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THE LIFTING PROPERTY FOR CROSSED PRODUCTS BY A DISCRETE GROUP

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Abstract. We prove that the lifting property of Kirchberg is preserved by the crossed product construction for actions of countable discrete groups. It follows that the full group C^*-algebra of a countable discrete group has the lifting property. In combination with Kirchberg's work this provides counterexamples to a series of conjectures formulated and shown by him to be equivalent. In particular, it follows that there exists a separable II_1-factor which is not a subfactor of an ultrapower of the hyperfinite II_1-factor.

1. Introduction

In a recent paper the author clarified the excision properties of equivariant KK-theory by proving a general result on G-extensions, which roughly says that a completely positive, but not necessarily equivariant, section for an equivariant surjective $*$-homomorphism between C^*-algebras, equipped with continuous actions of a locally compact group, can be exchanged with one which *is* equivariant, after the whole picture has been tensored with the regular representation. See [T] or the proof of Lemma 2.6 below for the precise statement. In this paper we will use this result to show that the lifting property (LP) of Kirchberg, [Ki1], is preserved under full crossed products by discrete groups. More precisely, our main result says that the full crossed product $A \times_\alpha G$ has the lifting property when G is a countable discrete group and A is a separable C^*-algebra with the lifting property. This result has several interesting consequences, but the most prominent is probably that it combines easily with Kirchberg's work in [Ki1] and [Ki2] to show that Connes' 'approximate embedding problem' from his work on injective factors, see p. 105 of [Co], has a negative answer: When H is a countable discrete group which does not have property (F) of Kirchberg, [Ki3], - and such groups exist by [Ki2] - then the central decomposition of the corresponding group von Neumann algebra must contain a II_1-factor which does not embed into an ultrapower of the hyperfinite II_1-factor.

2. Proofs

Definition 2.1. A C^*-algebra A has *the lifting property* when the following holds: For every C^*-algebra B with an ideal $J \subseteq B$, and every completely positive contraction $A \to B/J$, there is a completely positive contractive lift $A \to B$.

In a slightly different guise this notion was introduced by Kirchberg in [Ki1]. Since our definition is not explicitly the same as that of Kirchberg, we begin by proving that they agree, at least for separable C^*-algebras. In the following we let $\mathbb{K}$ denote the compact operators on a separable infinite dimensional Hilbert space.

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Lemma 2.2. *For separable C^*-algebras the lifting property as defined in Definition 2.1 is the same as the one considered by Kirchberg in* [Ki1].

Proof. It is fundamental to the argument that the unitization of a completely positive contraction is completely positive, cf. Lemma 3.9 of [CE]. First the case when A is non-unital: Assume that A satisfies Kirchbergs condition, cf. p. 453 in [Ki1]. Let $\varphi : A \to B/J$ be a completely positive contraction. By Kirchbergs condition the unitization $\varphi^+ : A^+ \to B^+/J$ admits a unital completely positive lift $s : A^+ \to B^+$. Then $s|_A$ is a completely positive contractive lift of φ. Conversely, assume that A satisfies the condition of Definition 2.1 and let $\varphi : A^+ \to B/J$ be a completely positive unital map, where B is also unital. Then $\varphi|_A$ admits a completely positive contractive lift $A \to B$. The unitization of this lift is also completely positive and hence a unital completely positive lift of φ.

