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SALEM BEN SAÏD AND BENT ØRSTED

ABSTRACT. Let $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ be a positive-real valued multiplicity function related to a root system $\mathscr{R}$, and Δ_k be the Dunkl-Laplacian operator. For $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, denote by $u_k(x,t)$ the solution to the deformed wave equation $\Delta_k^x u_k(x,t) = \partial_{tt} u_k(x,t)$, where the initial data belong to the Schwartz space on $\mathbb{R}^N$. We prove that for $k \geq 0$ and $N \geq 1$, the wave equation satisfies a weak Huygens' principle, and only if $(N-3)/2 + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \in \mathbb{N}$, a strict Huygens' principle holds. Here $\mathscr{R}^+ \subset \mathscr{R}$ is a subsystem of positive roots. As a particular case, if the initial data are supported in a closed ball of radius $R > 0$ about the origin, the strict Huygens' principle implies that the support of $u_k(x,t)$ is contained in the conical shell $\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |t| - R \leq \|x\| \leq |t| + R\}$. Our approach uses the representation theory of the group $SL(2,\mathbb{R})$, and Paley-Wiener theory for the Dunkl transform. Also, we show that the (t-independent) energy functional of u_k is, for large $|t|$, partitioned into equal potential and kinetic parts.

1. Introduction

In a series of lectures at Yale University, J. Hadamard formulated two different meanings of Huygens' principle which are nowadays known as Hadamard's major and minor premises [14]. A typical statement of the major premise is "every point on a wave front acts as a source of a new wave front, propagating radially outward". This statement is mainely the original principle proposed by Christiaan Huygens in the 17th centry [22], and it holds for a general class of wave propagations. In contrast to the major premise, the minor premise is a remarkable phenomena, that is valid only for very special equations, and never happens in even dimensional spaces. Mathematically, a second order hyperbolic equation satisfies Huygens' principle in the narrow sense ("minor premise"), if the solution of the corresponding Cauchy problem at some point x depends not on all the Cauchy data, but only on its part on the intersection of the characteristic conoid with vertex x with the Cauchy surface. This means that the fundamental solution of the corresponding Cauchy problem vanishes outside and inside the characteristic conoid, and thus must be located on it. Indeed, because of the Huygens' principle in the narrow sense that we can hear each other, one has a pure propagation without residual waves. This is not the case in the two dimensional space: when a pebble falls in water at a certain point x, the initial ripple on a circle around x will be followed by subsequent ripples. Thus a given point y will be hit by residual waves.

The problem of classifying all second order hyperbolic differential operators which obey Huygens' principle in the narrow sense, is known as the Hadamard's problem. This problem has received a good deal of attention and the literature is extensive [33, 25, 5, 13, 26, 34, 30, 27, 17, 1, 2, 6]. (Of course, this list of references is not complete). Nevertheless, this problem is still far from being fully solved. In the present paper, we shall treat a natural differential-difference operator of a similar hyperbolic nature.

We will use the terminology "weak Huygens' principle" for Hadamard's major premises, and "strict Huygens' principle" for Hadamard's minor premises.

The propagation of waves in $\mathbb{R}^N$ is governed by the wave equation

$$\Delta^x u(x,t) = \partial_{tt} u(x,t), \quad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{\textbf{L}}$$

Here Δ^x denotes the usual Laplacian operator in the x-variable, and the subscript t indicates differentiation in the t-variable. It is a well known fact that **(L)** possesses the weak Huygens' principle for all $N \geq 1$, and only for odd $N \geq 3$ where the strict Huygens' principle holds [5]. In this paper, we will investigate the validity of the weak and the strict Huygens' principle for **(L)** when the Laplacian Δ is replaced by the differential-difference Dunkl-Laplacian operator associated with Coxeter groups [7]. The main tools are the Paley-Wiener theory for the Dunkl transform (or the generalized Fourier transform) [24], and the representation theory of the group $SL(2,\mathbb{R})$.

To be more specific, let G be a finite Coxeter group on $\mathbb{R}^N$ with root system $\mathscr{R}$, and choose a positive subsystem $\mathscr{R}^+$ in $\mathscr{R}$. Let $k : \mathscr{R} \to \mathbb{R}^+$, $\alpha \mapsto k_\alpha$, be a multiplicity function. The Dunkl-Laplacian operator is given by

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathscr{R}^+} k_\alpha \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right\},$$

where Δ and ∇ are the usual Laplacian and gradient operators, $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product in $\mathbb{R}^N$, and r_α is the reflection on the hyperplane orthogonal to the root α.

Consider the following Cauchy problem

$$\Delta_k u_k(x,t) = \partial_{tt} u_k(x,t), \quad u_k(x,0) = f(x), \quad \partial_t u_k(x,0) = g(x), \tag{\textbf{D}}$$

where the Cauchy data f and g are two Schwartz functions on $\mathbb{R}^N$. The main results of this paper are:

Claim 1. (Weak Huygens' principle) Assume that $k \geq 0$, and $N \geq 1$. For a given $y \in \mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $f(x \circledast_k y)$ and $g(x \circledast_k y)$ for $\|y\| \leq |t|$. Here $\|y\|^2 = \sum_{j=1}^N y_j^2$, and $\circledast_k$ is a generalized translation. For $k \equiv 0$, $F(x \circledast_0 y) = F(x - y)$.

Claim 2. (Strict Huygens' principle) Assume that $k \geq 0$ and $N \geq 1$. If

$$\frac{N-3}{2} + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \in \mathbb{N},$$

then $u_k(x,t)$ depends only on the values of $f(x \circledast_k y)$ and $g(x \circledast_k y)$ (and their derivatives) for $\|y\| = |t|$.

In particular, if $x = 0$, then in Claim 1 (resp. Claim 2) the solution $u_k(0,t)$ will depend only on the values of $f(y)$ and $g(y)$ for $\|y\| \leq |t|$ (resp. $\|y\| = |t|$). Furthermore, if the Cauchy data (f,g) are supported in a closed ball of radius $R > 0$ about the origin, Claim 2 reads:

Claim 3. If $(N-3)/2 + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \in \mathbb{N}$, the support of $u_k(x,t)$ is contained in the conical shell

$$\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |t| - R \leq \|x\| \leq |t| + R\}.$$

We can also give a different proof for Claim 3 using another approach based only on de Jeu's Paley-Wiener theorem for the Dunkl transform [24]. See the end of Section 3 for a sketch of this approach; note that the details of this argument can be found in the last section, which deals with the principle of energy equipartition of a solution to **(D)**.

Here is the outline of our approach. We start by proving that there exists two tempered distributions $P^{(1)}_{k,t}$ and $P^{(2)}_{k,t}$, where the solution u_k to the Cauchy problem **(D)** is uniquely given by

$$u_k(x,t) = (P^{(1)}_{k,t} *_k f)(x) + (P^{(2)}_{k,t} *_k g)(x). \tag{1.1}$$

Here $*_k$ is a Dunkl-type generalized convolution. Based on a Paley-Wiener theorem [24], we show that $P^{(\ell)}_{k,t}$, for $\ell = 1, 2$, is supported inside the light cone $\mathscr{C} := \{(y,t) \mid \|y\| = \|t\|\}$, i.e. in the set $\{(y,t) \mid \|y\| \le \|t\|\}$. To prove the strict Huygens' principle, we use the representation theory of the group $SL(2,\mathbb{R})$. In the classical case, this approach goes back to R. Howe [18]. We show that $P^{(1)}_{k,t}$ and $P^{(2)}_{k,t}$ are supported on the light cone $\mathscr{C}$ if and only if $P^{(\ell)}_{k,t}$, for $\ell = 1, 2$, generates a finite-dimensional $\mathfrak{sl}(2,\mathbb{R})$-module of dimension

$$d_{k,\ell} = \frac{N+3}{2} - \ell + \sum_{\alpha\in\mathscr{R}^+} k_\alpha.$$

On the other hand, for $f \in \mathscr{C}^\infty(\mathbb{R}^N)$, denote by M_f the spherical mean operator, as first introduced in [28]

$$M_f(x,r) = d_k^{-1} \int_{S^{N-1}} f(x \circledast_k ry) w_k(y)\, d\omega(y), \quad x \in \mathbb{R}^N,\ r \ge 0.$$

Here d_k is a normalization constant, and w_k is the G-invariant weight function $w_k(x) := \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, x\rangle|^{2k_\alpha}$, for $x \in \mathbb{R}^N$. A key result in Rösler's paper [31], is that the spherical mean operator is positivity-preserving. Keeping in mind (1.1), and using the spherical mean operator for the Cauchy data (f, g), we prove that

$$\begin{aligned} u_k(x,t) = {} & d_k \frac{\sqrt{\pi}}{\Gamma(\gamma_k + N/2)} \int_0^{|t|} r^{2\gamma_k+N-1} \frac{d}{dt} \Big(\mathbb{S}_{-\gamma_k - \frac{N-3}{2}}(t^2 - r^2) \Big) M_f(r,x)\, dr \\ & + \operatorname{sign(t)} d_k \frac{\sqrt{\pi}}{\Gamma(\gamma_k + N/2)} \int_0^{|t|} r^{2\gamma_k+N-1} \mathbb{S}_{-\gamma_k - \frac{N-3}{2}}(t^2 - r^2) M_g(r,x)\, dr. \end{aligned}$$

Here $\gamma_k := \sum_{\alpha\in\mathscr{R}^+} k_\alpha$, and $\mathbb{S}_\lambda(x) := x_+^{\lambda-1}/\Gamma(\lambda)$ is the Riemann-Liouville distribution.

In the light of this integral representation of u_k, and Rösler results on the spherical mean operator, the two claims above become:

Claim 4. (Weak Huygens' principle) Assume that $k \ge 0$, and $N \ge 1$. For a given $y \in \mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $f(y)$ and $g(y)$ for $\|x\| - |t| \le \|y\| \le \|x\| + |t|$.

Claim 5. (Strict Huygens' principle) Assume that $k \ge 0$ and $N \ge 1$. If $(N-3)/2 + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \in \mathbb{N}$, then $u_k(x,t)$ depends only on the values of $f(y)$ and $g(y)$ (and their derivatives) for $\|y\| \ge \big|\|x\| - |t|\big|$.

In the last section we prove the energy equipartition theorem for the solution u_k. In this part we choose to work with smooth Cauchy data (f, g) supported in the closed ball of radius $R > 0$ about the origin. The advantage of this choice is to investigate, via Paley-Wiener theory for the Dunkl transform, the behavior of the difference between the kinetic and potential energy of a solution u_k to **(D)**. Indeed, if we denote by $\mathscr{K}_k[u_k](t)$ the kinetic energy, and by $\mathscr{P}_k[u_k](t)$ the potential energy, then $|\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t)|$ decays like $e^{-2s(|t|-R)}$, for all $t \in \mathbb{R}$ and for fixed $s > 0$. Thus the principle of energy equipartition holds for all $|t| > R$. However, if we work with the Cauchy data (f, g) in the Schwartz space, the principle of energy equipartition reads

$$\lim_{|t|\to\infty} \mathscr{K}_k[u_k](t) = \lim_{|t|\to\infty} \mathscr{P}_k[u_k](t) = \frac{\text{The total (t-independent) energy of } u_k}{2}.$$

This paper is organized as follows: In Section 2 we give an abbreviated background on the Dunkl theory. Section 3 is devoted to prove the main results, that is Claim 1, Claim 2, Claim 3, Claim 4, and Claim 5. In Section 4 we turn our attention to the energy equipartition theorem.

2. Background

Throughout the paper, $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^N$ as well as its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, denote by $\|x\| = \langle x, x\rangle^{1/2}$. Denote by $\mathscr{S}(\mathbb{R}^N)$ the Schwartz space of rapidly decreasing functions equipped with the usual Fréchet space topology.

Let G be a finite reflection group on $\mathbb{R}^N$ with root system $\mathscr{R}$, and fix a positive subsystem $\mathscr{R}^+$ of $\mathscr{R}$, normalized so that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in \mathscr{R}^+$.

