UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

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Preprint Series No.: 10

November 2007

2007/11/08

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Hypersurfaces in P^n with 1-parameter symmetry groups II

A. A. du Plessis and C. T. C. Wall

Introduction

We are interested in hypersurfaces $V \subset P^n(\mathbb{C})$ defined by homogeneous equations $f(x_0, \ldots, x_n) = 0$ of degree d. We say that V is *quasi-smooth* if V has isolated singularities and is not a cone. If V admits a subgroup G of PGL_{n+1} of symmetries with $r = \dim G \geq 1$, we call V $r-$symmetric.

In [4] we gave a detailed discussion of 1-symmetric quasi-smooth hypersurfaces in the case when G is semi-simple. The main object of this paper is to give a corresponding analysis when G is unipotent.

Our first main result Theorem 2.4 lists the possible cases. Let G be a unipotent group of type given by the sequence $R = \{r_1 \geq r_2 \geq \ldots\}$ (i.e. the Jordan blocks have sizes $r_i + 1$; we omit zeros in writing R). Then we have one of the following:

Case 2: $d \geq 3,\ R = \{2\}$, Case 3: $d = 4,\ R = \{3\}$,
Case 4: $d = 3,\ R = \{4\}$, Case 21: $d = 3,\ R = \{2, 1\}$.

We find that Case 21 splits into two: one a subcase of Case 2; the other we rename Case 5.

Our second main conclusion is the calculation of the total Milnor number $\mu(V)$ (the sum of the Milnor numbers $\mu_{P_i}(V)$ at all singular points P_i of V). The result is, where the V_i are auxiliary varieties defined ad hoc in each case:

Case	$\mu(V)$
2	$\frac{1}{2}(d-2)(2d-1)(d-1)^m + \mu(V_2) + \mu(V_3)$
3	22.3^m
4	$11.2^m + \mu(V_3)$
5	$25.2^m + \mu(V_3)$

There is a 'main' singular point P. Provided in Case 2 that V_2 is non-singular and in Case 5 that V_3 is, the Milnor number $\mu_P(V)$ is the first term of the sum and the singularity of V at P is semi-quasi-homogeneous.

The first two sections are devoted to preparation and the proof of Theorem 2.4. We then pause for a brief review of some important background results, holding for all quasi-smooth 1-symmetric hypersurfaces; in particular, we recall that $\tau(V) \leq (d-1)^{n-2}(d^2 - 3d + 3)$, and attains this value if and only if f is annihilated by vector fields ξ of degree 1 and η of degree $d-2$, not a multiple of ξ. We will call V *oversymmetric* in this case. Moreover, f is 2-symmetric if and only if it is oversymmetric with $d=3$, and is never 3-symmetric. We briefly recall the enumeration of oversymmetric hypersurfaces in the semi-simple case. We also give a number of auxiliary methods of calculation of Milnor numbers, so as not to interrupt the main discussion.

After a brief recall of the invariant theory of the nilpotent actions we discuss Cases 2-5 in successive sections; in each case we discuss the geometry of the action, show how to reduce f to a convenient normal form, analyse the conditions on f for V to be quasi-smooth, find the singular points, and study the total Milnor number $\mu(V)$ and the nature of the singularities presented. We proceed to discussion of the Tjurina number $\tau(V)$, and show that V is always oversymmetric in Cases 3 and 21, never in Case 4, while in Case 2 by Theorem 5.7 it occurs if and only if either (a) V_3 is a cone, or (b) after change of co-ordinates if necessary, $\partial\phi/\partial B$ and $\partial\phi/\partial X$ both vanish along $X=B=0$.

In a final section we recapitulate the complete list of the five 2-symmetric cases in more detail.

1 Unipotent actions on vector spaces and algebras

If N is a nilpotent endomorphism of a finite dimensional vector space K, we can choose co-ordinates to put N into Jordan canonical form, and count the sizes of the blocks. If the block sizes are $\lambda_1,\ldots,\lambda_t$, arranged in non-increasing order, then $n=\sum_i \lambda_i$. If we write $\nu_k := \operatorname{rank} N^{k-1} - \operatorname{rank} N^k$, then ν is the partition conjugate to λ, so both partitions are independent of the choice of co-ordinates. Our usual notation will be to set $r_i := \lambda_i - 1$ and let R be the sequence of r_i, with zeroes omitted.

We recall the representation theory of the Lie algebra sl_2. Denote the canonical basis vectors of sl_2 by

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$. Every (finite dimensional) sl_2-module M is a direct sum of irreducible modules, and any irreducible module of rank $s+1$ ($s=0,1,2,\ldots$) is isomorphic to the module K_s with basis $x_{-s}, x_{2-s}, \ldots, x_{s-2}, x_s$ and action given by

$$e.x_r = \tfrac{s-r}{2} x_{r+2}, \quad f.x_r = \tfrac{s+r}{2} x_{r-2}, \quad h.x_r = r x_r.$$

Thus the eigenvalues of h on M are all integers, and we can define a grading on M by assigning weight r to the eigenspace belonging to the eigenvalue r. Then for any $r \geq 0$, e^r gives an isomorphism of M_{-r} on M_r and f^r gives an isomorphism of M_r on M_{-r}.

Lemma 1.1. *For any sl_2-module M,*

(i) if $r \geq 0$, f^r gives an isomorphism from the weight space M_r to M_{-r},

(ii) if $r > 0$, then $\mathrm{Ker}(f|M_r) = 0$,

(iii) if $r < 0$, then $\dim \mathrm{Ker}(f|M_r) = \dim M_r - \dim M_{r-2}$,

(iv) if $x \in M$ and $f.x = h.x = 0$, then $e.x = 0$.

(i)–(iii) follow by inspection from the remarks above. It suffices to prove (iv) for each K_s. But if $s > 0$ then $f.x = h.x = 0$ for $x \in K_s$ implies $x = 0$, while if $s = 0$ then $e.x = h.x = f.x = 0$ for any $x \in K_0$.

Lemma 1.2. *The action of a nilpotent endomorphism N on a (finite dimensional) vector space K can be extended to an sl_2-action, with N acting as f.*

Proof. Express K as the direct sum of monogenic modules: say K'_s has basis $x, N.x, N^2.x, \ldots, N^s.x$ with $N^{s+1}.x = 0$. If we set, for $0 \leq i \leq s$, $N^i x =: \frac{s!}{(s-i)!} x_{s-2i}$, then we have $N x_{s-2i} = (s-i) x_{s-2i-2}$ or, writing $r = s - 2i$, $N x_r = \frac{s+r}{2} x_{r-2}$. We can now set, for each r, $e.x_r = \frac{s-r}{2} x_{r+2}$, $h.x_r = r x_r$. □

The action, and the grading it defines, are not determined solely by the nilpotent action. However, if we define the weight filtration by letting $F_v K$ be the sum of the eigenspaces of h belonging to eigenvalues $\leq v$, we have

$$F_v K = \sum_{p \in \mathbb{Z}} \mathrm{Ker}\, N^p \cap \mathrm{Im}\, N^{p-v-1}.$$

It suffices to check this on the modules K'_s. Then Ker N^p has basis $\{N^i x \,|\, i \geq s + 1 - p\}$ and Im N^{p-v-1} has basis $\{N^i x \,|\, i \geq p - v - 1\}$. Thus $N^i x \in F_v K$ if and only if, for some p, $i \geq \max(s + 1 - p, p - v - 1)$, i.e. $i + v + 1 \geq p \geq s + 1 - i$, thus if and only if $s - 2i \leq v$.

A linear operator $L(x_i) = \sum a_{i,j} x_j$ on a vector space K defines a linear differential operator $D_L := \sum a_{i,j} x_j \partial/\partial x_i$, which acts on the symmetric algebra $S(K)$ of K, and induces the action of L on K. We also regard D_L as a vector field on K, and then denote it by ξ_L.

Over a field of characteristic zero, we can also form the 1-parameter group $\{\mathrm{Exp}(tL)\}$ of automorphisms of K, which inherits an action on $S(K)$. If we have a Lie algebra $\mathfrak{g}$ of linear automorphisms of K, the exponentials generate a group G of automorphisms of K, and hence of $S(K)$, and the induced action of an element L of the Lie algebra is that of D_L.

We have seen how to extend a nilpotent operator N on K to an action of sl_2: this now extends to an action of sl_2 on the symmetric algebra $S(K)$ of K, which in turn we can restrict to the homogeneous part $M := S_d(K)$ of degree d.

If further K splits as a direct sum $K' \oplus K''$ with each of K', K'' invariant under N, extending as above to an sl_2-action leaves each summand sl_2-invariant. The induced actions of SL_2 and sl_2 on $S(K)$ now preserve each of the subspaces $M = S_{d'}(K') \otimes S_{d''}(K'')$. Corresponding remarks apply to a direct sum of three or more summands.

Applying Lemma 1.1 to M, we find

Theorem 1.3. *Let N be a nilpotent endomorphism of K. Then*

(i) for any $r \geq 0$, D_N^r gives an isomorphism from the weight space M_r to M_{-r},

(ii) if $w > 0$, then $\mathrm{Ker}(D_N|M_w) = 0$.

(iii) if $w < 0$, then $\dim \mathrm{Ker}(D_N|M_w) = \dim M_w - \dim M_{w-2}$.

(iv) $\mathrm{Ker}(D_N|\, M_0)$ *is the space of invariants of SL_2 acting on M.*

2 Restrictions on unipotent actions

Let K be a finite dimensional vector space over $\mathbb{C}$ with a nilpotent endomorphism N, of type given by the sequence $R = \{r_1 \geq r_2 \geq \cdots\}$. We consider homogeneous functions f on K, of degree d, annihilated by D_N, or equivalently, invariant under the 1-parameter group $G_N = \{\mathrm{Exp}(tN)\}$. We seek the conditions under which the hypersurface V in $P(K)$ defined by f is quasi-smooth. In this section we will enumerate the possibilities for $(d; R)$.

Let $\{x_i\}$ $(1 \leq i \leq n)$ be variables with assigned weights $w(x_i) = w_i$, arranged in non-decreasing order. Define a filtration of the polynomial ring $\mathbb{C}[x]$ by letting $f \in F(v)$ if f is a linear combination of monomials of weights $\leq v$.

Lemma 2.1. *Let $f_j, j = 1, \ldots, m$ be polynomials of degree D in the x_i with $f_j \in F(W_j)$; we suppose $W_1 \leq \cdots \leq W_m$. Suppose that the set Z of common zeroes of the f_j in affine n-space has dimension $\leq k$. Then $Dw_i \leq W_{i+m+k-n}$ for $i = 1, \ldots, n-k$.*

Proof. If all the f_j vanish on $\{x_1 = \cdots = x_{n-k-1} = 0\}$ then $\dim Z \geq k+1$; so one of the f_j, say f_{j_1}, contains a monomial in $x_{n-k}, \ldots, x_n$ alone, and so $Dw_{n-k} \leq W_{j_1}$. If all but f_{j_1} vanish on $\{x_1 = \cdots = x_{n-k-2}\}$ then $\dim Z \geq k+1$; so another of the f_j, say f_{j_2}, contains a monomial in $x_{n-k-1}, \ldots, x_n$, and so $Dw_{n-k-1} \leq W_{j_2}$. Continuing in this way we find distinct $j_1, j_2, \ldots, j_{n-k} \in \{1, \ldots, m\}$ s.t. $Dw_{n-k+1-i} \leq W_{j_i}$.

Since the numbers j_s for $1 \leq s \leq n-k-i+1$ are all distinct, at least one of them, say j_ℓ, is $\leq m-n+k+i$, by the pigeonhole principle. Hence $Dw_i \leq Dw_{n-k+1-\ell} \leq W_{j_\ell} \leq W_{m-n+k+i}$. □

Corollary 2.2. *Let f be homogeneous of degree d in the variables x_i; suppose $f \in F(W)$ and that the singular set of the variety V has dimension $\leq k-1$. Then $(d-1)w_i \leq W - w_{n+1-i-k}$ for $1 \leq i \leq n-k$. In particular, if $W = 0$ and the set of weights w_i is symmetric about 0, we have $(d-1)w_i \leq w_{i+k}$.*

For set $f_i := \partial f/\partial x_i$. Then f_i has degree $d-1$ and $f_i \in F(W - w_i)$. Rearranging these numbers in increasing order gives $W_j = W - w_{n+1-j}$. The singular set of V has dimension $\leq k-1$ if and only if the locus of common zeros of the f_i in affine space has dimension at most k. Applying the lemma shows that in this case, $(d-1)w_i \leq W_{i+k}$ for $i = 1, \ldots, n-k$, i.e. $(d-1)w_i \leq W - w_{n+1-i-k}$.

Lemma 2.3. *Let f be homogeneous of degree d in the variables p_i; suppose that each monomial occurring in f has weight ≤ 0; suppose also that the hypersurface $f = 0$ is quasi-smooth. Then*

(i) f contains a monomial of degree $d-1$ in the two variables of highest weight,

(ii) f contains two monomials, each of degree $d-1$ in the three variables of highest weight, with the other factors different.

Proof. (i) Write $\mathfrak{m}_2$ for the ideal generated by all variables other than the two of highest weight. If $f \in \mathfrak{m}_2^2$, the hypersurface $f = 0$ is singular along the line corresponding to these two co-ordinates. Otherwise, f must contain a monomial of degree $d-1$ in them and containing just one other co-ordinate.

(ii) Write $\mathfrak{m}_3$ for the ideal generated by all variables other than the three of highest weight. Each term in f not belonging to $\mathfrak{m}_3^2$ has degree $d-1$ in these and contains just one other co-ordinate. If this other co-ordinate is the same in all cases, say X, we can write $f = X\phi_{d-1}(p_1, p_2, p_3) + R$, with $R \in \mathfrak{m}_3^2$. But then the hypersurface $f = 0$ is singular along the curve $\phi_{d-1}(p_1, p_2, p_3) = 0$ in the plane defined by $\mathfrak{m}_3$. $\square$

We now apply these results to the problem of hypersurfaces invariant by a unipotent group.

Theorem 2.4. *Let $V : f = 0$ be a quasi-smooth hypersurface of degree $d > 2$ in projective space, which is invariant under the action of a unipotent group of type given by the sequence $R = \{r_1 \geq r_2 \geq \ldots\}$. Then we have one of the the following:*

Case 2:	$d \geq 3,\ R = \{2\},$	*Case 3:*	$d = 4,\ R = \{3\},$
Case 4:	$d = 3,\ R = \{4\},$	*Case 21:*	$d = 3,\ R = \{2, 1\}.$

Proof. For each i we have basis elements of weights $-r_i, 2 - r_i, \ldots, r_i$, and by Theorem 1.3, f is a linear combination of monomials of weight ≤ 0. Thus the hypotheses of the special case of Corollary 2.2 are satisfied.

If there is just one generator of positive weight, r_1 is 1 or 2, and other $r_i = 0$. If $r_1 = 1$, the ring of invariants is polynomial in the generators of weight ≤ 0, so is independent of x_n, and defines a cone. If $r_1 = 2$, we have Case 2 of the theorem.

By Corollary 2.2, we have $(d-1)w_{n-1} \leq w_n$. Now if $r_2 = r_1$, we have $w_{n-1} = w_n = r_1$, so $r_1 \geq r_1(d-1)$, a contradiction. If $r_2 = r_1 - 1$, we have $w_n = r_1, w_{n-1} = r_1 - 1$, so $r_1 \geq (r_1 - 1)(d-1)$ and $r_1 \leq \frac{d-1}{d-2}$. If $d > 3$ this implies $r_1 = 1$, a possibility we excluded above; if $d = 3$ we may also have $r_1 = 2$. If now $r_3 = 1$, we have $w_{n-2} = 1$, contradicting $(d-1)w_{n-2} \leq w_{n-1}$. Thus $r_3 = 0$, and we have Case 21 of the Theorem.

