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LATTICE GAUGE FIELD THEORY

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The inspiration for this thesis comes from mathematical physics, especially path integrals and the Chern-Simons action. Path integrals were introduced by Feynman in late 1940's and they have recently been applied to purely geometric problems. The work [33] of Edward Witten on the topological quantum field theory has been found very attractive by many enthusiastic mathematicians. Although the ideas of quantum field theory are far from mathematically understood, they seem to give unifying framework for many recent invariants in low dimensional topology. One of the important ingredients in formulating 3-dimensional topological gauge field theories is the Chern- Simons functional.

In quantum theory, for a compact Lie group G, we consider a G-bundle over an oriented 3-manifold M and a connection A which can be regarded as a Lie algebra valued one form on the G-bundle. We can define the Chern-Simons functional by

$$S(A) :=< S(A), [M] >= \int_M TP(A),$$

where $TP(A) = \frac{1}{8\pi^2}\mathrm{Tr}(AdA + \frac{2}{3}A \wedge A \wedge A)$ is the Chern-Simons form and Tr is an invariant quadratic form on $\}$. One can use this functional as the lagrangian of a quantum field theory. The Lagrangian formualtion of quantum theory leads us to "path integrals";

$$Z(M) = \int e^{2\pi i S(A)} dA$$

over the space of all connections on the 3-manifold.

In 1989 Witten proposed a formulation of a class of 3-manifold invariants as generalized Feynman integrals taking the form above (see also Dijkgraaf-Witten [8], Rabin[29]).

In Chern-Simons theory for which we follow Chern-Simons [7], the parameter which defines the "path" varies the connection. In our case, the path corresponds to a connection and the subdivision in the path corresponds to the subdivision in order to compute the variation of the Chern-Simons class for a connection.

Now, we recall Huebschmann [17] for the Chern-Simons function, let M be a smooth closed, oriented manifold and consider a G-bundle on $I \times M$ with a connection ω on M, parametrized by $I = [0, 1]$ or equivalently, as a path of connections. The Chern-Weil construction assigns to an invariant homogeneous degree k polynomial P on the Lie algebra $\mathfrak{g}$ the characteristic form $P(F_\omega)$ on $I \times M$ of degree $2k$. Integration along the fibers of the canonical projection $\pi : I \times M \to M$ yields the secondary Chern-Simons form $TP(\omega)$ on M of degree $2k - 1$ with

$$dTP(\omega) = P(F_\omega(1)) - P(F_\omega(0)),$$

where $P(F_\omega(0))$ and $P(F_\omega(1))$ are the characteristic forms on M induced by the embedding $i_0 : M \to I \times M$ and $i_1 : M \to I \times M$ defined with reference to the two end points of I. Given a $(2k - 1)$-cycle c of M, $\int_c TP(\omega)$ yields a real number which modulo 1 only depends on the gauge equivalence classes of the two connections ω_0 and ω_1. Then given a principal G-bundle ξ and a cycle c of M, after having fixed a connection ω_0 on ξ, on the space of base gauge equivalence classes of connections on ξ, we define a G- invariant function S with values in $\mathbb{R}/\mathbb{Z}$: Pick a path ω of connections from ω_0 to ω_1, assign to ω_0 on the fixed value $S(\omega_0)$ and given any ω_1 on ξ, define $S(\omega_1)$ by

$$S(\omega_1) = \int_c TP(\omega) + S(\omega_0) \bmod \mathbb{Z}.$$

Let $E \to M$ be a bundle and $\{U_i\}_{i\in I}$ be an open covering of M. We can define a connection on this bundle by pulling back the canonical connection on the universal bundle. Every trivial bundle has a flat connection. Local trivializations define flat connections ω_i in E_{U_i}. So $\omega_i \in \mathcal{A}^1(E_{U_i}, \})$ is a connection on E_{U_i}. By choosing a partition of unity $\{\varphi_i\}_{i\in I}$ and $\omega = \sum_{i\in I} \varphi_i \omega_i$ will be a connection on E. We want to see how many connections we can get in this way and we consider the space of such connections in order to see the analogy of path integration. Let's take a nerve of $\mathcal{U} = \{U_i\}_{i\in I}$ and its fat realization $\|NM_{\mathcal{U}}\| = \bigsqcup \Delta^p \times NM_{\mathcal{U}}(p)/\sim$. Now, look at the following diagram

$$\begin{tikzcd} \|NM_{\mathcal{U}}\| \arrow[dr, shift left, "h"] & E \arrow[d] \\ & M, \arrow[ul, shift left, "s"] \end{tikzcd}$$

here the section $s : M \to \Delta^p \times NM_{\mathcal{U}}$ defines a partition of unity since $\varphi_i(x) \neq 0$, $x \in M$, $h \circ s = \mathrm{id}$, hence varying $t \in \Delta^p$ corresponds to a simplicial construction $\bar{P}_{\cdot}S_{\cdot}$ closely related to the prismatic subdivision $\|| \bar{P}_{\cdot}S_{\cdot} \||$. This is the motivation how one can use the path integration and Chern-Simons functional by means of subdivision, that is, the path integration is interpreted as an integral over Δ^p. That is how our case can be thought as a model of the path integration. One can find infinitely many connections for each fix p. Finally we obtain formulas for the effect of the prismatic subdivision to the lattice gauge field for Chern-Simons theory. It would be an interesting problem to study the behaviour of the Chern-Simons functional when $p \to \infty$.

More precisely, the aim in this thesis is to find formulas for the variation of a Chern-Simons class for a given bundle $F \to |S|$, where S is a simplicial set, with a connection ω by using prismatic subdivision. I have been inspired by Phillips-Stone's article [27] in which they compute the Chern-Simons character of a lattice gauge field. In their work, they take a generic $SU(2)$-valued lattice gauge field u on a triangulation Λ of a manifold M of dimension ≥ 3. A G-valued lattice gauge field (l.g.f) on Λ is a collection $u = \{u_{ij}\}$ of group elements, one for each 1-simplex $< ij >$ of Λ, subject to the condition $u_{ji} = {u_{ij}}^{-1}$. They construct from u a principal $SU(2)$-bundle ξ over M and a piecewise smooth connection ω. They define a canonical connection follows from Dupont [9] in Milnor's universal G-bundle $\xi_\Delta G = (\pi_\Delta : E_\Delta G \to B_\Delta G)$ and a corresponding canonical Chern-Simons form. They extend l.g.f. to a G-valued parallel transport function (p.t.f.) over Λ which consists of a family V of piecewise smooth maps of cubes into G with the cocycle and compatibility conditions. This gives rise to define a classifying map $f : \Lambda \to B_\Delta G$ which leads to the construction of a pseudo section. They use the pseudo section to integrate the pulled-back Chern-Simons form from the universal bundle. By using the bar construction established by Eilenberg and MacLane [14], they compute the Chern-Simons character from each 3-simplex $\sigma \in \Lambda$. They compare the Chern-Simons class $S(\omega)$ with the Chern-Simons class for the canonical connection for $SU(2)$ and the second one is already zero. In our case, we do the variation of the Chern-Simons class for a connection for a general case. Since they work on a special case and $SU(2)$ is 2-connected, everything works without any problem. In our case there is no trivialization and such a cycle property to get a section. Therefore we would have to follow another way to calculate the related formula for the Chern-Simons character. For further information, one can see Freed [15], Huebschmann [17], Lüscher [19], Phillips-Stone [23],[24],[26],[28].

The Chern-Simons form and character have been introduced by S.S. Chern-J. Simons [7] and by taking this article as a base J. Cheeger and J. Simons have studied on differential characters in [6]. Lattice gauge fields were introduced by K. Wilson in [32] to represent classical field configurations in Monte Carlo evaluations of path integral solutions of quantum field theories. L.g.f.'s have been used to find a new way of computing the second Chern number (the topological charge) of a principal $SU(2)$-bundle ξ over a triangulated 4-manifold M by Phillips-Stone [25], if ξ has a connection.

The prismatic subdivision plays an important role in the thesis. It has also been used by McClure-Smith [21] to give a solution of Deligne's Conjecture. An affirmative answer to the Deligne's conjecture has been given by Kontsevich and Voronov. The prismatic subdivision was discovered independently and various times by e.g., Lisica-Mardesic [18] and Batanin [1], [2]. There is a related edgewise subdivision discovered by Quillen, Segal, Bökstedt and Goodwille and it has also been studied by Hsiang and Madsen. The edgewise subdivision of a simplex is a subdivision of the prismatic subdivision. A special case of the edgewise subdivision was given by Bökstedt-Brun-Dupont in an unpublished note [4] and applied by the same authors in [3]. It can be defined as a chain map from the singular chain complex of any topological space into itself. In the construction given in [4], one uses the Alexander-Whitney map and the definition is closely related with the Eilenberg-Whitney map (see MacLane [20] (chapter 8, p.238)). Although it is quite similar to the barycentric subdivision, it has better properties in the sense that it divides a simplex into simplices which are more precise. The construction in [4] was quite useful to give the geometric interpretion of the realization of the prism complex $P_pS_{q_0,\dots,q_p}$ and it motivated me to construct a canonical homeomorphism $L : \| \, |P_.S_.| \, \| \to \| \, |S_.| \, \|$ and Alexander-Whitney diagonal map. This homeomorphism is one of the main statements in the thesis. We would like to point out that we are not using the edgewise subdivision in our construction. In our case, it is enough to use prismatic subdivision , since we want to make a simplex as small as possible along the edges.

We have used also some other tools in this thesis. One of the most important of them is the simplicial currents given by Dupont-Just [13]. The complex $\mathcal{A}^*\|X\|$ of simplicial differential forms defined in Dupont [11] plays the role of the differential forms on a manifold. In their article, the aim is to introduce a complex $\Omega_*\|X\|$ of simplicial currents on a simplicial manifold X, with properties similar to the complex of currents on a manifold. We have used this article to define some required extensions in the simplicial currents and the chain homotopy between them.

Now, we are going to give a brief outline for the thesis:

The first two chapters deal with some definitions of simplicial constructions and Alexander-Whitney map and they contain some preliminaries.

Chapter 3 contains the edgewise subdivision construction given in [4] as a motivation to define the map

$$AW(\Delta) : |S| \to |E_pS_.| = |S_.| \times \dots \times |S_.| (p+1-\text{times})$$

by using the Alexander-Whitney map. This map is defined by means of a simplicial construction. The important property here is that $|P_pS_.|$ corresponds to a subdivision of the simplex in $|S_.|$. $l_p(t) : |P_pS_.| \to |S_.|$ leads us to find the chain map

$$aw : C_*(S) \to C_{*,*}(PS)$$

which is induced by $l_p(t)^{-1}$ for $t \in \overset{\circ}{\Delta}{}^p$.

In chapter 4, we examine the prism complex $|P_pS_.|$ and establish a canonical homeomorphism

$$L : \|\ |P_.S_.|\ \| \to \|\ |S_.|\ \|,$$

where $\|\ |S_.|\ \| = \|\Delta^\infty\| \times |S_.|$ and Δ^∞ is the simplicial set with one element $\iota_p = (0, ..., p)$ in each degree. Moreover, it is shown that there exists a well-defined chain homotopy

$$u_\delta : \|S_.\| \to \|\ |S_.|\ \|.$$

In chapter 5, an analogy between a nerve for a simplicial complex and a nerve for a simplicial set is given by defining $\bar{P}_pS_{q_0,...q_p}$. There is a projection $|\bar{P}_pS_{q_0,...q_p} \to |P_pS_{q_0,...q_p}|$. We point that $|\bar{P}_pS_{q_0,...q_p}|$ corresponds to the p-th nerve of $|P_pS_{q_0,...,q_p}|$ for the covering $|S_.|$ by stars $\{r^{-1}(\sigma) \mid \sigma \in S_0\}$, where $r : \bar{S}_. \to S_0$ is a retraction and $\bar{S}_q := S_{q+1}$ is the star complex. The reason is to construct the new complex $\bar{P}_pS_.$ is that there is no well-defined classifying map $\|\ |P_.S_.|\ \| \to BG$ and there is a homotopy equivalence $\|\ |\bar{P}_.S_.|\ \| \simeq \|S\|$. Nevertheless $\|\ |\bar{P}_.S_.|\ \| \simeq \|\ |P_.S_.|\ \|$ induced from the projection given above is a homotopy equivalence. Then we construct a classifying map

$$m : \|\ |\bar{P}_.S_.|\ \| \to BG.$$

At the end of the chapter in proposition 5.2, we show that $S_.$ is a deformation retract of $\bar{P}_pS_.$ with $r : \bar{P}_pS_. \to S_p$.

Chapter 6 deals with transition functions for a given bundle $F \to |S|$ and admissible trivializations. We also show that there is a bundle $F \to |S|$, S is a simplicial set, and trivializations with transition functions for a given set of transition functions satisfying some relations in proposition 6.8.

Chapter 7 includes the construction of the classifying map m for a given bundle $F \to |S|$ and admissible trivialization and also it contains the construction of a map $k : \|S\| \to BG$ defined with aid of transition functions.

Chapter 8, we work on the chain level in order to solve the problem of lifting to $|\bar{P}_.S_.|$. Lifting u_δ we get a lifting $H_\delta : \|S\| \to \|\ |P_.S_.|\ \|$ of the family u_δ defined in chapter 4. However on the space level, there is no any way to lift H_δ, but it can be done on the chain level which leads us to examine the double complex $C_{*,*}(PS)$ as a family of $\{C_{p,n}(PS)\}$ of modules with boundary maps given in proposition 8.1.

In chapter 9, we consider $\|\ |S_.|\ \|$ and show that there exists a map of bicomplexes

$$aw_0 : C_*(\Delta^\infty) \otimes C_*(S) \to C_*(PS),$$

where $C_*(\Delta^\infty)$ is the chain group with only one generator in each degree and aw_0 is an extension of u_0 on the chain level. We also define a chain map aw_1 as an extension of u_1.

In chapter 10, we briefly review some general information about spectral sequences and filtration. After that we examine the homology $H(C_*(PS)) = H(C_*(S))$ for the prism complex $P_.S_.$. In proposition 10.5 and proposition 10.6, we calculate the spectral sequences ${'E_{p,n}}^2$ and ${''E_{n,p}}^1$. We also give a remark about the similar calculation for $\bar{P}S$.

Chapter 11 deals with the lifting problem. First we show that the chain maps u_0 and u_1 given in chapter 8 are chain homotopic via chain homotopy s. In proposition 11.3, we find a lift $\tilde{u}_0$ for u_0.

Chapter 12 contains the chain homotopy T between the chain maps aw_0 and aw_1 given in chapter 9. In order to define this chain homotopy, we need to follow some steps given by lemma 12.2 and lemma 12.3. After defining T, we show that there is a lift $\tilde{aw}_0$ for aw_0.

In chapter 13, we start giving the definition of a dual simplicial de Rham complex as in Dupont [11]. It also deals with some definitions and properties for a complex $\Omega_*\|X\|$ of simplicial currents as in Dupont-Just [13] on a simplicial manifold X. This is used as a tool in this thesis.

Chapter 14 deals with the extensions of the chain maps aw_0 and aw_1 to the simplicial currents. We show that there exists a map of bicomplexes

$$aw_\Omega{}^0 : \Omega_*(\Delta^\infty) \otimes C_*(S) \to \Omega_{*,*}(PS)$$

and there exists a chain map

$$aw_\Omega{}^1 : \Omega_*(\Delta^\infty) \otimes C_*(S) \to \Omega_{*,0}(PS).$$

These two maps $aw_\Omega{}^0$ and $aw_\Omega{}^1$ correspond to the extensions of aw_0 and aw_1, respectively. At the end of the chapter, we show that there is a chain homotopy $s_\Omega{}'$ between the extensions $aw_\Omega{}^0$ and $aw_\Omega{}^1$. Moreover, we also show that there is a lift $\tilde{aw}_\Omega{}^0$ for $aw_\Omega{}^0$.

Chapter 15 reviews some general information about characteristic classes, Classical Chern-Weil theory and Chern-Simons theory. We also give the Chern-Simons form as a differential character due to Chegeer-Simons [6]. In corollary 15.11 (see Dupont-Kamber [12]), we give the Chern-Simons class $S_{P,u}(\omega) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$, where $P \in I^k(G)$ and $u \in H^{2k}(BG, \Lambda)$. At last, we give the difference of the evaluation of the Chern-Simons classes for two different connections ω_1 and ω_2 on a cycle as a difference form as follows;

$$< S_{P,u}(\omega_2), [M] > - < S_{P,u}(\omega_1), [M] >= \int_W P(F_{\tilde{\omega}}) = \int_M TP(\omega_1, \omega_2),$$

where $TP(\omega_1, \omega_2) = \int_0^1 i_{d/dt} P(F_{\tilde{\omega}})dt$, $\tilde{\omega} = (1-t)\omega_1 + t\omega_2$ in $M \times [0,1] = W^{2k}$, P is a polynomial of the two connections ω_1 and ω_2 defined on E.

The last chapter 16 includes the applications of the prismatic constructions to the gauge field. In this chapter, we start with the definition of a connection ω, induced from the canonical connection, in a simplicial bundle and as well as in the universal bundle $EG \to BG$. We then give a formula for the variation of the Chern-Simons class for a given bundle $F \to |S|$ with the connection ω. We evaluate the difference of the Chern-Simons classes for two connections on a cycle defined on $\Omega_{*,*}(\bar{P}S)$ which covers the cycle on $C_*(S)$. This can be expressed in a different way as follows; we evaluate a Chern-Simons class for the given bundle with ω on the difference of two cycles defined on $\Omega_{*,*}(\bar{P}S)$. We give the main formula for the variation with theorem 16.6. Furthermore, the lifting $\tilde{aw}_\Omega{}^0$ defines a cycle $\tilde{z}_l$ on $\Omega_{*,*}(\bar{P}S)$. Since the Chern-Simons class is related to the second Chern class of the bundle, we will give the variation in corollary 16.7 in terms of the Chern form for the bundle with the canonical connection on the cycle $\tilde{z}_l$ defined on $\Omega_{*,*}(\bar{P}S)$ via $\tilde{aw}_\Omega{}^0$.

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Contents

Chapter 1

SIMPLICIAL CONSTRUCTIONS AND CLASSIFYING SPACE

In order to understand the concept of a bundle over a simplicial set, we will first review some essential definitions of simplicial constructions. We give a brief review of classifying spaces of lie group G. Much more information of the ingredient in this chapter can be found in the mathematical literature, for example in Dijkgraaf-Witten [[8]], Dupont [[10]], Milnor [], Segal [].

Definition 1.1 (Standard n-Simplex) :

Let us consider Δ^n in $\mathbb{R}^{n+1}$, the convex hull of the set of canonical basis vectors $e_i = (0, ..., 1, ..., 0)$ with 1 on the i-th place, $i = 0, ..., n$. That is,

$$\Delta^n = \{t = (t_1, ..., t_n) \in \mathbb{R}^n | 1 \geq t_1 \geq ... \geq t_p \geq 0\}.$$

Let $\varepsilon^i : \Delta^{n-1} \to \Delta^n$, $i = 0, ..., n$, be the inclusion on the i-th face and $\eta^j : \Delta^{n+1} \to \Delta^n$ be the degeneracy map, $j = 0, ..., n$. The identities with the face and the degeneracy maps are as follows:

$$\varepsilon^j \varepsilon^i = \begin{cases} \varepsilon^i \varepsilon^{j-1} & : \quad i < j \\ \varepsilon^{i+1} \varepsilon^j & : \quad i \geq j, \end{cases}$$

$$\eta^j \eta^i = \begin{cases} \eta^i \eta^{j+1} & : \quad i \leq j \\ \eta^{i-1} \eta^j & : \quad i > j, \end{cases}$$

and

$$\eta^j \circ \varepsilon^i = \begin{cases} \varepsilon^i \circ \eta^{j-1} & : \quad i < j \\ \mathrm{id} & : \quad i = j, i = j+1 \\ \varepsilon^{i-1} \circ \eta^j & : \quad i > j+1, \end{cases}$$

Simplices are building blocks of a polyhedron. An n-simplex to represent an n-dimensional object, the vertices (p_i) must be geometrically independent, that is, no $(n-1)$-dimensional hyper plane contains all the $n+1$ points. A 0-simplex p_0 is a point, or a vertex, and a 1-simplex $(p_0 p_1)$ is a line, or an edge. A 2-simplex $(p_0 p_1 p_2)$ is defined to be a triangle with its interior included and a 3-simplex $(p_0 p_1 p_2 p_3)$ is a solid tetrahedron.

Definition 1.2 (Simplicial Set) :

A simplicial set S is a sequence $S = \{S_q\}$, $q = 0, 1, 2, ...$, of sets together with face operators $\varepsilon_i : S_q \to S_{q-1}$, $i = 0, ..., q$, and degeneracy operators $\eta_i : S_q \to S_{q+1}$, $i = 0, ..., q$, which satisfy the following identities

$$\varepsilon_i \varepsilon_j = \begin{cases} \varepsilon_{j-1}\varepsilon_i & : \quad i < j \\ \varepsilon_j \varepsilon_{i+1} & : \quad i \geq j, \end{cases}$$

$$\eta_i \eta_j = \begin{cases} \eta_{j+1}\eta_i & : \quad i \leq j \\ \eta_j \eta_{i-1} & : \quad i > j, \end{cases}$$

and

$$\varepsilon_i \circ \eta_j = \begin{cases} \eta_{j-1} \circ \varepsilon_i & : \quad i < j \\ \text{id} & : \quad i = j, i = j+1 \\ \eta_j \circ \varepsilon_{i-1} & : \quad i > j+1, \end{cases}$$

Definition 1.3 (Simplicial Map) :

A simplicial map is a map of simplicial sets which is a sequence of maps commuting with the face and degeneracy operators.

Example (Singular n-Simplex) :

Let M be a C^∞ manifold. A C^∞ singular n-simplex in M is a C^∞ map $\sigma : \Delta^n \to$M, where Δ^n is the standard n-simplex. Let $S_n^\infty(M)$ denote the set of all C^∞ singular n-simplices in M. Let $\varepsilon^i : \Delta^{n-1} \to \Delta^n$, $i = 0, ..., n$, be the inclusion on the i-th face. Define $\varepsilon_i : S_n^\infty(\mathrm{M}) \to S_{n-1}^\infty(\mathrm{M})$, $i = 0, ..., n$ by $\varepsilon_i(\sigma) = \sigma \circ \varepsilon^i$, $\sigma \in S_n^\infty(M)$.

Definiton 1.4 (Simplicial Space) :

Let $S = \{S_q\}$, $q = 0, 1, ...$ be a simplicial set and suppose that each S_q is a topological space such that all face and degeneracy operators are continuous. Then S is called a simplicial space and associated to this is so-called fat realization, the space $\|S\|$ is given by

$$\|S\| = \bigsqcup_{n \geq 0} \Delta^n \times S_n / \sim$$

with the identifications

$$(\varepsilon^i t, x) \sim (t, \varepsilon_i x), \ t \in \Delta^{n-1} \ , \ x \in S_n \ , \ \text{and} \ \ i = 0, .., n, \ n = 1, 2, ... \ .$$

We can give the geometric (thin) realization of S which is denoted by $|S|$ with the common identifications

$$(\eta^i t, x) \sim (t, \eta_i x), \ t \in \Delta^{n+1} \ , \ x \in S_n \ , \ \text{and} \ \ i = 0, .., n, \ n = 0, 1, ... \ .$$

Remark : A space whose only face operators defined is called Δ-space.

Definition 1.5 (Differential k-Form) :

Let $S = \{S_q\}$ be a simplicial set. A differential k-form φ on S is a family $\varphi = \{\varphi_\sigma\}$, $\sigma \in \bigsqcup_p S_p$ of k-forms such that

i) φ_σ is a k-form on the standard simplex Δ^p (i.e., every k- form on Δ^p can be expressible in the form $\sum_{0\leq i_0<...<i_k\leq p} a_{i_0...i_k} dt_{i_0} \wedge ... \wedge dt_{i_k}$ where $a_{i_0...i_k}$ are C^∞ functions on Δ^p, so the set $\{dt_1, ..., dt_p\}$ generates the set of k- forms $\mathcal{A}^*(\Delta^p)$ on Δ^p), where $\mathcal{A}^k(\Delta^p)$ is the set of k-forms on Δ^p, for $\sigma \in S_p$.

ii) $\varphi_{\varepsilon_i\sigma} = (\varepsilon^i)^* \varphi_\sigma$, $i = 0, ..., p$, $\sigma \in S_p$, $p = 1, 2, ...$, where $\varepsilon^i : \Delta^{p-1} \to \Delta^p$ is the i-th face map.

Example :

Let $S = S^\infty(M)$ be the set of all C^∞ singular simplices in M, for M a C^∞ manifold. Then w is a k-form on M, we get a k-form $\varphi = \{\varphi_\sigma\}$ on $S^\infty(M)$ by putting $\varphi_\sigma = \sigma^* w$ for $\sigma \in {S_p}^\infty(M)$.

Remark :

We have for any C^∞ manifold M a natural transformation $i : \mathcal{A}^*(M) \to \mathcal{A}^*(S^\infty(M))$ which is injective, so one can think of simplicial forms on $S^\infty(M)$ as some generalized kind of forms on M.

Definition 1.6 (Classifying Spaces and group cohomology) :

In order to define the cohomology of a topological group G, we need to introduce the concept of a classifying space. **A classifying space** BG is the base space of a principal G-bundle EG, the so-called universal bundle, with the following fundamental property: Any principal G-bundle E over a manifold M allows a bundle map into the universal bundle and any two such morphisms are smoothly homotopic. The classifying map is written as $\gamma : M \to BG$ for the induced map of the base manifolds. The topology of the bundle E is completely determined by the homotopy class of γ, that is, the different components of the space $\mathrm{Map}(M, BG)$ corresponds to the different bundles $E \to M$. It can be shown that up to homotopy BG is uniquely determined by requiring EG to be contractible, that is, any contractible space with a free action of G is a realization of EG.

Theorem 1.7 :

There is a topological space BG, called the classifying space for G such that the characteristic classes are in 1-1 correspondence with the cohomology classes in $H^*(BG)$.

The construction is as follows:

As usual $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the standard n-simplex with barycentric coordinates $t = (t_0, ..., t_n)$. Let $G^{n+1} = G \times ... \times G$ ($n + 1$-times) and let

$$EG = \bigsqcup_{n\geq 0} \Delta^n \times G^{n+1} / \sim$$

with the following identifications:

$$(\varepsilon^i t, (g_0, ..., g_n)) \sim (t, (g_0, ...\hat{g}_i, ..., g_n)) \ , \ t \in \Delta^{n-1} \ , \ g_0, ..., g_n \in G \ , \ i = 0, ..., n.$$

Now G acts on the right on EG by the action

$$(t, (g_0, ..., g_n))g = (t, (g_0 g, ..., g_n g)).$$

$\gamma_G : EG \to BG$ is a principal G-bundle and we let $BG = EG/G$ with $\gamma_G : EG \to BG$ the projection. The elements in $H^*(BG, \mathbb{Z})$ are called universal characteristic classes, since under γ^* they give rise to cohomology classes in $H^*(M, \mathbb{Z})$ that depends only on the topology of the bundle E.

Definiton 1.8 (Simplicial Manifold) :

A simplicial set $S = \{S_q\}$, $q = 0, 1, ...$, is called a simplicial manifold if all S_q are C^∞ manifolds and all face and degeneracy operators are C^∞ maps.

Remark :

Although the classifying space BG is not a manifold, it is the realization of a simplicial manifold, that is, a simplicial set where the set of p simplices constitute a manifold.

Definition 1.9 (Simplicial n-Form) :

A simplicial n-form φ on the simplicial manifold $S = \{S_p\}$ is a sequence $\varphi = \{\varphi^{(p)}\}$ of n-forms $\varphi^{(p)}$ on $\Delta^p \times S_p$, such that

$$(\varepsilon^i \times \mathrm{id})^* \varphi^{(p)} = (\mathrm{id} \times \varepsilon_i)^* \varphi^{(p-1)}$$

on $\Delta^{p-1} \times S_p$, $i = 0, ..., p$, $p = 0, 1, ...$ Notice that $\varphi = \{\varphi^{(p)}\}$ defines an n-form on $\bigsqcup_{p=0}^{\infty} \Delta^p \times S_p$ and the above condition is the natural condition for a form on $\|S\|$ in view of the necessary identifications given in definition 1.8. Let's denote the restriction $\varphi^{(p)}$ of φ to $\Delta^p \times S_p$ as φ.

Remark : When S is discrete, Definition 1.9 agrees with Definition 1.5.

Let $A^n(S)$ denote the set of simplicial n-forms on S. The exterior differential on $\Delta^p \times S_p$ defines a differential $d : A^n(S) \to A^{n+1}(S)$ and we have the exterior multiplication

$$\wedge : A^n(S) \otimes A^m(S) \to A^{n+m}(S)$$

satisfying the necessary identities which are associativity and graded commutativity. The complex $(A^*(S), d)$ is the total complex of a double complex $(A^{k,l}(S), d', d'')$. An n-form φ lies in $A^{k,l}(S)$, $k + l = n$ iff $\varphi_{\Delta^p \times S_p}$ is locally of the form

$$\varphi = \sum\nolimits_{a_{i_1 \dots i_k, j_1 \dots j_l}} dt_{i_1} \wedge ... \wedge dt_{i_k} \wedge dx_{j_1} \wedge ... \wedge dx_{j_l}$$

where $(t_0, ..., t_p) \in \Delta^p$ and $\{x_j\}$ are the local coordinates in S_p. So

$$A^n(S) = \bigsqcup\nolimits_{k+l=n} A^{k,l}(S)$$

and $d = d' + d''$, where d' is the exterior derivative with respect to the barycentric coordinates $(t_0, ..., t_p)$ and d'' is $(-1)^k$ times the exterior derivative with respect to the x-variables.

Definition 1.10 (Simplicial G-Bundle) :

Let G be a lie group with its lie algebra $\mathfrak{g}$. A simplicial G-bundle $\pi : F \to S$ is a sequence of differentiable G-bundles $\pi_p : F_p \to S_p$, where $F = \{F_p\}$, $S = \{S_p\}$ are simplicial manifolds. π is a simplicial differentiable map and the right group action is defined by $G_g : F \to F$, $g \in G$ which is simplicial.

A connection, θ, in this bundle is a 1-form on F with coefficients in $\mathfrak{g}$ such that $\Delta^p \times F_{p|_\theta}$ is a connection in the usual sense in the bundle $\Delta^p \times F_p \to \Delta^p \times S_p$.

Definition 1.11 (Simplicial Diagonal Map) :

For any simplicial set S, $\Delta s = s \times s$ defines a simplicial map $\Delta : S \to S \times S$ called the simplicial diagonal map.

Remark : All structures make sense as a Δ-space.

Definition 1.12 (Triviality of Bundles) :

If a fibre bundle can be written as a direct product of the base and the fibre then it is called as trivial.

Theorem 1.13 :

A principal bundle is trivial if and only if it admits a global section.

Proof :

$\Leftarrow$ Let (E, π, M, G) be a principal bundle and let $s \in \Gamma(M, E)$ be a global section. So there exists a homeomorphism $E \to M \times G$ by using s. Since the right action is transitive and free, any element $f \in E$ is uniquely written as $s(m).g$ for some $m \in M$ and $g \in G$.

Define a map $\psi : E \to M \times G$ by $\psi(s(m)g) = (m, g)$ which is a homeomorphism so E is trivial bundle $M \times G$.

$\Rightarrow$ Suppose $E \cong M \times G$. Let $\varphi : M \times G \to E$ be a trivialization and take a fixed element $g' \in G$. So $s_{g'} : M \to E$ which is defined by $s_{g'}(m) = \varphi(m, g')$ is a global section. $\square$

We would like to finish this chapter with the following remarks.

Remark :

The curvature of a trivial bundle is zero since a connection ω in a trivial G-bundle F is induced from the Maurer-Cartan connection in the principal G-bundle $G \to \mathrm{pt}$. This connection is called as flat connection.

Remark :

If we are given a covering $U = \{U_\alpha\}$ and a system of transition functions, one can construct a corresponding principal G- bundle:

The total space is defined as $\bigsqcup_\alpha U_\alpha \times G/\sim$ such that $(u, g) \in U_\alpha \times G$ identified with $(u, t_{\beta\alpha}(u).g) \in U_\beta \times G$, $\forall u \in U_\alpha \bigcup U_\beta$ and $g \in G$.

Chapter 2

ALEXANDER-WHITNEY MAP

It is well-known that for two simplicial modules A and B, one can define a chain map $C_*(A \times B) \to C_*(A) \otimes C_*(B)$. We want to do this for the simplicial space level in general. In order to do this, we need some motivation with Alexander-Whitney map and Eilenberg-Zilber theorem. So we give a few facts and theorems about Alexander-Whitney map without their proofs. As a general reference, one can see MacLane ([20]) (p. 239).

Theorem 2.1 (The Eilenberg-Zilber Theorem) :

If U and V are simplicial sets, then $U \times V$ is the simplicial set with $(U \times V)_n = U_n \times V_n$ the cartesian product of sets and

$$\begin{aligned} \varepsilon_i(u, v) &= (\varepsilon_i u, \varepsilon_i v) \\ \eta_i(u, v) &= (\eta_i u, \eta_i v), \end{aligned}$$

$i = 0, ..., n$ for $u \in U_n, v \in V_n$ and $n > 0$ in the case of ε_i (where $\varepsilon_i : S_n \to S_{n-1}$, $\eta_i : S_n \to S_{n+1}$, $i = 0, ..., n$ are the i-th face and the i-th degeneracy operators of S).

Let $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ be the projections on X and Y. Each singular simplex $\sigma : \Delta^n \to X \times Y$ is determined by its projections $\pi_1\sigma$ and $\pi_2\sigma$, while $\varepsilon_i\pi_j\sigma = \pi_j\varepsilon_i\sigma$, $\eta_i\pi_j\sigma = \pi_j\eta_i\sigma$. Hence,

$$\sigma \to (\pi_1\sigma, \pi_2\sigma)$$

provides an isomorphism

$$S(X \times Y) \cong S(X) \times S(Y)$$

of simplicial sets. The computation of the singular homology of $X \times Y$ is thus reduced to the computation of the homology of a cartesian product of simplicial sets.

There is a parallel product for simplicial modules A and B (i.e., a simplicial Λ-module is meant a simplicial set in the category of all Λ-modules) over a commutative ring. The cartesian product $A \times B$ is defined to be the simplicial module with $(A \times B)_n = A_n \times B_n$ and

$$\begin{aligned} \varepsilon_i(a \otimes b) &= \varepsilon_i a \otimes \varepsilon_i b, \\ \eta_i(a \otimes b) &= \eta_i a \otimes \eta_i b, \end{aligned}$$

$i = 0, ..., n$ for $a \in A_n, b \in B_n$ and $n > 0$ in the case of ε_i.

We will write $a \times b$ instead of $a \otimes b$ of $A_n \otimes B_n$ to avoid confusion. This insures that there is a natural isomorphism of simplicial modules

$$F(U \times V) \cong FU \times FV,$$

for $F(U \times V)$ in dimension n is the free module generated by the set $U_n \times V_n$ and $F(U \times V) \cong (FU_n) \otimes (FV_n)$.

Theorem 2.2 :

For simplicial modules A and B over a commutative ring Λ, C_* denotes the chain complex, there is a natural chain equivalence

$$C_*(A \times B) \rightleftharpoons C_*(A) \otimes C_*(B).$$

Lemma 2.3 :

For simplicial modules A and B, there exists a natural chain map

$$f : C_*(A \times B) \to C_*(A) \otimes C_*(B)$$

which is the identity in dimension zero. Any two such natural maps f are chain homotopic via a homotopy which is natural.

Proposition 2.4 :

For each non-negative integer n, $C_*(M^n)$ is acyclic.

Lemma 2.5 :

For simplicial modules A and B, there exists a natural chain map

$$g : C_*(A) \otimes C_*(B) \to C_*(A \times B)$$

which is the identity in dimension zero. Any two such natural maps g are homotopic by a chain homotopy natural in A and B. Denote the last face in the simplicial set S by $\tilde{\varepsilon}$, that is, for $a \in S_n$ set $\tilde{\varepsilon}a = \varepsilon_n a$. Thus, for any exponent $n-i$, $\tilde{\varepsilon}^{n-i}a = \varepsilon_{i+1}...\varepsilon_n a$.

Theorem 2.6 :

For any simplicial modules A and B, a natural chain map $f : C_*(A \times B) \to C_*(A) \otimes C_*(B)$ for the "Eilenberg-Zilber" theorem is given by

$$f(a \times b) = \sum_{i=0}^{n} \tilde{\varepsilon}^{n-i} a \otimes \varepsilon_0^i b,\ a \in A_n,\ b \in B_n.$$

f is known as the **"Alexander-Whitney map"**.

Theorem 2.7 :

For any simplicial modules A and B, a natural chain transformation g for the Eilenberg-Zilber theorem is given, for $a \in A_p$, $b \in B_q$, by (2.1)

$$g(a \otimes b) = \sum_{(v,w)} (-1)^{\varepsilon(v)} (\eta_{w_q}...\eta_{w_1} a \times \eta_{v_p}...\eta_{v_1} b),$$

where the sum is taken over all (p,q)-shuffles (v,w) and $\varepsilon(v)$ is the sign defined by $\sum_{i=0}^{p} v_i - (i-1)$. g is natural, $a \otimes b$ has dimension $p+q$, and so do $\eta_{w_q}...\eta_{w_1}$ and $\eta_{v_p}...\eta_{v_1}$.

Geometrically, g provides a "triangulation" of $\Delta^p \times \Delta^q$ of two simplices. Specifically, take $a \in M^p$ and $b \in M^q$, so "a" has vertices $(0,1,...,p)$. In this vertex notation,

$$\eta_{w_q}...\eta_{w_1} a = (i_0, ..., i_{p+q}),$$

with $0 = i_0 \leq i_1 \leq ... \leq i_{p+q} = p$, and $i_k = i_{k+1}$ precisely when k is one of $w_1, ..., w_q$.

Similarly, $\eta_{v_p}...\eta_{v_1} b = (j_0, ..., j_{p+q})$, with $j_k = j_{k+1}$ precisely when k is one of $v_1, ..., v_p$. The simplex displayed on the right of (2.1) then has the form

$$(i_0, ..., i_{p+q}) \times (j_0, ..., j_{p+q}),$$

where the first factor is degenerate at those indices k for which the second factor is not degenerate. This symbol may be read as the $(p+q)$-dimensional affine simplex with vertices (i_k, j_k) in $\Delta^p \times \Delta^q$. These simplices, for all (p,q)-shuffles, provide a simplicial subdivision of $\Delta^p \times \Delta^q$.

For example, if p=2, q=1, $\Delta^2 \times \Delta^1$ is a triangular prism and the three possible (2,1)-shuffles triangulate this prism into 3 simplices

$$(0122) \times (0001)\ ,\ (0112) \times (0011)\ ,\ (0012) \times (0111)\ ,$$

each of dimension 3.

Definition 2.8 (Simplicial Diagonal Map and Diagonal Map) :

For any simplicial set U, $\Delta u = u \times u$ defines a simplicial map $\Delta : U \to U \times U$ called the **simplicial diagonal map**. U determines the simplicial abelian group $F_Z(U)$ and hence the chain complex $C_*(F_Z U)$ which we write simply as $C_*(U)$; each $C_n(U)$ is the free abelian group generated by the set U_n, with $\partial = \sum (-1)^i \varepsilon_i$. The diagonal induces a chain transformation $C_*(U) \to C_*(U \times U)$; also denoted by Δ. If f is any one of the natural maps from the Eilenberg-Zilber theorem the composite

$$w = f\Delta : C_*(U) \to C_*(U \times U) \to C_*(U) \otimes C_*(U),$$

is called a **diagonal map** in $C_*(U)$. Since f is unique up to a (natural) chain homotopy, so is w. Since Δ is associative $((\Delta \times 1)\Delta = (1 \times \Delta)\Delta)$ and f is associative up to homotopy, there is a homotopy $(w \otimes 1)w \simeq (1 \otimes w)w$.

Chapter 3

PRISMATIC SUBDIVISION

In this chapter, we want to define $|S| \to |E_p S_.|$ by using Alexander-Whitney map for a simplicial space S. In order to understand this map, we would like to explain what $|E_p S_.|$ is and define $|P_p S_.|$ as follows:

Let $S_.$ be a simplicial set and $E_p S_. = \underbrace{S_. \times ... \times S_.}_{p+1-\text{times}}$. The projections correspond to the face operators and the repetitions correspond to the degeneracy operators. We will mention these operators explicitly in the next chapter.

First, let us define the $p+1$-prism complex $P_p S_{q_0,...,q_p}$ in a similar way used for a bisimplicial set $P_1 S_{q_0,q_1}$ as in Bökstedt-Brun-Dupont [4] by the following construction

$$P_p S_{q_0,q_1,...,q_p} = S_{q_0+q_1+...+q_p+p}.$$

As a motivation and in order to make the following remark clear, suppose S is a simplicial set with face operators $\varepsilon_i : S_n \to S_{n-1}$ and degeneracy operators $\eta_i : S_n \to S_{n+1}$, $i = 0, ..., n$. We can associate this to a bisimplicial set $P_. S_{..}$, where $P_1 S_{q_0,q_1}$ and $\varepsilon_i{}' = \varepsilon_i$, $\eta_i{}' = \eta_i$, $i = 0, ..., q_0$, $\varepsilon_j{}'' = \varepsilon_{j+q_0+1}$, $\eta_j{}'' = \eta_{j+q_0+1}$, $j = 0, ..., q_1$.

Note :

Sometimes we drop "." in the realizations unless a prism complex is considered. This should be clear from the context.

Now, we give the geometric interpration of $|P_p S_.|^{[1]}$ by the following remark;

Remark :

$$|P_p S_.|^{[1]} = \coprod_{q_0+...+q_p=1} \Delta^{q_0} \times ... \times \Delta^{q_p} \times S_{q_0+...+q_p+p}/\sim$$

when $p = 1$, then we get

$$\begin{aligned} |P_1 S_{q_0,q_1}|^{[1]} &= \coprod_{q_0+q_1=1} \Delta^{q_0} \times \Delta^{q_1} \times S_{q_0+q_1+1}/\sim \\ &\equiv \Delta^0 \times \Delta^1 \times S_2 \coprod \Delta^1 \times \Delta^0 \times S_2/\sim. \end{aligned}$$

Let's take $S = \Delta^2$ then

$$\begin{aligned} S_0 &= \{\{0\},\{1\},\{2\}\} \\ S_1 &= \{\{0,1\},\{0,2\},\{1,2\},\{0,0\},\{1,1\},\{2,2\}\} \\ S_2 &= \{\{0,0,1\},\{0,1,1\},\{0,0,2\},\{0,2,2\},\{1,1,2\},\{1,2,2\},\{0,0,0\},\{1,1,1\},\{2,2,2\},\{0,1,2\}\} \end{aligned}$$

here $\{0,1\},\{0,2\},\{1,2\}$ are the non-degenerate elements in S_1, since the others are reduced from the degeneracy operator

$$\begin{aligned} \eta_0 : S_0 &\rightarrow S_1 \\ \{0\} &\rightarrow \{0,0\} \\ \{1\} &\rightarrow \{1,1\} \\ \{2\} &\rightarrow \{2,2\}, \end{aligned}$$

and except $\{0,1,2\}$ the others are degenerate elements in S_2, they follow from η_1 and η_0 as follows;

$$\begin{aligned} \eta_1 : S_1 &\rightarrow S_2 \\ \{0,1\} &\rightarrow \{0,1,1\} \\ \{0,2\} &\rightarrow \{0,2,2\} \\ \{1,2\} &\rightarrow \{1,2,2\}, \end{aligned}$$

and

$$\begin{aligned} \eta_0 : S_1 &\rightarrow S_2 \\ \{0,1\} &\rightarrow \{0,0,1\} \\ \{0,2\} &\rightarrow \{0,0,2\} \\ \{1,2\} &\rightarrow \{1,1,2\} \\ \{0,0\} &\rightarrow \{0,0,0\} \\ \{1,1\} &\rightarrow \{1,1,1\} \\ \{2,2\} &\rightarrow \{2,2,2\}. \end{aligned}$$

If we check which elements are degenerate in the prism complex we need to use the degeneracy operator η_0''. By the definition $\eta_0'' = \eta_{0+0+1} : P_1 S_{0,0}{}^{[1]} \rightarrow P_1 S_{0,1}{}^{[1]}$. So $\{0,0,1\}$ is not a degenerate element in the prism complex. The boundary of $\Delta^0 \times \Delta^1 \times S_2$ for this element is (01) and (00) by using $\varepsilon_0'' = \varepsilon_{0+0+1} = \varepsilon_1$ and $\varepsilon_1'' = \varepsilon_{1+0+1} = \varepsilon_2$, respectively. Then, one can find that $\{0,0,1\},\{0,0,2\},\{1,1,2\},\{0,1,2\},\{0,1,1\},\{0,2,2\},\{1,2,2\}$ are non-degenerate elements in the prism complex so we will compute the boundaries for these elements corresponding to $(0,1),(0,0),(02),(00)$, $(12),(11)$, $(02),(01)$ come from ε_1, ε_2 and $(12),(02)$, $(11),(01)$, $(22),(02)$, $(22),(12)$ come from ε_0, ε_1, respectively.

Proposition 3.1 :

Let S be a simplicial set. One can construct the Alexander-Whitney diagonal map,

$$AW(\Delta) : |S| \rightarrow |E_p S_.| = |S_.| \times ... \times |S_.| (p+1-\text{times})$$

by $AW(\Delta)(l_{q_0,\dots,q_p}(t)(s), y) = \nu(s,y)$, where $l_{q_0,\dots,q_p}(t) : \Delta^{q_0} \times ... \times \Delta^{q_p} \to \Delta^{q_0+\dots+q_p+p}$ and $\nu : |P_pS_.| \to |E_pS_.|$.

Proof :

At first instance, let's define ν for $p = 1$ as follows;

The map ν

$$\nu : \Delta^{q_0} \times \Delta^{q_1} \times S_{q_0+q_1+1} \to \Delta^{q_0} \times \Delta^{q_1} \times S_{q_0} \times S_{q_1}$$

is defined by

$$\nu(s^0, s^1, y) = (s^0, s^1, \tilde{\varepsilon}^{q_1+1}y, {\varepsilon_0}^{q_0+1}y).$$

For $p = 2$, it is defined as $\nu(s^0, s^1, s^2, y) = (s^0, s^1, s^2, \tilde{\varepsilon}^{q_1+q_2+2}y, {\varepsilon_0}^{q_0+1}\tilde{\varepsilon}^{q_2+1}y, {\varepsilon_0}^{q_0+q_1+2}y)$.