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, let r_α be the reflection on the hyperplane $\langle \alpha\rangle^\perp$ orthogonal to α

$$r_\alpha(x) := x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha, \quad x \in \mathbb{R}^N.$$

Then G is a subgroup of the orthogonal group $O(N)$ generated by the reflections $\{r_\alpha \mid \alpha \in \mathscr{R}\}$, and is called a Coxeter group. A multiplicity function on $\mathscr{R}$ is a G-invariant function $k : \mathscr{R} \to \mathbb{C}$. Setting $k_\alpha := k(\alpha)$ for $\alpha \in \mathscr{R}$, we have $k_{h\alpha} = k_\alpha$ for all $h \in G$. The $\mathbb{C}$-vector space of multiplicity functions on $\mathscr{R}$ is denoted by $\mathscr{K}$. If $m := \sharp\{G\text{-orbits in } \mathscr{R}\}$, then $\mathscr{K} \cong \mathbb{C}^m$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathscr{K}$, in [7], Dunkl defined a family of first order differential-difference operators $T_\xi(k)$ that play the role of the usual partial differentiation. Dunkl's operators are defined by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \langle \alpha, \xi\rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x\rangle}, \quad f \in \mathscr{C}^1(\mathbb{R}^N).$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. The definition of $T_\xi(k)$ is independent of the choice of $\mathscr{R}^+$, and these operators mutually commute, i.e. $T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k)$. Further, if f and g are in $\mathscr{C}^1(\mathbb{R}^N)$, and at least one of them is G-invariant, then

$$T_\xi(k)[fg] = gT_\xi(k)f + fT_\xi(k)g. \tag{2.1}$$

We refer to [7, 10] for more details on the theory of Dunkl's operators.

The counterpart of the usual Laplacian is the Dunkl-Laplacian defined by

$$\Delta_k := \sum_{j=1}^{N} T_{\xi_j}(k)^2,$$

where $\{\xi_1, \ldots, \xi_N\}$ is an arbitrary orthonormal basis of $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$. By the normalization $\langle\alpha,\alpha\rangle = 2$, we can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in\mathscr{R}^+} k_\alpha \left\{ \frac{\langle\nabla f(x),\alpha\rangle}{\langle\alpha,x\rangle} - \frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle^2} \right\},$$

where Δ and ∇ denote the usual Laplacian and gradient, respectively. For the j-th basis vector ξ_j, we will use the abbreviation $T_{\xi_j}(k) = T_j(k)$.

Henceforth, $\mathscr{K}^+$ denotes the set of multiplicity functions $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ such that $k_\alpha \geq 0$ for all $\alpha \in \mathscr{R}$. For $k \in \mathscr{K}^+$, there exists a generalization of the usual exponential kernel $e^{\langle\cdot,\cdot\rangle}$ by means of the Dunkl system of differential equations.

Theorem 2.1. (cf. [8, 29]) *For $k \in \mathscr{K}^+$, there exists a unique holomorphic function E_k on $\mathbb{C}^N \times \mathbb{C}^N$ characterized by*

$$T_\xi(k)E_k(z,w) = \langle\xi,w\rangle E_k(z,w) \quad \textit{for all } \xi\in\mathbb{C}^N, \quad E_k(0,w)=1. \tag{2.2}$$

Further, the kernel E_k is symmetric in its arguments, and

$$E_k(\lambda z, w) = E_k(z,\lambda w), \qquad E_k(hz,w) = E_k(z,hw),$$

for $z,w\in\mathbb{C}^N$, $\lambda\in\mathbb{C}$, and $h\in G$.

For complex-valued k, there is a detailed investigation of (2.2) by Opdam [29]. Theorem 2.1 is a weak version of Opdam's result. For integral multiplicity, another proof for Theorem 2.1 can be found in [3], by means of shift operators. The function E_k is the so-called Dunkl kernel. When $k \equiv 0$, we have $E_0(z,w) = e^{\langle z,w\rangle}$ for $z,w\in\mathbb{C}^N$.

Let w_k denote the weight function on $\mathbb{R}^N$ defined by

$$w_k(x) := \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha,x\rangle|^{2k_\alpha}, \quad x\in\mathbb{R}^N.$$

It is G-invariant and homogeneous of degree $2\gamma_k$, with the index

$$\gamma_k := \sum_{\alpha\in\mathscr{R}^+} k_\alpha.$$

Notice that by G-invariance of k, the definition of w_k does not depend on the special choice of $\mathscr{R}^+$. Further, we denote by dx the Lebesgue measure corresponding to $\langle\cdot,\cdot\rangle$.

The Dunkl transform on the space $L^1(\mathbb{R}^N, w_k(x)dx)$ of integrable functions on $\mathbb{R}^N$ with respect to $w_k(x)dx$, is defined by

$$\mathscr{D}_k f(\xi) := c_k^{-1}\int_{\mathbb{R}^N} f(x)E_k(-ix,\xi)w_k(x)\,dx, \quad \xi\in\mathbb{R}^N,$$

where c_k denotes the Mehta-type constant

$$c_k := \int_{\mathbb{R}^N} e^{-\|x\|^2/2} w_k(x)\,dx. \tag{2.3}$$

Many properties of the Euclidean Fourier transform carry over to the Dunkl transform.

Theorem 2.2. (cf. [9, 23]) *For $k \in \mathscr{K}^+$, the following hold*

(i) *The Dunkl transform is a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$. Its inverse is given by $\mathscr{D}_k^{-1} f(\xi) = \mathscr{D}_k f(-\xi)$.*

(ii) (L^1-inversion) *If $f \in L^1(\mathbb{R}^N, w_k(x)dx)$, with $\mathscr{D}_k(f) \in L^1(\mathbb{R}^N, w_k(x)dx)$, then $f = \mathscr{D}_k^{-1}(\mathscr{D}_k(f))$ a.e.*

(iii) (Plancherel formula) *The Dunkl transform on $\mathscr{S}(\mathbb{R}^N)$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}^N, w_k(x)dx)$.*

In what follow we shall need a Paley-Wiener theorem for the Dunkl transform. For $R > 0$, denote by $\mathscr{C}_R^\infty(\mathbb{R}^N)$ the space of smooth functions on $\mathbb{R}^N$ with support contained in the closed metric ball of radius R about the origin. Denote by $\mathscr{H}_R(\mathbb{C}^N)$ the space of entire functions f on $\mathbb{C}^N$ with the property that for each integer $M > 0$, there exists a constant α_M such that

$$|f(z)| \leq \alpha_M (1 + \|z\|)^{-M} e^{R\|\mathrm{Im}(z)\|}, \quad z \in \mathbb{C}^N.$$

The following theorem can be found in [24].

Theorem 2.3. (Paley-Wiener Theorem) *Let G be a Coxeter group and suppose that $k \in \mathscr{K}^+$. Then the Dunkl transform $\mathscr{D}_k$ is a linear isomorphism between $\mathscr{C}_R^\infty(\mathbb{R}^N)$ and $\mathscr{H}_R(\mathbb{C}^N)$.*

The above theorem was proved in [24] for $\mathrm{Re}(k) \geq 0$, and its geometrical form was presented as a conjecture.

Another result needed in the sequel is a generalized translation operator. In [8], Dunkl proved that for $k \in \mathscr{K}^+$, there exists a linear isomorphism V_k that intertwines the algebra generated by the Dunkl's operators with the algebra of partial differential operators. The intertwining operator V_k is determined uniquely by

$$T_\xi(k) V_k = V_k \partial_\xi \quad \text{for all } \xi \in \mathbb{R}^N, \quad V_k \mathscr{P}_m(\mathbb{R}^N) \subset \mathscr{P}_m(\mathbb{R}^N), \quad V_k(1) = 1.$$

In [35], Trimèche used V_k to define a generalized translation operator on $\mathscr{C}^\infty(\mathbb{R}^N)$ by

$$f(x \circledast_k y) := V_k^x V_k^y (V_k^{-1} f)(x - y), \quad x, y \in \mathbb{R}^N.$$

Here the superscript denotes the relevant variable. When $k \equiv 0$, $f(x \circledast_0 y) = f(x-y)$. Further, in [35], the author defined a generalized convolution $*_k$ by

$$(f *_k g)(x) := \int_{\mathbb{R}^N} f(y) g(x \circledast_k y) w_k(y)\, dy.$$

By [35, Theorem 7.2]

$$\mathscr{D}_k(f *_k g)(\xi) = \mathscr{D}_k f(\xi) \mathscr{D}_k g(\xi) \qquad \text{and} \qquad f *_k g = g *_k f. \tag{2.4}$$

3. Wave equations associated with Dunkl operators

For a multiplicity function k in $\mathscr{K}^+$, consider the Cauchy problem for the wave equation associated with the Dunkl-Laplacian operator

$$\begin{aligned} \Delta_k u_k(x,t) &= \partial_{tt} u_k(x,t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ u_k(x,0) &= f(x), \ \ \partial_t u_k(x,0) = g(x). \end{aligned} \tag{3.1}$$

Here the functions f and g belong to $\mathscr{S}(\mathbb{R}^N)$. The subscript t indicates differentiation in the t-variable. Next, we shall prove the following statements:

($\mathcal{S}_1$) Let $k \in \mathscr{K}^+$ and $N \geq 1$. For a given y, the solution $u_k(x,t)$ depends on the values of $f(x \circledast_k y)$ and $g(x \circledast_k y)$ for $\|y\| \leq |t|$.

($\mathcal{S}_2$) Let $k \in \mathscr{K}^+$, $N \geq 1$, and $y \in \mathbb{R}^N$. Under a condition involving N and k, the solution $u_k(x,t)$ depends only on the values of $f(x \circledast_k y)$ and $g(x \circledast_k y)$ (and their derivatives) for $\|y\| = |t|$.

Another way of saying ($\mathcal{S}_1$) is that u_k is expressed as a sum of $*_k$-convolutions of f and g with distributions that vanish outside the ball of radius $|t|$ about the origin. Similarly, ($\mathcal{S}_2$) is equivalent to the fact that distributions we convolve f and g with also vanish inside the ball of radius $|t|$. In analogy with the classical case, i.e. when $k \equiv 0$, we shall say that (3.1) possesses the weak Huygens' principle if u_k satisfies ($\mathcal{S}_1$), and (3.1) obey the strict Huygens' principle if u_k satisfies ($\mathcal{S}_2$).

For time being, we only assume $k \in \mathscr{K}^+$ and $N \geq 1$. Set

$$U_k(x,t) := \begin{bmatrix} u_k(x,t) \\ \partial_t u_k(x,t) \end{bmatrix}. \tag{3.2}$$

Thus we may rewrite the wave equation in (3.1) as

$$\partial_t U_k(x,t) = \begin{bmatrix} 0 & 1 \\ \Delta_k & 0 \end{bmatrix} U_k(x,t). \tag{3.3}$$

Applying the Dunkl transform $\mathscr{D}_k$ to (3.3), in the x-variable, and using the fact that $\mathscr{D}_k(\Delta_k f)(\xi) = -\|\xi\|^2 \mathscr{D}_k(f)(\xi)$, we obtain

$$\partial_t \mathscr{D}_k(U_k(\cdot,t))(\xi) = \begin{bmatrix} 0 & 1 \\ -\|\xi\|^2 & 0 \end{bmatrix} \mathscr{D}_k(U_k(\cdot,t))(\xi) := \mathbb{A}\mathscr{D}_k(U_k(\cdot,t))(\xi). \tag{3.4}$$

Solving this ordinary differential equation, we get

$$\mathscr{D}_k(U_k(\cdot,t))(\xi) = e^{t\mathbb{A}} \mathscr{D}_k(U_k(\cdot,0))(\xi), \tag{3.5}$$

where

$$e^{t\mathbb{A}} = \begin{bmatrix} \cos(t\|\xi\|) & \sin(t\|\xi\|)/\|\xi\| \\ -\|\xi\|\sin(t\|\xi\|) & \cos(t\|\xi\|) \end{bmatrix}.$$

By the inversion formula for the Dunkl transform (Theorem 2.2(ii)), and the property (2.4) of the generalized convolution $*_k$, we have

$$\begin{aligned} U_k(x,t) &= \left\{P_{k,t} *_k U_k(\cdot,0)\right\}(x) \\ &:= \left\{ \begin{bmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{bmatrix} *_k U_k(\cdot,0) \right\}(x), \end{aligned} \tag{3.6}$$

where $P_{k,t} := \mathscr{D}_k^{-1}(e^{t\mathbb{A}})$, and

$$\begin{bmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{bmatrix} = \begin{bmatrix} \mathscr{D}_k^{-1}\left[\cos(t\|\cdot\|)\right] & \mathscr{D}_k^{-1}\left[\sin(t\|\cdot\|)/\|\cdot\|\right] \\ \mathscr{D}_k^{-1}\left[-\|\cdot\|\sin(t\|\cdot\|)\right] & \mathscr{D}_k^{-1}\left[\cos(t\|\cdot\|)\right] \end{bmatrix} \tag{3.7}$$

is the 2×2 matrix of tempered distributions on $\mathbb{R}^N$ obtained by applying the inverse of the Dunkl transform, in the sense of tempered distribution, entrywise to $e^{t\mathbb{A}}$. We shall call the distributions $P_{k,t}^{ij}$ the propagators of the deformed wave equation. We have proved:

Theorem 3.1. *The solution to the Cauchy problem (3.1) is given uniquely by*

$$u_k(x,t) = (P_{k,t}^{11} *_k f)(x) + (P_{k,t}^{12} *_k g)(x),$$

where, for a fixed t, $P_{k,t}^{11}$ *and* $P_{k,t}^{12}$ *are the tempered distributions on* $\mathbb{R}^N$ *given by*

$$P_{k,t}^{11} = \mathscr{D}_k^{-1}\left[\cos(t\|\cdot\|)\right], \quad P_{k,t}^{12} = \mathscr{D}_k^{-1}\left[\sin(t\|\cdot\|)/\|\cdot\|\right].$$

From the form of $\mathscr{D}_k(P_{k,t})$, one can observe that for each t, the function $x \mapsto u_k(x,t)$ belongs to $\mathscr{S}(\mathbb{R}^N)$.

