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A non-regular Gröbner fan

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Abstract

The Gröbner fan of an ideal $I \subset k[x_1, \dots, x_n]$, defined by Mora and Robbiano, is a complex of polyhedral cones in \mathbb{R}^n . The maximal cones of the fan are in bijection with the distinct monomial initial ideals of I as the term order varies. If I is homogeneous the Gröbner fan is complete and is the normal fan of the state polytope of I . In general the Gröbner fan is not complete and therefore not the normal fan of a polytope. We may ask if the *restricted* Gröbner fan, a subdivision of $\mathbb{R}_{\geq 0}^n$, is regular i.e. the normal fan of a polyhedron. The main result of this paper is an example of an ideal in $\mathbb{Q}[x_1, \dots, x_4]$ whose restricted Gröbner fan is not regular.

1 Introduction

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k and let $I \subset R$ be an ideal. The *Gröbner fan* and the *restricted Gröbner fan* of I are n -dimensional polyhedral fans defined in [11]. The main result of this paper is the following.

Theorem 1 *The restricted Gröbner fan of the two-dimensional ideal*

$$I = \langle acd + a^2c - ab, ad^2 - c, ad^4 + ac \rangle \subset \mathbb{Q}[a, b, c, d]$$

is not the normal fan of a polyhedron.

In contrast, when the ideal I is *homogeneous* its Gröbner fan and restricted Gröbner fan are known to be normal fans of polyhedra, see Section 2.

We recall the definition of a *fan* in \mathbb{R}^n . A *polyhedron* in \mathbb{R}^n is a set of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ where A is a matrix and b is a vector. Bounded polyhedra are called *polytopes*. If $b = 0$ the set is a *polyhedral cone*.

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Definition 2 A collection C of polyhedra in \mathbb{R}^n is a polyhedral complex if:

1. all proper faces of a polyhedron $P \in C$ are in C , and
2. the intersection of any two polyhedra $A, B \in C$ is a face of A and a face of B .

A polyhedral complex is a *fan* if it only consists of cones. A simple way to construct a fan is by taking the *normal fan* of a polyhedron.

Definition 3 Let $P \subset \mathbb{R}^n$ be a polyhedron. All non-empty faces of P are of the form

$$face_\omega(P) = \{p \in P : \langle \omega, p \rangle = \max_{q \in P} \langle \omega, q \rangle\}$$

for some $\omega \in \mathbb{R}^n$. For a face F of P we define its normal cone

$$N_P(F) := \overline{\{\omega \in \mathbb{R}^n : face_\omega(P) = F\}}$$

with the closure being taken in the usual topology. The normal fan of P is the fan consisting of the normal cones $N_P(F)$ as F runs through all non-empty faces of P .

If the union of all cones in a fan is \mathbb{R}^n , the fan is said to be *complete*. It is clear that the normal fan of a polytope is complete. Not all fans arise as the normal fan of a polyhedron. Those that do are called *regular*.

If the ideal I is homogeneous, its Gröbner fan is the normal fan of a polytope known as the *state polytope* of I ([2], [12, Chapter 2]). In the general case, no similar result exists as the Gröbner fan is not complete. However, we could ask if the *restricted* Gröbner fan of I , a fan in $\mathbb{R}_{\geq 0}^n$, is regular. Theorem 1 gives an example of an ideal in $\mathbb{Q}[x_1, \dots, x_4]$ whose Gröbner fan and restricted Gröbner fan are not regular.

The definitions of the Gröbner fan and the restricted Gröbner fan appear in Section 2, and the proof of Theorem 1 is given in Section 3. For the reader unfamiliar with Gröbner fans we provide the necessary background in Section 2. It is interesting to consider what happens if we homogenize the example ideal I and project its state polytope back into \mathbb{R}^4 . In Section 4 we will point out why the normal fan of this projection is not the Gröbner fan of I . In particular we conclude for this example that the third variant of the Gröbner fan, the *extended* Gröbner fan defined in [11], does not agree with the restricted fan in the positive orthant.

