THIELE CENTRE
FOR APPLIED MATHEMATICS IN NATURAL SCIENCE

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Research Report

No. 05 | May 2005

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This Thiele Research Report is also Research Report number 457 in the Stochastics Series at Department of Mathematical Sciences, University of Aarhus, Denmark.

Loss Rates for Lévy Processes with Two Reflecting Barriers

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May 9, 2005

Abstract

Let $\{X_t\}$ be a Lévy process which is reflected at 0 and $K > 0$. The reflected process $\{V_t^K\}$ is constructed as $V_t^K = V_0^K + X_t + L_t^0 - L_t^K$ where $\{L_t^0\}$ and $\{L_t^K\}$ are the local times at 0 and K, respectively. We consider the loss rate ℓ^K, defined by $\ell^K = \mathbb{E}_{\pi_K} L_1^K$, where $\mathbb{E}_{\pi_K}$ is the expectation under the stationary measure π_K. The main result of the paper is the identification of ℓ^K in terms of π_K and the characteristic triplet of $\{X_t\}$. We also derive asymptotics of ℓ^K as $K \to \infty$ when $\mathbb{E}X_1 < 0$ and the Lévy measure of $\{X_t\}$ is light-tailed.

Key words: Lévy process, reflection, Skorokhod problem, local time, loss rate, light tail, martingale, Lundberg equation, Cramér-Lundberg approximation, asymptotics.

1 Introduction

In this paper, we investigate a Lévy process $\{X_t\}$ reflected at 0 and at $K > 0$. We construct the reflected process $\{V_t^K\}$ as

$$V_t^K = V_0^K + X_t + L_t^0 - L_t^K \tag{1.1}$$

where L_t^0 and L_t^K are the local times at the respective boundaries, given as the solutions of a Skorokhod problem (see Section 2 for details). Among possible applications, we mention finite capacity dam models, buffer systems and queueing systems, see e.g. [2], [4], [9], [10], [15] and [24] and various telecommunication models, see e.g. [10], [13] and [25].

A first quantity of interest is of course the stationary distribution π_K. There are various more or less independent studies around, see in particular [7], [9], [14], [23], [24]. The simplest representation appears to be that of [14], [23], stating that

$$\overline{\pi}_K(y) = \pi_K[y, K] = \mathbb{P}\big(X_{\tau[y-K,y)} \geq y\big),\ 0 \leq y \leq K,$$

where $\tau[u, v) = \inf\{t > 0 : X_t \notin [u, v)\},\ u \leq 0 \leq v$, and this is the one we will use. A short self-contained derivation is given in [2] pp. 393–394; see also [1], [3], [22] and [2] IX.4 and XIV.3. In this paper we are concerned with the loss rate ℓ^K, defined as

$$\ell^K = \mathbb{E}_{\pi_K} L_1^K \tag{1.2}$$

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where $\mathbb{E}_{\pi_K}$ refers to the stationary situation. This can be interpreted as the overflow rate in a dam model and as the bit loss rate in (say) a finite data buffer. Due to this importance for applications, there is much literature studying the loss rate or similar quantities, e.g. [4], [10], [13], [15] and [25].

The main result of this paper, Theorem 3.6, is an identification of the loss rate ℓ^K in terms of known characteristics of $\{X_t\}$, more precisely the Lévy triplet (see below) and π_K. It is worth noting three related problems with an easy solution:

1. Discrete time two-sided reflected random walks given by the recursion

$$V_{n+1}^K = \min\big[K, \max\big(0, V_n^K + Y_n\big)\big]$$

with $Y_1, Y_2, \ldots$ i.i.d. Here

$$\ell^K = \int_0^K \mathbb{E}[x + Y - K]^+ \, \pi_K(\mathrm{d}x) \, .$$

2. The one-sided (at 0) reflected Lévy process $V_t^\infty = V_0^\infty + X_t + L_t^0$. Here clearly in stationarity L_t^0 has to balance the drift of X_t. This simple conservation law immediately gives $\mathbb{E}_{\pi_\infty} L_1^0 = \mathbb{E} X_1$.

3. Certain Lévy processes of a special structure, more precisely such that one of $L_t^{0,c}, L_t^{K,c}$ (the continuous parts) vanishes. Here combinations of the arguments for the two previous cases easily yield an expression for ℓ^K, as will be explained later.

However, the problem of identifying the loss rate ℓ^K appears non-trivial for a completely general Lévy process with two-sided reflection. Our approach is via four linear equations where the four unknowns are $\ell^{K,c} = \mathbb{E}_{\pi_K} L_1^{K,c}$, $\ell^{K,j} = \mathbb{E}_{\pi_K} L_1^{K,j}$ (here $L_t^{K,c}, L_t^{K,j}$ are the continuos part, resp. jump part, of L_t^K) and, in the obvious notation, $\ell^{0,c}, \ell^{0,j}$. Three of the equations are straightforward whereas the fourth involves martingale optional stopping. An additional non-trivial step of the analysis is the reduction of the resulting solution to a satisfying form, which essentially involves integrals w.r.t. π_K and the Lévy measure ν as well as terms relating to the drift and a possible Brownian component.

In Section 4 we proceed by obtaining asymptotics for the loss rate ℓ^K as $K \to \infty$, assuming negative drift so that $\ell^K \to 0$, and light tails which implies that the equation $\kappa(\alpha) = 0$ has a root $\gamma > 0$, where κ is the Lévy exponent of $\{X_t\}$, see Section 2. Our result states that (we use the customary notation $a(x) \sim b(x),\ x \to \infty$ to denote $a(x)/b(x) \to 1,\ x \to \infty$) $\ell^K \sim D\mathrm{e}^{-\gamma K},\ K \to \infty$, where the expression for D is in terms of quantities relating to Wiener-Hopf factorization and fluctuation theory for $\{X_t\}$. This part of the paper can be seen as a continuous-time analogue of [17] and as a strengthening of the logarithmic asymptotics of [13] to sharp asymptotics.

Finally, in Section 5 we show that if $\mathbb{E}X_1 > 0$ then $\ell^K \to \mathbb{E}X_1,\ K \to \infty$.