Before investigate the support of the solution u_k and of the propagators, let us make some observations regarding the estimate and the limit of $u_k(\cdot,t)$ in $L^2(\mathbb{R}^N, w_k(x)\,dx)$. We restrict our attention to the L^2-behaviors because these are the most physically interesting quantities. First, for all $t \in \mathbb{R}$, we have the following Strichartz-type inequality

$$\|u_k(\cdot,t)\|_k \leq \|f\|_k + \|(-\Delta_k)^{-1/2} g\|_k. \tag{3.8}$$

Here $\|\cdot\|_k$ denotes the norm in $L^2(\mathbb{R}^N, w_k(x)dx)$. Secondly, as $|t| \to \infty$, the function $t \mapsto \|u_k(\cdot,t)\|_k$ has a definite limit depending on the initial data

$$\lim_{|t|\to\infty} \|u_k(\cdot,t)\|_k^2 = \tfrac{1}{2}\|f\|_k^2 + \tfrac{1}{2}\|(-\Delta_k)^{-1/2} g\|_k^2. \tag{3.9}$$

It follows that if $\|u_k(\cdot,t)\|_k \to 0$ as $|t| \to \infty$, then

$$u_k \equiv 0.$$

To prove (3.8) and (3.9), we express $\int_{\mathbb{R}^N} |u_k(x,t)|^2 w_k(x)\,dx$ in terms of $\mathscr{D}_k(u_k(\cdot,t))(\xi)$ by means of the Plancherel formula. In view of

$$\mathscr{D}_k(u_k(\cdot,t))(\xi) = \cos(t\|\xi\|)\mathscr{D}_k f(\xi) + \frac{\sin(t\|\xi\|)}{\|\xi\|}\mathscr{D}_k g(\xi), \tag{3.10}$$

we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} & |u_k(x,t)|^2 w_k(x)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left\{ |\mathscr{D}_k f(\xi)|^2 + \frac{|\mathscr{D}_k g(\xi)|^2}{\|\xi\|^2} \right\} w_k(\xi)\,d\xi \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} |\mathscr{D}_k f(\xi)|^2 \cos(2t\|\xi\|) w_k(\xi)\,d\xi \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} \frac{|\mathscr{D}_k g(\xi)|^2}{\|\xi\|^2} \sin(2t\|\xi\|) w_k(\xi)\,d\xi \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} \frac{\mathscr{D}_k f(\xi)\overline{\mathscr{D}_k g(\xi)} + \overline{\mathscr{D}_k f(\xi)}\mathscr{D}_k g(\xi)}{\|\xi\|} \sin(2t\|\xi\|) w_k(\xi)\,d\xi.
\end{aligned}$$

Above we used the familiar trigonometric identities for double angles. Now the Strichartz-type inequality is clear. Equation (3.9) follows by using the classical Riemann-Lebesgue lemma for the Euclidean Fourier sine and cosine transforms.

Now we turn our attention to the statements $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$, stated in the beginning of this section. In terms of the propagators, the first statement amounts to the fact that $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are supported inside the light cone $\mathscr{C} := \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid \|x\|^2 - t^2 = 0\}$, i.e. in the set $\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid \|x\|^2 - t^2 \leq 0\}$. The second statement amounts to the fact that $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are supported on the light cone $\mathscr{C}$.

To prove $(\mathcal{S}_1)$, our method uses the Paley-Wiener Theorem 2.3 for the Dunkl transform.

The first key observation is that the functions $\cos(t\|x\|)$ and $\sin(t\|x\|)/\|x\|$ can be extended to entire functions on $\mathbb{C}^N$. Indeed, for $z \in \mathbb{C}$, the functions $\cos z$ and $\sin z/z$ are both even, and thus we may consider the functions $\cos(\sqrt{z})$ and $\sin(\sqrt{z})/\sqrt{z}$ which are entire analytic functions of z (even though $\sqrt{z}$ it is not single-valued). Thus, the analytic extensions of $\cos(t\|x\|)$ and $\sin(t\|x\|)/\|x\|$, respectively, are

$$\cos\left(t\langle z,z\rangle^{1/2}\right), \qquad \frac{\sin\left(t\langle z,z\rangle^{1/2}\right)}{\langle z,z\rangle^{1/2}},$$

since being the composition of analytic functions, and they coincide with the original functions when $z \in \mathbb{R}^N$. In order to apply the Paley-Wiener theorem, we need to show that

$$\left|\cos(t\langle z,z\rangle^{1/2})\right|, \quad \left|\frac{\sin(t\langle z,z\rangle^{1/2})}{\langle z,z\rangle^{1/2}}\right| \leq ce^{|t|\,\|\mathrm{Im}(z)\|}, \tag{3.11}$$

which turn out to be true. Indeed, if we write $\langle z,z\rangle^{1/2} = u+iv$, and use the fact that $|\cos(u+iv)|$ and $|\sin(u+iv)/(u+iv)|$ are both bounded by $e^{|v|}$, we obtain

$$\left|\cos(t\langle z,z\rangle^{1/2})\right|, \quad \left|\frac{\sin(t\langle z,z\rangle^{1/2})}{\langle z,z\rangle^{1/2}}\right| \leq ce^{|t|\,|v|}.$$

To get the inequalities in (3.11), one have to prove that $|v| \leq \|\mathrm{Im}(z)\|$. This follows as the following: As $\langle z,z\rangle = (u+iv)^2$, we have $u^2 - v^2 = \|\mathrm{Re}(z)\|^2 - \|\mathrm{Im}(z)\|^2$ and $uv = \langle \mathrm{Re}(z), \mathrm{Im}(z)\rangle$. Thus by Cauchy-Schwartz-Buniakowsly inequality $u^2v^2 \leq \|\mathrm{Re}(z)\|^2\|\mathrm{Im}(z)\|^2$. This together with $u^2-v^2 = \|\mathrm{Re}(z)\|^2-\|\mathrm{Im}(z)\|^2$, implies $v^2(v^2+\|\mathrm{Re}(z)\|^2 - \|\mathrm{Im}(z)\|^2) \leq \|\mathrm{Re}(z)\|^2\|\mathrm{Im}(z)\|^2$. This amounts to

$$\left(v^2 + \frac{\|\mathrm{Re}(z)\|^2 - \|\mathrm{Im}(z)\|^2}{2}\right)^2 \leq \left(\frac{\|\mathrm{Re}(z)\|^2 + \|\mathrm{Im}(z)\|^2}{2}\right)^2,$$

which yields $v^2 \leq \|\mathrm{Im}(z)\|^2$. Now, applying the Paley-Wiener Theorem 2.3, we conclude that the propagators $\mathscr{D}_k^{-1}[\cos(t\|\cdot\|)]$ and $\mathscr{D}_k^{-1}[\sin(t\|\cdot\|)/\|\cdot\|]$ are supported in the set $\|x\| \leq |t|$. We have proved:

Theorem 3.2. *For all $k \in \mathscr{K}^+$ and $N \geq 1$, the propagators $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are supported inside the light cone $\mathscr{C}$, i.e. in the set $\left\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid \|x\| \leq |t|\right\}$.*

Thus, the following weak Huygens' principle holds.

Theorem 3.3. (Weak Huygens' Principle) *Given a point $y \in \mathbb{R}^N$. If $k \in \mathscr{K}^+$ and $N \geq 1$, the solution $u_k(x,t)$ to the Cauchy problem (3.1) depends only on the values of $f(x \circledast_k y)$ and $g(x \circledast_k y)$ for $\|y\| \leq |t|$.*

Notice that the above theorem holds in all dimensions N. We shall now discuss the strict Huygens' principle which will holds only under a condition involving N and the multiplicity function k. Our approach uses the representation theory of the group $SL(2,\mathbb{R})$, following [20]

We start by investigate certain symmetries and invariance of the deformed wave equation, which are reflected in symmetries and invariance of the propagators. To see this, we define the 2×2 matrix

$$P_k = \begin{bmatrix} P_k^{11} & P_k^{12} \\ P_k^{21} & P_k^{22} \end{bmatrix}$$

of entrywise distributions on $\mathbb{R}^{N+1}$, where

$$P_k^{ij}(\psi_1 \otimes \psi_2) := \int_{\mathbb{R}} P_{k,t}^{ij}(\psi_1)\psi_2(t)\,dt, \quad i,j = 1,2$$

for $\psi_1 \in \mathscr{S}(\mathbb{R}^N)$ and $\psi_2 \in \mathscr{S}(\mathbb{R})$. Here we used the fact that $\psi_1 \otimes \psi_2 \in \mathscr{S}(\mathbb{R}^N) \otimes \mathscr{S}(\mathbb{R}) \cong \mathscr{S}(\mathbb{R}^{N+1})$. From the constructive proof of theorem 3.1, it follows that

$$\Delta_k P_k^{ij} = \partial_{tt} P_k^{ij}, \quad i,j = 1,2.$$

For $g \in G$, $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$, and for each $t \in \mathbb{R}$, denote by π_x the unitary action of G on $\psi(\cdot,t)$ given by

$$\pi_x(g)\psi(x,t) := \psi(g^{-1}\cdot x, t).$$

By duality, we get the action π_x^* of G on tempered distributions by the rule

$$\pi_x^*(g)(T)(\psi) = T(\pi_x(g)^{-1}\psi),$$

for $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$ and $T \in \mathscr{S}^*(\mathbb{R}^{N+1})$. Further, if τ is the operation of time-reflection $\tau(x,t) = (x,-t)$, denote by

$$\pi_t(\tau)\psi(x,t) := \psi(x,-t).$$

Similarly to π_x^*, we obtain the action π_t^* on distributions.