An interesting corollary of the restricted Gröbner fan being regular would be an easy proof that the memoryless reverse search algorithm ([1]) can be used for enumerating the maximal cones in the fan by exploiting the structure of the underlying polyhedron. In light of Theorem 1 the fact that the reverse search method can be used requires a non-trivial proof which will appear in [5].

2 The Gröbner fan of an ideal

For $\alpha \in \mathbb{N}^n$ we use the notation $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$. By a term order on R we mean a total ordering on the monomials in R such that:

1. For all $\alpha \in \mathbb{N}^n \setminus \{0\} : 1 < x^\alpha$ and
2. For $\alpha, \beta, \gamma \in \mathbb{N}^n : x^\alpha < x^\beta \Rightarrow x^\alpha x^\gamma < x^\beta x^\gamma$.

Let \prec be a term order. For a non-zero polynomial $f \in R$ we define its *initial term*, $in_{\prec}(f)$, to be the unique maximal term of f with respect to \prec . In the same way for $\omega \in \mathbb{R}^n$ we define the *initial form*, $in_{\omega}(f)$, to be the sum of all terms of f whose exponents maximize $\langle \cdot, \omega \rangle$. The *initial ideals* of an ideal I with respect to \prec and ω are defined as

$$in_{\prec}(I) = \langle in_{\prec}(f) : f \in I \setminus \{0\} \rangle \text{ and } in_{\omega}(I) = \langle in_{\omega}(f) : f \in I \setminus \{0\} \rangle.$$

Note that $in_{\prec}(I)$ is a monomial ideal while $in_{\omega}(I)$ might not be. A monomial in $R \setminus in_{\prec}(I)$ (with coefficient 1) is called a *standard monomial* of $in_{\prec}(I)$.

Definition 4 Let $I \subset R$ be an ideal and \prec a term order on R . A generating set $\mathcal{G} = \{g_1, \dots, g_m\}$ for I is called a *Gröbner basis* for I with respect to \prec if

$$in_{\prec}(I) = \langle in_{\prec}(g_1), \dots, in_{\prec}(g_m) \rangle.$$

The Gröbner basis \mathcal{G} is *minimal* if no polynomial can be left out. A *minimal Gröbner basis* is *reduced* if the initial term of every $g \in \mathcal{G}$ has coefficient 1 and all other monomials in g are standard monomials of $in_{\prec}(I)$.

For a term order \prec and an ideal I the reduced Gröbner basis is unique and depends only on I and $in_{\prec}(I)$. We denote it by $\mathcal{G}_{\prec}(I)$.

Given I a natural equivalence relation on \mathbb{R}^n is the one induced by taking initial ideals:

$$u \sim v \Leftrightarrow in_u(I) = in_v(I).$$

We introduce the following notation for the closures of the equivalence classes:

$$C_{\prec}(I) = \overline{\{u \in \mathbb{R}^n : in_u(I) = in_{\prec}(I)\}} \text{ and}$$

$$C_v(I) = \overline{\{u \in \mathbb{R}^n : in_u(I) = in_v(I)\}}.$$

A well known fact is that for a fixed ideal I there are only finitely many sets $C_{\prec}(I)$ and they cover $\mathbb{R}_{\geq 0}^n$, see [11]. Secondly, every initial ideal $in_{\prec}(I)$ is of the form $in_{\omega}(I)$ for some $\omega \in \mathbb{R}_{> 0}^n$. Consequently, every $C_{\prec}(I)$ is of the form $C_{\omega}(I)$. A third observation is that the equivalence classes are not convex in general since we allow the vectors to be anywhere in \mathbb{R}^n :

Example 5 Let $I = \langle x_1 - 1, x_2 - 1 \rangle$. The ideal I has five initial ideals: $\langle x_1 - 1, x_2 - 1 \rangle$, $\langle x_1, x_2 \rangle$, $\langle x_1, x_2 - 1 \rangle$, $\langle x_1 - 1, x_2 \rangle$ and $\langle 1 \rangle$. In particular, for $u = (-1, 3)^T$ and $v = (3, -1)^T$ we have $in_u(I) = in_v(I) = \langle 1 \rangle$ but $in_{\frac{1}{2}(u+v)}(I) = \langle x_1, x_2 \rangle$.

