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by Bruce Ebanks and Henrik Stetkær

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Ny Munkegade, Bldg. 1530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

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Bruce Ebanks
Department of Mathematics and Statistics
Mississippi State University
PO Drawer MA
Mississippi State, MS 39762, USA
ebanks@math.msstate.edu

Henrik Stetkær
Department of Mathematical Sciences
University of Aarhus
Ny Munkegade, Building 1530
DK-8000 Aarhus C, Denmark
stetkaer@imf.au.dk

Abstract

We provide an elementary way to compute continuous solutions of the 2-cocycle functional equation on solvable locally compact groups. Examples are given for certain linear groups. By "elementary" we mean that nothing is used from differential geometry, theory of Lie groups, or group cohomology.

1 Introduction

In an earlier paper [3], the first author introduced an elementary method of finding the general solution of the 2-cocycle equation on solvable groups. The 2-cocycle functional equation on a group G with values in an abelian group K (abbreviated the cocycle equation) is

$$F(x,y) + F(xy,z) = F(x,yz) + F(y,z), \quad x,y,z \in G. \tag{1.1}$$

A solution of (1.1), i.e. a map $F : G \times G \to K$ such that (1.1) holds for all $x, y, z \in G$, is called a 2-cocycle. In this paper we present an elementary way to compute the continuous solutions of this equation on locally compact solvable groups. The problem of computing continuous solutions is of particular interest for the linear groups that play important roles in quantum theory. Our main goal is to give continuous analogues of the results presented in [3]. Subsequent work will extend these results to other Lie groups. (In fact, it is on solvable groups where the most technical difficulties lie.)

Historically, the 2-cocycle functional equation plays a central role in the theory of projective representations (also known as ray representations) of groups in quantum mechanics, going back to the seminal paper of Bargmann [2] in 1954. Since all of our cocycles will be 2-cocycles, we shall omit the prefix 2. Cocycles are known by various other names in the quantum physics literature. For example, Bargmann [2] working with a Lie group G terms a continuous solution $F : G \times G \to \mathbb{R}$ of (1.1) an *exponent* if it also satisfies the normalizing condition $F(1,1) = 0$. Varadarajan [10] calls such a map a *multiplier* (or a *K-multiplier* if F takes values in an abelian locally compact group K).

Definition 1.1. Let G be a topological groups and K a topological abelian group. If $F : G \times G \to K$ is a continuous solution of (1.1), then we say that F is a *continuous cocycle on G into K*. The set of continuous cocycles on G into K is denoted $Z^2_{\mathcal{C}}(G,K)$.

There are two existing standard methods of finding continuous (normalized) cocycles in the quantum theory context. One method uses the cohomology theory of Lie groups and Lie algebras (see [2], [9]); the other uses the powerful coordinate-independent techniques of modern differential geometry (see [7]). Krause [8] introduced a simpler, coordinate-dependent version of the latter method. The new approach we introduce in this paper is much more elementary than any of those. Our approach uses only some basic elements from the theory of topological groups, combined with functional equations techniques. We present the main results in sections 4 and 5 and some examples in section 6.

To finish setting the stage, we introduce some further terminology and notation.

Definition 1.2. Given a group G, an abelian group K and a map $f : G \to K$, we shall call the map $\delta[f] : G \times G \to K$ defined by

$$\delta[f](x,y) := f(x) + f(y) - f(xy) \tag{1.2}$$

the *coboundary generated by f*. If G and K are topological groups then we define $B^2_{\mathcal{C}}(G,K) := \{\delta[f] \mid f \in C(G,K)\}$.

It is easy to see that any coboundary is a cocycle, so that $B^2_{\mathcal{C}}(G,K)$ is a subset of $Z^2_{\mathcal{C}}(G,K)$. Note however that a continuous coboundary may be generated by a discontinuous function. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is any discontinuous solution of the Cauchy functional equation $f(x+y) = f(x) + f(y)$, then $\delta[f] = 0$. Using Gajda [5] we derive in subsection 3.1 a condition under which any continuous coboundary is known to have a continuous generator.

We are mainly interested in the case of $K = X$, where X is a complex Banach space. In that case $Z^2_{\mathcal{C}}(G,X)$ is a complex vector space and $B^2_{\mathcal{C}}(G,X)$ is a subspace of it. The primary objective of this research is to determine explicit forms of continuous cocycles. A secondary objective is to find a basis of the vector space $H^2_{\mathcal{C}}(G,X) := Z^2_{\mathcal{C}}(G,X)/B^2_{\mathcal{C}}(G,X)$ for given G and X. That motivates the following definition.

Definition 1.3. Given two continuous cocycles F_1, F_2 on a group G into a complex Banach space X, we say that F_1 is *equivalent* to F_2, denoted $F_1 \simeq F_2$, if there exists a map $f \in C(G,X)$ such that $F_1 = F_2 + \delta[f]$.

In other words, two continuous cocycles F_1 and F_2 are equivalent if and only if they belong to the same coset determined by $B^2_{\mathcal{C}}(G,X)$. Our goal is to use this equivalence to exhibit the simplest possible form for a continuous cocycle F on G into X.

As mentioned above the first author [3] studied the general solution of the cocycle equation on solvable groups. However, that was done without any regularity assumptions, and it is not obvious how to obtain formulas for the continuous cocycles from the results of [3]. According to the theory of [3] a cocycle F can be written in a special form $F = \Psi_1 + \cdots + \Psi_r$. But knowing that F is continuous does not tell us that the individual terms $\Psi_1, \dots, \Psi_r$ are continuous. To infer continuity of them requires a separate investigation of the terms that we do in the present paper. We incorporate continuity in the set up from the beginning, so the present paper does not presuppose [3].

2 Notation and definitions

$\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields, respectively. We let $(z,w) \mapsto \langle z,w\rangle$ or just $z \cdot w$ denote the canonical bilinear form on $\mathbb{C}^n$: If $z = (z_1,\dots,z_n)^t \in \mathbb{C}^n$ and $w = (w_1,\dots,w_n)^t \in \mathbb{C}^n$, then $\langle z,w\rangle = z \cdot w = z_1w_1 + \cdots + z_nw_n$.

Group operations will be written multiplicatively, unless the group is abelian, in which case we often use $+$. Throughout the paper G will denote a group with neutral element 1 (in the abelian case 0).

Definition 2.1. Given two groups G, K, with K abelian, a map $\psi : G \times G \to K$ is called a *bi-morphism from G into K* if $\psi(xy,z) = \psi(x,z) + \psi(y,z)$ and $\psi(x,yz) = \psi(x,y) + \psi(x,z)$; it is called *skew-symmetric* if $\psi(x,y) = -\psi(y,x)$.

Definition 2.2. Let G be a group. If $g, n \in G$ then $n^g := gng^{-1}$ is called the *conjugate of n by g*.

Throughout this article, such exponent notation will always denote conjugation.

Without explicit mentioning, we will make subgroups of topological groups into topological groups by equipping them with the inherited topology.

By a locally compact group we mean a locally compact, Hausdorff topological group.

If A and B are topological spaces we let $C(A,B)$ denote the set of continuous functions from A to B, and we let $C(A) := C(A,\mathbb{C})$. If A is a manifold we let $C^\infty(A)$ denote the smooth complex-valued functions on A and $C_c^\infty(A)$ the compactly supported functions in $C^\infty(A)$.

3 On continuity

3.1 The question of continuous generators

We need conditions under which any continuous coboundary has a continuous generator. That is, if $\delta[f]$ is continuous, we wish to know whether there is a continuous generating function g for this coboundary. An answer can be deduced from the following theorem of Gajda [5].

Proposition 3.1. *Let G be a locally compact group, and let X be a complex Banach space. If $f : G \to X$ is such that for each $y \in G$ the function $x \mapsto f(xy) - f(x)$ is continuous on G and the function $x \mapsto f(yx) - f(x)$ is Borel measurable on G, then $f = h + g$, where $h : G \to X$ is a group homomorphism and $g : G \to X$ is continuous.*

From this we immediately get the following consequence.

Corollary 3.2. *Let G be a locally compact group, and let X be a complex Banach space. If $f : G \to X$ is such that the coboundary $\delta[f]$ is continuous on $G \times G$, then there exists a continuous $g : G \to X$ such that $\delta[f] = \delta[g]$.*

We shall not treat cocycles that take values in a general abelian topological group K as discussed in the introduction, because that leads to technical complications, but shall restrict ourselves to the case of K being a complex Banach space. Here Corollary 3.2 allows us to get by.

If G is a Lie group we even get differentiability. We use Lemma 3.3 in the special case of Example 6.6.