Begin with a solution $u_k(x,t)$ to the Cauchy problem (3.1) with Cauchy data (f,g). Then $\pi_x(h)u_k(x,t)$ solves the the wave equation with initial data $(\pi_x(h)f, \pi_x(h)g)$. The analogue of (3.6) reads

$$\pi_x(h)U_k(x,t) = \{P_{k,t} *_k \pi_x(h)U_k(\cdot,0)\}(x).$$

This amounts to

$$U_k(x,t) = \pi_x^*(h)\{P_{k,t} *_k \pi_x(h)U_k(\cdot,0)\}(x) = \{\pi_x^*(h)P_{k,t} *_k U_k(\cdot,0)\}(x),$$

which implies

$$\pi_x^*(h)P_{k,t}^{ij} = P_{k,t}^{ij}, \quad i,j = 1,2.$$

The G-invariance of $P_{k,t}^{ij}$ can also be observed directly from (3.7). Plugging this into the definition of P_k^{ij}, we conclude that

$$\pi_x^*(h)P_k^{ij} = P_k^{ij}, \quad i,j = 1,2.$$

For the operation of time-reflection, clearly $\pi_t(\tau)u_k(x,t) = u_k(x,-t)$ solves the Cauchy problem (3.1) with Cauchy data $(f,-g)$. Thus, the analogue of (3.6) reads

$$\begin{bmatrix} u_k(x,-t) \\ -(\partial_t u_k)(x,-t) \end{bmatrix} = \left\{ P_{k,t} *_k \begin{bmatrix} f \\ -g \end{bmatrix} \right\}(x),$$

which we may rewrite it as

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U_k(x,-t) = \left\{ P_{k,t} *_k \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U_k(\cdot,0) \right\}(x), \tag{3.12}$$

where $U_k(x,t)$ is the collum vector (3.2). On the other hand, from (3.6), it follows that $U_k(x,-t) = P_{k,-t} *_k U_k(x,0)$. Comparing this with equation (3.12), we obtain

$$P_{k,-t}^{ij} = (-1)^{i-j} P_{k,t}^{ij} \qquad \text{for } i,j = 1,2,$$

which implies

$$\pi_t^*(\tau)P_k^{ij} = (-1)^{i-j} P_k^{ij} \qquad \text{for } i,j = 1,2.$$

Remark 3.4. From the time-reflection action on the propagators, it is clear that time is reversible, except for a minus sign may appear when the second Cauchy datum g or its Dunkl transform are involved. So the past is determined by the present as well as the future.

Next, we shall investigate the symmetries of the propagators under a dilatation operator. This will inform us on the degree of the homogeneity of the distributions P_k^{ij}, with $i,j=1,2$.

For $\lambda>0$ and $\psi\in\mathscr{S}(\mathbb{R}^{N+1})$, denote by

$$S_\lambda^x\psi(x,t)=\psi(\lambda x,t),\qquad S_\lambda^t\psi(x,t)=\psi(x,\lambda t),$$

where the superscript denotes the relevant variable. Set $S_\lambda:=S_\lambda^x\circ S_\lambda^t$. By duality, the operators S_λ^x, S_λ^t, and S_λ act on distributions in the standard way.

We begin by looking to the symmetry properties of $P_{k,t}^{ij}$ under the dilatation S_λ. Observe that if $u_k(x,t)$ is a solution to (3.1) with initial data $(f(x),g(x))$, then $S_\lambda u_k(x,t)$ solves the wave equation with initial data $(S_\lambda^x f(x),\lambda S_\lambda^x g(x))$. Thus

$$S_\lambda U_k(x,t)=\Big\{P_{k,t}*_k\begin{bmatrix} S_\lambda^x f\\ \lambda S_\lambda^x g\end{bmatrix}\Big\}(x). \tag{3.13}$$

On the other hand

$$\begin{aligned} S_\lambda U_k(x,t)=\begin{bmatrix} S_\lambda u_k(x,t)\\ \partial_t\{S_\lambda u_k(x,t)\}\end{bmatrix}&=\begin{bmatrix} u_k(\lambda x,\lambda t)\\ \lambda\{\partial_t u_k\}(\lambda x,\lambda t)\end{bmatrix}\\ &=\begin{bmatrix} u_k\\ \lambda\partial_t u_k\end{bmatrix}(\lambda x,\lambda t)\\ &=\begin{bmatrix} 1&0\\ 0&\lambda\end{bmatrix}\begin{bmatrix} u_k\\ \partial_t u_k\end{bmatrix}(\lambda x,\lambda t)\\ &=\begin{bmatrix} 1&0\\ 0&\lambda\end{bmatrix}\Big\{P_{k,\lambda t}*_k\begin{bmatrix} f\\ g\end{bmatrix}\Big\}(\lambda x)\\ &=\begin{bmatrix} 1&0\\ 0&\lambda\end{bmatrix}S_\lambda^x\Big\{P_{k,\lambda t}*_k\begin{bmatrix} f\\ g\end{bmatrix}\Big\}(x). \end{aligned}$$

Using the fact that if $f_\lambda(x):=\lambda^{\gamma_k+N/2}f(\lambda x)$ then $\mathscr{D}_k(f_\lambda)(\xi)=\lambda^{-\gamma_k-N/2}\mathscr{D}_k(f)(\lambda\xi)$, one can check that S_λ^x preserves the convolution $*_k$. Therefore

$$\begin{aligned} S_\lambda U_k(x,t)&=\begin{bmatrix} 1&0\\ 0&\lambda\end{bmatrix}\Big\{S_\lambda^x P_{k,\lambda t}*_k\begin{bmatrix} S_\lambda^x f\\ S_\lambda^x g\end{bmatrix}\Big\}(x)\\ &=\begin{bmatrix} 1&0\\ 0&\lambda\end{bmatrix}\Big\{S_\lambda^x P_{k,\lambda t}*_k\begin{bmatrix} 1&0\\ 0&\lambda^{-1}\end{bmatrix}\begin{bmatrix} S_\lambda^x f\\ \lambda S_\lambda^x g\end{bmatrix}\Big\}(x). \end{aligned}\tag{3.14}$$

Comparing (3.13) with (3.14) gives $S_\lambda^x P_{k,\lambda t}^{ij}=\lambda^{j-i}P_{k,t}^{ij}$, for $i,j=1,2$. Now the symmetry properties of P_k^{ij} follow as the following: For $\psi_1\in\mathscr{S}(\mathbb{R}^N)$ and $\psi_2\in$

$\mathscr{S}(\mathbb{R})$, we have

$$\begin{aligned} S_\lambda(P_k^{ij})(\psi_1 \otimes \psi_2) &= P_k^{ij}(S^x_{\lambda^{-1}}(\psi_1) \otimes S^t_{\lambda^{-1}}(\psi_2)) \\ &= \int_{\mathbb{R}} P^{ij}_{k,t}(S^x_{\lambda^{-1}}(\psi_1)) S^t_{\lambda^{-1}}(\psi_2)(t)\, dt \\ &= \lambda \int_{\mathbb{R}} P^{ij}_{k,\lambda t}(S^x_{\lambda^{-1}}(\psi_1))\psi_2(t)\, dt \\ &= \lambda \int_{\mathbb{R}} S^x_\lambda(P^{ij}_{k,\lambda t}(\psi_1))\psi_2(t)\, dt \\ &= \lambda^{1+j-i} \int_{\mathbb{R}} P^{ij}_{k,t}(\psi_1)\psi_2(t)\, dt \\ &= \lambda^{1+j-i} P_k^{ij}(\psi_1 \otimes \psi_2). \end{aligned}$$

We summarize the above computations.

Proposition 3.5. *Let $k \in \mathscr{K}^+$.*

(i) *The distribution P_k^{ij} satisfies the deformed wave equation, i.e.*

$$\Delta_k P_k^{ij} = \partial_{tt} P_k^{ij}, \quad i,j = 1,2. \tag{3.15}$$

(ii) *If $g \in G$ and τ denotes the operation of time-reflection, then*

$$\pi_x^*(g) P_k^{ij} = P_k^{ij}, \qquad \pi_t^*(\tau) P_k^{ij} = (-1)^{i-j} P_k^{ij}, \quad i,j = 1,2.$$

(iii) *For $\lambda > 0$*

$$S_\lambda P_k^{ij} = \lambda^{1+j-i} P_k^{ij}, \quad i,j = 1,2.$$

Next, we will prove similar statements for what we shall call the Dunkl-Fourier transform of P_k^{ij}. For $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$, denote by

$$\mathscr{D}_k\mathscr{F}\psi(x,t) := (2\pi)^{-1/2} c_k^{-1} \int_{\mathbb{R}^{N+1}} \psi(x',t') E_k(-ix,x') e^{itt'} w_k(x')\, dx'\, dt'.$$

For a distribution T of compact support, we write

$$\mathscr{D}_k\mathscr{F}(T) = \widetilde{\mathscr{D}_k\mathscr{F}}(T)(x,t) w_k(x) dx dt$$

where

$$\widetilde{\mathscr{D}_k\mathscr{F}}(T)(x,t) := (2\pi)^{-1/2} c_k^{-1} T(E_k(-ix,x')e^{itt'}).$$

Since $E_k(g\cdot x, x') = E_k(x, g\cdot x')$, for $g \in G$, and w_k is G-invariant, thus, in the light of Proposition 3.5(ii), it follows that

$$\pi_x^*(g)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = \mathscr{D}_k\mathscr{F}(P_k^{ij}), \qquad \text{for all } g \in G$$

and

$$\pi_t^*(\tau)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = (-1)^{i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij}).$$

A crucial observation regarding $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ is that

$$(\|x\|^2 - t^2)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = 0, \quad i,j = 1,2. \tag{3.16}$$

This follows by taking the Dunkl-Fourier transform of (3.15) together with the fact that $\mathscr{D}_k\mathscr{F}(\Delta_k\psi)(x,t) = -\|x\|^2\mathscr{D}_k\mathscr{F}(\psi)(x,t)$ and $\mathscr{D}_k\mathscr{F}(\partial_{tt}\psi)(x,t) = -t^2\mathscr{D}_k\mathscr{F}(\psi)(x,t)$. Equation (3.16) says the distribution $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ is supported on the light cone $\mathscr{C} = \left\{(x,t) \in \mathbb{R}^{N+1} \mid \|x\| - t^2 = 0\right\}$, for $i,j = 1,2$.

Consider now the symmetry property of $\mathscr{D}_k\mathscr{F}(P_k^{ij})$. In view of Proposition 3.5(iii) and the fact that $E_k(\lambda x, x') = E_k(x, \lambda x')$, we have

$$\begin{aligned}
S_\lambda\left[\mathscr{D}_k\mathscr{F}(P_k^{ij})\right] &= S_\lambda\left[\widetilde{\mathscr{D}_k\mathscr{F}}(P_k^{ij})(x,t)w_k(x)dxdt\right] \\
&= S_\lambda\left[\widetilde{\mathscr{D}_k\mathscr{F}}(P_k^{ij})\right](x,t)S_\lambda\left[w_k(x)dxdt\right] \\
&= \lambda^{2\gamma_k+N+1}\widetilde{\mathscr{D}_k\mathscr{F}}(P_k^{ij})(\lambda x,\lambda t)w_k(x)dxdt \\
&= (2\pi)^{-1/2}c_k^{-1}\lambda^{2\gamma_k+N+1}P_k^{ij}(E_k(-ix',\lambda x)e^{i\lambda tt'})w_k(x)dxdt \\
&= (2\pi)^{-1/2}c_k^{-1}\lambda^{2\gamma_k+N+1}P_k^{ij}(E_k(-i\lambda x',x)e^{it\lambda t'})w_k(x)dxdt \\
&= (2\pi)^{-1/2}c_k^{-1}\lambda^{2\gamma_k+N+1}P_k^{ij}\left(S_\lambda\left[E_k(-ix',x)e^{itt'}\right]\right)w_k(x)dxdt \\
&= \lambda^{2\gamma_k+N+1}\mathscr{D}_k\mathscr{F}(S_{\lambda^{-1}}P_k^{ij}) \\
&= \lambda^{2\gamma_k+N+i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij}).
\end{aligned}$$

Similarly to Proposition 3.5, we get:

Proposition 3.6. *Let $k \in \mathscr{K}^+$.*

(i) *The distribution $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ is supported on the light cone $\mathscr{C}$, i.e.*

$$(\|x\|^2 - t^2)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = 0, \quad i,j = 1,2.$$

(ii) *If $g \in G$ and τ denotes the operation of time-reflection, then*

$$\begin{aligned}
\pi_x^*(g)\mathscr{D}_k\mathscr{F}(P_k^{ij}) &= \mathscr{D}_k\mathscr{F}(P_k^{ij}), \\
\pi_t^*(\tau)\mathscr{D}_k\mathscr{F}(P_k^{ij}) &= (-1)^{i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij})
\end{aligned}\quad, \quad i,j = 1,2.$$

(iii) *For $\lambda > 0$*

$$S_\lambda\left[\mathscr{D}_k\mathscr{F}(P_k^{ij})\right] = \lambda^{2\gamma_k+N+i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij}), \quad i,j = 1,2.$$

Next, we shall describe the structure of a representation of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ on $\mathscr{S}(\mathbb{R}^{N+1})$. This structure together with Proposition 3.5 and Proposition 3.6, allows to prove that the Cauchy problem (3.1) satisfies the strict Huygens' principle, under a condition involving N and k. We adapt the method of R. Howe for the classical wave equation, i.e. when $k \equiv 0$ (cf. [18, 21]).

