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THE CONNES-HIGSON CONSTRUCTION IS AN ISOMORPHISM

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Abstract. Let A and B be separable C^*-algebras, B stable. We show that the Connes-Higson construction gives rise to an isomorphism between the group of unitary equivalence classes of extensions of SA by B, modulo the extensions which are asymptotically split, and the homotopy classes of asymptotic homomorphisms from S^2A to B.

1. Introduction

The fundamental homotopy functors on the category of separable C^*-algebras are all based on extensions - either *a priori* or *a posteriori* . So also the E-theory of Connes and Higson; in the words of the founders: 'La E-theorie est ainsi le quotient par homotopie de la théorie des extensions', cf. [CH]. The connection between the asymptotic homomorphisms which feature explicitly in the definition of E-theory, and C^*-extensions, appears as a fundamental construction which associates an asymptotic homomorphism $SA \to B$ to a given extension of A by B. While it is easy to see that the homotopy class of the asymptotic homomorphism only depends on the homotopy class of the extension it is not so easy to decide if the converse is also true; if the extensions must be homotopic when the asymptotic homomorphisms which they give rise to via the Connes-Higson construction are. A part of the main result in the present paper asserts that this is the case when A is a suspension and B is stable. Rather unexpectedly it turned out that the methods we developed for this were also able to characterize E-theory as the quotient of all extensions of SA by B by an algebraic relation which is very similar to the algebraic relation which has been considered on the set of extensions since the way-breaking work of Brown, Douglas and Fillmore, [BDF]. Recall that in the BDF-approach two C^*-extensions are identified when they become unitarily equivalent after addition by extensions which are split, meaning that the quotient map admits a $*$-homomorphism as a right-inverse. In the algebraic relation, on the set of all C^*-extensions of SA by B, which we will show gives rise to E-theory, two extensions are identified when they become unitarily equivalent after addition by extensions which are asymptotically split, where we call an extension

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0$$

asymptotically split when there is an asymptotic homomorphism $\pi = (\pi_t)_{t\in[1,\infty)}$: $A \to E$ such that $p \circ \pi_t = \mathrm{id}_A$ for all t. We emphasize that with this algebraic relation all extensions of SA by B admit an inverse. In contrast, Kirchberg has shown, [Ki], that there are C^*-algebras A (e.g. $A = C_r^*(SL_2(\mathbb{Z}))$) for which the unitary equivalence classes of SA by $\mathcal{K}$, modulo the split extensions, do not form a group. Since our results show that the algebraic relation we have just described is the same as homotopy, our main result can also be considered as a result on

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homotopy invariance and it is therefore noteworthy that the proof is self-contained, and in particular does not depend on the homotopy invariance results of Kasparov.

Since there is also an equivariant version of E-theory, [GHT], which is being used in connection with the Baum-Connes conjecture, we formulate and prove our results in the equivariant case. With the present technology this does not require much additional work, but since some of the material which we shall build on does not explicitly consider the equivariant setting, notably [DL] and [H-LT], there are a few places where we leave the reader to check that the results from these sources can be adapted to the equivariant case.

2. An alternative to the BDF extension group

Let G be a locally compact, σ-compact group, and let A and B be separable G-algebras, i.e. separable C^*-algebras with a pointwise norm-continuous action of G by automorphisms. Assume also that B is weakly stable, i.e. that B is equivariantly isomorphic to $B \otimes \mathcal{K}$ where $\mathcal{K}$ denotes the compact operators of l_2 with the trivial G-action. Let $M(B)$ denote the multiplier algebra of B, $Q(B) = M(B)/B$ the corresponding corona algebra and $q_B : M(B) \to Q(B)$ the quotient map. Then G acts by automorphisms on both $M(B)$ and $Q(B)$[1]. It follows from [Th1] that we can identify the set of equivariant $*$-homomorphisms, $\mathrm{Hom}_G(A, Q(B))$, from A to $Q(B)$ with the set of G-extensions of A by B. Two G-extensions $\varphi, \psi : A \to Q(B)$ are *unitarily equivalent* when there is a unitary $w \in M(B)$ such that $q_B(w) \in Q(B)$ is G-invariant and $\operatorname{Ad} q_B(w) \circ \varphi = \psi$. Since B is weakly stable the set of unitary equivalence classes of extensions of A by B form a semi-group; the addition is obtained by choosing two G-invariant isometries $V_1, V_2 \in M(B)$ such that $V_1V_1^* + V_2V_2^* = 1$ and setting $\varphi \oplus \psi = q_B(V_1)\varphi q_B(V_1)^* + q_B(V_2)\psi q_B(V_2)^*$. A G-extension $\varphi : A \to Q(B)$ will be called *asymptotically split* when there is an asymptotic homomorphism $\pi = \{\pi_t\}_{t\in[1,\infty)} : A \to M(B)$ such that $q_B \circ \pi_t = \varphi$ for all t. All asymptotic homomorphisms we consider in this paper will be assumed to be equivariant in the sense that $\lim_{t\to\infty} g \cdot \pi_t(a) - \pi_t(g \cdot a) = 0$ for all $a \in A$ and $g \in G$. As in [MT2] we say that a G-extension $\varphi : A \to Q(B)$ is *semi-invertible* when there is a G-extension $\psi \in \mathrm{Hom}_G(A, Q(B))$ such that $\varphi \oplus \psi : A \to Q(B)$ is asymptotically split. Two semi-invertible extensions, φ, ψ, are called *stably unitary equivalent* when they become unitarily equivalent after addition by asymptotically split extensions, i.e. when there is an asymptotically split extension λ such that $\varphi \oplus \lambda$ is unitarily equivalent to $\psi \oplus \lambda$. This is an equivalence relation on the subset of semi-invertible extensions in $\mathrm{Hom}_G(A, Q(B))$ and the corresponding equivalence classes form an abelian group which we denote by $\mathrm{Ext}^{-1/2}(A, B)$. For any locally compact space X we consider $C_0(X) \otimes A$ as a G-algebra with the trivial G-action on the tensor factor $C_0(X)$. When $X =]0, 1]$ we denote $C_0]0, 1] \otimes A$ by $\mathrm{cone}(A)$. Similarly, we set $SA = C_0(0, 1) \otimes A$.

Lemma 2.1. *Let $\lambda : \mathrm{cone}(A) \to Q(B)$ be a G-extension. It follows that there is an asymptotic homomorphism $\pi = (\pi_t)_{t\in[1,\infty)} : \mathrm{cone}(A) \to M_2(M(B))$ such that*

$$q_{M_2(B)} \circ \pi_t = \begin{pmatrix} \lambda & \\ & 0 \end{pmatrix}$$

for all $t \in [1, \infty)$.

[1]These actions are not pointwise normcontinuous in general.

Proof. The proof is based on an idea of Voiculescu, cf. [V]. Let $\mu : \mathrm{cone}(A) \to M(B)$ be a continuous, self-adjoint and homogeneous lift of λ such that $\|\mu(x)\| \leq 2\|x\|$ for all $x \in \mathrm{cone}(A)$. μ exists by the Bartle-Graves selection theorem, cf. [L]. Define $\varphi_s : \mathrm{cone}(A) \to \mathrm{cone}(A)$ such that $\varphi_s(f)(t) = f((1-s)t), s \in [0,1]$. Choose continuous functions $f_i : [1,\infty) \to [0,1], i = 0,1,2,\cdots$, such that

1) $f_0(t) = 0$ for all $t \in [1,\infty)$,
2) $f_n \leq f_{n+1}$ for all n,
3) for each $n \in \mathbb{N}$, there is an $m_n \in \mathbb{N}$ such that $f_i(t) = 1$ for all $i \geq m_n$, and all $t \in [1, n+1]$,
4) $\lim_{t\to\infty} \max_i |f_i(t) - f_{i+1}(t)| = 0$.

Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be an increasing sequence of finite subsets with dense union in $\mathrm{cone}(A)$. Write $G = \bigcup_n K_n$ where $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ is a sequence of compact subsets of G such that $G = \bigcup_n K_n$. For each n, choose $m_n \in \mathbb{N}$ as in 3). We may assume that $m_{n+1} > m_n$. By Lemma 1.4 of [K] we can choose elements

$$X_0^n \geq X_1^n \geq X_2^n \geq \cdots$$

in B such that $0 \leq X_i^n \leq 1$ for all i and $X_i^n = 0$ for $i \geq m_n$, and

1') $X_i^n X_{i+1}^n = X_{i+1}^n$ for all i,
2') $\|X_i^n b - b\| \leq \frac{1}{n}$ for all $i = 0,1,2,\cdots, m_n - 1$, and all $b \in S_n$,
3') $\|X_i^n y - y X_i^n\| \leq \frac{1}{n}$ for all i and all $y \in L_n$,
4') $\|g \cdot X_i^n - X_i^n\| \leq \frac{1}{n}$, $g \in K_n$, for all i,
5') $\|X_i^n(g\cdot\mu(a) - \mu(g\cdot a)) - (g\cdot\mu(a) - \mu(g\cdot a))\| \leq \frac{1}{n}$, $g \in K_n, a \in F_n$, for all $i = 0,1,2,\cdots, m_n - 1$,

where L_n and S_n are the compact sets $L_n = \{\mu(\varphi_s(a)) : \ s \in [0,1], \ a \in F_n\}$ and

$$\begin{aligned} S_n = &\{\mu(\varphi_s(a)) + \mu(\varphi_s(b)) - \mu(\varphi_s(a+b)) : \ a,b \in F_n, \ s \in [0,1]\} \\ &\cup \{\mu(\varphi_s(ab)) - \mu(\varphi_s(a))\mu(\varphi_s(b)) : \ a,b \in F_n, \ s \in [0,1]\} . \end{aligned}$$

Since we choose the X's recursively we can arrange that $X_i^{n+1} X_k^n = X_k^n$ for all k and all $i \leq m_{n+1}$. By connecting first X_0^n to X_0^{n+1} via the straight line between them, then X_1^n to X_1^{n+1} via a straight line, then X_2^n to X_2^{n+1} etc., we obtain norm-continuous pathes, $X(t,i), t \in [n, n+1], i = 0,1,2,3,\cdots$, in B such $X(n,i) = X_i^n$, $X(n+1,i) = X_i^{n+1}$ for all i and

a) $X(t,i)X(t,i+1) = X(t,i+1)$, $t \in [n,n+1]$, for all i,
b) $\|X(t,i)b - b\| \leq \frac{1}{n}$ for all $i = 0,1,2,\cdots, m_n - 1$, $t \in [n,n+1]$ and all $b \in S_n$,
c) $\|X(t,i)y - yX(t,i)\| \leq \frac{1}{n}$ for all i, all $t \in [n,n+1]$ and all $y \in L_n$,
d) $\|g \cdot X(t,i) - X(t,i)\| \leq \frac{1}{n}, g \in K_n, t \in [n,n+1]$, for all i,
e) $\|X(t,i)(g\cdot\mu(a) - \mu(g\cdot a)) - (g\cdot\mu(a) - \mu(g\cdot a))\| \leq \frac{1}{n}$, $g \in K_n, a \in F_n, t \in [n,n+1]$, for all $i = 0,1,\cdots, m_n - 1$.

In addition, $X(t,i) = 0, i \geq m_{n+1}, t \in [n,n+1]$. Let $l_2(B)$ denote the Hilbert B-module of sequences $(b_1, b_2, b_3, \cdots)$ in B such that $\sum_{i=1}^\infty b_i^* b_i$ converges in norm. Writing an element $(b_1,b_2,b_3,\cdots) \in l_2(B)$ as the sum $\sum_{i=0}^\infty b_i e_i$ we define a representation V of G on $l_2(B)$ such that $V_g(\sum_{i=0}^\infty b_i e_i) = \sum_{i=0}^\infty (g \cdot b_i) e_i$. Then G acts by automorphisms on $\mathbb{L}(l_2(B))$ ($=$ the adjoinable operators on $l_2(B)$) such that

$g \cdot m = V_g m V_{g^{-1}}$. Set

$$T_t = \begin{pmatrix} \sqrt{1 - X(t,0)} & \sqrt{X(t,0) - X(t,1)} & \sqrt{X(t,1) - X(t,2)} & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{L}(l_2(B)).$$

Then $P_t = T_t^* T_t$ is a projection in $\mathbb{L}(l_2(B))$ since $T_t T_t^*$ clearly is. Note that P_t is tri-diagonal because of condition a) above, and that the entries of P_t are all in B, with the notable exception of the 1×1-entry which is equal to 1 modulo B. We define $\delta_t : \operatorname{cone}(A) \to \mathbb{L}(l_2(B))$ by

$$\delta_t(a)(\sum_{i=0}^{\infty} b_i e_i) = \sum_{i=0}^{\infty} \mu(\varphi_{f_i(t)}(a)) b_i e_i.$$

Set $\pi_t(a) = P_t \delta_t(a) P_t$ for $a \in \operatorname{cone}(A)$ and $t \in [1, \infty)$. We assert that $\pi = (\pi_t)_{t \in [1,\infty)}$ is an asymptotic homomorphism. Since the family of maps $a \mapsto \pi_t(a), t \in [1, \infty)$, is an equicontinuous family of self-adjoint and homogeneous maps, it suffices to take an n and elements $a, b \in F_n$, $g \in K_n$, and check that

$$\lim_{t \to \infty} P_t \delta_t(a) P_t \delta_t(b) P_t - P_t \delta_t(ab) P_t = 0,$$

$$\lim_{t \to \infty} P_t \delta_t(a + b) P_t - P_t \delta_t(a) P_t - P_t \delta_t(b) P_t = 0,$$

and

$$\lim_{t \to \infty} P_t \delta_t(g \cdot a) P_t - g \cdot (P_t \delta_t(a) P_t) = 0.$$

The first two limits are zero by 4), b) and c), the third by d) and e). For each a, t, $P_t \delta_t(a) P_t = \operatorname{diag}(\mu(a), 0, 0, \cdots)$ modulo $\mathbb{K}(l_2(B))$ (= the ideal of 'compact' operators on $l_2(B)$). Since B is weakly stable there is an equivariant isomorphism $l_2(B) \simeq B \oplus B$ of Hilbert B-modules which leaves the first coordinate invariant. We can therefore transfer π to an asymptotic homomorphism $\pi = (\pi_t)_{t \in [1,\infty)} : \operatorname{cone}(A) \to \mathbb{L}(B \oplus B) = M_2(M(B))$ with the stated property.

□

Two G-extensions $\varphi, \psi \in \operatorname{Hom}_G(A, Q(B))$ are *strongly homotopic* when there is a path $\mathbf{\Phi}_t \in \operatorname{Hom}_G(A, Q(B)), t \in [0, 1]$, such that $\mathbf{\Phi}_0 = \varphi, \mathbf{\Phi}_1 = \psi$ and $t \mapsto \mathbf{\Phi}_t(a)$ is continuous for all $a \in A$.

Theorem 2.2. *Let $\varphi : A \to Q(B)$ be a G-extension which is strongly homotopic to 0 in $\operatorname{Hom}_G(A, Q(B))$. It follows that there is an asymptotic homomorphism $\pi = (\pi_t)_{t \in [1,\infty)} : A \to M_2(M(B))$ such that*

$$q_{M_2(B)} \circ \pi_t = \begin{pmatrix} \varphi & \\ & 0 \end{pmatrix}$$

for all $t \in [1, \infty)$.

Proof. Since φ is strongly homotopic to 0 there is an equivariant $*$-homomorphism $\mu : A \to \operatorname{cone}(D)$, where $D \subseteq Q(B)$ is a separable G-algebra containing $\varphi(A)$, and an equivariant $*$-homomorphism $\lambda : \operatorname{cone}(D) \to Q(B)$ such that $\varphi = \lambda \circ \mu$. Apply Lemma 2.1 to λ. □

Corollary 2.3. *Every G-extension $\varphi : SA \to Q(B)$ is semi-invertible.*

Proof. Let $\alpha \in \operatorname{Aut} SA$ be the automorphism of SA given by $\alpha(f)(t) = f(1-t)$. It is wellknown that $\varphi \oplus \varphi \circ \alpha$ is strongly homotopic to 0. Hence $\varphi \oplus \varphi \circ \alpha \oplus 0$ is asymptotically split by Theorem 2.2. □

Because of Corollary 2.3 we drop the superscript $-1/2$ and write $\operatorname{Ext}(SA, B)$ instead of $\operatorname{Ext}^{-1/2}(SA, B)$.

Lemma 2.4. *Let $\varphi, \psi : SA \to Q(B)$ be two G-extensions which are strongly homotopic. It follows that φ and ψ are stably unitarily equivalent.*

Proof. This follows straightforwardly from Theorem 2.2 and Corollary 2.3. □

Set $IB = C[0,1] \otimes B$ and let $e_t : IB \to B$ denote evaluation at $t \in [0,1]$ and note that e_t defines a equivariant $*$-homomorphisms $M(IB) \to M(B)$ and $Q(IB) \to Q(B)$ which we again denote by e_t. Two G-extensions $\varphi, \psi \in \operatorname{Hom}_G(A, Q(B))$ are *homotopic* when there is a G-extension $\mathbf{\Phi} \in \operatorname{Hom}_G(A, Q(IB))$ such that $e_0 \circ \mathbf{\Phi} = \varphi$ and $e_1 \circ \mathbf{\Phi} = \psi$. As in [MT2] we denote the set of homotopy classes of G-extensions by $\operatorname{Ext}(A, B)_h$. In general this is merely an abelian semigroup, but $\operatorname{Ext}(SA, B)_h$ is a group.