Theorem 6 Let \prec be a term order and $v \in C_{\prec}(I)$ then for $u \in \mathbb{R}^n$

$$\text{in}_u(I) = \text{in}_v(I) \iff \forall g \in \mathcal{G}_{\prec}(I) : \text{in}_u(g) = \text{in}_v(g).$$

This theorem is a little more general than Proposition 2.3 in [12] as it allows the vectors to be negative. A proof will appear in [5]. Theorem 6 shows that the closures of the equivalence classes are polyhedral cones since for fixed \prec and fixed v each $g \in \mathcal{G}_{\prec}(I)$ introduces the equality $\text{in}_u(g) = \text{in}_v(g)$ which is equivalent to having u satisfy a set of linear equations and strict linear inequalities. The closure is taken by making the strict inequalities non-strict. Thus in particular, the set $C_v(I)$ is a convex polyhedral cone if it contains a strictly positive vector.

Definition 7 The Gröbner fan of an ideal $I \subset R$ is the set of the closures of all equivalence classes intersecting the positive orthant together with their proper faces.

This is a variation of the definitions appearing in the literature. The advantage of this variant is that it gives well-defined and *nice* fans in the homogeneous and non-homogeneous case simultaneously. By *nice* we mean that all cones in this fan are closures of equivalence classes. It is not clear a priori that the Gröbner fan is a polyhedral complex. The proof that the Gröbner fan is in fact a fan (polyhedral complex) will be deferred to [5]. The support of the Gröbner fan of I is called the *Gröbner region* of I .

For the purpose of this paper it is better to study the *restricted* Gröbner fan as we will see soon. Using the definition we already have together with the notion of *common refinements* of fans ([13]) it is straightforward to make a definition equivalent to the original one in [11].

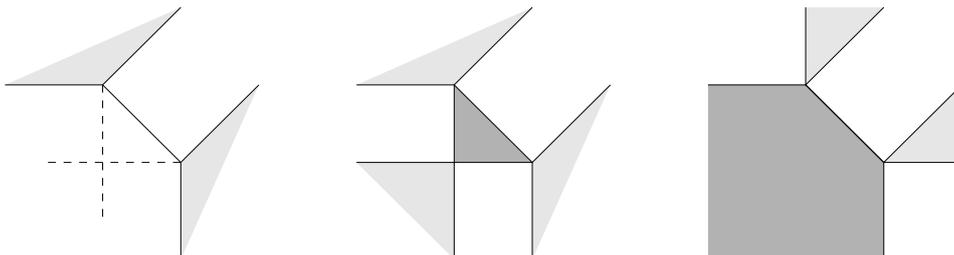
Definition 8 Let F and F' be two polyhedral fans in \mathbb{R}^n . Their common refinement is the polyhedral fan $F \wedge F' := \{C \cap C'\}_{(C,C') \in F \times F'}$.

Definition 9 (Definition 2.5 [11]) The restricted Gröbner fan of an ideal $I \subset R$ is the common refinement of the non-negative orthant $\mathbb{R}_{\geq 0}^n$ with its proper faces and the Gröbner fan of I .

The support of the restricted Gröbner fan is $\mathbb{R}_{\geq 0}^n$.

A fundamental question to ask is the following: *Is the Gröbner fan always the normal fan of a polytope?* The answer to this question is *no* since the Gröbner fan is not always complete. Even if we ask for a polyhedron instead, the answer is still *no* for trivial reasons as the following example shows.

Example 10 Let $f = x_1 + x_2 + 1$ and $I = \langle f \rangle \subset \mathbb{Q}[x_1, x_2]$. The picture to the left shows the two maximal cones in the Gröbner fan of I . Any polyhedron having these cones as normal cones will have at least a third normal cone (middle picture).



To the right the restricted Gröbner fan is shown. In this example it is the normal fan of an unbounded polyhedron.