Choose $x_1, x_2, \ldots, x_N$ as the usual system of coordinates on $\mathbb{R}^N$. Let

$$\mathbb{E}_{N,1} := \frac{1}{2}(\|x\|^2 - t^2), \quad \mathbb{F}_{N,1} := -\frac{1}{2}(\Delta_k - \partial_{tt}),$$

$$\mathbb{H}_{N,1} := \frac{N+1}{2} + \gamma_k + \sum_{j=1}^{N} x_j\partial_j + t\partial_t.$$

Using [16, Theorem 3.3], the following commutation relations hold

$$[\mathbb{E}_{N,1}, \mathbb{H}_{N,1}] = -2\mathbb{E}_{N,1}, \quad [\mathbb{F}_{N,1}, \mathbb{H}_{N,1}] = 2\mathbb{F}_{N,1}, \quad [\mathbb{E}_{N,1}, \mathbb{F}_{N,1}] = \mathbb{H}_{N,1}. \tag{3.17}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (3.17) gives raise to a representation Ω_k of $\mathfrak{sl}(2,\mathbb{R})$. On $\mathscr{S}(\mathbb{R}^{N+1})$, the representation Ω_k can be described as

$$\Omega_k(\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}) = \mathfrak{sl}_2^+ \oplus \mathfrak{sl}_2^0 \oplus \mathfrak{sl}_2^-, \tag{3.18}$$

where

$$\mathfrak{sl}_2^+ = \mathrm{Span}\{\mathbb{E}_{N,1}\}, \quad \mathfrak{sl}_2^0 = \mathrm{Span}\{\mathbb{H}_{N,1}\}, \quad \mathfrak{sl}_2^- = \mathrm{Span}\{\mathbb{F}_{N,1}\}.$$

The decomposition (3.18) is an instance of the Cartan decomposition

$$\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-,$$

where $\mathfrak{sl}_2^+ \simeq \Omega_k(\mathfrak{p}^+)$, $\mathfrak{sl}_2^0 \simeq \Omega_k(\mathfrak{k}_{\mathbb{C}})$, and $\mathfrak{sl}_2^- \simeq \Omega_k(\mathfrak{p}^-)$. Here $\mathfrak{k} = \mathfrak{u}(2)$, the Lie algebra of the compact group $U(2)$. The integrated form of the Lie algebra representation Ω_k is an analogue of the metaplectic representation of the universal covering $\widetilde{SL(2,\mathbb{R})}$ of the group $SL(2,\mathbb{R})$. If $(N+1)/2 + \gamma_k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, we obtain a representation of the double covering $Mp(2,\mathbb{R})$ of $SL(2,\mathbb{R})$, and if $(N+1)/2 + \gamma_k \in \mathbb{Z}$ we obtain a representation of $SL(2,\mathbb{R})$.

Remark 3.7. By [4], the Dunkl-Fourier transform is in $\widetilde{SL(2,\mathbb{R})}$, as generated above.

Recall that $(\mathcal{S}_2)$ is equivalent to the fact that the propagators P_k^{11} and P_k^{12} are supported on the light cone $\mathscr{C} = \{(x,t) \mid \|x\|^2 - t^2 = 0\}$. Next we will present our argument for the P_k^{ij}'s with $i,j = 1,2$. Since $\mathscr{C}$ is the locus of zeros of $\|x\|^2 - t^2$, then P_k^{ij} is supported on $\mathscr{C}$ if and only if

$$\mathbb{E}_{N,1}^m \cdot P_k^{ij} = 0$$

for some positive integer m, or

$$\mathbb{F}_{N,1}^m \cdot \mathscr{D}_k\mathscr{F}(P_k^{ij}) = 0$$

for some positive integer m. In the light of Proposition 3.5(i) (or Proposition 3.6(i)) this amounts to saying the distribution P_k^{ij} (or $\mathscr{D}_k\mathscr{F}(P_k^{ij})$) generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$-module. Thus the qualitative part of the strict Huygens' principle holds.

Theorem 3.8. *The strict Huygens' principle holds if and only if P_k^{ij} (or $\mathscr{D}_k\mathscr{F}(P_k^{ij})$) is supported on the light cone $\mathscr{C}$ if and only if P_k^{ij} (or $\mathscr{D}_k\mathscr{F}(P_k^{ij})$) generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$-module. In this case P_k^{ij} and $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ belong to the same module.*

***Claim* 3.9.** Strict Huygens' principle cannot hold when

$$\frac{N+1}{2} + \gamma_k \notin \mathbb{Z}.$$

To prove the claim, we need the following branching decomposition of $\mathscr{S}(\mathbb{R}^N)$ under the action of $G \times \widetilde{SL(2,\mathbb{R})}$. Those readers who are familiar with the theory of Howe reductive dual pairs [18, 19] will find that our formulation can be thought of as an analogue of Howe's theory.

Recall that $x_1, \ldots, x_N$ denotes the usual system of coordinates on $\mathbb{R}^N$. Set

$$\mathscr{H}_k := \frac{N}{2} + \gamma_k + \sum_{j=1}^N x_j\partial_j,$$

$$E := \frac{\mathscr{H}_k - \Delta_k/4 - \|x\|^2}{2}, \quad F := \frac{\mathscr{H}_k + \Delta_k/4 + \|x\|^2}{2}, \quad H := -\frac{\Delta_k}{4} + \|x\|^2.$$

Using again [16, Theorem 3.3], we can derive the following $\mathfrak{sl}(2,\mathbb{R})$-commutation relations

$$[E,H] = -2E, \qquad [F,H] = 2F, \qquad [E,F] = H. \tag{3.19}$$

What makes $\{E, F, H\}$ important is the fact that H is the infinitesimal generator of the maximal compact subgroup $SO(2, \mathbb{R})$ of $SL(2, \mathbb{R})$. Observe that $E^* = -F$ and $H^* = H$ in $L^2(\mathbb{R}^N, w_k(x)dx)$. This is a consequence of the fact that Δ_k is symmetric, while $\mathscr{H}_k^* = -\mathscr{H}_k$ as the bellow verification shows (you may require $k_\alpha \geq 1$, and after the formula is established the restriction can be dropped, i.e. back to $k_\alpha \geq 0$, by analytic continuation)

$$\begin{aligned}\int_{\mathbb{R}^N} \mathscr{H}_k f(x)g(x)w_k(x)\,dx = &-\int_{\mathbb{R}^N} f(x)\Big\{\sum_{j=1}^N x_j\partial_j g(x)\Big\}w_k(x)\,dx \\ &+\Big(\gamma_k - \frac{N}{2}\Big)\int_{\mathbb{R}^N} f(x)g(x)w_k(x)\,dx \\ &-\int_{\mathbb{R}^N} f(x)g(x)\Big\{\sum_{j=1}^N x_j\partial_j w_k(x)\Big\}\,dx,\end{aligned}$$

where, using the Parseval identity, we have

$$\sum_{j=1}^N x_j\partial_j w_k(x) = 2\sum_{j=1}^N x_j \sum_{\alpha\in\mathscr{R}^+} k_\alpha\Big(\big(\partial_j\langle x,\alpha\rangle\big)/\langle x,\alpha\rangle\Big)w_k(x) = 2\gamma_k w_k(x).$$

Equation (3.19), together with the observation above, gives raise to a unitary representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$. Similarly to Ω_k, we may describe this representation as

$$\omega_k(\mathfrak{p}^+) = E, \quad \omega_k(\mathfrak{k}_\mathbb{C}) = H, \quad \omega_k(\mathfrak{p}^-) = F.$$

Here $\mathfrak{k} = \mathfrak{so}(2)$, the Lie algebra of the compact group $SO(2,\mathbb{R})$.

For $g \in G$, denote by $\pi(g)$ the left regular action of G on $\mathscr{S}(\mathbb{R}^N)$

$$\pi(g)f(x) = f(g^{-1}x).$$

The action of G and $\mathfrak{sl}(2,\mathbb{R})$ on $\mathscr{S}(\mathbb{R}^N)$ commute.

To investigate the structure of the representation ω_k, note that for a polynomial $p \in \mathscr{P}(\mathbb{R}^N)$

$$e^{\nu\|x\|^2}p(-T_\xi(k))e^{-\nu\|x\|^2} = p(2\nu\langle\xi,\cdot\rangle - T_\xi(k)), \qquad \text{for } \nu\in\mathbb{R}.$$

This follows from the product rule (2.1). In particular, if $p(x) = \sum_{j=1}^N x_j^2$, we obtain

$$e^{\nu\|x\|^2}\Delta_k e^{-\nu\|x\|^2} = 4\|x\|^2 + \Delta_k - 4\nu\mathscr{H}_k, \qquad \text{for } \nu\in\mathbb{R}.$$

Thus, we may rewrite the representation ω_k as

$$\omega_k(\mathfrak{p}^+) = -\tfrac{1}{8}e^{\|x\|^2}\Delta_k e^{-\|x\|^2}, \tag{3.20}$$

$$\omega_k(\mathfrak{p}^-) = \tfrac{1}{8}e^{-\|x\|^2}\Delta_k e^{\|x\|^2}, \tag{3.21}$$

$$\omega_k(\mathfrak{k}) = e^{-\|x\|^2}\Big(-\frac{\Delta_k}{4} + \mathscr{H}_k\Big)e^{\|x\|^2}. \tag{3.22}$$

According to (3.21), the kernel of $\omega_k(\mathfrak{p}^-)$ consists of functions of the form $e^{-\|x\|^2}h(x)$ where h is a harmonic polynomial, i.e. $\Delta_k h = 0$. Now by (3.22), we get

$$\omega_k(\mathfrak{k})(e^{-\|x\|^2}h(x)) = e^{-\|x\|^2}\mathscr{H}_k h(x).$$

Thus $e^{-\|x\|^2}h(x)$ is an eigenvector for $w_k(\mathfrak{k})$ if and only if h is a homogeneous polynomial. If h has degree m, then

$$w_k(\mathfrak{k})\big(e^{-\|x\|^2}h(x)\big) = (m + \tfrac{N}{2} + \gamma_k)e^{-\|x\|^2}h(x).$$

Henceforth, for $m \in \mathbb{N}$, set $\mathscr{H}_m(k)$ to be the space of harmonic homogeneous polynomials on $\mathbb{R}^N$ of degree m.

On the other hand, for fixed $h \in \mathscr{H}_m(k)$, let $\mathscr{I}h := \{f(\|\cdot\|^2)h \mid f \in \mathscr{S}(\mathbb{R}^+)\}$. Since $g \circ \Delta_k \circ g^{-1} = \Delta_k$, the space $\mathscr{I}h$ is invariant under the action of G. Further, using (3.20), one can checks that $\omega_k(\mathfrak{p}^+)$ leaves $\mathscr{I}h$ invariant.

We summarize the consequences of the above computations.

Theorem 3.10. *Let $k \in \mathscr{K}^+$.*

(i) *For $N \geq 1$*

$$\mathscr{S}(\mathbb{R}^N) = \sum_{m=0}^{\infty}{}^{\oplus} \mathscr{H}_m(k) \otimes \mathscr{I},$$

where $\mathscr{I}$ denotes the space of G-invariant Schwartz functions.

(ii) *As a $G \times \widetilde{SL(2,\mathbb{R})}$-module, each space $\mathscr{H}_m(k) \otimes \mathscr{I}$ has the form*

$$\widetilde{\mathscr{H}_m}(k) \otimes \mathscr{W}_{m+\frac{N}{2}+\gamma_k},$$

where $\mathscr{W}_{m+\frac{N}{2}+\gamma_k}$ is the $\widetilde{SL(2,\mathbb{R})}$-representation of lowest weight $m + \frac{N}{2} + \gamma_k$, and $\widetilde{\mathscr{H}_m}(k) := e^{-\|x\|^2}\mathscr{H}_m(k)$. In particular, the summands are mutually orthogonal with respect to the inner product on $L^2(\mathbb{R}^N, w_k(x)dx)$.