A unital: Assume that A satisfies Kirchbergs condition, cf. p. 451 in [Ki1], and consider a completely positive contraction $\varphi : A \to B/J$. If we can lift the composition $A \to B/J \to B^+/J$ to a completely positive contraction $A \to B^+$, this lift will automatically take values in B, so we assume that B is unital. Consider the map $A \to (B/J) \otimes \mathbb{K}$ given by $a \mapsto \varphi(a) \otimes e_{11}$. By Kasparov's Stinespring theorem, cf. Theorem 3 of [Ka], there is a unital $*$-homomorphism $\pi : A \to M((B/J) \otimes \mathbb{K})$ and an element $W \in M((B/J) \otimes \mathbb{K}), \|W\| \leq 1$, such that $\varphi(\cdot) \otimes e_{11} = W^*\pi(\cdot)W$. By [OP] the map $M(B \otimes \mathbb{K}) \to M((B/J) \otimes \mathbb{K})$ induced by $B \otimes \mathbb{K} \to (B/J) \otimes \mathbb{K}$ is surjective so there is a completely positive unital lift $s : A \to M(B \otimes \mathbb{K})$ of π because A has the lifting property of Kirchberg. Let $X \in M(B \otimes \mathbb{K})$ be a lift of W, $\|X\| \leq 1$, and note that $(1_B \otimes e_{11})X^*s(\cdot)X(1_B \otimes e_{11}) : A \to B \otimes e_{11}$ gives us a completely positive contractive lift of φ. Assume next that A satisfies the condition in Definition 2.1 and consider a unital C^*-algebra B with ideal $J \subseteq B$ and a completely positive unital map $\varphi : A \to B/J$. By assumption there is a completely positive contractive lift $s : A \to B$ of φ. Let ω be a state of A and set $\tilde{s}(a) = s(a) + \omega(a)(1 - s(1))$. Then $\tilde{s}$ is a completely positive unital lift of φ.

□

The last proof suggests the following weakening of the condition for a separable C^*-algebra to have the lifting property.

Lemma 2.3. *A separable C^*-algebra A has the lifting property when the following holds: For every C^*-algebra B with an ideal $J \subseteq B$, and every $*$-homomorphism $A \to B/J$, there is a completely positive contractive lift $A \to B$.*

Proof. Assume that A meets the condition in the lemma and consider a completely positive contraction $s : A \to B/J$. Compose the unitization $s^+ : A^+ \to B^+/J$ with the canonical $*$-homomorphism $B^+/J \ni x \mapsto x \otimes e_{11} \in (B^+/J) \otimes \mathbb{K}$ to get the completely positive contraction $\tilde{s} : A^+ \to (B^+/J) \otimes \mathbb{K}$. From Kasparov's Stinespring theorem we get a $*$-homomorphism $\pi : A^+ \to M((B^+/J) \otimes \mathbb{K})$ and an element $W \in M((B^+/J) \otimes \mathbb{K})$, $\|W\| \leq 1$, such that $\tilde{s}(\cdot) = W^*\pi(\cdot)W$. It follows from [OP] that the $*$-homomorphism $M(B^+ \otimes \mathbb{K}) \to M((B^+/J) \otimes \mathbb{K})$ induced by the obvious map $B^+ \otimes \mathbb{K} \to (B^+/J) \otimes \mathbb{K}$ is surjective. Since A satisfies the condition of the lemma, there is a completely positive contraction $\varphi : A \to M(B^+ \otimes \mathbb{K})$ which lifts $\pi|_A$. Let $X \in M(B^+ \otimes \mathbb{K})$ be a lift of W, $\|X\| \leq 1$. By identifying B^+ with $B^+ \otimes e_{11} \subseteq B^+ \otimes \mathbb{K}$ we see that $(1_{B^+} \otimes e_{11})X^*\varphi(\cdot)X(1_{B^+} \otimes e_{11}) : A \to B^+$ is a completely positive contractive lift of s. Since $s(A) \subseteq B/J$, it follows that this lift takes values in B. □

Proposition 2.4. *Let A be a separable C^*-algebra. The following conditions are equivalent:*

1) *A has the lifting property.*
2) $\mathrm{Ext}(A, B \otimes \mathbb{K})$ *is a group for every σ-unital C^*-algebra B.*
3) *Every extension*

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \longrightarrow 0$$

is semi-split, i.e. there is a completely positive contraction $s : A \to E$ such that $p \circ s = \mathrm{id}_A$.