The Connes-Higson construction associates to any G-extension $\varphi \in \operatorname{Hom}_G(A, Q(B))$ an asymptotic homomorphism $CH(\varphi) : SA \to B$ in the following way, cf. [CH], [GHT]: By use of Lemma 1.4 of [K] or Lemma 5.3 of [GHT] there is a norm-continuous path $\{u_t\}_{t \in [1,\infty)}$ of elements in B such that $0 \leq u_t \leq 1$ for all t, $\lim_{t \to \infty} \|u_t b - b\| = 0$ for all $b \in B$, $\lim_{t \to \infty} \|u_t m - m u_t\| = 0$ for all $m \in q_B^{-1}(\varphi(A))$ and $\lim_{t \to \infty} \|g \cdot u_t - u_t\| = 0$ for all $g \in G$. From these data $CH(\varphi)$ is determined up to asymptotic equality as the equicontinuous[2] asymptotic homomorphism $CH(\varphi) : SA \to B$ which satisfies that

$$\lim_{t \to \infty} CH(\varphi)_t(f \otimes a) - f(u_t)x = 0, \qquad x \in q_B^{-1}(\varphi(a)),$$

for all $f \in C_0(0,1)$ and all $a \in A$. Let $[[SA, B]]$ denote the abelian group of homotopy classes of asymptotic homomorphisms, $SA \to B$, cf. [CH], [GHT]. The Connes-Higson construction defines in the obvious way a semi-group homomorphism $CH : \operatorname{Ext}(A, B)_h \to [[SA, B]]$. Since there is a canonical (semi-group) homomorphism $\operatorname{Ext}^{-1/2}(A, B) \to \operatorname{Ext}(A, B)_h$ we may also consider the Connes-Higson construction as a homomorphism $CH : \operatorname{Ext}^{-1/2}(A, B) \to [[SA, B]]$. Notice that $\operatorname{Ext}(SA, B)$ and $\operatorname{Ext}(SA, B)_h$ are both abelian groups and the canonical map $\operatorname{Ext}(SA, B) \to \operatorname{Ext}(SA, B)_h$ is a surjective group homomorphism by Corollary 2.3.

3. On equivalence of asymptotic homomorphisms

Lemma 3.1. *Let A and B be separable G-algebras, B weakly stable. Let $\varphi = (\varphi_t)_{t \in [1,\infty)} : A \to B$ be an asymptotic homomorphism which is homotopic to 0. It follows that there is an asymptotic homomorphism $\psi = (\psi_t)_{t \in [1,\infty)} : A \to B$ and a norm-continuous path $\{W_t\}_{t \in [1,\infty)}$ of G-invariant unitaries in $M(M_2(B))$ such that*

$$\lim_{t \to \infty} \begin{pmatrix} \varphi_t(a) & \\ & \psi_t(a) \end{pmatrix} - W_t \begin{pmatrix} 0 & \\ & \psi_t(a) \end{pmatrix} W_t^* = 0$$

for all $a \in A$.

[2] Equicontinuity of an asymptotic homomorphism $\pi = (\pi_t)_{t \in [1,\infty)} : A \to B$ means that $A \times G \ni (a, g) \mapsto g \cdot \pi_t(a), t \in [1, \infty)$, is an equicontinuous family of maps.

Proof. Let $\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\in[1,\infty)} : A \to IB$ be an asymptotic homomorphism such that $e_0 \circ \mathbf{\Phi}_t(a) = 0, e_1 \circ \mathbf{\Phi}_t(a) = \varphi_t(a)$ for all $t \in [1,\infty), a \in A$. We may assume that both φ and $\mathbf{\Phi}$ are equicontinuous, cf. Proposition 2.4 of [Th2]. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite subsets with dense union in A. For each n there is $\delta_n > 0$ with the property that

$$\|e_x \circ \mathbf{\Phi}_t(a) - e_y \circ \mathbf{\Phi}_t(a)\| < \frac{1}{n}$$

when $|x-y| < \delta_n, t \in [1,n], a \in F_n$. Choose then a sequence of functions $f_k : [1,\infty) \to [0,1]$ such that $f_1(t) = 1,\ f_k \geq f_{k+1},\ |f_k(t) - f_{k+1}(t)| < \delta_n,\ t \in [1,n]$ for all k, n and such that $f_k|_{[1,n]} = 0$ for all but finitely many k's for all n. Set $\lambda_t^n(a) = e_{f_n(t)} \circ \mathbf{\Phi}_t(a)$ for all $a \in A, n \in \mathbb{N}, t \in [1,\infty)$. Note that $\|\lambda_t^i(a) - \lambda_t^{i+1}(a)\| < \frac{1}{n}, a \in F_n,\ t \in [1,n]$, for all i and n. Then

$$\mu_t(a) = \operatorname{diag}(\varphi_t(a), \lambda_t^1(a), \lambda_t^2(a), \lambda_t^3(a), \cdots) \ \in \ \mathbb{K}(l_2(B))$$

and

$$\delta_t(a) = \operatorname{diag}(0, \lambda_t^1(a), \lambda_t^2(a), \lambda_t^3(a), \cdots) \ \in \ \mathbb{K}(l_2(B))$$

define asymptotic homomorphisms $\mu, \delta : A \to \mathbb{K}(l_2(B))$. By connecting appropriate permutation unitaries, acting on $l_2(B)$ by permutations of B-coordinates, we get a norm-continuous path of G-invariant unitaries $\{S_t\}_{t\in[1,\infty)} \subseteq \mathbb{L}(l_2(B))$ such that

$$S_t\delta_t(a)S_t^* = \operatorname{diag}(\lambda_t^1(a), \lambda_t^2(a), \lambda_t^3(a), \cdots)$$

for all a, t. Then $\lim_{t\to\infty} \mu_t(a) - S_t\delta_t(a)S_t^* = 0$ for all $a \in A$. Since B is weakly stable there is an isomorphism $l_2(B) \to B \oplus B$ of Hilbert B, G-algebras which fixes the first coordinate. Applying this isomorphism in the obvious way and remembering the identifications $\mathbb{K}(B \oplus B) = M_2(B)$ and $\mathbb{L}(B \oplus B) = M(M_2(B))$ gives the result. □

Theorem 3.2. *Let A and B be separable G-algebras, B weakly stable. Assume that $[[A,B]]$ is a group. Let $\varphi = (\varphi_t)_{t\in[1,\infty)},\ \psi = (\psi_t)_{t\in[1,\infty)} : A \to B$ be asymptotic homomorphisms which are homotopic. It follows that there is an asymptotic homomorphism $\lambda = (\lambda_t)_{t\in[1,\infty)} : A \to B$ and a norm-continuous path $\{W_t\}_{t\in[1,\infty)}$ of G-invariant unitaries in $M(M_2(B))$ such that*

$$\lim_{t\to\infty} \begin{pmatrix} \varphi_t(a) & \\ & \lambda_t(a) \end{pmatrix} - W_t \begin{pmatrix} \psi_t(a) & \\ & \lambda_t(a) \end{pmatrix} W_t^* = 0$$

for all $a \in A$.

Proof. This follows straightforwardly from Lemma 3.1. □

Lemma 3.3. *Let B be a weakly stable G-algebra and D_0 a separable G-subalgebra of $C_b([1,\infty), B)$. Let $V \in M(B)$ be a G-invariant isometry. There is then a weakly stable G-subalgebra D of $C_b([1,\infty), B)$ such that $VD \cup V^*D \cup D_0 \subseteq D$.*

Proof. Since B is weakly stable we can write $B = B \otimes \mathcal{K}$ with G acting trivially on the tensor-factor $\mathcal{K}$. We embed $\mathcal{K}$ into $M(B \otimes \mathcal{K})$ via $x \mapsto 1_B \otimes x$. Let $\{f_n\} \subseteq C_b([1,\infty), B \otimes \mathcal{K})$ be a dense sequence in D_0. For each $n \in \mathbb{N}$ there is a function $g_n \in C_b([1,\infty), \mathcal{K})$ such that $\|g_n f_n - f_n\| < \frac{1}{n}$. Let E_{00} be the C^*-algebra generated by $\{g_n\}_{n=1}^{\infty}$. Then $E_{00} \subseteq C_b([1,\infty), \mathcal{K}) \subseteq C_b([1,\infty), B^+ \otimes \mathcal{K})$. Consider a positive element $f \in E_{00}$ and an $\epsilon > 0$. Set $V_j =]j, j+2[\cap [1,\infty[, j = 0,1,2,\cdots$. We can then find a sequence $p_0 \leq p_1 \leq p_2 \leq \cdots$ of projections in $\mathcal{K}$ such that