Lemma 3.3. *Let G be a Lie group. If $f \in C(G)$ is such that $\delta[f] \in C^\infty(G \times G)$, then $f \in C^\infty(G)$.*

Proof. Choose a function $\phi \in C_c^\infty(G)$ such that $\int_G \phi(y)\,d\lambda(y) = 1$, where λ denotes a left Haar measure on G. Let $F := \delta[f]$. Multiplying the identity $F(x,y) = f(x) + f(y) - f(xy)$, $x, y \in G$, by $\phi(y)$ and integrating the result with respect to $d\lambda(y)$ we get that

$$f(x) = \int_G F(x,y)\phi(y)\,d\lambda(y) - \int_G f(y)\phi(y)\,d\lambda(y) + \int_G \phi(x^{-1}y)f(y)\,d\lambda(y)$$

from which the differentiability follows, because both F and $(x,y) \mapsto \phi(x^{-1}y)$ are smooth functions of two variables. □

3.2 On semidirect products of groups

A group G is the *semidirect product* of a normal subgroup N by another subgroup Q if $G = NQ$ and $N \bigcap Q = \{1\}$. In such a situation we use the notation $G = N \circledS Q$. Any $x \in G = N \circledS Q$ can be written uniquely in the form $x = n(x)q(x)$, where $n(x) \in N$ and $q(x) \in Q$.

In contrast to [3] we write the normal subgroup on the left because of the simplification this lends our treatment of examples in Section 6.

We shall need the connection between the algebraic structure and the topological one which is embodied in the next definition:

Definition 3.4. A semidirect product $G = N \circledS Q$ is a *topological semidirect product*, if G is a topological group such that the canonical maps $x \mapsto n(x)$ and $x \mapsto q(x)$ are continuous maps of G into G.

If $G = N \circledS Q$ is a topological semidirect product, and F is a continuous function on N then the function defined by $nq \mapsto F(n)$, $n \in N$, $q \in Q$, is continuous on G. Indeed, it is the function $x \to F(n(x))$ which is composed of two continuous maps. Similarly for a continuous function on Q instead of on N. It is indispensable for us that continuous functions on N or Q define continuous functions on G. That is the reason why we need to work with topological semidirect products instead of (algebraic) semidirect products.

According to Lemma 3.5 below a semidirect product under certain natural conditions is a topological semidirect product. The semidirect products in our examples are all of this type, so they are topological semidirect products.

Lemma 3.5. *Let G be a σ-compact, locally compact group which is the semidirect product $G = N \times_s Q$ of a closed normal subgroup N and a closed subgroup Q. Then $G = N \circledS Q$ is a topological semidirect product.*

Lemma 3.5 is the case $r = 2$ of the following result that we need in the proof of Theorem 5.2.

Lemma 3.6. *Let G be a σ-compact, locally compact group. Let $Q_1, Q_2, \ldots, Q_r$ be closed subgroups of G such that the product $Q_1 Q_2 \cdots Q_j$ is a normal subgroup of G for $j = 1, 2, \ldots r$. Assume furthermore that each element $x \in G$ can be decomposed in exactly one way as a product $x = q_1(x) q_2(x) \cdots q_r(x)$, where $q_j(x) \in Q_j$ for $j = 1, 2, \ldots, r$.*

Then

(a) The maps $x \to q_j(x)$, $j = 1, 2, \ldots r$, are continuous from G to Q_j.

(b) The product $Q_{i_1} Q_{i_2} \cdots Q_{i_s}$ is closed in G for any $1 \le i_1 < i_2 < \cdots < i_s \le r$.

Proof of Lemma 3.6. The subgroups $Q_1, Q_2, \ldots, Q_r$ are closed subgroups of G, so they, too, are σ-compact, locally compact groups. We equip the product $G_0 := Q_1 \times Q_2 \times \cdots \times Q_r$ with the product topology so that it becomes a σ-compact, locally compact space. The map $i : G_0 \to G$, defined by $i(q_1, q_2, \ldots, q_r) := q_1 q_2 \cdots q_r$, is continuous: If q_j is close to $q_j^{(0)}$ for each $j = 1, 2, \ldots r$ then $q_1 q_2 \cdots q_r$ is close to $q_1^{(0)} q_2^{(0)} \cdots q_r^{(0)}$ because the group operation in G is continuous.

By hypothesis i is a bijection, so we can make G_0 into a group by requiring $i : G_0 \to G$ to be an isomorphism. The essential point for the proof is that G_0 with this group structure is a topological group. The following result [4, Theorem 2] allows us to make a shortcut in the proof of this point:

Proposition. *Let X be a locally compact Hausdorff space with a group structure such that the maps $x \mapsto yx$ and $x \mapsto xy$ of X into X are continuous for all $y \in X$. Then X is a topological group.*

So it suffices to prove that the product map $G_0 \times G_0 \to G_0$ is continuous. This means that if we write

$$(q_1 q_2 \cdots q_r)(p_1 p_2 \cdots p_r) = s_1 s_2 \cdots s_r$$

then $s_j = s_j(q,p)$ depends continuously on $(q,p) \in G_0 \times G_0$ for each $j = 1, 2, \ldots, r$. Now,

$$(q_1 q_2 \cdots q_{r-1} q_r)(p_1 p_2 \cdots p_r) = (q_1 q_2 \cdots q_{r-1})(q_r p_1 p_2 \cdots p_{r-1} q_r^{-1})(q_r p_r)$$
$$= (q_1 q_2 \cdots q_{r-1})(p_1^{q_r} p_2^{q_r} \cdots p_{r-1}^{q_r})(q_r p_r).$$

Note that $p_j^{q_r} \in Q_j$, so that $p_1^{q_r} p_2^{q_r} \cdots p_{r-1}^{q_r} \in Q_1 \cdots Q_{r-1}$. In the same way as we moved q_r past $p_1 p_2 \cdots p_{r-1}$ we now move q_{r-1} past $p_1^{q_r} p_2^{q_r} \cdots p_{r-2}^{q_r}$ and up to $p_{r-1}^{q_r}$ and get

$$(q_1 q_2 \cdots q_{r-1} q_r)(p_1 p_2 \cdots p_r)$$
$$= (q_1 q_2 \cdots q_{r-2})((p_1^{q_r})^{q_{r-1}} (p_2^{q_r})^{q_{r-1}} \cdots (p_{r-1}^{q_r})^{q_{r-1}})(q_{r-1} p_{r-1}^{q_r})(q_r p_r).$$

Continuing in this way we get that each $s_j(q,p)$ is a product of $q_1, q_2, \ldots, q_r, p_1, p_2, \ldots, p_r$ and their inverses. Since the group operations in G are continuous we see that $s_j : G_0 \times G_0 \to G$ is a continuous map. But $s_j(G_0 \times G_0) \subseteq Q_j$ and Q_j has the topology from G, so $s_j : G_0 \times G_0 \to Q_j$ is a continuous map. This proves the claim, so G_0 is a topological group.

It follows from the open mapping theorem for groups ([6, Theorem 5.29]) that $i : G_0 \to G$ is an open map, and consequently that i^{-1} is continuous.

(a) The projection $\pi_j : G_0 \to Q_j$ on the j^{th} component is continuous for each $j = 1, 2, \ldots r$ by the definition of the product topology. We see from the formula $q_j = (\pi_j \circ i^{-1})(q_1 q_2 \cdots q_r)$ that q_j depends continuously on $q_1 q_2 \cdots q_r$.

(b) follows from $i : Q_1 \times Q_2 \cdots \times Q_r \to G$ being a homeomorphism. □

4 Continuous cocycles on semidirect products

4.1 Two auxiliary lemmas

In this subsection we prove two auxiliary lemmas that are used in our derivation of Theorem 4.3. In both lemmas we let G be a topological group such that $G = N \circledS Q$ is the topological semidirect product of a normal subgroup N by a subgroup Q. K is an abelian topological group. $F \in C(G \times G, K)$ is a cocycle, and we put $\kappa := F(1,1) \in K$. Then $F(1,x) = F(x,1) = \kappa$ for all $x \in G$, as follows from (1.1) by simple substitutions.

We define the function $f_0 \in C(G,K)$ by $f_0(x) := \kappa - F(n(x), q(x))$, $x \in G$. So $f_0(nq) = \kappa - F(n,q)$ for all $n \in N$ and $q \in Q$.

Lemma 4.1. *(a) $f_0(n) = f_0(q) = 0$ for all $n \in N$ and $q \in Q$.*

(b) $f_0(mx) - f_0(x) = F(m, n(x)) - F(m,x)$ for all $m \in N$ and $x \in G$.

Proof. (a) follows immediately from the definition of f_0.
(b) Applying (1.1) at the fourth equality sign below (with $x = m$, $y = n(x)$ and $z = q(x)$) we get for any $m \in N$ and $x \in G$:

$$f_0(x) - f_0(mx) = \{\kappa - F(n(x), q(x))\} - \{\kappa - F(n(mx), q(mx))\}$$
$$= F(n(mx), q(mx)) - F(n(x), q(x)) = F(mn(x), q(x)) - F(n(x), q(x))$$
$$= F(m,x) - F(m, n(x)).$$
□

The existence of an f satisfying Lemma 4.1(b) was proved in [3, Lemma 1]. The procedure here is much simpler.