Proof. 1) $\Rightarrow$ 3) is trivial. 3) $\Rightarrow$ 2) follows from [A], [Ka]. 2) $\Rightarrow$ 1) : Consider a $*$-homomorphism $\pi : A \to B/J$. To show that π admits a completely positive contractive lift we may assume that B is separable since A is. The pull-back along $\pi(\cdot) \otimes e_{11}$ of the extension $0 \to J \otimes \mathbb{K} \to B \otimes \mathbb{K} \to (B/J) \otimes \mathbb{K} \to 0$ is an extension of A by $J \otimes \mathbb{K}$ which is semi-split by assumption. It follows that there is a completely positive contractive map $s : A \to B \otimes \mathbb{K}$ which lifts $\pi(\cdot) \otimes e_{11}$. Then $(1_{M(B)} \otimes e_{11})s(\cdot)(1_{M(B)} \otimes e_{11})$ gives us a completely positive contractive lift of π. By Lemma 2.3, A has the lifting property. □

We need one more lemma on the lifting property.

Lemma 2.5. *Let A be a separable C^*-algebra with the lifting property. It follows that $A \otimes \mathbb{K}$ has the lifting property.*

Proof. Consider an extension

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \otimes \mathbb{K} \longrightarrow 0$$

By Propostion 2.4 we must produce a completely positive contractive section for p. Since A is separable we may assume that E is also. Let $P_n : A \otimes \mathbb{K} \to A \otimes M_n(\mathbb{C}) \subseteq A \otimes \mathbb{K}$ be a sequence of conditional expectations such that $\lim_{n\to\infty} P_n(x) = x$ for all $x \in A \otimes \mathbb{K}$. It follows from [A] that the set of liftable completely positive contractions is closed, so it suffices to lift each P_n to a completely positive contraction, or, alternatively, to find a completely positive contractive section for p over each $A \otimes M_n(\mathbb{C})$. Since A has the lifting property there is a completely positive contractive lift $s : A \to E$ of the identity over $A = A \otimes e_{11} \subseteq A \otimes \mathbb{K}$. By [OP] p extends to a surjective $*$-homomorphism $M(E) \to M(A \otimes \mathbb{K})$. Choose pre-images $d_i \in M(E)$ of $1_{M(A)} \otimes e_{1i} \in M(A \otimes \mathbb{K}), i = 1, 2, \cdots, n$, under this extension and set $S(\sum_{i,j} x_{ij} \otimes e_{ij}) = \sum_{i,j} d_i^* s(x_{ij}) d_j$. Then S is a completely positive section for p over $A \otimes M_n(\mathbb{C})$. By Remark 2.5 of [CS], we can find another which is contractive. □

Let G be a second countable locally compact group. We shall only need the following consequence of the main result in [T] when G is discrete, but the proof in the general case is the same.

Lemma 2.6. *Let*

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0 \tag{2.1}$$

be a G-extension, i.e. B, E and A are separable C^*-algebras equipped with continuous actions of G by automorphisms, and all maps are equivariant.

Assume that (2.1) is semi-split, i.e. that there is a completely positive section for the quotient map. It follows that the same is true for the extension

$$0 \longrightarrow B \rtimes G \longrightarrow E \rtimes G \longrightarrow A \rtimes G \longrightarrow 0 \tag{2.2}$$

of full crossed products.

Proof. Let λ be the left regular representation of G on $L^2(G)$. Consider the G-extension

$$0 \longrightarrow B \otimes \mathcal{K}(L^2(G)) \longrightarrow E \otimes \mathcal{K}(L^2(G)) \longrightarrow A \otimes \mathcal{K}(L^2(G)) \longrightarrow 0, \tag{2.3}$$

where all actions have been tensored with $\operatorname{Ad}\lambda$. The content of Theorem 1.1 in [T] is that the extension (2.3) admits a completely positive contractive section for the quotient map which is also equivariant. Thanks to the equivariant Stinespring construction this equivariant completely positive contraction induces a completely positive contraction $(A \otimes \mathcal{K}(L^2(G))) \rtimes G \to (E \otimes \mathcal{K}(L^2(G))) \rtimes G$ which is a section for the quotient map in the extension

$$0 \longrightarrow (B \otimes \mathcal{K}(L^2(G))) \rtimes G \longrightarrow (E \otimes \mathcal{K}(L^2(G))) \rtimes G \longrightarrow (A \otimes \mathcal{K}(L^2(G))) \rtimes G \longrightarrow 0. \tag{2.4}$$