In the same way, we get this map in general as;

$$\nu(s^0, ..., s^p, y) = (s^0, ..., s^p, \tilde{\varepsilon}^{q_1+\dots+q_p+p}y, {\varepsilon_0}^{q_0+1}\tilde{\varepsilon}^{q_2+\dots+q_p+p-1}y, ..., {\varepsilon_0}^{q_0+\dots+q_p+p}y)$$

and we have

$$\begin{tikzcd} {|P_pS_.|} \arrow[r, "\nu"] \arrow[dr, "l_p"'] & {|E_pS_.|} \\ & {|S|} \arrow[u, "AW(\Delta)"'] \end{tikzcd}$$

By the simplicial construction $P_pS_{q_0,\dots,q_p} = S_{q_0+\dots+q_p+p}$ we know that $|P_pS_.| \approx |S|$. So the $AW(\Delta)$ diagonal map is defined by

$$AW(l_{q_0,\dots,q_p}(t)(s), y) = \nu(s^0, ..., s^p, y),$$

where we have the following modification of $l_p(t)$ as in [4];

$$l_{q_0,\dots,q_p}(t) : \Delta^{q_0} \times ... \times \Delta^{q_p} \to \Delta^{q_0+\dots+q_p+p}.$$

induced by $l_p(t) : |P_pS_.| \to |S_.|$, where

$$\Delta^{q_i} = \{({s_1}^i, ..., {s_{q_i}}^i) \in \mathbb{R}^{q_i} \mid 1 \geq {s_1}^i \geq ... \geq {s_{q_i}}^i \geq 0\}$$

and

$$\Delta^p = \{(t_1, ..., t_p) \in \mathbb{R}^p \mid 1 \geq t_1 \geq ... \geq t_p \geq 0\}.$$

For $t = (t_1, ..., t_p) \in \Delta^p$, $l_{q_0,\dots,q_p}(t)$ is defined as:

$$\begin{aligned} l_{q_0,\dots,q_p}(t)(s^0, ..., s^p) = \quad & ({s_1}^0(1-t_1) + t_1, ..., {s_{q_0}}^0(1-t_1) + t_1, t_1, \\ & {s_1}^1(t_1 - t_2) + t_2, ..., {s_{q_1}}^1(t_1 - t_2) + t_2, t_2, \\ & , ..., \\ & {s_1}^{p-1}(t_{p-1} - t_p) + t_p, ..., {s_{q_{p-1}}}^{p-1}(t_{p-1} - t_p) + t_p, t_p, \\ & {s_1}^p t_p, ..., {s_{q_p}}^p t_p). \end{aligned}$$

These maps define a natural map of realizations $l_p(t) : |P_.S_.| \to |S_.|$, that is,

$$l_p(t) : \Delta^{q_0} \times ... \times \Delta^{q_p} \times P_{q_0,\dots,q_p} \to \Delta^{q_0+\dots+q_p+p} \times S_{q_0+\dots+q_p+p}.$$

Thus

$$l_p(t) = l_{q_0,\dots,q_p}(t) \times \mathrm{id}.$$

$\square$

Note :

We will only use the notation $l_p(t)$ instead of $l_{q_0,\dots,q_p}(t)$ to avoid the confusion.

Note :

We are using the prismatic subdivision in our construction. We could use the edgewise subdivision which can be defined by the Eilenberg-Zilber map $|S_.| \times \dots \times |S_.| \to |S_. \times \dots \times S_.|$, but the formulae would be complicated and it would not give us an advantage.

Proposition 3.2 :

1)For $t \in \overset{\circ}{\Delta}{}^p$ where $\overset{\circ}{\Delta}{}^p = \{(t_1, \dots, t_p) \mid 1 > t_1 > \dots > t_p > 0\}$, the map

$$l_p(t) : |P_p S_.| \to |S_.|$$

is a homeomorphism and $l_p(t)^{-1}$ is cellular.

2) For $t \in \overset{\circ}{\Delta}{}^p$, $l_p(t)^{-1}$ induces the map of cellular chain complexes $C_*(S) \to C_{*,*}(PS)$ is given by

$$aw(x) = \sum_{q_0+\dots+q_p=n} \eta_{q_0+\dots+q_{p-1}+p-1} \circ \dots \circ \eta_{q_0}(x)_{(q_0,\dots,q_p)},$$

where $x \in S_n$.

3) For the i-th face map $\varepsilon^i : \Delta^{p-1} \to \Delta^p$, we have $l_p(\varepsilon^i(t)) = l_{p-1}(t) \circ \pi_i$, where $\pi_i = \mathrm{proj}_i \times \mu_{(i)}$ and $\mu_{(i)} : P_p S_{q_0,\dots,q_p} \to P_{p-1} S_{q_0,\dots,\hat{q}_i,\dots,q_p}$ are the face operators corresponding to $\mu^{(i)} : \Delta^{q_0+\dots+\hat{q}_i+\dots+q_p+p-1} \to \Delta^{q_0+\dots+\dots+q_p+p}$ take $(e_0, \dots, e_{q_0+\dots+\hat{q}_i+\dots+q_p+p-1})$ to $(e_0, \dots, e_{q_0+\dots+q_p+p})$, deleting the elements $q_0 + \dots + q_{i-1} + i, \dots, q_0 + \dots + q_i + i$. It deletes $(q_i + 1)$- elements.

Proof :

1) In order to see that $l_p(t)$ is surjective, take an element from righthand-side and find its pre image on the left hand side. This procedure defines the inverse to $l_p(t)$.

Let us consider the case $p = 1$;

$$l_1(t) : \Delta^{q_0} \times \Delta^{q_1} \times P_{q_0,q_1} \to \Delta^{q_0+q_1+1} \times S_{q_0+q_1+1}.$$

Suppose that $(k, x) \in \Delta^n \times S_n$ and find q_0 such that ${s_1}^i \geq \dots \geq {s_{q_0}}^i \geq \frac{1}{2} \geq \dots \geq {s_{q_n}}^i \geq 0$. Then, for $s^0 = 2s_i - 1$ where $1 \leq i \leq q_0$ and $s^1 = 2s_{j+q_0+1}$, where $1 \leq j \leq q_1$, we have

$$l_1(t)(s^0, s^1, \eta_{q_0} x) = (l_{q_0,q_1}(t)(s^0, s^1), \eta_{q_0} x) \sim (\eta^{q_0} l_{q_0,q_1}(t)(s^0, s^1), x) = (k, x).$$

Then

$$l_1(t)^{-1} : \Delta^{q_0+q_1} \times S_{q_0+q_1} \to \Delta^{q_0} \times \Delta^{q_1} \times P_{q_0,q_1}$$

is defined by $l_1(t)^{-1}(k,x) = (s^0, s^1, \eta_{q_0}x)$.

In other words, $l_1(t)^{-1}$ sends $\Delta^n \times \{x\}$ to $\bigsqcup_{q_0=0}^{q_0+q_1} \Delta^{q_0} \times \Delta^{q_1} \times \{\eta_{q_0}x\}_{q_0,q_1}$.

For the case p=2;

$$l_2(t) : \Delta^{q_0} \times \Delta^{q_1} \times \Delta^{q_2} \times P_{q_0,q_1,q_2} \to \Delta^{q_0+q_1+q_2+2} \times S_{q_0+q_1+q_2+2}.$$

By following the same way as above, we can define $l_2(t)^{-1}$.

Suppose that $(k,x) \in \Delta^n \times S_n$, and find q_0, q_1 such that ${s_1}^i \geq ... \geq {s_{q_0}}^i \geq \frac{2}{3} \geq ... \geq {s_{q_1}}^i \geq \frac{1}{3} \geq ... \geq {s_{q_n}}^i \geq 0$. Then, for $s^0 = 3s_i - 2$, where $1 \leq i \leq q_0$, $s^1 = 3s_{j+q_0+1} - 1$, where $1 \leq j \leq q_1$ and $s^2 = 3s_{k+q_0+q_1+2}$ where $1 \leq k \leq q_2$, we have

$$\begin{aligned} l_2(t)(s^0, s^1, s^2, \eta_{q_0+q_1+1} \circ \eta_{q_0}x) &= (l_{q_0,q_1,q_2}(t)(s^0, s^1, s^2), \eta_{q_0+q_1+1} \circ \eta_{q_0}x) \\ &\sim (\eta^{q_0} \circ \eta^{q_0+q_1+1} l_{q_0,q_1,q_2}(t)(s^0, s^1, s^2), x) \\ &= (k,x). \end{aligned}$$

Thus $l_2(t)^{-1}$ sends $\Delta^n \times \{x\}$ to $\bigsqcup_{q_0+q_1+q_2=n} \Delta^{q_0} \times \Delta^{q_1} \times \Delta^{q_2} \times \{\eta_{q_0+q_1+1} \circ \eta_{q_0}x\}_{q_0,q_1,q_2}$.

In general,

$$l_p(t) : \Delta^{q_0} \times ... \times \Delta^{q_p} \times P_{q_0,...,q_p} \to \Delta^{q_0+...+q_p+p} \times S_{q_0+...+q_p+p}$$

is defined by

$$\begin{aligned} l_p(t)(s^0, ..., s^p, \eta_{q_0+...+q_{p-1}+p-1} \circ ... \circ \eta_{q_0}x) &= (l_{q_0,...,q_p}(t)(s^0, ..., s^p), \eta_{q_0+...+q_{p-1}+p-1} \circ ... \circ \eta_{q_0}x) \\ &\sim (\eta^{q_0} \circ ... \circ \eta^{q_0+...+q_{p-1}+p-1} l_{q_0,...,q_p}(t)(s^0, ..., s^p), x). \end{aligned}$$

Thus $l_p(t)^{(-1)}$ sends $\Delta^n \times \{x\}$ to $\bigsqcup_{q_0+...+q_p=n} \Delta^{q_0} \times ... \times \Delta^{q_p} \times \{\eta_{q_0+...+q_{p-1}+p-1} \circ ... \circ \eta_{q_0}x\}_{q_0,...,q_p}$.

By using the modification of $l_p(t)$, we can show that $l_p(t)$ is a homeomorphism for $t \in \overset{\circ}{\Delta}{}^p$.

We only do this for $p = 1$,

$$l_1(t) : \Delta^{q_0} \times \Delta^{q_1} \times S_{q_0+q_1+1} \to \Delta^{q_0+q_1+1} \times S_{q_0+q_1+1}$$

is defined by

$$l_1(t)(s^0, s^1, \eta_{q_0}x) = (l_1(t)(s^0, s^1), \eta_{q_0}x) \sim (\eta^{q_0} l_1(t)(s^0, s^1), x),$$

where $(\eta^{q_0} l_1(t)(s^0, s^1), x) \in \Delta^{q_0+q_1} \times S_{q_0+q_1}$.

The function of η^{q_0} here is to delete the (q_0+1)-st element, t_1, as

$$\eta^{q_0} l_1(t)(s^0, s^1) = \eta^{q_0}({s_1}^0(1-t_1) + t_1, ..., {s_{q_0}}^0(1-t_1) + t_1, t_1, {s_1}^1 t_1, ..., {s_{q_1}}^1 t_1)$$

then

$$\eta^{q_0} : \Delta^{q_0+q_1+1} \to \Delta^{q_0+q_1}$$

is defined by

$$\eta^{q_0}(l_1(t)(s^0, s^1)) = ({s_1}^0(1-t_1) + t_1, ..., {s_{q_0}}^0(1-t_1) + t_1, {s_1}^1 t_1, ..., {s_{q_1}}^1 t_1).$$

Thus

$$l_1(t)(s^0, s^1, \eta_{q_0}x) \sim (\eta^{q_0} l_1(t)(s^0, s^1), x).$$

One can easily show that $l_1(t)$ is a monomorphism.

If $l_1(t)(s^{0\prime}, s^{1\prime}, \eta_{q_0}x') = l_1(t)(s^0, s^1, \eta_{q_0}x)$ then we need to show that

$$(s^{0\prime}, s^{1\prime}, \eta_{q_0}x') = (s^0, s^1, \eta_{q_0}x).$$

By using the definition of $l_1(t)$, we get

$$(\eta^{q_0} l_1(t)(s^{0\prime}, s^{1\prime}), x') = (\eta^{q_0} l_1(t)(s^0, s^1), x).$$

It follows that ${s_1}^0 = {s_1}^{0\prime}$,...,${s_{q_0}}^0 = {s_{q_0}}^{0\prime}$,${s_1}^1 = {s_1}^{1\prime}$,..., ${s_{q_1}}^1 = {s_{q_1}}^{1\prime}$.

Moreover $l_1(t)$ is surjective:

For all $u^0 \in \Delta^{q_0+q_1+1}$, $\exists (s^0, s^1) \in \Delta^{q_0} \times \Delta^{q_1}$, $t \in \overset{\circ}{\Delta}{}^1$, such that

$$l_1(t)(s^0, s^1) = u^0 = (u_1, ..., u_{q_0+q_1+1}).$$

From the definition of $l_1(t)$, we can write

$$\begin{aligned} l_1(t)(s^0, s^1) &= u^0 = ({s_1}^0(1-t)+t, ..., {s_{q_0}}^0(1-t)+t, t, {s_1}^1 t, ..., {s_{q_1}}^1 t) \\ &= (u_1, ..., u_{q_0+q_1+1}) \end{aligned}$$

It follows by substituting t by u_{q_0+1} we get

$${s_1}^0 = \frac{u_1 - u_{q_0+1}}{1 - u_{q_0+1}}, ..., {s_{q_0}}^0 = \frac{u_{q_0} - u_{q_0+1}}{1 - u_{q_0+1}}, {s_1}^1 = \frac{u_{q_0+2}}{u_{q_0+1}}, ..., {s_{q_1}}^1 = \frac{u_{q_0+q_1+1}}{u_{q_0+1}}$$

here $s^0 \in \Delta^{q_0}$ and $s^1 \in \Delta^{q_1}$, since $u^0 = (u_1, ..., u_{q_0+q_1+1}) \in \Delta^{q_0+q_1+1}$ satisfies the following

$$1 \geq u_1 \geq ... \geq u_{q_0} \geq u_{q_0+1} \geq ... \geq u_{q_0+q_1+1} \geq 0.$$

Similarly one gets $1 \geq {s_1}^0 \geq ... \geq {s_{q_0}}^0 \geq 0$ and since $u_{q_0+1} \geq u_{q_0+2}$, we have $1 \geq {s_1}^1 \geq ... \geq {s_{q_0}}^1 \geq 0$.

Thus $\exists s^0 \in \Delta^{q_0}$, $s^1 \in \Delta^{q_1}$ for all $u^0 \in \Delta^{q_0+q_1+1}$, that is, $l_1(t)$ is surjective.

2) Now, consider the usual CW structure (see Bredon [5] (chapter 4, p.198)) on $|P_p S_.|$ and $|S_.|$. The map $l_p(t)^{-1}$ is cellular, since it converts the low dimensional cell in $|S_.|$ into the cell in $|P_p S_.|$, that is, $l_p(t)^{(-1)}(|S_.|^n) \subset |P_p S_.|^n$. So it induces a chain map of the associated cellular chain complexes. If we let P_* denote the total complex of the bicomplex associated to the bisimplicial set $P_. S_.$, then this map is given by the formula:

$$aw(x) = \sum\nolimits_{q_0=0}^{n} \eta_{q_0}(x)_{q_0, n-q_0},$$

and we have in general,

$$aw(x) = \sum_{q_0+...+q_p=n} \eta_{q_0+...+q_{p-1}+p-1} \circ ... \circ \eta_{q_0+q_1+1} \circ \eta_{q_0}(x).$$

Moreover aw is a chain map.

Let us check it for the case $p = 1$:

We have two differentials in the bicomplex $P_1 S_{q_0,q_1} \cong S_{q_0+q_1+1}$, $q_0, q_1 \geq 0$ given by

$$\begin{aligned} \varepsilon' &= \sum_{r=0}^{q_0} (-1)^r \varepsilon_r, \\ \varepsilon'' &= \sum_{r=0}^{q_1} (-1)^r \varepsilon_{q_0+r+1}. \end{aligned}$$

Let $x \in S_n$ so that $aw(x) \in \bigoplus_{q_0+q_1=n} S_{n+1}$. We compute the composition of aw with the differential ε. The component of $\varepsilon \circ aw(x) = (\varepsilon' + (-1)^{q_0}\varepsilon'')(\eta_{q_0}(x))$ in $P_1 S_{q_0,q_1}$ is,

$$\begin{aligned} \varepsilon \circ aw(x) &= \sum_{r=0}^{q_0+1} (-1)^r \varepsilon_r(\eta_{q_0+1}(x)) + (-1)^{q_0} \sum_{r=0}^{q_1+1} (-1)^r \varepsilon_{q_0+r+1}(\eta_{q_0}(x)) \\ &= (\varepsilon_0 - ... + (-1)^{q_0+1}\varepsilon_{q_0+1})\eta_{q_0+1}(x) + (-1)^{q_0}(\varepsilon_{q_0+1} - ... + (-1)^{q_1+1}\varepsilon_{q_0+q_1+2})\eta_{q_0}(x) \\ &= (\varepsilon_0 \circ \eta_{q_0+1} - ... + (-1)^{q_0+1}\varepsilon_{q_0+1} \circ \eta_{q_0+1})(x) + \\ &\quad (-1)^{q_0}(\varepsilon_{q_0+1} \circ \eta_{q_0} - ... + (-1)^{q_1+1}\varepsilon_{q_0+q_1+2} \circ \eta_{q_0})(x) \\ &= \eta_{q_0}(\varepsilon_0 - \varepsilon_1 + ... + (-1)^{q_0}\varepsilon_{q_0})(x) + (-1)^{q_0+1}\mathrm{id} + \\ &\quad (-1)^{q_0}(\mathrm{id} - \eta_{q_0}\varepsilon_{q_0+1} + ... + (-1)^{q_1+1}\eta_{q_0}\varepsilon_{q_0+q_1+1})(x) \\ &= \eta_{q_0} \sum_{r=0}^{q_0} (-1)^r \varepsilon_r(x) + \eta_{q_0}(-1)^{q_0} \sum_{r=0}^{q_1} (-1)^{r+1} \varepsilon_{r+q_0+1}(x) \\ &= \eta_{q_0}\Big(\sum_{r=0}^{q_0} (-1)^r \varepsilon_r(x) + (-1)^{q_0} \sum_{r=0}^{q_1} (-1)^{r+1} \varepsilon_{r+q_0+1}\Big)(x) \\ &= \eta_{q_0}\Big(\sum_{r=0}^{q_0+q_1+1} (-1)^r \varepsilon_r\Big)(x) \\ &= \eta_{q_0} \circ \varepsilon(x) \\ &= aw \circ \varepsilon(x) \end{aligned}$$

so that $\varepsilon \circ aw = aw \circ \varepsilon$, which means that aw is a chain map.

Let us show that it is a chain map for $p = 2$. We have to consider that we have three differentials in the multi-complex $P_2 S_{q_0,q_1,q_2} \cong S_{q_0+q_1+q_2+2}$, $q_0, q_1, q_2 \geq 0$ denoted by ε', ε'', ε'''.

Let $x \in S_n$ so that $aw(x) \in \bigoplus_{q_0+q_1+q_2=n} S_{n+2}$. We need to check

$$\varepsilon \circ aw(x) = aw \circ \varepsilon(x).$$

$$\varepsilon \circ aw(x) = (\varepsilon' + (-1)^{q_0}\varepsilon'' + (-1)^{q_0+q_1}\varepsilon''')(\eta_{q_0+q_1+1}\eta_{q_0}(x))$$

in P_{q_0,q_1,q_2}, where

$$\begin{aligned} \varepsilon' &= \sum_{r=0}^{q_0} (-1)^r \varepsilon_r, \\ \varepsilon'' &= \sum_{r=0}^{q_1} (-1)^r \varepsilon_{q_0+r+1}, \\ \varepsilon''' &= \sum_{r=0}^{q_2} (-1)^r \varepsilon_{q_0+q_1+r+2}. \end{aligned}$$

The components of $\varepsilon \circ \; aw(x)$ in $P_2S_{q_0,q_1,q_2}$ are

$$\begin{aligned}
\varepsilon \circ \; aw(x) &= \sum_{r=0}^{q_0+1}(-1)^r\varepsilon_r \circ \eta_{q_0+q_1+2} \circ \eta_{q_0+1}(x) + (-1)^{q_0}\sum_{r=0}^{q_1+1}(-1)^r\varepsilon_{q_0+r+1} \circ \eta_{q_0+q_1+2} \circ \eta_{q_0}(x) + \\
&\quad (-1)^{q_0+q_1}\sum_{r=0}^{q_2+2}(-1)^r\varepsilon_{q_0+q_1+r+2} \circ \eta_{q_0+q_1+1} \circ \eta_{q_0}(x) \\
&= (\varepsilon_0 - ... + (-1)^{q_0}\varepsilon_{q_0} + (-1)^{q_0+1}\varepsilon_{q_0+1})\eta_{q_0+q_1+2} \circ \eta_{q_0+1}(x) + \\
&\quad (-1)^{q_0}[\varepsilon_{q_0+1} - ... + (-1)^{q_1+1}\varepsilon_{q_0+q_1+2}]\eta_{q_0+q_1+2} \circ \eta_{q_0}(x) + \\
&\quad (-1)^{q_0+q_1}[\varepsilon_{q_0+q_1+2} - ... + (-1)^{q_2}\varepsilon_{q_0+q_1+q_2+2} + (-1)^{q_2+1}\varepsilon_{q_0+q_1+q_2+3} + \\
&\quad (-1)^{q_2+2}\varepsilon_{q_0+q_1+q_2+4}]\eta_{q_0+q_1+1} \circ \eta_{q_0}(x) \\
&= \eta_{q_0+q_1+1} \circ \eta_{q_0}(\varepsilon_0 - ... + (-1)^{q_0}\varepsilon_{q_0}) + (-1)^{q_0+1}\eta_{q_0+q_1+1}(x) + \\
&\quad (-1)^{q_0}[\eta_{q_0+q_1+1} + \eta_{q_0+q_1+1} \circ \eta_{q_0}(-\varepsilon_{q_0+1} + ... + (-1)^{q_1}\varepsilon_{q_0+q_1}) + (-1)^{q_1+1}\eta_{q_0}](x) + \\
&\quad (-1)^{q_0+q_1}[\eta_{q_0} - \eta_{q_0+q_1+1} \circ \eta_{q_0}(\varepsilon_{q_0+q_1+1} + ... + (-1)^{q_2+2}\varepsilon_{q_0+q_1+q_2+2})](x) \\
&= \eta_{q_0+q_1+1} \circ \eta_{q_0}\sum_{r=0}^{q_0+q_1+q_2+2}(-1)^r\varepsilon_r(x) + (-1)^{q_0+1}\eta_{q_0+q_1+1}(x) + \\
&\quad (-1)^{q_0}\eta_{q_0+q_1+1}(x) + (-1)^{q_0+q_1+1}\eta_{q_0}(x) + (-1)^{q_0+q_1}\eta_{q_0}(x) \\
&= \eta_{q_0+q_1+1} \circ \eta_{q_0}\sum_{r=0}^{q_0+q_1+q_2+2}(-1)^r\varepsilon_r(x) \\
&= \eta_{q_0+q_1+1} \circ \eta_{q_0} \circ \varepsilon(x) \\
&= aw \circ \varepsilon(x).
\end{aligned}$$

Thus, aw is a chain map for $p = 2$. It can be shown for general p.

3) For the i-th face map $\varepsilon^i : \Delta^{p-1} \to \Delta^p$, we have

$$l_p(\varepsilon^i(t)) = l_{p-1}(t) \circ \pi_i.$$

In other words, we have got a commutative diagram,

$$\begin{array}{ccc}
\Delta^{p-1} \times |P_pS_.| & \xrightarrow{\varepsilon^i \times id} & \Delta^p \times |P_pS_.| \\
\downarrow{\scriptstyle id \times \pi_i} & & \downarrow{\scriptstyle l_p} \\
\Delta^{p-1} \times |P_{p-1}S_.| & \xrightarrow{l_{p-1}} & |S_.|
\end{array}$$

It gives us the required equality. Let us see this commutativity for $p = 2,\ i = 2$:

The diagram becomes

$$\begin{array}{ccc}
\Delta^1 \times |P_2S_.| & \xrightarrow{\varepsilon^2 \times id} & \Delta^2 \times |P_2S_.| \\
\downarrow{\scriptstyle id \times \pi_2} & & \downarrow{\scriptstyle l_2} \\
\Delta^1 \times |P_1S_.| & \xrightarrow{l_1} & |S_.|
\end{array}$$

The maps are defined as follows;

$$\begin{aligned}
(\varepsilon^2 \times id)(t, s^0, s^1, s^2, x) &= ((t,0), s^0, s^1, s^2, x)\\
(id \times \pi_2)(t, s^0, s^1, s^2, x) &= (t, s^0, s^1, \mu_{(2)}x)\\
l_2((t,0), s^0, s^1, s^2, x) &= (l_2(t,0)(s^0, s^1, s^2), x)\\
l_1(t, s^0, s^1, \mu_{(2)}x) &= (l_1(t)(s^0, s^1), \mu_{(2)}x),
\end{aligned}$$

where $\pi_2 = \mathrm{pr}_2 \times \mu_{(2)}$. We need to take the equivalent element of $(l_1(t)(s^0, s^1), \mu_{(2)}x)$, since it is in $\Delta^{q_0+q_1+1} \times S_{q_0+q_1+1}$.

Then

$$(l_1(t)(s^0, s^1), \mu_{(2)}x) \sim (\mu^{(2)} l_1(t)(s^0, s^1), x).$$

We shall show that

$$(l_2(t,0)(s^0, s^1, s^2), x) = (\mu^{(2)} l_1(t)(s^0, s^1), x), \text{ since } l_2 \text{ is a monomorphism.}$$

i.e.,

$$l_2(t,0)(s^0, s^1, s^2) = \mu^{(2)} l_1(t)(s^0, s^1).$$

Then the left hand side is

$$\begin{aligned}
l_2(t,0)(s^0, s^1, s^2) &= ({s_1}^0(1-t)+t, ..., {s_{q_0}}^0(1-t)+t, t, {s_1}^1(t-0)+0, ..., {s_{q_1}}^1(t-0)+0, 0,\\
&\quad {s_1}^2.0, ..., {s_{q_2}}^2.0)\\
&= ({s_1}^0(1-t)+t, ..., {s_{q_0}}^0(1-t)+t, t, {s_1}^1 t, ..., {s_{q_1}}^1 t, 0, 0, ..., 0)
\end{aligned}$$

and the right hand side is

$$\begin{aligned}
\mu^{(2)} l_1(t)(s^0, s^1) &= \mu^{(2)}({s_1}^0(1-t)+t, ..., {s_{q_0}}^0(1-t)+t, t, {s_1}^1 t, ..., {s_{q_1}}^1 t)\\
&= ({s_1}^0(1-t)+t, ..., {s_{q_0}}^0(1-t)+t, t, {s_1}^1 t, ..., {s_{q_1}}^1 t, 0, 0, ..., 0),
\end{aligned}$$

where

$$\mu_{(2)} : S_{q_0+q_1+q_2+2} \longrightarrow S_{q_0+q_1+1}$$

is the face operator corresponding to

$$\mu^{(2)} : \Delta^{q_0+q_1+1} \longrightarrow \Delta^{q_0+q_1+q_2+2}$$

such that

$$\mu^{(2)}({s_1}^0(1-t)+t, ..., {s_{q_0}}^0(1-t)+t, t, {s_1}^1 t, ..., {s_{q_1}}^1 t) = ({s_1}^0(1-t)+t, ..., {s_{q_0}}^0(1-t)+t, t, {s_1}^1 t, ..., {s_{q_1}}^1 t, 0, 0, ..., 0).$$

Hence $l_2(t,0)(s^0, s^1, s^2) = \mu^{(2)} l_1(t)(s^0, s^1)$. It follows that the diagram commutes. $\square$

Chapter 4

SIMPLICIAL CONSTRUCTIONS, PRISM COMPLEXES AND REALIZATIONS

Let X be a topological space. This can be considered as a simplicial topological space by $X_p = X$ with the face and the degeneracy operators are the identity.

In this case $||X|| = \|\Delta^\infty\| \times X$, where $\|\Delta^\infty\| = \bigsqcup_{p\geq 0} \Delta^p/\sim$ denote the fat realization of X and $|X|$ denote the geometric (thin) realization of X. Also we have the simplicial topological space $E_. X$ where $E_p X = \underbrace{X \times ... \times X}_{p+1-\text{times}}$ for all p. Then the diagonal map

$$\Delta : X_p \to E_p X, \quad \text{for all } p,$$

defines a map of simplicial spaces, in particular a map of fat realizations

$$\|X\| \to \|E_p X_.\|.$$

In this chapter, we want to replace X by $|S_.|$ where $S_.$ is a simplicial set. For this, we must use instead the prism complex $|P_p S_.|$ defined before. Notice that the sequence of spaces $|P_p S_.|$ is not a simplicial space but only a Δ-space since one can not define sensible degeneracy operators. In particular, we are going to show that there is a canonical homeomorphism of fat realizations

$$L : \| \ |P_. S_.| \ \| \to \| \ |S_.| \ \| = \|\Delta^\infty\| \times |S_.|,$$

where $\|\Delta^\infty\| = \bigsqcup_{p\geq 0} \Delta^p/\sim$ given by $\varepsilon^i t \sim t$, $\forall t \in \Delta^{p-1}$, $i = 0, ..., p$, $p = 1, ...$.

Let us replace X by a simplicial space $|S_.|$ and define $(p+1)$-prism complex $E_p S_. = \underbrace{S_. \times ... \times S_.}_{p+1-\text{times}}$.

The (geometric) realization of $E_p S_.$ is defined by

$$\begin{aligned} |E_p S_.| = & \quad |S_. \times ... \times S_.| \\ = & \quad |S_.| \times ... \times |S_.| \\ = & \quad \bigsqcup_{q_0,...,q_p} \Delta^{q_0} \times ... \times \Delta^{q_p} \times S_{q_0} \times ... \times S_{q_p}/_\sim, \end{aligned}$$

with the necessary equivalence relation which follows from that the projections correspond to the face operators and the repetitions correspond to the degeneracy operators as follows:

$\pi_i : |E_p S_.| \to |E_{p-1} S_.|$ project on the i-th factor and $\delta_i : |E_p S_.| \to |E_{p+1} S_.|$ repeat the i-th factor. Although π_i's are cellular, i.e., $\pi_i(|E_p S_.|^{(n)}) \subset |E_{p-1} S_.|^{(n)}$, δ_i's are not cellular, since when we define $\delta_i : \Delta^{q_i} \to \Delta^{q_i} \times \Delta^{q_i}$, we see that δ_i do not convert the low cell in $|E_pS_.|$ into the cell in $|E_{p+1}S_.|$. That is why, we define the fat realization of $|E_pS_.|$ instead of defining the geometric realization of $|E_pS_.|$.

Lemma 4.1 :

Let's consider the $(p+1)$-prism complex $P_pS_{q_0,...,q_p}$. One can define the face operators on the sequences of space $\{|P_.S_.|\}$, hence obtain the fat realization:

$$\| \, |P_.S_.| \, \| = \bigsqcup \Delta^p \times |P_pS_.|/\sim,$$

where

$$|P_pS_.| = \bigsqcup \Delta^{q_0} \times ... \times \Delta^{q_p} \times S_{q_0+...+q_p+p}/\sim.$$

Let us define the face and the degeneracy operators on $\| \, |P_.S_.| \, \|$:

The face operators π_i on $|P_pS|$ are induced by

$$(\Delta^{q_0} \times ... \times \Delta^{q_p}) \times S_{q_0+...+q_p+p} \to (\Delta^{q_0} \times ... \times \hat{\Delta}^{q_i} \times \Delta^{q_p}) \times S_{q_0+...+\hat{q}_i+...+q_p+p-1}$$

where $\pi_i = (\mathrm{proj}_i) \times \mu_{(i)}$ and

$$\mu_{(i)} : P_pS_{q_0,...,q_p} \to P_{p-1}S_{q_0,...,\hat{q}_i,...q_p}$$

are the face operators corresponding to

$$\mu^{(i)} : \Delta^{q_0+...+\hat{q}_i+...+q_p+p-1} \to \Delta^{q_0+...+...+q_p+p}$$

take $(e_0, ..., e_{q_0+...+\hat{q_i}+...+q_p+p-1})$ to $(e_0, ..., e_{q_0+...+q_p+p})$, deleting the elements $q_0 + ... + q_{i-1} + i, ..., q_0 + ... + q_i + i$. It deletes $(q_i + 1)$- elements. $\mu_{(i)}$'s depend on $q_0, ..., q_p$ and i.

In contrary to this, there is no degeneracy operator. Since,

$$\eta_{(i)} : P_pS_{q_0,...,q_p} \to P_{p+1}S_{q_0,...,q_i,q_i,...,q_p}$$

are associated to the diagonal map

$$\Delta^{q_0} \times ... \times \Delta^{q_p} \to \Delta^{q_0} \times ... \times \Delta^{q_i} \times \Delta^{q_i} \times ... \times \Delta^{q_p}.$$

δ_i's are the degeneracy operators corresponding to

$$\eta^{(i)} : \Delta^{q_0+...+q_i+q_i+...+q_p+p+1} \to \Delta^{q_0+...+q_i+...+q_p+p}$$

which take $(e_0, ..., e_{q_0+...+2q_i+...+q_p+p+1})$ to $(e_0, ..., e_{q_0+...+q_i+...+q_p+p})$, repeating the sequence $q_0 + ... + q_{i-1} + i, ..., q_0 + ... + q_i + i$ which is not monotonely increasing. Therefore, there is no degeneracy operator. So the necessary equivalence relation is

$$(\varepsilon^i t, (s, y)) \sim (t, \pi_i(s, y))$$

for $\forall i$.

Lemma 4.2 :

Let S be a simplicial set. Then there is a map

$$L : ||\ |P_. S_.|\ || \to ||\ |S_.|\ || \text{ given via } l_p(t).$$

Proof :

We have $||\ |P_. S_.|\ || \xrightarrow{||\nu||} ||\ |E_. S_.|\ ||$, here

$$||\ |P_. S_.|\ || = \bigsqcup \Delta^p \times |P_p S_.|/_\sim$$

and by using the inverse of $l_p(t)$, we get

$$\bigsqcup \Delta^p \times |S_.|_{/\sim} \xrightarrow{\mathrm{id}\times l_p^{-1}(t)} \bigsqcup \Delta^p \times |P_p S_.|/_\sim,$$

since $l_p(t)$ is a homeomorphism.

In particular, we have a commutative diagram for each p and each n,

$$\begin{array}{ccccc}
\bigsqcup \Delta^p \times |S_.|/_\sim & \xrightarrow{\mathrm{id}\times l_p^{-1}(t)} & ||\ |P_. S_.|\ || & \xrightarrow{||\nu||} & ||\ |E_. S_.|\ || \\
\uparrow & & \uparrow & & \uparrow \\
\Delta^p \times |S_.^{(n)}|/_\sim & \xrightarrow{\mathrm{id}\times l_p^{-1}(t)} & ||\ |P_. S_.^{(n)}|\ || & \xrightarrow{||\nu||} & ||\ |E_. S_.^{(n)}|\ ||
\end{array}$$

For $n = 0$; the lower row becomes,

$$\Delta^p \times |S_.^{(0)}|/_\sim \xrightarrow{\mathrm{id}\times l_p^{-1}(t)} ||\ |P_. S_.^{(0)}|\ || \xrightarrow{||\nu||} ||\ |E_. S_.^{(0)}|\ ||$$

then we get

$$\Delta^p \times S_0 \xrightarrow{\mathrm{id}} \Delta^p \times S_0 \xrightarrow{\mathrm{id}\times\mathrm{diag}} \Delta^p \times S_0 \times \ldots \times S_0.$$

By extending this, one gets

$$L^{(p)} : \Delta^p \times |P_p S_.^{(p)}| \xrightarrow{\mathrm{id}\times l_p(t)} \Delta^p \times |S_.^{(p)}|.$$

□

Note :

The maps $l_p : |P_p S_.| \to |S_.|$ do not commute with the face operators π_i only up to homotopy. This can be seen by the following diagram.

$$\begin{tikzcd}
|P_p S_.| \arrow[r, "l_p"] \arrow[dr, "\pi_i"'] & |S_.| \\
 & |P_{p-1} S_.| \arrow[u, "l_{p-1}"']
\end{tikzcd}$$

here $l_{p-1} \circ \pi_i \sim l_p$.

Corollary 4.3 :

The maps $L_p : \Delta^p \times |P_p S_.|/_\sim \to \Delta^p \times |S_.|/_\sim$ given by $L_p(t,x) = (t, l_p(t)(x))$ induce a homeomorphism

$$L : \| \, |P_. S_.| \, \| \to \| \, |S_.| \, \| \tag{4.-2}$$

where the right hand side whose the face and the degeneracy operators are given by the identity.

Proof :

L is well-defined, that is, $L_p(\varepsilon^i t, x) \sim L_{p-1}(t, \pi_i x)$, in other words, $(\varepsilon^i t, x) \sim (t, \pi_i x)$.

$$L_p(\varepsilon^i t, x) = (\varepsilon^i t, l_p(\varepsilon^i t)(x)) \sim (t, l_{p-1}(t) \circ \pi_i(x)) = L_{p-1}(t, \pi_i x),$$

since $l_p(\varepsilon^i(t)) = l_{p-1}(t) \circ \pi_i$ for the i-th face map ε^i. $\square$

Corollary 4.4 :

$$L^{(p)} : \| \, |P_. S_.|^{(p)} \, \| \to \| \, |S_.|^{(p)} \, \|$$

is a homeomorphism.

Proof :

We can filter both sides of (4.1) by the p-skeletons, i.e., the images of $\Delta^p \times |P_p S_.|$ and $\Delta^p \times |S_.|$ as

$$L^{(p)} : \| \, |P_. S_.^{(p)}| \, \| \to \| \, |S_.^{(p)}| \, \|$$

and show that $L^{(p)}$ is a homeomorphism by using the fact that $L : \overset{\circ}{\Delta}{}^p \times |P_. S_.| \to \overset{\circ}{\Delta}{}^p \times |S_.|$ is a homeomorphism. This can be shown by using an induction on the skeleton. It is a homeomorphism for the zero skeleton and assume that $L^{(p-1)}$ is a homeomorphism and

$$\| \, |P_. S_.|^{(p)} \, \| = \bigsqcup \Delta^p \times |P_. S_.|/\sim .$$

$\bigsqcup \Delta^p \times |P_. S_.|$ can be written as $\bigsqcup \Delta^{p-1} \times |P_. S_.| \bigsqcup \overset{\circ}{\Delta}{}^p \times |P_. S_.|$.

Similarly $\| \, |S_.|^{(p)} \| = \bigsqcup \Delta^p \times |S_.|/\sim$ and $\bigsqcup \Delta^p \times |S_.| = \bigsqcup \Delta^{p-1} \times |S_.| \bigsqcup \overset{\circ}{\Delta}{}^p \times |S_.|$. We already know that $\overset{\circ}{\Delta}{}^p \times |P_. S_.| \to \overset{\circ}{\Delta}{}^p \times |S_.|$ is a homeomorphism and the first part $L^{(p-1)}$ is also a homeomorphism by the induction. Thus $L^{(p)}$ is a homeomorphism. $\square$

Corollary 4.5 :

$l : \| \, |P_. S_.| \, \| \to |S_.|$ is the composition of L and the projection. That is,

$$\| \, |P_. S_.| \, \| \xrightarrow{L} \| \, |S_.| \, \| \xrightarrow{\text{proj}} |S_.|.$$

Furthermore, it is a homotopy equivalence.

Note :

The map above is just induced by $\Delta^p \times |P_p S_.| \to |S_.|$ given by $(t, x) \to l_p(t)(x)$. A homotopy inverse is given by the inclusion $|S| = \Delta^0 \times |P_0 S_.| \subseteq \| \, |P_. S_.| \, \|$. On the other hand, we have another homotopy equivalence

$$u : \|S_.\| \to \| \, |S_.| \, \|$$

which is defined by

$$\begin{tikzcd}
\Delta^p \times S_p \arrow[r, "u"] \arrow[dr, "\mathrm{diag}\times\mathrm{id}"'] & \Delta^p \times |S_.| \\
& \Delta^p \times \Delta^p \times S_p \arrow[u, "\mathrm{id}\times\mathrm{inc}"']
\end{tikzcd}$$

takes (t, x) to (t, t, x), since

$$\|S_.\| \xrightarrow{u} \| \, |S_.| \, \| \xrightarrow{\mathrm{proj}} |S_.|$$

is a natural map.

Corollary 4.6 :

We can define a map v as a composition of L^{-1} and u as follows;

$$\| \, S_. \, \| \xrightarrow{u} \| \, |S_.| \, \| \xrightarrow{L^{-1}} \| \, |P_. S_.| \, \|.$$

This is a homotopy equivalence, since L is a homeomorphism then L^{-1} is continuous. u is a homotopy equivalence then $L^{-1} \circ u$ is a homeomorphism.

Note :

There is a commutative diagram

$$\begin{tikzcd}
\Delta^p \times S_p \arrow[r, "v"] \arrow[dr, "u"'] & \Delta^p \times (\Delta^0)^{p+1} \times (P_p S_{0,\dots,0}) \subseteq \Delta^p \times |P_p S_.| \arrow[d, "L_p"] \\
& \Delta^p \times |S_p|
\end{tikzcd}$$

where $v(t, x) = (t, (0, ..., 0), x)$. It follows that v induces the composite

$$L^{-1} \circ u : \|S_.\| \to \| \, |P_. S_.| \, \|.$$

Actually v is given by the map of Δ-spaces,

$$v : S_p \to |P_p S_.| \supseteq (\Delta^0)^{p+1} \times (P_p S_{0,\dots,0})$$

where S_p is discrete : $v(x) = ((0, ..., 0), x)$.

Note :

The composite of Δ-spaces

$$S_p \xrightarrow{v} |P_p S_.| \xrightarrow{\nu} |E_p S_.|$$

is given by $x \to (\nu_0(x), ..., \nu_p(x)) \in S_0 \times ... \times S_0$ where $\nu_i : S_p \to S_0$ is associated with $\{0\} \to \{0, ..., p\}$ sending 0 to i. Hence

$$\|\nu \circ v\| : \|S_.\| \to \| \ |E.S_.|^{(0)} \ \| \subseteq \| \ |E.S_.| \ \|.$$

We have a diagram

$$\begin{tikzcd} \Delta^p \times S_p \arrow[r, "u"] \arrow[dr, "\mathrm{inc}\times\mathrm{id}"'] & \Delta^p \times |S_.| \\ & \Delta^p \times \Delta^p \times S_p \arrow[u] \end{tikzcd}$$

Now we can define the homotopy u_δ:

$$u_\delta : ||S_.|| \to || \ |S_.| \ ||,$$

that is,

$$u_\delta : \Delta^p \times S_p \to \Delta^{p+1} \times \Delta^p \times S_p$$

is defined by

$$u_\delta(t_1, ..., t_p, x) = ([1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta], (t_1, ..., t_p, x))$$

for $0 < \delta < 1$. Here,

$$\begin{aligned} u_0(t, x) &= (1, ..., 1, t, x) \\ &= ({\varepsilon_0}^{p+1}(0), t, x) \\ &\sim (0, t, x) \in \Delta^0 \times |S_.| \end{aligned}$$

and

$$\begin{aligned} u_1(t, x) &= (t, 0, t, x) \\ &= (\varepsilon^{p+1} t, t, x) \\ &\sim (t, t, x) \\ &= u(t, x). \end{aligned}$$

In order to get a property for a good point, we need to find a relation among the points in Δ^p and the points in Δ^{p+1}. In this map, the gap lengths will be

$$(1 - t_1)\delta, (t_1 - t_2)\delta, ..., (t_{p-1} - t_p)\delta, t_p\delta, 1 - \delta.$$

We want to put the points in Δ^{p+1} between two points in Δ^p by a 1-1 order. $(t_1, ..., t_p)$ can fit into a single gap only if the gap length is $\geq t_1 - t_p$. Since $1 - (1 - t_2)\delta > t_1$ does not all fit into the first gap.

So only the last two are possible, i.e., either

$$1 - (1 - t_p)\delta \geq t_1 \geq t_p \geq 1 - \delta$$

or

$$1 - \delta \geq t_1$$

When $\delta \to 1$, $A_\delta = \{t \in \Delta^p | \text{both possibilities hold}\} \to \{0\}$. We can write a set of unwanted points as

$$C_\delta = \{t \in \Delta^p | \exists i \; 1 - (1 - t_{i+1})\delta \geq t_i\}.$$

Then $C_\delta \to \partial\Delta^p$ when $\delta \to 1$. So the wanted points' set is $\{t \in \Delta^p | \forall i \; 1 - (1 - t_i)\delta > t_i\}$.

Proposition 4.7 :

$u_\delta : \|S_.\| \to \| \, |S_.| \, \|$ is well-defined.

Proof :

Let's remember the necessary equivalence relations:

The equivalence relation $\sim$ on $||S_.||$ is;

$$(\varepsilon^i t, x) \sim (t, \varepsilon_i x).$$

The equivalence relation $\sim$ on $|| \, |S_.| \, ||$ is;

$$(\varepsilon^i t, (t, x)) \sim (t, \underline{\varepsilon_i}(t, x)).$$

The equivalence relations $\sim$ on $|S_.|$ are;

$$(t, x) = (\varepsilon^i t, x) \sim (t, \varepsilon_i x),$$

and

$$(\eta^i t, x) \sim (t, \eta_i x),$$

since $\varepsilon^i = \text{id}$.

Let's see it for the case $i = 0$,

$$\begin{aligned}
u_\delta(\varepsilon^0(t_1, ..., t_p), x) &= u_\delta((1, t_1, ..., t_p), x) = ((1, 1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta), (1, t_1, ..., t_p, x)) \\
&= (\varepsilon^0(1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta), (\varepsilon^0(t_1, ..., t_p), x)) \\
&\sim (1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta, \underline{\varepsilon_0}(\varepsilon^0(t_1, ..., t_p), x)) \\
&= (1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta, (t_1, ..., t_p), \varepsilon_0 x) \\
&\sim (1, 1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta, (\varepsilon^0(t_1, ..., t_p), x)) \\
&= (1, 1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta, (t_1, ..., t_p, x)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
u_\delta(t_1, ..., t_p, \varepsilon_0 x) &= (1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta, (t_1, ..., t_p, \varepsilon_0 x)) \\
&= (1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta, \underline{\varepsilon_0}(t_1, ..., t_p, x) \\
&\sim (\varepsilon^0(1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta), (t_1, ..., t_p, x)) \\
&= (1, 1 - (1 - t_1)\delta, ..., 1 - (1 - t_p)\delta, 1 - \delta, (t_1, ..., t_p), x)).
\end{aligned}$$

So

$$u_\delta(\varepsilon^0 t, x) \sim u_\delta(t, \varepsilon_0 x).$$

Similarly the same result is taken for all $i = 0, ..., p+1$.

Note that ε^i will repeat the i-th element $i = 1, ..., p$. Note that, it makes "one" the first element for $i = 0$ and makes "zero" the last element for $i = p+1$.

u_δ is well-defined. $\square$

Chapter 5

SIMPLICIAL SETS AND STAR COMPLEX

In this chapter, we will give an analogy between a nerve for a simplicial complex and a nerve for a simplicial set.

Let K be a simplicial complex and K' denote its barycentric subdivision consisting of simplices of the form $[\sigma_p \supseteq \sigma_{p-1} \supseteq \ldots \supseteq \sigma_0]$. This subdivision is the nerve of this simplicial complex considered as an ordered set and hence a category. This is also the nerve of the covering by the stars.

If we study with a simplicial set then the case will be different as follows;

For a simplicial set $S_.$, we will construct another simplicial set $\bar{S}_.$ and a retraction $r : \bar{S}_. \to S_0$ such that $\{r^{-1}(\sigma) \mid \sigma \in S_0\}$ corresponds to the covering of stars. If X is a topological space then we have a diagonal map

$$X \to X \times \ldots \times X.$$

But if we replace X by a simplicial set $S_.$, we have seen that we have to replace X_p by $P_pS_.$ but not S_p.

By a retraction $r : \bar{S}_. \to S_0$, the covering is $\{r^{-1}(\sigma) \mid \sigma \in S_0\}$. So the nerve of $|P_pS_.|$ covering $r^{-1}(\sigma)$ $(\sigma \in S_0)$ will correspond to $|\bar{P}_pS_.|$, where

$$\bar{P}_pS_{q_0,\ldots,q_p} := S_{q_0+\ldots+q_p+2p+1}.$$

In the simplicial complex case, X_p will be replaced with $|S_.|$. This will be mentioned more in proposition 5.2.