We define $C \in C(G \times G, K)$ by

$$C(x,y) := F(x,y) - F(n(x), n(y)^{q(x)}) - \delta[f_0](x,y), \; x, y \in G. \tag{4.1}$$

C was introduced in the proof of [3, Theorem 2]. Lemma 4.2 notes some of the properties of C that follow from F being a cocycle.

Lemma 4.2. *(a) $C(x,y) = C(q(x), y)$ for all $x, y \in G$.*

(b) If $n \in N$ and $q \in Q$, then $C(q,n) = F(q,n) - F(n^q, q)$.

(c) $C|_{Q\times Q} = F|_{Q\times Q} - \kappa$.

(d) $C(x,y) = C(x, n(y)) + C(x, q(y))$ for all $x, y \in G$.

(e) If $q_1, q_2 \in Q$ and $n \in N$, then

$$C(q_1 q_2, n) = C(q_1, n^{q_2}) + C(q_2, n). \tag{4.2}$$

(f) If $q \in Q$ and $n_1, n_2 \in N$, then

$$C(q, n_1 n_2) - C(q, n_1) - C(q, n_2) = F(n_1^q, n_2^q) - F(n_1, n_2). \tag{4.3}$$

Proof. (a) We shall prove that $C(nq, y) = C(q, y)$ for all $n \in N$, $q \in Q$ and $y \in G$. First observe that

$$\begin{aligned}
C(nq,y) - C(q,y) &= F(nq,y) - F(n, n(y)^q) - \delta[f_0](nq,y) \\
&\qquad - F(q,y) + F(1, n(y)^q) + \delta[f_0](q,y) \\
&= F(nq,y) - F(n,n(y)^q) - F(q,y) + \kappa - f_0(nq) + f_0(nqy) - f_0(qy),
\end{aligned}$$

where $f_0(q) = 0$ by Lemma 4.1(a). Here we use Lemma 4.1(b) on $f_0(nqy) - f_0(qy)$ to continue the computations:

$$\begin{aligned}
&= F(nq,y) - F(n,n(y)^q) - F(q,y) + F(n,q) + F(n, n(qy)) - F(n,qy) \\
&= F(nq,y) - F(n,n(y)^q) - F(q,y) + F(n,q) - F(n,qy) + F(n, n(y)^q) \\
&= F(nq,y) - F(q,y) + F(n,q) - F(n,qy),
\end{aligned}$$

which vanishes, F being a cocycle.

(b) For any $q \in Q$, $n \in N$,

$$\begin{aligned}
C(q,n) &= F(q,n) - F(n(q), n^q) - \delta[f_0](q,n) \\
&= F(q,n) - F(1,n^q) - 0 - 0 + f_0(n^q q) \\
&= F(q,n) - \kappa + [\kappa - F(n^q, q)] = F(q,n) - F(n^q, q).
\end{aligned}$$

(c) If $q_1, q_2 \in Q$ then

$$\begin{aligned}
C(q_1,q_2) &= F(q_1,q_2) - F(n(q_1), n(q_2)^{q_1}) - \delta[f_0](q_1,q_2) \\
&= F(q_1,q_2) - \kappa - 0 - 0 + 0 = F(q_1,q_2) - \kappa.
\end{aligned}$$

(d) We first compute $C(q, y)$ for $q \in Q$ and $y \in G$:

$$\begin{aligned}
\kappa + C(q, y) &= \kappa + F(q, y) - F(n(q), n(y)^q) - \delta[f_0](q, y) \\
&= \kappa + F(q, y) - \kappa - [0 + f_0(y) - f_0(qy)] \\
&= F(q, y) - [\kappa - F(n(y), q(y))] + [\kappa - F(n(qy), q(qy))] \\
&= F(q, y) + F(n(y), q(y)) - F(n(y)^q, qq(y)) \\
&= F(q, y) + F(n(y), q(y)) - [-F(q, q(y)) + F(n(y)^q, q) + F(n(y)^q q, q(y))] \\
&= F(q, y) + F(n(y), q(y)) + F(q, q(y)) - F(n(y)^q, q) - F(qn(y), q(y)) \\
&= F(q, y) + F(n(y), q(y)) + F(q, q(y)) - F(n(y)^q, q) \\
&\quad - [-F(q, n(y)) + F(q, y) + F(n(y), q(y))] \\
&= F(q, q(y)) - F(n(y)^q, q) + F(q, n(y)).
\end{aligned}$$

When we replace y first by $n(y)$ and then by $q(y)$ in this formula for $C(q, y)$, we find that $C(q, n(y)) = F(q, n(y)) - F(n(y)^q, q)$ and $C(q, q(y)) = F(q, q(y)) - \kappa$, from which it follows that $C(q, y) - C(q, n(y)) - C(q, q(y)) = 0$.

We finally get the desired result from the formula $C(x, y) = C(q(x), y)$, derived in (a).

(e) Applying first the formula $C(q, n) = F(q, n) - F(n^q, q)$ from (b) and then (1.1) we find that

$$\begin{aligned}
&C(q_1q_2, n) - C(q_1, n^{q_2}) - C(q_2, n) \\
&\quad = F(q_1q_2, n) - F(n^{q_1q_2}, q_1q_2) - F(q_1, n^{q_2}) + F(n^{q_1q_2}, q_1) - F(q_2, n) + F(n^{q_2}, q_2) \\
&\quad = [-F(q_1, q_2) + F(q_1, q_2n) + F(q_2, n)] \\
&\qquad - [-F(q_1, q_2) + F(n^{q_1q_2}, q_1) + F(n^{q_1q_2}q_1, q_2)] \\
&\qquad - F(q_1, n^{q_2}) + F(n^{q_1q_2}, q_1) - F(q_2, n) + F(n^{q_2}, q_2) \\
&\quad = F(q_1, q_2n) - F(q_1n^{q_2}, q_2) - F(q_1, n^{q_2}) + F(n^{q_2}, q_2),
\end{aligned}$$

which vanishes by (1.1), because $F(q_1, q_2n) = F(q_1, n^{q_2}q_2)$.

(f) We first compute $C(q, n_1n_2) = F(q, n_1n_2) - F(n_1^q n_2^q, q)$ using (1.1):

$$\begin{aligned}
F(q, n_1n_2) - F(n_1^q n_2^q, q) &= -F(n_1, n_2) + F(q, n_1) + F(qn_1, n_2) \\
&\quad - [-F(n_1^q, n_2^q) + F(n_1^q, n_2^q q) + F(n_2^q, q)] \\
&= -F(n_1, n_2) + F(q, n_1) + F(qn_1, n_2) + F(n_1^q, n_2^q) - F(n_1^q, n_2^q q) - F(n_2^q, q) \\
&= -F(n_1, n_2) + F(q, n_1) + F(qn_1, n_2) + F(n_1^q, n_2^q) - F(n_1^q, qn_2) - F(n_2^q, q) \\
&= -F(n_1, n_2) + F(q, n_1) + F(qn_1, n_2) + F(n_1^q, n_2^q) \\
&\quad - [-F(q, n_2) + F(n_1^q, q) + F(qn_1, n_2)] - F(n_2^q, q) \\
&= -F(n_1, n_2) + F(q, n_1) + F(n_1^q, n_2^q) + F(q, n_2) - F(n_1^q, q) - F(n_2^q, q),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&C(q, n_1n_2) - [F(n_1^q, n_2^q) - F(n_1, n_2)] \\
&\qquad = F(q, n_1) + F(q, n_2) - F(n_1^q, q) - F(n_2^q, q) = C(q, n_1) + C(q, n_2).
\end{aligned}$$

□

4.2 The structure of continuous cocycles on semidirect products

Theorem 4.3. *Let $G = N \circledS Q$ be the topological semidirect product of a normal subgroup N by a subgroup Q. Let K be an abelian topological group. Let finally $F \in C(G \times G, K)$. Then*

(a) F is a cocycle on G if and only if there exist cocycles $F_N \in C(N \times N, K)$, $F_Q \in C(Q \times Q, K)$, a map $f \in C(G, K)$ and a map $\Phi \in C(Q \times N, K)$ such that for all $x, y \in G$, $q, q_1, q_2 \in Q$, $n, n_1, n_2 \in N$:

$$F(x,y) = \delta[f](x,y) + F_N(n(x), n(y)^{q(x)}) + \Phi(q(x), n(y)) + F_Q(q(x), q(y)). \tag{4.4}$$

$$\Phi(q_1 q_2, n) = \Phi(q_1, n^{q_2}) + \Phi(q_2, n). \tag{4.5}$$

$$\Phi(q, n_1 n_2) - \Phi(q, n_1) - \Phi(q, n_2) = F_N(n_1^q, n_2^q) - F_N(n_1, n_2). \tag{4.6}$$

(b) Given a cocycle $F \in C(G \times G, K)$ we may choose the functions F_N, F_Q, f and Φ from (a) *as follows:*

$$f(x) := -F(n(x), q(x)), \; x \in G, \tag{4.7}$$

$$F_N := F|_{N \times N}, \tag{4.8}$$

$$\Phi(q, n) := F(q, n) - F(n^q, q), \; q \in Q, \; n \in N, \tag{4.9}$$

$$F_Q := F|_{Q \times Q}. \tag{4.10}$$

Proof. (a) and (b) To verify that a function F, defined by (4.4) and satisfying (4.5) and (4.6), is a cocycle for any cocycles $F_N \in C(N \times N, K)$, $F_Q \in C(Q \times Q, K)$ and maps $f \in C(G, K)$, $\Phi \in C(Q \times N, K)$, is a simple verification that we skip. We use the fact that G is a topological semidirect product to deduce that the individual terms of F in (4.4), and hence also F, are continuous functions on $G \times G$.