For any action α, the tensor product action $\alpha \otimes \operatorname{Ad}\lambda$ is obviously exterior equivalent to $\alpha \otimes \mathrm{id}$ and the full crossed products corresponding to the two actions are thus isomorphic in the natural way. There is therefore a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (B \otimes \mathcal{K}(L^2(G))) \rtimes G & \longrightarrow & (E \otimes \mathcal{K}(L^2(G))) \rtimes G & \longrightarrow & (A \otimes \mathcal{K}(L^2(G))) \rtimes G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (B \rtimes G) \otimes \mathcal{K}(L^2(G)) & \longrightarrow & (E \rtimes G) \otimes \mathcal{K}(L^2(G)) & \longrightarrow & (A \rtimes G) \otimes \mathcal{K}(L^2(G)) & \longrightarrow & 0,
\end{array}$$

where the vertical arrows are isomorphisms. It follows first that the lower extension is semi-split since (2.4) is, and then that (2.2) is semi-split. □

Lemma 2.6 is the analogue for full crossed products of Corollary 1.2 of [T]. What we really need is a version of Lemma 2.6 for twisted actions of discrete groups.

Assume now that G is discrete and countable. We shall in the following consider twisted actions of G in the sense of Packer and Raeburn, [PR]. Thus *a twisted action* of G on A is a pair (α, w) of maps $\alpha : G \to \operatorname{Aut} A$ and $w : G \times G \to UM(A)$ (= the unitary group of the multiplier algebra $M(A)$ of A), such that

1) $\alpha_e = \mathrm{id}, w(s,e) = w(e,s) = 1$ for $s \in G$,
2) $\alpha_s \circ \alpha_t = \operatorname{Ad} w(s,t) \circ \alpha_{st}$ for $s,t \in G$,
3) $\alpha_r(w(s,t))w(r,st) = w(r,s)w(rs,t)$ for $s,t \in G$.

Given such a twisted action (α, w) one defines the crossed product C^*-algebra $A \times_{\alpha,w} G$ as follows, cf. [PR]. Represent a function $f : G \to A$ of finite support as a formal sum $\sum_{t \in G} f(t)u_t$. We can then turn the finitely supported functions $G \to A$ into a $*$-algebra such that

$$\begin{aligned}
(\sum_{t\in G} f(t)u_t)(\sum_{t \in G} h(t)u_t) &= \sum_{s,t\in G} f(s)\alpha_s(g(t))w(s,t)u_{st} \\
&= \sum_{x \in G} \big[\sum_{s\in G} f(s)\alpha_s(g(s^{-1}x))w(s,s^{-1}x)\big]u_x
\end{aligned}$$

and

$$(\sum_{t\in G} f(t)u_t)^* = \sum_{t\in G} w(t,t^{-1})^*\alpha_t(f(t^{-1})^*)u_t.$$

The crossed product C^*-algebra $A \times_{\alpha,w} G$ is then by definition the enveloping C^*-algebra of this $*$-algebra, cf. [PR]. Note that when $w = 1$, α is a homomorphism and $A \times_{\alpha,w} G$ is just the ordinary (full) crossed product $A \times_\alpha G$.

An extension

$$0 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 0 \tag{2.5}$$

of C^*-algebras will be called a *twisted G-extension* when there are twisted actions (β^B, w^B), (β^E, w^E) and (β^A, w^A) on B, E and A, respectively, such that

1) $\beta^E_t|_B = \beta^B_t,\ t \in G,$
2) $p \circ \beta^E_t = \beta^A_t \circ p,\ t \in G,$
3) $w^E(s,t)b = w^B(s,t)b,\ b \in B, s,t \in G,$
4) $w^A(s,t)p(e) = p(w^E(s,t)e),\ e \in E, s,t \in G.$

In this case there are $*$-homomorphisms $\hat{i} : B \times_{\beta^B,w^B} G \to E \times_{\beta^E,w^E} \times G$ and $\hat{p} : E \times_{\beta^E,w^E} \times G \to A \times_{\beta^A,w^A} G$ given by

$$\hat{i}(\sum_{t\in G} f(t)u_t) = \sum_{t\in G} i(f(t))u_t$$

and

$$\hat{p}(\sum_{t\in G} f(t)u_t) = \sum_{t\in G} p(f(t))u_t,$$

in the notation from above.