2 Preliminaries

In this section we give a brief background on Lévy processes, included for easy reference. Standard references are [5] and [21]. We start with a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions, i.e. it is augmented and right-continuous. By a Lévy process $\{X_t\}$ (with respect to $\{\mathcal{F}_t\}$) we understand a real-valued process with $X_0 = 0$ and stationary independent increments which is continuous in probability, i.e. $X_{t+s} \xrightarrow{\mathbb{P}} X_t,\ s \to 0$. As a consequence of the continuity in probability, there always exists a cadlag version (right-continuous with left limits) of $\{X_t\}$. We shall throughout the paper always use the cadlag version of $\{X_t\}$. Also, $\{X_t\}$ is strong Markov. Let $\Theta = \{\alpha \in \mathbb{C} : \mathbb{E}e^{\Re(\alpha)X_1} < \infty\}$ and let, for each $\alpha \in \Theta$, $\kappa(\alpha) = \log \mathbb{E}e^{\alpha X_1}$. Then there exists $\theta \in \mathbb{R}$, $\sigma \geq 0$ and a nonnegative measure (the Lévy measure) ν on $\mathbb{R}$ with $\int_{-\infty}^{\infty}(1 \wedge y^2)\nu(\mathrm{d}y) < \infty$ such that

$$\kappa(\alpha) = \theta\alpha + \sigma^2\alpha^2/2 + \int_{-\infty}^{\infty}\left[e^{\alpha x} - 1 - \alpha x I\big(|x| \leq 1\big)\right]\nu(\mathrm{d}x).$$

For each t, the distribution of X_t is infinitely divisible and it holds that $\mathbb{E}e^{\alpha X_t} = e^{t\kappa(\alpha)}$. As usual, we will refer to $\kappa(\alpha)$ as the Lévy exponent and to (θ, σ, ν) as the characteristic triplet.

In the Skorokhod problem in (1.1), we require L_t^K to be non-decreasing and increasing only when $V_t^K = K$, and similarly for L_t^0; in addition, L_t^K, L_t^0 should be defined so that $V_t^K \in [0, K]$. Some axiomatic discussion in this framework is in [8] and [9]. We advocate here the more pragmatic view in XIV.3 in [2], to note that a two-sided reflection operator is easily constructed from the more standard one-sided one. More precisely, we need two such operators, one corresponding to upward reflection at 0 and one to downward reflection at K, and the two-sided reflected process together with its local times is then constructed by alternating between these two operators acoording to the epochs where 0 or K is hit.

A noteworthy property of $\{V_t^K\}$ is that it is strong Markov. This can be shown in a fashion similar to the proof of Corollary 2.8, p. 253 in [2].

In the sequel, martingales and stopping times are always with respect to $\{\mathcal{F}_t\}$.

3 Identification of the loss rate

With ℓ^K the loss rate as defined in (1.2), our aim is to express ℓ^K in terms of the stationary distribution π_K and the characteristic triplet.

Lemma 3.1 *If $\mathbb{E}|X_1| < \infty$, then*

$$\mathbb{E}\sup_{0\leq s\leq t} L_s^0 = \mathbb{E}L_t^0 < \infty$$

and

$$\mathbb{E}\sup_{0\leq s\leq t} L_s^K = \mathbb{E}L_t^K < \infty$$

for each t.

Proof. Since L_t^0 and L_t^K are increasing it is clear that $\sup_{0\leq s\leq t} L_s^0 = L_t^0$ and $\sup_{0\leq s\leq t} L_s^K = L_t^K$. In view of the assumption $\mathbb{E}|X_1| < \infty$ and (1.1) it is sufficient to show that $\mathbb{E}L_t^0 < \infty$. Without loss of generality, we may assume that $V_0^K = 0$. We define, recursively, $T_0 = 0$, $S_k = \inf\{t > T_{k-1} : V_t^K = K\},\ k = 1, 2, \ldots$ and $T_k = \inf\{t > S_k : V_t^K = 0\},\ k = 1, 2, \ldots$. We view $\{V_s^K\}$ as regenerative with $T_k,\ k = 0, 1, 2, \ldots$ as regeneration points, i.e. the cycles are $Y_k = [T_{k-1}, T_k)$. We split the kth cycle into two independent parts, $Y_k = Z_1 \cup Z_2$, where $Z_1 = [T_{k-1}, S_k)$ and $Z_2 = [S_k, T_k)$ (the independence follows if we apply the strong Markov property, see e.g. [5], p. 20, at S_k). It follows by Lemma 3.3, p. 256 in [2], that $\mathbb{E}L_{S_1} < \infty$ where L_t is the local time of the process which is one-sided reflected at 0. But clearly $L_{S_1}^0 = L_{S_1}$. The part Z_2 only contributes to the the local time at zero if a jump of V_t^K ends the cycle and increases L_t^0 by J, say. Then $\mathbb{E}J < \infty$ since $\mathbb{E}|X_1| < \infty$. Let N_t be the total number of cycles completed during $[0, t]$ and $\tilde{N}_t$ the number of cycles ended by a jump. Then N_t is less than or equal to the number of renewals in $[0,t]$ in a renewal process governed by the distribution of Z_2, and similarly for $\tilde{N}_t$ with Z_2 replaced by Z_1. Moreover, $\mathbb{E}Z_1, \mathbb{E}Z_2 > 0$ in view of $\mathbb{E}X_1 < \infty$. Since $\{L_t^0\}_{0\le t<S_1}, Z_2$ and J, Z_1 are pairwise independent it follows by Wald's equality, see e.g. [2], p. 414, that $\mathbb{E}L_t^0 < \infty$. □

As in the Introduction, we write

$$L_t^0 = L_t^{0,c} + L_t^{0,j} \text{ and } L_t^K = L_t^{K,c} + L_t^{K,j} \tag{3.1}$$

where $L_t^{K,c}$ is the continuous part of the local time at K, $L_t^{K,j}$ the jump part etc., i.e. $L_t^{K,j} = \sum_{0\le s\le t} \Delta L_s^K$ where $\Delta L_s^K = L_s^K - L_{s-}^K$ where $L_{s-}^K = \lim_{u\uparrow s} L_u^K$ and $L_t^{K,c} = L_t^K - L_t^{K,j}$ (note that L_t^0 and L_t^K can jump only if X_t does so and since $\{X_t\}$ is a cadlag process, $\{L_t^0\}$ and $\{L_t^K\}$ are also cadlag and can have only countably many jumps). Thus we have, in the obvious notation, $\ell^K = \ell_c^K + \ell_j^K$ and $\ell^0 = \ell_c^0 + \ell_j^0$. Let $\Delta X_s = X_s - X_{s-}$.

