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**Abstract**

The aim of the present paper is to study the semimartingale property of continuous time moving averages driven by Lévy processes. We provide necessary and sufficient conditions on the kernel for the moving average to be a semimartingale in the natural filtration of the Lévy process, and when this is the case we also provide a useful representation. Assuming that the driving Lévy process is of unbounded variation, we show that the moving average is a semimartingale if and only if the kernel is absolutely continuous with a density satisfying an integrability condition.

*Keywords:* semimartingales; moving averages; Lévy processes; bounded variation; absolutely continuity; stable processes; fractional processes

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## 1 Introduction

The present paper is concerned with the semimartingale property of moving averages which are driven by Lévy processes. More precisely, let $(X_t)_{t\geq 0}$ be a moving average on the form

$$X_t = \int_0^t \phi(t-s)\, dZ_s, \qquad t \geq 0, \tag{1.1}$$

where $(Z_t)_{t\geq 0}$ is a Lévy process and $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ is a deterministic function for which the integral exists. We are interested in when $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale, where $(\mathcal{F}_t^Z)_{t\geq 0}$ denotes the natural filtration of $(Z_t)_{t\geq 0}$.

Moving averages occur naturally in many different contexts e.g. in stochastic Volterra equations (see e.g. Protter (1985)) and in stochastic delay equations (see e.g. Reiß et al. (2007)), but also in finance, turbulence and telecommunication. Moreover, it is often important that the process of interest is a semimartingale, and in particular the following two properties are crucial: Firstly, if $(X_t)_{t\geq 0}$ models an asset price which is locally bounded and satisfies the No Free Lunch with Vanishing Risk condition then $(X_t)_{t\geq 0}$ has to be an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale (see Delbaen and Schachermayer (1994, Theorem 7.2)). Secondly, it possible to define a "reasonable"

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stochastic integral $\int_0^t H_s\,dX_s$ for all locally bounded $(\mathcal{F}_t^Z)_{t\geq 0}$-predictable processes $(H_t)_{t\geq 0}$ if and only if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale due to the Bichteler-Dellacherie Theorem (see Bichteler (1981, Theorem 7.6)).

Let $(Z_t)_{t\geq 0}$ denote a general semimartingale, $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ be absolutely continuous with a bounded density and let $(X_t)_{t\geq 0}$ be given by (1.1). Then by a stochastic Fubini result it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale, see e.g. Protter (1985, Theorem 3.3) or Reiß et al. (2007, Theorem 5.2). In the case where $(Z_t)_{t\in\mathbb{R}}$ is a two-sided Wiener process, $\phi \in L^2(\mathbb{R}_+, \lambda)$ and $(X_t)_{t\geq 0}$ is given by

$$X_t = \int_{-\infty}^t \phi(t-s)\,dZ_s, \qquad t \geq 0,$$

Knight (1992, Theorem 6.5) shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $\phi$ is absolutely continuous with a square integrable density ($\mathcal{F}_t^{Z,\infty} := \sigma(Z_s : -\infty < s \leq t)$). Related results can be found in Cherny (2001), Cheridito (2004) and Basse (2008b). Moreover, results characterizing when $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale are given in Jeulin and Yor (1993) and Basse (2008a).

The above presented results only provide sufficient conditions on $\phi$ or are only concerned with the Brownian case. In the present paper we study the case where $(Z_t)_{t\geq 0}$ is a Lévy process and we provide necessary and sufficient conditions on $\phi$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Assume $(Z_t)_{t\geq 0}$ is of unbounded variation and has characteristic triplet $(\gamma, \sigma^2, \nu)$. Our main result is the following:

$(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density $\phi'$ satisfying

$$\int_0^t \int_{[-1,1]} \big(|x\phi'(s)|^2 \wedge |x\phi'(s)|\big)\,\nu(dx)\,ds < \infty, \qquad \forall t > 0, \text{ if } \sigma^2 = 0, \quad (1.2)$$

$$\int_0^t |\phi'(s)|^2\,ds < \infty, \qquad \forall t > 0, \text{ if } \sigma^2 > 0. \quad (1.3)$$

In the case where $(Z_t)_{t\geq 0}$ is a symmetric $\alpha$-stable Lévy process, (1.2) corresponds to $\phi' \in L^\alpha([0,t],\lambda)$ for all $t > 0$ when $\alpha \in (1,2)$ and to $|\phi'|\log^+(|\phi'|) \in L^1([0,t],\lambda)$ for all $t > 0$ when $\alpha = 1$. When $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale it can be decomposed as

$$X_t = \phi(0)Z_t + \int_0^t \Big(\int_0^u \phi'(u-s)\,dZ_s\Big)\,du, \qquad t \geq 0. \quad (1.4)$$

As a corollary of (1.4) it follows that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation if and only if it is absolutely continuous, which is also equivalent to $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density satisfying (1.2)-(1.3) and $\phi(0) = 0$.