Otherwise we necessarily have $w_n = r_1$ and $w_{n-1} = r_1 - 2$, whence $r_1 \geq (r_1 - 2)(d-1)$ and $r_1 \leq \frac{2(d-1)}{d-2}$. This gives $r_1 \leq 4$ if $d = 3$, $r_1 \leq 3$ if $d = 4$ and $r_1 \leq 2$ if $d > 4$.

The cases $r_1 = 2$ were considered above. If $d = 4$, the remaining possibility is $r_1 = 3$, so that $w_n = 3, w_{n-1} = 1$. Since $3w_{n-2} \leq w_{n-1}$, we have $w_{n-2} = 0$, and Case 3 of the Theorem. It remains to consider the cases $d = 3$ and r_1 equal to 3 or 4.

If $r_1 = 3$, then $w_1 = 3$, $w_2 = 1$, so again $w_3 = 0$ and $r_2 = 0$. There is just one non-trivial Jordan block, which has size 4 and weights $-3, -1, 1, 3$: denote the corresponding variables by x_0, x_1, x_2, x_3, and write M for the space of homogeneous cubics in them. By Lemma 2.3, f must contain a term $x_0 x_2^2$, which has weight -1. Now apply Theorem 1.3 to M. Since $\dim M_{-3} = \dim M_{-1} = 3$, $\mathrm{Ker}(D|M_{-1}) = 0$. We thus have a contradiction. In fact the ring of invariants $Ker(D)$ is given explicitly in Lemma 4.1, and the homogeneous invariants of degree 3 are linear in both the variables of positive weight.

If $d = 3$ and $r_1 = 4$, then $w_n = 4$, $w_{n-1} = 2$, so $2w_{n-2} \leq w_{n-1} = 2$, hence $r_2 \leq 1$. If $r_2 = 0$, we have Case 4 of the theorem.

If $(d, r_1, r_2) = (3, 4, 1)$, we have co-ordinates of positive weights $4, 2, 1$, so by Corollary 2.2 no others, so $r_3 = 0$. Write $K = K_1 \oplus K_2 \oplus K_3$, where K_1 is the Jordan block of N of size 5, with co-ordinates of respective weights $-4, -2, 0, 2, 4$, which we denote x_0, x_1, x_2, x_3, x_4, K_2 the Jordan block of size 2, with co-ordinates y_0, y_1 of weights $-1, 1$; all the rest have weight 0.

By Lemma 2.3, f must contain the monomial $x_0x_3^2$ and the monomial $x_1y_1^2$. Write N_w for the vector space spanned by monomials of weight w of degree 1 on K_1 and degree 2 on K_2. Since each monomial of degree 2 in K_2 has weight 2, 0 or -2, there is a unique co-ordinate on K_1 with which we can multiply to attain weight 0; likewise to attain weight -2. Thus $\dim \operatorname{Ker}(D|N_0) = \dim N_0 - \dim N_{-2} = 0$, so no appropriate invariant function exists. □

It also follows from Lemma 2.3 that

Corollary 2.5. *If $r_1 = 3$ (Case 3), f must contain the monomial $x_2^3x_0$; if $r_1 = 4$ (Case 4), f must contain the monomial $x_3^2x_0$; and if $r_1 = 2, r_2 = 1$ (Case 21), f must contain the monomial $y_1^2x_0$.*

3 Toolkit

Before we start detailed investigation of the cases listed above, we first recall some general results, and then collect some methods of calculation of Milnor numbers, so as not to break the thread of exposition in the following sections .

Let V be quasi-smooth, with equation $f = 0$ of degree $d > 2$ in P^n. Recall that we call f, and the hypersurface V, *oversymmetric* if f is annihilated by vector fields ξ of degree 1 and η of degree $d - 2$ which is not a multiple of ξ. When $d = 3$ this is equivalent to requiring V to be 2-symmetric. We recall the important result

Theorem 3.1. *Suppose V quasi-smooth of degree d and ξ a vector field of degree $r \leq d - 2$ with $\xi(f) = 0$. Then $\tau(V) \leq (d-1)^n - r(d-1-r)(d-1)^{n-2}$, and equality holds if and only if there is a vector field η of degree $d - 1 - r$ with $\eta(f) = 0$ and independent of ξ. Moreover when this holds, any vector field annihilating f is a linear combination of ξ, η and Hamiltonian vector fields.*

This is the content of [6, Theorems 4.7, 4.9] when expressed in geometrical terms. Taking $r = 1$, we obtain

Theorem 3.2. *Suppose V quasi-smooth and 1-symmetric of degree d with $\xi(f) = 0$. Then $\tau(V) \leq (d-1)^{n-2}(d^2 - 3d + 3)$, and equality holds if and only if V is oversymmetric, with a second vector field η. When this holds, any vector field annihilating f is a linear combination of ξ, η and Hamiltonian vector fields.*

This gives the maximal value of τ for 1-symmetric, and conjecturally for all quasi-smooth hypersurfaces.

Corollary 3.3. *The hypersurface V cannot be 3-symmetric; it is 2-symmetric if and only if it is oversymmetric and $d = 3$.*

For by [6, Lemma 5.2], if f is annihilated by vector fields ξ, ξ' with $\xi \wedge \xi' \neq 0$, of degrees r, r' we must have $r + r' \geq d - 1$. If V is 2-symmetric, we have $r = r' = 1$, hence $d = 3$ and V is oversymmetric; the converse is immediate. It follows from the theorem that now any vector field annihilating f is a linear combination of ξ, ξ' and Hamiltonian vector fields; hence if linear, is a linear combination of ξ and ξ'.

The vector field ξ is the infinitesimal generator of a linear group G. The cases when G is semi-simple were discussed in our earlier paper [4], and the complete list of the oversymmetric cases was given in [6, §5.3], and more fully in our survey article [5]. The symmetry group may be taken to act diagonally, so is determined by its weights. Either the only two non-zero weights are ± 1, and the intersection of V with the zero weight space is a cone; or there are just three non-zero weights, and the set of weights is obtained by adding zeros to a set of four weights; these must admit the monomials x_1^d, either (B) $x_0 x_2^{d-1}$ or (C) $x_0 x_2^{d-2} x_3$, and either (λ_r) $x_0^r x_3^{d-r}$, (μ_r) $x_0^r x_2 x_3^{d-r-1}$, or (ν_r) $x_0^r x_1 x_3^{d-r-1}$ for some r.

In this article we complete the list by determining all the cases when G is unipotent.

* * *

We turn to calculations of Milnor numbers. We begin with Thom's splitting theorem (alias the Morse lemma with parameters). As we will need a precise version, we outline a proof.

Lemma 3.4. *(a) Let $f(x_1, \dots, x_r, y_1, \dots, y_k)$ have 2-jet a non-degenerate quadratic form in $x_1, \dots, x_r$. Suppose that (locally) the solution of the equations $\partial f/\partial x_i = 0$ ($1 \leq i \leq r$) is given by $x_i = \alpha_i(y_1, \dots, y_k)$ ($1 \leq i \leq r$). Then f is right-equivalent to $g(y) + \sum_1^r \pm z_i'^2$, where $g(y) = f(\alpha_1(y), \dots, \alpha_r(y), y_1, \dots, y_k)$.*

(b) Suppose $f(t, x, y_1, \dots, y_k)$ is singular at the origin, with non-zero coefficient of tx and that $\partial f/\partial t$ vanishes along $x = 0$. Then f is right-equivalent to $tx + f(0, 0, y_1, \dots, y_k)$.

Proof. (a) It follows from our hypothesis that the hypersurfaces $\partial f/\partial x_i = 0$ intersect transversely at O, so there is a solution of the form given. Substitute $x_i = z_i + \alpha_i(y)$ giving $f(x, y) = F(z, y)$, say. Then $\partial F/\partial z_i = \partial f/\partial x_i$ vanishes along $z_1 = \cdots = z_r = 0$, and $F(0, y) = g(y)$. Hence $F(z, y) - g(y) \in \langle z_1, \dots, z_r \rangle^2$. It follows in turn that we can write it as $\sum_1^r z_i h_i(y, z)$ with $h_i \in \langle z_1, \dots, z_r \rangle$, and as $\sum_{i,j=1}^r z_i z_j k_{i,j}(y, z)$, where it follows from our hypothesis that the matrix $k_{i,j}(0)$ is non-singular. Now by 'completing the square' r times we can write this in the form $\sum_1^r \pm z_i'^2$, where $(z_1', \dots, z_r', y_1, \dots, y_k)$ can be taken as local co-ordinates at O.

(b) Since $\partial f/\partial t$ vanishes along $x = 0$, we can write $\partial f/\partial t = xa(t, x, y)$ for some C^∞–function a. Hence $f(t, x, y) - f(0, x, y) = \int_0^t xa(t, x, y)dt$, hence has the form $xb(t, x, y)$ for some C^∞–function b. As also $f(0, x, y) - f(0, 0, y)$ is divisible by x, we can write $f(t, x, y) = xc(t, x, y) + f(0, 0, y)$. Now c vanishes at the origin and has non-zero coefficient of t; thus the co-ordinate change $t' = c(t, x, y)$ gives the desired equivalence. $\square$

Write $\chi_n(d)$ for the Euler characteristic of a smooth hypersurface of degree d in P^{n+1}: then (see e.g. [1, p. 152])

$$\chi_n(d) = n + 2 + \frac{(-1)^n}{d}((d-1)^{n+2} - (-1)^n). \tag{1}$$

When $n = -2, -1, 0$ this formula gives $0, 0, d$ respectively, so remains correct. The cone over such a hypersurface in P^{n+2} admits a $\mathbb{C}^*-$action which is free except at the fixed points, which consist of the hypersurface itself and an isolated point. Thus this cone has Euler characteristic $\chi_n(d)+1$. If V is a hypersurface of degree d in P^{n+1} with isolated singularities, then (see e.g. [1, p. 162]) $\chi(V) = \chi_n(d) + (-1)^{n-1}\mu(V)$.

In weighted projective space, suppose f, of degree d with respect to weights w_i $(0 \leq i \leq n+1)$ with sum W, defines a smooth hypersurface V, so $\dim V = n$. We have the following theorem of Steenbrink.

Theorem 3.5. *(see [1, Theorem B34] and [8]). The mixed Hodge numbers of the primitive cohomology of V are given by $h_0^{i,n-i}(V) = \dim M(f)_{d(i+1)-W}$, where $M(f)$ is the Milnor algebra*

$$M(f) = \mathbb{C}[x_0, \ldots, x_{n+1}]/\langle \partial f/\partial x_0, \ldots, \partial f/\partial x_{n+1}\rangle.$$

$M(f)$ has Euler-Poincaré polynomial $p(t) = \prod_{i=0}^{n+1}(1-t^{d-w_i})/(1-t^{w_i})$. Thus the primitive Betti number $h_0^n(V)$, which is the sum of the Hodge numbers $h_0^{i,n-i}(V)$, is equal to the sum of the coefficients of $p(t)$ in degrees congruent to $-W$ modulo d. This sum is given by $\frac{1}{d}\sum \epsilon^W p(\epsilon)$, where ϵ runs through the d^{th} roots of unity. It follows that

$$\chi(V) = n + 1 + \frac{(-1)^n}{d}\sum_{\epsilon^d=1} \epsilon^W p(\epsilon). \tag{2}$$

We now evaluate this in the two cases we will need.

Corollary 3.6. *A non-singular hypersurface V of degree d in weighted projective space with weights $w_0 = 2$ and $w_i = 1$ for $1 \leq i \leq n+1$ has*

$$\chi(V) = n + 2 + \tfrac{(-1)^n}{d}\{\tfrac{d-2}{2}(d-1)^{n+1} + (-1)^{n+1}\}$$

if d is odd, and is $\frac{1}{2}$ less than this if d is even. In particular, if $d = 3$ we have $n + 2 + \frac{(-1)^n}{3}\{2^n - (-1)^n\}$.

In this case $W = n+3$, so $p(t) = (1-t^{d-2})(1-t^{d-1})^{n+1}/(1-t^2)(1-t)^{n+1}$. If $\epsilon = 1$ we evaluate p by l'Hôpital's rule, obtaining $p(1) = \frac{1}{2}(d-2)(d-1)^{n+1}$. If $\epsilon = -1$, which is only possible if d is odd, we have $(1-\epsilon^{d-1})/(1-\epsilon) = 1$, so $p(-1) = (-1)^{n+1}\frac{1}{2}(d-2)$. Otherwise we have $(1-\epsilon^{d-1})/(1-\epsilon) = -\epsilon^{-1}$ and $(1-\epsilon^{d-2})/(1-\epsilon^2) = -\epsilon^{-2}$, so $p(\epsilon) = \epsilon^W(-\epsilon^{-2})(-\epsilon^{-1})^{n+1}$, which reduces to $(-1)^n$. We thus obtain $n+1+(-1)^n\frac{1}{d}\{\frac{1}{2}(d-2)(d-1)^{n+1} + (-1)^n(d-1)\}$ if d is odd, and $n+1+(-1)^n\frac{1}{d}\{\frac{1}{2}(d-2)(d-1)^{n+1} + (-1)^{n+1}\frac{1}{2}(d-2) + (-1)^n(d-2)\}$ if d is even, which reduces to the values stated.

If V has isolated singularities then, as above, we must add $(-1)^{n-1}\mu(V)$ to this expression. If the singularities occur at smooth points of the ambient weighted projective space, this is proved as before, using the additive nature of χ; at other points, we may take it as the definition of μ.

Corollary 3.7. *A non-singular hypersurface V of degree 6 in weighted projective space with weights $w_0 = w_1 = 1$ and $w_i = 2$ for $1 < i \leq n+1$ has*

$$\chi(V) = n + 1 + \tfrac{1}{6}(-1)^n(26.2^n + 4(-1)^n).$$

In this case, $W = 2n+4$ and $M(\phi)$ has Euler-Poincaré polynomial

$$p(t) = (1-t^5)^2(1-t^4)^n/(1-t)^2(1-t^2)^n = (1+t+t^2+t^3+t^4)^2(1+t^2)^n.$$

We have $p(1) = 5^2.2^n$ and $p(-1) = 2^n$. If $\epsilon^6 = 1$ and $\epsilon^2 \neq 1$, $(1-\epsilon^5)/(1-\epsilon) = -\epsilon^{-1}$ and $(1-\epsilon^4)/(1-\epsilon^2) = -\epsilon^{-2}$, so $p(\epsilon) = (-1)^n\epsilon^{-2-2n}$. Hence $\sum \epsilon^{2n-4}p(\epsilon) = 26.2^n + 2^n + 4(-1)^n$.

4 Invariants of unipotent actions

Now consider a nilpotent endomorphism N of a vector space K; we adopt as our standard notation $\xi_N = \sum_{i=1}^k x_{i-1}\partial/\partial x_i$. We write G_N for the Lie group obtained by exponentiating, $E_t = \text{Exp}(tN)$, then $t.\mathbf{x} = E_t\mathbf{x}$; thus a polynomial f on K is annihilated by ξ_N if and only if it is invariant under G_N. Write $\mathcal{I}_k$ for the ring of invariants of the group G_N (a subring of $\mathbb{C}[x_0, \ldots, x_k]$). If we have a second Jordan block, denote the variables $y_0, \ldots, y_l$, set $\xi = \sum_{i=1}^k x_{i-1}\partial/\partial x_i + \sum_{i=1}^l y_{i-1}\partial/\partial y_i$, and write $\mathcal{I}_{k,l}$ for the ring of invariants.