Thus we rephrase the question for restricted Gröbner fans: *Is the restricted Gröbner fan of an ideal always the normal fan of a polyhedron?*

We note that the Gröbner fan being regular is stronger than the restricted Gröbner fan being so. This is because the normal fan of the Minkowski sum of two polyhedra is the common refinement of their normal fans. The claim follows since $\mathbb{R}_{\geq 0}^n$ with its proper faces is the normal fan of $\mathbb{R}_{\leq 0}^n$.

The above question is known to have a positive answer in the following three special cases:

- If the ideal is homogeneous the answer is *yes* since the Gröbner fan is the normal fan of the *state polytope* of I introduced by Bayer and Morrison in [2]. We should mention that in [10] it is shown that the Gröbner fan is not the normal fan of the state polytope as it was defined in [2]. Instead we should use the construction in [12, Chapter 2]. We take the Minkowski sum of the state polytope with $\mathbb{R}_{\leq 0}^n$ to get a polyhedron having the restricted Gröbner fan as its normal fan.
- The *Newton polytope*, $New(f)$, of a polynomial f is defined to be the convex hull of the exponent vectors of the monomials in f . In the case of a *principal* ideal $I = \langle f \rangle$ the Newton polytope $New(f)$ will almost have the Gröbner fan as its normal fan since two vectors $u, v \in \mathbb{R}^n$ pick out the same initial ideal of I if and only if they are maximized on the same face of $New(f)$. The only thing that keeps $New(f)$ from having the Gröbner fan of I as its normal fan is that we have not included all equivalence classes in the Gröbner fan. However, the normal fan of the Minkowski sum of $New(f)$ and $\mathbb{R}_{\leq 0}^n$ is the restricted Gröbner fan.
- A third case where we have a similar result is for *zero-dimensional* ideals. The construction of a polytope is similar but simpler than the construction in the homogeneous case as there are only a finite number of standard monomials for each initial ideal. We claim, without proof, that the following construction works: For every term order \prec construct the vector v_{\prec} equal to the negative of the sum of all exponent vectors of all standard monomials of $in_{\prec}(I)$. Take the convex hull of all v_{\prec} as we vary the term order. The Minkowski sum of this polytope with $\mathbb{R}_{\geq 0}^n$ is a polyhedron whose normal fan is the restricted Gröbner fan.

In contrast to the above, we have Theorem 1.

3 The proof

This section contains a proof of Theorem 1. We start by deducing a necessary condition for a fan to be the normal fan of a polyhedron. We then show that the restricted Gröbner fan of the ideal in the theorem violates this condition. Finally we argue that the Gröbner fan has been computed correctly.

3.1 A necessary condition

Let F be a fan in \mathbb{R}^n . Suppose F is the normal fan of a polyhedron $P \subset \mathbb{R}^n$. The non-empty faces of P are in bijection with the cones in F by taking normal cones of the faces. Adjacency is preserved in the sense that two vertices of an edge of P map to cones in F having the normal fan of the edge as a common facet. Furthermore, the edge is perpendicular to the shared facet. If a set of normals of the shared facets in F are specified, then for every bounded edge the difference between its endpoints can be expressed as some scalar times the specified normal of its normal cone. The scalars are considered to be unknowns. Since the adjacency information of the vertices of P is present in F , the bounded edge graph of P can be deduced from F . A necessary condition for F to be the normal fan of P is that every combinatorial cycle in the edge graph is a geometric cycle in space. This condition gives rise to a feasible system of inequalities on the scalars dependent on F alone.

To be more specific about the inequality system, consider the adjacency graph of the n -dimensional cones in F , or equivalently the edge graph of the supposed polyhedron P . Let $V = \{1, \dots, m\}$ denote the vertices and a subset $E \subset \{(i, j) \in V \times V : i < j\}$ denote the edges in the graph. For each shared facet, choose a normal vector $d_{(i,j)} \in \mathbb{R}^n$ such that the i th cone is on the negative side of the hyperplane with normal vector $d_{(i,j)}$ and the j th cone is on the positive side. The graph (V, E) is considered to be undirected when we define its cycles. A vector $f \in \mathbb{R}^E$ is called a flow in (V, E) if

$$\forall j \in V : \sum_{(i,j) \in E} f_{(i,j)} = \sum_{(j,k) \in E} f_{(j,k)}.$$

