_1^2)| \leq \varepsilon, \tag{C.3.9}$$

and $c_2 = \frac{2\kappa_{n-1}}{((n-1)\mu_1+\mu_2)\kappa_n}$.

Algorithm TensorData*

Input: Let $A \in \mathcal{A}$ and $\varepsilon > 0$ be given.
Task: Find $E \in \mathcal{E}$ such that $d_H(E, \Phi^{-1}(A)) \leq \varepsilon$.
Action: Find $C \in \mathrm{SO}(n)$ such that $C^t AC = \mathrm{diag}(\mu_1[n-1], \mu_2)$.
(a) Find Put $\overline{z}_0 = 1 - \Big(\min\{c_1, \varepsilon/g_n(1-c_1^2)\}\Big)^2$.
(b) Find $\underline{z}_0 \in (-\infty, 0]$ such that $F(\underline{z}_0) \leq c_2\mu_2$,
(i) If $\frac{\mu_1}{\mu_2} \leq f_n(\overline{z}_0)$ put $a = g_n(1)$ and $b = 0$.
(ii) If $\frac{\mu_1}{\mu_2} > f_n(\overline{z}_0)$ determine $\overline{z}, \underline{z} \in \big[\underline{z}_0, \overline{z}_0\big]$
such that $f_n(\overline{z}) \leq \frac{\mu_1}{\mu_2} \leq f_n(\underline{z})$ and $|\overline{z} - \underline{z}| \leq c_3$,
where $c_3 = g_n(\underline{z})^{-1}\varepsilon \min(\frac{(n+2)F(\underline{z})}{F_+(\overline{z})\sqrt{1-\underline{z}_0}}, \sqrt{1-\overline{z}})$.
Put $z = \frac{\overline{z}+\underline{z}}{2}$, $a = g_n(z)$ and $b = a\sqrt{1-z}$.
Output: The ellipsoid $E = C\,\mathrm{diag}(a[n-1], b)B^n$.

Let a_0, b_0 be the lengths of the semi-axis of $E_0 = \Phi^{-1}(A)$ and let z_0 be its squared eccentricity. Note first that using the notation from the algorithm we have $z_0 \geq \underline{z}_0$. To see this define $\tilde{E}_0 = \mathrm{diag}(a_0[n-1], b_0)B^n$ and consider the circular disc $D = a_0 B^n \cap e_n^\perp$. As $D \subset \tilde{E}_0$ the monotonicity of the intrinsic volumes gives

$$a_0^{n-1}\kappa_{n-1} = V_{n-1}(D) \leq V_{n-1}(\tilde{E}_0) = 4\pi((n-1)\mu_1 + \mu_2).$$

as the trace of $\Phi(\tilde{E}_0)$ is $V_{n-1}(\tilde{E}_0)/(4\pi)$. Using (C.3.5), we conclude $c_2\mu_2 \leq F(z_0)$ and so $F(\underline{z}_0) \leq F(z_0)$. The monotonicity of F now yields $\underline{z}_0 \leq z_0$.

That **Algorithm TensorData*** yields an ellipsoid within the desired precision can be seen as follows. In the case where the algorithm used (ii), let a, b and z be the quantities determined in the algorithm. As $f_n(z_0) = \mu_1/\mu_2$, we have $f_n(\overline{z}) \leq f_n(z_0) \leq f_n(\underline{z})$ and the monotonicity of f_n shows

$$\underline{z} \leq z_0 \leq \overline{z}. \tag{C.3.10}$$

By construction, we also have $\underline{z} \leq z \leq \overline{z}$ so $|z - z_0| \leq \overline{z} - \underline{z} \leq c_3$. Hence, using the notation and result of Lemma C.1.2 we have