Some easy first observations are

$$0 = \kappa'(0) + \ell_1^{0,c} + \ell_1^{0,j} - \ell_1^{K,c} - \ell_1^{K,j}\,, \tag{3.2}$$

$$\ell^{0,j} = -\int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^{-x} (x+y)\,\nu(\mathrm{d}y)\,, \tag{3.3}$$

$$\ell^{K,j} = \int_0^K \pi_K(\mathrm{d}x) \int_{K-x}^{\infty} (x+y-K)\,\nu(\mathrm{d}y)\,. \tag{3.4}$$

Indeed, (3.2) follows just by taking expectations in (1.1), whereas e.g. (3.4) follows by noting that the inner integral is the contribution from an upward jump occuring at $V_t^K = x$. This shows that if at least one of $\ell_1^{0,c}, \ell_1^{K,c}$ disappears, then all four components of ℓ^K are explicit. However, if $\ell_1^{0,c}, \ell_1^{K,c}$ are both non-zero, we need one more equation, and this will be obtained by optional stopping of a martingale introduced in [12].

Remark 3.2 A simple example where $\ell^{K,c} = 0$ is a subordinator with an added negative drift. In general, contributions to $\ell^{K,c}$ may come from a drift term, a Brownian component or from the compensation scheme needed to construct the Lévy process when $\int_0^1 |x|\,\nu(\mathrm{d}x) = \infty$. Similar remarks apply to $\ell^{0,c}$. □

For technical reasons, the above outline of our approach needs, however, some modifications. One is that we have to treat small and large jumps separately. To this end, let L be a constant (not to be confused with L_t^K etc.!) satisfying $L > K \wedge 1$. The Lévy exponent $\kappa(\alpha)$ can then be rewritten as

$$\theta_L\alpha + \sigma^2\alpha^2/2 + \int_{-\infty}^{\infty} \bigl[\mathrm{e}^{\alpha x} - 1 - \alpha x I\bigl(|x|\le L\bigr)\bigr]\nu(\mathrm{d}x),\ \alpha\in\Theta, \tag{3.5}$$

where $\theta_L = \theta + \int_1^L x\nu(\mathrm{d}x) + \int_{-L}^{-1} x\nu(\mathrm{d}x)$. We split ΔL_t^K into two parts, $\underline{\Delta} L_t^K$ and $\overline{\Delta} L_t^K$, corresponding to $\Delta X_s \in [0,L]$ and $\Delta X_s \in (L,\infty)$, respectively, and ΔL_t^0 into $\underline{\Delta} L_t^0$ and $\overline{\Delta} L_t^0$ corresponding to $\Delta X_s \in [-L,0]$ and $\Delta X_s \in (-\infty,-L)$, respectively. Let $\underline{\ell}_j^K = \mathbb{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^K$, $\overline{\ell}_j^K = \mathbb{E}\sum_{0\le s\le 1}\overline{\Delta} L_s^K$, $\underline{\ell}_j^0 = \mathbb{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^0$ and $\overline{\ell}_j^0 = \mathbb{E}\sum_{0\le s\le 1}\overline{\Delta} L_s^0$. Then $\ell_j^K = \underline{\ell}_j^K + \overline{\ell}_j^K$ and $\ell_j^0 = \underline{\ell}_j^0 + \overline{\ell}_j^0$.

Proposition 3.3 *Assume that $\mathbb{E}|X_1| < \infty$. For each t, let M_t be the random variable*

$$\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V_s^K}\mathrm{d}s + \mathrm{e}^{\alpha V_0^K} - \mathrm{e}^{\alpha V_t^K} + \alpha\int_0^t \mathrm{e}^{\alpha V_s^K}\mathrm{d}L_s^{0,c}$$
$$+ \sum_{0\le s\le t} \mathrm{e}^{\alpha V_s^K}(1 - \mathrm{e}^{-\alpha\Delta L_s^0}) - \alpha\int_0^t \mathrm{e}^{\alpha V_s^K}\mathrm{d}L_s^{K,c} + \sum_{0\le s\le t} \mathrm{e}^{\alpha V_s^K}(1 - \mathrm{e}^{\alpha\Delta L_s^K}).$$

Then

$$M_t = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V_s^K}\mathrm{d}s + \mathrm{e}^{\alpha V_0^K} - \mathrm{e}^{\alpha V_t^K} + \alpha L_t^{0,c}$$
$$+ \sum_{0\le s\le t}(1-\mathrm{e}^{-\alpha\Delta L_s^0}) - \alpha\mathrm{e}^{\alpha K}L_t^{K,c} + \mathrm{e}^{\alpha K}\sum_{0\le s\le t}(1-\mathrm{e}^{\alpha\Delta L_s^K})$$

and for each $\alpha\in\Theta$, $\{M_t\}$ *is a zero mean martingale.*

Proof. The first claim follows immediately if we observe that $V_s^K = K$ just after a jump of $L_s^{K,j}$ and $V_s^K = 0$ just after a jump of $L_s^{0,j}$. By Lemma 3.1 it follows that $Y_t = L_t^0 - L_t^K$ is the difference between two increasing functions which are bounded a.s. and thus Y_t is a.s. of bounded variation. Also, $\{Y_t\}$ is cadlag and adapted to $\{\mathcal{F}_t\}$. It follows by Theorem 3.1, p. 255 in [2] or Theorem 1 in [12] that $\{M_t\}$ is a local martingale. The second claim follows by dominated convergence if we show that $\mathbb{E}\sup_{0\le s\le t} M_s < \infty$, see e.g. [19], Theorem 51, p. 38. Since $V_t^K \le K$, $\mathbb{E}L_t^{0,c} < \infty$ and $\mathbb{E}L_t^{K,c} < \infty$ (by the preceding lemma), it suffices to show that $\mathbb{E}\sum_{0\le s\le t}|1-\mathrm{e}^{-\alpha\Delta L_s^{0,j}}| < \infty$ and $\mathbb{E}\sum_{0\le s\le t}|1-\mathrm{e}^{\alpha\Delta L_s^{K,j}}| < \infty$ since both sums are increasing. The details for the two different barriers are similar so we treat only the one at K. It is clear that $\Delta L_s^K = \left(\Delta X_s - K + V_{s-}^K\right)^+$ (here $x^+ = x\vee 0$). Take $L > K\vee 1$ and let $\overline{\Delta} X_s$ consist of jumps of $\{X_t\}$ larger than L. We get

$$\mathbb{E}\sum_{0\le s\le t}|1-\mathrm{e}^{\alpha\Delta L_s^K}| = \mathbb{E}\sum_{0\le s\le t}|1-\mathrm{e}^{\alpha\underline{\Delta} L_s^K}| + \mathbb{E}\sum_{0\le s\le t}|1-\mathrm{e}^{\alpha(\overline{\Delta}X_s - K + V_{s-}^K)}|, \tag{3.6}$$

and for the first part of the right hand side of (3.6) we have

$$\mathbb{E}\sum_{0\le s\le t}|1-\mathrm{e}^{\alpha\underline{\Delta} L_s^K}| = |\alpha|\mathbb{E}\sum_{0\le s\le t}\mathrm{O}\big(\underline{\Delta} L_s^K\big) < \infty$$