In other words the flow entering j is the same as the flow leaving j . The set of flows is a subspace of \mathbb{R}^E . We introduce a vector $s \in \mathbb{R}_{>0}^E$ of unknown scalars such that the true vector from vertex i to vertex j is $s_{(i,j)}d_{(i,j)}$. Each cycle in the graph can be represented by a flow $f \in \mathbb{R}^E$ being 0 on the edges not appearing in the cycle and ± 1 elsewhere depending on the relative orientation of the cycle and the edge. For such an f the condition that the cycle forms a loop in space can be expressed as:

$$\sum_{(i,j) \in E} f_{(i,j)} s_{(i,j)} d_{(i,j)} = 0. \quad (1)$$

Note that (1) is a system of n equations – one for each coordinate of $d_{(i,j)}$. If F is the normal fan of a polyhedron P , there exist positive scalars $s_{(i,j)}$ satisfying (1) for every flow f since the cycle flows span the vector space of flows. By linearity this is equivalent to having the scalars satisfy (1) for a basis of the vector space of flows rather than the entire space. In matrix form we may express the necessary condition as the system

$$As = 0 \quad \text{and} \quad s_{(i,j)} > 0 \quad \text{for all} \quad (i, j) \in E \quad (2)$$

having a solution where A is a suitable $nl \times |E|$ matrix with l being the dimension of the vector space of flows.

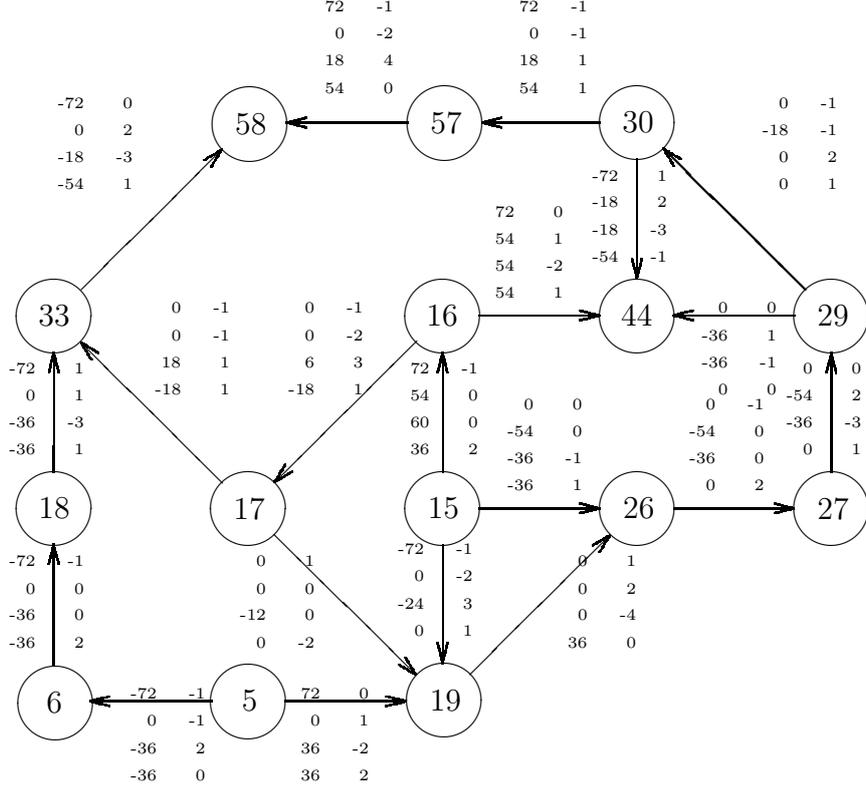


Figure 1: The certificate subgraph.