Let's define the star complex $\bar{S}_q := S_{q+1}$ with the face and degeneracy operators inherited from those of S_{q+1} as $\varepsilon_i : \bar{S}_q \to \bar{S}_{q-1}$ and $\eta_i : \bar{S}_q \to \bar{S}_{q+1}$, where $i = 0, \ldots, q$. Let $i : S_0 \hookrightarrow |\bar{S}_.|$ and $r : |\bar{S}_.| \to S_0$ be defined in degree q by $i(t, y) = (t, \eta_q \circ \ldots \circ \eta_0 y)$ and $r(t, x) = (0, \varepsilon_0 \circ \ldots \circ \varepsilon_q x)$, where $y \in S_0$ and $x \in S_{q+1}$. Then we have:

Proposition 5.1 :

This map $i : S_0 \hookrightarrow |\bar{S}_.|$ is a deformation retract with retraction $r : |\bar{S}_.| \to S_0$ by the following homotopy

$$H_\lambda : |\bar{S}_.| \to |\bar{S}_.|$$

defined by $H_\lambda(t,x) = ((1-\lambda)(t,0) + \lambda(1,...,1), \eta_q x)$ such that $H_1(t,x) \sim i \circ r(t,x)$ and $H_0(t,x) \sim \mathrm{id}_{|\bar{S}_q|}$.

Proof :

Let's take the homotopy as

$$H_\lambda : \bigsqcup \Delta^{q-1} \times \bar{S}_{q-1} \to \bigsqcup \Delta^q \times \bar{S}_q$$

Then

$$\begin{aligned} H_1(t,x) = (\underbrace{1,...,1}_{q-\text{times}}, \eta_q x) &= (\varepsilon^{q-1} \circ ... \circ \varepsilon^0(0), \eta_q x) \\ &\sim (0, \varepsilon_0 ... \varepsilon_{q-1} \eta_q x) \\ &= (0, \eta_0 \varepsilon_0 ... \varepsilon_{q-1} x) \\ &= i(0, \varepsilon_0 ... \varepsilon_{q-1} x) \\ &= i \circ r(t,x) \ \in S_1. \end{aligned}$$

where $\varepsilon^0(0) = 1$, $\varepsilon^1(0) = 0$.

$$\begin{aligned} H_0(t,x) = (t, 0, \eta_q x) &= (\varepsilon^q t, \eta_q x) \\ &\sim (t, \varepsilon_q \eta_q x) \\ &= (t,x) \\ &= \mathrm{id}_{|\bar{S}_q|}. \end{aligned}$$

Thus H_λ gives us a deformation retract S_0 of $|\bar{S}_q|$. $\square$

We would like to give some relevant remarks;

Remark:

1) Hence $|\bar{S}_.|$ has the same number of components as the number of vertices of S_0 and each is contractible even though $S_.$ is not contractible. On the other hand let's write a similar homotopy to see the relation between $|\bar{S}_q|$ and S_q. In order to compactify $|S|^q$, let's write a homotopy

$$H_\lambda : \Delta^{q-1} \times \bar{S}_{q-1} \to \Delta^q \times S_q \backslash \{\varepsilon^q \Delta^{q-1} \times S_q\}$$

where $\Delta^q \times S_q \backslash \{\varepsilon^q \Delta^{q-1} \times S_q\} = |S_q| \backslash |S_{q-1}|$. Instead of taking $|S_q|\backslash|S_{q-1}|$, we can take $|\bar{S}_{q-1}|$ which is compact. This is a way of saying how one can compactify $|S_q|$.

2) If K is a simplicial complex and K' is the barycentric subdivision considered as a simplicial set then $|\bar{K}'|$ is the union of the stars, $r^{-1}(\sigma)$, $\sigma \in {K_0}' = K_.$. For the construction of the corresponding nerve, the situation is more complicated, because the diagonal $X \to X \times ... \times X$ for a topological space is replaced for a simplicial set $S_.$ by the map $P_p S_. \to S_. \times ... \times S_.$. We shall therefore construct a similar complex $\bar{P}_p S$ corresponding to the p-th nerve of the covering by stars. Before giving the structure of $|\bar{P}_p S_.|$, let's express that it can be thought as a nerve of $|P_p S_.|$ for the covering of $|S_.|$ by the stars $\{r^{-1}(\sigma) \mid \sigma \in S_0\}$.

In the case of a manifold X, the nerve of a covering is the simplicial space defined by

$$NX_{\mathcal{U}_p} = \bigcup (U_{i_0} \bigcap ... \bigcap U_{i_p}).$$

where $\mathcal{U} = \{U_i\}_{i\in I}$ is the covering of X and the disjoint union is taken over all $(p+1)$-tuples $(i_0, ..., i_p)$ with $(U_{i_0} \bigcap ... \bigcap U_{i_p}) \neq \emptyset$.

In the case of a bundle over a manifold X, the classifying map is a map $\|NX_{\mathcal{U}.}\| \to BG$.

In our case of a simplicial set $S_.$, $NX_{\mathcal{U}_p}$ is replaced by $|\bar{P}_p S_.|$ which is homotopy equivalent to the set S_p. Then we will have the AW map,

$$|\bar{P}_p S_.| \to |\bar{S}_.| \times ... \times |\bar{S}_.|$$

and by the fact that $|\bar{S}_.|$ has the same homotopy type of S_0, we have $|\bar{P}_p S_.| \to |\bar{S}_.| \times ... \times |\bar{S}_.| \to S_0 \times ... \times S_0$. We can fix the last vertex and we have a trivialization. It gives us a trivial bundle over each $|\bar{S}_.|$ and then over $|\bar{P}_p S_.|$ by pullback via AW. On the other hand we have the following diagram

$$\begin{tikzcd}
l_* F \arrow[r] \arrow[d] & \bar{F} \times ... \times \bar{F} \arrow[r] \arrow[d] & \bar{F}_{S_0} \times ... \times \bar{F}_{S_0} \arrow[d] \\
\Delta^p \times |\bar{P}_p S_.| \arrow[r] & \Delta^p \times |\bar{S}_.| \times ... \times |\bar{S}_.| \arrow[r] & S_0 \times ... \times S_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & G \times ... \times G/G \arrow[d] \\
|\bar{P}_p S_.| \arrow[ur] \arrow[r] & * \times ... \times *.
\end{tikzcd}$$

It gives us a bundle over $|\bar{P}_p S_.|$ with trivialization for fix p and we have a canonical map m_t for all $t \in \Delta^p$, but when we consider $\|\ |\bar{P}_. S_.|\ \|$, we cannot have a bundle with trivialization since there is gluing appears.

By the homotopy equivalence $\|\ |\bar{P}_. S_.|\ \| \simeq \|S_.\|$, we then get the classifying map. Now let's give the structure of $|\bar{P}_p S_.|$ and $\bar{S}_.$:

$\bar{P}_p S_{q_0,...,q_p}$ is defined by $\bar{P}_p S_{q_0,...,q_p} := S_{q_0+...+q_p+2p+1}$, where $q_0 + ... + q_p = n$, with some face and degeneracy operators inherited from the face and degeneracy operators of S_{n+2p+1} as follows:

The face operator are

$$\varepsilon_j{}^{(i)} : S_{n+2p+1} \to S_{n+2p}$$

defined by

$$\varepsilon_j{}^{(i)} := \varepsilon_{q_0+...+q_{i-1}+j+2i},\ j = 0, ..., q_i \text{ but } j \neq 2i + 1 + \sum_{k=0}^{i} q_k.$$

So $\bar{P}_p S_{q_0,...,q_p}$ has only $n + p$ face operators, so we are skipping the $p + 1$ face operators

$$\{\varepsilon_{q_0+1}, \varepsilon_{q_0+q_1+3}, ..., \varepsilon_{q_0+...+q_p+2p+1}\}.$$

Similarly the degeneracy operators

$$\eta_j{}^{(i)} : S_{n+2p+1} \to S_{n+2p+2}$$

can be defined by

$$\eta_j{}^{(i)} := \eta_{q_0+\ldots+q_{i-1}+j+2i},\ j = 0, ..., q_i \text{ but } j \neq 2i + 1 + \sum_k^i q_k.$$

We have an inclusion

$$\Delta^q \times \bar{P}_p S_. \subseteq \Delta^{q+1} \times S_{n+2p+1}$$

with the elements in Δ^{q+1} which can be taken by putting zero at the last coordinate in the elements in Δ^q so the inclusion $\Delta^q \hookrightarrow \Delta^{q+1}$ takes $(s_1, ..., s_q)$ to $(s_1, ..., s_q, 0)$. It induces a surjective but not an injective map of realizations. So

$$|\bar{P}_p S_.| \to |P_p S_.|$$

is not in general a homotopy equivalence.

Now, let's define

$$\|\ |\bar{P}_. S_.|\ \| = \bigsqcup \Delta^p \times \Delta^{q_0} \times ... \times \Delta^{q_p} \times \bar{P}_p S_{q_0,...,q_p}/\sim$$

with the necessary equivalence relations:

$$(\varepsilon^i t, (s, y)) \sim (t, \pi_i(s, y)),$$

where $\pi_i = (\mathrm{proj}_i) \times \mu_{(i)}$ are the face operators on $|\bar{P}_p S_.|$.

$$\mu_{(i)} : S_{n+2p+1} \to S_{n+2p-q_i-1}$$

are the face operators corresponding to

$$\mu^{(i)} : \Delta^{n+2p-q_i-1} \to \Delta^{n+2p+1}$$

take $(e_0, ..., e_{q_0+...+\hat{q}_i+...+q_p+2p-1})$ to $(e_0, ..., e_{q_0+...+...+q_{n+2p+1}})$, by deleting the vectors with indices $(q_0 + ... + q_{i-1} + 2i, ..., q_0 + ... + q_i + 2i + 1)$. It deletes $q_i + 2$ elements. So

$$\mu_{(i)} := \varepsilon_{q_0+...+q_{i-1}+2i} \circ ... \circ \varepsilon_{q_0+...+q_i+2i+1},\ i = 0, ..., p.$$

There is no degeneracy operator on $|\bar{P}_p S_.|$, since it does not give a monotonely increasing sequence. We will consider the face and degeneracy operators on $\bar{P}_p S_.$ for the other equivalence relations as

$$(t, (\varepsilon^j s, y)) \sim (t, s, \varepsilon_j y) \text{ and } (t, (\eta^j s, y)) \sim (t, s, \eta_j y).$$

With the aid of $\Delta^{q_p} \subseteq \Delta^{q_p+1}$, we have an inclusion

$$\Delta^p \times \Delta^{q_0} \times ... \times \Delta^{q_p} \times \bar{P}_p S_{q_0,...,q_p} \hookrightarrow \Delta^p \times \Delta^{q_0+1} \times ... \times \Delta^{q_p+1} \times S_{n+2p+1}$$

Then this defines maps of realizations

$$\begin{tikzcd} \Delta^p \times |\bar{P}_p S_.| \arrow[r] \arrow[dr, "\bar{L}_p"'] & \Delta^p \times |P_p S_.| \arrow[d, "(\approx)"] \\ & \Delta^p \times |S_p| \end{tikzcd}$$

where L_p is the composition and

$$\begin{tikzcd} \| \, |\bar{P}_{.}S_{.}| \, \| \arrow[r, "f"] \arrow[dr, "\bar{L}_{.}"'] & \| \, |P_{.}S_{.}| \, \| \arrow[d, "\approx"] \\ & \| \, |S_{.}| \, \| \end{tikzcd}$$

We would like to give two interesting remarks;

Remark: Note that the horizontal map is also defined by

$$f : \Delta^{q_0} \times ... \times \Delta^{q_p} \times S_{n+2p+1} \longrightarrow \Delta^{q_0} \times ... \times \Delta^{q_p} \times S_{n+p}$$

which takes

$$(s^0, ..., s^p, x) \text{ to } (s^0, ..., s^p, \varepsilon_{q_0+1} \circ \varepsilon_{q_0+q_1+3} \circ ... \circ \varepsilon_{q_0+...+q_p+2p+1}x),$$

where $x \in S_{n+2p+1}$.

Remark: The simplicial set $\bar{P}_pS_{.}$ should not be confused with $P_p\bar{S}_{.}$ which are defined by $P_p\bar{S}_{q_0,...,q_p} := \bar{S}_{n+p} = S_{n+p+1}$ with face and degeneracy operators inherited from those of S_{n+p+1}.

The face operators

$${\varepsilon_j}^{(i)} : \bar{S}_{n+p} \longrightarrow \bar{S}_{n+p-1}$$

defined by

$${\varepsilon_j}^{(i)} := \varepsilon_{q_0+...+q_{i-1}+j},$$

$j = 0, ..., q_i$ but $j \neq q_p + 1$. Similarly, the degeneracy operators

$${\eta_j}^{(i)} : \bar{S}_{n+p} \longrightarrow \bar{S}_{n+p+1}$$

can be defined by

$${\eta_j}^{(i)} := \eta_{q_0+...+q_{i-1}+j},$$

$j = 0, ..., q_i$ but $j \neq q_p + 1$. So we have $n + p$ face and degeneracy operators. Since $|P_p\bar{S}_{.}|$ is contractible, we do not have a nice relation between $|P_p\bar{S}_{.}|$ and $|\bar{P}_pS_{.}|$. One can ask why we do not use $|P_p\bar{S}_{.}|$ instead since we know $|\bar{S}_{.}| \simeq S_0$. $|P_p\bar{S}_{.}| \simeq |\bar{S}_{.}| \simeq S_0$. Hence also $\| \, |P_{.}\bar{S}_{.}| \, \| \cong S_0$. $|P_p\bar{S}_{.}|$ is homotopy equivalent to $|\bar{S}_{.}|$ which is trivial. There is no point to define a classifying map on a bundle which is already trivial.

Proposition 5.2 :

Let $i : S_p \hookrightarrow \bar{P}_pS_{.}$ be an inclusion defined by $i(t, x) = (t, 0, \eta_0 \circ ... \circ \eta_p x) = (t, 0, \eta_{2p} \circ ... \circ \eta_0 x)$ and $r : \bar{P}_pS_{.} \to S_p$ be the retraction defined as $r(t, y) = (t, 0, \varepsilon_0 \circ ... \circ \varepsilon_{2p-2} \circ \varepsilon_{2p}y)$.

1) $i : S_p \hookrightarrow \bar{P}_pS_{.}$ is a deformation retract with the retraction r.

2) There is a diagram of homotopy equivalences

$$\begin{tikzcd}
\|S.\| \arrow[r, "i"] \arrow[dr] \arrow[ddr] & \| \, |\bar{P}.S.| \, \| \arrow[d, "f"] \\
& \| \, |P.S.| \, \| \arrow[d, "\approx"] \\
& \| \, |S.| \, \|
\end{tikzcd} \tag{5.-9}$$

3) There is a commutative diagram

$$\begin{tikzcd}
\| \, |\bar{P}.S.| \, \| \arrow[r] \arrow[d] & \| \, |E.\bar{S}.| \, \| \arrow[d] \\
\| \, |P.S.| \, \| \arrow[r] & \| \, |E.S.| \, \|
\end{tikzcd}$$

Proof:

1) Let's define the homotopy

$$H_\lambda : \Delta^p \times \Delta^{q_0-1} \times ... \times \Delta^{q_p-1} \times S_{n+p} \to \Delta^p \times \Delta^{q_0} \times ... \times \Delta^{q_p} \times S_{n+2p+1}$$

as

$$H_\lambda(t,s,y) = (t, (1-\lambda)(s^0, 0) + \lambda(1,...,1), ..., (1-\lambda)(s^p, 0) + \lambda(1,...,1), \eta_{n+2p} \circ ... \circ \eta_{q_0} y),$$

where $S_{n+p} = |\bar{P}_p S_{q_0-1,...,q_p-1}|$ and $S_{n+2p+1} = |\bar{P}_p S_{q_0,...,q_p}|$.

Then

$$\begin{aligned}
H_1(t,s,y) &= (t, \underbrace{1,...1}_{q_0-\text{times}}, ..., \underbrace{1,...,1}_{q_p-\text{times}}, \eta_{n+2p} \circ ... \circ \eta_{q_0} y) \\
&= (t, \varepsilon^{q_0-1}...\varepsilon^0(0), \varepsilon^{q_1-1}...\varepsilon^0(0), ..., \varepsilon^{q_p-1}...\varepsilon^0(0), \eta_{n+2p} \circ \eta_{n-q_p+2p-2} \cdots \eta_{q_0+q_1+2}\eta_{q_0} y) \\
&\sim (t, 0, \varepsilon_0 ... \varepsilon_{q_0-1}\varepsilon_{q_0+2} \cdots \varepsilon_{q_0+q_1+1}\varepsilon_{q_0+q_1+4} \cdots \varepsilon_{q_0+q_1+q_2+3}\varepsilon_{q_0+q_1+q_2+6} \cdots \varepsilon_{q_0+q_1+q_2+q_3+5}\varepsilon_{q_0+q_1+q_2+q_3+8} \cdots \\
&\quad \varepsilon_{q_0+...+q_{p-1}+2p-3}\varepsilon_{q_0+...+q_{p-1}+2p} \cdots \varepsilon_{n+2p-1}\eta_{n+2p}\eta_{n-q_p+2p-2} \cdots \eta_{q_0+q_1+2}\eta_{q_0} y) \\
&= (t, 0, \eta_{2p}\eta_{2p-2}\varepsilon_0 ... \varepsilon_{q_0-1}\varepsilon_{q_0+2} \cdots \varepsilon_{n-q_p-q_{p-1}+2p+1}\varepsilon_{n-q_p-q_{p-1}+2p-2} \cdots \varepsilon_{n-q_p+2p-3}\eta_{n-q_p-q_{p-1}+2p-4} \\
&\quad \varepsilon_{n-q_p+2p-1} \cdots \varepsilon_{n+2p-2}\eta_{n-q_p-q_{p-1}-q_{p-2}+2p-6} \cdots \eta_{q_0+q_1+q_2+q_3+6}\eta_{q_0+q_1+q_2+4}\eta_{q_0+q_1+2}\eta_{q_0} y) \\
&= (t, 0, \eta_{2p}\eta_{2p-2} \ldots \eta_6 \varepsilon_0 ... \varepsilon_{q_0-1}\varepsilon_{q_0+2} \cdots \varepsilon_{q_0+q_1+q_2+3}\varepsilon_{q_0+q_1+q_2+6} \cdots \varepsilon_{q_0+q_1+q_2+q_3+5} \\
&\quad \eta_{q_0+q_1+q_2+4}\eta_{q_0+q_1+2}\eta_{q_0}\varepsilon_{q_0+q_1+q_2+q_3+4} \cdots \varepsilon_{n+p-1} y) \\
&= (t, 0, \eta_{2p}\eta_{2p-2} \ldots \eta_6\eta_4\eta_2\varepsilon_0 ... \varepsilon_{q_0-1}\varepsilon_{q_0+2} \cdots \varepsilon_{q_0+q_1+1}\eta_{q_0}\varepsilon_{q_0+q_1+2} \cdots \varepsilon_{n+p-1} y) \\
&= (t, 0, \eta_{2p}\eta_{2p-2} \ldots \eta_6\eta_4\eta_2\eta_0\varepsilon_0 ... \varepsilon_{q_0-1}\varepsilon_{q_0+1} \cdots \varepsilon_{q_0+q_1}\varepsilon_{q_0+q_1+2} \cdots \varepsilon_{q_0+q_1+q_2+1} \cdots \varepsilon_{n-q_p+p} \cdots \varepsilon_{n+p-1} y) \\
&= (t, 0, \underline{\eta}\underline{\varepsilon} y) \\
&= i(t, 0, \underline{\varepsilon} y) \\
&= i \circ r(t, y) \in S_{2p+1},
\end{aligned}$$

where

$$\begin{aligned}\underline{\eta} &= \eta_{2p}...\eta_0 : S_{2p+1} \to S_p,\\ \underline{\varepsilon} &= \varepsilon_0...\varepsilon_{q_0-1}\varepsilon_{q_0+1}...\varepsilon_{q_0+q_1}\varepsilon_{q_0+q_1+2}...\varepsilon_{q_0+q_1+q_2+1}...\varepsilon_{n-q_p+p}...\varepsilon_{n+p-1} : S_{n+p} \to S_p.\end{aligned}$$

$$\begin{aligned}H_0(t,s,y) &= (t, s^0, 0, ..., s^p, 0, \eta_{n+2p} \circ ... \circ \eta_{q_0} y)\\ &= (t, \varepsilon^{q_0} s^0, ..., \varepsilon^{q_p} s^p, \eta_{n+2p} \circ \eta_{n-q_p+2p-2}...\eta_{q_0+q_1+2}\eta_{q_0} y)\\ &\sim (t, s, \varepsilon_{q_0}\varepsilon_{q_0+q_1+2}...\varepsilon_{n+2p}\eta_{n+2p}...\eta_{q_0} y)\\ &= (t,s,y)\\ &= \mathrm{id}_{\|\ |\bar{P}_p S_.|\ \|}.\end{aligned}$$

So, this homotopy gives that $\mathrm{id}_{|\bar{P}_p S_.|} \sim i \circ r$.

2) The first homotopy equivalence is induced by the inclusion given in 1).

We have defined $v : \|\ S_.\ \| \to \|\ |P_. S_.|\ \|$ as a composition of $u_\delta : \|S_.\| \to \|\ |S_.|\ \|$ and $L^{-1} : \|\ |S_.|\ \| \to \|\ |P_. S_.|\ \|$. Here L is a homeomorphism and L^{-1} is continuous. u_δ is a homotopy equivalence so v is a homotopy equivalence.

In the remark we already defined

$$\Delta^p \times |\bar{P}_. S_.| \to \Delta^p \times |P_. S_.|.$$

So it induces a homotopy equivalence

$$\|\ |\bar{P}_. S_.|\ \| \to \|\ |P_. S_.|\ \|.$$

Thus we have the required diagram of homotopy equivalences.

3) Let's see the following diagram is commutative;

$$\begin{array}{ccc}\Delta^p \times |\bar{P}_p S_.| & \xrightarrow{\mathrm{id}\times\mathrm{AW}} & \Delta^p \times |\bar{S}_.| \times ... \times |\bar{S}_.|\\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}}\\ \Delta^p \times |P_p S_.| & \xrightarrow{\mathrm{id}\times\mathrm{AW}} & \Delta^p \times |S_.| \times ... \times |S_.|\end{array}$$

$$\begin{aligned}(\mathrm{id}\times\mathrm{AW})(t, s^0, ..., s^p, y) &= (t, s^0, ..., s^p, \tilde{\varepsilon}^{q_1+...+q_p+2p} y,\\ &\quad {\varepsilon_0}^{q_0+2}\tilde{\varepsilon}^{q_2+...+q_p+2p-1} y, ...)\\ \bar{f}(t, s^0, ..., s^p, \tilde{\varepsilon}^{q_1+...+q_p+2p} y, {\varepsilon_0}^{q_0+2}\tilde{\varepsilon}^{q_2+...+q_p+2p-1} y, ...) &= (t, s^0, ..., s^p, \varepsilon_{q_0+1}\tilde{\varepsilon}^{q_1+...+q_p+2p} y,\\ &\quad \varepsilon_{q_1+1}{\varepsilon_0}^{q_0+2}\tilde{\varepsilon}^{q_2+...+q_p+2p-1} y, ...,)\end{aligned}$$

On the other hand

$$\begin{aligned}f(t, s^0, ..., s^p, y) &= (t, s^0, ..., s^p, \varepsilon_{n+p+1}...\varepsilon_{n+2p+1} y)\\ (\mathrm{id}\times\mathrm{AW})(t, s^0, ..., s^p, \varepsilon_{n+p+1}...\varepsilon_{n+2p+1} y) &= (t, s^0, ..., s^p, \tilde{\varepsilon}^{q_1+...+q_p+p}\varepsilon_{n+p+1}...\varepsilon_{n+2p+1} y,\\ &\quad {\varepsilon_0}^{q_0+1}\tilde{\varepsilon}^{q_2+...+q_p+p}\varepsilon_{n+p+1}...\varepsilon_{n+2p+1} y, ...)\end{aligned}$$

Let's check the first components;

$$\begin{aligned}
\varepsilon_{q_0+1}\tilde{\varepsilon}^{q_1+\ldots+q_p+2p} &= \tilde{\varepsilon}^{q_1+\ldots+q_p+p}\varepsilon_{n+p+1}\ldots\varepsilon_{n+2p+1}\\
\varepsilon_{q_0+1}\tilde{\varepsilon}^{n+2p+1-(q_0+1)} &= \tilde{\varepsilon}^{n+p-q_0}\varepsilon_{n+p+1}\ldots\varepsilon_{n+2p+1}\\
\varepsilon_{q_0+1}\varepsilon_{q_0+2}\ldots\varepsilon_{n+p}\varepsilon_{n+p+1}\ldots\varepsilon_{n+2p+1} &= \varepsilon_{q_0+1}\ldots\varepsilon_{n+p}\varepsilon_{n+p+1}\ldots\varepsilon_{n+2p+1}.
\end{aligned}$$

One can show that the other coordinates are the same.

So $\bar{f} \circ (\mathrm{id} \times \mathrm{AW}) = (\mathrm{id} \times \mathrm{AW}) \circ f$. □

Chapter 6

BUNDLES ON SIMPLICIAL SETS AND TRANSITION FUNCTIONS

In this chapter, we will define transition functions for a given bundle over a simplicial set and admissible trivialization. Conversely, we will also show that for a given set of transition functions satisfying some certain relations, there is a bundle over a realization of a simplicial set and trivializations with transition functions.

Definition 6.1 (A bundle over a simplicial set) :

A bundle over $|S|$ is a sequence of bundles over $\Delta^p \times \sigma$ for all p, where $\sigma \in S_p$ and with commutative diagrams;

$$\begin{array}{ccc} F_{\varepsilon_j \sigma} & \xrightarrow{\bar{\varepsilon}^j} & F_\sigma \\ \downarrow & & \downarrow \\ \Delta^{p-1} \times \varepsilon_j \sigma & \xrightarrow{\varepsilon^j} & \Delta^p \times \sigma \end{array} \tag{6.1}$$

and

$$\begin{array}{ccc} F_{\eta_j \sigma} & \xrightarrow{\bar{\eta}^j} & F_\sigma \\ \downarrow & & \downarrow \\ \Delta^{p+1} \times \eta_j \sigma & \xrightarrow{\eta^j} & \Delta^p \times \sigma \end{array} \tag{6.2}$$

with the compatibility conditions:

$$\bar{\varepsilon}^j \bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i \bar{\varepsilon}^{j-1} & : \quad i < j \\ \bar{\varepsilon}^{i+1} \bar{\varepsilon}^j & : \quad i \geq j, \end{cases} \tag{6.3}$$

$$\bar{\eta}^j \bar{\eta}^i = \begin{cases} \bar{\eta}^i \bar{\eta}^{j+1} & : \quad i \leq j \\ \bar{\eta}^{i-1} \bar{\eta}^j & : \quad i > j, \end{cases} \tag{6.4}$$

and

$$\bar{\eta}^j \bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i \bar{\eta}^{j-1} & : \quad i < j \\ 1 & : \quad i = j, i = j+1 \\ \bar{\varepsilon}^{i-1} \bar{\eta}^i & : \quad i > j+1. \end{cases} \tag{6.5}$$

Given a G-bundle $F \to |S|$, G a lie group, since Δ^p is contractible, we can choose a trivialization $\varphi_\sigma : F_\sigma \to \Delta^p \times \sigma \times G$ for a non-degenerate $\sigma \in S_p$. If σ is degenerate, that is, there exists τ such that $\sigma = \eta_i \tau$, then the trivialization of σ is defined as pullback of the trivialization of τ, that is, $\varphi_\sigma = {\eta^i}^*(\varphi_\tau)$.

Definition 6.2 (Admissible Trivialization) :

A set of trivializations is called admissible, in case φ_σ for $\sigma = \eta_i \tau$ is given by $\varphi_\sigma = {\eta^i}^*(\varphi_\tau)$.

We have thus proved:

Lemma 6.3 :

Admissible trivialization always exists.

Now, let us construct the transition functions for a simplex $\sigma \in S_p$ before giving the following proposition:

Given a bundle and a set of trivializations, we get for each face τ of say dim$\tau = q < p$ in σ, a transition function $v_{\sigma,\tau} : \Delta^q \to G$. E.g., if $\tau = \varepsilon_i \sigma$ then the transition function $v_{\sigma,\varepsilon_i\sigma} : \Delta^{p-1} \to G$ is given by the diagram

$$\begin{tikzcd} \Delta^{p-1} \times (\varepsilon_i\sigma) \times G \arrow[r, "\Theta"] \arrow[d] & \Delta^p \times (\sigma) \times G \arrow[d] \\ \Delta^{p-1} \times \varepsilon_i\sigma \arrow[r, "\varepsilon^i"] & \Delta^p \times \sigma \end{tikzcd}$$

where $\varepsilon_i \sigma = \tau$ and $\Theta = \varphi_\sigma \circ \bar{\varepsilon}^i \circ {\varphi_{\varepsilon_i\sigma}}^{-1}$. So

$$\{v_{\sigma,\tau} | \sigma \in S_p \text{ and } \tau \text{ is a face of } \sigma\}$$

are **the transition functions** for the bundle over $|S|$. Now, we can give the statement:

Proposition 6.4 :

Given a bundle on a simplicial set and admissible trivializations, the transition function $v_{\sigma,\tau}$, where τ is a face of σ, satisfies;

i) σ is nondegenerate: if $\gamma = \varepsilon_j \sigma$ and $\tau = \varepsilon_i \gamma$ then

$$v_{\sigma,\tau} = (v_{\sigma,\gamma} \circ \varepsilon^i).v_{\gamma,\tau}.$$

This is called the cocycle condition.

ii)σ is degenerate: If $\sigma = \eta_j \sigma'$ and $\tau = \varepsilon_i \sigma$ then when $i < j$ for $\tau = \eta_{j-1}\tau'$ one gets $\tau' = \varepsilon_i \sigma'$ and when $i > j+1$ for $\tau = \eta_j \tau'$ one gets $\tau' = \varepsilon_{i-1}\sigma'$. For the other cases, $i = j$ or $i = j+1$, $\tau = \sigma'$. Then the transition functions satisfy:

$$v_{\sigma,\tau} = \begin{cases} v_{\sigma',\tau'} \circ \eta^{j-1} & : \quad i < j \\ 1 & : \quad i = j, i = j+1 \\ v_{\sigma',\tau'} \circ \eta^j & : \quad i > j+1. \end{cases}$$

iii) If τ is a composition of face operators of σ, e.g., $\tau = \tilde{\varepsilon}^{p-(i-1)}\sigma$, $i = 1, ..., p$ then

$$v_{\sigma,\tau} = (v_{\sigma,\tilde{\varepsilon}^1\sigma} \circ (\varepsilon^i)^{p-i}).(v_{\tilde{\varepsilon}^1\sigma,\tilde{\varepsilon}^2\sigma} \circ (\varepsilon^i)^{p-i-1})...(v_{\tilde{\varepsilon}^{p-(i+1)}\sigma,\tilde{\varepsilon}^{p-i}\sigma} \circ \varepsilon^i).v_{\tilde{\varepsilon}^{p-i}\sigma,\tau}.$$

Proof :

i) If σ is non-degenerate:

$\gamma = \varepsilon_j\sigma$ and $\tau = \varepsilon_i\gamma$ then we can see the cocycle condition which is expressed in the proposition. In order to see this, let's take Δ^{p-2} and by using the trivializations look at the diagram below

$$\begin{tikzcd}
\Delta^{p-2} \times G \arrow[r] \arrow[d] & \Delta^{p-1} \times G \arrow[r] \arrow[d] & \Delta^p \times G \arrow[d] \\
\Delta^{p-2} \times \varepsilon_i\varepsilon_j\sigma \arrow[r, "\varepsilon^i"] & \Delta^{p-1} \times \varepsilon_j\sigma \arrow[r, "\varepsilon^j"] & \Delta^p \times \sigma
\end{tikzcd}$$

That is, the transition function $v_{\sigma,\tau} : \Delta^{p-2} \to G$ is the multiplication of pullback of the transition function $v_{\sigma,\varepsilon_j\sigma}$ via the face map $\varepsilon^i : \Delta^{p-2} \to \Delta^{p-1}$ with the transition function $v_{\varepsilon_j\sigma,\varepsilon_i\varepsilon_j\sigma} : \Delta^{p-2} \to G$. This can be written as

$$v_{\sigma,\tau} = v_{\sigma,\varepsilon_i\varepsilon_j\sigma} = (v_{\sigma,\varepsilon_j\sigma} \circ \varepsilon^i).v_{\varepsilon_j\sigma,\varepsilon_i\varepsilon_j\sigma}$$

then we get the result.

ii) If σ is degenerate:

If $\sigma = \eta_j\sigma'$ and $\tau = \varepsilon_i\sigma$ then $\tau = (\varepsilon_i \circ \eta_j)\sigma'$ and it is equal to

$$\tau = \varepsilon_i\eta_j\sigma' = \begin{cases} \eta_{j-1}\varepsilon_i\sigma' & : \quad i < j \\ 1\sigma' & : \quad i = j, i = j+1 \\ \eta_j\varepsilon_{i-1}\sigma' & : \quad i > j+1. \end{cases}$$

For the first case $i < j$:

If $\sigma = \eta_j\sigma'$ and $\tau = \varepsilon_i\sigma$ then

$$\tau = (\varepsilon_i \circ \eta_j)\sigma' = (\eta_{j-1} \circ \varepsilon_i)\sigma'$$

So $\varepsilon_{j-1}\tau = (\varepsilon_{j-1} \circ \eta_{j-1} \circ \varepsilon_i)\sigma'$, then there exists τ' such that $\varepsilon_{j-1}\tau = \tau'$. Thus for $\tau = \eta_{j-1}\tau'$, $\tau' = \varepsilon_i\sigma'$. Similarly the second part, $i > j+1$, can be shown.

By using the following diagram

$$\begin{tikzcd}
\Delta^{p-1} \times G \arrow[r] \arrow[d] & \Delta^p \times G \arrow[r] \arrow[d] & \Delta^{p-1} \times G \arrow[d] \\
\Delta^{p-1} \times \varepsilon_i\eta_j\sigma' \arrow[r, "\varepsilon^i"] & \Delta^p \times \eta_j\sigma' \arrow[r, "\eta^j"] & \Delta^{p-1} \times \sigma'
\end{tikzcd}$$

we get

$$v_{\sigma',\tau} = v_{\sigma',\varepsilon_i\eta_j\sigma'} = (v_{\sigma',\eta_j\sigma'} \circ \varepsilon^i).v_{\eta_j\sigma',\varepsilon_i\eta_j\sigma'},$$

where $\eta_j\sigma' = \sigma$, $\tau = \varepsilon_i\sigma$, $\tau' = \varepsilon_i\sigma'$ and $\sigma \in S_p$, $\sigma' \in S_{p-1}$, $\tau \in S_{p-1}$, $\tau' \in S_{p-2}$. Here $v_{\sigma',\eta_j\sigma'}$ is 1. So we have

$$v_{\sigma',\tau} = v_{\sigma,\tau}.$$

On the other hand when $i < j$ we have the following diagram;

$$\begin{array}{ccccc} \Delta^{p-1} \times G & \longrightarrow & \Delta^{p-2} \times G & \longrightarrow & \Delta^{p-1} \times G \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^{p-1} \times \eta_{j-1}\varepsilon_i\sigma' & \xrightarrow{\eta^{j-1}} & \Delta^{p-2} \times \varepsilon_i\sigma' & \xrightarrow{\varepsilon^i} & \Delta^{p-1} \times \sigma' \end{array}$$

Then we have

$$v_{\sigma',\tau} = v_{\sigma',\eta_{j-1}\varepsilon_i\sigma'} = (v_{\sigma',\varepsilon_i\sigma'} \circ \eta^{j-1}).v_{\varepsilon_i\sigma',\tau}.$$

The transition function, $v_{\varepsilon_i\sigma',\tau}$, is identity. So we have

$$v_{\sigma',\tau} = v_{\sigma',\varepsilon_i\sigma'} \circ \eta^{j-1} = v_{\sigma,\tau}$$

as required. That is, the transition function $v_{\sigma',\tau}$ is the pullback of the transition function $v_{\sigma',\tau'}$ via η^{j-1};

$$v_{\sigma,\tau} = \eta^{j-1*}(v_{\sigma',\tau'}).$$

The second case $i = j$, $i = j+1$

$$v_{\sigma',\sigma'} = (v_{\sigma',\eta_j\sigma'} \circ \varepsilon^j).v_{\eta_j\sigma',\sigma'} = 1.$$

since both parts on RHS are 1. For the last case $i > j+1$, we get

$$v_{\sigma',\tau} = (v_{\sigma',\varepsilon_{i-1}\sigma'} \circ \eta^j).v_{\varepsilon_{i-1}\sigma',\eta_j\varepsilon_{i-1}\sigma'}.$$

Similarly the second part is 1 so by the same idea used for the first case, we only have

$$v_{\sigma,\tau} = v_{\sigma',\tau'} \circ \eta^j.$$

Thus we have the required equalities.

iii) If $\tau = \tilde{\varepsilon}^{p-(i-1)}\sigma$, $i = 1, ..., p$, then

$$\Delta^{i-1} \times \tilde{\varepsilon}^i\sigma \xrightarrow{\varepsilon^i} \Delta^i \times \tilde{\varepsilon}^{p-i}\sigma \xrightarrow{\varepsilon^{i+1}} \ldots \xrightarrow{\varepsilon^{p-1}} \Delta^{p-1} \times \tilde{\varepsilon}^1\sigma \xrightarrow{\varepsilon^p} \Delta^p \times \sigma.$$

By using the same idea in ii), we get

$$\begin{aligned}
v_{\sigma,\tau} &= (v_{\sigma,\tilde{\varepsilon}^{p-i}\sigma} \circ \varepsilon^i).v_{\tilde{\varepsilon}^{p-i}\sigma,\tau} \\
&= (v_{\sigma,\tilde{\varepsilon}^{p-(i+1)}\sigma} \circ \varepsilon^{i+1} \circ \varepsilon^i).(v_{\tilde{\varepsilon}^{p-(i+1)}\sigma,\tilde{\varepsilon}^{p-i}\sigma} \circ \varepsilon^i).v_{\tilde{\varepsilon}^{p-i}\sigma,\tau} \\
&= ... \\
&= ... \\
&= ... \\
&= (v_{\sigma,\tilde{\varepsilon}^1\sigma} \circ (\varepsilon^{p-1} \circ ... \circ \varepsilon^i))...(v_{\tilde{\varepsilon}^{p-(i+1)}\sigma,\tilde{\varepsilon}^{p-i}\sigma} \circ \varepsilon^i).v_{\tilde{\varepsilon}^{p-i}\sigma,\tau} \\
&= (v_{\sigma,\tilde{\varepsilon}^1\sigma} \circ (\varepsilon^i)^{p-i}).(v_{\tilde{\varepsilon}^1\sigma,\tilde{\varepsilon}^2\sigma} \circ (\varepsilon^i)^{p-i-1})...(v_{\tilde{\varepsilon}^{p-(i+1)}\sigma,\tilde{\varepsilon}^{p-i}\sigma} \circ \varepsilon^i).v_{\tilde{\varepsilon}^{p-i}\sigma,\tau} \\
&= (v_{\sigma,\varepsilon_p\sigma} \circ (\varepsilon^i)^{p-i}).(v_{\varepsilon_p\sigma,\varepsilon_{p-1}\varepsilon_p\sigma} \circ (\varepsilon^i)^{p-i-1})...v_{\tilde{\varepsilon}^{p-i}\sigma,\tau}.
\end{aligned}$$

So $v_{\sigma,\tau}$ is defined in terms of the other transition functions with the composition of some face maps for $\tau = \tilde{\varepsilon}^{p-(i-1)}\sigma = (\varepsilon_i \circ .... \circ \varepsilon_p)\sigma$, $i = 1, ..., p$. □

Proposition 6.5 :

Assume that we have a bundle over $|S|$. Then

1) There exists admissible trivialization such that the transition function is given by

$$v_{\sigma,\varepsilon_i\sigma} = 1 \text{ if } i < p.$$

2) For $\tau = \tilde{\varepsilon}^{p-(i-1)}\sigma$, $i = 1, ..., p$, we get $v_{\sigma,\tau}$ as product of some transition functions:

$$v_{\sigma,\tau} = (v_\sigma \circ (\varepsilon^i)^{p-i}).(v_{\tilde{\varepsilon}^1\sigma} \circ (\varepsilon^i)^{p-i-1}).(v_{\tilde{\varepsilon}^2\sigma} \circ (\varepsilon^i)^{p-i-2})...(v_{\tilde{\varepsilon}^{p-(i+1)}\sigma} \circ (\varepsilon^i)^1).(v_{\tilde{\varepsilon}^{p-i}\sigma}).$$

3) The transition functions $v_{\sigma,\tau}$ satisfy the compatibility conditions:

$$v_\sigma \circ \varepsilon^i = \begin{cases} v_{\varepsilon_i\sigma} & : \ i < p-1 \\ v_{\varepsilon_{p-1}\sigma}.{v_{\varepsilon_p\sigma}}^{-1} & : \ i = p-1 \end{cases}$$

4) For a degenerate σ, we have

$$v_\sigma \circ \eta^j = v_{\eta_j\sigma}.$$

for $\forall j$.

Proof :

1) First of all let's define a special case of trivialization by induction on the dimension of the non-degenerate simplex.

One has a bundle over $|S|$ so $F \to \Delta^0 \times S_0$ is a 0-bundle. By choosing a trivialization we get $F \cong G$. One also has 1-bundle on $\Delta^1 \times S_1$ for the degeneracy elements and by admissible trivialization for the degeneracy elements in S_1, we get $\varphi_\sigma = {\eta_i}^*(\varphi_\tau)$. For the non-degenerate elements in S_1; we delete the first vertex $< 0 >$ then it is shrunk to $< 1 >$. So the identity on $< 0 >$ is extended to whole simplex.

The idea is the same for 2-simplex. We have a trivialization on each face but we can not extend this to the whole simplex because of the obstruction theory. Let us define the trivialization on the 2-simplex by the homotopy: $H_t : \Delta^2 \to \Delta^2$ defined by

$$\begin{aligned} H_t(\Delta^2 \backslash \varepsilon^2 \Delta^1) &= \mathrm{id}_{|_{\Delta^2 \backslash \varepsilon^2 \Delta^1}} \\ H_0(\Delta^2) &= \mathrm{id}_{|_{\Delta^2}} \\ H_1(\Delta^2) &= \Delta^2 \backslash \varepsilon^2 \Delta^1. \end{aligned}$$

The homotopy tells us that the identity on $\Delta^2 \backslash \varepsilon^2 \Delta^1$ is extended to the whole simplex.

In general: Let $A \subseteq X$ and $F \to X$ be a principal bundle. A is a deformation retract of X such that $H : X \times I \to X$, $H_0(X) = \mathrm{id}_X$, $H_1(X) = \mathrm{id}_A$ and $H_t(X) \subseteq A$. It induces an isomorphism ${H_0}^*(F) \cong {H_1}^*(F)$.

Then

$$F = {\mathrm{id}^*}_F = {H_0}^*(F) = {H_1}^*(F) \cong {H_1}^*(F_{|_A}) = {H_1}^*(A \times G) = X \times G.$$

A is contractible so each principal G bundle over A is trivial. Thus, one can define the trivialization on the whole simplex by the homotopy theory. So the homotopy $H_t : \Delta^p \to \Delta^p$, in general, is defined by

$$\begin{aligned} H_t(\Delta^p \backslash \varepsilon^p \Delta^{p-1}) &= \mathrm{id}_{|_{\Delta^p \backslash \varepsilon^p \Delta^{p-1}}} \\ H_0(\Delta^p) &= \mathrm{id}_{|_{\Delta^p}} \\ H_1(\Delta^p) &= \Delta^p \backslash \varepsilon^p \Delta^{p-1}. \end{aligned}$$

Thus we get the trivialization on the whole simplex extending the given trivialization on $\Delta^p \backslash \varepsilon^p \Delta^{p-1} = \bigcup_{i=0}^{p-1} \varepsilon^i \Delta^{p-1}$. Hence given trivializations of F_σ, for dimension of $\tau < p$, we obtain a special kind of trivializations $\varphi_\sigma : F_\sigma \to \Delta^p \times G$ for each non-degenerate σ extending those on $\sigma \times \varepsilon^i \Delta^{p-1}$, $i < p$.

If σ is degenerate say $\sigma = \eta_i \tau$ then we define the trivialization for F_σ by pullback via η^i from $\Delta^{p-1} \times \tau$. Now for these, we get the transition functions $v_{\sigma, \varepsilon_i \sigma}$ from the diagram,

$$\begin{array}{ccc} \Delta^{p-1} \times G & \xrightarrow{\varphi_\sigma \circ \bar{\varepsilon}^i \circ {\varphi_{\varepsilon_i \sigma}}^{-1}} & \Delta^p \times G \\ \downarrow & & \downarrow \\ \Delta^{p-1} \times \varepsilon_i \sigma & \xrightarrow{\varepsilon^i} & \Delta^p \times \sigma \end{array}$$

If $i < p$ then $\varphi_\sigma \circ \bar{\varepsilon}^i = \varphi_{\varepsilon_i \sigma}$, since $\Delta^{p-1} \hookrightarrow \Delta^p$ is an inclusion and we delete some first vertices. Hence

$$v_{\sigma, \varepsilon_i \sigma} = 1, \; i < p.$$

2) This is the reformulation of the previous proposition, since for $i = p$, we put $v_\sigma = v_{\sigma, \varepsilon_p \sigma}$. Then, for $\tau = \tilde{\varepsilon}^{p-(i-1)} \sigma$, $i = 1, ..., p$, we get $v_{\sigma, \tau}$ as given in the statement.

3) For $j = p$, we have $\gamma = \varepsilon_p \sigma$ and $\tau = \varepsilon_i \gamma$, and also $\gamma' = \varepsilon_i \sigma$ and $\tau = \varepsilon_{p-1} \gamma'$, when $(i < j)$, in proposition 6.4, put $\gamma = \varepsilon_p \sigma$ and $\tau = \varepsilon_i \gamma$ then

$$\tau = \varepsilon_i \varepsilon_p \sigma = \varepsilon_{p-1} \varepsilon_i \sigma.$$

Hence by the cocyle conditions;

$$\begin{aligned} v_{\sigma,\varepsilon_i\varepsilon_p\sigma} &= (v_{\sigma,\varepsilon_p\sigma}\circ\varepsilon^i).v_{\varepsilon_p\sigma,\varepsilon_i\varepsilon_p\sigma} \\ v_{\sigma,\varepsilon_{p-1}\varepsilon_i\sigma} &= (v_{\sigma,\varepsilon_i\sigma}\circ\varepsilon^{p-1}).v_{\varepsilon_i\sigma,\varepsilon_{p-1}\varepsilon_i\sigma} \end{aligned}$$

and $v_{\sigma,\varepsilon_p\sigma} = v_\sigma$ and when $i < p$, $v_{\sigma,\varepsilon_i\sigma}\circ\varepsilon^{p-1} = \mathrm{id}$, follows from 1), so the first equation becomes

$$v_{\sigma,\varepsilon_i\varepsilon_p\sigma} = (v_\sigma\circ\varepsilon^i).v_{\varepsilon_p\sigma,\varepsilon_i\varepsilon_p\sigma}.$$

The second one is,

$$v_{\sigma,\varepsilon_{p-1}\varepsilon_i\sigma} = v_{\varepsilon_i\sigma}.$$

Thus we get

$$v_{\varepsilon_i\sigma} = \begin{cases} (v_\sigma\circ\varepsilon^{p-1}).v_{\varepsilon_p\sigma} & : \quad i = p-1 \\ v_\sigma\circ\varepsilon^i & : \quad i < p-1 \end{cases}$$

In other words, we have

$$v_\sigma\circ\varepsilon^i = \begin{cases} v_{\varepsilon_i\sigma} & : \quad i < p-1 \\ v_{\varepsilon_{p-1}\sigma}.{v_{\varepsilon_p\sigma}}^{-1} & : \quad i = p-1 \end{cases}$$

4) From ii) in proposition 6.4, we put $i = p$ and we get

$$v_{\eta_j\sigma'} = \begin{cases} 1 & : \quad i < j \\ 1 & : \quad i = j, i = j+1 \\ v_{\sigma'}\circ\eta^j & : \quad i > j+1, \end{cases}$$

since only the last case gives $v_{\eta_j\sigma'} = v_{\sigma'}\circ\eta^j$ when $i = p$, $\sigma' \in S_{p-1}$. For the other cases $i < p-1$, we get $v_{\sigma',\varepsilon_i\sigma'}\circ\eta^{j-1} = v_{\eta_j\sigma',\varepsilon_i\eta_j\sigma'}$ or at most $p-1$, we get $v_{\sigma',\varepsilon_{i-1}\sigma'}\circ\eta^{j-1}$.That is why the transition functions will be 1. For the last case, we have $v_{\sigma',\varepsilon_{i-1}\sigma'}\circ\eta^j = v_{\sigma'}\circ\eta^j$. By interchanging σ' with σ, we get the required equalities. □

Note :

The transitions functions are generalized lattice gauge fields. Lattice gauge field is defined on 1-skeleton and one can extend this to $p-1$ simplex. So we get the transition functions on Δ^p.