$$\sup_{x \in \overline{V_j}} \|p_j f(x) p_j - f(x)\| < \epsilon.$$

Let $\{h_j\}$ be a partition of unity in $C_b[1,\infty)$ subordinate to the cover $\{V_j\}$ and set $g(t) = \sum_{j=0}^{\infty} h_j(t) p_j f(t) p_j$. Then $g \in C_b([1,\infty), \mathcal{K})$, $g \geq 0, \|g - f\| < \epsilon$. For each j we choose a partial isometry $v_j \in \mathcal{K}$ such that $v_j v_j^* = p_{j+2}$, $v_j^* v_j p_{j+2} = 0$ and $v_j^* v_j v_k^* v_k = 0, k < j$. Set $h(t) = \sum_{j=0}^{\infty} \sqrt{h_j(t)} v_j$. Then $hh^* g = g$ and $h^* h g = 0$. It follows that we can find a sequence $E_{00} = X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ of separable C^*-subalgebras of $C_b([1,\infty), \mathcal{K})$ and for each n have a dense sequence $\{f_1, f_2, \cdots\}$ in the positive part of X_n and elements $\{v_1, v_2, \cdots\}$ in X_{n+1} such that $\|f_k - v_k^* v_k\| < \frac{1}{k}$ and $v_k^* v_k v_k v_k^* = 0$ for all k. It follows then from Proposition 2.2 and Theorem 2.1 of [HR] that $E_0 = \overline{\bigcup_n X_n}$ is a separable stable C^*-subalgebra of $C_b([1,\infty), \mathcal{K})$ such that $E_{00} \subseteq E_0$. Note that E_0 contains a sequence $\{r_n\}$ with the property that $\lim_{n\to\infty} r_n x = x$ for all $x \in D_0$ since E_{00} does. By repeating this argument with D_0 substituted be the G-algebra D_1 generated by $D_0 \cup VD_0 \cup V^* D_0 \cup E_0 D_0$, we get a stable C^*-subalgebra $E_1 \subseteq C_b([1,\infty), \mathcal{K})$ which contains a sequence $\{r_n\}$ such that $\lim_{n\to\infty} r_n y = y$ for all $y \in D_1$. It is clear from the construction that we can arrange that $E_0 \subseteq E_1$. We can therefore continue this procedure to obtain sequences of separable G-algebras,

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$$

in $C_b([1,\infty), B \otimes \mathcal{K})$, and

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$$

in $C_b([1,\infty), \mathcal{K}) \subseteq C_b([1,\infty), B^+ \otimes \mathcal{K})$ such that each E_n is stable and contains a sequence $\{r_k\}$ such that $\lim_{k\to\infty} r_k x = x,\ x \in D_n$, and such that $D_n \cup VD_n \cup V^* D_n \cup E_n D_n \subseteq D_{n+1}$ for all n. Set $E_\infty = \overline{\bigcup_n E_n}$ and $D = \overline{\bigcup_n D_n}$. It follows from Corollary 4.1 of [HR] that E_∞ is stable. By construction $VD \cup V^* D \subseteq D$ and $E_\infty D \subseteq D$. The last property ensures that D is an ideal in the G-algebra E generated by E_∞ and D. There is therefore a $*$-homomorphism $\lambda : E_\infty \to M(D)$. By construction an approximate unit for E_∞ is also an approximate unit for D so λ extends to a $*$-homomorphism $\lambda : M(E_\infty) \to M(D)$ which is strictly continuous on the unit ball of $M(E_\infty)$. Since E_∞ is stable there is a sequence $P_i, i = 1,2,\cdots$, of orthogonal and Murray-von Neumann equivalent projections in $M(E_\infty)$ which sum to 1 in the strict topology. Then $Q_i = \lambda(P_i), i = 1,2,\cdots$, is a sequence of orthogonal and Murray-von Neumann equivalent projections in $M(D)$ which sum to 1 in the strict topology. Since E_∞ consists entirely of G-invariant elements it follows that all the Q_i's are G-invariant. Consequently $D \simeq Q_1 D Q_1 \otimes \mathcal{K}$ as G-algebras, proving that D is weakly stable.

□

Two asymptotic homomorphisms $\varphi = (\varphi_t)_{t\in[1,\infty)}, \psi = (\psi_t)_{t\in[1,\infty)} : A \to B$ will be called *equi-homotopic* when there is a family $\mathbf{\Phi}^\lambda = (\mathbf{\Phi}_t^\lambda)_{t\in[1,\infty)} : A \to B,\ \lambda \in [0,1]$, of asymptotic homomorphisms such that the family of maps, $[0,1] \ni \lambda \mapsto \mathbf{\Phi}_t^\lambda(a),\ t \in [1,\infty)$, is equicontinuous for each $a \in A$.

Theorem 3.4. *Let A and B be separable G-algebras, B weakly stable. Let $\varphi = (\varphi_t)_{t\in[1,\infty)}, \psi = (\psi_t)_{t\in[1,\infty)} : SA \to B$ be asymptotic homomorphisms. Then the following are equivalent:*

1) *φ and ψ are homotopic (i.e. $[\varphi] = [\psi]$ in $[[SA, B]]$).*
2) *φ and ψ are equi-homotopic.*
3) *There is an asymptotic homomorphism $\lambda = (\lambda_t)_{t\in[1,\infty)} : SA \to B$ and a norm-continuous path $\{W_t\}_{t\in[1,\infty)}$ of G-invariant unitaries in $M(M_2(B))$ such that*

$$\lim_{t\to\infty} \begin{pmatrix} \varphi_t(a) & \\ & \lambda_t(a) \end{pmatrix} - W_t \begin{pmatrix} \psi_t(a) & \\ & \lambda_t(a) \end{pmatrix} W_t^* = 0$$

for all $a \in A$.

Proof. The equivalence 1) $\Leftrightarrow$ 3) follows from Theorem 3.2 and the implication 2) $\Rightarrow$ 1) is trivial, so we need only prove that 2) $\Rightarrow$ 1). To this end, let $[[SA, B]]^e$ denote the set of equi-homotopy classes of asymptotic homomorphisms $SA \to B$. Choose G-invariant isometries $V_1, V_2 \in M(B)$ such that $V_1V_1^* + V_2V_2^* = 1$ and define a composition in $[[SA, B]]^e$ by

$$[\varphi] + [\psi] = [(V_1\varphi_tV_1^* + V_2\psi_tV_2^*)_{t\in[1,\infty)}] \ .$$

It follows from Lemma 3.3 that $[[SA, B]]^e$ is a group. It suffices therefore to show that the natural map $[[SA, B]]^e \to [[SA, B]]$ has trivial kernel. If φ is an asymptotic homomorphism representing an element in the kernel we conclude from Lemma 3.1 that there is a norm-continuous path $W_t, t \in [1,\infty)$, of G-invariant unitaries in $M_2(M(B))$ and an asymptotic homomorphism ψ such that

$$\lim_{t\to\infty} \begin{pmatrix} \varphi_t(a) & & \\ & \psi_t(a) & \\ & & 0 \\ & & & 0 \end{pmatrix} - \begin{pmatrix} W_t & \\ & W_t^* \end{pmatrix} \begin{pmatrix} 0 & & \\ & \psi_t(a) & \\ & & 0 \\ & & & 0 \end{pmatrix} \begin{pmatrix} W_t^* & \\ & W_t \end{pmatrix} = 0$$

for all $a \in SA$. By a standard rotation argument we can remove the unitaries $\begin{pmatrix} W_t & \\ & W_t^* \end{pmatrix}$ via an equi-homotopy and we see in this way that $[\varphi] + [\psi] = [\psi]$ in $[[SA, B]]^e$. Hence $[\varphi] = 0$ in $[[SA, B]]^e$.

□

Simple examples show that the implications 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) of Theorem 3.4 generally fail in $[[A, B]]$.

4. The main results

Let A and B be separable C^*-algebras. Set

$$M(B)_G = \{x \in M(B) : \ G \ni g \mapsto g \cdot x \text{ is norm-continuous}\}$$

and

$$Q(B)_G = \{x \in Q(B) : \ G \ni g \mapsto g \cdot x \text{ is norm-continuous}\}.$$

Then

$$0 \longrightarrow B \longrightarrow M(B)_G \longrightarrow Q(B)_G \longrightarrow 0 \tag{4.1}$$

is a short exact sequence of G-algebras. (This is not trivial - the surjectivity of the quotient map follows from Theorem 2.1 of [Th1].) We are going to construct a map $\alpha : [[SA, Q(B)_G \otimes \mathcal{K}]] \to \mathrm{Ext}(SA, B \otimes \mathcal{K})_h$. The key to this is another variant of the Voiculescu's tri-diagonal projection trick from [V]. Let b be a strictly positive element of $B \otimes \mathcal{K}$, $0 \leq b \leq 1$. A *unit sequence* in $B \otimes \mathcal{K}$ is a sequence $\{u_n\}_{n=0}^{\infty} \subseteq B \otimes \mathcal{K}$ such that

0) there is a continuous function $f_n : [0,1] \to [0,1]$ which is zero in a neighbourhood of 0 and $u_n = f_n(b)$,
1) $0 \leq u_n \leq 1$ for all $n = 0,1,2,3,\cdots$,
2) $u_{n+1}u_n = u_n$ for all n,
3) $\lim_{n\to\infty} u_n x = x, \ x \in B \otimes \mathcal{K}$,
4) $\lim_{n\to\infty} \|g \cdot u_n - u_n\| = 0, \ g \in G$.