Finally we study two-sided moving averages, i.e. where $(X_t)_{t\geq 0}$ is given by

$$X_t = \int_{-\infty}^t (\phi(t-s) - \psi(-s))\,dZ_s, \qquad t \geq 0,$$

$(Z_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process and $\phi, \psi\colon \mathbb{R} \to \mathbb{R}$ are deterministic functions for which the integral exists. The conditions on $\phi$ from the one-sided case translate into necessary conditions in the two-sided case. That is, if $(Z_t)_{t\in\mathbb{R}}$ is of unbounded variation and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale then $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density satisfying (1.2)-(1.3). Moreover, Knight (1992, Theorem 6.5) is extended from the Gaussian case to the $\alpha$-stable case with $\alpha \in (1,2]$. Several examples are considered, including fractional Lévy processes.

The paper is organized as follows. In Section 2 we collect some preliminary results. The main results are presented in Section 3. All proofs are given in Section 4. The two-sided case is considered in Section 5.

# 2 Preliminaries

Throughout the paper $(\Omega, \mathcal{F}, P)$ denotes a complete probability space. Let $(Z_t)_{t\geq 0}$ denote a Lévy process with characteristic triplet $(\gamma, \sigma^2, \nu)$, that is for $t \geq 0$, $E[e^{i\theta Z_t}] = e^{t\kappa(\theta)}$ for all $\theta \in \mathbb{R}$, where

$$\kappa(\theta) = i\gamma\theta - \sigma^2\theta^2/2 + \int \left(e^{i\theta s} - 1 - i\theta s 1_{\{|s|\leq 1\}}\right) \nu(ds), \qquad \theta \in \mathbb{R}.$$

For a general treatment of Lévy processes we refer to Sato (1999), Bertoin (1996) or Protter (2004). Let $f\colon \mathbb{R} \to \mathbb{R}$ denote a measurable function. Following Rajput and Rosiński (1989, page 460) we say that $f$ is $Z$-integrable if there exists a sequence of simple functions $(f_n)_{n\geq 1}$ such that $f_n \to f$ $\lambda$-a.s. and $\lim_n \int_A f_n(s)\, dZ_s$ exists in probability for all $A \in \mathcal{B}([0,t])$ and all $t > 0$ ($\lambda$ denotes the Lebesgue measure on $\mathbb{R}$). In this case we define $\int_0^t f(s)\, dZ_s$ as the limit in probability of $\int_0^t f_n(s)\, dZ_s$. By Rajput and Rosiński (1989, Theorem 2.7), $f$ is $Z$-integrable if and only if the following three conditions are satisfied for all $t > 0$:

$$\int_0^t f(s)^2\sigma^2\, ds < \infty, \tag{2.1}$$

$$\int_0^t \int \left(|xf(s)|^2 \wedge 1\right) \nu(dx)\, ds < \infty, \tag{2.2}$$

$$\int_0^t \left| f(s)\Big(\gamma + \int x(1_{\{|xf(s)|\leq 1\}} - 1_{\{|x|\leq 1\}})\, \nu(dx)\Big)\right| ds < \infty. \tag{2.3}$$

In this case $\int_0^t f(s)\, dZ_s$ is infinitely divisible with characteristic triplet $(\gamma_f, \sigma_f^2, \nu_f)$ given by

$$\begin{aligned}
\gamma_f &= \int_0^t f(s)\Big(\gamma + \int x(1_{\{|xf(s)|\leq 1\}} - 1_{\{|x|\leq 1\}})\, \nu(dx)\Big)\, ds, \\
\sigma_f^2 &= \int_0^t f(s)^2\sigma^2\, ds, \\
\nu_f(A) &= (\nu \times \lambda)((x,s) \in \mathbb{R} \times [0,t] : xf(s) \in A \setminus \{0\}), \qquad A \in \mathcal{B}(\mathbb{R}).
\end{aligned}$$