Proposition 6.6 :

Given a bundle, one can find the admissible trivializations such that the transition functions are determined by functions $v_\sigma : \Delta^{p-1} \to G$ for $\sigma \in S_p$ nondegenerate.

Proposition 6.7 :

Suppose given a set of functions

$$v_\sigma : \Delta^{p-1} \to G$$

for $\sigma \in S_p$ for all p, satisfying the compatibility conditions

$$v_\sigma \circ \varepsilon^i = \begin{cases} v_{\varepsilon_i\sigma} & : \quad i < p-1 \\ v_{\varepsilon_{p-1}\sigma}.v_{\varepsilon_p\sigma}{}^{-1} & : \quad i = p-1 \end{cases}$$

and

$$v_{\eta_j\sigma} = v_\sigma \circ \eta^j.$$

Then one can define for each $\sigma \in S_p$ and each lower dimensional face τ of σ, a function $v_{\sigma,\tau}$ such that i) and ii) in proposition 6.4 hold and such that

$$v_{\sigma,\tau} = \begin{cases} v_\sigma & : \quad i = p \\ 1 & : \quad i < p. \end{cases}$$

Proof :

For a lower dimensional face τ of σ write $\tau = \varepsilon_{i_1} \circ ... \circ \varepsilon_{i_k} \sigma$ and define $v_{\sigma,\tau}$ as in proposition 6.5 2). It is enough to show that $v_{\sigma,\tau}$ is well-defined for τ of codimension 2. That is, for $\tau = \varepsilon_i\varepsilon_j\sigma$ we must prove that

$$v_{\sigma,\varepsilon_i\varepsilon_j\sigma} = (v_{\sigma,\varepsilon_j\sigma} \circ \varepsilon^i).v_{\varepsilon_j\sigma,\varepsilon_i\varepsilon_j\sigma}$$

and

$$v_{\sigma,\varepsilon_{j-1}\varepsilon_i\sigma} = (v_{\sigma,\varepsilon_i\sigma} \circ \varepsilon^{j-1}).v_{\varepsilon_i\sigma,\varepsilon_{j-1}\varepsilon_i\sigma}$$

are equal when $i < j$.

We know that v_σ satisfies the compatibility conditions above. Now

$$\begin{aligned} v_{\sigma,\varepsilon_i\varepsilon_p\sigma} &= (v_\sigma \circ \varepsilon^i).v_{\varepsilon_p\sigma,\varepsilon_i\varepsilon_p\sigma} \\ v_{\sigma,\varepsilon_{p-1}\varepsilon_i\sigma} &= (v_{\sigma,\varepsilon_i\sigma} \circ \varepsilon^{p-1}).v_{\varepsilon_i\sigma,\varepsilon_{p-1}\varepsilon_i\sigma} \end{aligned}$$

By using the first and the second conditions given in the statement, we get

$$(v_\sigma \circ \varepsilon^i).v_{\varepsilon_p\sigma,\varepsilon_i\varepsilon_p\sigma} = \begin{cases} v_{\varepsilon_i\sigma}.1 = v_{\varepsilon_i\sigma} & : \quad i < p-1 \\ v_{\varepsilon_{p-1}\sigma}.v_{\varepsilon_p\sigma}{}^{-1}.v_{\varepsilon_p\sigma} = v_{\varepsilon_{p-1}\sigma} & : \quad i = p-1. \end{cases}$$

On the other hand, we have

$$(v_{\sigma,\varepsilon_i\sigma} \circ \varepsilon^{p-1}).v_{\varepsilon_i\sigma,\varepsilon_{p-1}\varepsilon_i\sigma} = \begin{cases} 1.v_{\varepsilon_i\sigma} & : \quad i < p-1 \\ 1.v_{\varepsilon_{p-1}\sigma} & : \quad i = p-1. \end{cases}$$

So

$$(v_{\sigma,\varepsilon_i\sigma} \circ \varepsilon^{p-1}).v_{\varepsilon_i\sigma,\varepsilon_{p-1}\varepsilon_i\sigma} = \begin{cases} v_{\varepsilon_i\sigma} & : \quad i < p-1 \\ v_{\varepsilon_{p-1}\sigma} & : \quad i = p-1. \end{cases}$$

When $i < j$ for $\varepsilon_i \varepsilon_j \sigma = \tau$ and $\varepsilon_{j-1}\varepsilon_i = \tau$, $v_{\sigma,\tau}$ gives the same compatibility conditions.

So $v_{\sigma,\tau}$ is well-defined. □

Proposition 6.8 :

Given a set of transition functions $v_{\sigma,\tau}$ satisfying i) and ii) in proposition 6.4, there is a bundle F over $|S|$ and trivializations with transition functions $v_{\sigma,\tau}$.

Proof :

Let's define

$$F_\sigma := \Delta^p \times \sigma \times G$$

where $\sigma \in S_p$, as a trivialization. Similarly, let $F_{\varepsilon_j \sigma} := \Delta^{p-1} \times \varepsilon_j \sigma \times G$, $\varepsilon_j \sigma \in S_{p-1}$. We want to define

$$\bar{\varepsilon}^j : \Delta^{p-1} \times (\varepsilon_j \sigma) \times G \to \Delta^p \times (\sigma) \times G$$

by using the transition functions and an appropriate face map as follows:

$$\begin{tikzcd}
\Delta^{p-1} \times (\varepsilon_j \sigma) \times G \arrow[r, "\bar{\varepsilon}^j"] \arrow[d] & \Delta^p \times (\sigma) \times G \arrow[d] \\
\Delta^{p-1} \times \varepsilon_j \sigma \arrow[r, "\varepsilon^j"] & \Delta^p \times \sigma.
\end{tikzcd}$$

Then

$$\bar{\varepsilon}^j (t, \varepsilon_j \sigma, g) = (\varepsilon^j(t), \sigma, v_{\sigma, \varepsilon_j \sigma}(t).g),$$

where $t \in \Delta^{p-1}$ and $v_{\sigma,\varepsilon_j\sigma} : \Delta^{p-1} \to G$. From the definition of trivialization, the diagram above is commutative. We need to show that the compatibility conditions hold:

$$\bar{\varepsilon}^j \bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i \bar{\varepsilon}^{j-1} & : \quad i < j \\ \bar{\varepsilon}^{i+1} \bar{\varepsilon}^j & : \quad i \geq j. \end{cases}$$

We have

$$\begin{tikzcd}
\Delta^{p-2} \times \varepsilon_i \varepsilon_j \sigma \times G \arrow[r, "\bar{\varepsilon}^i"] \arrow[d] & \Delta^{p-1} \times \varepsilon_j \sigma \times G \arrow[r, "\bar{\varepsilon}^j"] \arrow[d] & \Delta^p \times \sigma \times G \arrow[d] \\
\Delta^{p-2} \times \varepsilon_i \varepsilon_j \sigma \arrow[r, "\varepsilon^i"] & \Delta^{p-1} \times \varepsilon_j \sigma \arrow[r, "\varepsilon^j"] & \Delta^p \times \sigma
\end{tikzcd}$$

and

$$\begin{aligned}
\bar{\varepsilon}^j \circ \bar{\varepsilon}^i (t, \varepsilon_i \varepsilon_j \sigma, g) &= \bar{\varepsilon}^j (\bar{\varepsilon}^i (t, \varepsilon_i \varepsilon_j \sigma, g)) \\
&= \bar{\varepsilon}^j (\varepsilon^i(t), \varepsilon_j \sigma, v_{\varepsilon_j \sigma, \varepsilon_i \varepsilon_j \sigma}(t).g) \\
&= (\varepsilon^j \varepsilon^i(t), \sigma, (v_{\sigma,\varepsilon_j\sigma} \circ \varepsilon^i(t)).v_{\varepsilon_j \sigma, \varepsilon_i \varepsilon_j \sigma}(t).g) \\
&= (\varepsilon^i \varepsilon^{j-1}(t), \sigma, (v_{\sigma,\varepsilon_j\sigma} \circ \varepsilon^i(t)).v_{\varepsilon_j \sigma, \varepsilon_i \varepsilon_j \sigma}(t).g)
\end{aligned}$$

when $i < j$.

We know that $v_{\sigma,\tau}$ is well-defined from proposition 6.7, where $\tau = \varepsilon_i\varepsilon_j\sigma$. That is, $(v_{\sigma,\varepsilon_j\sigma} \circ \varepsilon^i(t)).v_{\varepsilon_j\sigma,\varepsilon_i\varepsilon_j\sigma}(t)$ and $(v_{\sigma,\varepsilon_i\sigma} \circ \varepsilon^{j-1}(t)).v_{\varepsilon_i\sigma,\varepsilon_{j-1}\varepsilon_i\sigma}(t)$ will give us the same compatibility conditions when $i < j$. Then

$$\begin{aligned}
\bar{\varepsilon}^j\bar{\varepsilon}^i(t,\varepsilon_i\varepsilon_j\sigma) &= (\varepsilon^i\varepsilon^{j-1}(t),\sigma,(v_{\sigma,\varepsilon_i\sigma}\circ\varepsilon^{j-1}(t)).v_{\varepsilon_i\sigma,\varepsilon_{j-1}\varepsilon_i\sigma}(t).g)\\
&= \bar{\varepsilon}^i(\varepsilon^{j-1}(t),\varepsilon_i\sigma,v_{\varepsilon_i\sigma,\varepsilon_{j-1}\varepsilon_i\sigma}(t).g)\\
&= \bar{\varepsilon}^i\bar{\varepsilon}^{j-1}(t,\varepsilon_{j-1}\varepsilon_i\sigma,g)\\
&= \bar{\varepsilon}^i\bar{\varepsilon}^{j-1}(t,\varepsilon_i\varepsilon_j\sigma,g).
\end{aligned}$$

When $i \geq j$, we get

$$\begin{aligned}
\bar{\varepsilon}^j\bar{\varepsilon}^i(t,\varepsilon_i\varepsilon_j\sigma) &= \bar{\varepsilon}^j(\bar{\varepsilon}^i(t,\varepsilon_i\varepsilon_j\sigma,g))\\
&= \bar{\varepsilon}^j(\varepsilon^i(t),\varepsilon_j\sigma,v_{\varepsilon_j\sigma,\varepsilon_i\varepsilon_j\sigma}(t).g)\\
&= (\varepsilon^j\varepsilon^i(t),\sigma,(v_{\sigma,\varepsilon_j\sigma}\circ\varepsilon^i(t)).v_{\varepsilon_j\sigma,\varepsilon_i\varepsilon_j\sigma}(t).g)\\
&= (\varepsilon^{i+1}\varepsilon^j(t),\sigma,(v_{\sigma,\varepsilon_{i+1}\sigma}\circ\varepsilon_j(t)).v_{\varepsilon_{i+1}\sigma,\varepsilon_j\varepsilon_{i+1}\sigma}(t).g).
\end{aligned}$$

From proposition 6.7, $(v_{\sigma,\varepsilon_j\sigma} \circ \varepsilon^i(t)).v_{\varepsilon_j\sigma,\varepsilon_i\varepsilon_j\sigma}(t)$ and $(v_{\sigma,\varepsilon_{i+1}\sigma} \circ \varepsilon^j(t)).v_{\varepsilon_{i+1}\sigma,\varepsilon_j\varepsilon_{i+1}\sigma}(t)$ give the same compatibility conditions when $i \geq j$. Thus

$$\begin{aligned}
\bar{\varepsilon}^j\bar{\varepsilon}^i(t,\varepsilon_i\varepsilon_j\sigma) &= \bar{\varepsilon}^{i+1}(\varepsilon^j(t),\varepsilon_{i+1}\sigma,v_{\varepsilon_{i+1}\sigma,\varepsilon_j\varepsilon_{i+1}\sigma}(t).g)\\
&= \bar{\varepsilon}^{i+1}\bar{\varepsilon}^j(t,\varepsilon_j\varepsilon_{i+1}\sigma,g)\\
&= \bar{\varepsilon}^{i+1}\bar{\varepsilon}^j(t,\varepsilon_i\varepsilon_j\sigma,g).
\end{aligned}$$

In order to get the compatibility conditions for the degeneracy operators let's define

$$\bar{\eta}^j : \Delta^{p+1} \times (\eta_j\sigma) \times G \to \Delta^p \times (\sigma) \times G$$

as follows:

$$\begin{tikzcd}
\Delta^{p+1} \times (\eta_j\sigma) \times G \arrow[r, "\bar{\eta}^j"] \arrow[d] & \Delta^p \times (\sigma) \times G \arrow[d] \\
\Delta^{p+1} \times \eta_j\sigma \arrow[r, "\eta^j"] & \Delta^p \times \sigma
\end{tikzcd}$$

$$\bar{\eta}^j(t,\eta_j\sigma,g) = (\eta^j(t),\sigma,1.g),$$

where $t \in \Delta^{p+1}$ and the transition function becomes identity, since

$$v_{\sigma,\tau} = 1 \; j < p+1 \text{ and } j = 0,...,p.$$

So we can get the required compatibility conditions

$$\bar{\eta}^j\bar{\eta}^i = \begin{cases} \bar{\eta}^i\bar{\eta}^{j+1} & : \; i \leq j \\ \bar{\eta}^{i-1}\bar{\eta}^j & : \; i > j \end{cases}$$

as follows:

$$\begin{tikzcd}
\Delta^{p+2} \times \eta_i\eta_j\sigma \times G \arrow[r, "\bar{\eta}^i"] \arrow[d] & \Delta^{p+1} \times \eta_j\sigma \times G \arrow[r, "\bar{\eta}^j"] \arrow[d] & \Delta^p \times \sigma \times G \arrow[d] \\
\Delta^{p+2} \times \eta_i\eta_j\sigma \arrow[r, "\eta^i"] & \Delta^{p+1} \times \eta_j\sigma \arrow[r, "\eta^j"] & \Delta^p \times \sigma
\end{tikzcd}$$

and

$$\begin{aligned}\bar{\eta}^j\bar{\eta}^i(t,\eta_i\eta_j\sigma,g) &= \bar{\eta}^j(\eta^i(t),\eta_j\sigma,1.g)\\ &= (\eta^j\eta^i(t),\sigma,1.1.g).\end{aligned}$$

Then

$$\bar{\eta}^j\bar{\eta}^i(t,\eta_i\eta_j\sigma,g) = (\eta^i\eta^{j+1}(t),\sigma,g)$$

when $i \leq j$. So finally we get

$$\bar{\eta}^j\bar{\eta}^i(t,\eta_i\eta_j\sigma,g) = \bar{\eta}^i\bar{\eta}^{j+1}(t,\eta_i\eta_j\sigma,g).$$

The other case can be shown similarly.

The last compatibility conditions are

$$\bar{\eta}^j\bar{\varepsilon}^i = \begin{cases} \bar{\varepsilon}^i\bar{\eta}^{j-1} & : \quad i < j\\ 1 & : \quad i = j, i = j+1\\ \bar{\varepsilon}^{i-1}\bar{\eta}^j & : \quad i > j+1.\end{cases}$$

We have a diagram

$$\begin{array}{ccccc} \Delta^{p-1}\times\varepsilon_i\eta_j\sigma\times G & \xrightarrow{\bar{\varepsilon}^i} & \Delta^p\times\eta_j\sigma\times G & \xrightarrow{\bar{\eta}^j} & \Delta^{p-1}\times\sigma\times G\\ \downarrow & & \downarrow & & \downarrow\\ \Delta^{p-1}\times\varepsilon_i\eta_j\sigma & \xrightarrow{\varepsilon^i} & \Delta^p\times\eta_j\sigma & \xrightarrow{\eta^j} & \Delta^{p-1}\times\sigma.\end{array}$$

So

$$\begin{aligned}\bar{\eta}^j\bar{\varepsilon}^i(t,\varepsilon_i\eta_j\sigma,g) &= \bar{\eta}^j(\varepsilon^i(t),\eta_j\sigma,v_{\eta_j\sigma,\varepsilon_i\eta_j\sigma}(t).g)\\ &= (\eta^j\varepsilon^i(t),\sigma,1.v_{\eta_j\sigma,\varepsilon_i\eta_j\sigma}(t).g).\end{aligned}$$

Then

$$\bar{\eta}^j\bar{\varepsilon}^i(t,\varepsilon_i\eta_j\sigma,g) = \begin{cases} (\varepsilon^i\eta^{j-1}(t),\sigma,1.g) & : \quad i < j\\ (1.t,\sigma,v_{\eta_j\sigma,\varepsilon_i\eta_j\sigma}(t).g) & : \quad i = j, i = j+1\\ (\varepsilon^{i-1}\eta^j(t),\sigma,v_{\eta_j\sigma,\varepsilon_i\eta_j\sigma}(t).g) & : \quad i > j+1.\end{cases}$$

From the first part of proposition 6.4 and the definition of $\bar{\varepsilon}^i$ and $\bar{\eta}^i$, we get

$$\bar{\eta}^j\bar{\varepsilon}^i(t,\varepsilon_i\eta_j\sigma,g) = \begin{cases} \bar{\varepsilon}^i\bar{\eta}^{j-1}(t,\eta_{j-1}\varepsilon_i\sigma,g) & : \quad i < j\\ (t,\varepsilon_{j+1}\eta_j\sigma,g) & : \quad i = j, i = j+1\\ \bar{\varepsilon}^{i-1}\bar{\eta}^j(t,\eta_j\varepsilon_{i-1}\sigma,v_{\eta_j\sigma,\eta_j\varepsilon_{i-1}\sigma}(t)g) & : \quad i > j+1.\end{cases}$$

At the last case $v_{\eta_j\sigma,\eta_j\varepsilon_{i-1}\sigma} = v_{\sigma,\varepsilon_{i-1}\sigma}\circ\eta^j$ and this is not identity when $i-1 < p$, but we can see that

$$\begin{aligned}\bar{\eta}^j\bar{\varepsilon}^i(t,\varepsilon_i\eta_j\sigma,g) &= (\varepsilon^{i-1}\eta^j(t),\sigma,v_{\sigma,\varepsilon_{i-1}\sigma}\circ\eta^j(t).g)\\ &= \bar{\varepsilon}^{i-1}(\eta^j t,\varepsilon_{i-1}\sigma,1.g)\\ &= \bar{\varepsilon}^{i-1}\bar{\eta}^j(t,\eta_j\varepsilon_{i-1}\sigma,g)\\ &= \bar{\varepsilon}^{i-1}\bar{\eta}^j(t,\varepsilon_i\eta_j\sigma,g).\end{aligned}$$

Thus we get the compatibility conditions as

$$\bar{\eta}^j\bar{\varepsilon}^i(t,\varepsilon_i\eta_j\sigma,g)=\begin{cases}\bar{\varepsilon}^i\bar{\eta}^{j-1}(t,\varepsilon_i\eta_j\sigma,g) & : \quad i<j\\ (t,\varepsilon_i\eta_j\sigma,g) & : \quad i=j, i=j+1\\ \bar{\varepsilon}^{i-1}\bar{\eta}^j(t,\varepsilon_i\eta_j\sigma,g) & : \quad i>j+1.\end{cases}$$

□

Corollary 6.9 :

Given a set of functions v_σ satisfying the compatibility conditions in proposition 6.7, one can construct a bundle F over $|S|$ and the trivializations with the transition functions $v_{\sigma,\varepsilon_p\sigma}=v_\sigma$ and $v_{\sigma,\varepsilon_i\sigma}=1$ when $i<p$ and $v_{\eta_i\sigma}=v_\sigma\circ\eta^i$ for a degenerate σ.

Proof :

It directly follows from Proposition 6.7 and Proposition 6.8. □

Chapter 7

TRANSITION FUNCTIONS, REALIZATIONS AND CLASSIFYING MAP

In this chapter, we construct a map $m : \| \, |\bar{P}_{\cdot}S_{\cdot}| \, \| \to BG$ for a given bundle F on $|S|$ and admissible trivialization and also we construct a map $k : \|S\| \to BG$ with aid of transition functions.

Now, look at the bundles which are defined over some simplicial sets in the following diagram;

$$\begin{tikzcd}
(p \circ L)^* F \arrow[r, "r"] \arrow[d] & \Delta^p \times F \arrow[r, "\text{proj}"] \arrow[d] & F \arrow[d, "\pi"] \\
\| \, |P_{\cdot}S_{\cdot}| \, \| \arrow[r, "L"] & \| \, |S_{\cdot}| \, \| \arrow[r, "p"] & |S_{\cdot}|
\end{tikzcd}$$

where $(p \circ L)^* F$ is the pullback of F by $p \circ L$.

Let $\tilde{F}$ be defined as a subset of

$$\{(t, s, f) \in \Delta^p \times \Delta^{q_0} \times ... \times \Delta^{q_p} \times F_{|_{\Delta^{n+p} \times S_{n+p}}} \mid \pi_y(f) = (l_p(t)(s), y)\}$$

where $t \in \Delta^p$, $s \in \Delta^{q_0} \times ... \times \Delta^{q_p}$, $f \in F_y$ and $y \in S_{n+p}$.

So $(p \circ L)^* F$ can be modelled by $\tilde{F}$ by the following proposition.

Proposition 7.1 :

There is a canonical isomorphism

$$(p \circ L)^* F \to \tilde{F}$$

where $(p \circ L)^* F$ is a bundle over $\| \, |P_{\cdot}S_{\cdot}| \, \|$. This map is defined as a projection on the last factor F. The equivalence relations $\tilde{F}$ are as follows;

$$i) \ (\varepsilon^i t, s, \bar{\mu}^{(i)} f) \sim (t, \pi_i s, f)$$

where $t \in \Delta^{p-1}$, $s \in \Delta^{q_0} \times ... \times \Delta^{q_p}$, $f \in F_{\mu_{(i)} y}$.

$$ii) \ (t, {\varepsilon_{(i)}}^j s', \bar{\varepsilon}^j_{(i)} f') \sim (t, s', f')$$

where $t \in \Delta^p$, $s' \in \Delta^{q_0} \times ... \times \Delta^{q_i - 1} \times ... \times \Delta^{q_p}$, $f' \in F_y$, $y \in S_{n+p-1}$.

$$iii) \ (t, {\eta_{(i)}}^j s', \bar{\eta}^j_{(i)} f') \sim (t, s', f')$$

where $t \in \Delta^p$, $s' \in \Delta^{q_0} \times ... \times \Delta^{q_i+1} \times ... \times \Delta^{q_p}$, $f' \in F_y$, $y \in S_{n+p+1}$.

Proposition 7.2 :

Let $r : (p \circ L)^* F \to \Delta^p \times F$ and the projection of $\Delta^p \times F$ on the second factor be given. Then the composition map

$$q : (p \circ L)^* F \to F$$

is a well-defined simplicial map.

Proof :

Let's see the equivalence relations under q;

$$i)\ (\varepsilon^i t, s, \bar{\mu}^{(i)} f) \sim (t, \pi_i s, f)$$

where

$$\bar{\mu}^{(i)} : F_{|_{\Delta^{n+p-q_i-1} \times S_{n+p-q_i-1}}} \to F_{|_{\Delta^{n+p} \times S_{n+p}}}$$

and

$$\mu_{(i)} : S_{n+p} \to S_{n+p-q_i-1}$$

i=0,...,p.

$$q(\varepsilon^i t, s, \bar{\mu}^{(i)} f) = \bar{\mu}^{(i)} f \quad \text{and} \quad q(t, \pi_i s, f) = f$$

We want to see that

$$\bar{\mu}^{(i)} f \sim f.$$

Then

$$\begin{aligned}
\pi_{\mu_{(i)} y}(\bar{\mu}^{(i)} f) &= (l_p(t)(\pi_i(s), \mu_{(i)} y) \\
&\sim (\mu^{(i)} l(t)(\pi_i(s)), y) \\
&= (l(\varepsilon^i t)(s), y) \\
&= \pi_y(f)
\end{aligned}$$

So

$$\bar{\mu}^{(i)} f \sim f.$$

ii) If

$$(t, {\varepsilon_{(i)}}^j s', {\bar{\varepsilon}_{(i)}}^j f') \sim (t, s', f')$$

where $t \in \Delta^p$, $s' \in \Delta^{q_0} \times ... \times \Delta^{q_i-1} \times ... \times \Delta^{q_p}$, $f' \in F_y$, $y \in S_{n+p-1}$ and

$${\bar{\varepsilon}_{(i)}}^j : F_{|_{\Delta^{n+p-1} \times S_{n+p-1}}} \to F_{|_{\Delta^{n+p} \times S_{n+p}}}$$

$$q(t, {\varepsilon_{(i)}}^j s', {\bar{\varepsilon}_{(i)}}^j f') = {\bar{\varepsilon}_{(i)}}^j f' \text{ and } q(t, s', f') = f'$$

We want to see

$$\bar{\varepsilon}_{(i)}{}^{j} f' \sim f'$$

Then

$$\pi_{\eta^{(i)}{}_j y}(\bar{\varepsilon}_{(i)}{}^{j} f') = (l_p(t)(\varepsilon_{(i)}{}^{j} s'), \eta_j{}^{(i)} y)$$

where $\eta^{(i)}{}_j y \in S_{n+p}$, $\bar{\varepsilon}_{(i)}{}^{j} f' \in F_{|_{\Delta^{n+p} \times S_{n+p}}}$.

$$\begin{array}{rcl} (l_p(t)(\varepsilon_{(i)}{}^{j} s'), \eta^{(i)}{}_j y) & \sim & (\eta^{j}{}_{(i)} l_p(t)(\varepsilon_{(i)}{}^{j} s', y) \\ & = & (l_p(t)(s'), y) \\ & = & \pi_y(f'). \end{array}$$

Thus

$$q(t, \varepsilon_{(i)}{}^{j} s', \bar{\varepsilon}_{(i)}{}^{j} f') \sim q(t, s', f')$$

iii) If

$$(t, \eta_{(i)}{}^{j} s', \bar{\eta}_{(i)}{}^{j} f') \sim (t, s', f')$$

where $t \in \Delta^p$, $s' \in \Delta^{q_0} \times ... \times \Delta^{q_i+1} \times ... \times \Delta^{q_p}$, $f' \in F_y$, $y \in S_{n+p+1}$, and

$$\bar{\eta}_{(i)}{}^{j} : F_{|_{\Delta^{n+p+1} \times S_{n+p+1}}} \to F_{|_{\Delta^{n+p} \times S_{n+p}}}.$$

$$q(t, \eta_{(i)}{}^{j} s', \bar{\eta}_{(i)}{}^{j} f') = \bar{\eta}_{(i)}{}^{j} f' \text{ and } q(t, s', f') = f'$$

Then

$$\pi_{\varepsilon_j{}^{(i)} y}(\bar{\eta}_{(i)}{}^{j} f') = (l_p(t)(\eta_{(i)}{}^{j} s'), \varepsilon_j{}^{(i)} y)$$

where $\varepsilon_j{}^{(i)} y \in S_{n+p}$, $\bar{\eta}_{(i)}{}^{j} f' \in F_{|_{\Delta^{n+p} \times S_{n+p}}}$.

Then

$$\begin{array}{rcl} (l_p(t)(\eta_{(i)}{}^{j} s'), \varepsilon_j{}^{(i)} y) & \sim & (\varepsilon_{(i)}{}^{j} l_p(t)(\eta_{(i)}{}^{j} s'), y) \\ & = & (l_p(t)(s'), y) \\ & = & \pi_y(f'). \end{array}$$

Thus

$$q(t, \eta_{(i)}{}^{j} s', \bar{\eta}_{(i)}{}^{j} f') \sim q(t, s', f').$$

□

Now, we can give a statement to define a classifying map $k : \|S\| \to BG$. We have a diagram

$$\begin{tikzcd}
\tilde{F} \arrow[r, dotted, "\tilde{m}"] \arrow[d] & EG \arrow[d] \\
\| \, |P_. S_.| \, \| \arrow[r, "m"] \arrow[d, "L"] & BG \\
\| \, |S_.| \, \| \arrow[d, "p"] & \\
|S_.| \arrow[uur, dotted] &
\end{tikzcd}$$

As we mentioned before that there is any well-defined map $\| \, |P_. S_.| \, \| \to BG$ in the sense we wish, so we can define a map $m : \| \, |\bar{P}_. S_.| \, \| \to BG$ by the following proposition:

Proposition 7.3 :

Given a bundle F on $|S|$ and admissible trivialization with the function $v_{y,\tau}$, where τ is a lower face of y, one can construct a map $m : \| \, |\bar{P}_. S_.| \, \| \to$ BG by

$$\begin{aligned}
m(t,s,y) \;=\; & (t, v_{\tilde{\mu}^{(p-1)}y,\tilde{\mu}^{(p)}y}(\rho^{(1)}(l_p(t)(s))^{-1}, ..., \\
& v_{\tilde{\mu}^{(2)}y,\tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l_p(t)(s))^{-1}, \\
& v_{\hat{\mu}_{(p)}y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l_p(t)(s)))^{-1}, v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l_p(t)(s)))^{-1}, 1).
\end{aligned}$$

Proof :

Let's start with a given bundle F on $|S|$ and an epimorphism $|\bar{P}_p S_.| \to |S_.|$, so

$$\begin{tikzcd}
\bar{F} \arrow[r] \arrow[d] & F \arrow[d] \\
|\bar{P}_. S_.| \arrow[r] & |S_.|
\end{tikzcd}$$

Transition functions which we will use to define $\tilde{m}$ will be taken from the bundle $F \to |S|$.

Let's take $y \in S_{n+2p+1}$ and $l_p(t)(s) \in \Delta^{n+2p+1}$. We already defined $l_p(t)(s)$ before:

$$\begin{aligned}
l_p(t)(s^0, 0, ..., s^{p-1}, 0, s^p, 0) \;=\; & ({s_1}^0(1-t_1)+t_1, ..., {s_{q_0}}^0(1-t_1)+t_1, t_1, t_1, \\
& {s_1}^1(t_1-t_2)+t_2, .., {s_{q_1}}^1(t_1-t_2)+t_2, t_2, t_2, \\
& ..., \\
& {s_1}^{p-1}(t_{p-1}-t_p)+t_p, .., {s_{q_{p-1}}}^{p-1}(t_{p-1}-t_p)+t_p, t_p, t_p, \\
& {s_1}^p t_p, ..., {s_{q_p}}^p t_p, 0).
\end{aligned}$$

For convenience, we drop p in $l_p(t)(s)$ and use $l(t)(s)$. So

$$\begin{aligned}
\rho^{(p)} l(t)(s) \;=\; & ({s_1}^0(1-t_1)+t_1, ..., {s_{q_0}}^0(1-t_1)+t_1, t_1, , t_1, \\
& {s_1}^1(t_1-t_2)+t_2, .., {s_{q_1}}^1(t_1-t_2)+t_2, t_2, t_2, \\
& ..., \\
& {s_1}^{p-1}(t_{p-1}-t_p)+t_p, ..., {s_{q_{p-1}}}^{p-1}(t_{p-1}-t_p)+t_p, t_p),
\end{aligned}$$

where $\rho^{(p)} := \eta^{n-q_p+2p-1} \circ ... \circ \eta^{n+2p}$ deletes $q_p + 2$ elements.

There is a fibre at $(l(t)(s), y)$, by using the trivialization $\varphi_y : F_y \to \Delta^{n+2p+1} \times y \times G$ and the projection on the last factor, we get $F_y \to G$. Let's denote this composition by $\bar{\varphi}_y(\tilde{f})$ where $\tilde{f} := (l(t)(s), y)$, $\tilde{f}_y \in F_{(l(t)(s),y)}$, $y \in S_{n+2p+1}$. On the other hand

$$\varphi_{\mu_{(p)}y} : F_{\mu_{(p)}y} \to \Delta^{n+2p-q_p-1} \times \mu_{(p)}y \times G$$

gives us

$$\bar{\varphi}_{\mu_{(p)}y} : F_{\mu_{(p)}y} \to G.$$

By the definition,

$$\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}_y) := v_{y,\mu_{(p)}y}(\rho^{(p)}l(t)(s)).\bar{\varphi}_{\mu_{(p)}y}(\tilde{f})_{\mu_{(p)y}}$$

where the transition function is

$$v_{y,\mu_{(p)}y} : \Delta^{n+2p-q_p-1} \to G,$$

where

$$\mu_{(p)} : S_{n+2p+1} \to S_{n+2p-q_p-1}$$

is defined by $\mu_{(p)} := \varepsilon_{n+2p-q_p} \circ ... \circ \varepsilon_{n+2p+1}$, deletes $q_p + 2$ elements.

The last component in $\tilde{m}(t, s, \tilde{f}_y)$ is defined via the trivialization $\varphi_y(\tilde{f})$ which is $\bar{\varphi}_y(\tilde{f})$. In order to find the p-th component, we will use the transition function $v_{y,\mu_{(p)}y}$. Then this component will be

$$v_{y,\mu_{(p)}y}(\rho^{(p)}l(t)(s))^{-1}.\bar{\varphi}_y(\tilde{f}).$$

To get the other coordinates in the image, we apply the same method several times. In general

$$\hat{\mu}_{(i)} : S_{n+2i+1-\sum_0^{p-i-1} q_{p-j}} \to S_{n+2i-1-\sum_0^{p-i} q_{p-j}}$$

$i = 1, ..., p$.

In general one can write

$$\begin{aligned}
\rho^{(i)}l(t)(s) \quad = \quad & ({s_1}^0(1-t_1)+t_1, ..., {s_{q_0}}^0(1-t_1)+t_1, t_1, t_1, \\
& {s_1}^1(t_1-t_2)+t_2, ..., {s_{q_1}}^1(t_1-t_2)+t_2, t_2, t_2, \\
& . \\
& . \\
& . \\
& {s_1}^{i-1}(t_{i-1}-t_i)+t_i, ..., {s_{q_{i-1}}}^{i-1}(t_{i-1}-t_i)+t_i, t_i),
\end{aligned}$$

$i = 1, ..., p$ and $\rho^{(i)}l(t)(s) = \eta^{q_0+...+q_{i-1}+2i-1} \circ ... \circ \eta^{n+2p}$.

Note : $\tilde{\mu}^{(1)} = \hat{\mu}_{(p)} = \mu_{(p)}$.

Then the p-1-st component is

$$v_{\hat{\mu}_{(p)}y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s)))^{-1}.\bar{\varphi}_{\hat{\mu}_{(p)}y}(\tilde{f})$$

where $\rho^{(p-1)}l(t)(s)$ deletes $q_{p-1}+2$-elements in $\rho^{(p)}l(t)(s)$, that is,

$$\rho^{(p-1)}l(t)(s) = \eta^{n-q_p-q_{p-1}+2p-3} \circ ... \circ \eta^{n+2p}.$$

Let's denote

$$\tilde{\mu}^{(p-i)} = \hat{\mu}_{(i+1)} \circ ... \circ \hat{\mu}_{(p)}$$

$i = 0, ..., p-1$, which deletes the elements $(q_0 + ... + q_i + 2i - 1, ..., n + 2p + 1)$. It deletes $q_{i+1} + ... + q_p + 2(p-i) = n - (q_0 + ... + q_i) + 2(p-i)$ elements.

Thus

$$\begin{aligned}
\tilde{m}(t, s, \tilde{f}_y) \;=\; & (t, v_{\tilde{\mu}^{(p-1)}y, \tilde{\mu}^{(p)}y}(\rho^{(1)}(l(t)(s)))^{-1} ... v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}l(t)(s))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& ..., \\
& v_{\tilde{\mu}^{(2)}y, \tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(t)(s)))^{-1}.v_{\hat{\mu}_{(p)}y, \tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s))^{-1}.v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}l(t)(s))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& v_{\hat{\mu}_{(p)}y, \tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s)))^{-1}.v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}(l(t)(s)))^{-1}.\bar{\varphi}_y(\tilde{f}), \\
& v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}(l(t)(s)))^{-1}.\bar{\varphi}_y(\tilde{f}), \\
& \bar{\varphi}_y(\tilde{f})).
\end{aligned}$$

In other words,

$$\begin{aligned}
\tilde{m}(t, s, \tilde{f}_y) \;=\; & (t, (v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}l(t)(s)) ... v_{\tilde{\mu}^{(p-1)}y, \tilde{\mu}^{(p)}y}(\rho^{(1)}(l(t)(s)))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& ..., \\
& (v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}l(t)(s)).v_{\hat{\mu}_{(p)}y, \tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s)).v_{\tilde{\mu}^{(2)}y, \tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(t)(s)))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& (v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}(l(t)(s))).v_{\hat{\mu}_{(p)}y, \tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s))))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}(l(t)(s)))^{-1}.\bar{\varphi}_y(\tilde{f}), \\
& \bar{\varphi}_y(\tilde{f})).
\end{aligned}$$

By using proposition 6.5 2), we can simplify $\tilde{m}$ as:

$$\begin{aligned}
\tilde{m}(t, s, \tilde{f}_y) \;=\; & (t, v_{y, \tilde{\mu}^{(p)}y}(\rho^{(1)}(l(t)(s)))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& ..., \\
& (v_{y, \tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(t)(s)))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& v_{y, \tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s)))^{-1}\bar{\varphi}_y(\tilde{f}), \\
& v_{y, \hat{\mu}_{(p)}y}(\rho^{(p)}(l(t)(s)))^{-1}.\bar{\varphi}_y(\tilde{f}), \\
& \bar{\varphi}_y(\tilde{f})).
\end{aligned} \tag{7.-41}$$

By the definition $EG = ||N\bar{G}||$ and $\gamma : N\bar{G} \to NG$, here

$$N\bar{G}(p) = \underbrace{G \times ... \times G}_{p+1-\text{times}} \text{ and } NG(p) = \underbrace{G \times ... \times G}_{p-\text{times}},$$

is given by

$$\gamma(g_0, ..., g_p) = (g_0 {g_1}^{-1}, ..., g_{p-1} {g_p}^{-1}). \tag{7.-43}$$

So we can identify $BG = ||NG||$. Then the required map is

$$\begin{aligned} m(t,s,y) \quad = \quad & (t, v_{\tilde{\mu}^{(p-1)}y,\tilde{\mu}^{(p)}y}(\rho^{(1)}(l(t)(s)))^{-1}, ..., \\ & v_{\tilde{\mu}^{(2)}y,\tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(t)(s)))^{-1}, \\ & v_{\hat{\mu}_{(p)}y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s)))^{-1}, v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l(t)(s)))^{-1}, 1). \end{aligned}$$

□

Proposition 7.4 :

Given a bundle F on $|S|$ and by using the previous proposition, one can show that the map $\tilde{m} : \tilde{F} \to EG$ is well-defined.

Proof :

For the first equivalence relation:

i) If

$$(\varepsilon^i t, (s, \bar{\mu}^{(i)}\tilde{f}_y)) \sim (t, \mathrm{proj}_i(s), \tilde{f}_{\mu_{(i)}y})$$

where $t \in \Delta^{p-1}$, $s = (s^0, 0, ..., s^p, 0)$, $y \in S_{n+2p+1}$, $\tilde{f}_{\mu_{(i)}y} \in F_{\mu_{(i)}y}$ and $\bar{\mu}^{(i)}\tilde{f}_y \in F_y$ then we will show that

$$\tilde{m}(\varepsilon^i t, (s, \bar{\mu}^{(i)}\tilde{f}_y)) \sim \tilde{m}(t, \mathrm{proj}_i(s), \tilde{f}_{\mu_{(i)}y})$$

When $i = p$, it follows from

$$\begin{aligned} \tilde{m}(\varepsilon^p t, s, \bar{\mu}^{(p)}\tilde{f}) \quad = \quad & (\varepsilon^p t, v_{y,\tilde{\mu}^{(p)}y}(\rho^{(1)}l(\varepsilon^p t)(s))^{-1}\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}), \\ & ..., \\ & (v_{y,\tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(\varepsilon^p t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}), \\ & v_{y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(\varepsilon^p t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}), \\ & v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l(\varepsilon^p t)(s)))^{-1}.\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}), \\ & \bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f})), \\ \sim \quad & (t, v_{y,\tilde{\mu}^{(p)}y}(\rho^{(1)}l(\varepsilon^p t)(s))^{-1}\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}), \\ & ..., \\ & (v_{y,\tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(\varepsilon^p t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}), \\ & v_{y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(\varepsilon^p t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}), \\ & v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l(\varepsilon^p t)(s)))^{-1}.\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f})). \end{aligned}$$

By the definition $v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l(\varepsilon^p t)(s)))^{-1}.\bar{\varphi}_y(\bar{\mu}^{(p)}\tilde{f}) := \bar{\varphi}_{\hat{\mu}_{(p)}y}(\tilde{f})$ and if we substitute in the equation above and using the necessary cocycle conditions we get

$$\begin{aligned} \tilde{m}(t, \mathrm{proj}_p(s), \tilde{f}_{\mu_{(p)}y}) \quad = \quad & (t, v_{\mu_{(p)}y,\tilde{\mu}^{(p)\prime}\mu_{(p)}y}(\rho^{(1)}\mu^{(p)}l(\varepsilon^p t)(s))^{-1}\bar{\varphi}_{\mu_{(p)}y}(\tilde{f}), \\ & ..., \\ & (v_{\mu_{(p)}y,\tilde{\mu}^{(3)\prime}\mu_{(p)}y}(\rho^{(p-2)}\mu^{(p)}(l(\varepsilon^p t)(s)))^{-1}\bar{\varphi}_{\mu_{(p)}y}(\tilde{f}), \\ & v_{\mu_{(p)}y,\tilde{\mu}^{(2)\prime}\mu_{(p)}y}(\rho^{(p-1)}\mu^{(p)}(l(\varepsilon^p t)(s)))^{-1}\bar{\varphi}_{\mu_{(p)}y}(\tilde{f}), \\ & \bar{\varphi}_{\mu_{(p)}y}(\tilde{f})). \end{aligned}$$

where $\mu^{(p)}l(\varepsilon^p t)(s) = l(t)(\mathrm{proj}_p s)$.

When $i = 0$, it follows from

$$\begin{aligned}
\tilde{m}(\varepsilon^0 t, s, \bar{\mu}^{(0)}\tilde{f}) &= (\varepsilon^0 t, v_{y,\tilde{\mu}^{(p)}y}(\rho^{(1)}l(\varepsilon^0 t)(s))^{-1}\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}), \\
&\quad ..., \\
&\quad (v_{y,\tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(\varepsilon^0 t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}), \\
&\quad v_{y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(\varepsilon^0 t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}), \\
&\quad v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l(\varepsilon^0 t)(s)))^{-1}.\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}), \\
&\quad \bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f})), \\
&\sim (t, v_{y,\tilde{\mu}^{(p)}y}(\rho^{(1)}l(\varepsilon^0 t)(s))^{-1}\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}), \\
&\quad ..., \\
&\quad (v_{y,\tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(\varepsilon^0 t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}), \\
&\quad v_{y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(\varepsilon^0 t)(s)))^{-1}\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}), \\
&\quad v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l(\varepsilon^0 t)(s)))^{-1}.\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f})).
\end{aligned}$$

By the definition $v_{y,\hat{\mu}_{(0)}y}(\mu^{(0)}(l(\varepsilon^0 t)(s)))^{-1}.\bar{\varphi}_y(\bar{\mu}^{(0)}\tilde{f}) := \bar{\varphi}_{\hat{\mu}_{(0)}y}(\tilde{f})$ and if we substitute in the equation above and using the necessary cocycle conditions

$$\begin{aligned}
v_{y,\mu_{(p)}{}'\mu_{(0)}y} &= v_{y,\mu_{(0)}y}.v_{\mu_{(0)}y,\mu_{(p)}{}'\mu_{(0)}y} \\
v_{y,\mu_{(0)}{}'\mu_{(p)}y} &= v_{y,\mu_{(p)}y}.v_{\mu_{(p)}y,\mu_{(0)}{}'\mu_{(p)}y}
\end{aligned}$$

we get

$$\begin{aligned}
\tilde{m}(t, \mathrm{proj}_0(s), \tilde{f}_{\mu_{(0)}y}) &= (t, v_{\mu_{(0)}y,\tilde{\mu}^{(p-1)}{}'\mu_{(0)}y}(\rho^{(2)}\mu^{(0)}l(\varepsilon^0 t)(s))^{-1}\bar{\varphi}_{\mu_{(0)}y}(\tilde{f}), \\
&\quad ..., \\
&\quad (v_{\mu_{(0)}y,\tilde{\mu}^{(2)}{}'\mu_{(0)}y}(\rho^{(p-1)}\mu^{(0)}(l(\varepsilon^0 t)(s)))^{-1}\bar{\varphi}_{\mu_{(0)}y}(\tilde{f}), \\
&\quad v_{\mu_{(0)}y,\mu_{(p)}{}'\mu_{(0)}y}(\rho^{(p)}\mu^{(0)}(l(\varepsilon^0 t)(s)))^{-1}\bar{\varphi}_{\mu_{(0)}y}(\tilde{f}), \\
&\quad \bar{\varphi}_{\mu_{(0)}y}(\tilde{f})),
\end{aligned}$$

where $\mu^{(0)}l(\varepsilon^0 t)(s) = l(t)(\mathrm{proj}_0 s)$.

For $i = 1, ..., p-1$, by using the same fact that we get the required equivalent elements.

ii) If

$$(t, \varepsilon^j{}_{(i)}s, \bar{\varepsilon}^j\tilde{f}) \sim (t, s, \tilde{f}_{\varepsilon_j{}^{(i)}y})$$

Then we will show that

$$\tilde{m}(t, \varepsilon^j{}_{(i)}s, \bar{\varepsilon}^j\tilde{f}) \sim \tilde{m}(t, s, \tilde{f}_{\varepsilon_j{}^{(i)}y}),$$

$j = 0, ..., q_i$.

For $i = p, j = q_p$, we get

$$\begin{aligned}\tilde{m}(t, \varepsilon^{q_p}{}_{(p)}s, \bar{\varepsilon}^{q_p}\tilde{f}) \;&=\; (t, v_{y,\tilde{\mu}^{(p)}y}(\rho^{(1)}l(t)(s))^{-1}\bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f})\\ &\quad ...,\\ &\quad (v_{y,\tilde{\mu}^{(3)}y}(\rho^{(p-2)}(l(t)(s)))^{-1}\bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f}),\\ &\quad v_{y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(s)))^{-1}\bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f}),\\ &\quad v_{y,\hat{\mu}_{(p)}y}(\rho^{(p)}(l(t)(s)))^{-1}.\bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f}),\\ &\quad \bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f})).\end{aligned}$$

On the other hand

$$\begin{aligned}\tilde{m}(t, s, \tilde{f}_{\varepsilon_{n+2p}y}) \;&=\; (t, v_{\varepsilon_{n+2p}y,\tilde{\mu}^{(p)\prime}\varepsilon_{n+2p}y}(\rho^{(1)\prime}l(t)(s))^{-1}\bar{\varphi}_{\varepsilon_{n+2p}y}(\tilde{f}),\\ &\quad ...,\\ &\quad (v_{\varepsilon_{n+2p}y,\tilde{\mu}^{(3)\prime}\varepsilon_{n+2p}y}(\rho^{(p-2)\prime}(l(t)(s)))^{-1}\bar{\varphi}_{\varepsilon_{n+2p}y}(\tilde{f}),\\ &\quad v_{\varepsilon_{n+2p}y,\tilde{\mu}^{(2)\prime}\varepsilon_{n+2p}y}(\rho^{(p-1)\prime}(l(t)(s)))^{-1}\bar{\varphi}_{\varepsilon_{n+2p}y}(\tilde{f}),\\ &\quad v_{\varepsilon_{n+2p}y,\hat{\mu}_{(p)}\prime\varepsilon_{n+2p}y}(\rho^{(p)\prime}(l(t)(s)))^{-1}.\bar{\varphi}_{\varepsilon_{n+2p}y}(\tilde{f}),\\ &\quad \bar{\varphi}_{\varepsilon_{n+2p}y}(\tilde{f})).\end{aligned}$$

From the definition $v_{y,\varepsilon_{n+2p}y}{}^{-1}\bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f}) := \bar{\varphi}_{\varepsilon_{n+2p}y}(\tilde{f})$ and $v_{y,\varepsilon_{n+2p}y} = \mathrm{id}$ and cocycle conditions, we get

$$(t, v_{y,\tilde{\mu}^{(p)}y}(\rho^{(1)}l(t)(s))^{-1}\bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f}), ..., \bar{\varphi}_y(\bar{\varepsilon}^{q_p}\tilde{f})),$$

what we wanted. For the other cases $i = 0, ..., p-1$, it can be shown similarly.

iii)If

$$(t, \eta^j{}_{(i)}s, \bar{\eta}^j\tilde{f}) \sim (t, s, \tilde{f}_{\eta_j{}^{(i)}y})$$

Then we will show that

$$\tilde{m}(t, \eta^j{}_{(i)}s, \bar{\eta}^j\tilde{f}) \sim \tilde{m}(t, s, \tilde{f}_{\eta_j{}^{(i)}y}),$$

$j = 0, ..., q_i$.