since $\mathbb{E}L_t^{K,j} < \infty$ and furthermore,

$$\begin{aligned}\mathbb{E}\sum_{0\le s\le t}|1-\mathrm{e}^{\alpha(\overline{\Delta}X_s - K + V_{s-}^K)}| &\le t\int_L^\infty \nu(\mathrm{d}y) + \mathbb{E}\sum_{0\le s\le t}\mathrm{e}^{\Re(\alpha)(-K+V_{s-}^K)}\mathrm{e}^{\Re(\alpha)\overline{\Delta}X_s}\\ &\le t\int_L^\infty \nu(\mathrm{d}y) + \sup_{x\in[-K,0]}\mathrm{e}^{\Re(\alpha)x}\mathbb{E}\sum_{0\le s\le t}\mathrm{e}^{\Re(\alpha)\overline{\Delta}X_s}\\ &= t\int_L^\infty \nu(\mathrm{d}y) + t\sup_{x\in[-K,0]}\mathrm{e}^{\Re(\alpha)x}\int_L^\infty \mathrm{e}^{\Re(\alpha)y}\nu(\mathrm{d}y) < \infty\end{aligned}$$

since $\alpha\in\Theta$ and $\int_L^\infty \nu(\mathrm{d}y) < \infty$. □

We need the following two lemmas for the proof of Theorem 3.6 below.

Lemma 3.4 *Let* $\alpha\in\Theta$. *Then* ℓ^K *satisfies the following equation:*

$$\begin{aligned}\alpha(1-\mathrm{e}^{\alpha K})\ell^K = &-\kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V_0^K} + \alpha\kappa'(0) - \alpha\mathrm{e}^{\alpha K}\overline{\ell}_j^K + \alpha\overline{\ell}_j^0\\ &+\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta}L_s^{K\,2} + \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta}L_s^{0\,2}\\ &-\mathrm{e}^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}L_s^K}) - \mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L_s^0}) + \mathrm{o}(\alpha^2)\end{aligned}\tag{3.7}$$

where $\mathrm{o}(\alpha^2)/\alpha^2 \to 0$ *if* $\alpha\to 0$ *through values in* Θ, *e.g. if* $\alpha = \mathrm{i}t,\ t\to 0$.

Proof. If we take $t = 1$ in $\{M_t\}$ in Proposition 3.3 and use the stationarity of $\{V_t^K\}$ we get

$$0 = \kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V_0^K} + \alpha\ell_c^0 + \mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\Delta L_s^0}) - \alpha\mathrm{e}^{\alpha K}\ell_c^K + \mathrm{e}^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\Delta L_s^K}). \tag{3.8}$$

We write

$$\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\Delta L_s^K}) = \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\underline{\Delta} L_s^K}) + \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta} L_s^K}) \tag{3.9}$$

$$\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\Delta L_s^0}) = \sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\underline{\Delta} L_s^0}) + \sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta} L_s^0}) \tag{3.10}$$

and apply the expansion

$$\mathrm{e}^{\alpha x} = 1 + \alpha x + \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^3}{6}\mathrm{e}^{\theta\alpha x},\ \ \theta\in[-1,1], \tag{3.11}$$

to the first parts of the right hand sides of (3.9) and (3.10) and get for the part in (3.9):

$$\begin{aligned}
\mathrm{e}^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\underline{\Delta} L_s^K}) &= \mathrm{e}^{\alpha K}\mathbb{E}\Big[-\alpha\sum_{0\le s\le 1}\underline{\Delta} L_s^K - \frac{\alpha^2}{2}\sum_{0\le s\le 1}\underline{\Delta} L_s^{K2}\Big] + \mathrm{o}(\alpha^2)\\
&= -\alpha\mathrm{e}^{\alpha K}\underline{\ell}_j^K - \mathrm{e}^{\alpha K}\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^{K2} + \mathrm{o}(\alpha^2)\\
&= -\alpha\mathrm{e}^{\alpha K}(\ell_j^K - \overline{\ell}_j^K) - \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^{K2} + \mathrm{o}(\alpha^2),
\end{aligned} \tag{3.12}$$

since $\mathbb{E}\sum_{0\le s\le 1}\alpha^3(\underline{\Delta} L_s^K)^3\mathrm{e}^{\theta\alpha\underline{\Delta} L_s^K}/6 = \mathrm{o}(\alpha^2)$, $\ell_j^K = \underline{\ell}_j^K + \overline{\ell}_j^K$ and $\mathrm{e}^{\alpha K}\alpha^2/2 = \alpha^2/2 + \mathrm{o}(\alpha^2)$. We proceed similarly for the part in (3.10) and get

$$\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\underline{\Delta} L_s^0}) = \alpha(\ell_j^0 - \overline{\ell}_j^0) - \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^{02} + \mathrm{o}(\alpha^2). \tag{3.13}$$

If we combine (3.8), (3.9), (3.10), (3.12) and (3.13) we get

$$\begin{aligned}
0 = {}& \kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V_0^K} + \alpha\ell^0 - \alpha\mathrm{e}^{\alpha K}\ell^K - \alpha\overline{\ell}_j^0 + \alpha\mathrm{e}^{\alpha K}\overline{\ell}_j^K\\
&- \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^{K2} - \frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta} L_s^{02} + \mathrm{e}^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta} L_s^K})\\
&+ \mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta} L_s^0}) + \mathrm{o}(\alpha^2).
\end{aligned}$$

The claim now follows if we use (1.1) to substitute $\ell^0 = \ell^K - \kappa'(0)$ and rearrange terms. □