Let $\{e_{ij}\}_{i,j=0}^{\infty}$ be the matrix units acting on $l_2(B \otimes \mathcal{K})$ in the standard way.

Lemma 4.1. *Let $\mathcal{U} = \{u_n\}$ be a unit sequence in $B \otimes \mathcal{K}$. Then*

$$\sqrt{u_0}e_{00} + \sum_{j=1}^{\infty} \sqrt{u_j - u_{j-1}}e_{0j}$$

converges in the strict topology to a partial isometry V in $\mathbb{L}(l_2(B \otimes \mathcal{K}))$ such that $VV^* = e_{00}$.

Proof. Let $b = (b_0, b_1, b_2, \cdots) = \sum_{i=0}^{\infty} b_i e_i \in l_2(B \otimes K)$. Then

$$\|\sum_{j=n}^{m} \sqrt{u_j - u_{j-1}}e_{0j}(b)\|^2 = \|\sum_{k,j=n}^{m} b_k^* \sqrt{u_k - u_{k-1}}\sqrt{u_j - u_{j-1}}b_j\|$$

$$= \|\sum_{k=n}^{m} b_k^*(u_k - u_{k-1})b_k + \sum_{k=n}^{m-1} b_k^* \sqrt{u_k - u_{k-1}}\sqrt{u_{k+1} - u_k}b_{k+1} + \sum_{k=n}^{m-1} b_{k+1}^* \sqrt{u_{k+1} - u_k}\sqrt{u_k - u_{k-1}}b_k\|$$

$$\leq \|\sum_{k=n}^{m} b_k^* b_k\| + 2\sqrt{\|\sum_{k=n}^{m-1} b_k^* b_k\|}\sqrt{\|\sum_{k=n+1}^{m} b_k^* b_k\|}$$

proving that $\sum_{j=1}^{\infty} \sqrt{u_j - u_{j-1}}e_{0j}(b)$ converges in $l_2(B \otimes \mathcal{K})$. And

$$\|(\sum_{j=n}^{m} \sqrt{u_j - u_{j-1}}e_{0j})^*(b)\|^2 = \|\sum_{j=n}^{m} b_0^*(u_j - u_{j-1})b_0\|,$$

proving that also $(\sum_{j=1}^{\infty} \sqrt{u_j - u_{j-1}}e_{0j})^*(b)$ converges in $l_2(B \otimes \mathcal{K})$. It follows that

$$V = \sqrt{u_0}e_{00} + \sum_{j=1}^{\infty} \sqrt{u_j - u_{j-1}}e_{0j}$$

exist as a strict limit in $\mathbb{L}(l_2(B \otimes \mathcal{K}))$. It it then straightforward to check that $VV^* = e_{00}$. □

Let $P_{\mathcal{U}} = V^*V$ and note that $P_{\mathcal{U}}$ is tri-diagonal with respect to the matrix units $\{e_{ij}\}$. Fix now a continuous and homogeneous section χ for the map $q_B \otimes \mathrm{id}_{\mathcal{K}} : M(B)_G \otimes \mathcal{K} \to Q(B)_G \otimes \mathcal{K}$. Consider an equicontinuous asymptotic homomorphism $\varphi = (\varphi_t)_{t\in[1,\infty)} : A \to Q(B)_G \otimes \mathcal{K}$. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite sets with dense union in A and $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ a sequence of compact subsets in G such that $\bigcup_n K_n = G$. It is easy to see that there is a unit sequence $\{u_n\}$ in $B \otimes \mathcal{K}$ with the following properties :

5) $\|u_n\chi(\varphi_t(a)) - \chi(\varphi_t(a))u_n\| \leq \frac{1}{n},\ a \in F_n, t \in [1, n+1]$,
6) $\|(1-u_n)(\chi(\varphi_t(ab)) - \chi(\varphi_t(a))\chi(\varphi_t(b)))\| \leq \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| + \frac{1}{n},\ t \in [1, n+1],\ a, b \in F_n$,
7) $\|(1-u_n)(\chi(\varphi_t(a+b)) - \chi(\varphi_t(a)) - \chi(\varphi_t(b)))\| \leq \|\varphi_t(a+b) - \varphi_t(a) - \varphi_t(b)\| + \frac{1}{n},\ t \in [1, n+1], a, b \in F_n$,
8) $\|(1-u_n)(g\cdot\chi(\varphi_t(a)) - \chi(\varphi_t(g\cdot a)))\| \leq \|g\cdot\varphi_t(a) - \varphi_t(g\cdot a)\| + \frac{1}{n},\ t \in [1, n], a \in F_n,\ g \in K_n$.

Let $\{\varphi_{t_n}\}_{n\in\mathbb{N}}$ be a discretization of φ, cf. Lemma 5.1 of [MT1], such that

9) $t_n \leq n$ for all $n \in \mathbb{N}$.

Set
$$\widetilde{\varphi}(a) = P_{\mathcal{U}}(\sum_{j=0}^{\infty} \chi(\varphi_{t_{j+1}}(a))e_{jj})P_{\mathcal{U}} .$$
Then $\widetilde{\varphi} : A \to \mathbb{L}(l_2(B\otimes\mathcal{K}))$ is an equivariant $*$-homomorphism modulo $\mathbb{K}(l_2(B\otimes\mathcal{K}))$. By identifying $\mathbb{L}(l_2(B \otimes \mathcal{K}))$ with $M(B \otimes \mathcal{K})$, $\mathbb{K}(l_2(B \otimes \mathcal{K}))$ with $B \otimes \mathcal{K}$ and the quotient $\mathbb{L}(l_2(B \otimes \mathcal{K}))/\mathbb{K}(l_2(B \otimes \mathcal{K}))$ with $Q(B \otimes \mathcal{K})$, we can consider $\widetilde{\varphi}$ as a map $\widetilde{\varphi} : A \to M(B \otimes \mathcal{K})$ with the property that $q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi} \in \mathrm{Hom}_G(A, Q(B \otimes \mathcal{K}))$.

Lemma 4.2. *The class of $q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}$ in $\mathrm{Ext}(A, B \otimes \mathcal{K})_h$ is independent of the choice of unit sequence, subject to the conditions 0)-8), and of the chosen discretization, subject to condition 9), and depends only on the class $[\varphi]$ of φ in $[[A, Q(B)_G \otimes \mathcal{K}]]$.*

Proof. Let $\{v_n\}$ be another unit sequence satisfying 0)-8). There is then a unit sequence $\{w_n\}$ in $B\otimes\mathcal{K}$ such that $w_nv_n = v_n, w_nu_n = u_n$ for all n. Connect u_0 to w_0 by a straight line, then u_1 to w_1 by a straight line, etc. This gives a path $\{w_n^t\}_{t\in[0,1[}$ of unit sequences. For each $t \in [0, 1[$ we get then a map $\mu_t : A \to M(B \otimes \mathcal{K})$ such that $q_{B\otimes\mathcal{K}} \circ \mu_t \in \mathrm{Hom}_G(A, Q(B \otimes \mathcal{K}))$ and $[q_{B\otimes\mathcal{K}} \circ \mu_0] = [q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}]$ in $\mathrm{Ext}(A, B \otimes \mathcal{K})$. Let $\delta : A \to M(B \otimes \mathcal{K})$ be the map obtained from φ as $\widetilde{\varphi}$ was, but by using $\{w_n\}$ instead of $\{u_n\}$. Then $\lim_{t\to1} \mu_t(a) = \delta(a)$ in the strict topology for all $a \in A$, and
$$\lim_{t\to1} \mu_t(a)\mu_t(b) - \mu_t(ab) = \delta(a)\delta(b) - \delta(ab),$$
$$\lim_{t\to1} \mu_t(a + \lambda b) - \mu_t(a) - \lambda\mu_t(b) = \delta(a + b) - \delta(a) - \lambda\delta(b),$$
$$\lim_{t\to1} \mu_t(a^*) - \mu_t(a)^* = \delta(a^*) - \delta(a)^*$$
$$\lim_{t\to1} \mu_t(g \cdot a) - g \cdot \mu_t(a) = \delta(g \cdot a) - g \cdot \delta(a)$$
in norm for all $a, b \in A, \lambda \in \mathbb{C}, g \in G$. Hence $[q_{B\otimes\mathcal{K}}\circ\delta] = [q_{B\otimes\mathcal{K}}\circ\widetilde{\varphi}]$ in $\mathrm{Ext}(A, B\otimes\mathcal{K})_h$. The same argument with the unit sequence $\{u_n\}$ replaced by $\{v_n\}$ shows that the class of $[q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}]$ in $\mathrm{Ext}(A, B \otimes \mathcal{K})_h$ is independent of the choice of unit sequence. Once this is established it is clear that a homotopy of asymptotic homomorphisms $A \to Q(B)_G\otimes\mathcal{K}$ gives rise, by an appropriate choice of unit sequence, to a homotopy which shows that $[q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}] \in \mathrm{Ext}(A, B \otimes \mathcal{K})_h$ only depends on the homotopy class of φ. That $[q_{B\otimes\mathcal{K}} \circ \tilde{\varphi}]$ is also independent of the discretization and only depends on the homotopy class of φ follows in the same way as in Lemma 5.3 and Lemma 5.4 of [MT1].