It is a classical theorem of Weitzenbock [11] that the ring of invariants is finitely generated. Weitzenbock also determined the localisation at x_0 of the ring of invariants. Indeed, if $x_0 \neq 0$, there is a unique choice $t_0 = -\frac{x_1}{x_0}$ of the parameter t such that $(t.\mathbf{x})_1 = 0$. Then all the $X_i = (t_0.x)_i$ for $2 \leq i \leq k$ are invariants, and clearly $\mathcal{I}_k[x_0^{-1}] = \mathbb{C}[x_0, X_2, \ldots, X_k, x_0^{-1}]$. The argument also applies if N has several Jordan blocks.

This remark can be used to compute the structure of the ring of invariants. Weitzenbock himself did this for $\dim K \leq 4$; a general algorithm was given by Tan [9], and a fuller account is in the book of Nowicki [7]. The results we need can be stated as follows.

Lemma 4.1. *We have rings of invariants*

$$\begin{aligned}
\mathcal{I}_2 &\cong \mathbb{C}[X, B], \\
\mathcal{I}_3 &\cong \mathbb{C}[X, B, C, \Delta / X^2\Delta + C^2 + B^3 = 0], \\
\mathcal{I}_4 &\cong \mathbb{C}[X, B, U, C, E / X^3E = 3X^2BU - B^3 - C^2], \\
\mathcal{I}_{2,1} &\cong \mathbb{C}[X, Y, T, B, S / XS = Y^2B + T^2],
\end{aligned}$$

where $X := x_0$*,* $Y := y_0$*, and*

$$\begin{aligned}
B &:= T^2_{x,x} = 2x_0x_2 - x_1^2, \\
C &:= 3x_0^2x_3 - 3x_0x_1x_2 + x_1^3, \\
\Delta &:= -9x_0^2x_3^2 + 18x_0x_1x_2x_3 - 8x_0x_2^3 + 3x_1^2x_2^2 - 6x_1^3x_3, \\
U &:= T^4_{x,x} = 2x_0x_4 - 2x_1x_3 + x_2^2, \\
E &:= 12x_0x_2x_4 - 9x_0x_3^2 + 6x_1x_2x_3 - 2x_2^3 - 6x_1^2x_4, \\
T &:= T^1_{x,y} = x_0y_1 - x_1y_0, \\
S &:= x_0y_1^2 - 2x_1y_0y_1 + 2x_2y_0^2.
\end{aligned}$$

Here $\mathcal{I}_3$ was given in [11], $\mathcal{I}_4$ in [9] and $\mathcal{I}_{2,1}$ in [7]. For Cases 2 and 3 we follow the notation of [4].

For the geometric problem, we have additional variables $w = (w_1, \ldots, w_m)$, all invariant. Thus the dimension $n = m+2$, $m+3$, $m+4$ or $m+4$ in our 4 cases respectively. Denote the corresponding elements of the ring of invariants by $W := (W_1, \ldots, W_m)$.

We can use changes of co-ordinates that are compatible with N to simplify our formulae.

Lemma 4.2. *In Case 4, the co-ordinate changes compatible with N are: $x_4' = \sum_0^4 a_i x_{4-i} + \sum e_j w_j$, $x_3' = \sum_0^3 a_i x_{3-i}$, $x_2' = \sum_0^2 a_i x_{2-i}$, $x_1' = \sum_0^1 a_i x_{1-i}$, $x_0' = a_0 x_0$, $w_i' = \sum p_{i,j} w_j + q_i x_0$, where $a_0 \neq 0$ and $(p_{i,j})$ is non-singular.*

For we have taken an arbitrary element of K for x_4'; then $x_3' = N x_4'$, x_2', x_1' and x_0' are determined. Since our change of co-ordinates must respect the filtration, w_i' must be as stated. For the formulae to define a co-ordinate change we must have $a_0 \neq 0$ and $(p_{i,j})$ non-singular.

The results in Cases 2 and 3 are almost the same, and Case 21 is very similar.

In the next four sections we give detailed discussions of the four cases of Theorem 2.4 in turn.

5 Case 2

We define a map $\pi : K \to L$ taking as target co-ordinates (W, X, B). This induces a map $\overline{\pi} : P(K) \to P(L)$, where $P(L)$ is the weighted projective space with all weights 1 except $w(B) = 2$. The map $\overline{\pi}$ is defined except on the set $\mathcal{E}$ where all co-ordinates other than x_2 vanish. Thus $\mathcal{E}$ is a point, which we also denote P. The space $P(L)$ has just one singular point, where all co-ordinates except B vanish: we denote it by Q.

As is usual for moduli spaces, we have a natural stratification. We define strata $\mathcal{S}_i$ in K and $\overline{\mathcal{S}}_i$ in L, compatible with each other under π and with passage to projective space.

$$\begin{array}{llll}
\mathcal{S}_0: & x_0 \neq 0; & \overline{\mathcal{S}}_0: & X \neq 0 \\
\mathcal{S}_1: & x_0 = 0,\ x_1 \neq 0; & \overline{\mathcal{S}}_1: & X = 0,\ B \neq 0 \\
\mathcal{S}_2: & x_0 = x_1 = 0; & \overline{\mathcal{S}}_2: & X = B = 0.
\end{array}$$

The set $\mathcal{F}$ of fixed points is defined by the vanishing of x_0, x_1, so coincides with $\mathcal{S}_2$. Each orbit of the action of G_N on K or on $P(K)$ outside $\mathcal{F}$ is isomorphic to an affine line; their degrees are 2,1 for $\mathcal{S}_0, \mathcal{S}_1$ respectively.

For any $(W, X, B) \in L$, we calculate $\pi^{-1}(W, X, B)$.

In each case, we have uniquely $w = W$, $x_0 = X$.

If $X \neq 0$, x_1 is free (i.e. can be chosen arbitrarily), and $x_2 = (B + x_1^2)/2X$: we have one orbit.

If $X = 0$, $x_1 = \pm\sqrt{(-B)}$, and x_2 is free. If $B \neq 0$, this gives two orbits, but if $B = 0$, a line of fixed points.

From this we infer (with some care) pre-images under $\overline{\pi}$; in each case, we tabulate the Euler characteristic of the pre-image.

Lemma 5.1. *[4, Lemma 6.4] The preimage $\overline{\pi}^{-1}(W, X, B)$ is as follows:*
($\overline{\mathcal{S}}_0$) one orbit, $\chi = 1$,
($\overline{\mathcal{S}}_1$) if $W \neq 0$, two orbits, $\chi = 2$; if $W = 0$ (the point Q), one orbit, $\chi = 1$,
($\overline{\mathcal{S}}_2$) infinitely many point orbits, $\chi = 1$.

Since the ring of invariants is a polynomial ring, any invariant function f is of the form $f = \phi \circ \pi$, where $\phi = \phi(W, X, B)$ is a polynomial function on L. Set $\phi_B := \frac{\partial \phi}{\partial B}$, $\phi_X := \frac{\partial \phi}{\partial X}$.

Denote by V the hypersurface in $P(K)$ defined by f, by V_1 the hypersurface in the weighted projective space $P(L)$ defined by ϕ, and by V_2 and V_3 the intersections of V_1 with $X = 0$ and with $X = B = 0$ respectively. As in similar cases below, our notation is chosen so that each V_r (also V_r^*, etc.) has dimension $m + 1 - r$.

Lemma 5.2. *(compare [4, Lemma 5.5]) V has isolated singular points if and only if V_1 has no singular points and V_3 has isolated singular points. The singular points of f are P and points P_i corresponding to the singular points Q_i of V_3 at which $\phi_B \neq 0$.*

Proof. At a critical point of f, the following vanish:

$$\frac{\partial f}{\partial w_i} = \frac{\partial \phi}{\partial W_i}, \quad \frac{\partial f}{\partial x_0} = \phi_X - 2x_2\phi_B, \quad \frac{\partial f}{\partial x_1} = 2x_1\phi_B, \quad \text{and} \quad \frac{\partial f}{\partial x_2} = -2x_0\phi_B.$$

If $\phi_B = 0$, we have a critical point of ϕ. If $W = X = B = 0$, the only corresponding point in $P(K)$ is P. Otherwise we have a singular point of V_1. Conversely, if we have a singular point of V_1, all the points in its pre-image are singular on V, so are non-isolated singular points of V.

For a critical point of f with $\phi_B \neq 0$, we have $x_0 = x_1 = 0$, hence $X = B = 0$, and a critical point of the restriction of ϕ to $X = B = 0$. If $W = 0$, we again have the point P. Otherwise we have a singular point of V_3. Conversely, if we have a singular point of V_3 at which $\phi_B \neq 0$, there is a unique corresponding value of x_2 giving a critical point of f, hence a unique corresponding singular point of V.

However, if we have a singular point of V_3 at which $\phi_B = 0$, then $\phi_X \neq 0$ as otherwise we would have a singular point of V_1; and as $\phi_X \neq 0$, there is no corresponding critical point of f. □

It also follows that if V has isolated singular points, so also has V_2. For if the singular locus of V_2 had positive dimension, it would have to intersect the hypersurface $\phi_X = 0$, and any point of intersection gives a singular point of V_1. Observe also that if d is even, $Q \notin V_2$, for otherwise Q would be a singular point of V_1. If d is odd, then there can be no term $B^{d/2}$, so $Q \in V_2$.

We can easily describe the singularities of V at points other than P.

Proposition 5.3. *The singularity at P_i of V corresponding to a singularity at Q_i of V_3 at which $\phi_B \neq 0$ is right-equivalent to a suspension of that singularity.*

Proof. We may suppose, after an allowable co-ordinate change, that at the singular point P_i we have $0 = x_2 = \phi_X$. Apply Lemma 3.4(a) to f with the variables (x_0, x_1, x_2). We observe that $\partial f/\partial x_0$, $\partial f/\partial x_1$, $\partial f/\partial x_2$ all vanish when $x_0 = x_1 = 0$ and $x_2 = \phi_X/2\phi_B$. Substituting these values gives $g(w) = f(w, 0, 0, 0)$. The result follows. □

We are now ready to calculate $\mu(V)$.

Theorem 5.4. *We have* $\mu(V) = \frac{1}{2}(d-2)(2d-1)(d-1)^m + \mu(V_2) + \mu(V_3)$.

Proof. First suppose d odd. Then as $Q \in V_1$, by Lemma 5.1 we have $\chi(V) = \chi(\mathcal{E}) + \chi(V_1 \setminus V_2) + 2\chi(V_2 \setminus V_3) - 1 + \chi(V_3) = \chi(V_1) + \chi(V_2) - \chi(V_3)$. For V and V_3 we apply (1); for V_1 we apply Corollary 3.6 with $n = m$; for V_2 the same, but with $n = m-1$, and amended for singularities. Thus

$$\begin{aligned}
\chi(V) &= \tfrac{(-1)^{m+1}}{d}\{(d-1)^{m+3} - (-1)^{m+1}\} + m + 3 + (-1)^m \mu(V),\\
\chi(V_1) &= \tfrac{(-1)^m}{d}\{\tfrac{d-2}{2}(d-1)^{m+1} + (-1)^{m+1}\} + m + 2,\\
\chi(V_2) &= \tfrac{(-1)^{m-1}}{d}\{\tfrac{d-2}{2}(d-1)^m + (-1)^m\} + m + 1 + (-1)^m \mu(V_2),\\
\chi(V_3) &= \tfrac{(-1)^m}{d}\{(d-1)^m - (-1)^m\} + m + (-1)^{m-3}\mu(V_3).
\end{aligned}$$

Since $\chi(V) - \chi(V_1) - \chi(V_2) + \chi(V_3)$ vanishes, we find that $\mu(V) - \mu(V_2) - \mu(V_3)$ is equal to $d^{-1}\{(d-1)^{m+3} + \frac{d-2}{2}(d-1)^{m+1} - \frac{d-2}{2}(d-1)^m - (d-1)^m\}$, which reduces to $\frac{1}{2}(d-2)(2d-1)(d-1)^m$.

In the case d even, as $Q \notin V_1$, we obtain $\chi(V) = \chi(V_1) + \chi(V_2) - \chi(V_3) + 1$, but the values of each of $\chi(V_1)$ and $\chi(V_2)$ are $\frac{1}{2}$ less than those above. Hence the formula in terms of d is the same as before. $\square$

If $m = 0$, the value of $\mu(V)$ is given by [2, Proposition 3.1]: the value $\frac{1}{2}(d-2)(2d-1)$ is correct if d is even; we must add $\frac{1}{2}$ if d is odd. Here if d is odd V_2 is necessarily singular: indeed, ϕ vanishes identically on $\overline{\mathcal{S}}_1$.

For the case $m = 1$, [4, Prop 6.6] gives $\mu(V) = \frac{1}{2}(d-2)(2d-1)(d-1) + k - N$, where $k = \lfloor \frac{d}{2} \rfloor$ and N is the number of distinct points of $V_2 \setminus V_3$ with $W \neq 0$. In this case, V_2 has dimension 0, so $\chi(V_2) = \#V_2$; by Corollary 3.6, if V_2 is smooth, $\chi(V_2) = \lfloor \frac{1}{2}(d+1) \rfloor$. To reconcile these we need to interpret $\mu(V_3)$ as 1 if $V_3 \neq \emptyset$, i.e. if the coefficient of W^d in ϕ is non-zero.

* * *

When does $\mu(V)$ take its maximal value? We expect this to occur if $\mu(V_2)$ and $\mu(V_3)$ are both as large as possible, hence when V_2 and V_3 are both cones. Geometry imposes restrictions as follows.

As already observed, $Q \in V_2$ if and only if d is odd.

If V_2 is a cone with vertex not in V_3, then it is a cone on V_3. Since it must have isolated singularities, V_3 must be non-singular.

If V_2 is a cone with vertex different from Q, then $Q \notin V_2$ (since the local geometry at Q differs from that elsewhere).

Thus if d is even, while $Q \notin V_2$, we cannot exclude the possibility that V_2 is a cone with vertex Q^* in V_3. When this holds, V_3 also is a cone, so indeed we expect $\mu(V)$ to be maximal. The singularity of V_2 at Q^* is equisingular to a sum of $(m-1)$ d^{th} powers and a $\frac{1}{2}d^{th}$ power, so $\mu(V_2) = \frac{1}{2}(d-1)^{m-1}(d-2)$; that of V_3 at Q^* is equisingular to a sum of d^{th} powers, so $\mu(V_3) = (d-1)^{m-1}$ and $\mu(V) = \frac{1}{2}(d-1)^{m-1}(2d^3 - 7d^2 + 8d - 2)$. Since $\phi_B(Q^*) = 0$, the only singular point of V is P.

This case does indeed occur for all $m \geq 1$ and even $d \geq 4$: we can take $\phi = X^d + XW_1^{d-1} + B^{d/2} + \sum_2^m W_i^d$. Then each of V_2 and V_3 is a cone, smooth except at the point Q^* where all co-ordinates except W_1 vanish; and V_1 is non-singular.

We believe these to give the maximal values of $\mu(V)$ for all even $d \geq 4, m \geq 1$: for $m = 1$ this follows from [4, Proposition 6.6].