It follows from ii) in proposition 6.4 and admissible trivialization. □

Proposition 7.5 :

Let $||S||$ be the fat realization of S. One can define

$$k : ||S|| \to BG$$

by

$$\begin{aligned}k(t,x) \;&=\; (t, [v_{\varepsilon_2...\varepsilon_p x}(0)]^{-1}, [v_{\varepsilon_3...\varepsilon_p x}(t_1)]^{-1}, ...,\\ &\quad [v_{\varepsilon_{p-1}\varepsilon_p x}(\eta^{p-3}\eta^{p-2}\eta^{p-1}t)]^{-1}, [v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}t)]^{-1},\\ &\quad [v_x(\eta^{p-1}t)]^{-1}).\end{aligned}$$

as a composition of m and an inclusion $i : \Delta^p \times S_p \hookrightarrow \Delta^p \times (\Delta^0)^{p+1} \times S_{2p+1}$.

Proof :

Let's take all $s^i = 0$. Then $n = 0$, $y \in S_{2p+1}$. The map $S_p \to S_{2p+1}$ is defined by $\eta_{2p} \circ \eta_{2p-2} \circ ... \circ \eta_0 x$, $x \in S_p$. In order to define the transition functions, we need to find the elements in Δ^{2p+1} by

$$l(t)(0, ..., 0) = (t_1, t_1, t_2, t_2, ..., t_p, t_p, 0).$$

$\mu_{(p)} : S_{2p+1} \to S_{2p-1}$ corresponds to $\mu_{(p)} := \varepsilon_{2p} \circ \varepsilon_{2p+1}$.

The element in Δ^{2p-1} follows from

$$\rho^{(p)} l(t)(0) = (t_1, t_1, t_2, t_2, ..., t_{p-1}, t_{p-1}, t_p),$$

where $\rho^{(p)} := \eta^{2p-1} \circ \eta^{2p}$.

In other words,

$$\rho^{(p)} l(t)(0) = \eta^{2p-1} \circ \eta^{2p} \circ \varepsilon^{2p+1} \circ \varepsilon^{2p-1} \circ ... \circ \varepsilon^1(t_1, ..., t_p).$$

For the p-th component, let's write $v_{y,\mu_{(p)}y}(\rho^{(p)}(l(t)(0))$ in terms of x:

$$\begin{aligned}
v_{y,\mu_{(p)}y}(\rho^{(p)}(l(t)(s))) &= v_{y,\varepsilon_{2p}\varepsilon_{2p+1}y}(\eta^{2p-1} \circ \eta^{2p} \circ \varepsilon^{2p+1} \circ \varepsilon^{2p-1} \circ ... \circ \varepsilon^1(t_1, ..., t_p)) \\
&= v_{y,\varepsilon_{2p+1}y}(\varepsilon^{2p} \circ \eta^{2p-1} \circ \eta^{2p} \circ \varepsilon^{2p+1} \circ ... \circ \varepsilon^1(t_1, ..., t_p)). \\
&\quad v_{\varepsilon_{2p+1}y,\varepsilon_{2p}\varepsilon_{2p+1}y}(\eta^{2p-1} \circ \eta^{2p} \circ \varepsilon^{2p+1} \circ \varepsilon^{2p-1} \circ ... \circ \varepsilon^1(t_1, ..., t_p)) \\
&= ... \\
&= v_y(\varepsilon^{2p}\varepsilon^{2p-3}...\varepsilon^1(t_1, ..., t_p)).v_{\varepsilon_{2p+1}y}(\eta^{2p-3} \circ ... \circ \varepsilon^1(t_1, ..., t_p)) \\
&= v_{\varepsilon_{2p}y}(\varepsilon^{2p-3}...\varepsilon^1(t_1, ..., t_p)).v_{\varepsilon_{2p+1}y}(\eta^{2p-3} \circ ... \circ \varepsilon^1(t_1, ..., t_p))^{-1}. \\
&\quad v_{\varepsilon_{2p+1}y}(\eta^{2p-3} \circ ... \circ \varepsilon^1(t_1, ..., t_p)) \\
&= v_{\varepsilon_{2p}y}(\varepsilon^{2p-3} \circ ... \circ \varepsilon^1(t_1, ..., t_p)) \\
&= ... \\
&= ... \\
&= v_{\eta_{p-1}\circ\varepsilon_1\circ...\circ\varepsilon_{2p-3}\circ\eta_{2p-4}\circ...\circ\eta_0 x}(t_1, ..., t_p) \\
&= v_x(\eta^{p-1}(t_1, ..., t_p)) \\
&= v_x(t_1, ..., t_{p-1}).
\end{aligned}$$

This transition function is defined on Δ^{p-1}.

For the $p-1$-st element, we will do the same thing for $v_{\mu_{(p)}y,\tilde{\mu}^{(2)}y}(\rho^{(p-1)}(l(t)(0))$ Then we get

$$v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}(t_1, ..., t_p)) = v_{\varepsilon_p x}(t_1, ..., t_{p-2}),$$

which is a transition function defined on Δ^{p-2}.

Similarly for the $p-2$-nd component, we get $v_{\varepsilon_{p-1}\varepsilon_p x}(t_1, ..., t_{p-3})$ defined on Δ^{p-3}.

One can write

$$\bar{\rho}^{(i)} := \eta^{2i-1} \circ ... \circ \eta^{2p},$$

$i = 1, ..., p$.

For the 2-nd component, we get $v_{\varepsilon_3...\varepsilon_p x}(t_1)$, and For the 1-st component, we get $v_{\varepsilon_2...\varepsilon_p x}(0)$.

Thus

$$\begin{aligned}\tilde{k}(t,\tilde{f}) &= (t, [v_x(\eta^{p-1}t)...v_{\varepsilon_2...\varepsilon_p x}(0)]^{-1}\bar{\varphi}_y(\tilde{f}),\\ &\quad ...,\\ &\quad [v_x(\eta^{p-1}t)v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}t)...v_{\varepsilon_{p-1}\varepsilon_p x}(\eta^{p-3}\eta^{p-2}\eta^{p-1}t)]^{-1}\bar{\varphi}_y(\tilde{f}),\\ &\quad [v_x(\eta^{p-1}t)v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}t)]^{-1}.\bar{\varphi}_y(\tilde{f}), [v_x(\eta^{p-1}t)]^{-1}.\bar{\varphi}_y(\tilde{f}),\\ &\quad \bar{\varphi}_y(\tilde{f})).\end{aligned}$$

So

$$\begin{aligned}k(t,x) &= (t, [v_{\varepsilon_2...\varepsilon_p x}(0)]^{-1}, [v_{\varepsilon_3...\varepsilon_p x}(t_1)]^{-1}, ...,\\ &\quad [v_{\varepsilon_{p-1}\varepsilon_p x}(\eta^{p-3}\eta^{p-2}\eta^{p-1}t)]^{-1}, [v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}t)]^{-1},\\ &\quad [v_x(\eta^{p-1}t)]^{-1}).\end{aligned} \tag{7.-113}$$

We can also show that the composition $k := m \circ i$ is well-defined.

If $(\varepsilon^j t, x) \sim (t, \varepsilon_j x)$ then we must see that

$$k(\varepsilon^j t, x) \sim k(t, \varepsilon_j x),$$

where $t \in \Delta^{p-1}$ and $x \in S_p$.

When $j = p$,

$$\begin{aligned}k(\varepsilon^p t, x) &= (\varepsilon^p t, [v_{\varepsilon_2...\varepsilon_p x}(\varepsilon^p 0)]^{-1},\\ &\quad [v_{\varepsilon_3...\varepsilon_p x}(\varepsilon^p t_1)]^{-1},\\ &\quad ...,\\ &\quad [v_{\varepsilon_{p-1}\varepsilon_p x}(\eta^{p-3}\eta^{p-2}\eta^{p-1}\varepsilon^p t)]^{-1},\\ &\quad [v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}\varepsilon^p t)]^{-1},\\ &\quad [v_x(\eta^{p-1}\varepsilon^p t)]^{-1})\\ &\sim (t, [v_{\varepsilon_2...\varepsilon_p x}(\varepsilon^p 0)]^{-1}, ..., v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}\varepsilon^p t)]^{-1})\\ &= (t, [v_{\varepsilon_2...\varepsilon_p x}(0)]^{-1}, ..., v_{\varepsilon_p x}(\eta^{p-2}t)]^{-1})\\ &= (t, [v_{\varepsilon_2...\varepsilon_{p-1}\varepsilon_p x}(0)]^{-1}, ..., [v_{\varepsilon_p x}(t_1, ..., t_{p-2})]^{-1}).\end{aligned}$$

On the other hand,

$$\begin{aligned}k := m \circ i(t, \varepsilon_p x) &= m(t, 0, ..., 0, \eta_{2p-2} \circ ... \circ \eta_0 \varepsilon_p x)\\ &= (t, [v_{\varepsilon_1...\varepsilon_{p-1}\varepsilon_p x}(0)]^{-1}, [v_{\varepsilon_2...\varepsilon_{p-1}\varepsilon_p x}(t_1)]^{-1}, ..., [v_{\varepsilon_p x}(\eta^{p-2}t)]^{-1})\\ &= (t, [v_{\varepsilon_2...\varepsilon_{p-1}\varepsilon_p x}(\varepsilon^1 0)]^{-1}, [v_{\varepsilon_3...\varepsilon_p x}(\varepsilon^2 t_1)]^{-1}, ..., [v_{\varepsilon_p x}(\eta^{p-1}t)]^{-1})\\ &= (t, [v_{\varepsilon_2...\varepsilon_{p-1}\varepsilon_p x}(0)]^{-1}, ..., [v_{\varepsilon_p x}(t_1, ..., t_{p-2})]^{-1}).\end{aligned}$$

When $j = 0$,

$$\begin{aligned}
k(\varepsilon^0 t, x) &= (\varepsilon^0 t, [v_{\varepsilon_2 \ldots \varepsilon_p x}(\varepsilon^0 0)]^{-1}, \\
&\quad [v_{\varepsilon_3 \ldots \varepsilon_p x}(\varepsilon^0 t_1)]^{-1}, \\
&\quad ..., \\
&\quad [v_{\varepsilon_{p-1}\varepsilon_p x}(\eta^{p-3}\eta^{p-2}\eta^{p-1}\varepsilon^0 t)]^{-1}, \\
&\quad v_{\varepsilon_p x}(\eta^{p-2}\eta^{p-1}\varepsilon^0 t)]^{-1}, \\
&\quad [v_x(\eta^{p-1}\varepsilon^0 t)]^{-1}) \\
&\sim (t, [v_{\varepsilon_3 \ldots \varepsilon_p x}(0, t_1)]^{-1}, ..., [v_{\varepsilon_p x}(\varepsilon^0 \eta^{p-3}\eta^{p-2} t)]^{-1}, [v_x(\varepsilon^0 \eta^{p-2} t)]^{-1}) \\
&= (t, [v_{\varepsilon_0 \varepsilon_3 \ldots \varepsilon_p x}(t_1)]^{-1}, ..., [v_{\varepsilon_0 \varepsilon_p x}(\eta^{p-3}\eta^{p-2} t)]^{-1}, [v_{\varepsilon_0 x}(\eta^{p-2} t)]^{-1}) \\
&= (t, [v_{\varepsilon_2 \varepsilon_3 \ldots \varepsilon_{p-1}\varepsilon_0 x}(t_1)]^{-1}, ..., v_{\varepsilon_{p-1}\varepsilon_0 x}(\eta^{p-3}\eta^{p-2} t)^{-1}, v_{\varepsilon_0 x}(\eta^{p-2} t)]^{-1}) \\
&= k(t, \varepsilon_0 x).
\end{aligned}$$

One can show this for the $k = 1, ..., p-1$. □

Proposition 7.6 :

One can see that

$$i : ||S|| \to ||\ |\bar{P}_{.}S_{.}|\ ||$$

is simplicial.

Proof :

$$i : \Delta^p \times S_p \to \Delta^p \times (\Delta^0)^{p+1} \times S_{2p+1}$$

is defined by

$$\begin{aligned}
i(\varepsilon^k t, x) &= (\varepsilon^k t, 0, ..., 0, y) \\
&\sim (t, 0, ..., 0, \mu_{(k)} y).
\end{aligned}$$

On the other hand

$$i(t, \varepsilon_k x) = (t, 0, ..., 0, \eta_{2p-2} \circ ... \circ \eta_0 \varepsilon_k x).$$

When $k = p$, one gets

$$\begin{aligned}
i(\varepsilon^p t, x) &= (\varepsilon^p t, 0, ..., 0, y) \\
&\sim (t, 0, ..., 0, \mu_{(p)} y) \\
&= (t, 0, ..., 0, \varepsilon_{2p}\varepsilon_{2p+1} y) \\
&= (t, 0, ..., 0, \eta_{2p-2} \circ ..\eta_2 \varepsilon_p x) \\
&= i(t, \varepsilon_p x).
\end{aligned}$$

Let's also do it for $k = 0$;

$$\begin{aligned}
i(\varepsilon^0 t, x) &= (\varepsilon^0 t, 0, ..., 0, y) \\
&\sim (t, 0, ..., 0, \mu_{(0)} y) \\
&= (t, 0, ..., 0, \varepsilon_0 \varepsilon_1 y) \\
&= (t, 0, ..., 0, \varepsilon_0 \varepsilon_1 \eta_{2p} \eta_{2p-2} ... \eta_2 \eta_0 x) \\
&= (t, 0, ..., 0, \eta_{2p-2} \circ .. \eta_0 \varepsilon_0 \varepsilon_1 \eta_0 x) \\
&= (t, 0, ..., 0, \eta_{2p-2} \circ .. \eta_0 \varepsilon_0 x) \\
&= i(t, \varepsilon_0 x).
\end{aligned}$$

One can show this for $k = 1, ..., p-1$. Thus the inclusion preserves the equivalence relation on $||S||$. $\square$

Chapter 8

ALGEBRAIC POINT OF VIEW

In this chapter , we are going to explain briefly why we need to work on the algebraic level and how the lifting problem will appear in this construction although we are going to mention the lifting problem later with details.

Recall the diagram (5.1) of homotopy equivalences given in the proposition 5.2,

$$\begin{tikzcd}
\|S.\| \arrow[r] \arrow[dr] \arrow[ddr, "u_\delta"'] & \| \, |\bar{P}.S.| \, \| \arrow[d, "f"] \\
& \| \, |P.S.| \, \| \arrow[d, "\approx"] \\
& \| \, |S.| \, \|.
\end{tikzcd}$$

We have defined the homotopy u_δ in the chapter 4 as follows;

$$u_\delta : \Delta^p \times S_p \to \Delta^{p+1} \times \Delta^p \times S_p$$

is defined by

$$u_\delta(t,x) = ([1-(1-t_1)\delta, ..., 1-(1-t_p)\delta, 1-\delta], (t,x))$$

such that $u_0(t,x) = (1,...,1,t,x) = (\varepsilon^p...\varepsilon^0(0),t,x) \sim (0,t,x) \in \Delta^0 \times |S.|$ and $u_1(t,x) = (t_1,...,t_p,0,t,x) = (t,0,t,x) = (\varepsilon^{p+1}t,t,x) \sim (t,t,x) \in \Delta^p \times |S.|$.

In the diagram above, there is a lifting H_δ of a family u_δ with the end points u_0 and u_1, where H_δ is a homotopy between u_0 and u_1 and $u_0 : S_p \to P_0S_{q_0}$, $u_1 : S_p \to P_pS_{0,...,0}$.

So in the following diagram

$$\begin{tikzcd}
& \| \, |P.S.| \, \| \arrow[d, "L"] \\
\|S.\| \arrow[ur, "H_\delta"] \arrow[r, "u_\delta"'] & \| \, |S.| \, \|,
\end{tikzcd}$$

u_δ lifts to H_δ but on the other hand in the following diagram

$$\begin{tikzcd}
& \| \, |\bar{P}.S.| \, \| \arrow[d, "f"] \\
\|S.\| \arrow[ur, "\tilde{u}_1"] \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_0"'] & \| \, |P.S.| \, \|,
\end{tikzcd}$$

although u_1 can be lifted to $\tilde{u}_1$, u_0 can not since when we take $f^{-1}(\Delta^0 \times |P_0 S_.|) = \Delta^0 \times |\bar{S}_.|$ which is contractible but $|S_.|$ is not. So there is no way to lift u_0. Therefore the homotopy H_δ can not be lifted. However it is possible to do this lifting on the chain level therefore we will work on the algebraic level. On the other hand on the space level there is a difference between $\|S_.\|$ and $|S_.|$, we need to carry the problem onto the chain level so there will be no difference in terms of the chain complex of a simplicial set although we consider two different realizations of S. So to be able to find this lifting, we switch the picture to that of the chain complex of $P_. S_.$.

In order to remedy the lifting problem, let's start by giving a definition of the chain complex of $P_. S_.$:

A bicomplex (a double complex) $C_{*,*}(PS)$ is a family of $\{C_{p,n}(PS)\}$ of modules with boundary maps $\partial' : C_{p,n}(PS) \to C_{p-1,n}(PS)$ and $\partial'' : C_{p,n}(PS) \to C_{p,n-1}(PS)$ such that

$$\partial'\partial' = 0,\ \partial''\partial'' = 0 \text{ and } \partial'\partial'' + \partial''\partial' = 0. \tag{8.-1}$$

Thus $C_{p,n}(PS)$ is a bigraded $\mathbb{Z}$-module with the tow differentials. Then the total complex $C_*(PS)$ is given by

$$C_m(PS) = \bigoplus_{p+n=m} C_{p,n}(PS)$$

with the boundary operator $\partial = \partial' + \partial''$ satisfying (8.1).

$$C_{p,n}(PS) = \bigoplus\nolimits_{q_0+\ldots+q_p=n} \mathbb{Z}[P_p S_{q_0,\ldots,q_p}] = \bigoplus\nolimits_{q_0+\ldots+q_p=n} \mathbb{Z}S_{n+p(q_0,\ldots,q_p)},$$

so there is a natural map $\bigoplus_{q_0+\ldots+q_p=n} \mathbb{Z}S_{n+p(q_0,\ldots,q_p)} \to C_{n+p}(S)$ and there is a map $C_{p,n}(PS) \to C_{p+n}(S)$. In order to get the right homology for $\|\,|P_. S_.|\,\|$, we need to define the differential ∂ on the double complex as a sum of two differentials, i.e., $\partial = \partial' + \partial''$.

On the other hand, we can find the sign convention by the following diagram

$$\begin{tikzcd} P_. S_. \arrow[r, "\nu"] \arrow[dr] & E_. S_. \arrow[d] \\ & S_. \end{tikzcd}$$

The sign conventions in $E_. S_.$ and in $S_.$ give a motivation to get the correct sign in $P_. S_.$ since the differentials are preserved under the maps in the diagram above. So the differential ∂ can be thought as $\sum_{j=0}^{n+p} (-1)^j \varepsilon_j$.

Proposition 8.1:

$C_{p,n}(PS)$ is a double complex with the external and the internal differentials, namely given $\partial' = \sum_{r=0}^{p} (-1)^{r+q_0+\ldots+q_{r-1}} \mu_{(r)}$, and $\partial'' = \sum_{i=1}^{p+1} (-1)^{i-1+\sum_{j=0}^{i-2} q_j} d^{(i)}$, where $d^{(i)}$ denotes the i-th differential. Thus the differential is defined as

$$\partial = \partial' + \partial''.$$

Proof :

Let's define the differentials;

For ∂', we are going to use the external differentials (horizontal differentials) $\mu_{(i)}$ given in chapter 5:

$$P_pS_{q_0,\ldots,q_p} \to P_{p-1}S_{q_0,\ldots,\hat{q_i},\ldots,q_p},$$

for ∂'', we are going to use the internal differentials (vertical differentials) ε_i as follows;

$$P_pS_{q_0,\ldots,q_p} \to P_pS_{q_0,\ldots,q_i-1,\ldots,q_p},$$

with $q_0 + \ldots + q_{i-1} + i \leq j \leq q_0 + \ldots + q_i + i$ for $q_i > 0$ (ε_i will be used, since $\mu_{(i)} = 0$) or for $q_i = 0$ ($\mu_{(i)}$ will be used since the degree for p will be reduced one) give ∂'' and ∂' respectively. If $q_i \neq 0$ then $\mu_{(i)} = 0$ so the differential will be only of the form of ∂''. If q_i's are not zero then in order to see that $\partial\partial = 0$, it is enough to see $\partial''\partial'' = 0$. It follows from that one deletes j-th and i-th coordinates when $i < j$ and $j < i$, since by interchanging and arranging this, one can see that they will be cancelled out. Then $\partial'\partial'' + \partial''\partial' = 0$ will also follow automatically, since ∂' is a chain map of degree -1 for the boundary ∂'' so $\partial''\partial' = -\partial'\partial''$. If $q_i = 0$ then for the differential ∂ we take ∂' which is defined by $\mu_{(i)}$ written in terms of only one ε_i. So $\partial\partial = 0$ will follow by using the same idea as above. Thus one can define the differentials carefully with the sign conventions;

$$\partial' = \sum_{r=0}^{p} (-1)^{r+q_0+\ldots+q_{r-1}} \mu_{(r)},$$

and

$$\partial'' = \sum_{i=1}^{p+1} (-1)^{i-1+\sum_{j=0}^{i-2} q_j} d^{(i)},$$

where $d^{(i)}$ denotes the i-th differential. Thus the differential is defined as

$$\partial = \partial' + \partial''.$$

□

Example :

As an example, we can take $p = 2$, since it is obvious for $p = 1$. We want to show that $(\partial' + \partial'') \circ (\partial' + \partial'')$ for $C_{2,0}(PS)$. First let us see $\partial' \circ \partial'$. From the definition, $\partial' : P_2S_{0,0} \to P_1S_{0,0}$ gives $\partial' = \mu_{(0)} - \mu_{(1)} + \mu_{(2)}$ (since q_i's are zero).

$$\begin{aligned}
\partial' \circ \partial' &= (\mu_{(0)} + \mu_{(1)})(\mu_{(0)} - \mu_{(1)} + \mu_{(2)}) \\
&= (\varepsilon_0 - \varepsilon_1)(\varepsilon_0 - \varepsilon_1 + \varepsilon_2) \\
&= \varepsilon_0\varepsilon_0 - \varepsilon_0\varepsilon_1 + \varepsilon_0\varepsilon_2 - \varepsilon_1\varepsilon_0 + \varepsilon_1\varepsilon_1 - \varepsilon_1\varepsilon_2 \\
&= 0.
\end{aligned}$$

Obviously $\partial'' = 0$ since $q_0 = q_1 = 0$. If q_i's are not zero then $\mu_{(i)} = 0$ so we only use ∂''. Let's see $\partial'' \circ \partial'' = 0$ for $C_{1,2}(PS)$, where $q_0 = q_1 = 1$. There are 2 internal differentials $d' = \sum_{r=0}^{q_0} (-1)^r \varepsilon_r$ and $d'' = \sum_{r=0}^{q_1} (-1)^r \varepsilon_{q_0+r+1}$. Then

$$\begin{aligned}
\partial'' \circ \partial'' &= (d' + (-1)^{q_0} d'')(d' + (-1)^{q_0+1} d'') \\
&= {d_0}'d' + (-1)^{q_0+1} {d_0}''d'' + (-1)^{q_0} {d_1}'d' + (-1)^{2q_0+1} {d_1}''d''.
\end{aligned}$$

comes from the definition of $\partial'' = d' + (-1)^{q_0+1}d''$, where $d' : P_1S_{1,1} \to P_1S_{0,1}$ and $d'' : P_1S_{1,1} \to P_1S_{1,0}$.

By substituting $q_0 = q_1 = 1$, we get

$$\begin{aligned}\partial'' \circ \partial'' &= -(\varepsilon_0 - \varepsilon_1)(\varepsilon_2 - \varepsilon_3) + (\varepsilon_1 - \varepsilon_2)(\varepsilon_0 - \varepsilon_1) \\ &= 0.\end{aligned}$$

One can show the same result for $p > 0$.

The last thing is to show

$$\partial' \circ \partial'' + \partial'' \circ \partial' = 0.$$

Let's show this by using some examples. First let's assume $q_0 = q_1 = q_2 = 0$. Then $\partial'' = 0$ and the required equality follows automatically. For the other possibilities, that is, some q_i's are not zero then the computation will be a little bit complicated. For example assume $q_0 = 1, q_1 = q_2 = 0$, then by using the definition of $\partial' = \mu_{(0)} - \mu_{(1)} + \mu_{(2)}$ and $\partial'' = \varepsilon_0 - \varepsilon_1$, we can write the boundary ∂ for $C_{2,1}(PS)$ as $\varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3$ and the required equality becomes

$$\begin{aligned}\partial' \circ \partial'' + \partial'' \circ \partial' &= (\varepsilon_0 - \varepsilon_1 + \varepsilon_2)(\varepsilon_0 - \varepsilon_1)(\varepsilon_2 - \varepsilon_3) \\ &= \varepsilon_0\varepsilon_0 - \varepsilon_0\varepsilon_1 - \varepsilon_1\varepsilon_0 + \varepsilon_1\varepsilon_1 + \varepsilon_2\varepsilon_0 - \varepsilon_2\varepsilon_1 + \varepsilon_0\varepsilon_2 - \varepsilon_0\varepsilon_3 - \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 \\ &= 0.\end{aligned}$$

Thus, one can see

$$\partial' \circ \partial'' + \partial'' \circ \partial' = 0.$$

for all $p, q_i \geq 0$.

Remark :

The differential ∂ fits together into $\sum_{j=0}^{n+p}(-1)^j\varepsilon_j$. This can be seen by the following examples;

For $p = 2$, if $q_0 = 0$ ($d' = 0$), $q_1 = 1$ ($\mu_{(1)} = 0$) and $q_2 = 1$ ($\mu_{(2)} = 0$) then

$$\begin{aligned}\partial' + \partial'' &= \varepsilon_0 - [\varepsilon_1 - \varepsilon_2] + (-1)^{2+0+1}[\varepsilon_3 - \varepsilon_4] \\ &= \varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 \\ &= \sum_{j=0}^{2+2}(-1)^j\varepsilon_j\end{aligned}$$

If $q_0 = 1$ ($\mu_{(0)} = 0$), $q_1 = 1$ ($\mu_{(1)} = 0$) and $q_2 = 0$ ($d''' = 0$) then

$$\begin{aligned}\partial' + \partial'' &= (-1)^4\varepsilon_4 + \varepsilon_0 - \varepsilon_1 + (-1)^{1+1}\varepsilon_2 - \varepsilon_3 \\ &= \varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 \\ &= \sum_{j=0}^{2+2}(-1)^j\varepsilon_j\end{aligned}$$

For $p = 3$, if $q_0 = 0$ $(d' = 0)$, $q_1 = 1$ $(\mu_{(1)} = 0)$ and $q_2 = 0$ $(d''' = 0)$, $q_3 = 1$ $(\mu_{(3)} = 0)$ then,

$$\begin{aligned}
\partial' + \partial'' &= \varepsilon_0 + [(-1)^{1+0}(\varepsilon_1 - \varepsilon_2)] + (-1)^{2+0+1}\varepsilon_3 + [(-1)^{3+1}(\varepsilon_4 - \varepsilon_5)] \\
&= \varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 \\
&= \sum_{j=0}^{2+2} (-1)^j \varepsilon_j
\end{aligned}$$

Thus the differential $\partial = \partial' + \partial''$.

Chapter 9

EXTENSIONS OF u_1 AND u_0

We have the following diagram

$$\begin{tikzcd} & \| \, |\bar{P}_{.}S| \, \| \arrow[d, "f"] \\ \|S\| \arrow[ur, dotted] \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] & \| \, |P_{.}S| \, \| \end{tikzcd}$$

In this chapter, we want to find the extensions aw_0 and aw_1 of u_0 and u_1, respectively in the following diagram

$$\begin{tikzcd} & C_{*,*}(\bar{P}S) \arrow[d, "f"] \\ C_*(\Delta^\infty) \otimes C_*(S) \arrow[ur, dotted] \arrow[r, "aw"] & C_{*,*}(PS) \end{tikzcd} \tag{9.1}$$

on the chain level.

Remark :

Let S be an arbitrary simplicial set. Suppose $S = \Delta[m]$ and $x \in S_m$, then there is a simplicial map $f_x : \Delta[m] \to S$ defined as $f_x(0, ..., m) = x$ for a generator $\iota_m = (0, ..., m) \in \Delta[m]$, that is, $f_x(0, ..., \hat{i}, ..., m) = \varepsilon_i x$ and $f_x(0, ..., i, i, ..., m) = \eta_i x$.

Remark :

By using the previous remark it is enough to construct chain maps for the case of S as a standard simplex. Let's take S as a standard m-simplex, $S_m = \Delta[m]$ and we can define the chain maps u_0, u_1 by $u_0 : S_p \to P_0(S_{q_0})_p = S_m = S_p$ and $u_1 : S_p \to P_p(S_{0,...,0}) = S_m = S_p$ (in this case $p = m$), where $m = n + p$, $n = q_0 + ... + q_p$. In general, the elements in $P_p S_{q_0,...,q_p}$ are denoted by $(\underline{i_0, ..., i_{q_0}}, \underline{i_{q_0+1}, ..., i_{q_0+q_1+1}}, ...)$ (increasing sequence). Thus u_0 and u_1 are defined as $u_0(0, ..., m) = (\underline{0, 1, ..., m})$ and $u_1(0, ..., m) = (\underline{0}, \underline{1}, ..., \underline{m})$, respectively.

Remark :

$\| \, |S_{.}| \, \| = \|\Delta^\infty\| \times |S_{.}|$, where Δ^∞ is the simplicial set with one element $\iota_p = (0, ..., p)$ in each degree and the boundary operators are defined as $\partial_i \iota_p = \iota_{p-1}$, $i = 0, ..., p$ and $\|\Delta^\infty\| = \bigsqcup_{p \geq o} \Delta^p / \sim$, where $\varepsilon^i t \sim t$, $t \in \Delta^{p-1}$. Hence $C_*(\Delta^\infty)$ is the chain group with only one generator in each degree as follows;

$\iota_p = (0, ..., p) \in C_p(\Delta^\infty)$ is a generator so

$$\begin{aligned} \partial \iota_p &= \sum_{i=0}^{p} (-1)^i (0, ..., \hat{i}, ..., p) \\ &\sim \sum_{i=0}^{p} (-1)^i (0, 1, ..., p-1) \\ &= \sum_{i=0}^{p} (-1)^i \iota_{p-1} \end{aligned}$$

Thus (9.2)

$$\partial \iota_p = \begin{cases} 0 & : \ p \text{ is odd or } p = 0 \\ \iota_{p-1} & : \ p = 2k, k = 1, ... \end{cases}$$

Lemma 9.1 :

Let S be a simplicial set. There exists a map of bicomplexes;

$$aw_0 : C_*(\Delta^\infty) \otimes C_*(S) \to C_*(PS)$$

defined for $S = \Delta[n]$ by (9.3)

$$aw_0(\iota_p \otimes (0, ..., n)) = \sum_{0 \le i_0 \le ... \le i_{p-1} \le n} (-1)^{pq_0 + (p-1)q_1 + ... + q_{p-1}} (\underline{0, 1, ..., i_0}, \underline{i_0, ..., i_1}, ..., \underline{i_{p-1}, ..., i_p})$$

Moreover aw_0 is a chain map on the total complexes.

Proof :

We need to show that

$$\partial' aw_0 = aw_0 \partial' \text{ and } \partial'' aw_0 = aw_0 \partial''$$

In order to show the first equality, let's take $p = 1 = n$ and see

$$\partial' aw_0(\iota_1 \otimes (0, 1)) = aw_0(\partial'(\iota_1 \otimes (0, 1))).$$

$$\begin{aligned} aw_0(\iota_1 \otimes (0, 1)) &= \sum_{0 \le i_0 \le 1} (-1)^{q_0} (\underline{0, 1, ..., i_0}, \underline{i_0, ..., 1}) \\ &= (-1)^0 (\underline{0}, \underline{0, 1}) + (-1)^1 (\underline{0, 1}, \underline{1}) \\ &= (\underline{0}, \underline{0, 1}) - (\underline{0, 1}, \underline{1}) \\ \partial' aw_0(\iota_1 \otimes (0, 1)) &= \partial'((\underline{0}, \underline{0, 1}) - (\underline{0, 1}, \underline{1})) \\ &= (\underline{0, 1}) - (\underline{0, 1}) \\ &= 0 \end{aligned}$$

On the other hand

$$\begin{aligned} aw_0(\partial'(\iota_1 \otimes (0,1))) &= aw_0(\sum_{i=0}^{1}(-1)^i \iota_0 \otimes (0,1)) \\ &= aw_0(0 \otimes (0,1)) \\ &= 0 \end{aligned}$$

In general, when p is odd and $n \geq 1$, it is obvious that $\partial' aw = aw\partial'$. On the other hand when $n > 1$ and p is even, it becomes complicated, so for simplicity take $p = 2$ and $n = 1$, we get

$$\begin{aligned} aw_0(\iota_2 \otimes (0,1)) &= (\underline{0}, \underline{0}, \underline{0,1}) - (\underline{0}, \underline{0,1}, \underline{1}) + (\underline{0,1}, \underline{1}, \underline{1}) \\ \partial' aw_0(\iota_2 \otimes (0,1)) &= -(\underline{0,1}, \underline{1}) + (\underline{0}, \underline{0,1}) \end{aligned}$$

On the other hand

$$\begin{aligned} aw_0(\partial'(\iota_2 \otimes (0,1))) &= aw_0(\iota_1 \otimes (0,1)) \\ &= (\underline{0}, \underline{0,1}) - (\underline{0,1}, \underline{1}) \end{aligned}$$

One can show that aw_0 commutes with ∂' for all n and even p.

For the second equality, let's take again $p = 1, n = 1$, then

$$\begin{aligned} aw_0(\iota_1 \otimes (0,1)) &= (\underline{0}, \underline{0,1}) - (\underline{0,1}, \underline{1}) \\ \partial'' aw_0(\iota_1 \otimes (0,1)) &= -(\underline{0}, \underline{1}) + (\underline{0}, \underline{0}) - (\underline{1}, \underline{1}) + (\underline{0}, \underline{1}) \\ &= (\underline{0}, \underline{0}) - (\underline{1}, \underline{1}) \\ aw_0\partial''(\iota_1 \otimes (0,1)) &= aw_0(\iota_1 \otimes (-1)((1) - (0))) \\ &= -(\underline{1}, \underline{1}) + (\underline{0}, \underline{0}) \end{aligned}$$

One can show that aw_0 commutes with ∂'' $\forall p, n$. Moreover, it directly follows from the first part that aw_0 is a chain map on the total complexes for all p. $\square$

Lemma 9.2 :

Let S be a simplicial set. There exists a chain map aw_1, for $S = \Delta[n]$, defined by (9.4)

$$aw_1(\iota_p \otimes (0, ..., n)) = \sum_{0 \leq i_0 \leq ... \leq i_{p-1} \leq n} (-1)^{pq_0 + ... + q_{p-1}} (\underline{0}, ..., \underline{i_0}, \underline{i_0}, ..., \underline{i_{p-1}}, ..., \underline{i_p}).$$

Proof :

We need to show that $\partial \circ aw_1 = aw_1 \circ \partial$. Let's take $p = 1 = n$ then

$$\begin{aligned} \partial aw_1(\iota_1 \otimes (0,1)) &= \partial((\underline{0}, \underline{0}, \underline{1}) - (\underline{0}, \underline{1}, \underline{1})) \\ &= (\underline{0}, \underline{0}) - (\underline{1}, \underline{1}) \\ aw_1\partial(\iota_1 \otimes (0,1)) &= aw_1(-\iota_1 \otimes (1) + \iota_1 \otimes (0)) \\ &= -(\underline{1}, \underline{1}) + (\underline{0}, \underline{0}) \end{aligned}$$

One can show that aw_1 is a chain map for all p. $\square$

Remark :

One can easily see that $aw_0(\iota_0 \otimes x) = u_0$, by taking $S = \Delta[n]$ as follows;

$$\begin{aligned} aw_0(\iota_0 \otimes (0,1,...,n)) &= \sum_{0 \le i_0 \le n} (\underline{0,1,...,i_0}) \\ &= (\underline{0,1,...,n}) \\ &= u_0(0,1,...,n) \in P_0 S_n. \end{aligned}$$

On the other hand the extension aw_1 of u_1 is defined as

$$aw_1(\iota_0 \otimes x) = u_1(x)$$

Chapter 10

SPECTRAL SEQUENCES

In this chapter, we are going to examine the homology for the prism complex $P_{.}S_{.}$ which is denoted by $H(C_*(PS))$. Therefore we are going to give some definitions and facts about spectral sequences followed from MacLane [20] (chapter 11, p.318) and a statement which explains the reason of the algebraic construction. For convenience we use the same indices as in the previous chapter.

A $\mathbb{Z}$-bigraded module is a family $E = \{E_{p,n}\}$ of modules, one for each pair of indices $p, n \in \mathbb{Z}$. A differential $d : E \to E$ of bidegree $(-r, r-1)$ is a family of homomorphisms $\{d : E_{p,n} \to E_{p-r,n+r-1}\}$, one for each p, n, with $d^2 = 0$. Then $H(E) = H(E, d)$ is the bigraded module $\{H_{p,n}(E)\}$ defined as

$$H_{p,n}(E) = \text{Ker}(d : E_{p,n} \to E_{p-r,n+r-1})/d(E_{p+r,n-r+1}).$$

If E is made into a $\mathbb{Z}$-graded module $E = \{E_m\}$ with total degree m by $E_m = \sum_{p+n=m} E_{p,n}$, d induces a differential $d : E_m \to E_{m-1}$ with the usual degree -1 and $H(\{E_m\}, d)$ is the graded module obtained from $H_{p,n}(E)$ as $H_m = \sum_{p+n=m} H_{p,n}$.

Definition 10.1 (Spectral Sequence) :

A spectral sequence $E = \{E^r, d^r\}$ is a sequence $E^2, E^3, ...$ of $\mathbb{Z}$-bigraded modules, each with a differential $d^r : {E_{p,n}}^r \to {E_{p-r,n+r-1}}^r$, $r = 2, 3, ...$ of bidegree $(-r, r-1)$ and with isomorphism

$$H(E^r, d^r) \cong E^{r+1}, \quad r = 2, 3, ...$$

Each E^{r+1} is the bigraded homology module of the preceding (E^r, d^r), that is,

$${E_{p,n}}^{r+1} = \text{Ker}d^r/d^r({E_{p+r,n-r+1}}^r).$$

Definition 10.2 (Filtration) :

A filtration $F_p = F_pC_*$ of chain complexes (C_*, ∂) of R-modules is given by $0 \subseteq F_0C_* \subseteq F_1C_* \subseteq ... \subseteq F_pC_* \subseteq ... \subseteq C_*$, where R is a commutative ring with unit. This filtration is always assumed to be finite in each degree. Then we may give a definition of an associated spectral sequence of a filtration.

Definition 10.3 (Associated Spectral Sequence) :

Assume we have have a filtration $F_p = F_pC_*$ as in the definition above. Then there is an associated spectral sequence $\{E_{p,n}{}^r\}$, $r = 0, 1, ..., \infty$, which is a sequence of bigraded chain complexes with differentials

$$d^r : E_{p,n}{}^r \to E_{p-r,n+r-1}{}^r \text{ such that } E_{p,n}{}^{r+1} = \mathrm{Ker}d^r / d^r(E_{p+r,n-r+1}{}^r),$$

where $p, n \geq 0$ and if for each k there exists a t such that $F_tC_i = C_i$ for $i \leq k$ then $d_{p,n}{}^r = 0$ for $r > t$ and $p + n \leq k$, so that $E_{p,n}{}^{t+1} = E_{p,n}{}^{t+2} = ... = E_{p,n}{}^{\infty}$. In particular

$$E_{p,n}{}^1 = H_{p+n}(F_p/F_{p-1})$$

and $d^1 : E_{p,n}{}^1 \to E_{p-1,n}{}^1$ is the boundary map in the exact sequence

$$0 \to F_{p-1}/F_{p-2} \to F_p/F_{p-2} \to F_p/F_{p-1} \to 0.$$

Moreover E^r converges to $H(C_*)$, that is, there is an induced filtration

$$F_p(H(C_*)) = \mathrm{Im}(H(F_{p,*}) \to H(C_*))$$

such that

$$0 \subseteq F_0H(C_*) \subseteq F_1H(C_*) \subseteq ... \subseteq F_pH(C_*) \subseteq ... \subseteq H(C_*)$$

and there is a canonical isomorphism

$$E_{p,n}{}^{\infty} \cong F_pH_{p+n}/F_{p-1}H_{p+n}.$$

We have given the total complex $C_*(PS)$ before. This chain complex has two filtrations

$$'F_pC_m = \bigoplus_{k \leq p} C_{k,m-k} \text{ and } ''F_nC_m = \bigoplus_{k \leq n} C_{m-k,k},$$

associated spectral sequences $\{'E_{p,n}{}^r, 'd^r\}$ and $\{''E_{n,p}{}^r, ''d^r\}$ both converging to $H(C_*)$. Here

$$'E_{p,n}{}^1 = H_n(C_{p,*}, \partial''), \quad ''E_{n,p}{}^1 = H_p(C_{*,n}, \partial'). \tag{10.1}$$

and

$$'E_{p,n}{}^2 = H_p(H_n(C_{*,*}, \partial''), \partial'), \quad ''E_{n,p}{}^2 = H_n(H_p(C_{*,*}, \partial'), \partial''). \tag{10.2}$$

Proposition 10.4 :

Suppose $H_p(C_{*,n}) = 0$ for $p > 0$ and $n = 0, 1, ...$ Let $C_{-1,n} = \mathrm{coker}(\partial' : C_{1,n} \to C_{0,n})$. Then the edge map e induced by the projection $C_m \to C_{0,m} \to C_{-1,m}$ is a homology isomorphism $e_* : H(C_*) \to H(C_{-1,*}, \partial'')$.

Proof :

By using (10.2), we get

$$''E_{n,p}{}^2 = \begin{cases} 0 & : \quad p > 0 \\ H_n(C_{-1,*}) & : \quad p = 0, \end{cases}$$

and for $r \geq 2$, $''E_{n,p}{}^2 = ''E_{n,p}{}^r$. On the other hand for $p = 0$

$$E_{m,0}{}^\infty \cong F_m H_m / F_{m-1} H_m = H_m(C_*).$$

Thus $H_m(C_*) \cong E_{m,0}{}^\infty \cong H_m(C_{-1,*})$, where $m = n + p$. □

Before giving the following proposition, let's recall some notations we are going to use. We have $\| \, |S| \, \| = \|\Delta^\infty\| \times |S_.|$ where $\|\Delta^\infty\| = \bigsqcup_{p \geq 0} \Delta^p / \sim$ and we know that $|P_.S|$ and $|S|$ are homeomorphic under l. The chain complex $C_*(\Delta^\infty)$ is the chain group with only one generator ι_p in each degree and the boundary operator ∂ is defined as before in (9.2).

Proposition 10.5 :

$$H_*(C_*(PS), \partial) \cong H_*(C_*(S)).$$

Proof :

There is a map between two bicomplexes

$$aw_0 : C_*(\Delta^\infty) \otimes C_*(S) \to C_*(PS) \tag{10.3}$$

We have showed that $aw_0 \circ \partial'' = \partial'' \circ aw_0$ in the previous chapter. On the left hand side we have a double complex and the differential ∂' is identity on $C_*(S)$ and for this double complex we have $'E_{p,n}{}^1 = H_n(C_{p,*}, \partial'')$ independent of p since in (10.3) $'E_{p,*}{}^1$ is the same for $\forall p$. By (10.2), we have

$$'E_{p,n}{}^2 = \begin{cases} 0 & : \quad p > 0 \\ H_n(C_*(S)) & : \quad p = 0, \end{cases}$$

Geometrically, $H_*(C_*(PS), \partial) \cong H_*(|PS|, \partial'')$ follows from the definition. $H_*(|PS|, \partial) \cong H_*(|S|)$ since l is a homeomorhism. Thus $H_*(C_*(PS)) \cong H_*(C_*(S))$. □

Proposition 10.6 :

We have for the double complex $''E_{n,p}{}^1 = H_p(C_{*,n}(PS), \partial')$

$$''E_{n,p}{}^1 = \begin{cases} 0 & : \quad n > 0 \\ H_p(C_*(S)) & : \quad n = 0, \end{cases}$$

Proof :

Before giving the proof, let's explain some notations:

We have given the double complex of PS in the chapter before as

$$C_{p,n}(PS) = \bigoplus_{q_0 + \ldots + q_p = n} \mathbb{Z}[P_p S_{q_0,\ldots,q_p}] = \bigoplus_{q_0 + \ldots + q_p = n} \mathbb{Z} S_{n+p(q_0,\ldots,q_p)}.$$

So when we take S as a standard m-simplex, $S_m = \Delta[m]$, the elements in the double complex are denoted by the following partition $(\underline{i_0, \ldots, i_{q_0}}, \underline{i_{q_0+1}, \ldots, i_{q_0+q_1+1}}, \ldots.)$, where $n + p = m$, here there are $p + 1$ groupings.

One can easily show that $H_1(C_{*,0}(PS)) = H(C_*(S))$ since $C_{p,0}(PS) = \bigoplus_{i=0}^{p} S_p$. On the other hand, one can also easily show that $H_1(C_{*,1}(PS)) = 0$. We have $C_{p,1}(PS) = \bigoplus_{i=0}^{p} S_{p+1}$. An element in this chain complex is denoted by $\underbrace{(0,...,1,...,0)}_{p+1-\text{times}}$, where 1 is in the i-th place.

Now we prove the proposition by introducing a filtration $F_p = F_pC_*(PS)$ of the total complex $C_*(PS)$ which is given by $0 \subseteq F_0C_*(PS) \subseteq F_1C_*(PS) \subseteq ... \subseteq F_pC_*(PS) \subseteq ... \subseteq C_*(PS)$. $F_0(S)$ is given by the partitions of form $(0,...,0,1)$,

$$S_{1(1)} \xleftarrow[\varepsilon_0]{} S_{2(0,1)} \xleftarrow[\varepsilon_0-\varepsilon_1]{} S_{3(0,0,1)} \xleftarrow[\varepsilon_0-\varepsilon_1+\varepsilon_2]{} S_{4(0,0,0,1)}$$

By the chain contraction η_0 we get $H(F_0(S)) = 0$. On the other hand $F_1(S)/F_0(S)$ is given by the partitions of the form $(0,...,1,0)$, i.e., the complex $F_1(S)$ is given by

$$\begin{array}{cccc} S_{1(1)} \leftarrow & S_{2(0,1)} \leftarrow & S_{3(0,0,1)} \leftarrow & S_{4(0,0,0,1)} \cdots \\ & S_{2(1,0)} \leftarrow & S_{3(0,1,0)} \leftarrow & S_{4(0,0,1,0)} \cdots \end{array}$$

$F_0(S)$ is a sub-complex of $F_1(S)$ and by the quotient group $F_1(S)/F_0(S)$, we have a short exact sequence

$$0 \to F_0(S) \to F_1(S) \to F_1(S)/F_0(S) \to 0.$$

The quotient gives us

$$S_{2(1,0)} \leftarrow S_{3(0,1,0)} \leftarrow S_{4(0,0,1,0)} \leftarrow ...$$

and by the contraction η_0 we get $H(F_1(S)/F_0(S)) = 0$. Thus $H(F_1(S)) = 0$.