$$\begin{aligned}
|a - a_0| &= \left(\frac{8\pi\mu_2}{\kappa_n}\right)^{\frac{1}{n-1}} |l_n(z) - l_n(z_0)| \\
&\leq \frac{1}{2(n+2)}\left(\frac{8\pi\mu_2}{\kappa_n}\right)^{\frac{1}{n-1}} \frac{F_+(\overline{z})}{F(\underline{z})^{\frac{n}{n-1}}}|z - z_0| \\
&\leq \frac{\varepsilon}{2\sqrt{1-\underline{z}_0}}
\end{aligned} \tag{C.3.11}$$

by the definition of c_3. This implies $|a - a_0| \leq \varepsilon/2$.
In addition, (C.3.10) and the monotonicity of g_n show $g_n(\underline{z}) \geq g_n(z_0) = a_0$. Combining this with (C.3.11) and the bound $|z - z_0| \leq c_3$, we see that

$$\begin{aligned}
|b - b_0| &= \left|a\sqrt{1-z} - a_0\sqrt{1-z_0}\right| \\
&\leq \sqrt{1-z}\left|a - a_0\right| + a_0\left|\sqrt{1-z} - \sqrt{1-z_0}\right| \\
&\leq \sqrt{1-\underline{z}_0}\left|a - a_0\right| + a_0\frac{1}{2\sqrt{1-\overline{\overline{z}}}}\left|z - z_0\right| \\
&\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, the Hausdorff distance of (any simultaneous rotations of) E_0 and E is not exceeding ε.
In the case where the algorithm determines E using (i) the following lemma ensures that $d_H(E, E_0) \leq \varepsilon$.

Lemma C.3.4. *Let $\varepsilon > 0$. Like in* ***Algorithm TensorData**** *choose $0 < c_1 < 1$ such that* (C.3.9) *holds and put $\bar{z}_0 = 1 - \left(\min\{c_1, \epsilon/g_n(1 - c_1^2)\}\right)^2$. If the eigenvalues μ_1, μ_2 of $\Phi(E_0)$, $E_0 \in \mathcal{E}$, are such that $\frac{\mu_1}{\mu_2} \leq f_n(\bar{z}_0)$, then the circular disc*

$$E = C^t \operatorname{diag}(g_n(1)[n-1], 0)B^n$$

satisfies $d_H(E_0, E) \leq \varepsilon$.

Proof. As before let a_0, b_0, z_0 be the lengths of the semi-axis and the squared eccentricity of E_0. By assumption, we have $f_n(\bar{z}_0) \geq \mu_1/\mu_2 = f_n(z_0)$, so Lemma C.1.1 implies $1 - c_1^2 \leq z_0 \leq 1$. Formula (C.3.5) gives $a_0 = g_n(z_0)$ and the monotonictity of g_n yields

$$g_n(1) \leq a_0 \leq g_n(1 - c_1^2).$$

The definition of $\bar{z}_0$ now gives

$$b_0 = \sqrt{1 - z_0}a_0 \leq \sqrt{1 - \bar{z}_0}\, g_n(1 - c_1^2) \leq \varepsilon$$

and this implies

$$\begin{aligned}
|h_{E_0}(u) - h_E(u)| &\leq \sqrt{(a_0 - g_n(1))^2(u_1^2 + \cdots + u_{n-1}^2) + b_0^2 u_n^2} \\
&\leq \max\{|a_0 - g_n(1)|, b_0\} \\
&\leq \max\{g_n(1 - c_1^2) - g_n(1), b_0\}, \\
&\leq \varepsilon
\end{aligned}$$

so the Hausdorff distance is not larger than ε. □

C.4 Associated ellipsoids

Let X be a stationary marked point process with marks K_i in the family $\mathcal{K}^n$ of convex bodies as defined in Section C.2. As the rank-2 volume tensor uniquely determines a centered ellipsoid (see [7]) an ellipsoidal approximation of the typical particle $\mathbf{K}_0$ of X can be introduced using this tensor. This was the starting point in [19] where the so-called Miles ellipsoid was defined in the three-dimensional case. In order to limit the dependence of the Miles ellipsoid from the position of the reference point, also the volume tensors of ranks zero and one play a role. We first give a definition of the Miles ellipsoid in general dimension.