Lemma 3.5 *Let $\alpha\in\Theta$ and define, for $x > 0$, $\overline{\nu}(x) = \nu\big((x,\infty)\big)$ and, for $x < 0$, $\underline{\nu}(x) = \nu\big((-\infty,x)\big)$. In stationarity it then holds that*

$$\begin{aligned}
\kappa(\alpha)\mathbb{E}\mathrm{e}^{\alpha V_t^K} = {}& \mathrm{o}(\alpha^2) + \int_{-\infty}^{\infty}(\mathrm{e}^{\alpha y}-1)I\big(|y|\ge L\big)\nu(\mathrm{d}y)\\
&+ \alpha\Big(\theta_L + \int_0^K x\pi_K(\mathrm{d}x)\int_{-\infty}^{\infty}(\mathrm{e}^{\alpha y}-1)I\big(|y|\ge L\big)\nu(\mathrm{d}y)\Big)\\
&+ \frac{\alpha^2}{2}\Big(2\theta_L + \sigma^2 + \int_0^K x^2\pi_K(\mathrm{d}x)\int_{-\infty}^{\infty}(\mathrm{e}^{\alpha y}-1)I\big(|y|\ge L\big)\nu(\mathrm{d}y) + \int_{-L}^{L}y^2\nu(\mathrm{d}y)\Big),
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
\mathrm{e}^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}L_s^K}) &= \mathrm{e}^{\alpha K}\int_0^K \pi_K(\mathrm{d}x)\int_L^\infty (1-\mathrm{e}^{\alpha(y-K+x)})\nu(\mathrm{d}y)\\
&= \mathrm{e}^{\alpha K}\overline{\nu}(L) - \int_0^K \mathrm{e}^{\alpha x}\pi_K(\mathrm{d}x)\int_L^\infty \mathrm{e}^{\alpha y}\nu(\mathrm{d}y) = (1+\alpha K+\alpha^2K^2/2)\overline{\nu}(L)\\
&\quad -\Big(1+\alpha\int_0^K x\pi_K(\mathrm{d}x)+\alpha^2/2\int_0^K x^2\pi_K(\mathrm{d}x)\Big)\int_L^\infty \mathrm{e}^{\alpha y}\nu(\mathrm{d}y)+\mathrm{o}(\alpha^2), \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L_s^0}) &= \int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^{-L}(1-\mathrm{e}^{\alpha(x+y)})\nu(\mathrm{d}y)\\
&= \underline{\nu}(-L) - \int_0^K \mathrm{e}^{\alpha x}\pi_K(\mathrm{d}x)\int_{-\infty}^{-L}\mathrm{e}^{\alpha y}\nu(\mathrm{d}y) = \underline{\nu}(-L)\\
&\quad -\Big(1+\alpha\int_0^K x\pi_K(\mathrm{d}x)+\alpha^2/2\int_0^K x^2\pi_K(\mathrm{d}x)\Big)\int_{-\infty}^{-L}\mathrm{e}^{\alpha y}\nu(\mathrm{d}y)+\mathrm{o}(\alpha^2), \qquad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
\alpha\mathrm{e}^{\alpha K}\overline{\ell}_j^K &= \alpha\mathrm{e}^{\alpha K}\int_0^K \pi_K(\mathrm{d}x)\int_L^\infty (y-K+x)\nu(\mathrm{d}y)\\
&= (\alpha+\alpha^2K)\int_0^K \pi_K(\mathrm{d}x)\int_L^\infty (y-K+x)\nu(\mathrm{d}y)+\mathrm{o}(\alpha^2), \qquad (3.17)
\end{aligned}
$$

$$
\alpha\overline{\ell}_j^0 = -\alpha\int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^{-L}(x+y)\nu(\mathrm{d}y), \qquad (3.18)
$$

$$
\alpha\kappa'(0) = \alpha\theta_L + \alpha\int_{-\infty}^{\infty} yI\big(|y|\ge L\big)\nu(\mathrm{d}y), \qquad (3.19)
$$

$$
\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta}L_s^{K\,2} = \frac{\alpha^2}{2}\int_0^K \pi_K(\mathrm{d}x)\int_{K-x}^{L}(y-K+x)^2\nu(\mathrm{d}y), \qquad (3.20)
$$

$$
\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}\underline{\Delta}L_s^{0\,2} = \frac{\alpha^2}{2}\int_0^K \pi_K(\mathrm{d}x)\int_{-L}^{-x}(x+y)^2\nu(\mathrm{d}y). \qquad (3.21)
$$

Proof. We have that $\kappa(\alpha) = \theta_L\alpha + \sigma^2\alpha^2/2 + \int_{-\infty}^{\infty}\big[\mathrm{e}^{\alpha x}-1-\alpha xI\big(|x|\le L\big)\big]\nu(\mathrm{d}x)$ and $\mathbb{E}\mathrm{e}^{\alpha V_s^K} = \int_0^K \mathrm{e}^{\alpha x}\pi_K(\mathrm{d}x)$. We split $(-\infty,\infty)$ into $(-\infty,-L)$, $[-L,L]$ and (L,∞) and apply the expansion in (3.11) to the integrands corresponding to the compact sets $[-L,L]$ and $[0,K]$. (3.14) then follows easily. For (3.15), (3.16), (3.17) and (3.18), we first use the stationarity of $\{V_t^K\}$ and that the domain of integration is bounded away from the origin and then (3.11) if necessary. (3.19) is obvious and (3.20) and (3.21) follow if we note that the processes $\sum_{0\le s\le t}\underline{\Delta}L_s^{K\,2}$ and $\sum_{0\le s\le t}\underline{\Delta}L_s^{0\,2}$ are of bounded variation and then use the stationarity once more. □

We are now prepared for the proof of our main result.

Theorem 3.6 *Let $\{X_t\}$ be a Lévy process and ℓ^K the loss rate as in (1.2). If $\int_1^\infty y\nu(\mathrm{d}y) = \infty$ then $\ell^K=\infty$ and otherwise*

$$
\ell^K = \frac{\kappa'(0)}{K}\int_0^K x\pi_K(\mathrm{d}x) + \frac{\sigma^2}{2K} + \frac{1}{2K}\int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^{\infty}\varphi_K(x,y)\nu(\mathrm{d}y) \qquad (3.22)
$$

where

$$\varphi_K(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y \le -x \\ y^2 & \text{if } -x < y < K-x \\ 2y(K-x)-(K-x)^2 & \text{if } y \ge K-x. \end{cases}$$

Proof. The first claim follows immediately if we note that the jumps larger than $L > K$ of $\{X_t\}$ must contribute to ℓ^K by

$$\int_0^K \pi_K(\mathrm{d}x)\int_L^\infty (y-K+x)\nu(\mathrm{d}y) \ge \int_L^\infty (y-K)\nu(\mathrm{d}y) = \infty$$

if $\int_1^\infty y\nu(\mathrm{d}y) = \infty$. If we apply Lemma 3.4 and Lemma 3.5 and identify the terms in the right hand side of (3.7), we get