If d is odd and V_2 is a cone, then the vertex of the cone is Q and V_3 is non-singular. In affine co-ordinates $B = 1$, ϕ is equisingular to a sum of m d^{th} powers, which suggests $\mu_Q(V_2) = (d-1)^m$, but since we must factor out the antipodal map on affine space we actually have $\mu_Q(V_2) = \frac{1}{2}(d-1)^m$, thus $\mu(V) = \frac{1}{2}(d-1)^{m+1}(2d-3)$. This case occurs for all $d = 2k+1$: we can take $\phi = XB^k + X^d + \sum_1^m W_i^d$. Again this gives the maximum value for μ if $m = 1$: we cannot show that this holds in general.

$*\quad *\quad *$

We now treat the case $d = 3$ in more detail. Here $f = a_1 B + a_3$, where a_1, a_3 are homogeneous functions of $x_0, w_1, \ldots, w_m$. If a_1 is not a multiple of x_0, we can make a change of co-ordinates to write $a_1 = w_1$; otherwise we can take $a_1 = x_0$ (if $a_1 \equiv 0$, V has non-isolated singularities). Denote by V_3^* the variety $w_1 = a_3 = 0$ in P^m and set $V_4 := V_3 \cap V_3^*$.

Lemma 5.5. *(i) If $a_1 = x_0$, V is quasi-smooth if and only if V_3 is non-singular. In this case, the only singular point is P.*

(ii) If $a_1 = w_1$, there is a bijection between singular points of V_2 and V_4; the singularity of the former is isomorphic to the suspension of the latter.

(iii) If $a_1 = w_1$, V_1 is non-singular if and only if V_3^* is non-singular.

Proof. (i) In this case, V_2 is a cone with vertex Q, so the result follows as above.

(ii) At a singular point $(W, 0, B)$ of V_2 we have $0 = \phi_B = W_1$ and $0 = \partial\phi/\partial W_i$ for $i \geq 2$, so $(W, 0, 0)$ is a singular point of V_4 (note that $(0, 0, B)$ is not a singular point of V_2 since $\partial\phi/\partial W_1$ does not vanish there).

Conversely, if $(W, 0, 0)$ is a singular point of V_4, the point $(W, 0, B)$ is singular on V_2 if and only if $B = -\partial a_3/\partial W_1$.

Now apply Lemma 3.4(a), taking B and W_1 as the preferred co-ordinates. The equations $\phi_B = \partial\phi/\partial W_1 = 0$ are solved by $W_1 = 0$, $B = -\partial a_3/\partial W_1$: substituting these in $\phi = w_1 B + a_3$ gives the restriction of a_3 to $W_1 = 0$.

(iii) The same argument as for (ii) applies here. □

If $a_1 = x_0$, V_2 is a cone with vertex Q, so $\mu(V) = 3.2^m$ as above. If $a_1 = w_1$, it follows that $\mu(V_2) = \mu(V_4)$, so by Theorem 5.4, $\mu(V) = 5.2^{m-1} + \mu(V_3) + \mu(V_4)$.

In some cases, we can determine the nature of the singularities.

Proposition 5.6. *Suppose V in Case 2 with V_2 non-singular. Then the singularity of V at P is semi-quasi-homogeneous with degree $2d$ and variables of weights 1, 4 and 2 (m times).*

Proof. We first give a direct argument, then an indirect method, which only determines the $\mu-$constant stratum, but will be usable in other cases below.

As in [4, Proposition 6.6], take local affine co-ordinates $x_2 = 1$ at P, and substitute $x_0 = \frac{1}{2}(B + x_1^2)$, so that f becomes $\phi(w, B, \frac{1}{2}(B + x_1^2))$. Now assign weights 4 to B, 1 to x_1 and 2 to the w_i. The terms of least weight $2d$ give $\phi(w, B, \frac{1}{2}x_1^2)$. We must check that this has an isolated critical point. At a singular point, $\partial\phi/\partial w_i$, ϕ_B and $x_1\phi_X$ vanish. Since V_1 is non-singular, $x_1 = 0$, so $(W, 0, B)$ is a singular point of V_2, contradicting our hypothesis.

For our second argument, we note that by Theorem 5.4, $\mu(V) - \mu(V_3)$ takes the same value for all these cases (with d and m fixed). Now $\mu(V) = \mu_P(V) + \sum_i \mu_{P_i}(V)$, and by Proposition 5.3, the values $\mu_{P_i}(V) = \mu_{Q_i}(V_3)$. Hence $\mu_P(V)$ is the same for all these cases, so all belong to the same $\mu-$constant stratum.

To determine this, first observe that we can adjoin a new variable w_{m+1}; then $f' := f + w_{m+1}^d$ again satisfies the conditions of Lemma 5.2, and the new singularity is obtained from the old one also by adjoining a new variable and adding its d^{th} power. Hence the $\mu-$constant type of the singularity can be deduced from the case with m decreased by 1.

If $m = 0$, as observed above, if d is odd, V_2 is necessarily singular; if d is even, the result holds. However the case $m = 1$ was analysed in [4, Proposition 6.6], where we showed directly that the singularity has the type stated. The result thus follows in general. $\square$

The second method can also be applied to the case when V_2 is a cone with vertex Q. We see that the singularity is equisingular to a sum of m d^{th} powers and the curve singularity occurring in the case $m = 0$, which can be taken to be $\prod_1^k (y^2 - 2x + 4c_i x^2)$ if $d = 2k$ and $x \prod_1^k (y^2 - 2x + 4c_i x^2)$ if $d = 2k + 1$ (with the c_i all distinct in each case). If $d \geq 5$, it is not quasi-homogeneous.

If $d = 3$, the cases arising when $a_1 = w_1$ can be enumerated in low dimensions by considering the varieties $V_4 \subset V_3$. We can determine the $\mu-$constant strata of the critical points of f using the fact that the terms of lowest weight are $\phi(w, B, \frac{1}{2}x_1^2)$, which reduces by splitting to $\phi(0, w_2, \ldots, w_m, 0, x_1^2)$, together with our calculation of μ.

For $m = 1$, we have A_5 at P, perhaps a further A_1.

For $m = 2$, we have the cubic curve $a_3(w_1, w_2, x_0) = 0$ meeting $x_0 = 0$ in V_3 and the point Q^* where $w_1 = x_0 = 0$ in V_4. Let w_1^r be the highest power of w_1 dividing $a_3(w_1, w_2, 0)$. If $r = 0$, $V_4 = \emptyset$, V_2 is non-singular and V has a $T_{2,3,6}$ at P and a further A_1 (or A_2) if V_3 has a repeated point (a 3-fold point). If $r \geq 1$, we apply the same substitution, but must now use the 2-jet Bw_1 and obtain the splitting by direct calculation. The first substitution gives the 4-jet $w_2^2 x_1^2 + \alpha w_2^4$, where $\alpha = 0$ if $r > 1$. Thus if $r = 1$ the singularity has $\mu = 11$, hence type $T_{2,4,6}$, and the other two points on V_3 could coincide, giving a further A_1. If $r = 2$ we have a singularity $T_{2,p,q}$ with $p, q \geq 5$ and $\mu = 12$, hence $p + q = 11$ so $(p, q) = (5, 6)$. In the case $r = 3$ we have $p + q = 12$, and need a further calculation to obtain the 5-jet, leading to $p = q = 6$. Thus in each case, we have $T_{2,3+r,6}$ at P.

For $m = 3$, if V_4 is non-singular (i.e. 3 points), $UT^1_{0,0,0}$ (in the notation of [10, p. 475]) together with $-$, A_1, A_2, $2A_1$, A_3, $3A_1$ or D_4. If V_4 is singular, we have non-reduced 3-jet (V or V' series) and the singularities do not have accepted names.

* * *

We turn to calculation of τ: here our results are much more partial. According to Lemma 3.2, τ takes its maximal value when f is oversymmetric. To find when this is applicable, we use the method of [4, §6].

Theorem 5.7. *A function in Case 2 is oversymmetric if and only if either (a) V_3 is a cone, or (b) after change of co-ordinates if necessary, ϕ_B and ϕ_X both vanish along $X = B = 0$.*

Proof. Since V_1 is non-singular, the sequence $\{\partial\phi/\partial W_1, \ldots, \partial\phi/\partial W_m, \phi_X, \phi_B\}$ is regular, and any vector field annihilating ϕ is a linear combination of the Hamiltonian fields $\partial(\phi,*)/\partial(W_i,W_j)$, $\partial(\phi,*)/\partial(W_i,X)$, $\partial(\phi,*)/\partial(W_i,B)$ and $\partial(\phi,*)/\partial(X,B)$. We seek a vector field η which is a lift of a linear combination of these. We are only interested in η modulo Hamiltonian fields: removing the corresponding linear combination of the $\partial(f,*)/\partial(w_i,w_j)$ and $\partial(f,*)/\partial(w_i,x_0)$, we can take $\sum C_i\partial(\phi,*)/\partial(W_i,B)$ $+ D\partial(\phi,*)/\partial(X,B)$. Since we seek η of degree $d-2$, we want the C_i and D to be constants. We now have

$$\eta = \textstyle\sum_1^m p_i\partial/\partial w_i + \sum_0^2 q_j\partial/\partial x_j,$$

where $p_i = -C_i\phi_B$, $q_0 = -D\phi_B$ and

$$2(x_2q_0 - x_1q_1 + x_0q_2) = \textstyle\sum C_i\partial\phi/\partial W_i + D\phi_X.$$

Thus

$$2(x_0q_2 - x_1q_1) = \sum C_i\partial\phi/\partial W_i + D\phi_X + 2x_2D\phi_B. \tag{3}$$

The right hand side of this equation must thus vanish identically along $X = B = 0$.

First suppose $D = 0$. Changing the w co-ordinates, we may suppose the vector field is $\partial/\partial W_1$. If we set $a_d(W) := \phi(W, X, B)$, we need $\partial a_d/\partial W_1 \equiv 0$, i.e. a_d is independent of W_1. Expressing the condition geometrically, it holds if and only if V_3 is a cone.

If $D \neq 0$, a suitable substitution $W_i' := W_i + \lambda_i X$, $X' := X$ reduces the C_i to zero, so it suffices to consider the vector field $\partial(\phi,*)/\partial(X,B)$. Here the condition reduces to requiring both ϕ_B and ϕ_X to vanish along $X = B = 0$. □

We could reformulate (b) as: there exist constants c_i such that ϕ_B and $\phi_X + \sum c_i\partial\phi/\partial W_i$ both vanish along $X = B = 0$.

This proof shows more generally that any vector field $\eta = \sum_1^m p_i\partial/\partial w_i + \sum_0^2 q_j\partial/\partial x_j$ annihilating f can be reduced modulo Hamiltonian vector fields to the lift of $\sum C_i\partial(\phi,*)/\partial(W_i,B) + D\partial(\phi,*)/\partial(X,B)$, where $p_i = -C_i\phi_B$, $q_0 = -D\phi_B$ and (3) holds. Moreover, we may suppose D and the C_i independent of x_0 and x_1. This can be used as the starting point for further calculations of τ. However, since the cases arising are diverse, we only consider $m = 0$, $m = 1$ and certain cases with $d = 3$.

In the case of curves ($m = 0$), the condition frequently holds, and then $\tau = d^2 - 3d + 3$ (see [2, Proposition 3.1]): otherwise, $\tau = d^2 - 3d + 2$.

The case of surfaces ($m = 1$) was treated in [4]. By Theorem 6.7 loc.cit., $\tau_{\mathrm{tot}}(V) = (d-1)(d^2 - 3d + 3)$ if $\alpha = 0$ or $\gamma = 0$, and $(d-1)(d^2 - 3d + 3) - 1$ otherwise; where α, γ are the coefficients of W^d and BW^{d-2} in ϕ. Moreover (Lemma 6.5 loc.cit.) P is the only singular point unless $\alpha = 0 \neq \gamma$, when there is one further singular point, of type A_1. The case $\alpha = 0$ corresponds to clause (a) of the Theorem; the case $\gamma = 0$ to clause (b) (here we appear to require $\beta = \gamma = 0$: the difference arises because of the above normalisation of co-ordinates).

We can calculate $\tau(V)$ ad hoc in further low dimensional cases. When $d = 3$, if $a_1 = w_1$ the values can be inferred from the above list of $\mu-$constant strata: we have $\tau = \mu$ for $T_{2,3,6}$ and $\tau = \mu - 1$ for $T_{2,p,q}$ with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. If $a_1 = x_0$ we have

Lemma 5.8. *If V is in Case 2, with $d = 3$ and $a_1 = x_0$, then*

$$\tau(V) = 2^{m+1} + \dim(\mathbb{C}[x_0, w_1, \ldots, w_m]/\langle x_0, \partial a_3/\partial x_0, \partial a_3/\partial w_1, \ldots, \partial a_3/\partial w_n\rangle).$$

Proof. Here $\phi = XB + a_3(W, X)$, so (3) reduces to

$$\textstyle\sum_i C_i \partial a_3/\partial W_i + D\partial a_3/\partial X \in \langle X\rangle.$$

By Lemma 5.5(i), in this case V_3 is non-singular. Hence the restrictions of the $\partial a_3/\partial W_i$ to $X = 0$ form a regular sequence in $\mathbb{C}[W]$, spanning an ideal J. The class of η modulo Hamiltonian fields and multiples of ξ is determined by the class of $D|_{X=0}$ modulo J.

The algebra $\mathbb{C}[W]/J$ is Gorenstein of dimension 2^m, with $\binom{m}{r}$ basis elements in degree r. If the ideal in it generated by the class of $\partial a_3/\partial X|_{X=0}$ has dimension e, its annihilator has dimension $2^m - e$. Since the space of multiples of ξ modulo Hamiltonian vector fields has dimension 2^{m+1}, we obtain $\tau(V) = 3.2^m - e$.

But $\mathbb{C}[x_0, w_1, \ldots, w_m]/\langle x_0, \partial a_3/\partial x_0, J\rangle \cong \mathbb{C}[W]/(J + \langle \partial a_3/\partial X|_{X=0}\rangle)$, so has dimension $2^m - e$. The result follows. □

If $m = 2$ we can take $a_3(W_1, W_2, 0) = W_1^3 + W_2^3$ and see easily that if the coefficient of W_1W_2X in a_3 is non-zero, $e = 0$ and $\tau = 12$: otherwise $e = 1$ and $\tau = 11$. If $m = 3$ we take $W_1^3 + W_2^3 + W_3^3 + 3\alpha W_1W_2W_3$: here either $e = 0$ or $e = 2$. For $m \geq 4$, cases are more numerous.

We observe that while there are numerous cases where $\tau(V)$ takes its maximal value (for given dimension n and degree d) but $\mu(V)$ does not, we do not know an example in the reverse direction. Indeed, μ maximal implies τ maximal for curves, surfaces of degree 4 or odd, and for cubic 3-folds. If V_2 is a cone with vertex in V_3 then τ is maximal; but if it is a cone with vertex Q, while ϕ_B vanishes along $X = B = 0$ we have no control on ϕ_X.

We now give a more detailed discussion of the 2-symmetric case $d = 3$, following the notation of the above proof.

Proposition 5.9. *In Case 2, f is 2-symmetric only in the following 3 cases:*

Case (b): we have $a_1 = X$. V_2 is a cone with vertex Q and V_3 is non-singular. After a suitable substitution $x_2' := x_2 + \frac{1}{2}b(w, x_0)$, a_3 is independent of x_0, and we may take $\eta = -2x_0\partial/\partial x_0 + x_1\partial/\partial x_1 + 4x_2\partial/\partial x_2$. The singularity has $\mu = 3.2^m$ and is quasi-homogeneous of degree 12 with respect to weights 3,6 and 4 (m times).