Definition C.4.1. *Let X satisfy* (C.2.3) *with $r = 2$.* The Miles ellipsoid *is the unique centered ellipsoid,* $\mathrm{e_M}(X)$, *with* $\mathrm{e_M}(X) = \{0\}$ *if* $\mathbb{E}V_n(\boldsymbol{K}_0) = 0$ *and otherwise*

$$\Psi^2(\mathrm{e_M}(X)) = \alpha_M \mathbb{E}\Psi^2\Big(\boldsymbol{K}_0 - \frac{\mathbb{E}\Psi^1(\boldsymbol{K}_0)}{\mathbb{E}V_n(\boldsymbol{K}_0)}\Big), \tag{C.4.1}$$

where $\alpha_M > 0$ is chosen such that $V_n(\mathrm{e_M}(X)) = \mathbb{E}V_n(\boldsymbol{K}_0)$.

Using [13, Equation (5.104)], which describes how Ψ^2 behaves under translations of the argument, (C.4.1) can also be written as

$$\Psi^2(\mathrm{e_M}(X)) = \alpha_M\Big(\mathbb{E}\Psi^2(\mathbf{K}_0) - \frac{(\mathbb{E}\Psi^1(\mathbf{K}_0))^2}{2\mathbb{E}V_n(\mathbf{K}_0)}\Big). \tag{C.4.2}$$

A stochastic interpretation of the Miles ellipsoid makes use of the so-called *particle cover density*

$$f_{\mathbf{K}_0}(x) = \frac{\mathbb{P}(x \in \mathbf{K}_0)}{\mathbb{E}V_n(\mathbf{K}_0)}, \quad x \in \mathbb{R}^n,$$

which can be defined whenever $\mathbb{E}V_n(\mathbf{K}_0) > 0$. This is indeed a probability density function as $f_{\mathbf{K}_0}$ is non-negative and integrates to 1.

Lemma C.4.2. *Assume that $\mathbb{E}V_n(\boldsymbol{K}_0) > 0$ and X satisfy* (C.2.3) *with $r = 2$. Let $Y \in \mathbb{R}^n$ be a multivariate random variable with density $f_{\mathbf{K}_0}$. Then the Miles ellipsoid is the unique centered ellipsoid satisfying*

$$\int_{\mathrm{e_M}(X)} x^2 dx = \alpha' \operatorname{Var}(Y) \tag{C.4.3}$$

where $\alpha' > 0$ is chosen such that $V_n(\mathrm{e_M}(X)) = \mathbb{E}V_n(\boldsymbol{K}_0)$.

Proof. The mean and the covariance matrix of Y are

$$\mathbb{E}Y = \int_{\mathbb{R}^n} x f_{\mathbf{K}_0}(x)\, dx = \frac{\mathbb{E}\Psi^1(\mathbf{K}_0)}{\mathbb{E}V_n(\mathbf{K}_0)}$$

and

$$\begin{aligned}\operatorname{Var}(Y) &= \int_{\mathbb{R}^n} x^2 f_{\mathbf{K}_0}(x) dx - \Big(\int_{\mathbb{R}^n} x f_{\mathbf{K}_0}(x) dx\Big)^2 \\ &= \frac{2}{\mathbb{E}V_n(\mathbf{K}_0)}\Big(\mathbb{E}\Psi^2(\mathbf{K}_0) - \frac{(\mathbb{E}\Psi^1(\mathbf{K}_0))^2}{2\mathbb{E}V_n(\mathbf{K}_0)}\Big),\end{aligned}$$

respectively. Hence, $\operatorname{Var}(Y)$ is proportional to the right hand side of (C.4.2) and the claim follows from the fact that a full-dimensional centered ellipsoid is uniquely determined by its rank-2 volume tensor. □

In [19] the Miles ellipsoid was defined in the three-dimensional setting using (C.4.3), so Lemma C.4.2 shows that Definition C.4.1 is an extension of this notion to arbitrary dimensions. We have added the proportionality constant α' in (C.4.3) which appears to be missing in [19]