$$\begin{aligned} \alpha(1-\mathrm{e}^{\alpha K})\ell^K &= -\theta_L\alpha^2\int_0^K x\pi_K(\mathrm{d}x) - \frac{\sigma^2\alpha^2}{2} - \frac{\alpha^2}{2}\int_0^K \pi_K(\mathrm{d}x)\int_0^{K-x} y^2\nu(\mathrm{d}y) \\ &- \frac{\alpha^2}{2}\int_0^K \pi_K(\mathrm{d}x)\int_{-x}^0 y^2\nu(\mathrm{d}y) + \frac{\alpha^2}{2}\int_0^K x^2\pi_K(\mathrm{d}x)\overline{\nu}(L) + \frac{\alpha^2}{2}\int_0^K x^2\pi_K(\mathrm{d}x)\underline{\nu}(-L) \\ &+ \frac{\alpha^2}{2}\int_0^K \pi_K(\mathrm{d}x)\int_{K-x}^L ((x-K)^2+2y(x-K))\nu(\mathrm{d}y) - \alpha^2 K\int_L^\infty y\nu(\mathrm{d}y) \\ &+ \frac{\alpha^2}{2}\int_0^K \pi_K(\mathrm{d}x)\int_{-L}^{-x}(x^2+2xy))\nu(\mathrm{d}y) + \alpha^2 K\int_0^K (K-x)\pi_K(\mathrm{d}x)\overline{\nu}(L) + \mathrm{o}(\alpha^2). \end{aligned} \tag{3.23}$$

We now divide both sides of (3.23) by $\alpha(1-\mathrm{e}^{\alpha K})$ and let first $\alpha \to 0$ and then $L \to \infty$ and get the limit (note that $\theta_L \to \kappa'(0),\ L \to \infty$)

$$\ell^K = \frac{\kappa'(0)}{K}\int_0^K x\pi_K(\mathrm{d}x) + \frac{\sigma^2}{2K} + \frac{1}{2K}\int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^\infty \varphi_K(x,y)\nu(\mathrm{d}y).$$

This completes the proof. □

Remark 3.7 In principle, the missing fourth equation complementing (3.2)–(3.4) is already available in the form $\mathbb{E}M_1 = 0$ once the martingale property of M_t has been established. However, the dependence on α introduces an arbitrariness in the resulting expression for ℓ^K which motivates considering the limit α to remove it. It is tempting instead to just note that $M_t^0 = \lim_{\alpha\to 0} M_t/\alpha$ should be a martingale so that the fourth equation could be taken as $\mathbb{E}M_1^0 = 0$. However, one easily gets

$$M_t^0 \ = \ t\kappa'(0) + V_0^K - V_t^K + L_t^0 - L_t^K \ = \ t\kappa'(0) - X_t$$

so that $\mathbb{E}M_1^0 = 0$ just is the uninformative standard formula for expressing $\mathbb{E}X_t$ in terms of $\kappa(\cdot)$. □

Next we consider the special case where $\{X_t\}$ is light-tailed (by light-tailed, we mean that $\Theta \cap \{z \in \mathbb{C} : \Re(z) > 0\} \neq \emptyset$). Then we have the following alternative expression:

Theorem 3.8 *Assume that $\{X_t\}$ is light-tailed with non-zero drift. Let γ be the real non-zero root of the equation $\kappa(\alpha) = 0$. Then*

$$\ell^K \ = \ \frac{\mathrm{e}^{\gamma K}}{\mathrm{e}^{\gamma K}-1}I_1 + \frac{1}{\mathrm{e}^{\gamma K}-1}I_2 + \frac{\gamma^{-1}\mathrm{e}^{\gamma K}}{\mathrm{e}^{\gamma K}-1}I_3 + \frac{\gamma^{-1}}{\mathrm{e}^{\gamma K}-1}I_4 - \frac{\kappa'(0)}{\mathrm{e}^{\gamma K}-1}$$

where

$$\begin{aligned}
I_1 &= \int_0^K \pi_K(\mathrm{d}x) \int_{K-x}^{\infty} (y-K+x)\nu(\mathrm{d}y) < \infty \\
I_2 &= \int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^{-x} (x+y)\nu(\mathrm{d}y) < \infty \\
I_3 &= \int_0^K \pi_K(\mathrm{d}x) \int_{K-x}^{\infty} (1-\mathrm{e}^{\gamma(y-K+x)})\nu(\mathrm{d}y) < \infty \\
I_4 &= \int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^{-x} (1-\mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y) < \infty.
\end{aligned}$$

Proof. Let $\varepsilon > 0$. Then the contribution to ℓ^K of the jumps larger than ε is

$$I_1^\varepsilon = \int_0^K \pi_K(\mathrm{d}x) \int_{(K-x)\vee\varepsilon}^{\infty} (y-K+x)\nu(\mathrm{d}x).$$

If $I_1 = \infty$, it follows by monotone convergence that $I_1^\varepsilon \to \infty$, $\varepsilon \downarrow 0$ which contradicts $\ell^K < \infty$. A similar argument applies to

$$I_2^\varepsilon = \int_0^K \pi_K(\mathrm{d}x) \int_{-\infty}^{-(x\vee\varepsilon)} (x+y)\nu(\mathrm{d}y)$$

to show that $I_2 < \infty$. Let $I_3^\varepsilon, I_4^\varepsilon$ be defined in the obvious way. We truncate the Lévy measure at ε and $-\varepsilon$. The jumps of $\{X_t\}$ then correspond to either $(-\infty, -\varepsilon)$, $[-\varepsilon, 0]$, $[0, \varepsilon]$ or (ε, ∞). By arguing precisely as when we derived (3.7), we get, if we take $\alpha = \gamma$,

$$\gamma(\mathrm{e}^{\gamma K} - 1)\ell^K = -\gamma\kappa'(0) + \gamma\mathrm{e}^{\gamma K} I_1^\varepsilon + \gamma I_2^\varepsilon + \mathrm{e}^{\gamma K} I_3^\varepsilon + I_4^\varepsilon + \mathrm{O}(\varepsilon).$$

If we let $\varepsilon \downarrow 0$ and apply monotone convergence once again, we get that $I_3, I_4 < \infty$ and also that

$$\gamma(\mathrm{e}^{\gamma K} - 1)\ell^K = -\gamma\kappa'(0) + \gamma\mathrm{e}^{\gamma K} I_1 + \gamma I_2 + \mathrm{e}^{\gamma K} I_3 + I_4,$$

from which the claim follows easily. □

Theorem 3.8 plays a crucial rôle when we derive asymptotics in the next section.