If $f$ is locally square integrable it is easily shown that (2.1)–(2.3) are satisfied and hence $\int_0^t f(s)\, dZ_s$ is well-defined for all $t \geq 0$. Note also that (2.3) is satisfied if $(Z_t)_{t\geq 0}$ is symmetric. Recall that $(Z_t)_{t\geq 0}$ is a symmetric $\alpha$-stable Lévy process with $\alpha \in (0,2]$ if $\gamma = \sigma^2 = 0$ and $\nu$ has density $s \mapsto c|s|^{-1-\alpha}$ for some $c > 0$ when

$\alpha \in (0,2)$, and $\nu = 0$ and $\gamma = 0$ when $\alpha = 2$. In this case (2.1)–(2.3) reduce to $f \in L^\alpha([0,t],\lambda)$ for all $t > 0$.

Let $I \subseteq \mathbb{R}$ denote an interval (not necessarily bounded) and $f$ be a function from $I$ into $\mathbb{R}$; $f$ is said to be of bounded variation if on each finite interval $[a,b] \subseteq I$ the total variation of $f$ is finite, that is

$$\mathrm{V}_{a,b}(f) := \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty, \tag{2.4}$$

where the sup is taken over all partitions $a = t_0 < \cdots < t_n = b$, $n \geq 1$ of $[a,b]$. Note that a Lévy process $(Z_t)_{t\geq 0}$ is of bounded variation if and only if $\int_{[-1,1]} |s|\,\nu(ds) < \infty$ and $\sigma^2 = 0$ (see e.g. Sato (1999, Theorem 21.9)). In addition, $f$ is said to be absolutely continuous if there exists a locally integrable function $h$ such that

$$f(t) - f(u) = \int_u^t h(s)\,ds, \qquad \forall\, u,t \in I,\ u \leq t,$$

and in this case $h$ is called the density of $f$. If $f$ is assumed measurable and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is another measurable function, then $f$ is said to have locally $g$-moment if

$$\int_u^t g(|f(s)|)\,ds < \infty, \qquad \forall\, u,t \in I,\ u \leq t. \tag{2.5}$$

If (2.5) is satisfied with $g(x) = x^\alpha$ for some $\alpha > 0$ then $f$ is said to have locally $\alpha$-moment.

An increasing family of $\sigma$-algebras $(\mathcal{F}_t)_{t\geq 0}$ is called a filtration if it satisfies the usual conditions of right-continuity and completeness. For each process $(Y_t)_{t\geq 0}$ we let $(\mathcal{F}_t^Y)_{t\geq 0}$ denote its the natural filtration, i.e. $(\mathcal{F}_t^Y)_{t\geq 0}$ is the least filtration for which $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t^Y)_{t\geq 0}$-adapted. Let $(\mathcal{F}_t)_{t\geq 0}$ denote a filtration. We say that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if it admits the following representation

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0,$$

where $(M_t)_{t\geq 0}$ is a càdlàg local $(\mathcal{F}_t)_{t\geq 0}$-martingale starting at 0 and $(A_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted, càdlàg, of bounded variation and starting at 0, and $X_0$ is $\mathcal{F}_0$-measurable. (Recall that càdlàg means right-continuous with left-hand limits).

We need the following standard notation: For functions $f,g\colon \mathbb{R} \to (0,\infty)$ we write $f(x) \approx g(x)$ as $x \to \infty$ if $f/g$ is bounded above and below on some interval $(K,\infty)$, where $K > 0$. Furthermore we write $f(x) = o(g(x))$ as $x \to \infty$ if $f(x)/g(x) \to 0$ as $x \to \infty$. A similar notation is used as $x \to 0$.

Assume $\nu$ has positive mass on $[-1,1]$. Similar to Marcus and Rosiński (2001) we let $\xi\colon [0,\infty) \to [0,\infty)$ be given by

$$\xi(x) = \int_{[-1,1]} \Big(|sx|^2 \wedge |sx|\Big)\,\nu(ds), \qquad x \geq 0. \tag{2.6}$$

Note that $\xi$ is 0 at 0, continuous and increasing and satisfies:

(i) $\xi(x)/x \to \int_{[-1,1]} |s| \, \nu(ds) \in (0,\infty]$ as $x \to \infty$,

(ii) If $\int_{[-1,1]} |s|^\alpha \, \nu(ds) < \infty$ for $\alpha \in (1,2]$ then $\xi(x) = o(x^\alpha)$ as $x \to \infty$.