The Miles ellipsoid is scaled in such a way that its volume equals the mean volume of the typical particle. By (C.4.1) it is proportional to the mean rank-2 volume tensor of the typical particle centered at

$$\bar{c} = \frac{\mathbb{E}\Psi^1(\mathbf{K}_0)}{\mathbb{E}V_n(\mathbf{K}_0)}.$$

If the typical particle is deterministic and coincides with the full-dimensional set $K \in \mathcal{K}^n$, the vector $\bar{c}$ is just the center of mass $c(K)$ of K and $e_M(X)$ does not change when K is translated. However, if $\mathbf{K}_0$ is random and such that $c(\mathbf{K}_0) \neq 0$ with positive probability (this is usually the case in applications) then the 'ratio-of-means' estimator $\bar{c}$ will typically be different from $c(\mathbf{K}_0)$, and even different from $\mathbb{E}c(\mathbf{K}_0)$. As a consequence $e_M(X)$ not only depends on the 'shape' of $\mathbf{K}_0$, but also on its position and these two kinds of properties are inseparably entangled in $e_M(X)$. If the mean ellipsoid is supposed to summarize the mean shape of $\mathbf{K}_0$, this sensitivity of $e_M(X)$ to the position of the typical particle, and, in practice, the position of the reference point, is problematic.

We therefore suggest another mean ellipsoid. It is also based on the rank-2 volume tensor of the typical particle, but now each realization is centered individually before taking the average tensor.

Definition C.4.3. *Let the stationary marked point process X satisfy* (C.2.3) *with* $r = 2$. *The* inertia ellipsoid *is the unique centered ellipsoid*, $e_I(X)$, *with* $e_I(X) = \{0\}$ *if* $\mathbb{E}V_n(\boldsymbol{K}_0) = 0$ *and otherwise*

$$\Psi^2(e_I(X)) = \alpha_I \mathbb{E}\Psi^2\Big(\boldsymbol{K}_0 - \frac{\Psi^1(\boldsymbol{K}_0)}{V_n(\boldsymbol{K}_0)}\Big), \tag{C.4.4}$$

with $\alpha_I > 0$ *such that* $V_n(e_I(X)) = \mathbb{E}V_n(\boldsymbol{K}_0)$.

Clearly, (C.4.4) can be rewritten as

$$\Psi^2(e_I(X)) = \alpha_I \mathbb{E}\Psi^2\Big(\mathbf{K}_0 - c(\mathbf{K}_0)\Big),$$

showing that we now average particles with their centers of mass at $\mathbf{0}$. Both, the Miles ellipsoid and the inertia ellipsoid are based on mean volume tensors. Section C.3 allows us to define an ellipsoid based on the mean surface tensor if X is L-restricted isotropic with $L \in G(n,1)$.

Definition C.4.4. *Let $L \in G(n,1)$ and let X be a stationary, L-restricted isotropic marked point process with* $\mathbb{E}V_{n-1}(\boldsymbol{K}_0) > 0$. The Blaschke ellipsoid *is the unique centered ellipsoid of revolution such that*

$$\Phi^2(e_B(X)) = \mathbb{E}\Phi^2(\boldsymbol{K}_0). \tag{C.4.5}$$

Note that the right hand side of (C.4.5) is well-defined due to (C.2.1). Seen as a matrix, it is an element of $\mathcal{A}$, so Theorem C.3.1 implies existence and uniqueness of the centered ellipsoid of revolution $e_B(X)$ satisfying (C.4.5). In view of (C.2.5) the Blaschke ellipsoid could alternatively be defined by

$$\Phi^2(e_B(X)) = \Phi^2(B_s(X)),$$

stating that the $e_B(X)$ and the Blaschke shape have the same rank-2 surface tensor. Taking the trace on both sides of (C.4.5) we see

$$V_{n-1}(e_B(X)) = \mathbb{E}V_{n-1}(\mathbf{K}_0),$$

which can be considered as the natural proportionality relation and shows that rescaling with a proportionally constant α, like in the definitions of Miles- and inertia ellipsoids are not necessary.