It is left to prove the converse, i.e. that any cocycle $F \in C(G \times G, K)$ can be written in this form. It suffices to prove that the functions defined in (b) work. Clearly, these functions are continuous, and F_N and F_Q are cocycles, being restrictions of the cocycle F to subgroups. In the proof we use the results of Lemma 4.2 without explicit mentioning. In particular note that $\Phi = C|_{Q \times N}$, where C is defined by (4.1).

We begin by proving (4.4):

$$\begin{aligned}
&F(x,y) - \delta[f](x,y) - F_N(n(x), n(y)^{q(x)}) - \Phi(q(x), n(y)) - F_Q(q(x), q(y)) \\
&\quad = F(x,y) - \delta[f_0 - \kappa](x,y) - F(n(x), n(y)^{q(x)}) \\
&\qquad - C(q(x), n(y)) - F(q(x), q(y)) \\
&\quad = C(x,y) - C(q(x), n(y)) - C(q(x), q(y)) \\
&\quad = C(x,y) - C(x, n(y)) - C(x, q(y)) = 0.
\end{aligned}$$

The formulas (4.5) and (4.6) follow from (4.2) and (4.3). □

The next result specializes the previous theorem to the case where either one or both of N, Q are abelian.

Theorem 4.4. *Let $G = N \circledS Q$ be as in Theorem 4.3, let X be a complex Banach space, and suppose the map $F : G \times G \to X$ is a continuous cocycle on G into X.*

1. *If N is abelian, then F has the form*

$$\begin{aligned} F(n_1q_1, n_2q_2) &= \delta[f](n_1q_1, n_2q_2) \\ &\quad + \Psi_N(n_1, n_2^{q_1}) + \phi(q_1, n_2) + F_Q(q_1, q_2), \end{aligned} \tag{4.11}$$

for a continuous map $f : G \to X$, a continuous, skew-symmetric bi-morphism Ψ_N on N into X such that

$$\Psi_N(n_1^q, n_2^q) = \Psi_N(n_1, n_2), \; q \in Q, \tag{4.12}$$

a continuous function $\phi : Q \times N \to X$ satisfying (4.5) and

$$\phi(q, n_1n_2) = \phi(q, n_1) + \phi(q, n_2), \tag{4.13}$$

and a continuous cocycle F_Q on Q into X.

2. *If Q is abelian, then F has the form*

$$\begin{aligned} F(n_1q_1, n_2q_2) &= \delta[f](n_1q_1, n_2q_2) \\ &\quad + F_N(n_1, n_2^{q_1}) + \Phi(q_1, n_2) + \Psi_Q(q_1, q_2), \end{aligned} \tag{4.14}$$

for a continuous map $f : G \to X$, a continuous cocycle F_N on N into X, a continuous function $\Phi : Q \times N \to X$ satisfying and (4.5) *and* (4.6)*, and a continuous, skew-symmetric bi-morphism Ψ_Q on Q into X.*

3. *If both N and Q are abelian, then F has the form*

$$\begin{aligned} F(n_1q_1, n_2q_2) &= \delta[f](n_1q_1, n_2q_2) \\ &\quad + \Psi_N(n_1, n_2^{q_1}) + \phi(q_1, n_2) + \Psi_Q(q_1, q_2), \end{aligned} \tag{4.15}$$

where f, Ψ_N, ϕ are as in part 1 and Ψ_Q is as in part 2.

Proof. Our starting point is the decomposition

$$F(n_1q_1, n_2q_2) = \delta[g](n_1q_1, n_2q_2) + F_N(n_1, n_2^{q_1}) + F_Q(q_1, q_2) + \Phi(q_1, n_2),$$

provided by Theorem 4.3. If N is abelian, then it is by now classical (see, e.g. [1]) that $F_N = \delta[f_N] + \Psi_N$ for some map $f_N : N \to X$ and a skew-symmetric bi-morphism Ψ_N on N into X. Moreover, since F_N is continuous, both its symmetric and skew-symmetric parts are continuous. Hence $\delta[f_N]$ and Ψ_N are continuous on $N \times N$. It follows from Corollary 3.2 that we may take f_N continuous. Now inserting this form of F_N into (4.6) gives

$$\begin{aligned} &\Phi(q, n_1n_2) - \Phi(q, n_1) - \Phi(q, n_2) \\ &\quad = (\delta[f_N] + \Psi_N)(n_1^q, n_2^q) - (\delta[f_N] + \Psi_N)(n_1, n_2), \end{aligned}$$

which upon defining ϕ by

$$\phi(q, n) := \Phi(q, n) + f_N(n^q) - f_N(n) \tag{4.16}$$

reduces to $\phi(q, n_1n_2) - \phi(q, n_1) - \phi(q, n_2) = \Psi_N(n_1^q, n_2^q) - \Psi_N(n_1, n_2)$. Since the left side of this equation is symmetric in n_1, n_2 while the right side is skew-symmetric, both sides are zero. Thus we get (4.12) and (4.13). Also (4.16) and (4.5) for Φ yield (4.5) for ϕ. Defining f by $f(nq) := g(nq) + f_N(n)$, we have the desired result in part 1.

If Q is abelian, we apply the same procedure to F_Q that we just applied to F_N. If N and Q are both abelian, then we get part 3 by combining the results of parts 1 and 2. □

Remark 4.5. The converse of Theorem 4.4 holds: If $F : G \times G \to X$ satisfies the conditions of part 1, then F is a continuous cocycle. Similarly for the parts 2 and 3. This can be proved by elementary computations.

Note that we could summarize (4.15) in the form

$$F(n_1q_1, n_2q_2) \simeq \Psi_N(n_1, n_2^{q_1}) + \phi(q_1, n_2) + \Psi_Q(q_1, q_2).$$

That is the case $r = 2$ of Theorem 5.2, which is the topological analogue of [3, Theorem 5]. The case $r = 1$ of Theorem 5.2 is the classical result that $F(x, y) \simeq \Psi(x, y)$ if G is abelian.

5 Continuous cocycles on solvable groups

Assuming that G is solvable, it must have an invariant normal series. That means each subgroup in the series is not only normal in the preceding subgroup of the series but also normal in G. The assumptions of the next theorem guarantee that we can view G as being built up through a sequence of semidirect products. To get a result in a convenient form for solvable groups of rank $r \geq 3$ we need to assume a bit more about the subgroup structure. If the factor groups determined by the invariant series do not satisfy the additional conditions postulated in the next lemma, then one has to proceed step-by-step using Theorem 4.4 repeatedly and the job is more difficult and involved. The additional conditions are satisfied by any semidirect product (which is the case of $r = 2$) and by all our examples in Section 6.

Lemma 5.1. *Suppose a group G can be written as $G = Q_1Q_2 \cdots Q_r$ where each Q_j is a subgroup of G. Assume that*

$$Q_j^{Q_{j+1}\cdots Q_r} \subseteq Q_j \text{ for all } j = 1, \ldots, r-1. \tag{5.1}$$

Then

(a) $Q_jQ_{j+1} \cdots Q_r$ is a subgroup of G for each $j = 1, 2, \ldots, r$.

(b) $Q_1Q_2 \cdots Q_j$ is a normal subgroup of G for each $j = 1, 2, \ldots, r$.

(c) If Q_j is abelian for each $j = 1, 2, \ldots, r$, then G is solvable.

Proof. (a) Let $p_j \cdots p_r, q_j \cdots q_r \in Q_j Q_{j+1} \cdots Q_r$ for some j. Then

$$(p_j \cdots p_r)(q_j \cdots q_r) = (p_j q_j^{p_{j+1} \cdots p_r}) \cdots (p_{r-1} q_{r-1}^{p_r})(p_r q_r)$$

shows that $Q_j Q_{j+1} \cdots Q_r$ is closed under multiplication. $Q_j Q_{j+1} \cdots Q_r$ is closed under inversion since each Q_j is a group and $Q_r Q_{r-1} \cdots Q_j \subseteq Q_j Q_{j+1} \cdots Q_r$. To see this, observe that

$$q_r q_{r-1} \cdots q_{j+1} q_j = ((\cdots (q_j^{q_{j+1}})^{q_{j+2}} \cdots)^{q_r}) \cdots (q_{r-1}^{q_r}) q_r.$$

(b) is proved by induction on j.