Case (a1): we have $a_1 = W_1$, V_3 is a cone with vertex not on $W_1 = 0$ and V_4 non-singular. After a further substitution $x_2' := x_2 + \frac{1}{2}b(w, x_0)$, may suppose a_3 independent of w_1. Then f is invariant by $\eta = x_1\partial/\partial x_1 + 2x_2\partial/\partial x_2 - 2w_1\partial/\partial w_1$. There are two singularities, with Milnor numbers 5.2^{m-1} and 2^{m-1}; both quasi-homogeneous of degree 6, the first with respect to weights 1 and 2 ($m - 1$ times); the second with respect to weights 3 and 2 ($m - 1$ times).

Case (a2): we have $a_1 = W_1$; V_3, V_4 are cones with vertex on $W_1 = 0$. After a further substitution $w_2' := b(w, x_0)$, we may suppose $a_3 - x_0w_2^2$ independent of w_2. Then f is invariant by $\eta = w_2\partial/\partial x_2 - w_1\partial/\partial w_2$. The singularity has $\mu = 13.2^{m-2}$, and is in the same $\mu-$constant stratum as $x^6 + x^2y^2 + y^6 + \sum_2^m w_i^3$.

Proof. The enumeration is given in Theorem 5.7. For Case (b), we must have $a_1 = X$; it follows that V_2 is a cone with vertex Q, V_3 is non-singular and the singularity was determined above. Now write $a_3(w, x_0) = x_0^2 b_1(w, x_0) + x_0 b_2(w) + b_3(w)$. Since ϕ_X vanishes along $X = B = 0$, b_2 vanishes identically, so the substitution $x_2' = x_2 + \frac{1}{2} b_1(w, x_0)$ reduces a_3 to b_3, independent of x_0. That f is now invariant under η (so we can take $D = 2$ and all $C_i = 0$) follows by inspection. We could also infer the singularities from the semi-simple group action.

If Case (a) (V_3 is a cone) occurs, a_1 cannot be x_0 (else V_3 would be non-singular), so can be taken as w_1. We must distinguish according as the vertex of the cone does or does not lie on $w_1 = 0$.

In Case (a1) it does not, so the intersection V_4 of the cone with $w_1 = 0$ is non-singular, hence so is V_2. The description of the singularities now follows from Propositions 5.3 and 5.6, or again from the group action.

After adjusting the w co-ordinates, we may suppose $a_3(w, 0)$ independent of w_1. Then we can write $a_3 = b_3 + x_0 w_1 b_1(w, x_0)$, with b_3 independent of w_1. Again the substitution $x_2' = x_2 + \frac{1}{2} b_1(w, x_0)$ reduces a_3 to b_3. Now by inspection, $\eta f = 0$ (so we may take $C_1 = 2$, $C_i = 0$ for $i \neq 1$).

In case (a2), we may suppose $a_3(w, 0)$ independent of w_2, and hence that $a_3 = b_3 + x_0 w_2 b_1(w, x_0)$, with b_3 independent of w_2. Now if the coefficient of w_2 in b_1 were zero, the point where all co-ordinates except w_2 vanish would be singular on V_3^*. Hence we can write $b_1 = c^2 w_2 + c_1$, and substitute $w_2' = c w_2 + \frac{1}{2} c^{-1} c_1$, which reduces a_3 to the form $b_3' + x_0 w_2^2$, with b_3' independent of w_2. Thus $\eta f = 0$, where $\eta = w_2 \partial/\partial x_2 - w_1 \partial/\partial w_2$ (so we may take $C_2 = 2$, $C_i = 0$ for $i \neq 2$).

To describe the singularity, as in Proposition 5.6, it suffices to consider the case $m = 2$. Here since V_2 and V_3 are cones with the same vertex, we must select $T_{2,6,6}$ from the above list. $\square$

6 Case 3

We define the map $\pi : K \to L$ by $\pi(w, x_0, x_1, x_2, x_3) = (W, X, B, \Delta)$ in the notation of Lemma 4.1. This induces $\overline{\pi} : P(K) \to P(L)$, where $P(L)$ is the weighted projective space with all weights 1 except $w(B) = 2, w(\Delta) = 4$; the map $\overline{\pi}$ is defined except on the set $\mathcal{E}$ where all co-ordinates except x_2 and x_3 vanish: thus $\mathcal{E}$ is a projective line and $\chi(\mathcal{E}) = 2$; it contains the point P where all co-ordinates except x_3 vanish. We define strata by

$$\begin{array}{llll} \mathcal{S}_0 : & x_0 \neq 0; & \overline{\mathcal{S}}_0 : & X \neq 0 \\ \mathcal{S}_1 : & x_0 = 0,\ x_1 \neq 0; & \overline{\mathcal{S}}_1 : & X = 0,\ B \neq 0 \\ \mathcal{S}_2 : & x_0 = x_1 = 0; & \overline{\mathcal{S}}_2 : & X = B = 0. \end{array}$$

The set $\mathcal{F}$ of fixed points is given by the vanishing of x_0, x_1, x_2. Each orbit of the action of G_N on $K \setminus \mathcal{F}$ (or on $P(K) \setminus \mathcal{F}$) is isomorphic to an affine line; their degrees are 3,2,1 for $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2 \setminus \mathcal{F}$ respectively.

We now describe the pre-image under π of any $(W, X, B, \Delta) \in L$. In each case, $w = W$.

($\mathcal{S}_0$) if $X \neq 0$, $x_0 = X$, x_1 is free, $x_2 = (B + x_1^2)/2X$, $x_3 = (C + 3Xx_1x_2 - x_1^3)/3X^2$ where $C = \pm\surd(-X^2\Delta - B^3)$.

($\mathcal{S}_1$) if $X = 0, B \neq 0$, $x_1 = \pm\surd(-B)$, x_2 is free, and $x_3 = (3x_1^2x_2^2 - \Delta)/6x_1^3$.

($\mathcal{S}_2$) if $X = B = 0$: if $\Delta \neq 0$, the pre-image is empty; if $\Delta = 0$, $x_1 = 0$ and x_2, x_3 are arbitrary. If $x_2 \neq 0$ we have a non-trivial orbit; if $x_2 = 0$ we have fixed points.

From this we infer (again with some care)

Lemma 6.1. *The preimage $\overline{\pi}^{-1}(W, X, B, \Delta)$ is as follows:*

($\mathcal{S}_0$) if $B^3 + X^2\Delta = 0$, one orbit, $\chi = 1$, if not, two orbits, $\chi = 2$,

($\mathcal{S}_1$) if $W = 0$, one orbit, $\chi = 1$, if not, two orbits, $\chi = 2$,

($\mathcal{S}_2$) if $\Delta = 0$, a plane, $\chi = 1$, if not, the empty set, $\chi = 0$.

A priori the map f need not factor through π. However, we have

Lemma 6.2. *There is an allowable change of co-ordinates which puts f in the form $f = \phi \circ \pi$. More precisely, we may take $\phi = \Delta + a_0B^2 + a_2B + a_4$, where a_i is homogeneous of degree i in W, X.*

Proof. Since $d = 4$, we can write $f = a_0'\Delta + a_0B^2 + a_1C + a_2B + a_4$, where a_i is homogeneous of degree i in the invariant co-ordinates $w_1, \ldots, w_m, x_0$.

By Corollary 2.5, for V to be quasi-smooth, f must contain the monomial $x_2^3x_0$; so we must have $a_0' \neq 0$. We may thus suppose $a_0' = 1$. Substituting $x_3' := x_3 - \frac{1}{6}a_1(w, x)$ gives an expression of the same form but with $a_1 = 0$. This gives $f = \Delta + a_0B^2 + a_2B + a_4$, which is indeed of the form $\phi \circ \pi$. □

Denote by V the hypersurface $f = 0$ in $P(K)$, by V_0 the hypersurface $\phi = 0$ in $P(L)$, by V_1 its intersection with $X = 0$, and by V_3 its intersection with $X = B = \Delta = 0$. Write also V_1^* and V_2^* for the respective intersections of V_0 and V_1 with $B^3 + X^2\Delta = 0$. Write L' for the vector space with co-ordinates W, X, Z (all of degree 1), $P(L')$ for the corresponding projective space, $\psi_1(W, X, Z) := -XZ^3 + a_0X^2Z^2 + a_2XZ + a_4 = 0$, V_1' for the hypersurface defined by ψ_1 in $P(L')$, and V_2' for its intersection with $X = 0$.

Lemma 6.3. *The singular points of V are isolated iff the hypersurfaces V_1', V_3 are both non-singular; and then the only singular point of V is P.*

Proof. By Lemma 6.2, we can take $f = \Delta + a_0B^2 + a_2B + a_4$. Since $\partial f/\partial x_3 = \partial\Delta/\partial x_3 = -6C$, C vanishes at all critical points of f.

First consider critical points of f in $x_0 \neq 0$. Since each such point lies in a non-trivial orbit, and f is invariant, it follows that if f has isolated critical points, there can be none with $x_0 \neq 0$. Now in this region, the critical points of f are the same as those of $x_0^2 f$, which is equal to $-C^2 - B^3 + x_0^2(a_0B^2 + a_2B + a_4)$. These coincide with the critical points of $\phi_0 := -B^3 + x_0^2(a_0B^2 + a_2B + a_4)$ lying in $C = 0$. Now regard ϕ_0 as a function ψ_0 of the variables W, X, B. If this has a critical point with $X \neq 0$, we certainly have a critical point of ϕ_0. Conversely, if we have a critical point of ϕ_0, we have $0 = \partial\phi_0/\partial x_2 = 2x_0\partial\psi_0/\partial B$, so $0 = \partial\psi_0/\partial B$, and in view of this, $\partial\psi_0/\partial X = \partial\phi_0/\partial x_0$ vanishes, and so do the $\partial\psi_0/\partial W_i$; so we have a critical point of ψ_0. Finally, in $X \neq 0$ we may make the substitution $Z := X^{-1}B$. The critical points correspond, and since $\psi_0(W, X, XZ) = X^2\psi_1(W, X, Z)$, they correspond to those of ψ_1. Thus f has no critical points in $x_0 \neq 0$ if and only if ψ_1 has none in $X \neq 0$.

Now consider critical points of f with $x_0 = 0$. As $\partial f/\partial x_3 = -6C$, and C reduces to $-x_1^3$, we must also have $x_1 = 0$, hence $B = C = \Delta = 0$. It follows that $\partial f/\partial x_2 = \partial f/\partial x_1 = 0$. There remain the conditions

$$0 = \partial f/\partial w_i = \partial a_4/\partial w_i, \quad 0 = \partial f/\partial x_0 = \partial a_4/\partial x_0 + 2x_2 a_2 - 8x_2^3.$$

If we have a singular point of V_3, there is only one further equation to determine both x_2 and x_3 so we have a non-isolated singularity of $f = 0$. Thus (ii) is a necessary condition for f to have isolated singularities. If it holds, then for any critical points in $x_0 = x_1 = 0$ we have $w = 0$, and now the remaining equation implies $x_2 = 0$, giving the unique critical point P.

It remains to consider singular points of V_1' lying in $X = 0$. Here $\partial\psi_1/\partial Z$ vanishes, $\partial\psi_1/\partial X = -Z^3 + a_2 Z + \partial a_4/\partial X$, and $\partial\psi_1/\partial W_i = \partial a_4/\partial W_i$. Since we are now assuming that V_3 has no singular points, the vanishing of the $\partial\psi_1/\partial W_i$ implies $W = 0$ and that of $\partial\psi_1/\partial X$ then gives $Z = 0$, showing that there are indeed no such singular points. □

Theorem 6.4. *For V quasi-smooth in Case 3, $\mu(V) = 22.3^m$.*

Proof. As before, we calculate $\chi(V)$ by decomposing $\overline{\pi}(V)$ according to the stratification, calculating the Euler characteristic of each piece, inferring those of the pre-images, and adding up.

In $\overline{\mathcal{S}}_2$, Δ vanishes on the image of π and f reduces to $a_4(W, 0)$. The zero locus is thus the hypersurface V_3. Hence $\chi(V \cap \mathcal{S}_2) = \chi(V_3) + \chi(\mathcal{E}) = \chi_{m-2}(4) + 2$.

In $\overline{\mathcal{S}}_1$, we can assign W and B and solve $\Delta = -(a_0 B^2 + a_2 B + a_4)$. The set where $W = 0$ is a single point, so contributes $\chi = 1$, and the set $W \neq 0$ is the product of the punctured B-plane and the punctured W space, so has $\chi = 0$. Hence $\chi(V \cap \mathcal{S}_1) = 1$.

In $\overline{\mathcal{S}}_0$, we can normalise co-ordinates by $X := 1$. Projecting $V_0 \setminus V_1$ onto (W, B) space is an isomorphism, since $\Delta = -(a_0 B^2 + a_2 B + a_4)$ on V_0. Thus $\chi(V_0 \setminus V_1) = 1$. Restricting to the subset where $B^3 + \Delta = 0$ we obtain an isomorphism of $V_1^* \setminus V_2^*$ onto the set of (W, B) where $B^3 = a_0 B^2 + a_2 B + a_4$, which we can identify in turn (replacing B by Z) with the subset $V_1' \setminus V_2'$ of V_1' with $X = 1$. Thus $\chi(V_1^*) - \chi(V_2^*) = \chi(V_1') - \chi(V_2')$. But V_1' is non-singular, so $\chi(V_1') = \chi_m(4)$, and V_2' is the cone on V_3, so $\chi(V_2')$ is equal to $\chi(V_3) + 1 = \chi_{m-2}(4) + 1$. We thus have

$$\chi(V_1^* \setminus V_2^*) = \chi_m(4) - \chi_{m-2}(4) - 1$$

and hence $\chi(V \cap \mathcal{S}_0)$ is twice $\chi(V_0 \setminus V_1)$ minus this, i.e. $3 - \chi_m(4) + \chi_{m-2}(4)$.

Adding these up, we find $\chi(V) = -\chi_m(4) + 2\chi_{m-2}(4) + 6$. Now V has just one singular point and $\dim V = m + 2$, so $\chi(V) = \chi_{m+2}(4) + (-1)^{m-1}\mu(V)$. Thus finally $\mu(V) = (-1)^m\{\chi_{m+2}(4) + \chi_m(4) - 2\chi_{m-2}(4) - 6\}$, which reduces to 22.3^m. □

We can also calculate τ.

Proposition 6.5. *Any f in Case 3 is oversymmetric. Hence $\tau_{tot}(V) = 3^{m+1}.7$.*

Proof. We can use essentially the same formula as in [4, Proposition 6.3]. We have $f = \Delta + a_0 B^2 + a_2 B + a_4$, with a_i homogeneous of degree i in x_0 and the w_i. Set $\eta' := x_1^2 \partial/\partial x_1 + (3x_1x_2 - 3x_0x_3)\partial/\partial x_2 + (4x_2^2 - 3x_1x_3)\partial/\partial x_3$. Then $\eta'\Delta = 0$ and $\partial\Delta/\partial x_3 = 3\eta' B$. Hence f is annihilated by $\eta' - \frac{1}{3}(2a_0B + a_2)\partial/\partial x_3$. □

We contrast $\mu(V) = 22.3^m$ with $\tau(V) = 7.3^{m+1} = 21.3^m$. The values $\mu = 22, \tau = 21$ were obtained in [4, Prop 6.3] for the case $m = 0$.