In order to make the characterization of the filtration clear, we need some notations:

Let the type of elements in $P_pS_{q_0,...,q_p}$, where $q_0 + ...q_p = 1$, be denoted by $(q_0,...,q_p)$. $F_i(S)$ will be characterized by putting at most i zeros after 1 in the sequence, i.e., at most one is 1 and the remainders are 0. Thus one can see $H(C_*(PS)) = 0$ by using an induction on the filtration, since $C_n(PS) = \bigsqcup_{i=0}^{\infty} F_iC_n(PS)$ and $H(C_*(PS)) = \bigoplus_i H(F_i(C_*(PS))$. Therefore $H(C_{*,1}(PS)) = 0$.

We can also show that $H(C_{*,2}(PS)) = 0$. Although $H(C_{*,n}(PS)) = 0$ for $n > 2$, it is quite complicated to write the filtration. Therefore, we will also show that this result is true when $n = 2$ and one can follow the same idea for $n > 2$.

We know that $C_{p,2}(PS) = \bigoplus_{p=0}^{\infty} S_{p+2} = C_{p,2}(PS) = C^{(2)}(PS) \oplus C^{(1,1)}(PS)$, here we have two different types of elements $(0,...,2,...0)$ and $(0,...,0,1,1)$ come from $C^{(2)}(PS)$ and $C^{(1,1)}(PS)$, respectively. For the first type of elements, we will exactly follow the same idea as before when $n = 1$. The filtration $F_i(S)$ will be characterized by putting at most i zeros after 2 in the sequence. The rest follows from the case when $n = 1$. Thus $H(C^{(2)}(PS)) = 0$. For the second type of elements $C^{(1,1)}(PS)$, let's start with $F_0(S)$ given by the partition of the forms $(0,...,1,1)$ as

$$S_{3(1,1)} \xleftarrow[\varepsilon_0]{} S_{4(0,1,1)} \xleftarrow[\varepsilon_0-\varepsilon_1]{} S_{5(0,0,1,1)} \xleftarrow[\varepsilon_0-\varepsilon_1+\varepsilon_2]{} S_{6(0,0,0,1,1)},$$

here the chain contraction is η_0 so we get $H(F_0(S)) = 0$. On the other hand $F_1(S)/F_0(S)$ is given by the partitions $(0, ..., 1, 0, 1)$ and $(0, ..., 0, 1, 1, 0)$ then $F_1(S)$ is

$$\begin{array}{llll} S_{3(1,1)} \leftarrow & S_{4(0,1,1)} \leftarrow & S_{5(0,0,1,1)} \leftarrow & S_{6(0,0,0,1,1)} \cdots \\ & S_{4(1,0,1)} \leftarrow & S_{5(0,1,0,1)} \leftarrow & S_{6(0,0,1,0,1)} \cdots \\ & S_{4(1,1,0)} \leftarrow & S_{5(0,1,1,0)} \leftarrow & S_{6(0,0,1,1,0)} \cdots \end{array}$$

By the same idea as before $H(F_1(S)/F_0(S)) = 0$ and $H(F_0(S)) = 0$ then by the short exact sequence we get $H(F_1(S)) = 0$. By using an induction on i in F_iS, one can see that $H(F_iC^{(1,1)}(PS)) = 0$.

In general we can determine the filtration F_iS by putting at most

i zeros after 1 in the q_0-th, 0 zero after 1 in the $q_{i+1}-$st
$i-1$ zeros after 1 in the q_0-th, 1 zero after 1 in the q_i-th
$i-2$ zeros after 1 in the q_0-th, 2 zeros after 1 in the $q_{i-1}-$st
$i-3$ zeros after 1 in the q_0-th, 3 zeros after 1 in the $q_{i-2}-$nd
...,
...,
...,
3 zeros after 1 in the q_0-th, $i-3$ zeros after 1 in the q_4-rd
2 zeros after 1 in the q_0-th, $i-2$ zeros after 1 in the q_3-rd
1 zero after 1 in the q_0-th, $i-1$ zeros after 1 in the q_2-nd
0 zero after 1 in the q_0-th, i zeros after 1 in the q_1-st

places in $F_iS/F_{i-1}S$, respectively. Thus we get $H(C^{(1,1)}(PS)) = 0$.

We already know $C_{p,2}(PS) = C^{(2)}(PS) \oplus C^{(1,1)}(PS)$ and $H(C_{*,2}(PS) = H(C^{(2)}(PS)) \oplus H(C^{(1,1)}(PS)) = 0$.

By using the same idea for the filtration, one can see $H(C_{*,n}(PS)) = 0$ for $n > 2$. ☐

We would like to close this chapter with one interesting remark.

Remark :

One can try to compute the same thing for $C_{*,n}(\bar{P}S)$. This computation is quite complicated but one can give an idea as follows;

$$\begin{tikzcd}
S_1 \arrow[r, shift left, "-\eta_2\eta_1"] \arrow[d, shift left, "\varepsilon_0"] & S_3 \arrow[l, shift left, "\mu_{(0)}-\mu_{(1)}"] \arrow[r, shift left, "\tilde{S}"] \arrow[d, shift left] & S_5 \arrow[l, shift left] \arrow[d, shift left] \\
S_0 \arrow[u, shift left, "\eta_0"] & S_1 \arrow[l] \arrow[u, shift left] & S_2 \arrow[l, "r"'] \arrow[u, shift left]
\end{tikzcd}$$

The first diagram commutes;

$$\begin{aligned} (\varepsilon_0 - \varepsilon_1)(\varepsilon \circ \varepsilon_2) &= \varepsilon_0\varepsilon_0\varepsilon_2 - \varepsilon_1\varepsilon_0\varepsilon_2 \\ &= \varepsilon_0\varepsilon_0\varepsilon_1 - \varepsilon_0\varepsilon_2\varepsilon_3 \\ &= \varepsilon_0\varepsilon_0\varepsilon_2 - \varepsilon_1\varepsilon_0\varepsilon_2 \\ &= \varepsilon_0(\varepsilon_0\varepsilon_1 - \varepsilon_2\varepsilon_3). \end{aligned}$$

There are various ways of defining the map $S_1 \to S_3$ which are $\eta_0\eta_0$, $\eta_2\eta_0$ and $\eta_1\eta_1$. The chain homotopy between id and $\eta_0\varepsilon_0$ is $-{\eta_1}^2$. That is, (10.4)

$$(\varepsilon_0\varepsilon_1 - \varepsilon_2\varepsilon_3)(-{\eta_1}^2) = \mathrm{id} - \eta_0\varepsilon_0.$$

It is not so clear what the chain homotopy $\tilde{S}$ is. $H(S)$ is the direct sum of $H(C(\bar{P}S))$ but $H_p(C_{*,0}(\bar{P}S))$ could be likely larger then $H(S)$ since $H_p(C_{*,0}(\bar{P}S)) = H_p(S) + \mathrm{Ker}\varepsilon$. Let's denote $H(C^1)$ for the homology for the first sequence and $H(C^2)$ for the second sequence. Let's take an element $y \in H(C^1)$ and $y' = y - \eta\varepsilon y \in \mathrm{Ker}\varepsilon$ since $\varepsilon(y - \eta\varepsilon y) = \varepsilon y - \varepsilon y = 0$. We have $\mathrm{Ker}\varepsilon \to H(C^0) \to H(C^1)$. So $y = y' + \eta\varepsilon y$ and η is injective. We know that $E^\infty = H(S)/\mathrm{Ker}\varepsilon$ and $E^\infty = H(S)$ so $H_0(C_{*,0}(\bar{P}S) = H(S) = H_0$.Then some of the homologies may be killed by some for $n > 0$.

Chapter 11

THE LIFTING PROBLEM

We are going to give the lifting problem explicitly and by considering the diagram (5.1), we will examine the lift of u_0. First we need a motivation for this.

Let C_* and $\tilde{C}_*$ be chain complexes and suppose that there is a chain complex D_* and chain maps $u_0, u_1 : D_* \to C_*$ in the following diagram:

$$\begin{tikzcd} & \tilde{C}_* \arrow[d, "f"] \\ D_* \arrow[ur, "\tilde{u}_1"] \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_0"'] & C_* \end{tikzcd}$$

such that u_1 lifts to $\tilde{u}_1$ and such that a sequence of homomorphisms $s_i : D_i \to C_{i+1}$ is the chain homotopy, i.e., $\partial \circ s + s \circ \partial = u_1 - u_0$.

Lemma 11.1 :

Suppose f is surjective and D_* is a free chain complex in each degree. Then s and u_0 lift to $\tilde{s}$ and $\tilde{u}_0$, respectively. In fact, lift s to $\tilde{s} : D_i \to \tilde{C}_{i+1}$ arbitrarily such that $f \circ \tilde{s} = s$. Define $\tilde{u}_0 = \tilde{u}_1 - (\partial \circ \tilde{s} + \tilde{s} \circ \partial)$. Then $\tilde{u}_0$ is a chain map and $\partial \circ \tilde{s} + \tilde{s} \circ \partial = \tilde{u}_1 - \tilde{u}_0$.

Proof :

Define $\tilde{u}_0 = \tilde{u}_1 - (\partial \circ \tilde{s} + \tilde{s} \circ \partial)$. Let's show that $\tilde{u}_0$ is a chain map and $f \circ \tilde{u}_0 = u_0$. For the first one, we need to see that $\partial \circ \tilde{u}_0 = \tilde{u}_0 \circ \partial$.

$$\begin{aligned} \partial \circ \tilde{u}_0 &= \partial \circ \tilde{u}_1 - \partial \circ \tilde{s} \circ \partial \\ \tilde{u}_0 \circ \partial &= \tilde{u}_1 \circ \partial - \partial \circ \tilde{s} \circ \partial \end{aligned}$$

It follows from that $\tilde{u}_1$ is a chain map. For the second part of the proof, we have

$$\begin{aligned} f \circ \tilde{u}_0 &= f \circ \tilde{u}_1 - (f \circ \partial \circ \tilde{s} + f \circ \tilde{s} \circ \partial) \\ &= u_1 - (\partial \circ f \circ \tilde{s} + s \circ \partial) \\ &= u_1 - (\partial \circ s + s \circ \partial) \\ &= u_1 - (u_1 - u_0) \\ &= u_0 \end{aligned}$$

□

Now let's recall some of the remarks in the chapter 9;

Remark :

Let S be an arbitrary simplicial set. Suppose $S = \Delta[m]$ and $x \in S_m$, then there is a simplicial map $f_x : \Delta[m] \to S$ defined as $f_x(0, ..., m) = x$ for a generator $\iota_m = (0, ..., m) \in \Delta[m]$, that is, $f_x(0, ..., \hat{i}, ..., m) = \varepsilon_i x$ and $f_x(0, ..., i, i, ..., m) = \eta_i x$.

Remark :

By using the previous remark it is enough to construct a chain homotopy for the case of S as a standard simplex. Let's take S as a standard m-simplex, $S_m = \Delta[m]$ and we can define the chain maps u_0, u_1 by $u_0 : S_p \to P_0(S_{q_0})_p = S_m = S_p$ and $u_1 : S_p \to P_p(S_{0,...,0}) = S_m = S_p$ (in this case $p = m$), where $m = n+p$, $n = q_0+...+q_p$. In general, the elements in $P_pS_{q_0,...,q_p}$ are denoted by $(\underline{i_0, ..., i_{q_0}}, \underline{i_{q_0+1}, ..., i_{q_0+q_1+1}}, ...)$ (increasing sequence). Thus u_0 and u_1 are defined as $u_0(0, ..., m) = (\underline{0, 1, ..., m})$ and $u_1(0, ..., m) = (\underline{0}, \underline{1}, ..., \underline{m})$, respectively.

Proposition 11.2 :

Let u_0 and u_1 be given as above. Then these chain maps u_0 and u_1 are chain homotopic.

Proof :

We can define $s_m : C_m(S) \to C_{m+1}(PS)$ arbitrarily.

For $m = 0$, $u_0 = u_1 : S_0 \to \mathbb{Z}S_0 = C_0(PS)$. Let's find the chain homotopy s_m in general. We have $P_pS_{q_0,...,q_p}$, where $q_0 + ... + q_p = n$, and the elements in $P_pS_{q_0,...,q_p}$ are denoted by $(\underline{i_0, ..., i_{q_0}}, \underline{i_{q_0+1}, ..., i_{q_0+q_1+1}}, ...)$. Now, let's define

$$\begin{aligned}
s_0(0) &= (\underline{0}, \underline{0}) \in P_1S_{0,0} \\
s_1(0,1) &= (\underline{0}, \underline{0}, \underline{1}) - (\underline{0,1}, \underline{1}) \in P_2S_{0,0,0} - P_1S_{1,0} \\
\partial s_0(0) &= (\partial' + \partial'')s_0(0) = (\varepsilon_0 - \varepsilon_1 + 0)(\underline{0}, \underline{0}) = (\underline{0}) - (\underline{0}) = 0 \\
s_0\partial(0,1) &= s_0(1-0) = s_0(1) - s_0(0) \\
&= (\underline{1}, \underline{1}) - (\underline{0}, \underline{0}) \\
\partial s_1(0,1) &= (\partial' + \partial'')s_1(0,1) = (\varepsilon_0 - \varepsilon_1 + \varepsilon_2 + 0)(\underline{0}, \underline{0}, \underline{1}) - (\varepsilon_2 + \varepsilon_0 - \varepsilon_1)(\underline{0,1}, \underline{1}) \\
&= (\underline{0}, \underline{1}) - (\underline{0}, \underline{1}) + (\underline{0}, \underline{0}) - (\underline{0,1}) - (\underline{1}, \underline{1}) + (\underline{0}, \underline{1}) \\
&= -s_0\partial(0,1) + u_1(0,1) - u_0(0,1)
\end{aligned}$$

Thus

$$\partial s_1(0,1) + s_0\partial(0,1) = u_1(0,1) - u_0(0,1).$$

Let's do this one more step;

$$\begin{aligned}
\partial(0,1,2) &= (1,2)-(0,2)+(0,1)\\
s_1\partial(0,1,2) &= s_1(1,2)-s_1(0,2)+s_1(0,1)\\
&= (\underline{1},\underline{1},\underline{2})-(\underline{1,2},\underline{2})-(\underline{0},\underline{0},\underline{2})+(\underline{0,2},\underline{2})+(\underline{0},\underline{0},\underline{1})-(\underline{0,1},\underline{1})\\
s_2(0,1,2) &= (\underline{0},\underline{0},\underline{1},\underline{2})-(\underline{0,1}\underline{1},\underline{2})+(\underline{0,1,2},\underline{2})\in P_3S_{0,0,0,0}-P_2S_{1,0,0}+P_1S_{2,0}\\
\partial s_2(0,1,2) &= (\partial'+\partial'')s_2(0,1,2)\\
&= (\varepsilon_0-\varepsilon_1+\varepsilon_2-\varepsilon_3+0)(\underline{0},\underline{0},\underline{1},\underline{2})-(\varepsilon_2-\varepsilon_3+\varepsilon_0-\varepsilon_1)(\underline{0,1},\underline{1},\underline{2})+\\
&\quad (-\varepsilon_3+\varepsilon_0-\varepsilon_1+\varepsilon_2)(\underline{0,1,2},\underline{2})\\
&= (\underline{0},\underline{1},\underline{2})-(\underline{0},\underline{1},\underline{2})+(\underline{0},\underline{0},\underline{2})-(\underline{0},\underline{0},\underline{1})-\\
&\quad (\underline{0,1},\underline{2})+(\underline{0,1},\underline{1})-(\underline{1},\underline{1},\underline{2})+(\underline{0},\underline{1},\underline{2})-\\
&\quad (\underline{0,1,2})+(\underline{1,2},\underline{2})-(\underline{0,2},\underline{2})+(\underline{0,1},\underline{2})
\end{aligned}$$

Thus

$$(s_1\partial+\partial s_2)(0,1,2)=(\underline{0},\underline{1},\underline{2})-(\underline{0,1,2})=u_1(0,1,2)-u_0(0,1,2).$$

One can write a general formula for the chain homotopy s_m as;

$$s_m(0,1,....,m)=\sum_{i=0}^{m}(-1)^i(\underline{0,1,...,i},\underline{i},...,\underline{m}).$$

One can easily check that $s_m\partial+\partial s_{m+1}=u_1-u_0$;

$$\begin{aligned}
s_m\partial(0,1,...,m+1) &= \sum_{j=0}^{m+1}(-1)^j s_m(0,...,\hat{j},...,m+1)\\
&= \sum_{j=0}^{m+1}[\sum_{0\le i<j}(-1)^{i+j}(\underline{0,...,i},\underline{i},.....,\underline{\hat{j}},....,\underline{m+1})+\\
&\qquad \sum_{j<i\le m+1}(-1)^{i+j-1}(\underline{0,...,\hat{j},...,i},\underline{i},.....,\underline{m+1})]\\
\partial s_{m+1}(0,1,....,m+1) &= \sum_{i=0}^{m+1}(-1)^i\partial(\underline{0,1,...,i},\underline{i},....,\underline{m+1})\\
&= \sum_{i=0}^{m+1}[(\sum_{0\le j\le i}(-1)^{i+j}(\underline{0,...,\hat{j},...,i},\underline{i},....,\underline{m+1})+\\
&\qquad \sum_{i\le j\le m+1}(-1)^{i+j+1}(\underline{0,...,...,i},\underline{i},....,\underline{\hat{j}},...,\underline{m+1})]
\end{aligned}$$

So

$$\begin{aligned}
(s_m\partial+\partial s_{m+1})(0,....,m+1) &= \sum_{i=0}^{m+1}(\underline{0,...,i-1},\underline{i},...,\underline{m+1})-(\underline{0,...,i},\underline{i+1},...,\underline{m+1})\\
&= (\underline{0},\underline{1},...,\underline{m+1})-(\underline{0,...,m+1})\\
&= u_1(0,...,m+1)-u_0(0,...,m+1).
\end{aligned}$$

Thus, for a general S the formula is (11.1)

$$s_m(x) = \sum_{i=0}^{m} (-1)^i \eta_i(x)_{(i,0,\ldots,0)} \in P_{m-i+1} S_{(i,0,\ldots,0)}$$

where $s_m : C_m(S) \to C_{m+1}(PS)$, $m = n + p$, $(p = m - i + 1)$.

Then

$$s_m \partial + \partial s_{m+1} = u_1 - u_0 : C_{m+1}(S) \to C_{m+1}(PS).$$

□

Note : The formula (11.1) for a general S will be used in the future references.

Proposition 11.3 :

In the following diagram

$$\begin{tikzcd} & C_*(\bar{P}S) \arrow[d, "f"] \\ C_*(S) \arrow[r, "u_0"] \arrow[ur, dotted, "\tilde{u}_0"] & C_*(PS) \end{tikzcd}$$

one can find a lift for u_0.

Proof :

By using Lemma 11.1 for an arbitrary lift to s such that $f \circ \tilde{s} = s$ one can find a lift $\tilde{u}_0$ by defining $\tilde{u}_0 = \tilde{u}_1 - (\partial \circ \tilde{s} + \tilde{s} \circ \partial)$. One can find a lifting $\tilde{s}$ with the following computation. Although after the projection on $C_*(PS)$, we get the same components, $\tilde{s}$ may not cover the elements coming from s. For instance, look at the following case:

$$\begin{array}{rcl}
\tilde{s}_0(0) & = & (\underline{0,0},\underline{0,0}) \in \bar{P}_1 S_{0,0} \\
\tilde{s}_1(0,1) & = & (\underline{0,0},\underline{0,0},\underline{1,1}) - (\underline{0,1,1},\underline{1,1}) \in \bar{P}_2 S_{0,0,0} - \bar{P}_1 S_{1,0} \\
\partial\tilde{s}_1(0,1) & = & (\partial' + \partial'')\tilde{s}_1(0,1) = (\varepsilon_0 - \varepsilon_1 + \varepsilon_2 + 0)(\underline{0,0},\underline{0,0},\underline{1,1}) - (\varepsilon_0 - \varepsilon_1 + \varepsilon_2)(\underline{0,1,1},\underline{1,1}) \\
& = & (\underline{0,0},\underline{1,1}) - (\underline{0,0},\underline{1,1}) + (\underline{0,0},\underline{0,0}) - (\underline{1,1},\underline{1,1}) + (\underline{0,1},\underline{1,1}) - (\underline{0,1,1}) \\
& = & -\tilde{s}_0\partial(0,1) - (\underline{0,1,1}) + (\underline{0,1},\underline{1,1}).
\end{array}$$

In the last line, one should get $(\underline{0,0},\underline{1,1})$ instead of $(\underline{0,1},\underline{1,1})$. Although when we apply ε^1, we have $(\underline{0},\underline{1})$ but $(\underline{0,0},\underline{1,1}) \neq (\underline{0,1},\underline{1,1})$. One should consider a difference to get the right covering as follows;

$$\begin{array}{rcl}
\partial\tilde{s}_1(0,1) + \tilde{s}_0\partial(0,1) & = & (\underline{0,1},\underline{1,1}) - (\underline{0,1,1}) + (\underline{0,0},\underline{1,1}) - (\underline{0,0},\underline{1,1}) \\
& = & \tilde{u}_1(0,1) - [(\underline{0,0},\underline{1,1}) - (\underline{0,1},\underline{1,1}) + (\underline{0,1,1})]
\end{array}$$

Thus $\tilde{u}_0(0,1) = (\underline{0,0},\underline{1,1}) - (\underline{0,1},\underline{1,1}) + (\underline{0,1,1}) = \tilde{u}_1(0,1) - (\partial\tilde{s}_1(0,1) + \tilde{s}_0\partial(0,1))$.

$$\begin{aligned}
\tilde{s}_2(0,1,2) &= (\underline{0,0},\underline{0,0},\underline{1,1},\underline{2,2}) - (\underline{0,1,1},\underline{1,1},\underline{2,2}) + (\underline{0,1,2,2},\underline{2,2}) \in \\
&\quad \bar{P}_3 S_{0,0,0,0} - \bar{P}_2 S_{1,0,0} + \bar{P}_1 S_{2,0} \\
\partial\tilde{s}_2(0,1,2) &= (\partial' + \partial'')\tilde{s}_2(0,1,2) \\
&= (\varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + 0)(\underline{0,0},\underline{0,0},\underline{1,1},\underline{2,2}) - \\
&\quad (\varepsilon_2 - \varepsilon_3 + \varepsilon_0 + \varepsilon_1)(\underline{0,1,1},\underline{1,1},\underline{2,2}) + \\
&\quad (-\varepsilon_3 + \varepsilon_0 - \varepsilon_1 + \varepsilon_2)(\underline{0,1,2,2},\underline{2,2}) \\
&= (\underline{0,0},\underline{1,1},\underline{2,2}) - (\underline{0,0},\underline{1,1},\underline{2,2}) + (\underline{0,0},\underline{0,0},\underline{2,2}) - \\
&\quad (\underline{0,0},\underline{0,0},\underline{1,1}) - (\underline{0,1,1},\underline{2,2}) + (\underline{0,1,1},\underline{1,1}) - \\
&\quad (\underline{1,1},\underline{1,1},\underline{2,2}) + (\underline{0,1},\underline{1,1},\underline{2,2}) - (\underline{0,1,2,2}) + \\
&\quad (\underline{1,2,2},\underline{2,2}) - (\underline{0,2,2},\underline{2,2}) + (\underline{0,1,2,2},\underline{2,2}) \\
\partial(0,1,2) &= (1,2) - (0,2) + (0,1) \\
\tilde{s}_1\partial(0,1,2) &= (\underline{1,1},\underline{1,1},\underline{2,2}) - (\underline{1,2,2},\underline{2,2}) - (\underline{0,0},\underline{0,0},\underline{2,2}) + \\
&\quad (\underline{0,2,2},\underline{2,2}) + (\underline{0,0},\underline{0,0},\underline{1,1}) - (\underline{0,1,1},\underline{1,1}) \\
\partial\tilde{s}_2(0,1,2) + \tilde{s}_1\partial(0,1,2) &= (\underline{0,1},\underline{1,1},\underline{2,2}) - (\underline{0,1,1},\underline{2,2}) + (\underline{0,1,2},\underline{2,2}) - (\underline{0,1,2,2}) \\
&= \tilde{u}_1(0,1,2) - [(\underline{0,0},\underline{1,1},\underline{2,2}) - (\underline{0,1},\underline{1,1},\underline{2,2}) + \\
&\quad (\underline{0,1,1},\underline{2,2}) - (\underline{0,1,2},\underline{2,2}) + (\underline{0,1,2,2})]
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{u}_0(0,1,2) &= (\underline{0,0},\underline{1,1},\underline{2,2}) - (\underline{0,1},\underline{1,1},\underline{2,2}) + (\underline{0,1,1},\underline{2,2}) - (\underline{0,1,2},\underline{2,2}) + (\underline{0,1,2,2}) \\
&= \tilde{u}_1(0,1,2) - (\partial\tilde{s}_2(0,1,2) + \tilde{s}_1\partial(0,1,2)).
\end{aligned}$$

If we do it one more step, we will get

$$\begin{aligned}
\tilde{u}_0(0,1,2,3) &= (\underline{0,0},\underline{1,1},\underline{2,2},\underline{3,3}) - (\underline{0,1},\underline{1,1},\underline{2,2},\underline{3,3}) + (\underline{0,1,1},\underline{2,2},\underline{3,3}) - \\
&\quad (\underline{0,1,2},\underline{2,2},\underline{3,3}) + (\underline{0,1,2,2},\underline{3,3}) - (\underline{0,1,2,3},\underline{3,3}) + (\underline{0,1,2,3,3}) \\
&= \tilde{u}_1(0,1,2,3) - (\partial\tilde{s}_3(0,1,2,3) + \tilde{s}_2\partial(0,1,2,3)).
\end{aligned}$$

We can write a formula for the lifting as

$$\tilde{s}_m(0,...,m) = \sum_{i=0}^{m} (-1)^i (\underline{0,1,....,i,i},\underline{i,i},...,\underline{m,m}).$$

One can check that $\tilde{s}_m\partial + \partial\tilde{s}_{m+1} = \tilde{u}_1 - \tilde{u}_0$;

$$\begin{aligned}
\tilde{s}_m\partial(0,1,...,m+1) &= \sum_{j=0}^{m+1}(-1)^j\tilde{s}_m(0,...,\hat{j},...,m+1) \\
&= \sum_{j=0}^{m+1}[\sum_{0\le i<j}(-1)^{i+j}(\underline{0,1,...,i,i},\underline{i,i},.....,\underline{\hat{j,j}},....,\underline{m+1,m+1}) + \\
&\quad \sum_{j<i\le m+1}(-1)^{i+j-1}(\underline{0,...,\hat{j},...,i,i},\underline{i,i},.....,\underline{m+1,m+1})] \\
\partial\tilde{s}_{m+1}(0,1,....,m+1) &= \sum_{i=0}^{m+1}(-1)^i\partial(\underline{0,1,...,i,i},\underline{i,i},....,\underline{m+1,m+1}) \\
&= \sum_{i=0}^{m+1}[(\sum_{0\le j\le i}(-1)^{i+j}(\underline{0,...,\hat{j},...,i,i},\underline{i,i},....,\underline{m+1,m+1}) + \\
&\quad \sum_{i\le j\le m+1}(-1)^{i+j+1}(\underline{0,...,...,i,i},\underline{i,i},....,\underline{\hat{j,j}},...,\underline{m+1,m+1})]
\end{aligned}$$

So

$$\begin{aligned}
(\tilde{s}_m\partial + \partial\tilde{s}_{m+1})(0,....,m+1) &= \sum_{i=0}^{m+1}[(\underline{0,...,i-1,i},\underline{i,i},...,\underline{m+1,m+1}) - \\
&\quad (\underline{0,...,i,i},\underline{i+1,i+1},...,\underline{m+1,m+1})] \\
&= (\underline{0,0},\underline{1,1},...,\underline{m+1,m+1}) - (\underline{0,0},\underline{1,1},...,\underline{m+1,m+1}) + \\
&\quad (\underline{0,1},\underline{1,1},...,\underline{m+1,m+1}) - (\underline{0,1,1},\underline{2,2},...,\underline{m+1,m+1}), \\
&, \quad ...,+ \\
&\quad (\underline{0,1,...,m-1,m},\underline{m,m},\underline{m+1,m+1}) - \\
&\quad (\underline{0,...,m,m},\underline{m+1,m+1}) + \\
&\quad (\underline{0,...,m,m+1},\underline{m+1,m+1}) - (\underline{0,...,m+1,m+1}) \\
&= \tilde{u}_1(0,...,m+1) - [\tilde{u}_1(0,...,m+1) - (\tilde{s}_m\partial + \partial\tilde{s}_{m+1})(0,....,m+1)].
\end{aligned}$$

where

$$\begin{aligned}
\tilde{u}_1(0,...,m+1) &= (\underline{0,0},\underline{1,1},...,\underline{m+1,m+1}) - (\underline{0,1},\underline{1,1},...,\underline{m+1,m+1}) + \\
&\quad (\underline{0,1,1},\underline{2,2},...,\underline{m+1,m+1}),...,-(\underline{0,1,...,m-1,m},\underline{m,m},\underline{m+1,m+1}) + \\
&\quad (\underline{0,...,m,m},\underline{m+1,m+1}) - (\underline{0,...,m,m+1},\underline{m+1,m+1}) + \\
&\quad (\underline{0,...,m+1,m+1})
\end{aligned}$$

So

$$\tilde{u}_0(0,...,m) = \sum_{i=0}^{m}(\underline{0,...,i,i},\underline{i+1,i+1},...,\underline{m,m}) - (\underline{0,...,i,i+1},\underline{i+1,i+1},...,\underline{m,m})$$

These can be written in terms of degeneracy operators as follows;

$$\tilde{s}_m(x) = \sum_{i=0}^{m}(-1)^i{\eta_i}^2\circ\eta_i\circ\eta_{i+1}\circ...\circ\eta_m(x)_{(i,0,...,0)} \in \bar{P}_{m-i+1}S_{(i,0,...,0)}$$

and

$$\tilde{u}_0(x) = \sum_{i=0}^{m} [\eta_i \circ ... \circ \eta_m - {\eta_{i+1}}^2 \circ ... \circ \eta_m](x)_{(i,0,...,0)} \in \bar{P}_{m-i} S_{(i,0,...,0)}$$

and

$$\tilde{u}_1(x) = \eta_0 \circ ... \circ \eta_m(x)_{(0,0,...,0)} \in \bar{P}_m S_{(0,...,0)}$$

□

Chapter 12

THE CHAIN HOMOTOPY T AND THE REQUIRED LIFTING $a\tilde{w}_0$

So far we have considered the following diagram

$$\begin{tikzcd} & \| \, |\bar{P}.S.| \, \| \arrow[d, "f"] \\ \|S\| \arrow[ur, "\tilde{u}_0"] \arrow[r, "u_0"] & \| \, |P.S.| \, \| \end{tikzcd}$$

and found the lift of u_0. Moreover, we found the extensions aw_0 and aw_1 of u_0 and u_1 on the chain level, respectively. Now look at the diagram

$$\begin{tikzcd} & C_{*,*}(\bar{P}S) \arrow[d, "f"] \\ C_*(\Delta^\infty) \otimes C_*(S) \arrow[ur, dotted, "\tilde{aw}"] \arrow[r, shift left, "aw"] & C_{*,*}(PS) \arrow[l, shift left, "L"] \end{tikzcd}$$

on the chain level, we look for the required lifting $\tilde{aw}$.

Proposition 12.1 :

There is a chain homotopy T

$$T(\iota_p \otimes x) = \begin{cases} aw_0(\iota_1 \otimes x) - aw_1(\iota_1 \otimes x) + s(x) \in C_{n+1}(PS) & : \quad p = 0 \\ (-1)^p [aw_0(\iota_{p+1} \otimes x) - aw_1(\iota_{p+1} \otimes x)] \in C_{n+p+1}(PS) & : \quad p > 0. \end{cases} \tag{12.1}$$

between aw_1 and aw_0 which can be found as a composition of two chain homotopies.

In order to make this statement clear, we will examine this into some steps:

Lemma 12.2 :

The chain map $\epsilon : C_*(\Delta^\infty) \to C_*(\Delta^\infty)$ is the augmentation which is zero for $p > 0$ and id otherwise. Then in $C_*(\Delta^\infty)$ there is a chain homotopy D between ϵ and id such that $\partial D + D\partial = \epsilon - \text{id}$.

Proof :

We have $C_*(\Delta^\infty) \to C_*(\Delta^\infty)$. D is defined by $D(\iota_p) = (-1)^p \iota_{p+1}$. One can easily check that this gives us a chain homotopy:

$$\begin{aligned}(\partial D + D\partial)(\iota_p) &= \partial((-1)^p \iota_{p+1}) + D(\partial\iota_p) \\ &= (-1)^p \sum_{i=0}^{p+1} (-1)^i \iota_p + D(\sum_{i=0}^{p} (-1)^i \iota_{p-1}) \\ &= \sum_{i=0}^{p+1} (-1)^i (-1)^p \iota_p + \sum_{i=0}^{p} (-1)^i (-1)^{p-1} \iota_p.\end{aligned}$$

If $p > 0$ then

$$\begin{aligned}(\partial D + D\partial)(\iota_p) &= (-1)^p \iota_p + (-1)^{p-1} \iota_p + ... + (-1)^p (-1)^p \iota_p + (-1)^p (-1)^{p-1} \iota_p + (-1)^{p+1} (-1)^p \iota_p \\ &= -\iota_p.\end{aligned}$$

If $p = 0$ then

$$\begin{aligned}(\partial D + D\partial)(\iota_0) &= (\partial D)(\iota_0) + (D\partial)(\iota_0) \\ &= \partial(D\iota_0) + D(\partial\iota_0) \\ &= \partial(-1)^0 \iota_1 \\ &= \sum_{i=0}^{1} (1)^i \iota_0 \\ &= \iota_0 - \iota_0 = 0.\end{aligned}$$

So if $p > 0$ then $\epsilon(\iota_p) = 0$, that is, $\partial D + D\partial = -\mathrm{id}$, and if $p = 0$ then

$$\begin{aligned}(\partial D + D\partial)(\iota_0) &= \epsilon(\iota_0) \otimes \mathrm{id}(\iota_0) - (\mathrm{id} \otimes \mathrm{id})(\iota_0) \\ &= (\mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id})(\iota_0) \\ &= 0.\end{aligned}$$

Thus D is a chain homotopy between ϵ and id. □

Now, we want to get a chain homotopy $\bar{s}$ between v_0 and id via D.

Lemma 12.3 :

There is a chain homotopy $\bar{s} : C_*(\Delta^\infty) \otimes C_*(S) \to C_{*+1}(\Delta^\infty) \otimes C_*(S)$ which is defined by $D \otimes \mathrm{id}$ such that $\partial\bar{s} + \bar{s}\partial = \mathrm{id} \otimes \mathrm{id} - (\mathrm{id} \otimes \mathrm{id} \circ \epsilon \otimes \mathrm{id}) = \mathrm{id} \otimes \mathrm{id} - \epsilon \otimes \mathrm{id}$.

Proof :

One can directly check that $\partial\bar{s} + \bar{s}\partial = \mathrm{id} \otimes \mathrm{id} - \epsilon \otimes \mathrm{id}$.

$$\begin{aligned}\partial\bar{s} + \bar{s}\partial &= (\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial)(D \otimes \mathrm{id}) + (D \otimes \mathrm{id})(\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial) \\ &= (-1)^0 \partial D \otimes \mathrm{id} + (-1)^{1.-1} D \otimes \partial + (-1)^0 D\partial \otimes \mathrm{id} + (-1)^0 D \otimes \partial \\ &= (\partial D + D\partial) \otimes \mathrm{id} \ (D \text{ is a chain homotopy}) \\ &= (\mathrm{id} - \epsilon) \otimes \mathrm{id} \\ &= \mathrm{id} \otimes \mathrm{id} - \epsilon \otimes \mathrm{id}.\end{aligned}$$

□

Now we can explicitly give the chain homotopies T_1, T_2 which are used for the chain homotopy T between aw_1 and aw_0.

Proof of Proposition 12.1 :

By using the chain homotopy T_1 between $(aw_0 - aw_1) \circ \mathrm{id}$ and $(aw_0 - aw_1) \circ v_0$ by the composition of $\bar{s}$ with $aw_0 - aw_1$:

$$T_1 := (aw_0 - aw_1) \circ \bar{s} : C_*(\Delta^\infty) \otimes C_*(S) \to C_{*+1}(\Delta^\infty) \otimes C_*(S) \to C_{*+1}(PS).$$

So

$$\begin{aligned} T_1(\iota_p \otimes x) &= ((aw_0 - aw_1) \circ \bar{s})(\iota_p \otimes x) \\ &= (aw_0 - aw_1)(D(\iota_p) \otimes x) \\ &= (-1)^p[aw_0(\iota_{p+1} \otimes x) - aw_1(\iota_{p+1} \otimes x)]. \end{aligned}$$

We have

$$aw_0 - aw_1 = (aw_0 - aw_1) \circ \mathrm{id} \overset{T_1}{\sim} (aw_0 - aw_1) \circ v_0,$$

and

$$\begin{tikzcd} C_*(\Delta^\infty) \otimes C_*(S) \arrow[d, shift left, "\mathrm{pr}"] \\ C_*(S) \arrow[u, shift left, "i"] \end{tikzcd}$$

and $v_0 = i \circ \mathrm{pr} := (\mathrm{id} \otimes \mathrm{id}) \circ (\epsilon \otimes \mathrm{id}) = \epsilon \otimes \mathrm{id}$ and ϵ induces a homomorphism $\epsilon_* : H_0(C_*(\Delta^\infty)) \to \mathbb{Z}$ and Δ^∞ is contractible.

From the diagram above, we have $v_0 = i \circ \mathrm{pr}$ and $aw_0 \circ i \circ \mathrm{pr} = u_0 \circ \mathrm{pr}$. In the previous chapter we have defined the chain homotopy $s : C_*(S) \to C_{*+1}(PS)$ between u_0 and u_1. Then by using the same idea as above, we can find a chain homotopy T_2 between $u_0 \circ \mathrm{pr}$ and $u_1 \circ \mathrm{pr}$ as follows:

$$T_2 := s \circ \mathrm{pr} : C_*(\Delta^\infty) \otimes C_*(S) \to C_*(S) \to C_{*+1}(PS)$$

which is defined by

$$T_2(\iota_p \otimes x) = (s \circ \mathrm{pr})(\iota_p \otimes x) = \begin{cases} s(x) \in C_{n+1}(PS) & : \quad p = 0 \\ 0 & : \quad p > 0. \end{cases}$$

Finally, we end up by getting the composition of these two chain homotopies T_1 and T_2 in order to find the chain homotopy T. T is defined by $T = T_1 + T_2$ which is the chain homotopy between aw_0 and aw_1 and we get

$$T(\iota_p \otimes x) = \begin{cases} aw_0(\iota_1 \otimes x) - aw_1(\iota_1 \otimes x) + s(x) \in C_{n+1}(PS) & : \quad p = 0 \\ (-1)^p[aw_0(\iota_{p+1} \otimes x) - aw_1(\iota_{p+1} \otimes x)] \in C_{n+p+1}(PS) & : \quad p > 0. \end{cases}$$

If we take $S = \Delta[n]$, $x \in S$,

$$T(\iota_p \otimes (0, ..., n)) = \begin{cases} aw_0(\iota_1 \otimes (0, ...n)) - aw_1(\iota_1 \otimes (0, ..., n) + s(0, ..., n) \in C_{n+1}(PS) & : \quad p = 0 \\ (-1)^p[aw_0(\iota_{p+1} \otimes (0, ..., n)) - aw_1(\iota_{p+1} \otimes 0, ..., n)] \in C_{n+p+1}(PS) & : \quad p > 0. \end{cases}$$

Now, let's show that T is a chain homotopy between aw_1 and aw_0, i.e., there exists T such that

$$\partial T + T\partial = aw_1 - aw_0.$$

For $p = 0$, $S = \Delta[1]$, we have

$$\begin{aligned}
T(\iota_0 \otimes (0,1)) &= aw_0(\iota_1 \otimes (0,1)) - aw_1(\iota_1 \otimes (0,1)) + s(0,1) \\
&= (\underline{0}, \underline{0,1}) - 2(\underline{0,1}, \underline{1}) + (\underline{0}, \underline{1}, \underline{1}) \\
\partial T(\iota_0 \otimes (0,1)) &= (\underline{0}, \underline{0}) - (\underline{1}, \underline{1}) + (\underline{0}, \underline{1}) - (\underline{0,1})
\end{aligned}$$

On the other hand

$$\begin{aligned}
T\partial(\iota_0 \otimes (0,1)) &= T(\iota_0 \otimes (1 - 0)) \\
&= aw_0(\iota_1 \otimes ((1) - (0))) - aw_1(\iota_1 \otimes ((1) - (0))) + s(1) - s(0) \\
&= (\underline{1}, \underline{1}) - (\underline{0}, \underline{0}).
\end{aligned}$$

Then

$$\begin{aligned}
(\partial T + T\partial)(\iota_0 \otimes (0,1)) &= (\underline{0}, \underline{1}) - (\underline{0,1}) = u_1(0,1) - u_0(0,1) \\
&= aw_1(\iota_0 \otimes (0,1)) - aw_0(\iota_0 \otimes (0,1)).
\end{aligned}$$

For $p = 0$ and $n > 0$, one can easily show the same equality as follows;

$$\begin{aligned}
\partial T(\iota_0 \otimes x) &= \partial aw_0(\iota_1 \otimes x) - \partial aw_1(\iota_1 \otimes x) + \partial s(x) \\
T\partial(\iota_0 \otimes x) &= T(\iota_0 \otimes \partial x) \\
&= aw_0(\iota_1 \otimes \partial x) - aw_1(\iota_1 \otimes \partial x) + s(\partial x) \\
(\partial T + T\partial)(\iota_0 \otimes x) &= \partial aw_0(\iota_1 \otimes x) - \partial aw_1(\iota_1 \otimes x) + aw_0(\iota_1 \otimes \partial x) - aw_1(\iota_1 \otimes \partial x) + \\
&\quad \partial s(x) + s\partial x \\
&= aw_0\partial(\iota_1 \otimes x) - aw_1\partial(\iota_1 \otimes x) + aw_0(\iota_1 \otimes \partial x) - aw_1(\iota_1 \otimes \partial x) + \\
&\quad (aw_1 - aw_0)(\iota_0 \otimes x) \\
&= -aw_0(\iota_1 \otimes \partial x) + aw_1(\iota_1 \otimes \partial x) + aw_0(\iota_1 \otimes \partial x) - aw_1(\iota_1 \otimes \partial x) + \\
&\quad (aw_1 - aw_0)(\iota_0 \otimes x) \\
&= aw_1(\iota_0 \otimes x) - aw_0(\iota_0 \otimes x).
\end{aligned}$$

For $p > 0$, one can show the same result as follows;

$$\begin{aligned}
(\partial T + T\partial)(\iota_p \otimes x) &= \partial[(-1)^p(aw_0(\iota_{p+1} \otimes x) - aw_1(\iota_{p+1} \otimes x))] + T(\partial\iota_p \otimes x + (-1)^p\iota_p \otimes \partial x) \\
&= (-1)^p aw_0(\partial\iota_{p+1} \otimes x) + (-1)^{2p+1} aw_0(\partial\iota_{p+1} \otimes x) - (-1)^p aw_1(\partial\iota_{p+1} \otimes x) - \\
&\quad (-1)^{2p+1} aw_1(\iota_{p+1} \otimes \partial x) + T(\partial\iota_p \otimes x) + \\
&\quad (-1)^{2p}[aw_0(\iota_{p+1} \otimes \partial x) - aw_1(\iota_{p+1} \otimes \partial x)] \\
&= aw_1(\iota_p \otimes x) - aw_0(\iota_p \otimes x)
\end{aligned}$$

since when p is odd $\partial\iota_p = 0$ and $\partial\iota_{p+1} = \iota_p$, and when p is even $\partial\iota_p = \iota_{p-1}$ and $\partial\iota_{p+1} = 0$. $\square$

Corollary 12.4 :

There is a lift $a\tilde{w}_0$ in the diagram

$$\begin{tikzcd} & C_*(\bar{P}S)| \arrow[d, "f"] \\ C_*(\Delta^\infty) \otimes C_*(S) \arrow[ur, "a\tilde{w}_0"] \arrow[r, "aw_0"] & C_*(PS) \end{tikzcd}$$

Proof :

f is surjective and aw_1 has a lift. Then by lemma 11.1, T and aw_0 lift to $\tilde{T}$ and $a\tilde{w}_0$, respectively. In fact, lift T to $\tilde{T} : C_*(\Delta^\infty) \otimes C_*(S) \to C_{*+1}(\bar{P}S)$ arbitrarily such that $f \circ \tilde{T} = T$. Define $a\tilde{w}_0 = a\tilde{w}_1 - (\partial\tilde{T} + \tilde{T}\partial)$. Then $a\tilde{w}_0$ is a chain map and

$$\partial\tilde{T} + \tilde{T}\partial = a\tilde{w}_1 - a\tilde{w}_0,$$

follows from lemma 11.1. $\square$

Chapter 13

SIMPLICIAL CURRENTS

In this chapter, we will start giving the definition of a dual complex to the de Rham complex on a simplicial set as in Dupont [11]. We aim to give the definition f a complex $\Omega_*\|X\|$ of simplicial currents as in Dupont-Just [13] on a simplicial manifold X, with similar properties to the complex of currents on a manifold. Let's start with the definition of a dual simplicial complex. $(\mathcal{A}^*(\Delta^q), d)$ is the usual de Rham complex on Δ^q. Then the simplicial de Rham complex $(A^*(S), d)$ consists in degree k of the simplicial k- forms, i.e., $\phi = \{\phi_\sigma\}$, $\sigma \in \bigsqcup_p S_p$, such that

i) $\phi_\sigma \in \mathcal{A}^k(\Delta^p)$ for $\sigma \in \bigsqcup_p S_p$ and

ii) $\phi_{\varepsilon_i\sigma} = (\varepsilon^i)^* \phi_\sigma$, $i = 0, ..., p$, $\sigma \in S_p$, $p = 1, ...$

Furthermore $(d\phi)_\sigma = d\phi_\sigma$.