To show (i)–(ii) let

$$H(x) = x \int\limits_{x^{-1} \le |s| \le 1} |s| \, \nu(ds) \qquad \text{and} \qquad K(x) = x^2 \int\limits_{|s| < x^{-1}} s^2 \, \nu(ds),$$

and note that $\xi(x) = H(x) + K(x)$ for $x > 1$. We have

$$\int\limits_{x^{-1} \le |s| \le 1} |s| \, \nu(ds) \le \xi(x) x^{-1} \le \int\limits_{[-1,1]} |s| \, \nu(ds), \qquad x > 1, \tag{2.7}$$

where the first inequality follows from $H \le \xi$ and the second from (2.6) since $|xs|^2 \wedge |xs| \le |xs|$. Hence by (2.7) and monotone convergence (i) follows. To show (ii) assume $\int_{[-1,1]} |s|^\alpha \, \nu(ds) < \infty$ for some $\alpha \in (1,2]$. For all $\epsilon > 0$ we have

$$\limsup_{x \to \infty} H(x) x^{-\alpha} \le \int\limits_{[-\epsilon,\epsilon]} |s|^\alpha \, \nu(ds),$$

and

$$K(x) x^{-\alpha} \le \int\limits_{|s| < x^{-1}} |s|^\alpha \, \nu(ds),$$

which shows $\xi(x) x^{-\alpha} \to 0$ as $x \to \infty$ and completes the proof of (ii).

Assume $\nu$ is absolutely continuous in a neighborhood of zero with a density $f$ satisfying $f(x) \approx |x|^{-\alpha-1}$ as $x \to 0$ for some $\alpha \in (0,2)$ (this is satisfied in the $\alpha$-stable case). An easy calculation shows:

(1) $\xi(x) \approx x^\alpha$ as $x \to \infty$ if $\alpha \in (1,2)$,

(2) $\xi(x) \approx x \log(x)$ as $x \to \infty$ if $\alpha = 1$,

(3) $\int_{[-1,1]} |s| \, \nu(ds) < \infty$ if $\alpha \in (0,1)$.

# 3 Main results

First let $(Z_t)_{t \ge 0}$ denote a nondeterministic Lévy process with characteristic triplet $(\gamma, \sigma^2, \nu)$ and $\phi \colon \mathbb{R}_+ \to \mathbb{R}$ be a measurable function which is $Z$-integrable (see (2.1)–(2.3)). Throughout this section we let $(X_t)_{t \ge 0}$ be the moving average

$$X_t = \int_0^t \phi(t-s) \, dZ_s, \qquad t \ge 0. \tag{3.1}$$

Theorem 3.1 below is the main result of the paper. It provides a complete characterization of when $(X_t)_{t \ge 0}$ is an $(\mathcal{F}_t^Z)_{t \ge 0}$-semimartingale. Recall the definition of the function $\xi$ in (2.6).

**Theorem 3.1.** *Assume $(Z_t)_{t\geq 0}$ is of unbounded variation. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density $\phi'$ which is locally square integrable when $\sigma^2 > 0$ and has locally $\xi$-moment when $\sigma^2 = 0$ (that is, $\phi'$ satisfies (1.2)–(1.3)).*

*Assume $(Z_t)_{t\geq 0}$ is of bounded variation. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if it is of bounded variation which is also equivalent to $\phi$ is of bounded variation.*

In particular, if $\sigma^2 = 0$, $\int_{[-1,1]} |x|^\alpha \, \nu(dx) < \infty$ for some $\alpha \in (1,2]$ and $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density having locally $\alpha$-moment then it follows by (ii) on page 5 and the above theorem that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. In the case where $(X_t)_{t\geq 0}$ is a semimartingale the next proposition provides a useful representation of this process.

**Proposition 3.2.** *Assume $(Z_t)_{t\geq 0}$ is of unbounded variation and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Then*

$$X_t = \phi(0) Z_t + \int_0^t \Big( \int_0^u \phi'(u-s)\, dZ_s \Big)\, du, \qquad t \geq 0,$$

*where $\phi'$ denotes the density of $\phi$ and $(\int_0^u \phi'(u-s)\, dZ_s)_{u\geq 0}$ is chosen measurable.*

Hence we obtain the following corollary.