The assumption of L-restricted isotropy with $L \in G(n,1)$ is needed in the definition of Blaschke ellipsoid due to the lack of a general result stating that the rank-2 surface tensor induces a uniquely determined centered ellipsoid. However for $n = 2$ this assumption can be omitted as Theorem B.2.1 in Paper B implies the needed uniqueness.

Example C.4.5. We give an example illustrating the dependency between the Miles ellipsoid and the choice of reference points. Let $K = \mathrm{diag}(a[n-1], b)B^n$ be a centered ellipsoid of revolution with lengths of its semi-axes $a > 0$ and $b > 0$. Put $\mathbf{K}_0 = K + Te_n$, where T is a uniformly distributed stochastic variable in $[-b, b]$. There is a stationary marked point process X with typical particle $\mathbf{K}_0$, for instance an independently marked Poisson process with intensity $\gamma = 1$. By construction X is L-restricted isotropic.
Now $\mathbb{E}\Psi^1(\mathbf{K}_0) = \mathbf{0}$ so by [7, page 266]

$$\begin{aligned}\Psi^2(e_M(X)) &= \alpha\mathbb{E}\Psi^2(\mathbf{K}_0)\\ &= \frac{\alpha\kappa_n}{2(n+2)}a^{n-1}b\,\mathrm{diag}(a^2[n-1], b^2 + (n+2)\mathbb{E}T^2)\\ &\propto \mathrm{diag}(a^2[n-1], \tfrac{n+5}{3}b^2).\end{aligned}$$

For $a = \sqrt{\frac{n+5}{3}}b \neq b$ the Miles ellipsoid becomes a ball even though the typical particle is always a translation of the same ellipsoid K, which is not a ball. In contrast, both the inertia and the Blaschke ellipsoid coincide with K.

This example illustrates how the Miles ellipsoid may yield poor results if the reference points are 'far away' from the centers of mass of the particles. In a practical application using $e_M(X)$ it is therefore recommended to chose the reference points of the individual particles close to the respective centers of mass, if possible.

For practical applications we need estimators of the right hand sides of (C.4.1), (C.4.4) and (C.4.5). We will from now on assume that the integrability condition (C.2.3) with $r = 2$ is satisfied when talking about the Miles ellipsoid and the inertia ellipsoid. When talking about the Blaschke ellipsoid we will assume that there is $L \in G(n,1)$ such that X is L-restricted isotropic and $\mathbb{E}V_{n-1}(\mathbf{K}_0) > 0$. We now consider estimators of these mean ellipsoids from observations of X in a full-dimensional window $W \in \mathcal{K}^n$. To avoid treating boundary effects, we follow the approach in [19] assuming that all particles with reference point in W can be observed even if they are not completely contained in W. If this assumption is not satisfied, estimators of the average rank-2 surface tensor of the typical particle solely based on the particle process $\tilde{X} \cap W$ in W can be found in [15].

Let N_W be the number of particles with reference point in W. For $r \in \mathbb{N}_0$ define

$$\hat{\Psi}^r = \frac{1}{N_W} \sum_{(x,K)\in X, x\in W} \Psi^r(K).$$

Let

$$\hat{\Psi}_M(W) = \hat{\Psi}^2 - \frac{(\hat{\Psi}^1)^2}{2\hat{\Psi}^0}$$

be an estimator of the unscaled right side of (C.4.2) (and thus of (C.4.1)),

$$\hat{\Psi}_I(W) = \frac{1}{N_W} \sum_{(x,K)\in X, x\in W} \Psi^2\Big(K - \frac{\Psi^1(K)}{\Psi^0(K)}\Big)$$

be an estimator of of the unscaled right side of (C.4.4) and lastly

$$\hat{\Phi}_B(W) = \frac{1}{N_W} \sum_{(x,K)\in X, x\in W} \Phi^2(K)$$

be an estimator of (C.4.5).