Definition 13.1 :

Let S be a simplicial set. A dual complex to the de Rham complex S is defined by

$$A_n(S) = \bigoplus_{p=0}^{\infty} \mathcal{A}^p(\Delta^{n+p}) \otimes C_{n+p}(S), \tag{13.1}$$

where $C_{n+p}(S)$ is the usual $n+p$-th chain group of S, with the boundary operator

$$\partial : A_n(S) \to A_{n-1}(S)$$

defined by

$$\partial(w \otimes \sigma) = (-1)^n dw \otimes \sigma + \sum_{i=0}^{n+p} (-1)^i (\varepsilon^i)^* w \otimes \varepsilon_i \sigma,$$

where $w \in \mathcal{A}^p(\Delta^{n+p})$, $\sigma \in S_{n+p}$. One can easily check that $\partial \circ \partial = 0$ as follows;

$$\begin{aligned}
\partial \circ \partial(w \otimes \sigma) &= \partial[(-1)^n dw \otimes \sigma + \sum_{i=0}^{n+p} (-1)^i (\varepsilon)^* w \otimes \varepsilon_i \sigma] \\
&= (-1)^{n-1}(-1)^n ddw \otimes \sigma + (-1)^n \sum_{i=0}^{n+p} (-1)^i (\varepsilon^i)^* dw \otimes \varepsilon_i \sigma + \\
&\quad (-1)^{n-1} \sum_{i=0}^{n+p} (-1)^i d((\varepsilon^i)^* w) \otimes \varepsilon_i \sigma + \sum_{i=0}^{n+p} (-1)^i (\varepsilon^i)^* w \otimes \varepsilon_i \circ \varepsilon_i \sigma \\
&= (-1)^n \sum_{i=0}^{n+p} (-1)^i (\varepsilon^i)^* dw \otimes \varepsilon_i \sigma (1-1) \\
&= 0
\end{aligned}$$

□

Lemma 13.2 :

The natural inclusion $\mathcal{I}_* : C_*(S) \to A_*(S)$ given by $\mathcal{I}_*(\sigma) = 1 \otimes \sigma$, $\sigma \in S_n$, is a chain map. It induces an isomorphism in homology. We have chain maps

$$\mathcal{I}^* : A^n(S) \to C^n(S) \ , \ \mathcal{E}^* : C^n(S) \to A^n(S)$$

given by $\mathcal{I}^*(\varphi)_\sigma = \int_{\Delta^n} \varphi_\sigma$, $\varphi \in A^n(S), \sigma \in S_n$. We have a pairing

$$A^n(S) \otimes A_n(S) \to \mathbb{R}$$

given by $< \phi, w \otimes \sigma >= \int_{\Delta^{n+p}} \phi_\sigma \wedge w$, where $\phi \in A^n(S)$, $w \in \mathcal{A}^p(\Delta^{n+p})$, $\sigma \in S_{n+p}$. The second chain map is given by $\mathcal{E}^*(c)_\sigma = n! \sum_{|I|=n} w_I . c_{\mu_I(\sigma)}$, $\sigma \in S_p$, $c \in C^n(S)$, where $I = (i_0, ..., i_n)$ with $i \leq i_0 < i_1 < ... < i_n \leq p$, $\mu_I : S_p \to S_n$ is the face operator corresponding to the inclusion $\mu^I : \Delta^n \to \Delta^p$ onto the n-dimensional face spanned by $\{e_{i_0}, ..., e_{i_n}\}$, and $w_I = \sum_{s=0}^n (-1)^s t_{i_s} dt_{i_0} \wedge ... \wedge d\hat{t}_{i_s} \wedge ... \wedge dt_{i_n}$.

Lemma 13.3 :

There is a chain map

$$\mathcal{E}_* : A_*(S) \to C_*(S)$$

defined by

$$\mathcal{E}_*(w \otimes \sigma) = n! \sum_{|I|=n} \left[\int_{\Delta^{n+p}} w_I \wedge w \right] \mu_I(\sigma),$$

where $w \in \mathcal{A}^p(\Delta^{n+p})$, $\sigma \in S_{n+p}$. This induces an isomorphism in homology

$$\mathcal{E}_* : H(A_*(S)) \to H(C_*(S))$$

which is inverse to $\mathcal{I}_*$.

Lemma 13.4 :

A simplicial map $f : S \to S'$ induces a chain map $f_* : A_n(S) \to A_n(S')$ given by $f_*(w \otimes \sigma) = w \otimes f(\sigma)$, $w \in \mathcal{A}^p(\Delta^{n+p})$, $\sigma \in S_{n+p}$. Its dual

$$f^* : A^n(S') \to A^n(S)$$

is given by $(f^*\phi)_\sigma = \phi_{f(\sigma)}$, that is, $< f^*\phi, T >=< \phi, f_*T >$, $\phi \in A^n(S')$, $T \in A_n(S)$.

After this preparation, we can give the definition of simplicial currents and some necessary facts. The space $\Omega_*\|X\|$ will correspond to the dual space of the complex $A^*\|X\|$ of simplicial differential forms which is an associative, graded commutative algebra. In this case the de Rham theorem gives a quasi-isomorphism from the complex $C_*(X)$ of singular chains on X to the complex $\Omega_*(X)$ of compactly supported currents on X. One needs to equip the space $A^*\|X\|$ with a natural Frechét topology in order to give a definition of the simplicial currents. One can see the monograph [31] for the theory of Frechét spaces. (p. 85).

Definition 13.5:

The simplicial n-forms have been defined in definition 1.9 as the space

$$A^n\|X\| = \{\phi \in \Pi_k A^n(\Delta^k \times X_k) \mid (\varepsilon^i \times \mathrm{id})^*\phi^{(k)} = (\mathrm{id} \times \varepsilon_i)\phi^{(k-1)}\}.$$

The space of simplicial n-currents is the dual space

$$\Omega_n\|X\| = A^n\|X\|'$$

with the necessary topology.

Remark : Another construction of a complex of simplicial currents follows from the definition 13.1, in the case of a discrete simplicial set. This complex embeds in $\Omega_*\|X\|$ as a dense sub-complex with the same homology. The extended complex of (13.1) to general simplicial manifolds gives a complex formula (13.2)

$$\mathcal{A}_n(X) = \bigoplus_{k=0} \bigoplus_{l=0} \mathcal{A}^{k-l}(\Delta^k) \otimes \Omega_{n-l}(X_k)$$

with differential

$$\partial(w \otimes T) = (-1)^l dw \otimes T + \sum_{i=0}^{k} (-1)^i (\varepsilon^i)^* w \otimes (\varepsilon_i)_* T + (-1)^l w \otimes \partial T$$

In order to see (13.1), we take $k = n + p, n = l$ in (13.2), then

$$\begin{aligned} A_n(S) &= \bigoplus_{p=0}^{\infty} \mathcal{A}^p(\Delta^{n+p}) \otimes \Omega_0(S) \\ &= \bigoplus_{p=0}^{\infty} \mathcal{A}^p(\Delta^{n+p}) \otimes C_{n+p}(S). \end{aligned}$$

For the future reference when we take $X_k = \Delta^\infty$, it gives

$$A_n(\Delta^\infty) = \bigoplus_{k=0}^{\infty} \mathcal{A}^{k-n}(\Delta^k) \otimes C_0(\Delta^\infty).$$

We give the simplicial de Rham theorem for currents as follows:

Corollary 13.6 :

There is a natural isomorphism

$$H(\Omega_*\|X\|) \cong H_*(\|X\|).$$

We can give another definition of $\Omega_*\|X\|$ as follows:

Definition 13.7 :

There is an alternative definition of $\Omega_n\|X\|$ as the following quotient space

$$\tilde{\Omega}_*\|X\| = \bigoplus_k \Omega_*(\Delta^k \times X_k)/\overline{\mathrm{span}}_{\mathbb{C}}\{(\varepsilon^i \times \mathrm{id})_* U - (\mathrm{id} \times \varepsilon_i)_* U \mid U \in \Omega_n(\Delta^{k-1} \times X_k)\}.$$

Theorem 13.8 :

There is a natural, continuous isomorphism

$$\Omega_*\|X\| \cong \tilde{\Omega}_*\|X\|.$$

Theorem 13.9 :

There is a quasi-isomorphism

$$\mathcal{A}_*(X) \to \Omega_*\|X\|,$$

where $\mathcal{A}_*(X)$ is embedded as a dense subspace and $\Omega_*\|X\|$ is regarded as a completion of $\mathcal{A}_*(X)$ with a suitable topology.

Before giving the following corollary, let's observe bicomplexes $A^{*,*}\|X\|$ and $\Omega_{*,*}\|X\|$. Let

$$A^{*,*}\|X\| = (\Pi_k A^{p,q}(\Delta^k \times X_k)) \bigcap A^n\|X\|,$$

$\Omega_{p,q}\|X\| = A^{p,q}\|X\|' = \{U \in \Omega_n\|X\| \mid U(A^{r,s}\|X\|) = 0 \text{ unless } (r,s) = (p,q)\}$. We get bicomplexes $(A^{*,*}\|X\|, d_\Delta, d_x)$ and $(\Omega_{*,*}\|X\|, \partial'_\Delta, \partial''_\Omega)$ with total complexes $(A^*\|X\|, d)$ and $(\Omega_*\|X\|, \partial_\Omega)$. We can also give the simplicial deRham theorem for currents for a simplicial manifold X.

Corollary 13.10 :

For each q the chain complexes $(\Omega_{*,q}\|X\|, \partial_\Omega{}')$ and $(C_{*,q}(X), \partial')$ are naturally chain homotopy equivalent. In fact there are natural maps of bicomplexes

$$\mathcal{I}' : C_{*,*}(X) \to \Omega_{*,*}\|X\|,\ \mathcal{E}' : \Omega_{*,*}\|X\| \to C_{*,*}(X)$$

such that $\mathcal{E}' \circ \mathcal{I}' = \mathrm{id}$, and chain homotopies

$$s' : \Omega_{p,q}\|X\| \to \Omega_{p+1,q}\|X\|$$

such that

$$\mathcal{I}' \circ \mathcal{E}' - \mathrm{id} = s'\partial_\Omega{}' + \partial_\Omega{}'s', \; s'\partial_\Omega{}'' = \partial_\Omega{}''s',$$

here

$$\begin{aligned} \mathcal{I}'(T) &= 1_{\Delta^p} \times T \quad , \quad T \in C_{p,q}(X) \\ \mathcal{E}'(T) &= p! \sum_{|I|=p} (\mu_I)_*(T \wedge \alpha_I), \quad , \quad T \in \Omega_{p,q}(\Delta^k \times X_k), k \geq p \\ s'(T) &= \sum_{0 \leq |I| \leq p} |I|! {h_I}'(T \wedge \alpha_I). \end{aligned}$$

We will always consider the dual to the Poincaré lemma operator ${h_j}'$ which increase the degree by $k+1$, $I = (i_0, i_1, ..., i_{|I|})$ denotes a sequence of integers such that $0 \leq i_0 < i_1 < ... < i_{|I|} \leq k$,

$$\alpha_I = \sum_{j=0}^{|I|} (-1)^j t_{i_j} dt_{i_0} \wedge ... \wedge d\hat{t}_{i_j} \wedge ... \wedge dt_{i_{|I|}}.$$

We have the following isomorphisms

$$\begin{aligned} H^*(\|X\|) &\cong H(\mathcal{A}^*\|X\|, d) \\ H_*(\|X\|) &\cong H(\Omega_*\|X\|, \partial). \end{aligned}$$

Now, we are ready to find the extensions $a{w_\Omega}^0$ and $a{w_\Omega}^1$ of aw_0 and aw_1, respectively, in the simplicial currents.

Chapter 14

EXTENSIONS OF aw_0 AND aw_1

We have defined the maps aw_0 and aw_1 in the chapter 9 on the chain level. In this chapter we want to extend them to the simplicial currents. So (9.1) becomes

$$\begin{tikzcd}
 & \Omega_{*,*}(\bar{P}S) \arrow[d, "f_\Omega"] \\
\Omega_*(\Delta^\infty) \otimes C_*(S) \arrow[ur, dotted] \arrow[r, "aw_\Omega"] & \Omega_{*,*}(PS)
\end{tikzcd}$$

Now we are going to define the chain maps ${aw_\Omega}^0$ and ${aw_\Omega}^1$.

Lemma 14.1 :

Let S be a simplicial set. There exists a map of bicomplexes;

$${aw_\Omega}^0 : \Omega_*(\Delta^\infty) \otimes C_*(S) \to \Omega_{*,*}(PS)$$

defined for $S = \Delta[n]$ by

$${aw_\Omega}^0(U \otimes (0, ..., n)) = \sum_{q_0+...+q_{p+l}=n} (-1)^{(p+l)q_0+...+q_{p+l-1}} U \times 1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}} \otimes (\underline{0, ..., i_0}, \underline{i_0, ..., i_1}, ..., \underline{..., n}),$$

where $U \in \Omega_p(\Delta^\infty)$, that is, $U : \mathcal{A}^p(\Delta^{p+l}) \to \mathbb{R}$, $1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}} \in \Omega_n(PS)$.

Proof :

For simplicity, let's take $p = 1$, $l = 0$, $n = 1$ and show that

$$\partial_\Omega' {aw_\Omega}^0 = {aw_\Omega}^0 \partial_\Omega' \text{ and } \partial_\Omega'' {aw_\Omega}^0 = {aw_\Omega}^0 \partial_\Omega''.$$

$$\begin{aligned}
{aw_\Omega}^0(U \otimes (0,1)) &= \sum_{q_0+q_1=1} (-1)^{q_0} U \times 1_{\Delta^{q_0}\times\Delta^{q_1}} \otimes (\underline{0, ..., i_0}, \underline{i_0, ..., 1}) \\
&= U \times 1_{\Delta^0\times\Delta^1} \otimes (\underline{0}, \underline{0,1}) - U \times 1_{\Delta^1\times\Delta^0} \otimes (\underline{0,1}, \underline{1}) \\
\partial_\Omega' {aw_\Omega}^0(U \otimes (0,1)) &= \partial U \times 1_{\Delta^0\times\Delta^1} \otimes (\underline{0}, \underline{0,1}) - \partial U \times 1_{\Delta^1\times\Delta^0} \otimes (\underline{0,1}, \underline{1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\partial_\Omega{}'(U\otimes(0,1)) &= \partial U\otimes(0,1)\\
aw_\Omega{}^0(\partial_\Omega{}'(U\otimes(0,1))) &= aw_\Omega{}^0(\partial U\otimes(0,1))\\
&= \sum_{q_0+q_1=1}(-1)^{q_0}\partial U\times 1_{\Delta^{q_0}\times\Delta^{q_1}}\otimes(\underline{0,...,i_0},\underline{i_0,...,1})\\
&= \partial U\times 1_{\Delta^0\times\Delta^1}\otimes(\underline{0},\underline{0,1})-\partial U\times 1_{\Delta^1\times\Delta^0}\otimes(\underline{0,1},\underline{1}).
\end{aligned}$$

Then

$$\partial_\Omega{}'aw_\Omega{}^0 = aw_\Omega{}^0\partial_\Omega{}'.$$

For the second equality, we have

$$\begin{aligned}
\partial_\Omega{}''aw_\Omega{}^0(U\otimes(0,1)) &= U\times 1_{\partial(\Delta^0\times\Delta^1)}\otimes(\underline{0},\underline{0,1})-U\times 1_{\partial(\Delta^1\times\Delta^0)}\otimes(\underline{0,1},\underline{1})\\
&= -U\times 1_{\Delta^0\times\partial\Delta^1}\otimes(\underline{0},\underline{0,1})-U\times 1_{\partial\Delta^1\times\Delta^0}\otimes(\underline{0,1},\underline{1})\\
&\sim -U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{0},\underline{1})+U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{0},\underline{0})-\\
&\quad U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{1},\underline{1})+U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{0},\underline{1})\\
&= U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{0},\underline{0})-U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{1},\underline{1}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
aw_\Omega{}^0\partial_\Omega{}''(U\otimes(0,1)) &= aw_\Omega{}^0(U\otimes -(1-0))\\
&= -aw_\Omega{}^0(U\otimes(1))+aw_\Omega{}^0(U\otimes(0))\\
&= -U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{1},\underline{1})+U\times 1_{\Delta^0\times\Delta^0}\otimes(\underline{0},\underline{0}).
\end{aligned}$$

So

$$aw_\Omega{}^0\partial_\Omega{}'' = aw_\Omega{}^0\partial_\Omega{}''.$$

In general,

$$\begin{aligned}
\partial_\Omega{}'aw_\Omega{}^0(U\otimes(0,...,n)) &= \sum_{q_0+...+q_{p+l}=n}(-1)^{(p+l)q_0+...+q_{p+l-1}}\partial U\times 1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}}\otimes(\underline{0,...,i_0},...,\underline{....,n})\\
&= aw_\Omega{}^0(\partial U\otimes(0,...,n))\\
&= aw_\Omega{}^0\partial_\Omega{}'(U\otimes(0,...,n)).
\end{aligned}$$

and

$$\begin{aligned}
\partial_\Omega{}''aw_\Omega{}^0(U\otimes(0,...,n)) &= \sum_{q_0+...+q_{p+l}=n}(-1)^{(p+l)q_0+...+q_{p+l-1}}U\times\partial(1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}})\otimes(\underline{0,...,i_0},...,\underline{....,n})\\
&\sim \sum_{q_0+...+q_{p+l}=n}(-1)^{(p+l)q_0+...+q_{p+l-1}}U\times 1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}}\otimes\partial''(\underline{0,...,i_0},...,\underline{....,n})\\
&= aw_\Omega{}^0(U\otimes\partial(0,...,n))\\
&= aw_\Omega{}^0\partial_\Omega{}''(U\otimes(0,...,n)).
\end{aligned}$$

$\square$

Before giving the following lemma, we need some preparation for this. We have a homeomorphism

$$l : \Delta^{p+l} \times \Delta^{q_0} \times ... \times \Delta^{q_{p+l}} \to \Delta^{p+l+n}$$

which gives a map between the spaces of differential forms

$$l^* : \mathcal{A}^{p+n}(\Delta^{p+l+n}) \to \mathcal{A}^{p+n}(\Delta^{p+l} \times \Delta^{q_0} \times ... \times \Delta^{q_{p+l}}).$$

By duality, this induces a map between the simplicial currents

$$l_* : \Omega_{p+n}(\Delta^{p+l} \times \Delta^{q_0} \times ... \times \Delta^{q_{p+l}}) \to \Omega_{p+n}(\Delta^{p+l+n}).$$

Let $U \in \Omega_p(\Delta^\infty)$, i.e., $U : \mathcal{A}^p(\Delta^{p+l}) \to \mathbb{R}$, and $U \times 1_{\Delta^{q_0} \times ... \times \Delta^{q_{p+l}}} \in \Omega_{p+n}(\Delta^{p+l} \times \Delta^{q_0} \times ... \times \Delta^{q_{p+l}})$ then $l_*(U \times 1_{\Delta^{q_0} \times ... \times \Delta^{q_{p+l}}}) \in \Omega_{p+n}(\Delta^{p+l+n})$, where $1_{\Delta^{q_0} \times ... \times \Delta^{q_{p+l}}} \in \Omega_n(\Delta^{q_0} \times ... \times \Delta^{q_{p+l}})$.

Lemma 14.2 :

Let S be a simplicial set. There exists a chain map,

$$aw_\Omega{}^1 : \Omega_*(\Delta^\infty) \otimes C_*(S) \to \Omega_{*,0}(PS)$$

defined for $S = \Delta[n]$ by

$$aw_\Omega{}^1(U \otimes (0, ..., n)) = \sum_{q_0 + ... + q_{p+l} = n} (-1)^{(p+l)q_0 + ... + q_{p+l-1}} l_*(U \times 1_{\Delta^{q_0} \times ... \times \Delta^{q_{p+l}}}) \otimes (\underline{0}, ..., \underline{i_0}, \underline{i_0}, ..., \underline{n}).$$

Proof :

We will show that

$$\partial_\Omega aw_\Omega{}^1 = aw_\Omega{}^1 \partial_\Omega.$$

The proof of this statement is much more complicated than the proof of the previous lemma so let's take $p = 1$, $n = 2$ and $l = 0$.

$$\begin{aligned}
aw_\Omega{}^1(U \otimes (0,1,2)) &= l_*(U \times 1_{\Delta^0 \times \Delta^2}) \otimes (\underline{0}, \underline{0}, \underline{1}, \underline{2}) - l_*(U \times 1_{\Delta^1 \times \Delta^1}) \otimes (\underline{0}, \underline{1}, \underline{1}, \underline{2}) + \\
&\quad l_*(U \times 1_{\Delta^2 \times \Delta^0}) \otimes (\underline{0}, \underline{1}, \underline{2}, \underline{2}) \\
\partial aw_\Omega{}^1(U \otimes (0,1,2)) &= l_*(\partial U \times 1_{\Delta^0 \times \Delta^2} + (-1)^2 U \times 1_{\partial\Delta^0 \times \Delta^2} + (-1)^3 U \times 1_{\Delta^0 \times \partial\Delta^2}) \otimes (\underline{0}, \underline{0}, \underline{1}, \underline{2}) - \\
&\quad l_*(\partial U \times 1_{\Delta^1 \times \Delta^1} + (-1)^2 U \times 1_{\partial\Delta^1 \times \Delta^1} + (-1)^4 U \times 1_{\Delta^1 \times \partial\Delta^1}) \otimes (\underline{0}, \underline{1}, \underline{1}, \underline{2}) + \\
&\quad l_*(\partial U \times 1_{\Delta^2 \times \Delta^0} + (-1)^2 U \times 1_{\partial\Delta^2 \times \Delta^0} + (-1)^5 U \times 1_{\Delta^2 \times \partial\Delta^0}) \otimes (\underline{0}, \underline{1}, \underline{2}, \underline{2}) \\
&= l_*(\partial U \times 1_{\Delta^0 \times \Delta^2}) \otimes (\underline{0}, \underline{0}, \underline{1}, \underline{2}) - l_*(\partial U \times 1_{\Delta^1 \times \Delta^1}) \otimes (\underline{0}, \underline{1}, \underline{1}, \underline{2}) + \\
&\quad l_*(\partial U \times 1_{\Delta^2 \times \Delta^0}) \otimes (\underline{0}, \underline{1}, \underline{2}, \underline{2}) + \\
&\quad l_*(U \times 1_{\Delta^0 \times \Delta^1}) \otimes (\underline{0}, \underline{0}, \underline{2}) - l_*(U \times 1_{\Delta^0 \times \Delta^1}) \otimes (\underline{0}, \underline{0}, \underline{1}) - \\
&\quad l_*(U \times 1_{\Delta^0 \times \Delta^1}) \otimes (\underline{1}, \underline{1}, \underline{2}) + l_*(U \times 1_{\Delta^1 \times \Delta^0}) \otimes (\underline{0}, \underline{1}, \underline{1}) + \\
&\quad l_*(U \times 1_{\Delta^1 \times \Delta^0}) \otimes (\underline{1}, \underline{2}, \underline{2}) - l_*(U \times 1_{\Delta^1 \times \Delta^0}) \otimes (\underline{0}, \underline{2}, \underline{2}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
{aw_\Omega}^1(\partial(U\otimes(0,1,2)) &= {aw_\Omega}^1(\partial U\otimes(0,1,2)-U\otimes(1,2)+U\otimes(0,2)-U\otimes(0,1))\\
&= l_*(\partial U\times 1_{\Delta^0\times\Delta^2})\otimes(\underline{0},\underline{0},\underline{1},\underline{2})-l_*(\partial U\times 1_{\Delta^1\times\Delta^1})\otimes(\underline{0},\underline{1},\underline{1},\underline{2})+\\
&\quad l_*(\partial U\times 1_{\Delta^2\times\Delta^0})\otimes(\underline{0},\underline{1},\underline{2},\underline{2})-l_*(U\times 1_{\Delta^0\times\Delta^1})\otimes(\underline{1},\underline{1},\underline{2})+\\
&\quad l_*(U\times 1_{\Delta^1\times\Delta^0})\otimes(\underline{1},\underline{2},\underline{2})+l_*(U\times 1_{\Delta^0\times\Delta^1})\otimes(\underline{0},\underline{0},\underline{2})-\\
&\quad l_*(U\times 1_{\Delta^1\times\Delta^0})\otimes(\underline{0},\underline{2},\underline{2})-l_*(U\times 1_{\Delta^0\times\Delta^1})\otimes(\underline{0},\underline{0},\underline{1})+\\
&\quad l_*(U\times 1_{\Delta^1\times\Delta^0})\otimes(\underline{0},\underline{1},\underline{1})\\
&= \partial {aw_\Omega}^1(U\otimes(0,1,2)).
\end{aligned}$$

Thus

$$\partial {aw_\Omega}^1={aw_\Omega}^1\partial.$$

We can however show this equality for general p,n,l;

$$\begin{aligned}
{aw_\Omega}^1\partial_\Omega(U\otimes(0,...,n)) &= {aw_\Omega}^1((\partial'+\partial'')(U\otimes(0,...,n)))\\
&= {aw_\Omega}^1(\partial U\otimes(0,...,n)+(-1)^pU\otimes\partial(0,...,n))
\end{aligned}$$

On the other hand

$$\begin{aligned}
\partial_\Omega {aw_\Omega}^1(U\otimes(0,...,n)) &= {\partial_\Omega}'{aw_\Omega}^1(U\otimes(0,...,n))\\
&= \sum_{q_0+...+q_{p+l}=n}(-1)^{(p+l)q_0+....+q_{p+l-1}}\partial l_*(U\times 1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}})\otimes(\underline{0},...,\underline{i_0},\underline{i_0},...,\underline{n})\\
&= \sum_{q_0+...+q_{p+l}=n}(-1)^{(p+l)q_0+....+q_{p+l-1}}l_*(\partial(U\times 1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}}))\otimes(\underline{0},...,\underline{i_0},\underline{i_0},...,\underline{n})\\
&= \sum_{q_0+...+q_{p+l}=n}(-1)^{(p+l)q_0+....+q_{p+l-1}}\big[l_*(\partial U\times 1_{\Delta^{q_0}\times...\times\Delta^{q_{p+l}}}+\\
&\quad (-1)^{p+1}U\times 1_{\partial\Delta^{q_0}\times...\times\Delta^{q_{p+l}}}+(-1)^{p+q_0+2}U\times 1_{\Delta^{q_0}\times\partial\Delta^{q_1}\times...\times\Delta^{q_{p+l}}}+\\
&\quad ...+\\
&\quad (-1)^{p+q_0+...+q_{p+l-2}+p+l}U\times 1_{\Delta^{q_0}\times...\times\partial\Delta^{q_{p+l}}})\big]\otimes(\underline{0},...,\underline{i_0},\underline{i_0},...,\underline{n})\\
&= {aw_\Omega}^1(\partial U\otimes(0,...,n)+(-1)^pU\otimes\partial(0,...,n)).
\end{aligned}$$

□

Now, we can show that these two maps are homotopic via ${s_\Omega}'$.

Proposition 14.3 :

There is a homotopy ${s_\Omega}'$ between ${aw_\Omega}^1$ and ${aw_\Omega}^0$ defined by

$${s_\Omega}'=({aw_\Omega}^1-{aw_\Omega}^0)\circ s',$$

and

$${s_\Omega}':\Omega_p(\Delta^\infty)\otimes C_n(S)\to\Omega_{p+1}(\Delta^\infty)\otimes C_n(S)\to\Omega_{n+p+1}(PS)$$

such that the formula (14.1)

$$\partial_\Omega {s_\Omega}'+{s_\Omega}'\partial_\Omega={aw_\Omega}^0-{aw_\Omega}^1.$$

Proof :

We can show that ${s_\Omega}'$ is a homotopy such that (16.1) holds as follows;

$$\begin{aligned}
(\partial_\Omega {s_\Omega}' + {s_\Omega}'\partial_\Omega)(U\otimes x) &= \partial_\Omega({aw_\Omega}^1 - {aw_\Omega}^0)s'(U\otimes x) + ({aw_\Omega}^1 - {aw_\Omega}^0)s'\partial_\Omega(U\otimes x)\\
&\overset{(1)}{=} ({aw_\Omega}^1\partial_\Omega - {aw_\Omega}^0\partial_\Omega)s'(U\otimes x) + ({aw_\Omega}^1 - {aw_\Omega}^0)s'\partial_\Omega(U\otimes x)\\
&= ({aw_\Omega}^1 - {aw_\Omega}^0)\partial_\Omega s'(U\otimes x) + ({aw_\Omega}^1 - {aw_\Omega}^0)s'\partial_\Omega(U\otimes x)\\
&= ({aw_\Omega}^1 - {aw_\Omega}^0)(\partial_\Omega s' + s'\partial_\Omega)(U\otimes x)\\
&\overset{(2)}{=} ({aw_\Omega}^1 - {aw_\Omega}^0)(\mathcal{I}'\circ\mathcal{E}' - \mathrm{id})(U\otimes x)\\
&\overset{(3)}{=} (\mathcal{I}'\circ aw_1\circ\mathcal{E}' - \mathcal{I}'\circ aw_0\circ\mathcal{E}')(\mathcal{I}'\circ\mathcal{E}' - \mathrm{id})(U\otimes x)\\
&\overset{(4)}{=} \big[\mathcal{I}'(aw_1 - aw_0)\mathcal{E}' - ({aw_\Omega}^1 - {aw_\Omega}^0)\big](U\otimes x)\\
&\overset{(5)}{=} (-{aw_\Omega}^1 + {aw_\Omega}^0)(U\otimes x).
\end{aligned}$$

where (1) follows from lemma 14.1 and lemma 14.2, (2) follows from that s' is a chain homotopy between $\mathcal{I}'\circ\mathcal{E}'$ and id, (3) follows from the following diagram

$$\begin{tikzcd}
C_p(\Delta^\infty)\otimes C_n(S) \arrow[r, "aw_0", "aw_1"'] \arrow[d, shift left, "\mathcal{I}'"] & C_{p,n}(PS) \arrow[d, shift left, "\mathcal{I}'"] \\
\Omega_p(\Delta^\infty)\otimes C_n(S) \arrow[r, "{aw_\Omega}^0", "{aw_\Omega}^1"'] \arrow[u, shift left, "\mathcal{E}'"] & \Omega_{p,n}(PS) \arrow[u, shift left, "\mathcal{E}'"]
\end{tikzcd}$$

(4) follows from corollary 14.4 and (5) follows from that $aw_0 \overset{T}{\sim} aw_1$.

Thus ${aw_\Omega}^0 \overset{{s_\Omega}'}{\sim} {aw_\Omega}^1$. $\square$

Corollary 14.4 :

There is a lift $\tilde{aw}_\Omega^0$ follows from corollary 12.4 in the following diagram

$$\begin{tikzcd}
 & \Omega_{*,*}(\bar{P}S) \arrow[d, "f_\Omega"] \\
\Omega_*(\Delta^\infty)\otimes C_*(S) \arrow[r, "aw_\Omega"] \arrow[ur, dotted, "\tilde{aw}_\Omega^0"] & \Omega_{*,*}(PS).
\end{tikzcd}$$

Chapter 15

CHARACTERISTIC CLASSES AND CHERN-SIMONS THEORY

In this chapter, we want to mention Chern-Weil Theory for a differentiable principal G-bundle, G a Lie group. First let's start by giving the Chern-Weil construction as in Dupont [10] (chapter 4).

Let G be a lie group with lie algebra $\mathfrak{g}$ and $S^k(\mathfrak{g}^*)$ be the set of k-linear symmetric forms. Suppose $P : \mathfrak{g} \times ... \times \mathfrak{g} \to \mathbb{R}$ is a k-linear symmetric form. It is determined by the corresponding homogeneous polynomial of degree k:

$$\begin{array}{ccc} X & \to & P(X, ..., X) \\ \mathfrak{g} & \to \mathbb{R} & \end{array}$$

is the polynomial function. Then the adjoint representation induces an action of G on $S^k(\mathfrak{g}^*)$ for $\forall k$:

$$(gP)(X_1, ..., X_k) = P(Ad(g^{-1})X_1, ..., Ad(g^{-1})X_k),$$

$\forall X_1, ..., X_k \in \mathfrak{g}$, $g \in G$. Let $I^k(G)$ be the G-invariant part of $S^k(\mathfrak{g}^*)$. $I^*(G) = \bigoplus_k I^k(G)$ is a ring with unit $1 \in I^0(G) = \mathbb{R}$.

For a principal G-bundle $\xi = (E, \pi, M)$ with connection ω, where M is a differentiable manifold, ${F_\omega}^k \in \mathcal{A}^{2k}(E, \mathfrak{g}^{\bigotimes k})$ is the curvature of the connection ω, $P({F_\omega}^k) \in \mathcal{A}^{2k}(M)$. $P({F_\omega}^k)$ is horizontal (i.e., ${R^*}_g P({F_\omega}^k) = P({F_\omega}^k)$) since ${F_\omega}^k$ is horizontal and $P({F_\omega}^k)$ is an invariant horizontal (i.e., ${F_\omega}^k(X, Y) = 0$ for $\forall X$ vertical) $2k$-form since ${F_\omega}^k$ is equivariant and P is invariant.

Theorem 15.1 :

i) $P({F_\omega}^k)$ is basic, i.e., the pullback of a unique $2k$-form is also denoted by $P({F_\omega}^k) \in \mathcal{A}^{2k}(M)$.

ii) For a bundle map $(\bar{f}, f) : E' \to E$ and ω is a connection in $\xi = (E, \pi, M)$, $\omega' = \bar{f}^*\omega$ is a connection in $\xi' = (E', \pi', M')$ and $P({F_{\omega'}}^k) = f^* P({F_\omega}^k)$.

Corollary 15.2 :

Let $w_E(P) \in H^{2k}(\mathcal{A}^*(M))$ be the corresponding cohomology class for $P \in I^k(G)$. This defines a multiplicative homomorphism

$$w_E : I^*(G) \to H(\mathcal{A}^*(M))$$

called the Chern-Weil homomorphism.

Definition 15.3 : (Characteristic Class)

For $P \in I^*(G)$, $w_E(P)$ is called the characteristic class of E corresponding to P. That is $w(P) = [P({F_\omega}^k)]$.

$P({F_\omega}^k)$ is the integrand and notice that the de Rham isomorphism to singular cohomology

$$\mathfrak{I} : {H^*}_{\mathrm{DR}}(M) \to H^*(M, \mathbb{R})$$

is induced by the integration map

$$\mathfrak{I} : \mathcal{A}^*(M) \to C^*(M, \mathbb{R})$$

defined by $< \mathfrak{I}(\alpha), c >= \int_c \alpha$, where $c \in C_*(M)$ is a singular chain. That is

$$< \mathfrak{I}([P({F_\omega}^k)]), c >= \int_c P({F_\omega}^k).$$

Definition 15.4 :

Let's consider the category of topological principal G-bundles and bundle maps for a fixed lie group and let $H^*(-, \mathbb{R})$ be the real singular cohomology functor for a given coefficient ring $\mathbb{R}$. A characteristic class $c(-)$ with coefficients in $\mathbb{R}$ associates to every topological principal G-bundle $\xi = (E, \pi, X)$ a cohomology class $c(\xi) \in H^*(M, \mathbb{R})$ which is natural with respect to bundle maps, i.e.,

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

then $c(\xi') = f^* c(\xi)$.

Now if we want to give Chern-Weil theory for a differentiable principal G-bundle, G a lie group, we will observe that BG is the realization of a simplicial manifold and if $\xi = (E, \pi, M)$ is a differentiable principal G-bundle, $\bar{E}_./G$ is a simplicial manifold, where $\bar{E}_p = \underbrace{E \times ... \times E}_{p+1-\text{times}}$.

Let's establish the de Rham theorem for a simplicial manifold M. Let $A^n(M)$ denote the set of compatible simplicial n-forms. Since the exterior differential preserves compatible forms then we have a de Rham complex $A^*(M)$ for M. The usual singular cohomology $H^*(\| M \|, R) \cong H(C^*(M, R))$, where $C^*(M_., R) = \mathrm{Tot}(C^{*,*}(M_., R))$, $C^{p,q}(M_.) = C^q(M_p, R)$, where $C^*(M_p, R)$ is the cochain complex based on smooth singular simplices and the horizontal differential is $\delta' = \sum (-1)^i {\varepsilon_i}^*$. Now $A^*(M_.)$ is a double complex, i.e., $A(M_.) = \bigoplus_{k+l=n} A^{k,l}(M_.)$ with respect to the product $\Delta^p \times M_p$ and we have an integration map

$$\mathfrak{I} : A^{p,q}(M_.) \to C^{p,q}(M_.)$$

defined by $\mathfrak{I}(\alpha)_\sigma = \int_{\Delta^p \times \Delta^q} (\mathrm{id} \times \sigma)^* \alpha$, $\alpha \in \mathrm{Map}(\Delta^q, M_p)$. $\mathcal{I}$ induces an isomorphism of cohomology rings ${H_{\mathrm{DR}}}^*(M_.) = H(A^*(M_.)) \cong H^*(\| M_. \|, \mathbb{R})$.

Example : (The Pontrjagin Classes)

Let's take the group of non-singular $n \times n$ matrices, $G = Gl(n, \mathbb{R})$ with its lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of all matrices with lie bracket $[A, B] = AB - BA$. For $\forall A \in \mathfrak{gl}(n, \mathbb{R})$ and $g \in G$, we have $\mathrm{Ad}(g)(A) = gAg^{-1}$. Let $P_{k/2}$ be the homogeneous polynomial of degree $k \in \mathbb{Z}^+$ which is the coefficient of λ^{n-k} in the polynomial in λ

$$\det(\lambda.1 - \frac{1}{2\pi}A) = \sum_k P_{k/2}(A, ..., A)\lambda^{n-k},$$

where $A \in \mathfrak{gl}(n, \mathbb{R})$. $P_{k/2} \in I^k(Gl(n, \mathbb{R}))$ and it is called the $k/2$-th Pontrjagin polynomial and the Chern-Weil images $w_E(P_{k/2})$ are called the Pontrjagin classes.

Example : (The Chern Classes)

If we take $G = Gl(n, \mathbb{C})$ with its lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ and consider the complex valued invariant polynomials c_k which are the coefficients to λ^{n-k} in the polynomial λ

$$\det(\lambda.1 - \frac{1}{2\pi i}A) = \sum_k C_k(A, ..., A)\lambda^{n-k},$$

where A is an $n \times n$ matrix of complex numbers. The Chern-Weil image of these polynomials $w_E(C_k)$ gives the characteristic classes with complex coefficients as called the Chern classes. $C_{k|_{\mathfrak{gl}(n,\mathbb{R})}}$ satisfy $i^k C_k(A, ..., A) = P_{k/2}(A, ..., A)$, $A \in \mathfrak{gl}(n, \mathbb{R})$. It follows that the l-th Pontrjagin class of a $Gl(n, \mathbb{R})$-bundle is $(-1)^l$ times the $2l$-th Chern class of the complexification.

Now, let M be a manifold and $\pi : E \to M$ be a principal G-bundle. We are going to give a definition of Chern-Simons form on the total space E.

Corollary 15.5 :

For $P \in I^*(G)$, $w(P) = w(P)(\gamma) \in H^*(BG, \mathbb{R})$ defines a characteristic class such that for any G-bundle $\xi = (E, \pi, M)$ with connection ω, the class

$$w(P)(\xi) = \mathfrak{J}[P({F_\omega}^k)] \in H^{2k}(M, \mathbb{R}),$$

where $\mathfrak{J} : {H_{\mathrm{DR}}}^*(M) \to H^*(M, \mathbb{R})$, is the characteristic class for ξ corresponding to $w(P)$. It is independent of the connection and depends only on the topological G-bundle ξ.

If $\pi : E \to M$ is an ordinary differentiable G-bundle then the usual Chern-Weil homomorphism is $w : I^*(G) \to H^*(BG, \mathbb{R})$ such that $P \in I^k(G)$, $w(P)$ is represented in $\mathcal{A}^{2k}(NG)$ by $P({F_\omega}^k)$, where ${F_\omega}^k$ is the curvature form of ω defined by $\omega = {t_0}'\omega_0 + ... + {t_p}'\omega_p$, where ${t_0}', ..., {t_p}'$ are the barycentric coordinates in Δ^p. The structural equation states that $d\omega = F_\omega - \frac{1}{2}[\omega, \omega]$. For $P \in I^k(G)$, $w(P)(.)$ is the corresponding characteristic class.

Definition 15.6 (Chern-Simons Form) :

Let $\pi : E \to M$ be a principal G-bundle and ω be a $\mathfrak{g}$-valued 1-form on E. When $G = SU(2)$, the real valued 4-form $\tilde{p}$ on E given by

$$\tilde{p} = \frac{1}{8\pi^2}\mathrm{Tr}(F_\omega \wedge F_\omega),$$

where F_ω is the curvature of ω and $\mathrm{Tr} : \mathfrak{g} \bigotimes \mathfrak{g} \to \mathbb{R}$, is the lift of a unique 4-form p on M, that is, $\pi^*(p) = \tilde{p}$.

$$TP(\omega) = \frac{1}{8\pi^2}\mathrm{Tr}(\omega \wedge F_\omega - \frac{1}{3}\omega \wedge \omega \wedge \omega)$$

is called **the Chern-Simons form**.

In Chern-Simons [7] the authors considered the forms TP defined in E by

$$TP(\omega) = k \int_0^1 P(\omega \wedge \varphi_t^{k-1}) dt,$$

where $P \in I^k(G)$, $\varphi_t = tF_\omega + \frac{1}{2}(t^2 - t)[\omega, \omega]$. TP is a real-valued invariant (2k-1)-form on the total space E.

Corollary 15.7 :

Let $\pi : E \to M$ be a principal G-bundle as given in the previous definition. The form p represents the second Chern class of the bundle.

Proof :

It can be shown as follows:

Let's take $G = SU(2)$ and denote $P(F_\omega) = p$ which is called as "the topological charge density" of the gauge field or the second Chern class of ξ when dim M=4. Therefore (ξ, ω) is an SU(2)-valued gauge field on a manifold M of dimension ≥ 3. It means that $\xi = (E, M, \omega)$ is a principal SU(2)-bundle and ω is an $\mathfrak{su}(2)$-valued 1-form on E satisfying the connection conditions. Define the real-valued 4-form $\tilde{p}$ on E by

$$\tilde{p} = \frac{1}{8\pi^2} Tr(F_\omega \wedge F_\omega),$$

where F_ω is the curvature of ω, is the lift of a unique 4-form p on M; that is, $\pi^*(p) = \tilde{p}$ and $\mathrm{Tr} : \mathfrak{g} \bigotimes \mathfrak{g} \to \mathbb{R}$ given by $\mathrm{Tr}(B, A) = \mathrm{Tr}BA$. Tr is a real-valued map since $\mathrm{Tr}\overline{BA} = \mathrm{Tr}BA$. One can show this as follows:

$$\mathrm{Tr}\overline{BA} = \mathrm{Tr}\bar{B}\bar{A} = \mathrm{Tr}B^t A^t = \mathrm{Tr}(AB)^t = \mathrm{Tr}BA.$$

Now, let's show that the form p represents the second Chern class of $\xi(\omega)$:

$G = SU(2) \subseteq Gl(2, \mathbb{C})$. Let's take an element g from SU(2), that is, $g^{-1} = g^t = \bar{g}^t$, since $SU(2) = \{g \in Gl(2, \mathbb{C}) | gg^t = I\}$. Let $\mathfrak{su}(2)$ denote the lie algebra of SU(2) and it is defined by

$$\mathfrak{su}(2) = \{A \in \mathfrak{gl}(2, \mathbb{C}) | A + \bar{A}^t = 0 \text{ or } \mathrm{Tr}A = 0\}$$

where $\mathfrak{su}(2) \subseteq \mathfrak{M}(2, \mathbb{C}) = \mathfrak{gl}(2, \mathbb{C})$ and $\mathfrak{gl}(2, \mathbb{C})$ is the lie algebra of $Gl(2, \mathbb{C})$.

Therefore $\det(\lambda.1 - \frac{1}{2\pi i}A) = \sum C_k(A, ..., A)\lambda^{n-k}$ where A is defined by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

since in general; $G = Gl(n, \mathbb{C})$ has the lie algebra $\mathfrak{gl}(n, \mathbb{C}) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. The complex valued invariant polynomials C_k which are the coefficients to λ^{n-k} in the formula (15.1)

$$\det(\lambda.1 - \frac{1}{2\pi i}A) = \sum_k C_k(A, ..., A)\lambda^{n-k},$$

where A is an $n \times n$ matrix of complex numbers. The Chern-Weil image of these polynomials give characteristic classes with complex coefficients and they are called the Chern classes given in this chapter before. $c_k|_{\mathfrak{gl}(\mathfrak{n},\mathbb{R})}$ satisfies

$$i^k C_k(A, ..., A) = P_{k/2}(A, ..., A)\ ,\ A \in \mathfrak{gl}(n, \mathbb{R}).$$

After diagonalizing A, it becomes

$$A = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

We know that A$\in \mathfrak{su}(2)$ so $\mu_1 + \mu_2 = 0$ then

$$\begin{aligned}
\det(\lambda 1 - \frac{1}{2\pi i}A) &= \det(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \frac{1}{2\pi i}\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}) \\
&= \det(\begin{pmatrix} \lambda - \frac{\mu_1}{2\pi i} & 0 \\ 0 & \lambda - \frac{\mu_2}{2\pi i} \end{pmatrix}) \\
&= (\lambda - \frac{\mu_1}{2\pi i})(\lambda - \frac{\mu_2}{2\pi i}) \\
&= \lambda^2 - \lambda(\frac{\mu_1 + \mu_2}{2\pi i}) - \frac{\mu_1\mu_2}{4\pi^2}.
\end{aligned}$$

By using (15.1) we get

$$\sum_{k=0}^{2} C_k(\xi)\lambda^{n-k} = C_0(\xi)\lambda^2 + C_1(\xi)\lambda + C_2(\xi),$$

and

$$C_0(\xi) = 1\ , C_1(\xi) = 0\ , C_2(\xi) = \frac{\mu_1^2}{4\pi^2} \text{ since } -\mu_2 = \mu_1.$$

We want to show that

$$C_2(\xi) = \tilde{p}.$$

It can be shown as follows;

$$\begin{aligned}
C_2(\xi) &= \frac{1}{8\pi^2}\mathrm{Tr}(A^2) \\
&= \frac{1}{8\pi^2}\mathrm{Tr}(\begin{pmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{pmatrix} \\
&= \frac{1}{8\pi^2}(\mu_1^2 + \mu_2^2) \\
&= \frac{\mu_1^2}{4\pi^2}.
\end{aligned}$$

Thus $C_2(\xi) = \tilde{p} = \frac{1}{8\pi^2}\mathrm{Tr}(F_\omega \wedge F_\omega)$ as required. $\square$

Corollary 15.8 :

The term $\tilde{p}$ is always (locally) exact, no matter what the bundle ξ may be. In fact, a simple calculation shows that it is the coboundary of the 3-form $TP(\omega)$ defined on E by

$$TP(\omega) = \frac{1}{8\pi^2}\mathrm{Tr}(\omega \wedge F_\omega - \frac{1}{3}\omega \wedge \omega \wedge \omega).$$

Proof :

By Poincare's lemma, the closed form $\mathrm{Tr}(F_\omega \wedge F_\omega)$ is locally exact with $d : C^3 \to C^4$ given by

$$\delta TP(\omega) = \tilde{p} = \frac{1}{8\pi^2}\mathrm{Tr}(F_\omega \wedge F_\omega),$$

since

$$\begin{aligned} \delta TP(\omega) &= \frac{1}{8\pi^2} d\mathrm{Tr}(\omega \wedge F_\omega - \frac{1}{3}\omega \wedge \omega \wedge \omega) \\ &= \frac{1}{8\pi^2} d(\mathrm{Tr}(\omega d\omega + \frac{2}{3}\omega^3)), \end{aligned}$$

by using $F_\omega = d\omega + \omega \wedge \omega$, we get

$$\mathrm{Tr}(\omega \wedge d\omega + \omega \wedge \omega \wedge \omega - \frac{1}{3}\omega \wedge \omega \wedge \omega) = \mathrm{Tr}(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega).$$

We can show that $\delta TP(\omega) = \tilde{p}$, that is, we show that

$$\frac{1}{8\pi^2} d\mathrm{Tr}(\omega d\omega + \frac{2}{3}\omega^3) = \frac{1}{8\pi^2}\mathrm{Tr}(F_\omega \wedge F_\omega),$$

as follows;

$$\begin{aligned} \delta TP(\omega) &= \frac{1}{8\pi^2}\mathrm{Tr}[(d\omega)^2 + \frac{2}{3}(d\omega\omega^2 - \omega d\omega\omega + \omega^2 d\omega)] \\ &= \frac{1}{8\pi^2}\mathrm{Tr}[(F_\omega - \omega^2)^2 + \frac{2}{3}\{(F_\omega - \omega^2)\omega^2 - \omega(F_\omega - \omega^2)\omega + \omega^2(F_\omega - \omega^2)\}] \end{aligned}$$

where $F_\omega = d\omega + \omega \wedge \omega = d\omega + \omega^2$ since $[\omega, \omega] = 2(\omega \wedge \omega)$ and then

$$\delta TP(\omega) = \frac{1}{8\pi^2}\mathrm{Tr}[F_\omega^2 - \omega^2 F_\omega - F_\omega\omega^2 + \omega^4 + \frac{2}{3}(F_\omega\omega^2 - \omega F_\omega\omega + \omega^2 F_\omega - \omega^4)].$$

We know that $\mathrm{Tr}\omega^4 = 0$ and $\mathrm{Tr}\omega F_\omega\omega = -\mathrm{Tr}\omega^2 F_\omega = -\mathrm{Tr}F_\omega\omega^2$. Therefore we get

$$\begin{aligned} \delta TP(\omega) &= \frac{1}{8\pi^2}\mathrm{Tr}(F_\omega^2 - \omega^2 F_\omega - F_\omega\omega^2 + \frac{2}{3}(F_\omega\omega^2 + \frac{1}{2}(F_\omega\omega^2 + \omega^2 F_\omega) + \omega^2 F_\omega)] \\ &= \frac{1}{8\pi^2}\mathrm{Tr}(F_\omega^2) \\ &= \frac{1}{8\pi^2}\mathrm{Tr}(F_\omega \wedge F_\omega) \\ &= \tilde{p}. \end{aligned}$$

□

Now we need some preparation for the next chapter. In order to compute a related formula for the variation of Chern-Simons class for a given bundle $F \to |S_{.}|$ with a connection ω, we need to find the difference Chern-Simons form as a difference of the characters. In the following definition, we study on a certain graded ring $\hat{H}^*(M)$ for a smooth manifold M. This is the ring of differential characters on M. If $\Lambda \subset \mathbb{R}$ is a proper subgroup, a differential character (mod Λ) is a homomorphism f from the group of smooth singular k-cycles to $\mathbb{R}/\Lambda$, whose coboundary is the modΛ reduction of some (necessarily closed) differential form $w \in \mathcal{A}^{k+1}(M)$. One can see that f uniquely determines not only w, but a class $u \in H^{k+1}(M, \Lambda)$ whose real image is cohomologous to the de Rham class of w.