(c) Clearly $[G, G] = [Q_1 \cdots Q_{r-1} Q_r, Q_1 \cdots Q_{r-1} Q_r] \subseteq Q_1 \cdots Q_{r-1}$, and the statement follows by downwards induction on r. Alternatively, if we define $G_j := Q_1 Q_2 \cdots Q_j$ for $j = 1, 2, \ldots r$, then $G = G_r \trianglerighteq G_{r-1} \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq \{1\}$ is a normal series for G. □

Theorem 5.2. *Let G be a σ-compact, locally compact group, let $Q_1, Q_2, \ldots, Q_r$ be closed abelian subgroups of G, and let X be a complex Banach space. Suppose that any element $g \in G$ can be written uniquely as*

$$g = q_1 \cdots q_r, \text{ where } q_j \in Q_j \text{ for } j = 1, ..., r,$$

and that condition (5.1) is satisfied.

Then a map $F : G \times G \to X$ is a continuous cocycle on G into X if and only if there exist continuous skew-symmetric bi-morphisms Ψ_i on Q_i into X and continuous maps $\phi_j : (Q_{j+1} \cdots Q_r) \times Q_j \to X$ such that

$$F(q_1 \cdots q_r, p_1 \cdots p_r) \simeq \sum_{i=1}^{r} \Psi_i(q_i, p_i^{q_{i+1} \cdots q_r}) + \sum_{j=1}^{r-1} \phi_j(q_{j+1} \cdots q_r, p_j), \tag{5.2}$$

$$\Psi_i(q_i^k, p_i^k) = \Psi_i(q_i, p_i), \tag{5.3}$$

$$\phi_i(k, q_i p_i) = \phi_i(k, q_i) + \phi_i(k, p_i), \tag{5.4}$$

$$\phi_i(kl, q_i) = \phi_i(k, q_i^l) + \phi_i(l, q_i), \tag{5.5}$$

for all $q_i, p_i \in Q_i$; $k, l \in Q_{i+1} \cdots Q_r$.

Proof. The case $r = 1$ is classical ($F \simeq \Psi_1$) and the case $r = 2$ is covered by Theorem 4.4. We proceed by induction on r. Assume the statement is true for some positive integer $r \geq 1$, and let G be solvable of rank $r + 1$. Observe that $Q_2 \cdots Q_{r+1}$ is a group by condition (5.1) and Lemma 5.1. Observe also that it is closed in G by Lemma 3.6(b) so that it, too, is a σ-compact and locally compact group. Each $g \in G$ can be written uniquely as $g = q_1 \cdots q_{r+1}$ with $q_i \in Q_i$, where each Q_i is abelian and Q_1 is normal in G. For any $x, y \in G$, write $x = q_1(q_2 \cdots q_{r+1})$, $y = p_1(p_2 \cdots p_{r+1})$. Applying part 1 of Theorem 4.4 we have

$$\begin{aligned} F(x, y) = \delta[f_1](q_1(q_2 \cdots q_{r+1}), p_1(p_2 \cdots p_{r+1})) + \Psi_1(q_1, p_1^{q_2 \cdots q_{r+1}}) \\ + \phi_1(q_2 \cdots q_{r+1}, p_1) + F_Q(q_2 \cdots q_{r+1}, p_2 \cdots p_{r+1}), \end{aligned}$$

where Ψ_1, ϕ_1 are as desired and F_Q is a continuous cocycle on the subgroup $Q := Q_2 \cdots Q_{r+1}$ into X. By the induction hypothesis, F_Q has the form

$$F_Q(q_2 \cdots q_{r+1}, p_2 \cdots p_{r+1}) = \delta[f_Q](q_2 \cdots q_{r+1}, p_2 \cdots p_{r+1}) + \sum_{i=2}^{r+1} \Psi_i(q_i, p_i^{q_{i+1} \cdots q_{r+1}}) + \sum_{j=2}^{r} \phi_j(q_{j+1} \cdots q_{r+1}, p_j).$$

Inserting this into the equation above and defining $f : G \to X$ by

$$f(q_1 q_2 \cdots q_{r+1}) := f_1(q_1 q_2 \cdots q_{r+1}) + f_Q(q_2 \cdots q_{r+1}),$$

we obtain the desired form for F.

The proof of the converse consists of direct computations to verify that $F = \Psi_i$ and $F = \phi_i$ satisfy (1.1). □

Remark 5.3. If G has the discrete topology then the assumption in Theorem 5.2 about G being σ-compact can be deleted. The assumption is only used via Lemma 3.6 to ensure that certain maps are continuous. And any map on a discrete group is continuous.

6 Some examples

In this section we illustrate Theorem 5.2 with some examples. We give detailed expositions of them, because our results, which are very explicit, apparently cannot be found in the literature.

We begin with two very simple, but useful lemmas about continuous skew-symmetric bi-additive functions.

Lemma 6.1. *(a) Let n be a positive integer and suppose $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous skew-symmetric bi-additive function. Then there exists a complex skew-symmetric $n \times n$-matrix A such that $\Psi(x, y) = \langle x, Ay \rangle$ for all $x, y \in \mathbb{R}^n$.*

(b) In particular, if $\Psi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ is a continuous skew-symmetric bi-additive function, then there exists a constant $c \in \mathbb{C}$ such that

$$\Psi((x_1, x_2), (y_1, y_2)) = c(x_1 y_2 - y_1 x_2) \text{ for all } (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

Proof. Since Ψ is continuous and additive in each component it is a bilinear form, so there exists a complex $n \times n$-matrix A such that $\Psi(x, y) = \langle x, Ay \rangle$ for all $x, y \in \mathbb{R}^n$. The skew-symmetry of Ψ implies that of A. □

Lemma 6.2. *Let X be a Hausdorff topological vector space over $\mathbb{R}$ or $\mathbb{C}$.*

(a) The only separately continuous, skew-symmetric bi-additive function $\Psi : \mathbb{R} \times \mathbb{R} \to X$ is 0.

(b) The only separately continuous, skew-symmetric bi-morphism $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to X$ is 0.

Proof. (a) By the additivity we get that $\Psi(q,y) = q\Psi(1,y)$ for all $q \in \mathbb{Q}$ and $y \in \mathbb{R}$. By the continuity in the first variable we then get that $\Psi(x,y) = x\Psi(1,y)$ for all $x, y \in \mathbb{R}$. Arguing in the same way on the second variable we get that $\Psi(x,y) = xy\Psi(1,1)$ for all $x, y \in \mathbb{R}$. This expression shows that Ψ is symmetric. But it is also by assumption skew-symmetric. Hence $\Psi = 0$.
(b) follows immediately from (a) when you apply (a) to the map $(x,y) \mapsto \Psi(e^x, e^y)$ from $\mathbb{R}^+ \times \mathbb{R}^+$ to X. □

Example 6.3. We consider the $(ax+b)$-group

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}^+,\ b \in \mathbb{R} \right\},$$

and a complex Banach space X. We claim that any continuous cocycle $F : G \times G \to X$ has the form $F = \delta[f]$, where $f \in C(G,X)$.

To prove this claim we start by noting that G is the semidirect product $G = N \circledS Q$, where

$$N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}, \text{ and } Q := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}^+ \right\}.$$

Below we identify N with $(\mathbb{R}, +)$ and Q with $(\mathbb{R}^+, \cdot)$ in the obvious way.