Lemma 2.7. *Assume that (2.5) is a twisted G-extension. It follows that*

$$0 \longrightarrow B \times_{\beta^B,w^B} G \xrightarrow{\hat{i}} E \times_{\beta^E,w^E} \times G \xrightarrow{\hat{p}} A \times_{\beta^A,w^A} G \longrightarrow 0 \tag{2.6}$$

is an extension, which is semi-split when (2.5) is.

Proof. By Theorem 3.4 of [PR] there is a genuine, untwisted action α^E of G on $E \otimes \mathcal{K}(L^2(G))$ and a map $v^E : G \to UM(E \otimes \mathcal{K}(L^2(G)))$ such that $\alpha^E_s = \operatorname{Ad} v^E_s \circ (\beta^E_s \otimes \mathrm{id})$ and $v^E_s(\beta^E_s \otimes \mathrm{id})(v^E_t)(w^E(s,t) \otimes 1)v^{E*}_{st} = 1_{M(E\otimes\mathcal{K}(L^2(G)))}$ for all $s,t \in G$. It follows that we can define actions α^B and α^A on $B \otimes \mathcal{K}(L^2(G))$ and $A \otimes \mathcal{K}(L^2(G))$, respectively, such that $\alpha^B_s = \alpha^E_s|_B$ and $\alpha^A_s \circ (p \otimes \mathrm{id}) = (p \otimes \mathrm{id}) \circ \alpha^E_s$ for all $s \in G$. Similarly, there are maps $v^B : G \to UM(B \otimes \mathcal{K}(L^2(G)))$ and $v^A : G \to UM(A \otimes \mathcal{K}(L^2(G)))$ such that $v^B_s b = v^E_s b$ for all $b \in B \otimes \mathcal{K}(L^2(G))$ and $(p \otimes \mathrm{id})(v^E_s e) = v^A_s(p \otimes \mathrm{id})(e)$ for all $e \in E \otimes \mathcal{K}(L^2(G)), s \in G$. There is then a $*$-isomorphism $\varphi_E : (E \otimes \mathcal{K}(L^2(G))) \times_{\beta^E\otimes\mathrm{id},w^E\otimes 1} G \to (E \otimes \mathcal{K}(L^2(G))) \times_{\alpha^E,1} G$ such that

$$\varphi_E(\sum_{t\in G} f(t)u_t) = \sum_{t\in G} f(t)v^{E*}_t u_t. \tag{2.7}$$

cf. Lemma 3.3 of [PR]. Since we also have the identities $v^B_s(\beta^B_s \otimes \mathrm{id})(v^B_t)(w^B(s,t) \otimes 1)v^{B*}_{st} = 1_{M(B\otimes\mathcal{K}(L^2(G)))}$ and $v^A_s(\beta^A_s \otimes \mathrm{id})(v^A_t)(w^A(s,t) \otimes 1)v^{A*}_{st} = 1_{M(A\otimes\mathcal{K}(L^2(G)))}$ for all $s,t \in G$, there are also $*$-isomorphisms $\varphi_B : (B \otimes \mathcal{K}(L^2(G))) \times_{\beta^B\otimes\mathrm{id},w^E\otimes 1} G \to (B \otimes$

$\mathcal{K}(L^2(G))) \times_{\alpha^B,1} G$ and $\varphi_A : (A \otimes \mathcal{K}(L^2(G))) \times_{\beta^A \otimes \mathrm{id}, w^A \otimes 1} G \to (A \otimes \mathcal{K}(L^2(G))) \times_{\alpha^A,1} G$ given by the formulas analogous to (2.7). Then