□

It follows that we have the desired map $\alpha : [[A, Q(B)_G \otimes \mathcal{K}]] \to \mathrm{Ext}(A, B \otimes \mathcal{K})_h$ which is easily seen to be a semi-group homomorphism.

Lemma 4.3. *Let $\varphi : SA \to Q(B) \otimes \mathcal{K}$ be an equivariant $*$-homomorphism which we consider as a (constant) asymptotic homomorphism. Let X be a compact subset with dense span in SA and choose a unit sequence $\mathcal{U} = \{u_n\}$ in $B \otimes \mathcal{K}$ such that*

$$\|\sqrt{u_n - u_{n-1}}\chi(\varphi(a)) - \chi(\varphi(a))\sqrt{u_n - u_{n-1}}\| < 2^{-n} \tag{4.2}$$

for all $a \in X$ and

$$\sum_{j=1}^{\infty} \|g \cdot \sqrt{u_j - u_{j-1}} - \sqrt{u_j - u_{j-1}}\|^2 < \infty \tag{4.3}$$

for all $g \in G$. Then $[q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}] = [\iota \circ \varphi]$ in $\mathrm{Ext}(SA, B \otimes \mathcal{K})$, where $\iota : Q(B)_G \otimes \mathcal{K} \to Q(B \otimes \mathcal{K})_G$ is the natural embedding.

Proof. $\widetilde{\varphi}$ has the form

$$\widetilde{\varphi}(a) = P_{\mathcal{U}}(\sum_{j=0}^{\infty} \chi(\varphi(a))e_{jj})P_{\mathcal{U}} \ .$$

Let $V \in \mathbb{L}(l_2(B \otimes \mathcal{K}))$ be the partial isometry defining $P_{\mathcal{U}}$ and note that $g \cdot V - V \in \mathbb{K}(l_2(B \otimes \mathcal{K}))$ for all $g \in G$ because of (4.3). Thus

$$\begin{pmatrix} V & 1 - VV^* \\ 1 - V^*V & -V^* \end{pmatrix}$$

is a unitary in $M_2(\mathbb{L}(l_2(B \otimes \mathcal{K})))$ which is G-invariant modulo $M_2(\mathbb{K}(l_2(B \otimes \mathcal{K})))$ and satisfies that

$$\begin{pmatrix} V & 1 - VV^* \\ 1 - V^*V & -V^* \end{pmatrix} \begin{pmatrix} \widetilde{\varphi} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^* & 1 - V^*V \\ 1 - VV^* & -V \end{pmatrix} = \begin{pmatrix} \varphi_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\varphi_0(a) = (\sqrt{u_0}\chi(\varphi(a))\sqrt{u_0} + \sum_{j=1}^{\infty} \sqrt{u_j - u_{j-1}}\chi(\varphi(a))\sqrt{u_j - u_{j-1}})e_{00}.$$

Thanks to (4.2) the have that

$$\sum_{j=1}^{\infty} \|\sqrt{u_j - u_{j-1}}\chi(\varphi(a))\sqrt{u_j - u_{j-1}} - (u_j - u_{j-1})\chi(\varphi(a))\| < \infty$$

for all $a \in X$. Since $\sum_{j=1}^{\infty}(u_j - u_{j-1})\chi(\varphi(a)) + u_0\chi(\varphi(a)) = \chi(\varphi(a))$ (with convergence in the strict topology) we find that $\varphi_0(a) = \chi(\varphi(a))e_{00}$ modulo $\mathbb{K}(l_2(B \otimes \mathcal{K}))$ for all $a \in X$, and hence in fact for all $a \in SA$. This proves the lemma.

□

Since A is separable, $[[SA, X \otimes \mathcal{K}]] = \varinjlim_D [[SA, D \otimes \mathcal{K}]]$ for any G-algebra X, when we take the limit over all separable G-subalgebras D of X. It follows from [DL] that the suspension map $S : [[SA, X \otimes \mathcal{K}]] \to [[S^2A, SX \otimes \mathcal{K}]]$ is an isomorphism.[3] Hence

[3] Dadarlat and Loring did not consider the equivariant theory in [DL], but it is easy to check that their arguments carry over unchanged.

$[[SA, - \otimes \mathcal{K}]]$ is a homotopy invariant and half-exact functor on the category of G-algebras (and not only separable G-algebras). There is therefore a map

$$\partial : [[SA, SQ(B)_G \otimes \mathcal{K}]] \to [[SA, B \otimes \mathcal{K}]]$$

arising as the boundary map coming from the extension (4.1), cf. e.g. [GHT]. Well-known arguments from the K-theory of C^*-algebras, cf. [Bl], show that $[[SA, SM(B)_G \otimes \mathcal{K}]] = [[SA, M(B)_G \otimes \mathcal{K}]] = 0$, so the six-terms exact sequence obtained by applying $[[SA, - \otimes \mathcal{K}]]$ to (4.1) shows that ∂ is an isomorphism. For any G-algebra D we let $s : D \to D \otimes \mathcal{K}$ be the stabilising $*$-homomorphism given by $s(d) = d \otimes e$ for some minimal projection $e \in \mathcal{K}$. Since B is weakly stable there is an equivariant $*$-homomorphism $\gamma_0 : B \otimes \mathcal{K} \to B$ such that $s \circ \gamma_0 : B \otimes \mathcal{K} \to B \otimes \mathcal{K}$ is equivariantly homotopic to $\mathrm{id}_{B \otimes \mathcal{K}}$. Let $\gamma : Q(B \otimes \mathcal{K})_G \to Q(B)_G$ the $*$-isomorphism induced by γ_0.

Lemma 4.4. *The composition of the maps*

$$[[S^2A, B \otimes \mathcal{K}]] \xrightarrow{\partial^{-1}} [[S^2A, SQ(B)_G \otimes \mathcal{K}]]$$

$$\xrightarrow{\quad S^{-1} \quad} [[SA, Q(B)_G \otimes \mathcal{K}]] \xrightarrow{\alpha} \mathrm{Ext}(SA, B \otimes \mathcal{K})_h \xrightarrow{CH} [[S^2A, B \otimes \mathcal{K}]]$$

is the identity.

Proof. We are going to use Theorem 2.3 of [H-LT].[4] Let $x = s_*([\mathrm{id}_{SB}]) \in [[SB, SB \otimes \mathcal{K}]]$, where $[\mathrm{id}_{SB}] \in [[SB, SB]]$ is the element represented by the identity map of SB and $s : SB \to SB \otimes \mathcal{K}$ is the stabilising $*$-homomorphism. By Theorem 2.3 of [H-LT] it suffices to identify the image of x under the Bott-periodicity isomorphism $[[SB, SB \otimes \mathcal{K}]] \simeq [[S^2B, B \otimes \mathcal{K}]]$ and show that the image of that element is not changed under the map we are trying to prove is always the identity. This is what we do. Under the isomorphism $[[SB, SB \otimes \mathcal{K}]] \simeq [[S^2B, B \otimes \mathcal{K}]]$, coming from Bott-periodicity, the image of x is represented by the asymptotic homomorphism $S^2B \to B \otimes \mathcal{K}$ arising by applying the Connes-Higson construction to the Toeplits extension tensored with B :

$$0 \longrightarrow B \otimes \mathcal{K} \longrightarrow T_0 \otimes B \longrightarrow SB \longrightarrow 0. \tag{4.4}$$

In other words, if $\varphi : SB \to Q(B \otimes \mathcal{K})$ is the Busby invariant of (4.4) the image of x in $[[S^2B, B \otimes \mathcal{K}]]$ is $[CH(\varphi)]$. For each separable G-subalgebra $D \subseteq Q(B)_G$ we let $\iota_D : D \to Q(B)_G$ denote the inclusion. Then the boundary map $\partial : [[S^2B, SQ(B)_G \otimes \mathcal{K}]] \to [[S^2B, B \otimes \mathcal{K}]]$ is given by

$$\partial(z) \ = \ \lim_D [CH(\iota_D) \otimes \mathrm{id}_{\mathcal{K}}] \bullet z,$$

where $\bullet$ denote the composition product in E-theory. Hence $\partial^{-1}[CH(\varphi)]$ is the element $z \in [[S^2B, SQ(B)_G \otimes \mathcal{K}]]$ with the property that

$$\lim_D [CH(\iota_D) \otimes \mathrm{id}_{\mathcal{K}}] \bullet z = [CH(\varphi)]$$

for all large enough D. Let $\iota : Q(B)_G \otimes \mathcal{K} \to Q(B \otimes \mathcal{K})_G$ be the natural embedding. By the naturality of the Connes-Higson construction,

$$[CH(\iota_D) \otimes \mathrm{id}_{\mathcal{K}}] \bullet S([s \circ \gamma \circ \varphi]) = [CH(\iota \circ s \circ \gamma \circ \varphi)]$$