From part 3 of Theorem 4.4 we get

$$F(n_1q_1, n_2q_2) = \delta[f](n_1q_1, n_2q_2) + \Psi_N(n_1, n_2^{q_1}) + \phi(q_1, n_2) + \Psi_Q(q_1, q_2),$$

where $f \in C(G,X)$, $\Psi_N : N \times N \to X$ and $\Psi_Q : Q \times Q \to X$ are continuous, skew-symmetric bi-morphisms, and $\phi : \mathbb{R}^+ \times \mathbb{R} \to X$ is a continuous map such that

$$\phi(q, n_1 + n_2) = \phi(q, n_1) + \phi(q, n_2), \text{and} \tag{6.1}$$

$$\phi(q_1q_2, n) = \phi(q_1, q_2n) + \phi(q_2, n) \tag{6.2}$$

for all $q, q_1, q_2 \in \mathbb{R}^+$ and $n, n_1, n_2 \in \mathbb{R}$. According to Lemma 6.2 $\Psi_N = 0$ and $\Psi_Q = 0$, so it just remains to produce a function $h \in C(G,X)$ such that

$$\delta[h](n_1q_1, n_2q_2) = \phi(q_1, n_2) \text{ for all } n_1, n_2 \in N,\ q_1, q_2 \in Q.$$

The function $h(nq) := -\phi(2,n)$, $n \in N$, $q \in Q$, may be used. To see this we observe that the left hand side of (6.2) is symmetric in q_1 and q_2 so that

$$\phi(q_2, q_1n) + \phi(q_1, n) = \phi(q_1, q_2n) + \phi(q_2, n).$$

Here we replace q_2 by 2 and n by n_2 to get

$$\phi(2, q_1n_2) + \phi(q_1, n_2) = \phi(q_1, 2n_2) + \phi(2, n_2) = 2\phi(q_1, n_2) + \phi(2, n_2),$$

which means that $\phi(2, q_1n_2) - \phi(2, n_2) = \phi(q_1, n_2)$. Using this to get the last equality sign below we find

$$\begin{aligned} \delta[h](n_1q_1, n_2q_2) &= h(n_1q_1) + h(n_2q_2) - h(n_1q_1n_2q_2) \\ &= -\phi(2, n_1) - \phi(2, n_2) + \phi(2, n_1 + q_1n_2) \\ &= -\phi(2, n_1) - \phi(2, n_2) + \phi(2, n_1) + \phi(2, q_1n_2) \\ &= \phi(2, q_1n_2) - \phi(2, n_2) = \phi(q_1, n_2). \end{aligned}$$

Example 6.4. Let H_1 be the Heisenberg group (in polarized form) with elements represented as matrices

$$\begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

or as elements of $\mathbb{R}^3$ with multiplication

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1 y_2)$$

Then the equivalence classes of the two maps $B_1 : H_1 \times H_1 \to \mathbb{C}$ and $B_2 : H_1 \times H_1 \to \mathbb{C}$, defined by

$$\begin{aligned} B_1((x_1, y_1, t_1), (x_2, y_2, t_2)) &:= y_1(t_2 + x_1 y_2) - t_1 y_2, \\ B_2((x_1, y_1, t_1), (x_2, y_2, t_2)) &:= x_1(t_2 + x_1 y_2/2), \end{aligned}$$

form a basis for $H^2_{\mathcal{C}}(H_1, \mathbb{C})$.

Proof. H_1 is the semidirect product $NⓢQ$, where $N = \{(0, y, t)\}$ and $Q = \{(x, 0, 0)\}$ are closed abelian subgroups of H_1 with N normal. Observe that conjugation takes the form

$$n^q = (0, y, t)^{(x,0,0)} = (0, y, t + xy).$$

Below we identify N with $\mathbb{R}^2$ and Q with $\mathbb{R}$ whenever convenient.

In the notation of Theorem 4.4, part 3 any continuous cocycle $F : H_1 \times H_1 \to \mathbb{C}$ has the form

$$\begin{aligned} &F((x_1, y_1, t_1), (x_2, y_2, t_2)) \\ &\quad \simeq \Psi_1((y_1, t_1), (y_2, t_2 + x_1 y_2)) + \Psi_2(x_1, x_2) + \phi((x_1, 0, 0), (0, y_2, t_2)). \end{aligned}$$

We find the first term on the right by Lemma 6.1(b) and note that the second term on the right vanishes by Lemma 6.2(a). This gives us that

$$F((x_1, y_1, t_1), (x_2, y_2, t_2)) \simeq c_1[y_1(t_2 + x_1 y_2) - t_1 y_2] + \phi(x_1, (y_2, t_2)),$$

where $c_1 \in \mathbb{C}$ is a constant and where $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is a continuous function which is additive in its second (vector) component and satisfies

$$\phi(x_1 + x_2, (y_2, t_2)) = \phi(x_1, (y_2, t_2 + x_2 y_2)) + \phi(x_2, (y_2, t_2)).$$

The continuity and additivity yield the existence of continuous maps ϕ_1, ϕ_2 such that

$$\phi(x_1, (y_2, t_2)) = \phi_1(x_1) y_2 + \phi_2(x_1) t_2,$$

and substituting this into the previous equation we get

$$\begin{aligned} &\phi_1(x_1 + x_2) y_2 + \phi_2(x_1 + x_2) t_2 \\ &\quad = \phi_1(x_1) y_2 + \phi_2(x_1)(t_2 + x_2 y_2) + \phi_1(x_2) y_2 + \phi_2(x_2) t_2. \end{aligned}$$

Comparing coefficients of t_2 we see that ϕ_2 is additive, hence linear. With $\phi_2(x) = c_2 x$, we now have

$$\phi_1(x_1 + x_2) - \phi_1(x_1) - \phi_1(x_2) = c_2 x_1 x_2,$$

which implies $\phi_1(x) = bx + c_2x^2/2$. In conclusion, we have

$$F((x_1, y_1, t_1), (x_2, y_2, t_2))$$
$$\simeq c_1[y_1(t_2 + x_1y_2) - t_1y_2] + [bx_1 + c_2x_1^2/2]y_2 + c_2x_1t_2.$$

Observe also that the map $((x_1, y_1, t_1), (x_2, y_2, t_2)) \mapsto bx_1y_2$ is a continuous coboundary with generator $g(x, y, t) := -bt$, therefore F has the asserted form.

On the other hand, elementary calculations (or a reference to the general theory) show B_1 and B_2 are cocycles.

We shall finally show that if $c_1B_1 + c_2B_2 = \delta[h]$, where $c_1, c_2 \in \mathbb{C}$ and where $h : H_1 \to \mathbb{C}$, then $c_1 = c_2 = 0$. To do so we note the general fact that if A is an abelian group then $\delta[h](a_1, a_2) = \delta[h](a_2, a_1)$ for all $a_1, a_2 \in A$. Thus we get here for any elements a_1 and a_2 of an abelian subgroup A of H_1 that $c_1B_1(a_1, a_2) + c_2B_2(a_1, a_2) = c_1B_1(a_2, a_1) + c_2B_2(a_2, a_1)$. Taking $A = N$ we get $c_1 = 0$. Taking $A := \{(x, 0, t) \mid x, t \in \mathbb{R}\}$ we get $c_2 = 0$. □

The next two examples generalize the previous one in different directions. The first illustrates the need for Theorem 5.2, because the group in it is not a semidirect product of two abelian groups. So we cannot refer to the simpler case of Theorem 4.4.

Example 6.5. Let UT_4 be the group of upper triangular 4×4 matrices

$$\begin{bmatrix} 1 & x & t & s \\ 0 & 1 & y & u \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

over $\mathbb{R}$ with ones on the main diagonal.

The vector space $H^2_{\mathcal{C}}(UT_4, \mathbb{C})$ is five-dimensional with the set of equivalence classes of the following five functions T_1, T_2, T_3, T_4, T_5 as a basis.

$$T_1\left(\begin{bmatrix}1 & x_1 & t_1 & s_1\\ 0 & 1 & y_1 & u_1\\ 0 & 0 & 1 & z_1\\ 0 & 0 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & x_2 & t_2 & s_2\\ 0 & 1 & y_2 & u_2\\ 0 & 0 & 1 & z_2\\ 0 & 0 & 0 & 1\end{bmatrix}\right):=u_1z_2-z_1(u_2+y_1z_2),$$

$$T_2\left(\begin{bmatrix}1 & x_1 & t_1 & s_1\\ 0 & 1 & y_1 & u_1\\ 0 & 0 & 1 & z_1\\ 0 & 0 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & x_2 & t_2 & s_2\\ 0 & 1 & y_2 & u_2\\ 0 & 0 & 1 & z_2\\ 0 & 0 & 0 & 1\end{bmatrix}\right):=t_1y_2-y_1(t_2+x_1y_2),$$

$$T_3\left(\begin{bmatrix}1 & x_1 & t_1 & s_1\\ 0 & 1 & y_1 & u_1\\ 0 & 0 & 1 & z_1\\ 0 & 0 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & x_2 & t_2 & s_2\\ 0 & 1 & y_2 & u_2\\ 0 & 0 & 1 & z_2\\ 0 & 0 & 0 & 1\end{bmatrix}\right):=y_1(u_2+y_1z_2/2),$$

$$T_4\left(\begin{bmatrix}1 & x_1 & t_1 & s_1\\ 0 & 1 & y_1 & u_1\\ 0 & 0 & 1 & z_1\\ 0 & 0 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & x_2 & t_2 & s_2\\ 0 & 1 & y_2 & u_2\\ 0 & 0 & 1 & z_2\\ 0 & 0 & 0 & 1\end{bmatrix}\right):=x_1(t_2+x_1y_2/2),$$

$$T_5\left(\begin{bmatrix}1 & x_1 & t_1 & s_1\\ 0 & 1 & y_1 & u_1\\ 0 & 0 & 1 & z_1\\ 0 & 0 & 0 & 1\end{bmatrix},\begin{bmatrix}1 & x_2 & t_2 & s_2\\ 0 & 1 & y_2 & u_2\\ 0 & 0 & 1 & z_2\\ 0 & 0 & 0 & 1\end{bmatrix}\right):=x_1z_2.$$

Proof. Straightforward computations show that $T_1,\ldots,T_5$ are continuous cocycles. That they are linearly independent modulo $B^2_{\mathcal{C}}(UT_4,\mathbb{C})$ can be proved using the same idea as in Example 6.4. It is left to show that any continuous cocycle F is equivalent to a linear combination of $T_1,\ldots,T_5$.