Example 3.9 Let us consider Brownian motion with drift μ and variance 1, i.e. $\kappa(\alpha) = \mu\alpha + \alpha^2/2$. Then $\gamma = -2\mu$ and Theorem 3.8 gives us that $\ell^K = -\mu/(\mathrm{e}^{-2\mu K} - 1)$. On the other hand, it is known, see e.g. [2], p. 394, that π_K has density $2\mu\mathrm{e}^{2\mu x}/(\mathrm{e}^{2\mu K} - 1)$, $0 \leq x \leq K$, and Theorem 3.6 gives us

$$\begin{aligned}
\ell^K &= \frac{1}{2K} + \frac{\mu}{K}\int_0^K x\frac{2\mu\mathrm{e}^{2\mu x}}{\mathrm{e}^{2\mu K} - 1}\mathrm{d}x = \frac{1}{2K} + \frac{\mu}{(\mathrm{e}^{2\mu K} - 1)}\left[\mathrm{e}^{2\mu K} - \frac{\mathrm{e}^{2\mu K} - 1}{2\mu K}\right] \\
&= \frac{1}{2K} + \frac{\mu\mathrm{e}^{2\mu K}}{\mathrm{e}^{2\mu K} - 1} - \frac{1}{2K} = \frac{-\mu}{\mathrm{e}^{-2\mu K} - 1}.
\end{aligned}$$

□

4 Asymptotic loss rate in the presence of light tails

In the following, we assume negative drift, i.e. $\mathbb{E}X_1 = \kappa'(0) < 0$, and let

$M_t = \sup_{0 \leq s \leq t} X_s$, $M = M_\infty = \sup_{0 \leq t < \infty} X_t$.

$m_t = \inf_{0\le s\le t} X_s$, $m = m_\infty = \inf_{0\le t<\infty} X_t$.

$\tau_+(u) = \inf\{t > 0 : X_t > u\}$, $\tau_+^w(u) = \inf\{t > 0 : X_t \ge u\},\ u \ge 0.$

$\tau_-^s(-y) = \inf\{t > 0 : X_t < -y\},\ y \ge 0.$

The overshoot of level u, $B(u) = X_{\tau_+(u)} - u,\ u \ge 0.$

The weak overshoot of level u, $B^w(u) = X_{\tau_+^w(u)} - u,\ u \ge 0$

$B(\infty)$ a r.v. having the limiting distribution (if it exists) of $B(u)$ as $u \to \infty$.

Further, we will assume that the Lundberg equation $\kappa(\gamma) = 0$ has a solution $\gamma > 0$ with $\kappa'(\gamma) < \infty$. We let $\mathbb{P}_L$ and $\mathbb{E}_L$ correspond to a measure which is exponentially tilted by γ, i.e.

$$\mathbb{P}(G) = \mathbb{E}_L\big[\mathrm{e}^{-\gamma X_\tau}; G\big] \tag{4.1}$$

when τ is a stopping time and $G \in \mathcal{F}_\tau,\ G \subseteq \{\tau < \infty\}$ where $\mathcal{F}_\tau$ is the stopping time σ-field, see [2], Ch. XIII. Note that $\mathbb{E}_L X_1 = \kappa'(\gamma) > 0$ by convexity.

Lemma 4.1 *Let I_1, I_2, I_3 and I_4 be as in Theorem 3.8. Then*

$$\begin{aligned} I = I_1 + \gamma^{-1} I_3 = &\int_K^\infty \nu(\mathrm{d}y)((y-K) + \gamma^{-1}(1-\mathrm{e}^{\gamma(y-K)})) \\ &+ \int_0^K \overline{\pi}_K(x)\mathrm{d}x \int_{K-x}^\infty (1-\mathrm{e}^{\gamma(y-K+x)})\nu(\mathrm{d}y) \end{aligned}$$

and

$$\begin{aligned} J = I_2 + \gamma^{-1} I_4 = &\int_{-\infty}^0 \nu(\mathrm{d}y)(y + \gamma^{-1}(1-\mathrm{e}^{\gamma y})) \\ &+ \int_0^K \overline{\pi}_K(x)\mathrm{d}x \int_{-\infty}^{-x} (1-\mathrm{e}^{\gamma(x+y)}\nu(\mathrm{d}y). \end{aligned}$$

Proof. Change order of integration and perform partial integration. Then switch order of integration once more. □

Lemma 4.2 *Assume that $\{X_t\}$ is not compound Poisson with lattice jump distribution and that $\kappa'(0) < 0$ and $\kappa'(\gamma) < \infty$. Then, for each $y \ge 0$,*

$$\mathbb{P}\big(\tau_-^s(-y) > \tau_+^w(u)\big) \sim \mathrm{e}^{-\gamma u}\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\mathbb{P}_L\big(\tau_-^s(-y) = \infty\big),\ u \to \infty.$$

Proof. We first note that $\tau_+^w(u)$ is a stopping time and that $\{\tau_-^s(-y) > \tau_+^w(u)\} \in \mathcal{F}_{\tau_+^w(u)}$. Then (4.1) gives

$$\begin{aligned} \mathbb{P}\big(\tau_-^s(-y) > \tau_+^w(u)\big) &= \mathbb{E}_L\left[\mathrm{e}^{-\gamma X_{\tau_+^w(u)}}; \tau_-^s(-y) > \tau_+^w(u)\right] \\ &= \mathrm{e}^{-\gamma u}\mathbb{E}_L\left[\mathrm{e}^{-\gamma B(u)}; \tau_-^s(-y) > \tau_+(u)\right]\mathbb{P}_L\big(\tau_+^w(u) = \tau_+(u)\big) \\ &\quad + \mathrm{e}^{-\gamma u}\mathbb{P}_L\left[\tau_-^s(-y) > \tau_+^w(u) | \tau_+^w(u) \ne \tau_+(u)\right]\mathbb{P}_L\big(\tau_+^w(u) \ne \tau_+(u)\big). \end{aligned}$$

By the Blumenthal zero-one law, see e.g. [5], p. 19, it follows that $\mathbb{P}\big(\tau_+(0) = 0\big)$ has to be either 0 or 1. In the first case the sample paths of $\{M_t\}$ are step functions a.s. and the result follows in the same way as Lemma 2.3 in [17]. In the second case it follows by the strong Markov property applied at $\tau_+^w(u)$ that $\mathbb{P}_L\big(\tau_+^w(u) = \tau_+(u)\big) = 1$ and $\mathbb{P}_L\big(\tau_+^w(u) \ne \tau_+(u)\big) = 0$ for every $u > 0$. In any case, $\mathbb{P}_L\big(\tau_+^w(u) \ne \tau_+(u)\big) \to 0$

and $\mathbb{P}_L\big(\tau_+^w(u) = \tau_+(u)\big) \to 1,\ u \to \infty$. From [6] it follows that $B(u) \to B(\infty)$ in $\mathbb{P}_L$-distribution (assuming only $\kappa'(\gamma) < \infty$). Furthermore, $\{\tau_-^s(-y) > \tau_+(u)\} \uparrow \{\tau_-^s(-y) = \infty\}$ in $\mathbb{P}_L$-distribution. We finish the proof by applying the argument used in the proof of Corollary 5.9, p. 368 in [2], saying that $B(u)$ and $\{\tau_-^s(-y) > \tau_+(u)\}$ are asymptotically independent. $\square$