3.2 The certificate

Proof of Theorem 1: The restricted Gröbner fan of the ideal

$$I = \langle acd + a^2c - ab, ad^2 - c, ad^4 + ac \rangle \subset \mathbb{Q}[a, b, c, d]$$

has 81 full dimensional cones each corresponding to a monomial initial ideal. Their adjacency graph (V, E) has 163 edges, with each edge having an edge direction equal to a specified normal of the shared facet. We present a certificate that the fan is not the normal fan of a polyhedron. Only the subgraph in Figure 1 is needed to describe it. Two vectors are written for each edge in the subgraph. The vector to the right is the edge direction $d_{(i,j)}$ and the vectors to the left describe four flows in the subgraph.

Let V' be the set of vertices appearing in the subgraph and E' the edges. Let f^1, f^2, f^3 and f^4 denote the flows above. Suppose the restricted Gröbner fan was the normal fan of a polyhedron P . Equality system (1) implies

$$\forall (r, t) \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} : \sum_{(i,j) \in E'} f_{(i,j)}^r s_{(i,j)} d_{(i,j)}^t = 0. \quad (3)$$

In particular, the sum of the equations in (3) for $(r, t) = (1, 1), (2, 2), (3, 3), (4, 4)$ is zero. Therefore,

$$0 = \sum_{r=1}^4 \sum_{(i,j) \in E'} s_{(i,j)} d_{(i,j)}^r f_{(i,j)}^r = \sum_{(i,j) \in E'} s_{(i,j)} \sum_{r=1}^4 d_{(i,j)}^r f_{(i,j)}^r.$$

5	(10, 2, 5, 3)	18	(4, 3, 5, 4)	30	(15, 1, 3, 11)
6	(14, 4, 11, 5)	19	(5, 1, 2, 2)	33	(3, 1, 2, 3)
15	(7, 6, 5, 3)	26	(7, 1, 2, 3)	44	(7, 5, 4, 4)
16	(7, 11, 8, 4)	27	(17, 1, 4, 9)	57	(7, 1, 2, 7)
17	(5, 2, 3, 3)	29	(10, 1, 2, 6)	58	(7, 1, 3, 8)
5	6 (3,1,2,1)	16	44 (5,7,5,3)	29	30 (8,1,2,5)
5	19 (8,4,5,3)	17	19 (4,1,2,2)	29	44 (9,3,3,5)
6	18 (2,1,2,1)	17	33 (6,1,3,4)	30	44 (6,5,4,4)
15	16 (6,8,6,3)	18	33 (4,1,3,4)	30	57 (13,1,3,11)
15	19 (5,3,3,2)	19	26 (10,1,3,4)	33	58 (6,1,3,7)
15	26 (9,2,3,3)	26	27 (18,2,5,9)	57	58 (10,1,3,11)
16	17 (8,15,11,5)	27	29 (13,1,3,7)		

Figure 2: Representative weight vectors for cones in the certificate.

The local contribution at each edge except the edge (29,30) is zero because $d_{(i,j)} \cdot (f_{(i,j)}^1, f_{(i,j)}^2, f_{(i,j)}^3, f_{(i,j)}^4)^T = 0$ (check this in the picture). Consequently,

$$0 = s_{(29,30)} d_{(29,30)} \cdot f_{(29,30)} = 18s_{(29,30)}$$

implying $s_{(29,30)} = 0$. Hence the vertices 29 and 30 have the same coordinates which contradicts that P is a polyhedron with the required edge graph. \square

Remark 11 Another way to argue is by observing that we have applied the trivial direction of Farkas' lemma to (3). With A' being the 16×20 matrix representing the equalities in (3) a variant of Farkas' lemma says:

$$\exists y : y^T A' \geq 0 \text{ and } y^T A' \neq 0 \iff \nexists s > 0 : A' s = 0.$$

In our case $y = (1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1)^T$ where the four nonzero components correspond to the equations (1, 1), (2, 2), (3, 3) and (4, 4).

3.3 Correctness of the subgraph

For completeness, a positive interior point in each of the 15 maximal cones of the restricted Gröbner fan leading to the inconsistency is given in the top part of Figure 2. Further, a positive vector in the relative interior of every shared facet is given in the bottom part.