UT_4 is solvable of rank $r=3$, and each element q can be decomposed uniquely as $q=q_1q_2q_3$ with $q_i\in Q_i$, where

$$Q_1=\left\{\begin{bmatrix}1 & 0 & 0 & s\\ 0 & 1 & 0 & u\\ 0 & 0 & 1 & z\\ 0 & 0 & 0 & 1\end{bmatrix}: z,u,s\in\mathbb{R}\right\},$$

$$Q_2=\left\{\begin{bmatrix}1 & 0 & t & 0\\ 0 & 1 & y & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}: t,y\in\mathbb{R}\right\},$$

$$Q_3=\left\{\begin{bmatrix}1 & x & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}: x\in\mathbb{R}\right\}.$$

Specifically, we write two arbitrary elements of UT_4 as $q=q_1q_2q_3$ and $p=p_1p_2p_3$, with decompositions ($i=1$ corresponding to q, $i=2$ to p)

$$\begin{bmatrix} 1 & x_i & t_i & s_i \\ 0 & 1 & y_i & u_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & s_i \\ 0 & 1 & 0 & u_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_i & 0 \\ 0 & 1 & y_i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We shall for $i=1,2,3$ identify Q_i with $(\mathbb{R}^{4-i},+)$ in the obvious way.

It is easy to check that UT_4 satisfies the conditions of Lemma 5.1. Applying Theorem 5.2 we obtain from (5.2) that

$$F(q_1q_2q_3, p_1p_2p_3) \simeq \sum_{i=1}^{3} \Psi_i(q_i, p_i^{q_{i+1}\cdots q_3}) + \sum_{j=1}^{2} \phi_j(q_{j+1}\cdots q_3, p_j),$$

where each Ψ_i is a continuous skew-symmetric bi-morphism satisfying additionally (5.3) and each ϕ_j fulfills (5.4) and (5.5). In detail, by Lemma 6.1 we have

$$\begin{aligned} \Psi_1(q_1,p_1) &= a_1[s_1u_2 - s_2u_1] + a_2[s_1z_2 - s_2z_1] + a_3[u_1z_2 - u_2z_1], \\ \Psi_2(q_2,p_2) &= a_4[t_1y_2 - t_2y_1],\ \Psi_3(q_3,p_3) = 0, \end{aligned}$$

for arbitrary constants $a_1, ..., a_4 \in \mathbb{C}$. Moreover, taking into account the additional condition (5.3) we find after some calculations that $a_1 = a_2 = 0$. Thus we have

$$\sum_{i=1}^{3} \Psi_i(q_i, p_i^{q_{i+1}\cdots q_3}) = a_3T_1(q,p) + a_4T_2(q,p).$$

Turning to the ϕ terms, first by (5.4) we see that there exist continuous maps $\phi_{11}, \phi_{12}, \phi_{13}, \phi_{21}, \phi_{22}$ such that

$$\begin{aligned} \phi_1(q_2q_3,p_1) &= \phi_{11}(q_2q_3)s_2 + \phi_{12}(q_2q_3)u_2 + \phi_{13}(q_2q_3)z_2, \\ \phi_2(q_3,p_2) &= \phi_{21}(q_3)t_2 + \phi_{22}(q_3)y_2. \end{aligned}$$

Next, by (5.5) we have also

$$\begin{aligned} \phi_1(kl,p_1) &= \phi_1(k,p_1^l) + \phi_1(l,p_1), \\ \phi_2(mn,p_2) &= \phi_2(m,p_2^n) + \phi_2(n,p_2), \end{aligned}$$

for all $k,l \in Q_2Q_3$ and $m,n \in Q_3$. Letting

$$k = \begin{bmatrix} 1 & x_3 & t_3 & 0 \\ 0 & 1 & y_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad l = \begin{bmatrix} 1 & x_4 & t_4 & 0 \\ 0 & 1 & y_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$m = \begin{bmatrix} 1 & x_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad n = \begin{bmatrix} 1 & x_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we calculate that

$$\begin{aligned}
&\phi_{11}(kl)s_2 + \phi_{12}(kl)u_2 + \phi_{13}(kl)z_2 \\
&\quad = \phi_{11}(k)(s_2 + x_4u_2 + t_4z_2) + \phi_{12}(k)(u_2 + y_4z_2) + \phi_{13}(k)z_2 \\
&\qquad + \phi_{11}(l)s_2 + \phi_{12}(l)u_2 + \phi_{13}(l)z_2, \\
&\phi_{21}(mn)t_2 + \phi_{22}(mn)y_2 \\
&\quad = \phi_{21}(m)(t_2 + x_4y_2) + \phi_{22}(m)y_2 + \phi_{21}(n)t_2 + \phi_{22}(n)y_2.
\end{aligned}$$

Comparing coefficients of s_2, u_2, z_2, t_2, y_2 yields respectively

$$\begin{aligned}
\phi_{11}(kl) &= \phi_{11}(k) + \phi_{11}(l), \\
\phi_{12}(kl) &= \phi_{11}(k)x_4 + \phi_{12}(k) + \phi_{12}(l), \\
\phi_{13}(kl) &= \phi_{11}(k)t_4 + \phi_{12}(k)y_4 + \phi_{13}(k) + \phi_{13}(l), \\
\phi_{21}(mn) &= \phi_{21}(m) + \phi_{21}(n), \\
\phi_{22}(mn) &= \phi_{21}(m)x_4 + \phi_{22}(m) + \phi_{22}(n).
\end{aligned}$$

The continuous solution of this system of equations is, by a lengthy but elementary computation, given by

$$\begin{aligned}
\phi_{11}(\hat{k}) = 0, \quad \phi_{12}(\hat{k}) &= b_3x + b_4y, \\
\phi_{13}(\hat{k}) &= b_5x + b_6y + b_3t + b_4y^2/2, \\
\phi_{21}(\hat{m}) = c_1x,\ \phi_{22}(\hat{m}) &= c_2x + c_1x^2/2,
\end{aligned}$$

for arbitrary constants $b_3, ..., b_6, c_1, c_2 \in \mathbb{C}$, where

$$\hat{k} = \begin{bmatrix} 1 & x & t & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \hat{m} = \begin{bmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus we have

$$\begin{aligned}
\phi_1(q_2q_3, p_1) &= b_3(x_1u_2 + t_1z_2) + b_4T_3(q,p) + b_5T_5(q,p) + b_6y_1z_2, \\
\phi_2(q_3, p_2) &= c_1T_4(q,p) + c_2x_1y_2.
\end{aligned}$$

Observe now that the maps $(q,p) \mapsto x_1u_2 + t_1z_2,\ y_1z_2,\ x_1y_2$ are continuous coboundaries with respective generators

$$\begin{bmatrix} 1 & x & t & s \\ 0 & 1 & y & u \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto -s,\ -u,\ -t.$$

Therefore we conclude that

$$\sum_{j=1}^{2} \phi_j(q_{j+1} \cdots q_3, p_j) \simeq b_4T_3(q,p) + b_5T_5(q,p) + c_1T_4(q,p),$$

yielding the required form for F. □

The next example generalizes Example 6.4 to higher-dimensional Heisenberg groups.

Example 6.6. Let $n \geq 2$ and let $H_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Heisenberg group with the group composition

$$(x_1, y_1, t_1)(x_2, y_2, t_2) := (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1 \cdot y_2),$$
$$\text{for } (x_1, y_1, t_1), (x_2, y_2, t_2) \in H_n.$$

(a) For each continuous cocycle $F : H_n \times H_n \to \mathbb{C}$ there exist complex skew-symmetric $n \times n$ matrices S_1 and S_2 and a complex $n \times n$ matrix A with $\operatorname{tr} A = 0$ such that

$$F((x_1, y_1, t_1), (x_2, y_2, t_2)) \simeq \langle x_1, S_1 x_2 \rangle + \langle y_1, S_2 y_2 \rangle + \langle x_1, A y_2 \rangle \tag{6.3}$$

for all $(x_1, y_1, t_1), (x_2, y_2, t_2) \in H_n$.

(b) Conversely, any function F of the form (6.3) is a continuous cocycle.

(c) The functions of the form (6.3) form a basis of $H^2_C(H_n, \mathbb{C})$: If $F = \delta[h]$ for some $h \in C(H_n)$, then $S_1 = S_2 = A = 0$.