Lemma 6.6. *Suppose V in Case 3. Then the singularity of V is semi-quasi-homogeneous of degree* 12 *in variables of weights* 1, 4, 6 *and* 3 *(m times).*

Proof. Recall that $\Delta := -9x_0^2x_3^2 + 18x_0x_1x_2x_3 - 8x_0x_2^3 + 3x_1^2x_2^2 - 6x_1^3x_3$. When $x_3 = 1$, we can rewrite this as $\Delta = -(3x_0 - 3x_1x_2 + \frac{4}{3}x_2^3)^2 + 6(\frac{2}{3}x_2^2 - x_1)^3$. This suggests setting $q := x_0 - x_1x_2 + \frac{4}{9}x_2^3$, $p := x_1 - \frac{2}{3}x_2^2$, so we substitute $x_2 := 3y$, $x_1 := p + 6y^2$, $x_0 := q + 3py + 6y^3$. This gives $\Delta = -9q^2 - 6p^3$, $B = 6qy - p^2 + 6py^2$ and so $f = -9q^2 - 6p^3 + a_0(6qy - p^2 + 6py^2)^2 + a_2(6qy - p^2 + 6py^2) + a_4$, where a_i is homogeneous of degree i in $w_1, \dots, w_r, q + 3py + 6y^3$.

Now assign weight 1 to y, 3 to each w_i, 4 to p and 6 to q. The term of least weight in x_0 is $6y^3$, of weight 3; the term of least weight in B is $6py^2$, of weight 6. Hence each term in f has weight at least 12, and the terms of degree 12 give the sum of a term $-9q^2$, which we can ignore, and $g := -6p^3 + 36a_0p^2y^4 + 6a_2py^2 + a_4$, where a_i is homogeneous of degree i in $w_1, \dots, w_r, 6y^3$. It remains to show that g has an isolated singularity.

We compare g with the function $\psi_1(W, X, Z) := -XZ^3 + a_0X^2Z^2 + a_2XZ + a_4$, and observe that formally $g(w, p, y) = \psi_1(w, 6y^3, py^{-1})$. Since by Lemma 6.3, the hypersurface V_1' defined by $\psi_1 = 0$ is non-singular, g has no singular points with $y \neq 0$. But if $y = 0$, the condition $\partial g/\partial p = 0$ forces $p = 0$; and the restriction to $p = y = 0$ defines the hypersurface V_3 which, by the same result, is also non-singular. Hence indeed g has an isolated singularity, and the result follows. □

7 Case 4

Here we define $\pi : K \to L$ by $\pi(w, x_0, x_1, x_2, x_3, x_4) = (W, X, B, U, E)$ in the notation of Lemma 4.1. The induced map $\overline{\pi} : P(K) \to P(L)$ is defined except on the set $\mathcal{E}$ where all co-ordinates except x_3 and x_4 vanish: $\mathcal{E}$ is a projective line containing the point P where all co-ordinates except x_4 vanish and one other orbit, and $\chi(\mathcal{E}) = 2$. We define strata by

i	0	1	2	3
$\mathcal{S}_i$	$x_0 \neq 0$	$x_0 = 0,\ x_1 \neq 0$	$x_0 = x_1 = 0,\ x_2 \neq 0$	$x_0 = x_1 = x_2 = 0$
$\overline{\mathcal{S}}_i$	$X \neq 0$	$X = 0,\ B \neq 0$	$X = B = 0,\ U \neq 0$	$X = B = U = 0.$

The set $\mathcal{F}$ of fixed points is given by the vanishing of x_0, x_1, x_2, x_3. Each orbit of the action of G_N on $K \setminus \mathcal{F}$ (or on $P(K) \setminus \mathcal{F}$) is isomorphic to an affine line; their degrees are 4,3,2,1 for $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \setminus \mathcal{F}$ respectively. The closure of each orbit in $P(K) \setminus \mathcal{F}$ is obtained by adjoining the point P.

We now describe the pre-image under π of any $(W, X, B, U, E) \in L$. In each case, $w = W$ and $x_0 = X$.

($\mathcal{S}_0$) x_1 is free, $x_2 = (B + x_1^2)/2X$, $x_3 = (C + 3Xx_1x_2 - x_1^3)/3X^2$ where $C = \pm\sqrt{(}3X^2BU - X^3E - B^3)$, and $x_4 = (U + 2x_1x_3 - 2x_2^2)/2X$;

($\mathcal{S}_1$) $x_1 = \pm\sqrt{(}-B)$, x_2 is free, $x_3 = (x_2^2 - U)/2x_1$, and $x_4 = (6x_1x_2x_3 - 2x_2^3 - E)/6x_1^2$;

($\mathcal{S}_2$) if $E^2 \neq 4U^3$ the pre-image is empty; otherwise, $x_1 = 0$, $x_2 = -E/2U$, x_3 and x_4 are free;

($\mathcal{S}_3$) if $E \neq 0$ the pre-image is empty; otherwise, $x_1 = x_2 = 0$, x_3 and x_4 are free (if $x_3 \neq 0$ we have a non-trivial orbit but if $x_3 = 0$ we have fixed points).

From this we infer pre-images under $\overline{\pi}$.

Lemma 7.1. *For $(W, X, B, U, E) \in \overline{\mathcal{S}}_i$, the value $\chi(\overline{\pi}^{-1}(W, X, B, U, E))$ is given by:*

i	*Condition*	ψ_i	$\chi(if\ \psi_i = 0)$	$\chi(if\ \psi_i \neq 0)$
0	$X \neq 0$	$B^3 + X^3E - 3X^2BU$	1	2
1	$X = 0,\ B \neq 0$	W	1	2
2	$X, B = 0,\ U \neq 0$	$4U^3 - E^2$	1	0
3	$X,\ B,\ U = 0$	E	1	0

Lemma 7.2. *Suppose f, invariant under the group, defines a hypersurface V with isolated singularities. Then there is an allowable change of co-ordinates which puts f in the form $f = E + 3a_1U + a_3$, where a_i is homogeneous of degree i in the invariant co-ordinates.*

Proof. We have $d = 3$, and so can write $f = b_0C + a_0E + b_1B + 3a_1U + a_3$, where a_i, b_i are homogeneous of degree i in w, x_0. By Corollary 2.5, for V to be quasi-smooth, f must contain the monomial $x_3^2x_0$, so we must have $a_0 \neq 0$. We may thus take $a_0 = 1$. Now substitute $x_4 = x_4' + \frac{b_0}{6}x_1 - \frac{b_1}{6}$ and $x_4 = x_3' + \frac{b_0}{6}x_0$. This reduces b_0 and b_1 to 0 at the expense of adding terms to a_3. We thus have $f = E + 3a_1U + a_3$, of the desired form. □

It will be convenient to write a_3^* for $a_1^3 + a_3$, and to give names to varieties as follows. We define $V_2 \subset P^m$ by $a_3(W, X) = 0$, V_2^* by $a_3^*(W, X) = 0$, their respective intersections with $X = 0$ by V_3, V_3^*, and $V_3 \cap V_3^*$ by V_4. In weighted projective space with coordinates (W, X, Z, U) (where U has weight 2) write $\psi_1 := -Z^3 + 3UZ + 3a_1(W, X)U + a_3(W, X)$; denote the hypersurface $\psi_1 = 0$ by V_0, and its intersection with $X = 0$ by V_1.

Lemma 7.3. *Suppose $f = E + 3a_1U + a_3$ as above. Then the singular points of V are isolated iff*

(i) V_0, or equivalently V_2^* is non-singular,

(ii) V_1, or equivalently V_3^* is non-singular, and

(iii) V_3 has isolated singular points.

The singular points of f are then P and points P_i corresponding to the singular points Q_i of V_3.

Proof. Since $x_1\partial f/\partial x_4 + x_0\partial f/\partial x_3 = x_1\partial E/\partial x_4 + x_0\partial E/\partial x_3 = -6C$, C again vanishes at all critical points of f.

Each singular point of V in $x_0 \neq 0$ lies in a non-trivial orbit, and V is invariant, so V can have no singular point, and f can have no critical point with $x_0 \neq 0$. In this region, the critical points of f are the same as those of $x_0^3 f$, which is equal to $3x_0^2BU - B^3 - C^2 + x_0^3(3a_1U + a_3)$. These coincide with the critical points of $\phi_0 := 3x_0^2BU - B^3 + x_0^3(3a_1U + a_3)$ lying in $C = 0$. Now regard ϕ_0 as a function ψ_0 of the variables W, X, B, U. If ψ_0 has a critical point with $X \neq 0$, we certainly have a critical point of ϕ_0. Conversely, if we have a critical point of ϕ_0, set

$$w_i := W_i, \quad x_0 := X, \quad x_1 := t, \quad x_2 := (B + t^2)/2X,$$
$$x_3 := (3x_0x_1x_2 - x_1^3)/3x_0^2, \quad x_4 := (U + 2x_1x_3 - x_2^2)/2x_0,$$

(so $C = 0$). Then $0 = \partial\phi_0/\partial x_4 = 2x_0\partial\psi_0/\partial U$, so $0 = \partial\psi_0/\partial U$, and $0 = \partial\phi_0/\partial x_2 = 2x_0\partial\psi_0/\partial B + 2x_2\partial\psi_0/\partial U$, so $0 = \partial\psi_0/\partial B$, and hence again, $\partial\psi_0/\partial X = \partial\phi_0/\partial x_0$ and the $\partial\psi_0/\partial W_i$ all vanish; so we have a critical point of ψ_0.

In $X \neq 0$ we may make the substitution $Z := X^{-1}B$, then $\psi_0(W, X, XZ) = X^3\psi_1(W, X, Z)$. Then the critical points of ψ_0 correspond to those of ψ_1. Thus f has no critical points in $x_0 \neq 0$ if and only if ψ_1 has none in $X \neq 0$; equivalently, $V_0 \setminus V_1$ is non-singular.

For a critical point of ψ_1, $0 = \partial\psi_1/\partial U = 3(Z + a_1)$, so $Z = -a_1$ and $0 = \partial\psi_1/\partial Z = 3U - 3Z^2$, so $U = Z^2 = a_1^2$. With this value of U, the partial derivatives of ψ_1 with respect to X and the W_i coincide with those of $a_1^3 + a_3$. Thus f has no critical point in $x_0 \neq 0$ if and only if a_3^* has none in $X \neq 0$.

For singular points on $x_0 = 0$, we have $0 = \partial f/\partial x_4 = -6x_1^2$, so $x_1 = 0$ also, and hence $U = x_2^2$, $E = -2x_2^3$, so f reduces to $-2x_2^3 + 3x_2^2a_1(w, 0) + a_3(w, 0)$. We now have $\partial f/\partial x_4 = \partial f/\partial x_3 = 0$, and

$$\partial f/\partial x_2 = 6x_2(a_1(w, 0) - x_2), \quad \partial f/\partial x_1 = -6x_3(a_1(w, 0) - x_2),$$
$$\partial f/\partial x_0 = 12x_2x_4 - 9x_3^2 + 6x_4a_1(w, 0) + 3x_2^2\partial a_1/\partial x_0 + \partial a_3/\partial x_0(w, 0).$$

If $x_2 = a_1$, then $\partial f/\partial w_i = \partial(a_1^3 + a_3)/\partial w_i = \partial a_3^*/\partial w_i$. If the restriction of a_3^* to $x_0 = 0$ has a critical point, we can assign this value to the w_i, set $x_2 = a_1$, and then only have one further equation in x_3 and x_4: thus f has non-isolated critical points. Thus condition (ii) is necessary. If it holds, then if $x_2 = a_1$ we have $w = 0$, hence in turn $x_2 = x_3 = 0$ and we have the unique critical point P_4. Note also that (ii) implies that a_3^* has no critical point on $X = 0$, thus completing the proof of the necessity of (i).

For a critical point of f with $x_2 \neq a_1$, we must have $x_2 = x_3 = 0$. Then $\partial f/\partial w_i$ reduces to $\partial a_3/\partial w_i$ and $\partial f/\partial x_0$ to $6x_4a_1 + \partial a_3/\partial x_0$. Thus we have a critical point of the restriction of a_3 to $x_0 = 0$; since $a_1 \neq x_2 = 0$, each such critical point yields a unique value of x_4 and hence critical point of f. □

Theorem 7.4. *For V quasi-smooth in Case 4, we have $\mu(V) = 11.2^m + \mu(V_3)$.*

Proof. As before, we calculate the $\chi(V \cap \mathcal{S}_i)$ using Lemma 7.1.

For $\mathcal{S}_3$ we only have to consider $a_3(w, 0) = 0$, which defines V_3. Hence $\chi(V \cap \mathcal{S}_3) = \chi(V_3) + \chi(\mathcal{E})$.

We know that $\chi(V \cap \mathcal{S}_2)$ is equal to the Euler characteristic of the set of (W, U, E) with $\phi(W, 0, 0, U, E) = 0$, $4U^3 = E^2$ and $U \neq 0$. Since we can solve $\phi = 0$ for E, it suffices to consider the set of (W, U) where $4U^3 = (3a_1 U + a_3)^2$ and $U \neq 0$. We cannot have $W = 0$ here, as this would imply $U = 0$. We can thus project on the space P^{m-1} with co-ordinates W. The fibre consists of the roots of the cubic equation in U, which has discriminant $16a_3^3 a_3^*$.

In the following table, the first column defines the subset of P^{m-1}, the second gives its Euler characteristic, the third is the number of points in the fibre with $U \neq 0$, and the fourth the contribution to

$$\chi(\{(W, U) \mid 4U^3 = (3a_1 U + a_3)^2,\ U \neq 0\}.$$

$P^{m-1} \setminus (V_3 \cup V_3^*)$	$m - \chi(V_3) - \chi(V_3^*) + \chi(V_4)$	3	$3m - 3\chi(V_3) - 3\chi(V_3^*) + 3\chi(V_4)$
$V_3 \setminus V_3^*$	$\chi(V_3) - \chi(V_4)$	1	$\chi(V_3) - \chi(V_4)$
$V_3^* \setminus V_3$	$\chi(V_3^*) - \chi(V_4)$	2	$2\chi(V_3^*) - 2\chi(V_4)$
V_4	$\chi(V_4)$	0	0

Hence $\chi(V \cap \mathcal{S}_2) = 3m - 2\chi(V_3) - \chi(V_3^*)$.

In $\mathcal{S}_1$ $(X = 0, B \neq 0)$ we can again assign W, B and U and solve for E. Since the other conditions are independent of B, which runs through $\mathbb{C}^*$, we have $\chi = 0$ in each case, except when $W = U = 0$ which leads to the unique point with $E = 0$ also, and hence to $\chi(V \cap \mathcal{S}_1) = 1$.