[4] The equivariant theory was not explicitly considered in [H-LT], but all arguments carry over unchanged.

for all separable G-subalgebras $D \subseteq Q(B)_G$ which contains $\gamma \circ \varphi(SB)$. Since $s \circ \gamma_0$ is equivariantly homotopic to the identity map, we have that

$$[CH(\iota \circ s \circ \gamma \circ \varphi)] = (s \circ \gamma_0)_*[CH(\varphi)] = [CH(\varphi)],$$

so we conclude that $\partial^{-1}[CH(\varphi)] = S([s \circ \gamma \circ \varphi])$. Hence $\alpha \circ S^{-1} \circ \partial^{-1}[CH(\varphi)] = [\iota \circ s \circ \gamma \circ \varphi]$ by Lemma 4.3. Thus the image of $[CH(\varphi)]$ in $[[S^2B, B \otimes \mathcal{K}]]$ under the composite map is $CH[\iota \circ s \circ \gamma \circ \varphi] = (s \circ \gamma_0)_*[CH(\varphi)] = [CH(\varphi)]$. The proof is complete. □

Lemma 4.5. *Let $\lambda \in \operatorname{Ext}(SA, B \otimes \mathcal{K})$. Then $\varphi = s \circ \gamma \circ \lambda$ is an equivariant $*$-homomorphism $\varphi : SA \to Q(B)_G \otimes \mathcal{K}$ such that $\alpha[\varphi] = s_* \circ \gamma_*[\lambda]$ in $\operatorname{Ext}(SA, B \otimes \mathcal{K})_h$ and such that $[\varphi] = 0$ in $[[SA, Q(B)_G \otimes \mathcal{K}]]$ implies that $[\lambda] = 0$ in $\operatorname{Ext}(SA, B \otimes \mathcal{K})$.*

Proof. If $[\varphi] = 0$ in $[[SA, Q(B)_G \otimes \mathcal{K}]]$, there is a path μ^s, $s \in [0,1]$, of asymptotic homomorphisms $SA \to Q(B)_G \otimes \mathcal{K}$ such that $\mu^0 = \varphi$ and $\mu^1 = 0$ and a unit sequence $\mathcal{U} = \{u_n\}$ in $B \otimes \mathcal{K}$ such that

$$q_{B\otimes\mathcal{K}} \circ \widetilde{\mu^s}, \;\; s \in [0,1], \tag{4.5}$$

connects $q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}$ to 0. By Theorem 3.4 we may assume that μ is an equi-homotopy and it is then easy to see that (4.5) is a strong homotopy. By Lemma 2.4 we conclude from this that $[q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}] = 0$ in $\operatorname{Ext}(SA, B \otimes \mathcal{K})$. But $[q_{B\otimes\mathcal{K}} \circ \widetilde{\varphi}] = [\varphi]$ in $\operatorname{Ext}(SA, B \otimes \mathcal{K})$ by Lemma 4.3. Hence $\alpha[\varphi] = s_* \circ \gamma_*[\lambda]$ in $\operatorname{Ext}(SA, B \otimes \mathcal{K})_h$ and $[\varphi] = 0 \Rightarrow s_* \circ \gamma_*[\lambda] = 0$ in $\operatorname{Ext}(SA, B \otimes \mathcal{K})$. To complete the proof it suffices to show that $s_* \circ \gamma_* : \operatorname{Ext}(SA, B \otimes \mathcal{K}) \to \operatorname{Ext}(SA, B \otimes \mathcal{K})$ is injective. However, γ is an equivariant $*$-homomorphism and therefore γ_* is an isomorphism. The injectivity of $s_* : \operatorname{Ext}(SA, B) \to \operatorname{Ext}(SA, B \otimes \mathcal{K})$ follows from the weak stability of B : There is a G-invariant isometry $V \in M(B \otimes \mathcal{K})$ such that $x \mapsto V^*s(x)V$ is an equivariant $*$-automorphism $B \otimes \mathcal{K} \to B \otimes \mathcal{K}$ and $s(x) = \operatorname{Ad} V(V^*s(x)V)$. Since $\operatorname{Ad} V$ induces the identity map on $\operatorname{Ext}(SA, B \otimes \mathcal{K})$ we see that $s_* : \operatorname{Ext}(SA, B) \to \operatorname{Ext}(SA, B \otimes \mathcal{K})$ is an isomorphism.

□

Lemma 4.6. *The map $CH : \operatorname{Ext}(SA, B) \to [[S^2A, B]]$ is injective.*

Proof. Consider an extension $\lambda \in \operatorname{Ext}(SA, B \otimes \mathcal{K})$ and assume that $[CH(\lambda)] = 0$ in $[[S^2A, B \otimes \mathcal{K}]]$. With the notation from Lemma 4.5 we find that

$$CH \circ \alpha[\varphi] = CH[s \circ \gamma \circ \lambda] = s_* \circ \gamma_*[CH(\lambda)] = 0.$$

But then Lemma 4.4 implies that $[\varphi] = 0$ in $[[SA, Q(B)_G \otimes \mathcal{K}]]$. By Lemma 4.5 this yields the conclusion that $[\lambda] = 0$ in $\operatorname{Ext}(SA, B \otimes \mathcal{K})$. Thus $CH : \operatorname{Ext}(SA, B \otimes \mathcal{K}) \to [[S^2A, B \otimes \mathcal{K}]]$ is injective. But B is weakly stable so the result follows. □

The surjectivity of $CH : \operatorname{Ext}(SA, B) \to [[S^2A, B]]$ follows from Lemma 4.4. Furthermore, it follows from Lemma 4.6 that α is well-defined as a map $\alpha : [[SA, Q(B)_G \otimes \mathcal{K}]] \to \operatorname{Ext}(SA, B \otimes \mathcal{K})$ and then Lemma 4.4 tells us that

$$CH^{-1} = \alpha \circ S^{-1} \circ \partial^{-1}.$$

Another description of CH^{-1} can be obtained from [MT2]. The crucial construction for this is the map E which was considered in [MT1] and [MT2], inspired by [MM] and [MN]. However, in [MT1] and [MT2] we only defined E as a map into homotopy classes of extensions, so to see that the E-construction can also invert the CH-map of Lemma 4.6 we must show that it is well-defined as a map from homotopy classes of asymptotic homomorphisms to stable unitary equivalence classes of extensions. Let us therefore review the construction.

Given an equicontinuous asymptotic homomorphism $\varphi = \{\varphi_t\}_{t\in[1,\infty)} : A \to B$ we choose a discretization $\{\varphi_{t_i}\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty}\sup_{t\in[t_i,t_{i+1}]}\|\varphi_t(a) - \varphi_{t_i}(a)\| = 0$ for all $a \in A$. Since G is σ-compact (and φ equicontinuous) we can also arrange that

$$\lim_{i\to\infty}\sup_{t\in[t_i,t_{i+1}]}\sup_{g\in K}\|g\cdot\varphi_t(a) - \varphi_t(g\cdot a)\| = 0$$

for all $a \in A$ and all compact subsets $K \subseteq G$. To define from such a discretization a map $\mathbf{\Phi} : A \to \mathbb{L}(l_2(\mathbb{Z})\otimes B)$ we introduce the standard matrix units $e_{ij}, i, j \in \mathbb{Z}$, which act on the Hilbert B-module $l_2(\mathbb{Z})\otimes B$ in the obvious way. Then

$$\mathbf{\Phi}(a) = \sum_{i\geq 1}\varphi_{t_i}(a)e_{ii}$$

defines a map $\mathbf{\Phi} : A \to \mathbb{L}(l_2(\mathbb{Z})\otimes B)$. As in the proof of Lemma 2.1 we can define a representation of G on $l_2(\mathbb{Z})\otimes B$ and in this way obtain a representation of G as automorphisms of $\mathbb{L}(l_2(\mathbb{Z})\otimes B)$. Since B is weakly stable we can identify B with $\mathbb{K}(l_2(\mathbb{Z})\otimes B))$, the B-compact operators in $\mathbb{L}(l_2(\mathbb{Z})\otimes B)$. Observe that $\mathbf{\Phi}$ is then an equivariant $*$-homomorphism modulo B. Furthermore, $\mathbf{\Phi}(a)$ commutes modulo B with the two-sided shift $T = \sum_{j\in\mathbb{Z}} e_{j,j+1}$ which is G-invariant. So we get in this way a G-extension

$$E(\varphi) : A \to Q(B) = \mathbb{L}(l_2(\mathbb{Z})\otimes B)/\mathbb{K}(l_2(\mathbb{Z})\otimes B)$$

such that

$$E(\varphi)(f\otimes a) = f(\underline{T})\underline{\mathbf{\Phi}(a)}$$

for all $f \in C(\mathbb{T}), a \in A$. Here and in the following we denote by $\underline{S}$ the image in $Q(B) = \mathbb{L}(l_2(\mathbb{Z})\otimes B)/\mathbb{K}(l_2(\mathbb{Z})\otimes B)$ of an element $S \in \mathbb{L}(l_2(\mathbb{Z})\otimes B)$.