One can give the Chern-Simons form as a differential character by using a lift of Weil homomorphism due to Cheeger-Simons [6].

Definition 15.9 :

Let $\xi = (E, M, \omega)$ be a principal G-bundle, where G is a lie group. Let $\varepsilon(G)$ be the category of these objects which are triples ξ with morphisms being connection-preserving bundle maps, i.e., if $\xi = (E, M, \omega)$ and $\hat{\xi} = (\hat{E}, \hat{M}, \hat{\omega})$ are two bundles and $\varphi : \{E, M\} \to \{\hat{E}, \hat{M}\}$ is a bundle map then $\varphi : \xi \to \hat{\xi}$ is a morphism if $\varphi^*(\hat{\omega}) = \omega$. Let $\mathcal{A}_{cl}{}^*(M)$ denote the set of closed forms. The Weil homomorphism constructs a homomorphism $w : I^k(G) \to H^{2k}(BG, \mathbb{R})$ and a natural transformation $W : I^k(G) \to \mathcal{A}^{2k}(M)$ such that the following diagram of natural transformation commutes,

$$\begin{array}{ccccc} I^*(G) & \xrightarrow{w} & H^*(BG, \mathbb{R}) & \xleftarrow{r} & H^*(BG, \Lambda) \\ \downarrow W & & \downarrow C_{\mathbb{R}} & & \downarrow C_{\Lambda} \\ \mathcal{A}_{cl}{}^*(M) & \xrightarrow{DR} & H^*(M, \mathbb{R}) & \xleftarrow{r} & H^*(M, \Lambda) \end{array}$$

here C_{Λ}, $C_{\mathbb{R}}$ are provided by the theory of characteristic classes and DR is the de Rham homomorphism. If $P \in I^k(G)$, $u \in H^*(BG, \Lambda)$ and F_{ω} is the curvature form of $\xi \in \varepsilon$ then $W(P) = P(\underbrace{F_{\omega}, ..., F_{\omega}}_{k})$, and $C_{\Lambda}(u) = u(\xi)$, is the characteristic class. Set

$$K^{2k}(G, \Lambda) = \{(P, u) \in I^k(G) \times H^{2k}(BG, \Lambda) \mid w(P) = r(u)\}.$$

Set

$$R^k(M, \Lambda) = \{(w, u) \in \mathcal{A}_0{}^k \times H^k(M, \Lambda) \mid r(u) = [w]\}$$

and $R^*(M, \Lambda) = \bigoplus R^k(M, \Lambda)$. Here $r : H^k(M, \Lambda) \to H^k(M, \mathbb{R})$ and $[w]$ is the de Rham class of w and $\mathcal{A}_0{}^k$ is written for the closed k-forms with periods lying in Λ. $R(M)$ has an obvious ring structure $(u, w).(v, \phi) = (u \cup v, w \wedge \phi)$.

$K^*(G, \Lambda) = \bigoplus K^{2k}(G, \Lambda)$ forms a graded ring. The previous diagram induces $W \times C_{\Lambda} : K^*(G, \Lambda) \to R^*(M, \Lambda)$.

On the other hand we have a ring which is given by

$$\hat{H}^k(M, \mathbb{R}/\Lambda) = \{f \in \mathrm{Hom}(Z_k, \mathbb{R}/\Lambda) \mid f \circ \partial \in \mathcal{A}^{k+1}\}.$$

We set $\hat{H}^{-1}(M,\Lambda)=\Lambda$. $\hat{H}^*(M,\mathbb{R}/\Lambda)=\bigoplus \hat{H}^k(M,\mathbb{R}/\Lambda)$ is a graded Λ-module whose objects are called **differential characters**. So the result can be expressed as saying that there exists a unique natural transformation

$$S: K^*(G,\Lambda)\to \hat{H}^*(M,\Lambda)$$

such that the diagram

$$\begin{tikzcd} & \hat{H}^*(M,\mathbb{R}/\Lambda) \arrow[d, "\delta_1,\delta_2"] \\ K^*(G,\Lambda) \arrow[ur, "S"] \arrow[r, "W\times C_\Lambda"'] & R^*(M,\Lambda) \end{tikzcd}$$

commutes. $S_{P,u}\in \hat{H}^{2k-1}(M,\mathbb{R}/\Lambda)$ is called **the Chern-Simons character**.

Theorem 15.10 :

Let $(P,u)\in K^{2k}(G,\Lambda)$. For each $\xi\in\varepsilon(G)$ there exists a unique $S_{P,u}\in \hat{H}^{2k-1}(M,\mathbb{R}/\Lambda)$ satisfying;

1)$\delta_1(S_{P,u}(\omega))=P(F_\omega)$
2)$\delta_2(S_{P,u}(\omega))=u(\xi)$
3)If $\hat{\xi}\in\varepsilon(G)$ and $\phi:\xi\to\hat{\xi}$ is a morphism then $\phi^*(S_{P,u}(\hat{\omega}))=S_{P,u}(\omega)$.

For a pair $([P],u)$, we construct the Chern-Simons classes for $\xi=(E,M,\omega)$ with ${F_\omega}^{k+1}=0$.

Corollary 15.11 : (Dupont-Kamber [12])

Suppose $P(F_\omega)=0$. Then

1)$S_{P,u}(\omega)\in H^{2k-1}(M,\mathbb{R}/\Lambda)$ is the Chern-Simons class.
2)$B(S_{P,u}(\omega))=-u(\xi)$, where $B:H^{k-1}(M,\mathbb{R}/\Lambda)\to H^k(M,\Lambda)$ is the Bockstein-homomorphism. $u(\xi)=\psi^*(u)$ is the characteristic class associated to ξ.

Proof : Let us look at the following diagram of simplicial bundles

$$\begin{tikzcd} E \arrow[r, "\bar{\psi}"] \arrow[d] & \bar{E}_{.} \arrow[d] & \bar{G}_{.} \arrow[l, "\bar{\eta}"'] \arrow[d] \\ M \arrow[r, "\psi"] & \bar{E}_{.}/G & BG \arrow[l, "\eta"'] \end{tikzcd}$$

and consider $\bar{E}_{.}/G$ is approximately BG, where $\psi: M\to BG$ is the classifying map. Let $\bar{u}\in C^{2k}(\bar{E}_{.}/G,\mathbb{Z})$ be a cochain represents $(\eta^*)^{-1}u$. Then $J(P(F_{\bar{\omega}}))-r\bar{u}=\delta\bar{s}$ (*) for some cochain $\bar{s}\in C^{2k-1}(\bar{E}_{.}/G,\mathbb{R})$. The reduction mod $\mathbb{Z}$ of the cochain $\psi^*\bar{s}\in C^{2k-1}(M,\mathbb{R})$ is a cocycle. Here $\psi^*\bar{s}\in C^{2k-1}(M,\mathbb{R}/\mathbb{Z})$ is called **the Chern-Simons character**. In fact it defines a cohomology class

$$\begin{aligned} \delta\psi^*\bar{s}=\psi^*\delta\bar{s} &= \psi^*(J(P(F_{\bar{\omega}}))-r\bar{u}) \\ &= J(P(F_\omega))-r\psi^*\bar{u} \\ &= -r\psi^*\bar{u}\equiv 0 \bmod \mathbb{Z}, \end{aligned}$$

since $P(F_\omega) = 0$ when ${F_\omega}^{2k+1} = 0$. Therefore $\psi^* \bar{s}$ mod $\mathbb{Z}$ defines a cohomology class in $H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ which is called **the Chern-Simons class** and denoted by $S_{P,u}(\omega)$. □

Proposition 15.12 : (Dupont-Kamber [12])

$S_{P,u}(\omega) = [\psi^* \bar{s}] \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$.

1) $S_{P,u}(\omega)$ is well-defined.
2) $S_{P,u}(\omega)$ is natural with respect to bundle maps preserving connections.
3) $S_{P,u}(\omega_{MC}) = 0$.

Proof :

1) Suppose $J(F_{\bar{\omega}}) - r\bar{u}_1 = \delta(\bar{s}_1)$ is another choice then $\bar{u}_1 - \bar{u} = \delta t$ for $t \in C^{2k-1}(\bar{E}_\cdot/G, \mathbb{Z})$ and by using both choices, that is (*) and the one given above, we get $\delta(\bar{s} - \bar{s}_1 + rt) = 0$, i.e., $\bar{s} - \bar{s}_1 + rt = c$, so that $\bar{s} - \bar{s}_1 + rt$ represents an element $[c] \in H^{2k-1}(\bar{E}_\cdot/G, \mathbb{R}) \cong H^{2k-1}(BG, \mathbb{R})$. Therefore $\psi^* \bar{s} - \psi^* \bar{s}_1 \equiv \psi^*(\bar{s} - \bar{s}_1 + rt)$ mod $\mathbb{Z}$ represents $\psi^* c \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ since

$$\begin{aligned} \psi^*(\bar{s} - \bar{s}_1 + rt) &\overset{\psi^*\text{linear}}{=} \psi^* \bar{s} - \psi^* \bar{s}_1 + r\psi^*(t) \\ &\overset{t \in \mathbb{Z}}{=} \psi^* \bar{s} - \psi^* \bar{s}_1 \\ &\equiv \psi^* c. \end{aligned}$$

$[c] \in H^{2k-1}(BG, \mathbb{R}) = 0$ then $\psi^* \bar{s} - \psi^* \bar{s}_1 \cong \psi^* c \sim 0$.

2) follows from the previous diagram.

3) follows from 2) since ω_{MC} is induced from $G \to *$.

Proposition 15.13 :

We can give more geometric definition for the Chern-Simons class. Suppose $M^{2k-1} = \partial W^{2k}$ oriented and ξ extends to $\hat{\xi}$ over W. Let $\hat{\omega}$ be any extension of ω to a connection in $\hat{E}$. Setting $\hat{\xi} = \{\hat{E}, W, \hat{\omega}\}$ and we have the morphism $\xi \to \hat{\xi}$. Thus $S_{P,u}(\hat{\omega})|_{M^{2k-1}} = S_{P,u}(\omega)$. Moreover

$$< S_{P,u}(\omega), [M] >= \int_W P(F_{\hat{\omega}}) \text{ mod } \mathbb{Z}.$$

Proof :

If we replace E by $\hat{E}$ in the last diagram, we get

$$\begin{aligned} < S_{P,u}(\omega), [M] > &= < \psi^* \bar{s}, [M] > \\ &= < \psi^* \bar{s}, [\partial W] > \\ &= < \delta\psi^* \bar{s}, [W] > \\ &= < J(P(F_{\hat{\omega}})), [W] > \\ &= \int_W P(F_{\hat{\omega}}) \end{aligned}$$

Here $P(F_{\hat{\omega}})$ is the difference form, that is the Chern-Simons form. □

Corollary 15.14 :

For two given connections ω_1 and ω_2 in a bundle $E \to M$, we have $\tilde{\omega} = (1-t)\omega_1 + t\omega_2$ in $M \times [0,1] = W^{2k}$. The difference of the characters must be the reduction of a form. In other words, the difference of the evaluations of the Chern-Simon classes on a cycle is given as a difference form (15.2),

$$< S_{P,u}(\omega_2), [M] > - < S_{P,u}(\omega_1), [M] >= \int_W P(F_{\tilde{\omega}}) = \int_M TP(\omega_1, \omega_2),$$

where $TP(\omega_1, \omega_2) = \int_0^1 i_{d/dt} P(F_{\tilde{\omega}})dt.$ On the other hand $\dot{\tilde{\omega}} = \omega_2 - \omega_1$, for fix t $\frac{d\tilde{\omega}}{dt} = (1-t)d\omega_1 + td\omega_2$. So we can find the curvature $F_{\tilde{\omega}} = \frac{d\tilde{\omega}}{dt} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$. Then

$$S_{P,u}(\omega_2) - S_{P,u}(\omega_1) = k \int_0^1 P(\dot{\tilde{\omega}} \wedge {F_{\tilde{\omega}}}^{k-1})_{|Z_{2k-1}(M)} = TP(\omega_1, \omega_2).$$

P is a polynomial of the two connections ω_1, ω_2.

Example :

Suppose $E = M \times G$ and assume $M = M^{2k-1}$ closed oriented $P \in I^k(G)$. Put $W = M \times [0,1]$. For any connection ω in E and ω_{MC} we write

$$\tilde{\omega} = (1-t)\omega_{MC} + t\omega = \omega_{MC} + tA$$

connection in $\tilde{E} = W \times G$ where $A = \omega - \omega_{MC}$. By the previous proposition

$$< S_{P,u}(\omega), [M] > - < S_{P,u}(\omega_{MC}), [M] >= \int_{M \times [0,1]} P(F_{\tilde{\omega}})$$

since $S_{P,u}(\omega_{MC}) = 0$. Hence

$$< S_{P,u}(\omega), [M] >= \int_M TP(A),$$

where $TP(A) \stackrel{\text{def}}{=} \int_0^1 i_{d/dt} P(F_{\tilde{\omega}})dt$, $i_{d/dt}$ is the usual interior product in the t-variable. $TP(A)$ is an algebraic expression in A, for example for $k = 2$, $TP(A) = P(A \wedge dA + \frac{1}{3}A \wedge [A, A])$.

In the chapter 8, we started giving the background for finding a lifting in order to find the relation with Chern-Simons theory. Although one can define the Chern-Simons form on $m^*EG \to \| \, |\bar{P}_. S_.| \, \|$ by pulling back the canonical one on the universal bundle, we can not define the Chern-Simons form in the same way on the bundle defined over $\|S\|$. In the following chapter we will see this with the details.

Chapter 16

APPLICATIONS TO THE GAUGE FIELD

In this chapter, we give the variation of a Chern-Simons class for a given bundle $F \to |S|$ with a connection ω induced by the canonical connection ω_Δ via $\tilde{k}^*$. We will consider the evaluation of a difference of Chern-Simons classes for two connections on a cycle defined on $\Omega_{*,*}(\bar{P}S)$. In other words, we consider the evaluation of a Chern-Simons class for a given bundle with the connection ω on the difference of two cycles on $\Omega_{*,*}(\bar{P}S)$. So do to this, we use the difference Chern-Simons form given in the previous chapter. On the other hand the Chern-Simons class is related in an appropriate way to the second Chern class of the bundle. When we pull down the Chern-Simons form on the total space of a principal bundle with its connection to the base space, they give rise to the Chern-Simons class.

We will also give a definition of a connection in a simplicial bundle due to Dupont [9] and a corresponding Chern-Simons form. By starting with a canonical connection ω_Δ in the total space of a universal G-bundle $\pi_\Delta : EG \to BG$, one can write the corresponding canonical Chern-Simons form. After this by pulling back the canonical connection via $\tilde{m}$ defined by trivialization and the classifying map $m : \| \, |\bar{P}_. S_.| \, \| \to BG$, one can get the Chern-Simons form on the total space of the bundle $m^*EG \to \| \, |\bar{P}_. S_.| \, \|$. Since we can not define a section in our case, we can not use the definition for the Chern-Simons character given by global section but instead we can use the other definition given by differential forms and cohomology class. We have already given this definition at the end of the previous chapter and we are going to apply this to the gauge field in this chapter.

Let S be a simplicial set and $F \to |S|$ be a simplicial bundle. We have given the definition of a simplicial bundle as a sequence of bundles over $\Delta^p \times \sigma$ for all p, where $\sigma \in S_p$ with some commutative diagrams (6.1), (6.2) and the compatibility conditions (6.3), (6.4) and (6.5) given in chapter 6. Now, let's give the definition of a connection in a simplicial bundle.

Definition 16.1 :

A connection in $F \to |S|$ is a family of 1-forms $\theta = \{\theta_\sigma \mid \sigma \in S_p\}$ on F_σ with coefficients in $\mathfrak{g}$ such that $\theta_{|F_\sigma}$ is a connection in the usual sense in the bundle $F_\sigma \to \Delta^p \times \sigma$, $\sigma \in S_p$ such that

$$\begin{array}{ccc} F_{\varepsilon_i \sigma} & \xrightarrow{\bar{\varepsilon}^i} & F_\sigma \\ \downarrow & & \downarrow \\ \Delta^{p-1} \times \varepsilon_i \sigma & \xrightarrow{\varepsilon^i} & \Delta^p \times \sigma \end{array}$$

$\theta_{\varepsilon_i \sigma} = \bar{\varepsilon}^{i*} \theta_\sigma$, $i = 0, ..., p$.

Remark :

There is a similar definition of a simplicial bundle and connection over simplicial manifolds which includes the case of $EG \to BG$, (cf. chapter 17) where the p-th total space is $EG(p) = \Delta^p \times N\bar{G}(p)$. In the definition of a simplicial bundle given in chapter 6, in $F_\sigma \to \Delta^p \times \sigma$, F_σ can not canonically be written in general as a product of Δ^p with something as in the case of the universal bundle. Thus let G be a lie group and consider the simplicial space $N\bar{G}(p) = G \times ... \times G$ ($p+1$- times). In $N\bar{G}$

$$\varepsilon_i(g_0, ..., g_p) = (g_0, ..., \hat{g}_i, ..., g_p) \text{ and } \eta_i(g_0, ..., g_p) = (g_0, ..., g_i, g_i, ..., g_p),\ i = 0, ..., p.$$

By the definition

$$EG = \parallel N\bar{G}(p) \parallel = \bigsqcup \Delta^p \times G^{p+1}/\sim \text{ and } BG = EG/G = \parallel NG(p) \parallel = \bigsqcup \Delta^p \times G^p/\sim.$$

If we take the simplicial G-bundle $\gamma : N\bar{G} \to NG$, we get the universal bundle $\gamma_G : EG \to BG$. Let's give the construction of a canonical connection in this bundle.

Let ω_0 be the Maurer-Cartan connection in the bundle $G \to \text{pt}$ and $q_i : \Delta^p \times N\bar{G}(p) \to G$ be the projection onto the i-th factor in G^{p+1}, $i = 0, ..., p$, and let $\omega_i = {q_i}^* \omega_0$. Then ω is given over $\Delta^p \times N\bar{G}(p)$ by

$$\omega = {t_0}'\omega_0 + {t_1}'\omega_1 + ... + {t_p}'\omega_p,$$

where ${t_0}', ..., {t_p}'$ are the barycentric coordinates in Δ^p such that $\sum_{i=0}^{p} {t_i}' = 1$. Any convex combination of connections is again a connection. So $\omega_{|_{\Delta^p \times N\bar{G}(p)}}$ is a connection in the usual sense in $\Delta^p \times N\bar{G}(p) \to \Delta^p \times G$ and the universal connection $\omega_\Delta = \sum_{i=0}^{p} {t_i}' {x_i}^{-1} dx_i \in \mathcal{A}^1(\Delta^p \times G^{p+1}, \mathfrak{g})$, where ${x_i}^{-1}dx_i$ denotes the pullback of the Maurer-Cartan form by the projection onto the i-th coordinate of G^{p+1}.

We have a diagram

$$\begin{tikzcd}
(m \circ \tilde{u}_1)^* EG \arrow[r] \arrow[d] & \tilde{F} \arrow[r, "\tilde{m}"] \arrow[d] & EG \arrow[d] \\
\|S\| \arrow[r, "\tilde{u}_1"] & \| \, |\bar{P}_{.}S_{.}| \, \| \arrow[r, "m"] & BG
\end{tikzcd}$$

where $\tilde{F} = \{(t, s, f) \in \Delta^p \times \Delta^{q_0+1} \times ... \times \Delta^{q_p+1} \times F_{|_{\Delta^{n+2p+1} \times S_{n+2p+1}}} \mid \pi_y(f) = (l_p(t)(s), y)\}$ and $t \in \Delta^p$, $s \in \Delta^{q_0+1} \times ... \times \Delta^{q_p+1}$, $f \in F_y$, $y \in S_{n+2p+1}$.

One can write **the connection** $\tilde{\omega}_\Delta$ in $\tilde{F} \to \| \, |\bar{P}_{.}S_{.}| \, \|$ via the connection ω_Δ in the universal bundle as follows;

$$\begin{aligned}
\tilde{\omega}_\Delta = \tilde{m}^*(\omega_\Delta) &= \tilde{m}^*(\sum_{i=0}^{p} {t_i}' {x_i}^{-1}(t, s, \tilde{f}) dx_i(t, s, \tilde{f})) \\
&= \sum_{i=0}^{p} {t_i}' {x_i}^{-1}(t, s, g) dx_i(t, s, g)
\end{aligned}$$

by setting $\bar{\varphi}_y(\tilde{f}) = g$. Here $\tilde{f} := (l(t)(s), y) \in F_y$, $y \in S_{n+2p+1}$.

The right action on EG is given by

$$\begin{array}{rcl} EG \times G & \rightarrow & EG. \\ (x_0, ..., x_p, x) & \rightarrow & (x_0 x, ..., x_p x) \end{array}$$

On the other hand by using (7.2) we have $\gamma(x_0, ..., x_p) = (x_0 {x_1}^{-1}, ..., x_{p-1} {x_p}^{-1})$ corresponds to $(g_1, ..., g_p)$. By setting $x_p = g = \bar{\varphi}_y(\tilde{f})$ (fixing the last trivialization), we get

$$\begin{array}{rcl} x_{p-1} {x_p}^{-1} & = & g_p \Rightarrow x_{p-1} = g_p g \\ x_{p-2} {x_{p-1}}^{-1} & = & g_{p-1} \Rightarrow x_{p-2} = g_{p-1} g_p g \\ . & = & . \\ . & = & . \\ . & = & . \\ x_0 x_1^{-1} & = & g_1 \Rightarrow x_0 = g_1 ... g_p g. \end{array}$$

We can also write x_i's in EG in terms of g's and transition functions as

$$(g_1 ... g_p g, g_2 ... g_p g, ..., g_{p-1} g_p g, g) \in EG.$$

We have defined $\tilde{m}$ in the proposition 7.3 with (7.1) so

$$\begin{array}{rcl} x_p & = & g = \bar{\varphi}_y(\tilde{f}) \\ x_{p-1} & = & g_p g = v_{y, \hat{\mu}_{(p)} y}(\rho^{(p)} l(t)(s)^{-1} \bar{\varphi}_y(\tilde{f}) \\ . & = & . \\ . & = & . \\ . & = & . \\ x_0 & = & g_1 ... g_p g \overset{\text{cocycle condition}}{=} v_{y, \tilde{\mu}^{(p)} y}(\rho^{(1)} l(t)(s)^{-1} \bar{\varphi}_y(\tilde{f}). \end{array}$$

Thus the required connection is (16.1)

$$\begin{array}{rcl} \tilde{\omega}_\Delta & = & {t_0}'[(v_{y, \tilde{\mu}^{(p)} y}(\rho^{(1)} l(t)(s))^{-1} g)^{-1} d(v_{y, \tilde{\mu}^{(p)} y}(\rho^{(1)} l(t)(s))^{-1} g)] + \\ & & {t_1}'[(v_{y, \tilde{\mu}^{(p-1)} y}(\rho^{(2)} l(t)(s))^{-1} g)^{-1} d(v_{y, \tilde{\mu}^{(p-1)} y}(\rho^{(2)} l(t)(s))^{-1} g)] + \\ & & ... + \\ & & {t_{p-1}}'[(v_{y, \hat{\mu}_{(p)} y}(\rho^{(p)} l(t)(s))^{-1} g)^{-1} d(v_{y, \hat{\mu}_{(p)} y}(\rho^{(p)} l(t)(s))^{-1} g)] + \\ & & {t_p}' g^{-1} dg. \end{array}$$

Since $\rho^{(i)} l(t)(s) = \eta^{q_0 + ... + q_{i-1} + 2i - 1} \circ ... \circ \eta^{n+2p}$, $i = 1, ..., p$, then

$$\begin{aligned}
\tilde{\omega}_\Delta &= t_0'[\mathrm{Ad}_{g^{-1}}(v_{y,\tilde{\mu}^{(p)}y}(s^0,t_1)d_{y,\tilde{\mu}^{(p)}y}(s^0,t_1)^{-1}) + g^{-1}dg] + \\
&\quad ... + \\
&\quad t_{p-2}'[\mathrm{Ad}_{g^{-1}}(v_{y,\tilde{\mu}^{(2)}y}(s^0,...,s^{p-2},t_1,...,t_{p-1})d_{y,\tilde{\mu}^{(2)}y}(s^0,...,s^{p-2},t_1,...,t_{p-1})^{-1}) + g^{-1}dg] + \\
&\quad t_{p-1}'[\mathrm{Ad}_{g^{-1}}(v_{y,\hat{\mu}^{(p)}y}(s^0,...,s^{p-1},t_1,...,t_p)d_{y,\hat{\mu}^{(p)}y}(s^0,...,s^{p-1},t_1,...,t_p)^{-1}) + g^{-1}dg] + \\
&\quad t_p' g^{-1}dg.
\end{aligned}$$

Now, we will give the evaluation of a difference of Chern-Simons classes for two connections on a cycle defined on $\|\ |\bar{P}_{.}S_{.}|\ \|$. In other words, we consider the evaluation of Chern-Simons class for the given bundle with a connection on the difference of two cycles on $\|\ |\bar{P}_{.}S_{.}|\ \|$.

Let $F \to |S|$ be given with a connection ω. In the gauge theory, the aim is to start with a connection and deform it. For, example, we can start with the connection over $|S_{.}|$ and deform it on $\Delta^\infty \times |S_{.}|$ with respect to t variables. When t is close to the one of the end points then the connection in this bundle corresponds to the one comes from $\tilde{u}_1$, that is, $\tilde{\omega}_{\tilde{u}_1} = \tilde{m}^*(\omega_{\tilde{u}_1})$. We evaluate this at the vertices of the triangulation. We have

$$\begin{array}{c}
\Delta^\infty \times |P_{.}S_{.}| \\
\downarrow^{\approx}\, l \\
\Delta^\infty \times |S_{.}|.
\end{array}$$

We want to find the variation of the Chern-Simons class for the bundle $F \to |S|$ by using prismatic subdivision. Although we have a homeomorphism l, there is no a well-defined map $\|\ |P_{.}S_{.}|\ \| \to BG$. Therefore we had to construct $\|\ |\bar{P}_{.}S_{.}|\ \|$ and define a well-defined map $m : \|\ |\bar{P}_{.}S_{.}|\ \| \to BG$. So we can use $\|\ |\bar{P}_{.}S_{.}|\ \|$ instead for the required variation for the Chern-Simons class. There is a family of connections over $|\bar{P}_{.}S_{.}|$. For $\forall t \in \Delta^p$, the connections over $|P_{.}S_{.}|$ are changed. This follows that the Chern-Simons class for $F \to |S|$ is deformed by varying t's, since for $\forall t \in \Delta^p$, there are trivializations on $|S_{.}|$ in $\|\ |P_{.}S_{.}|\ \|$. The aim is to vary the Chern-Simons class for the bundle over $|S|$ and it can be done by varying the connections defined over $\|\ |\bar{P}_{.}S_{.}|\ \|$. In other words, the bundle over $\|\ |P_{.}S_{.}|\ \|$ enables us to find the variation of the Chern-Simons class for the bundle over $|S|$ by using the bundle over $\|\ |\bar{P}_{.}S_{.}|\ \|$.

The reason why we are studying on the simplicial currents level is that we can only increase p up til n on the chain level but by using the simplicial currents, one can take arbitrary large p. By making enough subdivisions, we approximate the given connection by piecewise flat connections.

Remark :

Now we have to find the two connections over $\| \, |\bar{P}_. S_.| \, \|$ by using the following diagram (16.2):

$$\begin{tikzcd}
\|S_.\| \arrow[r, "\tilde{u}_1"] \arrow[dr] \arrow[ddr] \arrow[dddr] & \| \, |\bar{P}_. S_.| \, \| \arrow[r, "m"] \arrow[d, "f"] & BG \\
& \| \, |P_. S_.| \, \| \arrow[d, "L"', "\approx"] & \\
& \| \, |S_.| \, \| \arrow[d, "pr"] & \\
& |S_.| \arrow[uuur] &
\end{tikzcd}$$

here the big diagram commutes and the right side of it is commutative up to homotopy, since S is a deformation retract of $|\bar{P}_. S_.|$. There is a unique way of defining the lift for u_1. On the other hand, there is no way to pull back $\tilde{u}_0$ to $\|S\|$ as connection. $\tilde{u}_0$ depends on the choice of lifting, i.e., there is no any canonical way to do that. As done in proposition 11.3, we can find $\tilde{u}_0$ for an arbitrary lift $\tilde{s}$ to s. $\tilde{u}_0$ and $\tilde{u}_1$ are chain maps and there exists a simplicial map which induces $\tilde{u}_1$. On the other hand if we think the same thing for $\tilde{u}_0$, this does not give the same conclusion. Since when $p = 0$, $|\bar{P}_0 S|$ is going to be contractible and its image under f composed with L will go to a point, then the map $\|S_.\| \to |S_.|$ is supposed to take an element to a point but the canonical map doesn't go to a point. In other words, these two maps are supposed to be homotopic. The canonical map is not homotopic to a point. Then there is no a simplicial map which induces $\tilde{u}_0$. Therefore there is no way to pull back a connection by $\tilde{u}_0$.

We are going to compare the connections on the bundle $m^*EG \to \| \, |\bar{P}_. S_.| \, \|$. The first connection comes from by pulling back the canonical connection on the universal bundle, let's denote it by $\tilde{\omega}_\Delta = \tilde{m}^*(\omega_\Delta)$ in (16.1). The second one comes from by pulling back the connection over $|S_.|$ via $L \circ f$. A connection over $\|S_.\|$ induces a connection over $|S_.|$ but this is not true in general. Let's explain this with the following remark;

Remark :

When we consider a bundle over $\|S_.\|$ and a connection ω_1 in this bundle, $\bar{\varepsilon}^{i*}\omega_{1_\sigma} = \omega_{1_{\varepsilon_i \sigma}}$, $i = 0, ..., p$, follows from definition 18.1. In our case one can see that

$$\bar{\eta}^{i*}\omega_{1_\sigma} = \omega_{1_{\eta_i \sigma}},$$

$i = 0, ..., p$, which follows from $k : \|S\| \to BG$ given as (7.3). So this connection induces a connection in the bundle over $|S_.|$, that is, $\tilde{k}^*(\omega_\Delta) = \omega$.

One can show that $k(\eta^i t, x) \sim k(t, \eta_i x)$, $i = 0, ..., p$, where $t \in \Delta^{p+1}$, $x \in S_p$, as follows. We do it only for $i = 0$ and $i = p$:

For $i = 0$;

$$\begin{aligned}
k(\eta^0 t, x) &= (\eta^0 t, [v_{\varepsilon_2 \ldots \varepsilon_p x}(\eta^0 t_1)]^{-1}, [v_{\varepsilon_3 \ldots \varepsilon_p x}(\eta^0 (t_1, t_2))]^{-1}, ..., [v_{\varepsilon_p x}(\eta^{p-1}\eta^p\eta^0 t)]^{-1}, [v_x(\eta^p\eta^0 t)]^{-1}) \\
&\sim (t, 1, [v_{\varepsilon_2 \ldots \varepsilon_p x}(0)]^{-1}, [v_{\varepsilon_3 \ldots \varepsilon_p x}(t_2)]^{-1}, ..., v_{\varepsilon_p x}(t_2, ..., t_{p-1})]^{-1}, v_x(t_2, ..., t_p)]^{-1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
k(t,\eta_0 x) &= (t,[v_{\varepsilon_2\ldots\varepsilon_{p+1}\eta_0 x}(0)]^{-1},[v_{\varepsilon_3\ldots\varepsilon_{p+1}\eta_0 x}(t_1)]^{-1},[v_{\varepsilon_4\ldots\varepsilon_{p+1}\eta_0 x}(t_1,t_2)]^{-1},\ldots,\\
&\quad [v_{\varepsilon_{p+1}\eta_0 x}(t_1,\ldots,t_{p-1})]^{-1},[v_{\eta_0 x}(t_1,\ldots,t_p)]^{-1})\\
&= (t,[v_{\eta_0\varepsilon_1\ldots\varepsilon_p x}(0)]^{-1},[v_{\eta_0\varepsilon_2\ldots\varepsilon_p x}(t_1)]^{-1},[v_{\eta_0\varepsilon_3\ldots\varepsilon_p x}(t_1,t_2)]^{-1},\ldots,\\
&\quad [v_{\eta_0\varepsilon_p x}(t_1,\ldots,t_{p-1})]^{-1},[v_x(t_2,\ldots,t_p)]^{-1})\\
&= (t,[v_{\varepsilon_1\ldots\varepsilon_p x}(0)]^{-1},[v_{\varepsilon_2\ldots\varepsilon_p x}(0)]^{-1},[v_{\varepsilon_3\ldots\varepsilon_p x}(t_2)]^{-1},\ldots,\\
&\quad [v_{\varepsilon_p x}(t_2,\ldots,t_{p-1})]^{-1},[v_x(t_2,\ldots,t_p)]^{-1})\\
&= k(\eta^0 t,x),
\end{aligned}$$

since $v_{\varepsilon_1\ldots\varepsilon_p x}(0)=1$.

For $i=p$, the same result follows from $v_{\eta_p x}(t)=1$. So the other cases for i can be shown similarly. The connection pulled back of the canonical connection over BG is the same as the one pulled back along the bundle over $\|S_.\| \to |S_.| \to BG$ which preserves the degeneracy operators. Therefore the connection over $\|S_.\|$ is induced by the one over $|S_.|$. In other words, admissible trivialization and the map k follow that $\tilde{k}^*(\omega_\Delta)=\omega$.

The following lemma will help us to see the required connections.

Lemma 16.2 :

We have the diagram (16.2), let's call $\tilde{U}_1 := \tilde{u}_1 \circ pr \circ L \circ f$ and $\mathrm{id}_{\|\,|\bar{P}_.S_.|\,\|}$. So there exists a chain homotopy

$$\bar{s}: C_*(\bar{P}S) \to C_{*+1}(\bar{P}S)$$

such that $\partial\bar{s}+\bar{s}\partial = \tilde{U}_1 - \mathrm{id}$.

Corollary 16.3 :

Let $F \to |S|$ be given with a connection $\omega \in \mathcal{A}^1(F,\mathcal{G})$. By extending the lemma above to the currents we can define two connections over $\|\,|\bar{P}_.S_.|\,\|$, namely $\tilde{\omega}_\Delta$ and $\overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta) = (\overline{\tilde{u}_1 \circ pr \circ L \circ f})^*(\omega)$, where

$$\begin{tikzcd}
\tilde{U}_1(m^*EG) \arrow[r, "\overline{\tilde{U}_1}"] \arrow[dr] & m^*EG \arrow[d] \\
 & \|\,|\bar{P}_.S_.|\,\|
\end{tikzcd}$$

Now we can continue with the simplicial currents and we have already found the extensions $aw_\Omega{}^0$ and $aw_\Omega{}^1$.

Note :

We point out that u_0 is only used to extend the chain complexes to the simplicial currents and to find the chain homotopy T. It does not have a direct role in the evaluation of Chern-Simons classes. We have the diagram

$$\begin{tikzcd} & \Omega_{*,*}(\bar{P}S) \arrow[d] \\ \Omega_*(\Delta^\infty) \otimes C_*(S) \arrow[ur] \arrow[r] & \Omega_{*,*}(PS). \end{tikzcd}$$

We used the lift of $a w_\Omega{}^1$ in order to get the lift of $a w_\Omega{}^0$. After getting this lift, we can give the evaluation of the Chern-Simons classes in terms of l-subdivisions.

Lemma 16.4 :

Let z be a cycle on $C_n(S)$ and $f \in \mathcal{A}^0(\Delta^l)$, $v_l = dt_1 \wedge ... \wedge dt_l \in \mathcal{A}^l(\Delta^l)$. Let us call a monomorphism

$$U : \mathcal{A}^l(\Delta^l) \to \Omega_0(\Delta^l)$$

defined by $U_{v_l}(f) = \int_{\Delta^l} f v_l$. Then $l! U_{v_l} \otimes z$ will be a cycle on $\Omega_0(\Delta^\infty) \otimes C_n(S)$.

Proof :

z is a cycle on $C_n(S)$ then $\partial z = 0$. We need to show that (16.3)

$$\partial(l! U_{v_l} \otimes z) = 0.$$

$$\begin{aligned} \partial(l! U_{v_l} \otimes z) &= l! U_{dv_l} \otimes z + (-1)^0 l! U_{v_l} \otimes \partial z \\ &= 0. \end{aligned}$$

□

Note : Let $\tilde{z}_l$ be a cycle on $\Omega_{*,*}(\bar{P}S)$. It covers the cycle $a w_\Omega{}^0(l! U_{v_l} \otimes z)$. So as an example, we can take $\tilde{z}_l = a\tilde{w}_\Omega^0(l! U_{v_l} \otimes z)$. Both $a\tilde{w}_\Omega^0(l! U_{v_l} \otimes z)$ and $a\tilde{w}_\Omega^1(l! U_{v_l} \otimes z)$ cover the cycle $z \in C_n(S)$.

Definition 16.5 :

The variation of a Chern-Simons class $S(\omega)$ for a given bundle $F \to |S|$ with a connection ω is given by

$$< S(\omega), z > - < S(\tilde{\omega}_\Delta), \tilde{z}_l >,$$

where $\tilde{\omega}_\Delta$ is the connection on $m^* EG \to \| \, |\bar{P}_{.} S_{.}| \, \|$ which is pull back of the canonical connection on the universal bundle, z is a cycle on $C_*(S)$ and $\tilde{z}_l$ is a cycle on $\Omega_{*,*}(\bar{P}S)$.

Theorem 16.6 :

Let $F \rightarrow |S|$ be given with a connection ω and let $z \in C_*(S)$ be a cycle, $\tilde{z}_l \in \Omega_{*,*}(\bar{P}S)$ be a cycle covers the cycle $z \in C_*(S)$. Then one can compute the variation of Chern-Simons class for the bundle $F \rightarrow |S|$ with ω in the following ways;

i) $< S(\omega), z > - < S(\tilde{\omega}_\Delta), \tilde{z}_l >=< S(\overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta)), \tilde{z}_l > - < S(\tilde{\omega}_\Delta), \tilde{z}_l >= \int_{\tilde{z}_l} TP(\overline{\tilde{U}_f}^*(\omega), \tilde{\omega}_\Delta)$,

ii) $< S(\omega), z > - < S(\tilde{\omega}_\Delta), \tilde{z}_l >=< S(\tilde{\omega}_\Delta), \tilde{U}_1(\tilde{z}_l) - \tilde{z}_l >$, where $\tilde{U}_f := pr \circ L \circ f$.

Proof :

i) The evaluation of the difference of the Chern-Simons classes for the bundle with two connections can be computed on a cycle which covers z as follows;

From the corollary 16.3, there are two connections in the bundle $\tilde{F} \rightarrow \| \, |\bar{P}_{.}S_{.}| \, \|$ which are $\tilde{\omega}_\Delta$ and $\overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta)$. Let

$$(\tilde{\omega}_\Delta, \overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta)) = (1-t)(\tilde{\omega}_\Delta) + t\overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta)$$

be the linear combination of these two connections. $\tilde{z}_l$ is a cycle on $\Omega_{*,*}(\bar{P}S)$ then by (15.2)

$$\begin{aligned} < S(\overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta)) - S(\tilde{\omega}_\Delta), \tilde{z}_l > &= < S(\overline{\tilde{U}_1}^*(\omega_\Delta)), \tilde{z}_l > - < S(\tilde{\omega}_\Delta), \tilde{z}_l > \\ &= \int_{\tilde{z}_l} TP(\overline{\tilde{U}_f}^*(\omega), \tilde{\omega}_\Delta), \end{aligned}$$

where $S(-)$ stands for the Chern-Simons class for the bundle and $TP(-)$ stands for the difference form.

This formula can be written in terms of given connection ω and the cycle z since

$$\begin{aligned} < S(\overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta)), \tilde{z}_l > &= < S((\overline{\tilde{u}_1 \circ pr \circ L \circ f})^*(\tilde{\omega}_\Delta)), \tilde{z}_l > \\ &= < S(\overline{\tilde{u}_1} \circ \tilde{U}_f{}^*(\tilde{\omega}_\Delta)), \tilde{z}_l > \\ &= < S(\overline{\tilde{u}_1}^*(\tilde{\omega}_\Delta)), \tilde{U}_f(\tilde{z}_l) > \\ &= < S(\omega), z >, \end{aligned}$$

where $(pr_* \circ L_* \circ f_*)\tilde{z}_l = z$, since $\tilde{z}_l$ covers z and $\overline{\tilde{u}_1}^*(\tilde{\omega}_\Delta) = \omega$ since the connection over $\|S_{.}\|$ induces a connection over $|S_{.}|$. We substitute this in the formula above so we get by using the definition 16.5

$$< S(\omega), z > - < S(\tilde{\omega}_\Delta), \tilde{z}_l >= \int_{\tilde{z}_l} TP(\overline{\tilde{U}_f}^*(\omega), \tilde{\omega}_\Delta).$$

ii) One can compute the evaluation of the Chern-Simons class for the bundle with the canonical connection on the difference of two different cycles which cover z. For this we can use the formula shown in i) and the variation of the Chern-Simons class for the bundle with the canonical connection on the difference of the two cycles is given by

$$\begin{aligned} < (\tilde{U}_1{}^* - \mathrm{id})S(\tilde{\omega}_\Delta), \tilde{z}_l > &= < S(\tilde{\omega}_\Delta), (\tilde{U}_1 - \mathrm{id})\tilde{z}_l > \\ &= < S(\tilde{\omega}_\Delta), \tilde{U}_1(\tilde{z}_l) - \tilde{z}_l > \\ &= < S(\tilde{\omega}_\Delta), \tilde{U}_1(\tilde{z}_l) > - < S(\tilde{\omega}_\Delta), \tilde{z}_l > \\ &= \int_{\tilde{z}_l} TP(\overline{\tilde{U}_f}^*(\omega), \omega_\Delta). \end{aligned}$$

$\square$

Example :

As an example, one can take $\tilde{z}_l = a\tilde{w}^0_\Omega(l!U_{v_l}(f) \otimes z)$ and give the variation as in i) and ii) in the theorem 16.6. By using the canonical lifting $a\tilde{w}^1_\Omega$ and choosing a lift $\tilde{s'}_\Omega$, we find a lift $a\tilde{w}^0_\Omega$. The images $a\tilde{w}^1_\Omega(l!U_{v_l} \otimes z)$, $a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z)$ will give two cycles on $\Omega_{*,*}(\bar{P}S)$. Finally we can take $\tilde{U}_1(\tilde{z}_l) = a\tilde{w}^1_\Omega(l!U_{v_l} \otimes z)$ and give the variation as in i) and ii) in terms of l-subdivision. The first formula becomes

$$\begin{aligned} < S(\omega), z > - < S(\tilde{\omega}_\Delta), a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z) > &= < (\tilde{U_1}^* - \mathrm{id})S(\tilde{\omega}_\Delta), a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z) > \\ &= < S(\overline{\tilde{U}_1}^*(\tilde{\omega}_\Delta)) - S(\tilde{\omega}_\Delta), a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z) > \\ &= \int_{\tilde{z}_l} TP(\overline{\tilde{U}_f}^*(\omega), \tilde{\omega}_\Delta). \end{aligned}$$

The second formula becomes (16.4)

$$\begin{aligned} < S(\omega), z > - < S(\tilde{\omega}_\Delta), a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z) > &= < S(\tilde{\omega}_\Delta), a\tilde{w}^1_\Omega(l!U_{v_l} \otimes z) - a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z) > \\ &= < S(\tilde{\omega}_\Delta), (\tilde{U}_1 - \mathrm{id})\tilde{z}_l > \\ &= \int_{\tilde{z}_l} TP(\overline{\tilde{U}_f}^*(\omega), \tilde{\omega}_\Delta). \end{aligned}$$

Corollary 16.7 :

The variation is

$$< S(\omega), z > - < S(\tilde{\omega}_\Delta), a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z) > = < P(F_{\tilde{\omega}_\Delta}), \tilde{s'}_\Omega(l!U_{v_l} \otimes z) >,$$

when $\tilde{z}_l = a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z)$.

Proof :

One can find the evaluation of the Chern form $\delta S(\tilde{\omega}_\Delta)$ for the bundle with the canonical connection on the cycle defined on $\Omega_{*,*}(\bar{P}S)$.

Moreover we can replace the difference as $a\tilde{w}^1_\Omega - a\tilde{w}^0_\Omega = \partial\tilde{s'}_\Omega + \tilde{s'}_\Omega\partial$ since $a\tilde{w}^1_\Omega \sim a\tilde{w}^0_\Omega$. Then

$$\begin{aligned} < S(\omega), z > - < S(\tilde{\omega}_\Delta), a\tilde{w}^0_\Omega(l!U_{v_l} \otimes z) > &\overset{ii)16.6}{=} < S(\tilde{\omega}_\Delta), \tilde{U}_1(\tilde{z}_l) - \tilde{z}_l > \\ &\overset{(16.4)}{=} < S(\tilde{\omega}_\Delta), (a\tilde{w}^1_\Omega - a\tilde{w}^0_\Omega)(l!U_{v_l} \otimes z) > \\ &= < S(\tilde{\omega}_\Delta), (\partial\tilde{s'}_\Omega + \tilde{s'}_\Omega\partial)(l!U_{v_l} \otimes z) > \\ &\overset{(18.3)}{=} < S(\tilde{\omega}_\Delta), \partial\tilde{s'}_\Omega(l!U_{v_l} \otimes z) > \\ &= < \delta S(\tilde{\omega}_\Delta), \tilde{s'}_\Omega(l!U_{v_l} \otimes z) > \\ &= < P(F_{\tilde{\omega}_\Delta}), \tilde{s'}_\Omega(l!U_{v_l} \otimes z) > \end{aligned}$$

is the evaluation of the Chern form $\delta S(\tilde{\omega}_\Delta)$ for the bundle with the canonical connection on the cycle defined on $\Omega_{*,*}(\bar{P}S)$. $\square$

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