Proposition C.4.6. *Let X be an ergodic stationary marked point process and W be a convex body in $\mathbb{R}^n$ with $V_n(W) > 0$. For i) and ii) we assume* (C.2.3) *with $r = 2$, whereas for iii) we assume that X is L-restricted isotropic with respect to a line L.*

i) *Let $\hat{e}_M$ be the unique centered ellipsoid with*

$$\Psi^2(\hat{e}_M) = \hat{\alpha}_M \hat{\Psi}_M(W),$$

where $\hat{\alpha}_M > 0$ is given such that $V_n(\hat{e}_M) = \hat{\Psi}^0$. The ellipsoid $\hat{e}_M$ is a consistent estimator of $\mathrm{e_M}(X)$.

ii) *Let $\hat{e}_I$ be the unique centered ellipsoid with*

$$\Psi^2(\hat{e}_I) = \hat{\alpha}_I \hat{\Psi}_I(W),$$

where $\hat{\alpha}_I > 0$ is given such that $V_n(\hat{e}_I) = \hat{\Psi}^0$. The ellipsoid $\hat{e}_I$ is a consistent estimator of $\mathrm{e_I}(X)$.

iii) *The unique ellipsoid $\hat{e}_B$ in $\mathcal{E}$ with*

$$\Phi^2(\hat{e}_B) = \hat{\Phi}_B(W)$$

is a consistent estimator of $\mathrm{e_B}(X)$.

Proof. Let $\{W_m\}$ be an increasing sequence of convex bodies such that $r(W_n) \to \infty$ with

$$r(W_n) = \max\{r \geq 0 : W_m \text{ contains a ball of radius r}\}.$$

As X is assumed to be ergodic it follows from [4, Cor. 12.2.V] that

$$\hat{\Psi}^r \xrightarrow{a.s.} \mathbb{E}\Psi^r(\mathbf{K}_0).$$

Hence

$$\hat{\Psi}_M(W_m) \xrightarrow{a.s.} \mathbb{E}\Psi^2(\mathbf{K}_0) - \frac{\mathbb{E}\Psi^1(\mathbf{K}_0)^2}{2\mathbb{E}\Psi^0(\mathbf{K}_0)}.$$

As Ψ^2 is a homeomorphism from the set of all full-dimensional centered ellipsoids into the set of all positive definite symmetric $n \times n$-matrices, and $\hat{\alpha}_M$ is a composition of $(\Psi^2)^{-1}$, $\hat{\Psi}^r$ and $\hat{\Psi}_M(W_m)$ we conclude that $\hat{\alpha}_M$ converges to α_M almost surely implying that $\hat{\mathrm{e}}_M$ converges almost surely to $\mathrm{e}_{\mathrm{M}}(X)$ in the Hausdorff metric making $\hat{\mathrm{e}}_M$ a consistent estimator.
Again by [4, Cor. 12.2.V] it follows that

$$\hat{\Psi}_I(W_m) \xrightarrow{a.s.} \mathbb{E}\Big(\Psi^2\Big(\mathbf{K}_0 - \frac{\Psi^1(\mathbf{K}_0)}{\Psi^0(\mathbf{K}_0)}\Big)\Big).$$

Hence by a similar argument as before then $\hat{\mathrm{e}}_I$ converges almost surely to $\mathrm{e}_{\mathrm{I}}(X)$ in the Hausdorff metric.
Now [4, Cor. 12.2.V] also yields

$$\hat{\Phi}_B(W_m) \xrightarrow{a.s.} \mathbb{E}\Phi^2(\mathbf{K}_0)$$

and by Theorem C.3.1 the sets $\hat{\mathrm{e}}_B$ converges almost surely to $\mathrm{e}_{\mathrm{B}}(X)$ in the Hausdorff metric. This shows the last consistency claim. □

Note that Campbell's theorem (see for instance [16, Theorem 3.5.5]) implies that $\hat{\Phi}_B$ is in fact a ratio unbiased estimator of (C.4.5).