Remark 3.11. The decomposition in (ii) could as well formulated for $L^2(\mathbb{R}^N, w_k(x)dx)$ as for the Schwartz space.

The following is then immediate.

Corollary 3.12. *Under the action of $\widetilde{SL(2,\mathbb{R})}$, the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ decomposes as*

$$\mathscr{S}(\mathbb{R}^N) = \bigoplus_{m=0}^{\infty} \dim(\widetilde{\mathscr{H}_m}(k)) \mathscr{W}_{m+\frac{N}{2}+\gamma_k},$$

where

$$\dim(\widetilde{\mathscr{H}_m}(k)) = \begin{pmatrix} m+N-1 \\ N-1 \end{pmatrix} - \begin{pmatrix} m+N-3 \\ N-1 \end{pmatrix}.$$

If $N > 1$, this is always nonzero, but if $N = 1$, it is zero for $m \geq 2$.

Clearly now the Claim 3.9 holds, since the spectrum of $\omega_k(\mathfrak{k})$ (or its dual) acting on $\mathscr{S}(\mathbb{R}^{N+1})$ (or $\mathscr{S}^*(\mathbb{R}^{N+1})$) is $(N+1)/2 + \gamma_k + \mathbb{Z}_+$, whilst the spectrum of $\omega_k(\mathfrak{k})$ (or its dual) in finite dimensional modules is contained in $\mathbb{Z}$. Thus the following is proved.

Theorem 3.13. *Strict Huygens' principle must fail when*

$$\frac{N+1}{2} + \gamma_k \notin \mathbb{Z}.$$

The above theorem leaves the possibility that Huygens' principle may holds when $(N+1)/2 + \gamma_k \in \mathbb{Z}$ by investigate whether the necessary finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$-module exist, which it turn out to be true. Indeed, using Proposition

3.5(iii) and Proposition 3.6(iii), we have

$$\begin{cases} \Big\{\sum_{\ell=1}^{N} x_\ell\partial_\ell + t\partial_t\Big\}(P_k^{ij}) = (1+j-i)P_k^{ij}, \\ \Big\{\sum_{\ell=1}^{N} x_\ell\partial_\ell + t\partial_t\Big\}(\mathscr{D}_k\mathscr{F}(P_k^{ij})) = (2\gamma_k+N+i-j)\mathscr{D}_k\mathscr{F}(P_k^{ij}), \end{cases} \quad i,j=1,2$$

and therefore

$$\begin{cases} \mathbb{H}_{N,1}P_k^{ij} = -\left(\dfrac{N+1}{2}+\gamma_k+i-j-1\right)P_k^{ij}, \\ \mathbb{H}_{N,1}\mathscr{D}_k\mathscr{F}(P_k^{ij}) = \left(\dfrac{N+1}{2}+\gamma_k+i-j-1\right)\mathscr{D}_k\mathscr{F}(P_k^{ij}). \end{cases} \quad i,j=1,2$$

Thus, if we assume $(N-1)/2+\gamma_k+i-j\in\mathbb{N}$, with $i,j=1,2$, and keeping in mind

$$\mathbb{F}_{N,1}\cdot P_k^{ij}=0, \qquad \mathbb{E}_{N,1}\cdot\mathscr{D}_k\mathscr{F}(P_k^{ij})=0,$$

we can conclude that each distribution P_k^{ij}, with $i,j=1,2$, generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathscr{S}^*(\mathbb{R}^{N+1})$ of highest weight $(N-1)/2+\gamma_k+i-j$. It is worthwhile to recall that for a finite-dimensional representation V of $SL(2,\mathbb{R})$, the operator $\mathbb{F}_{N,1}^{(\dim V-1)}$ convert a highest weight vector to a lowest weight, up to a constant [15, 36]. We now summarize all the above computations and discussions.

Theorem 3.14. *Under the assumption*

$$\frac{N-1}{2}+\gamma_k+i-j\in\mathbb{N}, \tag{3.23}$$

the tempered distribution P_k^{ij} generates an $\mathfrak{sl}(2,\mathbb{R})$-module of dimension

$$d_{i,j}(k)=\frac{N-1}{2}+\gamma_k+i-j+1, \quad i,j=1,2,$$

with highest wight vector $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ of highest weight $\left(\frac{N-1}{2}+\gamma_k+i-j\right)$. Further, for each i and j, there exists a constant $\alpha_{i,j}$ such that

$$P_k^{ij}=\alpha_{i,j}\mathbb{F}_{N,1}^{d_{i,j}(k)-1}\cdot\mathscr{D}_k\mathscr{F}(P_k^{ij}),$$

which is equivalent to

$$\mathscr{D}_k\mathscr{F}(P_k^{ij})=(-1)^{(N-1)/2+\gamma_k}\alpha_{i,j}\mathbb{E}_{N,1}^{d_{i,j}(k)-1}\cdot P_k^{ij}.$$

By taking into account the condition (3.23) for both P_k^{11} and P_k^{12}, we obtain (recall Theorem 3.8):

Theorem 3.15. (Strict Huygens' Principle) *Let u_k be the solution to the Cauchy problem (3.1) with the Cauchy data $(f,g)\in\mathscr{S}(\mathbb{R}^N)\times\mathscr{S}(\mathbb{R}^N)$. If*

$$\frac{N-3}{2}+\gamma_k\in\mathbb{N},$$

then $u_k(x,t)$ depends only on the values of $f(x\circledast_k y)$ and $g(x\circledast_k y)$ (and their derivatives) for $\|y\|=|t|$.

Now, let us consider the following Cauchy problem

$$\Delta_k u_k(x,t) = \partial_{tt} u_k(x,t), \quad u_k(x,0) = f(x), \quad \partial_t u_k(x,0) = g(x), \tag{3.24}$$

$f, g \in \mathscr{C}_R^\infty(\mathbb{R}^N)$, where $\mathscr{C}_R^\infty(\mathbb{R}^N)$ stands for the set of smooth functions with support contained in the closed ball of radius $R > 0$ about the origin. In these circumstances, Theorem 3.15 reads:

Theorem 3.16. *If $(N-3)/2 + \gamma_k \in \mathbb{N}$, the support of the solution $u_k(x,t)$ to the Cauchy problem (3.24) is contained in the conical shell*

$$\left\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |t| - R \le \|x\| \le |t| + R\right\}, \tag{3.25}$$

which is the union

$$\bigcup_{\|y\| \le R} \mathscr{C}_y \tag{3.26}$$

where $\mathscr{C}_y$ is the light cone

$$\mathscr{C}_y = \{(x,t) \mid \|x-y\| = |t|\}.$$

We start by the proof of the right hand side inequality in (3.25). Using the Paley-Wiener Theorem 2.3 for the function f, one can prove that for each $M \in \mathbb{N}$ there exists a constant α_M such that, the entire function $\xi \mapsto D_k(f(\cdot \circledast_k y))(\xi)$ satisfies

$$|D_k(f(\cdot \circledast_k y))(\xi)| \le \alpha_M (1 + \|\xi\|)^{-M} e^{\|\mathrm{Im}(\xi)\|(R + \|y\|)}.$$

Thus, $f(\cdot \circledast_k y)$ is supported in the closed ball of radius $R + \|y\|$ about the origin. Similarly for $g(\cdot \circledast_k y)$. In view of Theorem 3.2 and Theorem 3.3, we conclude that for all $k \in \mathscr{K}^+$ and $N \ge 1$, the support of the solution $u_k(x,t)$ to (3.24) is contained in $\{\ (x,t) \mid \|x\| \le R + |t|\ \}$. Next, we prove the left hand side inequality in (3.25), which holds only if $(N-3)/2 + \gamma_k \in \mathbb{N}$. By Theorem 3.15, the solution $u_k(0,t)$ depends only the values of $f(y)$ and $g(y)$ for $\|y\| = |t|$. That is

$$u_k(0,t) = 0 \qquad \text{for } |t| > R. \tag{3.27}$$

By abuse of notation we write $\tau_z(k)f(x)$ for $f(x \circledast_k -z)$. If $k \equiv 0$, $\tau_z(0)f(x) = f(x+z)$. One can checks that $\tau_z(k)$ commutes with $\Delta_k - \partial_{tt}$. Thus, if $u_k(x,t)$ is a solution to the Cauchy problem (3.24) with the Cauchy data (f,g), then $\tau_z(k)u_k(x,t)$ solves (3.24) with initial data $(\tau_z(k)f, \tau_z(k)g)$. Since $\tau_z(k)f$ and $\tau_z(k)g$ have support contained in $\overline{B(\mathbf{o}, R + \|z\|)}$, (3.27) implies that $\tau_z(k)u_k(0,t) = 0$ for $|t| > R + \|z\|$, i.e.

$$u_k(z,t) = 0 \qquad \text{for } |t| > R + \|z\|.$$

Finally, the set (3.25) coincides with the union (3.26) since: if $(x,y) \in \mathscr{C}_y$ with $\|y\| \le R$, then $\|x-y\| = |t|$ so $\|x\| \le \|x-y\| + \|y| \le |t| + R$ and $|t| = \|x-y\| \le \|x\| + R$, implies (3.25). Conversely, if (x,t) satisfies (3.25), then $(x,t) \in \mathscr{C}_y$ with $y = x - |t|\frac{x}{\|x\|} = \frac{x}{\|x\|}(\|x\| - |t|)$ which has norm less or equal to R.

However, we can prove Theorem 3.16 using another approach involving only the Paley-Wiener Theorem 2.3. We shall sketch this approach at the end of this section, and its details will be illustrate in the next section to prove the principle of energy equipartition.

Next, we go back to the Cauchy problem (3.1) where the Cauchy data $(f,g) \in \mathscr{S}(\mathbb{R}^N) \times \mathscr{S}(\mathbb{R}^N)$. It is natural to think about some connection between solutions to wave equations and spherical mean type operators. As in the classical case, we shall express the solution u_k to (3.1) in terms of what is commonly called the Dunkl-type spherical mean operator [28].

For $f \in \mathscr{C}^\infty(\mathbb{R}^N)$, denote by $f \mapsto M_f$ the Dunkl-type spherical mean operator defined by

$$M_f(x,r) := \frac{1}{d_k}\int_{S^{N-1}} f(x \circledast_k ry) w_k(y)\, d\omega(y), \quad x \in \mathbb{R}^N, r \geq 0,$$

where $d_k := \int_{S^{N-1}} w_k(x)\, d\omega(x)$. According to [31, Theorem 4.1], there exists a unique compactly supported probability measure $\sigma^k_{x,r}$ such that

$$M_f(x,r) = \int_{\mathbb{R}^N} f(\xi)\, d\sigma^k_{x,r}(\xi),$$

and

$$\operatorname{supp}(\sigma^k_{x,r}) \subseteq \bigcup_{g\in G} \{\xi \in \mathbb{R}^N \mid \|\xi - gx\| \leq r\}.$$

A sharper statement on the support of $\sigma^k_{x,r}$ is given in [31, Corollary 5.2]

$$\operatorname{supp}(\sigma^k_{x,r}) \subseteq \big\{\xi \in \mathbb{R}^N \mid \|\xi\| \geq \big|\|x\| - r\big|\big\}. \tag{3.28}$$

Before expressing the solution u_k in terms of the spherical mean operator, let us recall few known facts about the Riemann-Liouville distributions on the real line [12].