$$\begin{array}{ccc} (B \otimes \mathcal{K}(L^2(G))) \times_{\beta^B \otimes \mathrm{id}, w^E \otimes 1} G & \xrightarrow{\varphi_B} & (B \otimes \mathcal{K}(L^2(G))) \times_{\alpha^B} G \\ \downarrow & & \downarrow \\ (E \otimes \mathcal{K}(L^2(G))) \times_{\beta^E \otimes \mathrm{id}, w^E \otimes 1} G & \xrightarrow{\varphi_E} & (E \otimes \mathcal{K}(L^2(G))) \times_{\alpha^E} G \\ \downarrow & & \downarrow \\ (A \otimes \mathcal{K}(L^2(G))) \times_{\beta^A \otimes \mathrm{id}, w^A \otimes 1} G & \xrightarrow{\varphi_A} & (A \otimes \mathcal{K}(L^2(G))) \times_{\alpha^A} G \end{array}$$

commutes. Since there is a natural isomorphism $(E \otimes \mathcal{K}(L^2(G))) \times_{\beta^E \otimes \mathrm{id}, w^E \otimes 1} G \simeq (E \times_{\beta^E, w^E} G) \otimes \mathcal{K}(L^2(G))$, cf. Corollary 3.7 of [PR], we conclude that there is commuting diagram

$$\begin{array}{ccc} (B \times_{\beta^B, w^B} G) \otimes \mathcal{K}(L^2(G)) & \longrightarrow & (B \otimes \mathcal{K}(L^2(G))) \times_{\alpha^B} G \\ \downarrow & & \downarrow \\ (E \times_{\beta^E, w^E} G) \otimes \mathcal{K}(L^2(G)) & \longrightarrow & (E \otimes \mathcal{K}(L^2(G))) \times_{\alpha^E} G \\ \downarrow & & \downarrow \\ (A \times_{\beta^A, w^A} G) \otimes \mathcal{K}(L^2(G)) & \longrightarrow & (A \otimes \mathcal{K}(L^2(G))) \times_{\alpha^A} G \ , \end{array}$$

where the horizontal maps are $*$-isomorphisms. Since we are talking about full crossed products, the column on the right hand side is an extension. It follows from Lemma 2.6 that this extension is semi-split when (2.5) is. The conclusion follows straightforwardly from this. □

Theorem 2.8. *Let A be a separable C^*-algebra with the lifting property and let $\alpha : G \to \mathrm{Aut}\, A$ be an action of the countable discrete group G by automorphisms of A. It follows that the full crossed product C^*-algebra, $A \times_\alpha G$, has the lifting property.*

Proof. Consider an extension

$$0 \longrightarrow B \longrightarrow E \xrightarrow{p} A \times_\alpha G \longrightarrow 0. \tag{2.8}$$

By Proposition 2.4 we must show that there is a completely positive contractive section for p. Since $A \times_\alpha G$ is separable we can assume without loss of generality that E is separable. By tensoring through with $\mathbb{K}$ we may also assume that B, E, A and $A \times_\alpha G$ are all stable, thanks to Lemma 2.5. Set $E_0 = p^{-1}(A) \subseteq E$. Let $u_g \in M(A \times_\alpha G)$, $g \in G$, be the canonical unitaries arising from the crossed product. By [CH] the unitary group of the multiplier algebra of a stable separable C^*-algebra is contractible in norm. In particular, the unitary group of $M(A \times_\alpha G)$ is norm-connected. Furthermore, the $*$-homomorphism $M(E) \to M(A \times_\alpha G)$ induced by p is surjective by [OP]. For each g we can therefore choose a unitary lift $v_g \in M(E)$ of u_g. Of course, we choose $v_e = 1$. Then $\beta'_g = \mathrm{Ad}\, v_g|_{E_0}$ is an automorphism of E_0. Since $v_g v_h v_{gh}^* E_0 \subseteq E_0$, $E_0 v_g v_h v_{gh}^* \subseteq E_0$, we can define $w' : G \times G \to M(E_0)$ by

$$w'(g,h) = v_g v_h v_{gh}^*.$$

Then (β', w') is a twisted action of G on E_0. This twisted action induces in the obvious way twisted actions (β, w) and (β'', w'') of G on $B \subseteq E_0$ and $A = p(E_0)$, respectively, in such a way that