Lemma 4.7. *$E(\varphi)$ is a semi-invertible G-extension, and up to stable unitary equivalence it does not depend on the chosen discretization of φ.*

Proof. Consider another discretization $(\varphi_{s_i})_{i\in\mathbb{N}}$ of φ and define $\mathbf{\Psi} : A \to \mathbb{L}(l_2(\mathbb{Z})\otimes B)$ by

$$\mathbf{\Psi}(a) = \sum_{i\leq 0}\varphi_{s_{-i+1}}(a)e_{ii}.$$

There is then a G-extension $-E(\varphi) : C(\mathbb{T})\otimes A \to \mathbb{L}(l_2(\mathbb{Z})\otimes B)/\mathbb{K}(l_2(\mathbb{Z})\otimes B)$ such that $-E(\varphi)(f\otimes a) = f(\underline{T})\underline{\mathbf{\Psi}(a)}$. It suffices to show that $-E(\varphi)\oplus E(\varphi)$ is unitarily equivalent to an asymptotically split G-extension. Define $\Lambda : A \to \mathbb{L}(l_2(\mathbb{Z})\otimes B)$ such that

$$\Lambda(a) = \sum_{i\geq 1}\varphi_{t_i}(a)e_{ii} + \sum_{i\leq 0}\varphi_{s_{-i+1}}(a)e_{ii}.$$

There is then a G-extension $\pi_0 : C(\mathbb{T})\otimes A \to \mathbb{L}(l_2(\mathbb{Z})\otimes B)/\mathbb{K}(l_2(\mathbb{Z})\otimes B)$ such that $\pi_0(f\otimes a) = f(\underline{T})\underline{\Lambda(a)}$. $-E(\varphi)\oplus E(\varphi)$ is clearly unitarily equivalent (via a G-invariant unitary) to $\pi_0\oplus 0$, so it suffices to show that π_0 is asymptotically split. For each n we define $\Lambda_n : A \to \mathbb{L}(l_2(\mathbb{Z})\otimes B)$ by

$$\Lambda_n(a) = \sum_{i>n}\varphi_{t_i}(a)e_{ii} + \sum_{1\leq i\leq n}\varphi_{t_n}(a)e_{ii} + \sum_{\{i\leq 0:\ s_{-i+1}\leq t_n\}}\varphi_{t_n}(a)e_{ii} + \sum_{\{i\leq 0:\ s_{-i+1}>t_n\}}\varphi_{s_i}(a)e_{ii}.$$

Then $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a discrete asymptotic homomorphism such that $\lim_{n\to\infty}\|\Lambda_n(a)-\Lambda_{n+1}(a)\|=0$, $\lim_{n\to\infty}\|g\cdot\Lambda_n(a)-\Lambda_n(g\cdot a)\|=0$, $g\in G$, $\lim_{n\to\infty}\|T\Lambda_n(a)-\Lambda_n(a)T\|=0$ and $\Lambda_n(a)=\Lambda(a)$ modulo $\mathbb{K}(l_2(\mathbb{Z})\otimes B)$. By convex interpolation and an obvious application of the C^*-algebra

$$\{f\in C_b([1,\infty),M(B))\ :\ q_B(f(t))=q_B(f(1)),t\in[1,\infty)\}/C_0([1,\infty),B)$$

we get an asymptotic homomorphism $(\pi_t)_{t\in[1,\infty)}:C(\mathbb{T})\otimes A\to M(B)=\mathbb{L}(l_2(\mathbb{Z})\otimes B)$ such that $\pi_0=q_B\circ\pi_t$ for all t.

□

Theorem 3.4 and Lemma 4.7 in combination show that there is group homomorphism $E:[[SA,B]]\to\mathrm{Ext}(C(\mathbb{T})\otimes SA,B)$ such that $E[\varphi]=[E(\varphi)]$ for any equicontinuous asymptotic homomorphism $\varphi:SA\to B$. By pulling extensions back along the inclusion $S^2A\subseteq C(\mathbb{T})\otimes SA$ we can also consider E as a map $E:[[SA,B]]\to\mathrm{Ext}(S^2A,B)$. Let $\chi:SA\to S^3M_2(A)$ be a $*$-homomorphism which is invertible in KK-theory. By weak stability of B there is also an isomorphism $\beta:[[S^2A,B]]\to[[S^2M_2(A),B]]$. Let $\xi:S^2\to\mathcal{K}$ be the asymptotic homomorphism which arises from the Connes-Higson construction applied to the Toeplits extension. By changing χ 'by a sign' we may assume that the composite map

$$[[S^2A,B]]\xrightarrow{\beta}[[S^2M_2(A),B]]\xrightarrow{[\varphi]\mapsto[\xi\otimes\varphi]}[[S^4M_2(A),B]]\xrightarrow{(S\chi)^*}[[S^2A,B]]$$

is the identity. Consider the diagram

$$\begin{tikzcd}
\mathrm{Ext}(SA,B) \arrow[d, "CH"'] & \mathrm{Ext}(S^3M_2(A),B) \arrow[l, "\chi^*"'] \arrow[d, "CH"] \\
{[[S^2A,B]]} \arrow[ur, "E\circ\beta"] & {[[S^4M_2(A),B]].} \arrow[l, "(S\chi)^*"]
\end{tikzcd} \tag{4.6}$$

The square commutes by the naturality of the Connes-Higson construction, and it follows from Lemma 2.3 of [MT2] (or Lemma 5.5 of [MT1]) that $(S\chi)^*\circ CH\circ E\circ\beta=\mathrm{id}$. We conclude therefore that $CH\circ\chi^*\circ E\circ\beta=\mathrm{id}$. We have now obtained our main results :

Theorem 4.8. *Let A and B be separable G-algebras, B weakly stable. $CH:\mathrm{Ext}(SA,B)\to[[S^2A,B]]$ is an isomorphism with inverse $\chi^*\circ E\circ\beta$.*

Theorem 4.9. *Let A and B be separable G-algebras, B weakly stable, and let $\varphi,\psi:SA\to Q(B)$ be two G-extensions. The following conditions are equivalent :*

1) $[\varphi]=[\psi]$ *in* $\mathrm{Ext}(SA,B)$ *(i.e. φ and ψ are stably unitarily equivalent).*
2) *There is an asymptotically split G-extension $\lambda\in\mathrm{Hom}_G(SA,Q(B))$ such that $\varphi\oplus\lambda$ is strongly homotopic to $\psi\oplus\lambda$.*
3) *φ and ψ are homotopic.*

Proof. 1) $\Rightarrow$ 2) follows from Lemma 6.1 of [Th1]. 2) $\Rightarrow$ 3) follows from the easily established fact that an asymptotically split G-extension is homotopic to zero. 3) $\Rightarrow$ 1) follows from Theorem 4.8. □

It is not so clear how much of these results survive when the quotient C^*-algebra is not a suspension. It may be that $CH : \mathrm{Ext}^{-1/2}(A,B) \to [[SA,B]]$ is an isomorphism, but if one wants to be able to handle all extensions it is necessary to work with a suspended C^*-algebra as the quotient algebra. Indeed, the construction of Anderson, [A], of the first separable C^*-algebra for which the approach of [BDF] does not give a group can easily be seen to give an extension of a separable C^*-algebra A by $\mathcal{K}$ which is not semi-invertible.

Remark 4.10. Our results combine nicely with the work of Kirchberg from [Ki] to shed new light on the relation between E-theory and KK-theory. Recall that Connes and Higson constructed a natural transformation $KK(A,B) \to E(A,B)$ which is an isomorphism when A is nuclear. In [S] Skandalis demonstrated that the map can fail to be injective by exhibiting a separable C^*-algebra A for which $E(A,A) = 0$ while $KK(A,A) \neq 0$.

It follows from Theorem 4.8 that the natural transformation $KK(A,B) \to E(A,B)$ of Connes and Higson can be identified with the obvious map $\mathrm{Ext}^{-1}(SA, B\otimes\mathcal{K}) \to \mathrm{Ext}(SA, B\otimes\mathcal{K})$. Hence the examples of Kirchberg, [Ki], show that $KK(A,\mathbb{C}) \to E(A,\mathbb{C})$ can fail to be surjective. Specifically, let G be a countable discrete and non-amenable subgroup of a connected Lie-group. Kirchberg constructed in [Ki] an extension of $SC^*_r(G)$ by $\mathcal{K}$ which is not semi-split. This means that $\mathrm{Ext}^{-1}(SC^*_r(G),\mathcal{K}) \to \mathrm{Ext}(SC^*_r(G),K)$ is not surjective and hence $KK(C^*_r(G),\mathbb{C}) \to E(C^*_r(G),\mathbb{C})$ is not surjective. It follows therefore that the functor $KK(C^*_r(G),-)$ is not half-exact, thus answering a question raised by Skandalis in [S].

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