Let $\Lambda = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > 0\}$. Consider the locally integrable function on $\mathbb{R}$ defined for $\lambda \in \mathbb{C}$ by

$$x_+^{\lambda-1} := \begin{cases} x^{\lambda-1} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

For $\psi \in \mathscr{D}(\mathbb{R})$, the corresponding regular distribution

$$\langle x_+^{\lambda-1}, \psi\rangle = \int_0^\infty x^{\lambda-1}\psi(x)\, dx$$

is a holomorphic $\mathscr{D}^*(\mathbb{R})$-valued function with respect to the variable $\lambda \in \Lambda$. It admits an analytic continuation into the domain $\Lambda^* = \{\lambda \in \mathbb{C} \mid \lambda \neq 0, 1, 2, 3 \ldots\}$, where

$$\operatorname{Res}_{\lambda\to m} x_+^{\lambda-1} = \frac{(-1)^m}{m!}\delta^{(m)}(x), \qquad \text{for } m = 0, 1, 2, 3, \ldots.$$

To eliminate these poles, one can divide $x_+^{\lambda-1}$ by $\Gamma(\lambda)$. Therefore, we may define an entire $\mathscr{D}^*(\mathbb{R})$-valued function, on the complex plane, by

$$\mathbb{C} \ni \lambda \mapsto \mathbb{S}_\lambda(x) := \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathscr{D}^*(\mathbb{R}).$$

This distribution is nowadays known as the Riemann-Liouville distribution. In particular

$$\begin{aligned} \mathbb{S}_{-m}(x) &= \delta^{(m)}(x), \qquad \text{for all } m = 0, 1, 2, 3, \ldots \\ \frac{d}{dx}\mathbb{S}_\lambda(x) &= \mathbb{S}_{\lambda-1}(x). \end{aligned} \tag{3.29}$$

Now we turn our attention to the relation between u_k and the spherical mean operator. By Theorem 3.1, we know that

$$u_k(x,t) = \int_{\mathbb{R}^N} P^{11}_{k,t}(y) f(x \circledast_k y) w_k(y)\, dy + \int_{\mathbb{R}^N} P^{12}_{k,t}(y) g(x \circledast_k y) w_k(y)\, dy. \tag{3.30}$$

Since $P^{ij}_{k,-t} = (-1)^{i-j} P^{ij}_{k,t}$, we shall present proofs valid for $t > 0$, and make the suitably altered statements for $t \in \mathbb{R}$.

By [32], if $F(x) = F_0(\|x\|)$ where $F_0 : \mathbb{R}^+ \to \mathbb{C}$, then $\mathscr{D}_k F(\xi) = H_{\gamma_k+N/2-1}F_0(\|\xi\|)$, where H_α denotes the Hankel transform defined by

$$H_\alpha F_0(r) := \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^\infty F_0(s) \frac{J_\alpha(rs)}{(rs)^\alpha} s^{2\alpha+1}\, ds.$$

Here J_α denotes the Bessel function of the first kind. Thus, in terms of the spherical mean operator, we may rewrite (3.30) as

$$\begin{aligned} u_k(x,t) &= \int_0^\infty r^{2\gamma_k+N-1} \int_{S^{N-1}} P^{11}_{k,t}(ry') f(x \circledast_k ry') w_k(y') d\omega(y')\, dr \\ &\quad + \int_0^\infty r^{2\gamma_k+N-1} \int_{S^{N-1}} P^{12}_{k,t}(ry') g(x \circledast_k ry') w_k(y')\, d\omega(y')\, dr \\ &= d_k \int_0^\infty r^{2\gamma_k+N-1} H_{\gamma_k+N/2-1} F_t(r) M_f(r,x)\, dr \\ &\quad + d_k \int_0^\infty r^{2\gamma_k+N-1} H_{\gamma_k+N/2-1} G_t(r) M_g(r,x)\, dr, \end{aligned}$$

where $F_t(s) = \cos(ts)$ and $G_t(s) = \sin(ts)/s$. On the other hand, we have

$$\begin{aligned} H_\alpha F_t(r) &= \frac{1}{2^\alpha \Gamma(\alpha+1) r^\alpha} \int_0^\infty \cos(ts) J_\alpha(rs) s^{\alpha+1}\, ds \\ &= \begin{cases} \dfrac{2\sqrt{\pi}}{\Gamma(\alpha+1)} t \dfrac{(t^2-r^2)^{-\alpha-\frac{3}{2}}}{\Gamma(-\alpha-\frac{1}{2})} & \text{if } 0 < r < t \\ 0 & \text{if } 0 < t < r \end{cases} \qquad \text{(cf. [11, p. 32,formula (4)])} \\ &= \frac{2\sqrt{\pi}}{\Gamma(\alpha+1)} t \mathbb{S}_{-\alpha-\frac{1}{2}}(t^2-r^2) \\ &= \frac{\sqrt{\pi}}{\Gamma(\alpha+1)} \frac{d}{dt} \left(\mathbb{S}_{-\alpha+\frac{1}{2}}(t^2-r^2) \right). \end{aligned}$$

Similarly for G_t, we have

$$\begin{aligned} H_\alpha G_t(r) &= \begin{cases} \dfrac{\sqrt{\pi}}{\Gamma(\alpha+1)} \dfrac{(t^2-r^2)^{-\alpha-\frac{1}{2}}}{\Gamma(-\alpha+\frac{1}{2})} & \text{if } 0 < r < t \\ 0 & \text{if } 0 < t < r \end{cases} \qquad \text{(cf. [11, p. 36, formula (28)])} \\ &= \frac{\sqrt{\pi}}{\Gamma(\alpha+1)} \mathbb{S}_{-\alpha+\frac{1}{2}}(t^2-r^2). \end{aligned}$$

We summarize the above computations.

Theorem 3.17. *For all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$*

$$\begin{aligned} u_k(x,t) &= d_k \frac{\sqrt{\pi}}{\Gamma(\gamma_k+N/2)} \int_0^{|t|} r^{2\gamma_k+N-1} \frac{d}{dt} \left(\mathbb{S}_{-\gamma_k-\frac{N-3}{2}}(t^2-r^2) \right) M_f(r,x)\, dr \\ &\quad + \mathrm{sign(t)} d_k \frac{\sqrt{\pi}}{\Gamma(\gamma_k+N/2)} \int_0^{|t|} r^{2\gamma_k+N-1} \mathbb{S}_{-\gamma_k-\frac{N-3}{2}}(t^2-r^2) M_g(r,x)\, dr. \end{aligned}$$

Keeping in mind Rösler's results on the support of the measure $\sigma^k_{x,r}$ associated with M_f and M_g, Theorem 3.3 implies the following:

Theorem 3.18. (Weak Huygens' Principle) *Let $k \in \mathscr{K}^+$, $N \geq 1$, and given a point $y \in \mathbb{R}^N$.*

(i) *The solution $u_k(x,t)$ to the Cauchy problem (3.1) depends only on the values of $f(y)$ and $g(y)$ in the union*
$$\bigcup_{g\in G} \left\{ y \in \mathbb{R}^N \;\mid\; \|y - gx\| \leq |t| \right\}.$$

(ii) *A slightly weaker variant of (i) says: The solution $u_k(x,t)$ to the Cauchy problem (3.1) depends only on the values of $f(y)$ and $g(y)$ in the set*
$$\left\{ y \in \mathbb{R}^N \;\mid\; \|x\| - |t| \leq \|y\| \leq \|x\| + |t| \right\}.$$

Similarly, by (3.28), Theorem 3.15 yields to:

Theorem 3.19. (Strict Huygens' Principle) *Let $k \in \mathscr{K}^+$ and $N \geq 1$. If*
$$\frac{N-3}{2} + \gamma_k \in \mathbb{N},$$
then the solution $u_k(x,t)$ to the Cauchy problem (3.1) will depend only on the values of $f(y)$ and $g(y)$ in the set
$$\left\{ y \in \mathbb{R}^N \;\mid\; \|y\| \geq \big| \|x\| - |t| \big| \right\}.$$

Remark 3.20. (i) Note that, if the initial data (f,g) are supported inside a closed ball of radius R about the origin, then, by means of Theorem 3.19, we recover Theorem 3.16.

(ii) Let G_1 and G_2 be two finite Coxeter groups on $\mathbb{R}^N$ and $\mathbb{R}^M$, with root systems $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively. Set k_1 and k_2 to be the multiplicity functions on $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively. Consider the generalized wave equation
$$\Delta^x_{k_1} u_{k_1,k_2}(x,y) = \Delta^y_{k_2} u_{k_1,k_2}(x,y) \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M,$$
where Δ_{k_1} (resp. Δ_{k_2}) denotes the Dunkl-Laplacian operator associated with G_1 (resp. G_2). Here the superscript indicates the relevant variable. If $\frac{N-M}{2} + \gamma_{k_1} - \gamma_{k_2} - 1 \in \mathbb{N}$, then there exists a distribution T on $\mathbb{R}^N \times \mathbb{R}^M$ with singular support, i.e. T is supported on the set $\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^M \mid \sum_{i=1}^N x_i^2 = \sum_{i=1}^M y_i^2\}$, so that $(\Delta_{k_1} - \Delta_{k_2})T = \delta$.

We close this section by making the following comment. As we mentioned before, we can prove Theorem 3.16 using another method involving only the Paley-Wiener Theorem 2.3. We sketch this approach and its details will be illustrate in the next section to prove the principle of energy equipartition.

Using (3.5) and the inversion formula of the Dunkl transform, we may rewrite u_k as
$$u_k(x,t) = \int_0^\infty \left\{ \Phi_k(r,x)\cos(tr) + \frac{\Psi_k(r,x)}{r}\sin(tr) \right\} dr, \tag{3.31}$$
where
$$\Phi_k(r,x) = r^{2\gamma_k+N-1} \int_{S^{N-1}} \mathscr{D}_k f(r\xi') E_k(ix, r\xi') w_k(\xi')\, d\omega(\xi'),$$
$$\Psi_k(r,x) = r^{2\gamma_k+N-1} \int_{S^{N-1}} \mathscr{D}_k g(r\xi') E_k(ix, r\xi') w_k(\xi')\, d\omega(\xi').$$

If $(N-1)/2+\gamma_k \in \mathbb{N}$, then, for fixed x, the integral formulas for $\Phi_k(r,x)$ and $\Psi_k(r,x)$ continue analytically to even functions for $r \in \mathbb{C}$. In these circumstances, (3.31) becomes

$$u_k(x,t) = \frac{1}{2}\int_{\mathbb{R}} \left\{ \Phi_k(r,x) + \operatorname{sign}(t)\frac{\Psi_k(r,x)}{ir} \right\} e^{ir|t|}\, dr.$$

Let $r = a+ib \in \mathbb{C}$. The holomorphic extensions Φ_k and Ψ_k satisfy

$$|\Phi_k(r,x)| \le c_0(k)|r|^{2\gamma_k+N-1} e^{|b|\|x\|} \sup_{\xi' \in S^{N-1}} |\mathscr{D}_k f(r\xi')|,$$

$$\left|\frac{\Psi_k(r,x)}{r}\right| \le c_0(k)|r|^{2\gamma_k+N-2} e^{|b|\|x\|} \sup_{\xi' \in S^{N-1}} |\mathscr{D}_k g(r\xi')|.$$

If $(N-1)/2+\gamma_k = 0$, the last estimate gives a problem at $r=0$. Thus we shall exclude this case, and the condition $(N-1)/2+\gamma_k \in \mathbb{N}$ becomes $(N-3)/2+\gamma_k \in \mathbb{N}$. Indeed, the condition $(N-1)/2+\gamma_k = 0$ is equivalent to $N=1$ and $k \equiv 0$, which corresponds to the rank one classical wave equation, where it is well known that the strict Huygens' principle fails.

Applying the Paley-Wiener theorem to the Cauchy data (f,g), we conclude that, for fixed $s>0$, there exists a constant c depending only on k and the Cauchy data of u_k such that

$$|u_k(x,t)| \le c e^{-s(|t|-\|x\|-R)}, \qquad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Now the left hand side inequality in (3.25) is rather clear.

4. Energy equipartition theorem

Energy is defined in physics as the ability to do work. "Kinetic energy" corresponds to energy in the form of motion, and "potential energy" corresponds to energy in a form stored for later use. These are defined below for our wave equation (we shall not comment on any physical significance).

In this section, we show that, under an assumption involving k and N, the difference between the kinetic and potential energies of a solution to (3.1) decays like $e^{-2|t|s}$, for fixed $s>0$. Thus, the energy equipartition theorem holds. The equipartition says when $|t|$ is large, the kinetic and potential energies are both equal to the half of the (t-independent) total energy.

For the time being, we only assume $k \in \mathscr{K}^+$ and $N \ge 1$.