$$0 \longrightarrow B \longrightarrow E_0 \xrightarrow{p} A \longrightarrow 0$$

becomes a twisted G-extension. Since A has the lifting property we conclude from Lemma 2.7 that

$$0 \longrightarrow B \times_{\beta,w} G \longrightarrow E_0 \times_{\beta',w'} G \longrightarrow A \times_{\beta'',w''} G \longrightarrow 0 \tag{2.9}$$

is a semi-split extension. By the universal property of the twisted crossed product there is a $*$-homomorphism $\chi : E_0 \times_{\beta',w'} G \to E$ such that

$$\chi(\sum_{t\in G} f(t)u_t) = \sum_{t\in G} f(t)v_t.$$

This map sends $B \times_{\beta,w} G \subseteq E_0 \times_{\beta',w'} G$ to $B \subseteq E$ and hence it induces a $*$-homomorphism $\chi' : A \times_{\beta'',w''} G \to A \times_\alpha G$ such that

$$\begin{tikzcd} E_0 \times_{\beta',w'} G \arrow[r] \arrow[d, "\chi"'] & A \times_{\beta'',w''} G \arrow[d, "\chi'"] \\ E \arrow[r, "p"] & A \times_\alpha G \end{tikzcd} \tag{2.10}$$

commutes. But now note that $\beta'' = \alpha$ and that $w'' = 1$ by construction. Diagram (2.10) is therefore the square on the right hand side of a commuting diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & B \times_{\beta,u} G \arrow[r] \arrow[d] & E_0 \times_{\beta',u'} G \arrow[r] \arrow[d, "\chi"'] & A \times_\alpha G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E \arrow[r, "p"] & A \times_\alpha G \arrow[r] & 0 . \end{tikzcd}$$

Since the upper extension is semi-split, it follows that the lower one is also.

□

Since $\mathbb{C}$ obviously has the lifting property we obtain

Corollary 2.9. *$C^*(G)$ has the lifting property for any countable discrete group G.*

In particular, $\mathrm{Ext}(C^*(G))$ is always a group. The same is true for $\mathrm{Ext}(SC^*(G))$ and $\mathrm{Ext}(\mathrm{cone}(C^*(G)))$, in contrast with what is the case for the reduced group C^*-algebras, cf. [Ki1].

Now we consider the conjectures (B1)-(B7) from [Ki1]. Conjecture (B6) says that the local lifting property (LLP) implies the weak expectation property (WEP) of Lance, cf. [L]. Since the local lifting property is weaker than the lifting property itself, (B6) in combination with Corollary 2.6 implies that $C^*(G)$ has the WEP for any discrete countable group G. By combining Corollary 2.6 with Proposition 1.1(i) of [Ki1] we conclude from this that

$$C^*(G) \otimes_{min} C^*(G) = C^*(G) \otimes_{max} C^*(G)$$

for every countable discrete group G, if (B6) holds. In particular, it follows that any countable discrete group has property (F), a property originally introduced by Kirchberg in [Ki3], provided (B6) is true. However, according to [Ki2] there exist countable discrete groups which do not have property (F). Consequently (B6) can not be true. Since Kirchberg has shown that (B1)-(B7) are all equivalent we have obtained the following result.

Corollary 2.10. *The conjectures (B1)-(B7) of* [Ki1] *are all false. In particular, there exists a separable II_1-factor which is not a subfactor of an ultrapower R_ω of the hyperfinite II_1-factor R.*

There are other ways to use Corollary 2.6 as the key to obtain Corollary 2.10 from the web of implications established by Kirchberg in [Ki1]. For example, one can now deduce from Theorem 4.1, Proposition 1.3 and Corollary 3.7 in [Ki1], that when H is a countable discrete group without property (F), the group von Neumann algebra $VN(H)$ will contain a separable II_1-factor in its central decomposition which is not a subfactor of the ultrapower of the hyperfinite II_1-factor. By Theorem 4.1 of [Ki1] the same II_1-factor is an example of a C^*-algebra which is not the quotient of a C^*-algebra with the WEP.

Finally, we observe that we have also falsified conjectures (C1), C(2) and (F) from [Ki1].

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