Besides inference from observations in full-dimensional windows, estimators from lower dimensional sections are also of great practical interest. For instance, a stereological estimator of the rank-2 mean *surface* particle tensor is derived in [15] where Crofton's section formula with hyperplanes for surface tensors yields an estimator from sections of the entire germ-grain process $\tilde{X}$ with hyperplanes. A similar estimator from linear sections can be derived from the corresponding Crofton formula in [9]. As the mean *volume* particle tensors depend on the choice of reference points they cannot be derived from lower-dimensional affine sections of $\tilde{X}$ without additional information outside the sectioning flat. However, local stereological estimators of these tensors are available as the sections are here taken individually through the reference point of each particle of the process. An example of such an estimator is the slice estimator in [11].
Under the further assumption of X being L-restricted isotropic, Theorem C.2.2 allows for a generalization of the section estimator defined for $n = 3$ and $k = 1$ in [11]: if $L = \mathrm{span}(e_{n-k+1}, \dots, e_n)$ then a ratio unbiased estimator of $\mathbb{E}\Psi^r(\mathbf{K}_0)$ is

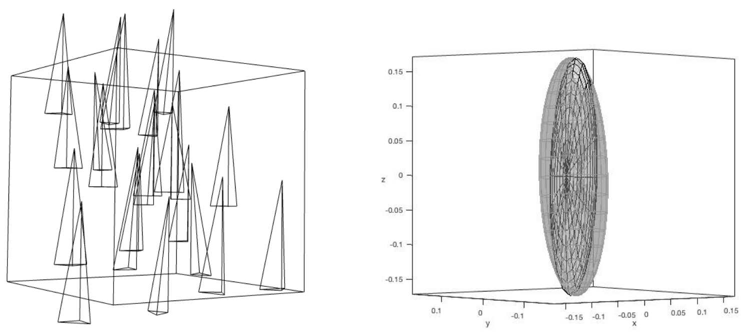
$$\frac{\sum_{K \in X, x(K) \in W} \tilde{\Psi}^r(K - x(K))}{N(W)}$$

where

$$\tilde{\Psi}^r(K)_{i_1,\dots,i_n} = \frac{(r+n-k-1)!}{r!} c_{i_1,\dots,i_{n-k}} \Psi_M^{r+n-k-1}(K \cap M_+)_{i, i_{n-k+1},\dots,i_n}$$

only depends on lower-dimensional sections of the particles. However, in view of the remark after Corollary C.2.7 then Theorem C.2.4 does not allow for a similar ratio unbiased estimator of $\mathbb{E}\Phi^2(\mathbf{K}_0)$.

In Figure C.1(a) we see a realization of a stationary particle process in $W = [0,1]^3$ induced by the stationary independently marked Poisson point process X in $\mathbb{R}^3$, where all marks are uniform rotations about the z-axis of a deterministic 3-simplex S. The center of mass of S coincides with the origin. By construction, X is L-restricted isotropic with $L = \text{span}\{e_3\} \in G(3,1)$. In Figure C.1(b) the above defined estimators of Miles (black) and Blaschke (grey) ellipsoids are depicted. As the reference points were chosen to be the centers of mass of the particles, the inertia and Miles ellipsoids coincide. We note that all three estimators capture the elongation and the orientation of S.



(a) (b)

Figure C.1: Simulation in $W = [0,1]^3$ of the particle process induced by a stationary marked point process with intensity $\gamma = 30$. The marks are rotations of a deterministic 3-simplex with vertices at the origin, at $(1/10,0,0)$, $(0,1/10,0)$ and at $(0,0,-1/2)$, rotated uniformly about the z-axis after translating its center of mass at the origin. The reference points are formed by a stationary Possion point process. In b. the Miles and inertia ellipsoids are depicted in black (coinciding) and the Blaschke ellipsoid in grey.

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