Let $u_k(x,t)$ be a solution to the Cauchy problem (3.1). Define the kinetic and potential energies by

$$\mathscr{K}_k[u_k](t) := \frac{1}{2}\int_{\mathbb{R}^N} |\partial_t u_k(x,t)|^2 w_k(x)\, dx,$$

$$\mathscr{P}_k[u_k](t) := \frac{1}{2}\int_{\mathbb{R}^N} \sum_{j=1}^N |T_j^x(k) u_k(x,t)|^2 w_k(x)\, dx.$$

Here the superscript x denotes the relevant variable. The total energy of u_k is by definition $\mathscr{E}_k[u_k](t) := \mathscr{K}_k[u_k](t) + \mathscr{P}_k[u_k](t)$.

Before investigate the difference between the kinetic and potential energies, we notice that $\mathscr{E}_k[u_k](t)$ is a conserved quantity, i.e. $\mathscr{E}_k[u_k](t)$ is independent of t. To see this, we express the total energy in terms of $\mathscr{D}_k(u_k(\cdot,t))(\xi)$. Since

$$\mathscr{D}_k(T_j^x(k)u_k(\cdot,t))(\xi) = -i\xi_j\mathscr{D}_k(u_k(\cdot,t))(\xi),$$

by means of the Plancherel formula, we obtain

$$\mathscr{E}_k[u_k](t) = \frac{1}{2}\int_{\mathbb{R}^N}\Big\{|\partial_t\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2 + \|\xi\|^2|\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2\Big\}w_k(\xi)\,d\xi.$$

On the other hand, since

$$\mathscr{D}_k(u_k(\cdot,t))(\xi) = \cos(t\|\xi\|)\mathscr{D}_kf(\xi) + \frac{\sin(t\|\xi\|)}{\|\xi\|}\mathscr{D}_kg(\xi), \qquad \text{for all } \ t\in\mathbb{R},$$

we compute

$$\begin{aligned}|\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2 &= \cos^2(t\|\xi\|)|\mathscr{D}_kf(\xi)|^2 + \frac{\sin^2(t\|\xi\|)}{\|\xi\|^2}|\mathscr{D}_kg(\xi)|^2 \\ &\quad + 2\frac{\cos(t\|\xi\|)\sin(t\|\xi\|)}{\|\xi\|}\mathrm{Re}\left(\mathscr{D}_kf(\xi)\overline{\mathscr{D}_kg(\xi)}\right),\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}|\partial_t\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2 &= \cos^2(t\|\xi\|)|\mathscr{D}_kg(\xi)|^2 + \|\xi\|^2\sin^2(t\|\xi\|)|\mathscr{D}_kf(\xi)|^2 \\ &\quad - 2\|\xi\|\cos(t\|\xi\|)\sin(t\|\xi\|)\mathrm{Re}\left(\mathscr{D}_kf(\xi)\overline{\mathscr{D}_kg(\xi)}\right).\end{aligned} \tag{4.2}$$

Thus we have proved

$$\begin{aligned}\mathscr{E}_k[u_k](t) &= \frac{1}{2}\int_{\mathbb{R}^N}\Big\{\|\xi\|^2|\mathscr{D}_kf(\xi)|^2 + |\mathscr{D}_kg(\xi)|^2\Big\}w_k(\xi)\,d\xi \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\Big\{\sum_{j=1}^N|T_j^x(k)f(x)|^2 + |g(x)|^2\Big\}w_k(x)\,dx,\end{aligned}$$

which is independent of the variable t.

Consider now the mater of the energy equipartition. Using (4.2) and repeating the argument used above to prove the conservation of $\mathscr{E}_k[u_k]$, we may rewrite the kinetic energy as

$$\begin{aligned}\mathscr{K}_k[u_k](t) &= \frac{1}{4}\|\mathscr{D}_k(g)\|_k^2 + \frac{1}{4}\|\langle\cdot,\cdot\rangle^{1/2}\mathscr{D}_k(f)\|_k^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N}\Big[|\mathscr{D}_kg(\xi)|^2 - \|\xi\|^2|\mathscr{D}_kf(\xi)|^2\Big]\cos(2t\|\xi\|)w_k(\xi)\,d\xi \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^N}\Big[\overline{\mathscr{D}_kf(\xi)}\mathscr{D}_kg(\xi) + \overline{\mathscr{D}_kg(\xi)}\mathscr{D}_kf(\xi)\Big]\|\xi\|\sin(2t\|\xi\|)w_k(\xi)\,d\xi,\end{aligned}$$

using the familiar trigonometric identities for double angles. Here $\|\cdot\|_k$ denotes the norm in $L^2(\mathbb{R}^N, w_k(x)dx)$. Similarly, by (4.1) we obtain

$$\begin{aligned}\mathscr{P}_k[u_k](t) &= \frac{1}{4}\|\mathscr{D}_k(g)\|_k^2 + \frac{1}{4}\|\langle\cdot,\cdot\rangle^{1/2}\mathscr{D}_k(f)\|_k^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N}\Big[\|\xi\|^2|\mathscr{D}_kf(\xi)|^2 - |\mathscr{D}_kg(\xi)|^2\Big]\cos(2t\|\xi\|)w_k(\xi)\,d\xi \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N}\Big[\overline{\mathscr{D}_kf(\xi)}\mathscr{D}_kg(\xi) + \overline{\mathscr{D}_kg(\xi)}\mathscr{D}_kf(\xi)\Big]\|\xi\|\sin(2t\|\xi\|)w_k(\xi)\,d\xi.\end{aligned}$$

Now the difference between the kinetic and potential energies is given by

$$\begin{aligned}\mathscr{K}_k[u_k](t)-\mathscr{P}_k[u_k](t)&=\frac{1}{2}\int_{\mathbb{R}^N}\left[|\mathscr{D}_kg(\xi)|^2-\|\xi\|^2|\mathscr{D}_kf(\xi)|^2\right]\cos(2t\|\xi\|)w_k(\xi)\,d\xi\\&-\frac{1}{2}\int_{\mathbb{R}^N}\left[\overline{\mathscr{D}_kf(\xi)}\mathscr{D}_kg(\xi)+\overline{\mathscr{D}_kg(\xi)}\mathscr{D}_kf(\xi)\right]\\&\qquad\times\|\xi\|\sin(2t\|\xi\|)w_k(\xi)\,d\xi.\end{aligned}\tag{4.3}$$

Using the spherical-polar coordinates $\xi=r\xi'$, we get

$$\mathscr{K}_k[u_k](t)-\mathscr{P}_k[u_k](t)=\frac{1}{2}\int_0^\infty\{\Phi_k(r)\cos(2tr)-\Psi_k(r)r\sin(2tr)\}\,dr,$$

where

$$\Phi_k(r)=r^{2\gamma_k+N-1}\int_{S^{N-1}}\Big\{|\mathscr{D}_kg(r\xi')|^2-r^2|\mathscr{D}_kf(r\xi')|^2\Big\}w_k(\xi')\,d\omega(\xi')$$

$$\Psi_k(r)=r^{2\gamma_k+N-1}\int_{S^{N-1}}\Big\{\mathscr{D}_kf(r\xi')\overline{\mathscr{D}_kg(r\xi')}+\overline{\mathscr{D}_kf(r\xi')}\mathscr{D}_kg(r\xi')\Big\}\,w_k(\xi')\,d\omega(\xi').$$

Henceforth, we will choose to work with solutions to (3.1) where the Cauchy data (f,g) belong to $\mathscr{C}^\infty(\mathbb{R}^N)$ and supported in the closed ball of radius $R>0$ about the origin. Further, by Remark 3.4, we shall often presenting proofs valid for $t>0$, and making the suitably altered statement for all t, without comment.

Since $\overline{E_k(z,w)}=E_k(\overline{z},\overline{w})$, it follows that $\xi\mapsto\overline{\mathscr{D}_kf(-\xi)}$ is the Dunkl transform of $\overline{f}$. Thus $\overline{\mathscr{D}_kf(\xi)}$, similarly $\overline{\mathscr{D}_kg(\xi)}$, belongs to the Paley-Wiener space $\mathscr{H}_R(\mathbb{C}^N)$. In particular, they can be extended to entire analytic functions on $\mathbb{C}^N$. Since $w_k(\xi')d\omega(\xi')$ is (-1)-invariant, the following lemma holds.

Lemma 4.1. *If $\frac{N-1}{2}+\gamma_k\in\mathbb{N}$, the functions Φ_k and Ψ_k continue analytically to even functions of r.*

In the light of the above lemma, we may rewrite $\mathscr{K}_k[u_k](t)-\mathscr{P}_k[u_k](t)$ as

$$\frac{1}{4}\int_{\mathbb{R}}\left[\Phi_k(r)+ir\Psi_k(r)\right]e^{2itr}\,dr.\tag{4.4}$$

Now, by the Paley-Wiener Theorem 2.3, and since S^{N-1} is compact, we conclude that for any $M\in\mathbb{N}$ there exist two constants α_M and β_M such that

$$\begin{aligned}|\Phi_k(p)|&\leq c_0(k)\alpha_M(1+|p|)^{-M}e^{2R|\mathrm{Im}(p)|},\\|p\Psi_k(p)|&\leq c_0(k)\beta_M(1+|p|)^{-M}e^{2R|\mathrm{Im}(p)|},\end{aligned}\tag{4.5}$$

with $p\in\mathbb{C}$.

Fix $s>0$. To find a bound for $\mathscr{K}_k[u_k](t)-\mathscr{P}_k[u_k](t)$, we shift the contour in the integral (4.4) from $\mathbb{R}$ to $\mathbb{R}+is$. This idea was inspired by [2]. Thus

$$\begin{aligned}\mathscr{K}_k[u_k](t)-\mathscr{P}_k[u_k](t)&=\frac{1}{4}\int_{\mathbb{R}}\{\Phi_k(r)+ir\Psi_k(r)\}\,e^{2irt}\,dr\\&=\frac{e^{-2ts}}{4}\int_{\mathbb{R}}\{\Phi_k(r+is)+i(r+is)\Psi_k(r+is)\}\,e^{2irt}\,dr.\end{aligned}$$

In view of (4.5), there exists a constant $\chi_M(k)$ such that

$$\left|\mathscr{K}_k[u_k](t)-\mathscr{P}_k[u_k](t)\right|\leq\chi_M(k)e^{-2ts}e^{2Rs}\int_{\mathbb{R}}(1+|r|)^{-M}\,dr,$$

and the following holds:

Theorem 4.2. *For* $k \in \mathscr{K}^+$ *and* $N \geq 1$, *assume that*

$$\frac{N-1}{2} + \gamma_k \in \mathbb{N}.$$

Let u_k *be a solution to the Cauchy problem (3.1), where the Cauchy data (f,g) are supported in the closed ball of radius* $R > 0$ *about the origin. Fix* $s > 0$. *Then there exists a constant* C *depending on* k *and* (f, g) *but not on* s, *such that*

$$\Big|\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t)\Big| \leq Ce^{-2s(|t|-R)}, \qquad \textit{for all } \ t \in \mathbb{R}.$$

The following is then immediate.

Theorem 4.3. (Energy Equipartition Theorem) *Under the same assumptions as in the previous theorem, we have*

$$\mathscr{K}_k[u_k](t) = \mathscr{P}_k[u_k](t) = \frac{\mathscr{E}_k[u_k](R)}{2} \qquad for \ \ |t| \geq R.$$

We close this section by making two comments. First, in the theorem above we did not exclude the case $N = 1$ if $k \equiv 0$, since the classical wave equation on $\mathbb{R} \times \mathbb{R}$ has an equipartitioned energy.

Second, it is possible to prove the energy equipartition theorem when the Cauchy data (f, g) are two Schwartz functions on $\mathbb{R}^N$. In this case Theorem 4.3 reads

$$\lim_{|t|\to\infty} \mathscr{K}_k[u_k](t) = \lim_{|t|\to\infty} \mathscr{P}_k[u_k](t) = \frac{\mathscr{E}_k[u_k](0)}{2}.$$

To see this one needs to show that the integrals in (4.3) tend to zero as $|t| \to \infty$. This follows by means of the classical Riemann-Lebesgue lemma for the Euclidean Fourier sine and cosine transforms.

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