In particular $\dim H^2_C(H_n, \mathbb{C}) = 2n^2 - n - 1$.

Proof. (a) H_n is the semidirect product $N \circledS Q$, where $N = \{(0, y, t)\}$ and $Q = \{(x, 0, 0)\}$ are closed abelian subgroups with N normal. Conjugation takes the form

$$n^q = (0, y, t)^{(x,0,0)} = (0, y, t + x \cdot y).$$

Below we will identify N with $(\mathbb{R}^{n+1}, +)$, and Q with $(\mathbb{R}^n, +)$, whenever convenient.

Any continuous cocycle F has according to Theorem 4.4, part 3, the form

$$F((x_1, y_1, t_1), (x_2, y_2, t_2)) \simeq \Psi_N((y_1, t_1), (y_2, t_2 + x_1 \cdot y_2)) + \Psi_Q(x_1, x_2)$$
$$+ \phi(x_1, (y_2, t_2)),$$

where $\Psi_N : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{C}$, $\Psi_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ are continuous, skew-symmetric, bi-additive maps, with

$$\Psi_N((y_1, t_1 + x \cdot y_1), (y_2, t_2 + x \cdot y_2)) = \Psi_N((y_1, t_1), (y_2, t_2)), \tag{6.4}$$

and where $\phi : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{C}$ is continuous, additive in its second component, and satisfies

$$\phi(x_1 + x_2, (y_2, t_2)) = \phi(x_1, (y_2, t_2 + x_2 \cdot y_2)) + \phi(x_2, (y_2, t_2)). \tag{6.5}$$

By Lemma 6.1 we get

$$\Psi_N((y_1, t_1), (y_2, t_2)) = \langle y_1, S_2 y_2 \rangle + c \cdot [t_1 y_2 - t_2 y_1],$$
$$\Psi_Q(x_1, x_2) = \langle x_1, S_1 x_2 \rangle,$$

for skew-symmetric matrices S_1, S_2 and a constant vector $c \in \mathbb{C}^n$. Now (6.4) requires that also

$$c \cdot [(x \cdot y_1)y_2 - (x \cdot y_2)y_1] = 0.$$

Since $n \geq 2$ this is impossible unless $c = 0$, hence

$$\Psi_N((y_1, t_1), (y_2, t_2 + x_1 \cdot y_2)) + \Psi_Q(x_1, x_2) = \langle x_1, S_1 x_2 \rangle + \langle y_1, S_2 y_2 \rangle.$$

Next, the continuity and additivity of ϕ in its second component yield the existence of continuous maps $V : \mathbb{R}^n \to \mathbb{C}^n$, $d : \mathbb{R}^n \to \mathbb{C}$ such that

$$\phi(x, (y, t)) = V(x) \cdot y + d(x)t,$$

and substituting this into (6.5) we find that

$$\begin{aligned} &V(x_1 + x_2) \cdot y_2 + d(x_1 + x_2)t_2 \\ &\qquad = V(x_1) \cdot y_2 + d(x_1)(t_2 + x_2 \cdot y_2) + V(x_2) \cdot y_2 + d(x_2)t_2. \end{aligned}$$

Comparing coefficients of t_2 we see that d is additive, hence $d(x) = d \cdot x$ for some constant vector $d \in \mathbb{C}^n$. Now the preceding equation becomes

$$[V(x_1 + x_2) - V(x_1) - V(x_2)] \cdot y_2 = (d \cdot x_1)(x_2 \cdot y_2).$$

Since the left hand side is symmetric in x_1 and x_2, so is the right hand side. Again, since $n \geq 2$ this cannot happen unless $d = 0$, and we arrive at the conclusion that V is additive. Being continuous it is linear. Hence we have $V(x) \cdot y = \langle x, Ay \rangle$ for some matrix A. Therefore $\phi(x, (y, t)) = \langle x, Ay \rangle$, giving F the claimed form, except for the fact that A may be chosen with zero trace. To see this consider $\alpha \in \mathbb{C}$ and define $h(x, y, t) := -\alpha t$. We find that $\delta[h]((x_1, y_1, t_1), (x_2, y_2, t_2)) = \alpha \langle x_1, y_2 \rangle = \langle x_1, \alpha I y_2 \rangle$ which implies that modulo coboundaries we may replace A by $A - \alpha I$ for any $\alpha \in \mathbb{C}$. In particular by the traceless matrix $A - n^{-1}(\operatorname{tr} A)I$.

(b) The verification consists of simple computations that we skip.

(c) Assume that $F = \delta[h]$ for some $h \in C(H_n)$ where F has the form (6.3). Proceeding as at the end of Example 6.4 we infer that $S_1 = S_2 = 0$, so what remains is that

$$\begin{aligned} h(x_1, y_1, t_1) + h(x_2, y_2, t_2) - h(x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1 \cdot y_2) = \langle x_1, A y_2 \rangle \\ \text{for all } (x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}. \end{aligned} \tag{6.6}$$

Since the right hand side of (6.6) is $C^\infty(H_n \times H_n)$ we get by Lemma 3.3 that $h \in C^\infty(H_n)$.

A differentiation of (6.6) with respect to t_1 yields

$$\frac{\partial h}{\partial t}(x_1, y_1, t_1) = \frac{\partial h}{\partial t}(x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1 \cdot y_2).$$

Choosing x_2, y_2 and t_2 judiciously we get

$$\frac{\partial h}{\partial t}(x_1, y_1, t_1) = \frac{\partial h}{\partial t}(0, 0, 0)$$

which implies that

$$h(x,y,t) = \alpha t + H(x,y) \text{ for all } (x,y,t) \in H_n,$$

where $\alpha := \partial h/\partial t(0,0,0) \in \mathbb{C}$ is a constant, and $H \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Substituting this result back into (6.6) gives us that

$$H(x_1,y_1) + H(x_2,y_2) - H(x_1+x_2, y_1+y_2) = x_1 \cdot (A + \alpha I)y_2$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}^n$. Differentiating this with respect to y_1 and then choosing x_2 and y_2 suitably we get $H(x,y) = \beta \cdot y + K(x)$ for all $x, y \in \mathbb{R}^n$, where $\beta \in \mathbb{C}^n$ is a constant and $K \in C^\infty(\mathbb{R}^n)$. Substituting this back into the identity for H yields

$$K(x_1) + K(x_2) - K(x_1+x_2) = x_1 \cdot (A + \alpha I)y_2 \text{ for all } x_1, x_2, y_2 \in \mathbb{R}^n.$$

The left hand side is independent of y_2, hence so is the right hand side:

$$x_1 \cdot (A + \alpha I)y_2 = x_1 \cdot (A + \alpha I)0 = 0.$$

This holds for all $x_1, y_2 \in \mathbb{R}^n$, so $A + \alpha I = 0$, i.e. $A = -\alpha I$. But $\operatorname{tr} A = 0$, so $\alpha = 0$ and hence $A = 0$. □

The results are quite different in the Examples 6.4 and 6.6. We used in an essential way during our discussion of Example 6.6 that $n \geq 2$. If we nevertheless take $n = 1$ in the conclusion of Example 6.6 we get from (6.3) that $F((x_1,y_1,t_1),(x_2,y_2,t_2)) = x_1 A y_2$. During our discussion of Example 6.4 we found that this function is a coboundary on H_1. So the solutions we get by taking $n = 1$ in Example 6.6 are all $\simeq 0$. That does not fit with Example 6.4 in which the cohomology space has dimension 2.

References

[1] J. Aczél, The general solution of two functional equations by reduction to functions additive in two variables and with the aid of Hamel bases, *Glasnik Mat.-Fiz. Astronom. (2)* 20 (1965), 65–73.

[2] V. Bargmann, On unitary ray representations of continuous groups, *Ann. of Math. (2)* 59 (1954), 1–46.

[3] B. Ebanks, General solution of the 2-cocycle functional equation on solvable groups, *Aequationes Math.* 73 (2007), 260–279.

[4] Ellis, R., *Locally compact transformation groups.* Duke Math. J. **24** (1957), 119–125.

[5] Z. Gajda, Difference properties of higher orders for continuity and Riemann integrability, *Colloq. Math.* 53 (1987), 275–288.

[6] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, vol. I*, Springer Verlag, Göttingen/Heidelberg, 1963.

[7] J. C. Houard, On invariance groups and Lagrangian theories, *J. Math. Phys.* **18** (1977), 502–516.

[8] J. Krause, Pedestrian approach to two-cocycles for unitary ray representations of Lie groups, *J. Math. Phys.* **28** (1987), 2268–2279.

[9] J.-M. Lévy-Leblond, The pedagogical role and epistemological significance of group theory in quantum mechanics, *Riv. Nuovo Cimento (2)* **4** (1974), 99–143.

[10] V. S. Varadarajan, *Geometry of Quantum Theory*, Second edition, Springer-Verlag, New York, 1985.