Finally, for $\mathcal{S}_0$, while we again solve uniquely for E, so that $\chi(\phi^{-1}(0) \cap \overline{\mathcal{S}}_0) = 1$, we have to distinguish according as $B^3 + X^3E - 3X^2BU = 0$ or not, hence according as $0 = B^3 - 3X^2BU - X^3(3a_1U + a_3)$. As before, since here X is non-zero, we can replace B by $Z = B/X$, so obtain $0 = Z^3 - 3ZU - (3a_1U + a_3)$, giving V_0. By Lemma 7.3, both V_0 and its intersection V_1 with $X = 0$ are non-singular. Applying again Lemma 7.1, we obtain, since $\chi(\overline{\mathcal{S}}_0) = 1$,

$$\chi(V \cap \mathcal{S}_0) = (\chi(V_0) - \chi(V_1)) + 2(1 - \chi(V_0) + \chi(V_1)) = 2 - \chi(V_0) + \chi(V_1).$$

Recall that by Corollary 4.4, if H^n is a smooth hypersurface of dimension n where one of the weights is 2, then $\chi(H^n) = n + 2 + \frac{1}{3}\{(-2)^n - 1\}$. Since V_0, V_1 have respective dimensions $m + 1, m$,

$$\chi(V_0) - \chi(V_1) = 1 + \tfrac{1}{3}\{(-2)^{m+1} - (-2)^m\} = 1 - (-2)^m.$$

Adding up, $\chi(V)$ is equal to

$$\chi(V_3) + \chi(\mathcal{E}) + 3m - 2\chi(V_3) - \chi(V_3^*) + 1 + 2 - (1 - (-2)^m),$$

and to $\chi_{m+3}(3) + (-1)^m\mu(V)$. We can substitute $\chi(\mathcal{E}) = 2$, $\chi(V_3^*) = \chi_{m-2}(3)$ and $\chi(V_3) = \chi_{m-2}(3) + (-1)^{m-1}\mu(V_3)$, so that

$$\mu(V) - \mu(V_3) = (-1)^{m-1}(\chi_{m+3}(3) + 2\chi_{m-2}(3) - 3m - 5) + (2^m - (-1)^m).$$

Substituting $\chi_n(3) = \frac{(-1)^n}{3}(2^{n+2} - (-1)^{n+2}) + n + 2$, this reduces to 11.2^m. $\square$

As before, we can determine the singularities.

Proposition 7.5. *Suppose V in Case 4. Then the singularity of V corresponding to a singularity of V_3 is right-equivalent to a suspension of that singularity.*

Proof. Suppose Q_i a singular point of V_3. Then a_1 cannot be a multiple of x_0, for otherwise $V_3 = V_3^*$ would be non-singular. We may thus set $w_1 = a_1$. By the arguments above, a_1 does not vanish at Q_i. We may thus work in affine co-ordinates $w_1 = 1$.

Now apply Lemma 3.4(a) to f with the variables $(x_0, x_1, x_2, x_3, x_4)$. We observe that $\partial f/\partial x_0$, $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$, $\partial f/\partial x_4$ all vanish when $x_0 = x_1 = x_2 = x_3 = 0$ and $x_4 = -\frac{1}{2}\partial a_3/\partial x_0$. Substituting these values gives $g(w) = a_3(w, 0, 0, 0)$. The result follows. $\square$

Lemma 7.6. *Suppose V in Case 4. Then the singularity of V at P is semi-quasi-homogeneous of degree* 12 *in variables of weights* 1, 6, 6, 6 *and* 4 *(m times).*

Proof. If we substitute $x_4 = 1$, $x_3 = 2z$, $x_2 = y + 3z^2$, and $x_1 = x + 3yz + 3z^3$, we obtain $E = 6Uy - 8y^3 - 6x^2$; thus $f = -6x^2 + 6Uy - 8y^3 + 3Ua_1 + a_3$.

We also obtain $U = 2x_0 - 4xz - 6yz^2 - 3z^4 + y^2$. Substitute $x_0 = \frac{1}{2}U + 2xz - \frac{1}{2}y^2 + 3yz^2 + \frac{3}{2}z^4$ in f, and assign weights 1 to z, 4 to y and to the w_i, 6 to x and 8 to U. Then in the expression for x_0, all terms have weight > 4 except for $\frac{3}{2}z^4$, of weight 4. Hence all terms in f have weight at least 12, and those of exactly this weight are obtained by substituting $\frac{3}{2}z^4$ for x_0 in a_1 and a_3.

It thus remains only to show that the result of this substitution has an isolated singularity. Here we can ignore the summand $-6x^2$; the rest is obtained from $\psi_1(W, X, Z, U)$ by the substitution $Z = 2y$, $X = \frac{3}{2}z^4$. But by Lemma 7.3, the hypersurface V_0 given by $\psi_1 = 0$ is non-singular. The result follows. $\square$

We observe that we also have $f = -6x^2 + (U - \frac{4}{3}y^2 + \frac{2}{3}a_1 y - a_1^2)(2y + a_1) + a_1^3 + a_3$, which we can write as $-6x^2 + 6U'y' + a_3^*(w, x_0)$, though in view of the substitution $x_0 = \frac{1}{2}U' + 2zx + 3z^2y' - \frac{3}{2}z^2a_1 + \frac{3}{2}z^4 + \frac{1}{6}y'^2 - \frac{1}{2}y'a_1 + \frac{3}{8}a_1^2$ we must make for x_0, the simplicity of this form is misleading.

In certain cases, we can also determine τ.

Proposition 7.7. *Suppose f, in the normal form for Case 4, satisfies also*

(i) a_1 is a multiple of x_0, and

(ii) $\partial a_3/\partial x_0$ vanishes when $x_0 = 0$.

Then the singularity of f at P is quasi-homogeneous, so $\tau_P(V) = \mu_P(V) = 11.2^m$.

Proof. Write f as $f = E + ax_0U + ux_0^2 + cx_0^3 + C(w)$, where u is a non-zero linear combination of the w_i. We now define a number of vector fields. In the table, the left column gives the name, the next defines the field, and the last gives its effect on f. Here ∂_i denotes $\partial/\partial x_i$ and $R = (\partial_4 E\partial_0 - \partial_3 E\partial_1 + \partial_2 E\partial_2 - \partial_1 E\partial_3 + \partial_0 E\partial_4)/6$.

H	$-2x_0\partial_0 - x_1\partial_1 + x_3\partial_3 + 2x_4\partial_4$	$-2ax_0U - 2x_0^2(2u + 3cx_0)$
M	$x_1\partial_1 + (3/2)x_2\partial_2 + (3/2)x_3\partial_3 + x_4\partial_4$	$ax_1U + 2x_0x_1(2u + 3cx_0)$
P	$x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4$	$3E + 3ax_0U + x_0^2(2u + 3cx_0)$
Q	$x_0\partial_2 + x_1\partial_3 + x_2\partial_4$	$2x_0(3U + aB)$
R		$6U^2 + a(x_0E + BU) + x_0B(2u + 3cx_0)$
S	$(\partial_4 E\partial_2 - \partial_3 E\partial_3 + \partial_2 E\partial_4)/12$	$3x_0E - 3BU + ax_0^2U$

Linear combinations of these give

Y_1	$2x_4H + x_3M$	$-(4x_0x_4 - x_1x_3)(aU + x_0(2u + 3cx_0))$
Y_2	$-x_1M + U\partial_2 + R$	$ax_0E + 6(4x_0x_4 - x_1x_3)U$
		$+ 2aBU + 2x_0(x_0x_2 - x_1^2)(2u + 3cx_0)$
Y_3	$-5S + 4x_0P - \frac{7}{2}U\partial_4$	$-3x_0E - 6BU + 4x_0^3(2u + 3cx_0)$
Y_4	$3Q + 3x_0\partial_2 - 2ax_0\partial_4$	$18x_0(4x_0x_4 - x_1x_3) - 6ax_0(x_0x_2 - x_1^2) - 4a^2x_0^3$

Thus the vector field $Z = Y_1 + \frac{a}{6}Y_2 + \frac{a^2}{18}Y_3 + \frac{2bu+3cx_0}{18}Y_4$ kills f, and at the point P, Z reduces to $2\partial/\partial x_4$. The result now follows by Saito's criterion. □

Condition (i) is invariant under allowed changes of co-ordinates. An invariant version of (ii) is that substituting $x_0 = 0$ in $\partial a_3/\partial x_0$ gives a function in the Jacobian ideal of $a_3(w, 0)$. We believe both these conditions to be necessary for the result.

Since $\tau(V) \leq \mu(V) = 11.2^m < 12.2^m$, no function in Case 4 can be oversymmetric.

8 **Case** 21

First we normalise co-ordinates.

Lemma 8.1. *There is an allowable change of co-ordinates which puts f in the form $f = S + a_1B + a_3$, where a_i is homogeneous of degree i in w, x_0, y_0. Moreover, we may suppose that either $a_1 = x_0$ or $a_1 = w_1$.*

Proof. Here $d = 3$ and f has the form $a_0S + a_1B + b_1T + a_3$, where a_i (and b_i) denotes a homogeneous function of degree i in w, x_0, y_0. It follows from Corollary 2.5 that f must contain the monomial $y_1^2x_0$. Hence we must have $a_0 \neq 0$, and can take $a_0 = 1$. Now substitute $y_1' := y_1 + \frac{1}{2}b_1$ to reduce b_1 to 0 (the extra terms introduced can be absorbed in a_3), and so f to $S + a_1B + a_3$.

If a_1 involves any of the w co-ordinates, we can make a linear substitution among the w's to reduce a_1 to the form $qx_0 + py_0 + w_1$, and then change again to achieve $a_1 \equiv w_1$. Otherwise, we can write $a_1 = 2py_0 + qx_0$ and use the substitution $y_1' = y_1 - px_1$, $y_0' = y_0 - px_0$. This transforms S to $S + (p^2x_0 - 2py_0)B$, so a_1 is changed to $(q - p^2)x_0$. We may thus suppose that either $a_1 \equiv x_0$ or $a_1 \equiv 0$. However, if $a_1 \equiv 0$, V is singular along the plane $w = y_0 = y_1 = 0$. □

Proposition 8.2. *In Case 21, V is always 2-symmetric.*

Proof. The function f is annihilated by the vector fields $\xi = x_0\partial/\partial x_1 + x_1\partial/\partial x_2 + y_0\partial/\partial y_1$ and $\xi' := y_0\partial/\partial x_1 + y_1\partial/\partial x_2 - a_1\partial/\partial y_1$. □

There are significant differences between the two cases. We now rename the case with $a_1 = x_0$ as 21_0, and the case $a_1 = w_1$ as Case 5.

In Case 21_0, the operators $D \pm D'$ each fall into Case 2 (other linear combinations are all in Case 21_0). To see this, substitute

$$x_0 = u_0 + v_0, \quad y_0 = u_0 - v_0, \quad x_1 = u_1 + v_1, \quad y_1 = u_1 - v_1,$$

Then f reduces to $-8x_2u_0v_0+4u_0v_1^2+4v_0u_1^2+a_3$, which we can write as $4u_0(v_1^2-2v_0x_2)$ added to a cubic in w, u_0, v_0 and v_1. Conversely, this is essentially the normal form for Case 2 where the cubic has the form $4v_0u_1^2$ added to a cubic in w, u_0, v_0, and thus can be identified with Case (a2) of Proposition 5.9.

For the remainder of this section we consider only Case 5; here all non-zero linear combinations of D and D' are in Case 5. We re-name a_1 as z_0. In this notation, co-ordinates are $(w_1, \ldots, w_m, x_0, y_0, z_0, x_1, y_1, x_2)$, and we re-group the terms in f as $J + a_3(x_0, y_0, z_0, w)$, where

$$J := 2x_2(y_0^2 - x_0z_0) + x_1^2z_0 - 2x_1y_1y_0 + y_1^2x_0.$$

It is now more natural to treat all the differential operators on the same footing, and consider f as invariant under the 2-dimensional group G whose Lie algebra is spanned by D and D'. We see that J is invariant under G, and it seems very likely that the ring of invariants coincides with the polynomial ring $\mathbb{C}[w, x_0, y_0, z_0, J]$ (clearly it contains this, and we can show that the localisation at $\langle x_0 \rangle$ is correct), but we will not use the precise assertion.

We define a new stratification,
$\mathcal{S}_0$: $y_0^2 - x_0z_0 \neq 0$,
$\mathcal{S}_1$: $y_0^2 - x_0z_0 = 0$ but $(x_0, y_0, z_0) \neq (0, 0, 0)$,
$\mathcal{S}_2$: $(x_0, y_0, z_0) = (0, 0, 0)$.

Lemma 8.3. *A point in $\mathcal{S}_2$ is fixed under G; otherwise the dimension of the orbit is equal to the rank of* $\left(\begin{smallmatrix} x_0 & y_0 & x_1 \\ y_0 & z_0 & y_1 \end{smallmatrix}\right)$.

For the fixed points of $A\xi_0 + B\xi_1 = (Ax_0 + By_0)\partial/\partial x_1 + (Ay_0 + Bz_0)\partial/\partial y_1 + (Ax_1 + By_1)\partial/\partial x_2$ are those where the three coefficients vanish; this holds for some $(A, B) \neq (0, 0)$ if and only if the rank of the matrix drops.

We define a projection $\pi : K \to L$ by $\pi(w, x_0, y_0, z_0, x_1, y_1, x_2) = (w, x_0, y_0, z_0)$, and again write $\overline{\pi} : P(K) \to P(L)$ for the induced map of projective spaces. The exceptional set where $\overline{\pi}$ is undefined is the projective plane where $w = x_0 = y_0 = z_0 = 0$, and the pre-image of any point in the target is isomorphic to affine 3-space. Write $\overline{\pi}_0$ for the restriction of $\overline{\pi}$ to the hypersurface V defined by $f = 0$.

Lemma 8.4. *The fibres of $\overline{\pi}_0$ are as follows:*
in $\overline{\mathcal{S}}_0$ each fibre is a quadric isomorphic to an affine plane,
in $\overline{\mathcal{S}}_1$, one or two affine planes according as $a_3(w, x_0, y_0, z_0) = 0$ or $\neq 0$,
in $\overline{\mathcal{S}}_2$, affine 3-space or the empty set according as $a_3(w, 0, 0, 0) = 0$ or $\neq 0$.

Proof. The assertions are trivial except for $\overline{\mathcal{S}}_1$. Here we may write $(x_0, y_0, z_0) = (t^2, tu, u^2)$ for some t, u not both zero. Then J reduces to $x_1^2z_0 - 2x_1y_1y_0 + y_1^2x_0 = (ux_1 - ty_1)^2$. Thus we have the plane(s) given by $ux_1 - ty_1 = \pm\sqrt{(-a_3)}$. □

We have two hypersurfaces in $P(L)$: the cone $\overline{\mathcal{S}}_1 \cup \overline{\mathcal{S}}_2 = C : y_0^2 - x_0z_0 = 0$ and the variety V_0 defined by $a_3(w, x_0, y_0, z_0) = 0$. Write $V_1 := V_0 \cap C$, and $V_3 := V_0 \cap \overline{\mathcal{S}}_2$ for the variety $a_3(w, 0, 0, 0) = 0$. Also define V_1^* as the variety $\phi(w, t, u) := a_3(w, t^2, tu, u^2) = 0$ in weighted projective space $P(2^m1^2)$.

It follows from the lemma that

$$\chi(V) = 3 + \chi(\overline{\mathcal{S}}_0) + 2\chi(\overline{\mathcal{S}}_1) - \chi(\overline{\mathcal{S}}_1 \cap V_0) + \chi(\overline{\mathcal{S}}_2 \cap V_0). \quad (4)$$

Proposition 8.5. *The variety V has isolated singularities if and only if*

(i) for no (w, x_0, y_0, z_0) in $\mathcal{S}_1$ do we have $\partial a_3/\partial w_i = 0$ for each i, and the matrix A of rank 1, where

$$A := \begin{pmatrix} \partial a_3/\partial x_0 & \partial a_3/\partial y_0 & \partial a_3/\partial z_0 \\ z_0 & -2y_0 & x_0 \end{pmatrix},$$

(ii) for any singular point of V_3 we have

$$(\partial a_3/\partial y_0)^2 \neq 4(\partial a_3/\partial x_0)(\partial a_3/\partial z_0).$$

In particular, singular points of V_3 are isolated.