We are now in a position to prove the following result:

Theorem 4.3 *Assume that $\{X_t\}$ satisfies the conditions in Lemma 4.2. Then*

$$\ell^K \sim D\mathrm{e}^{-\gamma K},\ K \to \infty,$$

where

$$\begin{aligned} D = & -\kappa'(0) + \mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)} \int_0^\infty \mathrm{e}^{\gamma x}\mathbb{P}\big(\tau_-^s(-x) = \infty\big)\int_x^\infty (1-\mathrm{e}^{\gamma(y-x)})\nu(\mathrm{d}y)\mathrm{d}x \\ & + \int_{-\infty}^0 (y+\gamma^{-1}(1-\mathrm{e}^{\gamma y}))\nu(\mathrm{d}y) + \int_0^\infty \mathbb{P}\big(\tau_+^w(x)<\infty\big)\int_{-\infty}^{-x}(1-\mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y)\mathrm{d}x. \end{aligned}$$

Proof. We shall use the representation of ℓ^K in Theorem 3.8. We write, in the obvious notation, $I = I^1 + I^2, J = J^1 + J^2$, see Lemma 4.1. It follows directly from $\kappa(\gamma), \kappa'(\gamma) < \infty$ that $\mathrm{e}^{\gamma K} I^1 \to 0$. In I^2 we make the change of variables $z = K - x$ and take $u = K - z$ in Lemma 4.2:

$$\begin{aligned} \mathrm{e}^{\gamma K} I^2 = & \int_0^K \mathrm{e}^{\gamma z}\mathrm{e}^{\gamma(K-z)}\mathbb{P}\big(\tau_-^s(-z) > \tau_+^w(K-z)\big)\mathrm{d}z \int_z^\infty (1-\mathrm{e}^{\gamma(y-z)})\nu(\mathrm{d}y) \\ & \to \mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)} \int_0^\infty \mathrm{e}^{\gamma z}\mathbb{P}_L\big(\tau_-^s(-z) = \infty\big)\mathrm{d}z \int_z^\infty (1-\mathrm{e}^{\gamma(y-z)})\nu(\mathrm{d}y),\ K \to \infty. \end{aligned}$$

The convergence follows from the pointwise convergence in Lemma 4.2 and dominated convergence, which is applicable since $\mathrm{e}^{\gamma K}\overline{\pi}_K(K-x)I\big(x \le K\big) \le \mathrm{e}^{\gamma K}\mathbb{P}\big(M \ge K-x\big)I\big(x \le K\big) \le \mathrm{e}^{\gamma x}$ and

$$\int_0^\infty \mathrm{e}^{\gamma x}\mathrm{d}x \int_x^\infty (1-\mathrm{e}^{\gamma(y-x)})\nu(\mathrm{d}y) = \int_0^\infty (\gamma^{-1}\mathrm{e}^{\gamma y} - y\mathrm{e}^{\gamma y} - \gamma^{-1})\nu(\mathrm{d}y) < \infty.$$

In J^2 we bound $\overline{\pi}_K(x)I\big(x \le K\big)$ by 1 and note that

$$\int_0^\infty \mathrm{d}x \int_{-\infty}^{-x}(1-\mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y) = \int_{-\infty}^0 (-y-\gamma^{-1}+\gamma^{-1}\mathrm{e}^{\gamma y})\nu(\mathrm{d}y) < \infty.$$

Then it follows by $\overline{\pi}_K(x) \to \mathbb{P}\big(\tau_+^w(x) < \infty\big)$ and dominated convergence that

$$J^2 \to \int_0^\infty \mathbb{P}\big(\tau_+^w(x)<\infty\big)\int_{-\infty}^{-x}(1-\mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y).$$

The assertion now follows from Theorem 3.8. $\square$

5 Asymptotic loss rate in the case of positive drift

For the sake of completeness we include the following result.

Theorem 5.1 *Assume that $\mathbb{E}X_1 > 0$. Then $\ell^K \to \mathbb{E}X_1,\ K \to \infty$.*

Proof. If $\mathbb{E}X_1 = \infty$ there is nothing to prove, so we concentrate on the case $0 < \mathbb{E}X_1 < \infty$. First we note that as $K \to \infty$, $\pi_K\big([0,a)\big) \to 0$ for all $a > 0$. The proof of this is straightforward:

$$\begin{aligned}\overline{\pi}_K(a) &= \mathbb{P}\big(X_{\tau[a-K,a)} \geq a\big) \geq \mathbb{P}\big(M \geq a, m \geq a - K\big)\\ &= \mathbb{P}\big(m \geq a - K\big) \to 1,\ K \to \infty,\end{aligned}$$

since $\mathbb{E}X_1 > 0$. We now use (1.1) to obtain $\ell^K = \mathbb{E}X_1 + \ell^0$. We define, for some arbitrary $\varepsilon > 0$,

$$T = \int_0^1 I\big(V_t^K \in [0,\varepsilon)\big)\mathrm{d}t \text{ and } A = \big\{V_t^K \in [0,\varepsilon) \text{ for some } t \in [0,1]\big\}.$$

Then $\mathbb{E}T = \mathbb{E}\big[T|A\big]\mathbb{P}\big(A\big)$ and $\mathbb{E}\big[T|A\big] > 0$ since $\mathbb{E}X_1 < \infty$. But on the other hand, $\mathbb{E}T = \pi_K\big([0,\varepsilon)\big)$ and thus $\mathbb{P}\big(A\big) \to 0$, $K \to \infty$. Now the argument in the proof of Lemma 3.1 tells us that $\mathbb{E}\big[L_1^0|A\big] < \infty$ and the claim follows directly. □

Remark 5.2 From (1.1) we get that $\ell^0 = \ell^K - \mathbb{E}X_1$ and if $\mathbb{E}X_1 > 0$ it follows by an easy sign reversion argument that if $\kappa(\alpha) = 0$ has a root $\underset{\sim}{\gamma} < 0$ with $\kappa'(\gamma) < \infty$, then we may apply Theorem 4.3 to $\{-X_t\}$ to obtain $\ell^K - \mathbb{E}X_1 \sim \widetilde{D}\mathrm{e}^{\gamma K}$, $K \to \infty$. □

If we take $\mu > 0$ in Example 3.9, we see that $\ell^K \to \mu$ and that the asymptotic behaviour of $\ell^K - \mu$ squares with the remark above, as should be.

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