To verify the correctness of the certificate the following procedure is suggested: It is straightforward to check that the flows are flows and that the dot products of flows and listed directions are 0 except for the edge (29,30). The question is how to check the correctness of the edge subgraph and the listed directions. For each of the listed edges (i, j) with $i < j$ compute the corresponding reduced Gröbner bases \mathcal{G}_i and \mathcal{G}_j and use Theorem 6 to compute their cones C_i and C_j . Check that the

listed facet vector for the edge (i, j) is in the closure of both cones C_i and C_j and that the listed direction vector non-strictly separates C_i and C_j with C_j being on the non-negative side. Checking that the listed facet vector is in the relative interior of a facet of C_r completes the verification. The non-straightforward part of this test was implemented as a 230 line script in Singular [7]. The script itself is available on the internet, see [9].

4 Further remarks

4.1 Homogenizing the ideal

In [11] a complete fan in \mathbb{R}^n called the *extended* Gröbner fan is defined for any (not necessarily homogeneous) ideal $I \subset R$. This is done by homogenizing the ideal with a new variable. The *extended* Gröbner fan is defined as the Gröbner fan of the homogenized ideal intersected with \mathbb{R}^n . It is clear that the extended Gröbner fan is regular as the Gröbner fan of the homogenized ideal is regular and the normal fan of the projection of its polytope to $\mathbb{R}^n \times \{0\}$ is the intersection of the Gröbner fan of the homogenized ideal with $\mathbb{R}^n \times \{0\}$. Therefore our example shows that the restricted Gröbner fan of an ideal and its extended Gröbner fan need not agree in $\mathbb{R}_{>0}^n$.

In our example the procedure works as follows. We homogenize the ideal I using the variable “e” to get

$${}^hI = \langle cd^2 + ace, -c^2e + c^2d + abd, c^2e + c^3 - bce - bcd - abd + abc, -ce^2 + ad^2, \\ -c^2e + acd - abe, c^2e - bce + ac^2 - abd, c^2e + a^2c, bce + a^2b \rangle.$$

The Gröbner fan of the new ideal is a complete fan in \mathbb{R}^5 . Intersecting this fan with $\mathbb{R}^4 \times \{0\}$ we get the extended Gröbner fan, a regular fan that almost equals the Gröbner fan of I in the positive orthant. The subgraph listed for I is valid for the extended fan on all edges except the edge connecting vertex 57 and vertex 58. The vector $(10, 1, 3, 11)^T$ listed as a relative interior facet vector in the Gröbner fan of I is not in the boundary of the cone containing $(7, 1, 2, 7)^T$ in the extended fan.

4.2 A program for finding the example

A C++ program was written for finding non-regular Gröbner fans. The input for the program is a set of generators for an ideal I and the output is either a coordinatization of a polyhedron with the restricted Gröbner fan as its normal fan or a certificate for its non-existence. The program works in two steps.

- In step 1 it calls a software package being developed by the author for computing Gröbner fans of polynomial ideals. This work will appear in [5]. The package computes the maximal cones (n -dimensional) of the Gröbner fan of I storing all facets ($n - 1$ -dimensional). This is done using exhaustive search on the graph whose vertices are the maximal cones of the fan, with two maximal cones being connected if they share a facet. At each maximal cone the reduced Gröbner basis is known, its facets are computed using linear programming and

the Gröbner bases of its neighbors are computed using the local basis change procedure in [3]. A specialized implementation for toric ideals was worked out in [8].

- From the Gröbner fan computed above the inequality system (2) is deduced. Linear programming methods are used for checking its feasibility. The result is either positive scalars leading to a coordinatization of the vertices of the polyhedron or a certificate for its non-existence.

The software libraries [6] and [4] were used for doing the arithmetic and solving linear programming problems, respectively.

Knowing that we should avoid homogeneous, zero-dimensional and principal ideals, it was not hard to find the example when the C++ program had been written. A practical issue is that we are restricted to ideals with not too complex Gröbner fans as the entire edge graph must be handled by the LP-code. In looking for a 3-variable example this seems to be an unfortunate restriction as nothing interesting happens in the small manageable examples we have tried.

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