When this holds, P is the only singular point of V.

Proof. Since $\partial f/\partial x_2 = 2(y_0^2 - x_0 z_0)$, there are no singularities in $\mathcal{S}_0$.

In $\mathcal{S}_1$, again write $(x_0, y_0, z_0) = (t^2, tu, u^2)$ for t, u not both zero. Then

$$\partial f/\partial x_1 = 2u(ux_1 - ty_1), \quad \partial f/\partial y_1 = -2t(ux_1 - ty_1),$$

$\partial f/\partial x_0 = y_1^2 + \partial a_3/\partial x_0$, $\partial f/\partial y_0 = -2x_1y_1 + \partial a_3/\partial y_0$, $\partial f/\partial z_0 = x_1^2 + \partial a_3/\partial z_0$, and $\partial f/\partial w_i = \partial a_3/\partial w_i$. It follows that for a critical point of f, $ux_1 = ty_1$, and hence that the matrix A has rank 1.

Conversely, given (w, x_0, y_0, z_0) in $\mathcal{S}_1$ such that $\partial a_3/\partial w_i = 0$ for each i, and A has rank 1, we can take $(x_0, y_0, z_0) = (t^2, tu, u^2)$, and let the upper row of A equal $-v^2$ times the lower. Then if $x_1 = tv$, $y_1 = uv$ and x_2 is arbitrary, we have a critical point of f: none of these critical points is isolated.

In $\mathcal{S}_2$ we have identically $\partial f/\partial x_2 = \partial f/\partial x_1 = \partial f/\partial y_1 = 0$. For a critical point of f we have a critical point of a_3, and 3 further equations, for x_1^2, x_1y_1 and y_1^2, which are inconsistent unless also $(\partial a_3/\partial y_0)^2 = 4(\partial a_3/\partial x_0)(\partial a_3/\partial z_0)$.

Conversely, if there is a critical point of a_3 at which this identity holds, we can solve for x_1 and y_1 and take an arbitrary value for x_2, again obtaining a non-isolated singularity of f.

It remains to consider the case $w = x_0 = y_0 = z_0 = 0$. Here the only critical point is $x_1 = y_1 = 0$, which is indeed isolated. □

For a singular point of the intersection $V_1 := V_0 \cap C$, Lagrange's multiplier rule tells us that the $\partial a_3/\partial w_i$ vanish and the matrix A has rank at most 1: for singular points with $(x_0, y_0, z_0) \neq (0, 0, 0)$, this condition is necessary and sufficient. Thus (i) is equivalent to the condition that $V_1 \cap \overline{\mathcal{S}}_1$, or equivalently the open set of V_1^* where $(t, u) \neq (0, 0)$, be non-singular.

Now V_1 is always singular along $V_1 \cap \overline{\mathcal{S}}_2$. For V_1^* on the other hand, it is singular at a point on $t = u = 0$ only if $\partial a_3/\partial w_i = 0$ for each i, i.e. at a singular point of V_3. Thus V_1^* is non-singular if and only if V_3 is.

The singularities of V_1^* and V_3 are related as follows. If we expand ϕ as a Taylor series, the second order terms in t and u are $2((\partial a_3/\partial x_0)t^2 + (\partial a_3/\partial y_0)tu + (\partial a_3/\partial z_0)u^2)$, a form which is non-singular if and only if (ii) holds. However, since the ambient weighted projective space is singular at this point, we cannot say that one singularity is the suspension of the other.

Theorem 8.6. *If V_3 is non-singular, $\mu(V) = 25.2^m$. Moreover, the singularity of V is semi-quasi-homogeneous of degree* 6 *in variables of weights* 1, 1, 3, 3, 3 *and* 2 *(m times).*

Proof. Although the first assertion follows from the second, we give an independent proof.

For the stratification of $P(L)$ we have $\chi(\overline{\mathcal{S}}_2) = m$, $\chi(\overline{\mathcal{S}}_1) = 2$ and $\chi(\overline{\mathcal{S}}_0) = 1$, since $\overline{\mathcal{S}}_2$ is a projective space, and forgetting the w_i defines a projection of the others to the projective plane on x_0, y_0, z_0 with contractible fibres. Substituting in (4), and using the notations V_i thus gives $\chi(V) = 8 - \chi(V_1) + 2\chi(V_3)$.

Now $\chi(V) = \chi_{m+4}(3) + (-1)^{m-1}\mu(V)$ and since V_3 is non-singular, $\chi(V_3) = \chi_{m-2}(3)$. Since the natural projection $V_1^* \to V_1$ is bijective, $\chi(V_1) = \chi(V_1^*)$, while by Corollary 3.7, $\chi(V_1^*) = m + 1 + \frac{1}{6}(-1)^m(26.2^m + 4(-1)^m)$.

Putting the above results together, we obtain

$$\begin{aligned}\mu(V) &= \tfrac{1}{3}(2^{m+6} - (-1)^m) + (-1)^m(m+6)\\ &\quad + \tfrac{1}{6}(26.2^m + 4(-1)^m) + (-1)^m(m+1)\\ &\quad - \tfrac{2}{3}((2)^m - (-1)^m) + (-1)^{m-1}2m + 8(-1)^{m-1}\\ &= 25.2^m.\end{aligned}$$

Set $x_2 = 1/2$ and rewrite J as $(y_0 - x_1y_1)^2 - (x_0 - x_1^2)(z_0 - y_1^2)$. Substitute $x' := x_0 - x_1^2$, $y' := y_0 - x_1y_1$ and $z' := z_0 - y_1^2$: then $f = (y'^2 - x'z') + a_3(x' + x_1^2, y' + x_1y_1, z' + y_1^2, w)$. Now assign weights 1 to x_1, y_1, 2 to the w_i and 3 to x', y', z'. Then the terms of weight 6 give $g = (y'^2 - x'z') + a_3(x_1^2, x_1y_1, y_1^2, w)$. Since V_3 is non-singular, so is V_1^*, so g has an isolated singularity, and the result follows. □

If V_1^* is singular, we do not have a formula for its Euler characteristic, so must proceed differently: in fact, we resolve the singularity of C. Define $\widehat{P}$ as the subvariety of $P^1 \times P(L)$, where P^1 has co-ordinates $(t_0 : t_1)$, given by $t_1x_0 = t_0y_0$, $t_1y_0 = t_0z_0$. If $L_0 := \mathcal{S}_2$ denotes the subspace $x_0 = y_0 = z_0 = 0$ of L, then the projection $\widehat{P} \to P(L)$ has image C; it is bijective over $C \setminus P(L_0)$, but over $P(L_0)$ is the projection $P^1 \times P(L_0) \to P(L_0)$.

Define $\widehat{V} \subset \widehat{P}$ to be the subvariety given by $a_3(w, x_0, y_0, z_0) = 0$: thus it is a complete intersection of multi-degree (1,1), (1,1), (0,3). The natural projection $\pi : \widehat{V} \to V_1$ is an isomorphism outside V_3, but a product $P^1 \times V_3$ over it. In particular, $\chi(\widehat{V}) = \chi(V_1) + \chi(V_3)$. By Proposition 8.5(i), $V_1 \setminus V_3$, hence its pre-image, is non-singular. Also, any singularity of $\widehat{V}$ projects to a singular point P_i of V_3.

Lemma 8.7. *Above each singular point P_i of V_3 there are just two singular points of $\widehat{V}$, and the singularity at each is isomorphic to a suspension of the singularity of V_3 at P_i.*

Proof. To study a neighbourhood of P_i, it is convenient to make a linear change of the co-ordinates (x_0, y_0, z_0), preserving the quadratic $y_0^2 - x_0z_0$, so that $\partial a_3/\partial x_0 = \partial a_3/\partial z_0 = 0$, $\partial a_3/\partial y_0 \neq 0$ at P_i (here we use Proposition 8.5(ii)). Then $\widehat{V}$ is non-singular at all points of $\pi^{-1}(P_i)$ except those where $(t_0 : t_1)$ is $(0 : 1)$ or $(1 : 0)$: it suffices to consider the first.

Take affine co-ordinates in $\widehat{V}$ with $t_1 = 1$ and $w_1 = 1$. Then $y_0 = t_0z_0$, $x_0 = t_0y_0 = t_0^2z_0$. Thus a_3 lifts to $\alpha_3 := a_3(1, w_2, \dots, w_r, t_0^2z_0, t_0z_0, z_0)$. At the point $t_0 = z_0 = 0$, the 2-jet has a non-zero coefficient of t_0z_0. We now apply Lemma 3.4(b): this we can do since $\partial\alpha_3/\partial t_0$ is divisible by z_0. It thus follows that we have a suspension of the restriction to $t_0 = z_0 = 0$, which is just the intersection with V_3. □

We can now show

Theorem 8.8. *If V is quasi-smooth in Case 5, then $\mu(V) = 25.2^m + \mu(V_3)$.*

Proof. Since $\widehat{V}$ is a complete intersection with isolated singularities, its Euler characteristic is obtained from that of a smooth complete intersection of the same multidegrees by adding $(-1)^{m-1}$ times the sum of the Milnor numbers.

While we could calculate the default value of $\chi(\widehat{V})$ directly, we can also obtain it from the above calculations in the case when V_3 is non-singular. In the proof of Theorem 8.6 we calculated values of $\chi(V_1)$ and $\chi(V_3)$: denote them for now by c_1 and c_3. Thus $c_1 = m + 1 + \frac{1}{6}(-1)^m(26.2^m + 4(-1)^m)$ and $c_3 == \chi_{m-2}(3)$. In the case V_3 non-singular, we have $\chi(\widehat{V}) = c_1 + c_3$ and $\chi(V) = 8 - c_1 + 2c_3$, whereas $\chi(V) = \chi_{m+4}(3) + (-1)^{m-1}\mu(V)$, leading to $\mu(V) = 25.2^m$.

In the general case, by Lemma 8.7 there are two singular points of $\widehat{V}$ in $\pi^{-1}(P_i)$, each a suspension of the singularity of V_3 at P_i; hence $\mu(\widehat{V}) = 2\mu(V_3)$. Now $\chi(V_3) = c_3 + (-1)^{m-1}\mu(V_3)$. Also, by the remark just made,

$\chi(\widehat{V}) = c_1 + c_3 + (-1)^{m-1}2\mu(V_3)$. As before, we have $\chi(V) = 8 - \chi(V_1) + 2\chi(V_3) = 8 - \chi(\widehat{V}) + 3\chi(V_3)$, which now equals $8 - [c_1 + c_3 + (-1)^{m-1}2\mu(V_3)] + 3[c_3 + (-1)^{m-1}\mu(V_3)]$, i.e. $8 - c_1 + 2c_3 + (-1)^{m-1}\mu(V_3)$. Substituting this value in $\chi(V) = \chi_{m+4}(3) + (-1)^{m-1}\mu(V)$ gives $\mu(V) = 25.2^m + \mu(V_3)$ as desired. □

Observe, however, that unlike the other cases, here there is just one singular point P, and the values $\tau = 25.2^m$ and $\mu = 25.2^m + \mu(V_3)$ both hold for the singularity at this point.

9 2-symmetric cases

Finally we list 2-symmetric hypersurfaces. This is just the subcase $d = 3$ of the list of oversymmetric cases: in the semisimple case, the weights are obtained from one of $[-1, 0, 1]$, $[-2, 1, 2]$ and $[-2, 1, 4]$ by adding zeros; in the unipotent case, the subcases of Case 2 when an additional action exists were analysed in Proposition 5.9: we had three cases (a1), (a2), (b); and Case 21 splits into two subcases: Case 21_0 and Case 5. Since some cases arise more than once by using different 1-parameter subgroups, it is better to give the list separately.

Theorem 9.1. *If f, of degree ≥ 3, such that $f = 0$ is quasi-smooth, is 2-symmetric, then f belongs to one of the following 5 cases (A)–(E).*

(A) $f = x_0x_1x_2 + a_3(x_3, \dots, x_n)$ $(n \geq 2)$, where $a_3 = 0$ is non-singular, with the 2-parameter action $(\lambda, \mu).(x_0, x_1, x_2, \dots, x_n) = (\lambda^{-1}x_0, \mu^{-1}x_1, \lambda\mu x_2, \dots, x_n)$, and 3 singular points, mutually isomorphic. We have 1-parameter subgroups with weights

$[-a, -b, a+b]$ for any a and b. There are 3 singularities, all isomorphic, each with $\mu = 2^{n-2}$ and homogeneous of degree 3 with respect to weights 1 ($n-1$ times) and 2.

(B) $f = x_0x_1^2 + x_0x_2x_3 + a_3(x_3, \dots, x_n)$ $(n \geq 3)$, where $a_3 = 0$ is non-singular, and has non-singular intersection with $x_3 = 0$. This is annihilated by $-2x_0\partial_0 + x_1\partial_1 + 2x_2\partial_2$ with non-zero weights $[-2, 1, 2]$, and by $x_3\partial/\partial x_1 - 2x_1\partial/\partial x_2$, which is in case (a1) of Proposition 5.9. There are two singularities, with Milnor numbers 5.2^{n-3} and 2^{n-3}; both homogeneous of degree 6, the first with respect to weights 1 and 2 ($n-3$ times); the second with respect to weights 3 and 2 ($n-3$ times).

(C) $f = x_0(2x_0x_2 - x_1^2) + a_3(x_3, \dots, x_n)$ $(n \geq 2)$, where $a_3 = 0$ is non-singular. This is annihilated by $-2x_0\partial_0 + x_1\partial_1 + 4x_2\partial_2$ with non-zero weights $[-2, 1, 4]$, and by $x_0\partial/\partial x_1 + x_1\partial/\partial x_2$, which is in case (b) of Proposition 5.9. The singularity has $\mu = 3.2^{n-2}$ and is homogeneous of degree 12 with respect to weights 3,6 and 4 ($n-2$ times).

(D) $f = x_3(2x_0x_2 - x_1^2) + x_0x_4^2 + a_3(x_0, x_3, x_5, \dots, x_n)$ $(n \geq 4)$, with a_3 non-singular. We have vector fields $x_0\partial/\partial x_1 + x_1\partial/\partial x_2$ in case (a2) of Proposition 5.9 and $x_4\partial/\partial x_2 - x_3\partial/\partial x_4$ in Case 21_0. The singularity has $\mu = 13.2^{n-4}$, and is in the same $\mu-$constant stratum as $x^6 + x^2y^2 + y^6 + \sum_2^{n-4} w_i^3$.

(E) $f = 2x_2(y_0^2 - x_0z_0) + x_1^2z_0 - 2x_1y_1y_0 + y_1^2x_0 + a_3(x_0, y_0, z_0, w_1, \dots, w_m)$ $(n \geq 5)$, satisfying the conditions of Proposition 8.5. This is invariant by $x_0\partial/\partial x_1 + x_1\partial/\partial x_2 + y_0\partial/\partial y_1$ and $y_0\partial/\partial x_1 + y_1\partial/\partial x_2 + a_1\partial/\partial y_1$; any non-zero linear combination of these is in Case 5. If V_3 is non-singular, $\mu(V) = 25.2^{n-5}$ and the singularity of V is semi-quasi-homogeneous of degree 6 in variables of weights 1, 1, 3, 3, 3 and 2 ($n-5$ times). In general, we have $\mu(V) = 25.2^{n-5} + \mu(V_3)$.

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