**Corollary 3.3.** *Assume $(Z_t)_{t\geq 0}$ is of unbounded variation. Then the following four statements are equivalent:*

*(a) $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation,*

*(b) $(X_t)_{t\geq 0}$ is absolutely continuous,*

*(c) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale and $\phi(0) = 0$,*

*(d) $\phi$ is absolutely continuous with a density satisfying (1.2) ″=(1.3) and $\phi(0) = 0$.*

In the symmetric $\alpha$-stable case with $\alpha \in (1,2)$ the equivalence between (b) and (d) follows by Rosiński (1986, Theorem 6.1). Braverman and Samorodnitsky (1998) studies, among other things, processes $(Y_t)_{t\geq 0}$ on the form $Y_t = \int_0^t f(t,s)\, dZ_s$, where $(Z_t)_{t\geq 0}$ is a symmetric Lévy process and $f$ is a deterministic function. Their Theorem 5.1 provides necessary and sufficient conditions on $f(t,s)$ for $(X_t)_{t\geq 0}$ to be absolutely continuous. In Marcus and Rosiński (2003) and Kwapień et al. (2006) necessary and sufficient conditions on $\phi$ are obtained for $(X_t)_{t\geq 0}$ to have locally bounded or continuous sample paths.

The next corollary follows by Theorem 3.1 and the estimates on $\xi$ given in (1)–(3) on page 5.

**Corollary 3.4.** *Assume $\sigma^2 = 0$ and $\nu$ is absolutely continuous in a neighborhood of zero with a density $f$ satisfying $f(x) \approx |x|^{-\alpha-1}$ as $x \to 0$ for some $\alpha \in (0,2)$ (this is satisfied in the $\alpha$-stable case with $\alpha \in (0,2)$). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if* *(i)* $\phi$ *is absolutely continuous with a density having locally* $\alpha$*-moment when* $\alpha \in (1,2)$*,*

*(ii)* $\phi$ *is absolutely continuous with a density having locally* $x\log^+(x)$*-moment when* $\alpha = 1$*,*

*(iii)* $\phi$ *is of bounded variation when* $\alpha \in (0,1)$*.*

Here $\log^+$ denotes the positive part of log, i.e. $\log^+(x) = \log(x)$ for $x \geq 1$ and 0 otherwise.

In the following let $(X_t)_{t\geq 0}$ be the Riemann-Liouville fractional integral given by

$$X_t = \int_0^t (t-s)^\gamma \, dZ_s, \qquad t \geq 0, \tag{3.2}$$

where $\gamma$ is such that the integral exists. If $(Z_t)_{t\geq 0}$ is a Wiener process and $\gamma > -\frac{1}{2}$, $(X_t)_{t\geq 0}$ is called a Lévy fractional Brownian motion (see Mandelbrot and Van Ness (1968, page 424)). Assume $(Z_t)_{t\geq 0}$ has no Brownian component (i.e. $\sigma^2 = 0$). For $(X_t)_{t\geq 0}$ to be well-defined it is necessary that one of the following (a)-(c) are satisfied:

(a) $\gamma > -\frac{1}{2}$,

(b) $\gamma = -\frac{1}{2}$ and $\int_{[-1,1]} x^2 |\log|x|| \, \nu(dx) < \infty$,

(c) $\gamma < -\frac{1}{2}$ and $\int_{[-1,1]} |x|^{-1/\gamma} \, \nu(dx) < \infty$.

Condition (a) is also sufficient for $(X_t)_{t\geq 0}$ to be well-defined and when $(Z_t)_{t\geq 0}$ is symmetric, the conditions (a)-(c) are both necessary and sufficient for $(X_t)_{t\geq 0}$ to be well-defined. When $\gamma = 0$, $(X_t)_{t\geq 0} = (Z_t)_{t\geq 0}$; thus let us assume $\gamma \neq 0$. As a consequence of Theorem 3.1 we have the following.

**Corollary 3.5.** *Let* $(X_t)_{t\geq 0}$ *be given by* (3.2) *and assume* $(Z_t)_{t\geq 0}$ *has no Brownian component. Then* $(X_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^Z)_{t\geq 0}$*-semimartingale if and only if one of the following* (1)-(3) *is satisfied:*

*(1)* $\gamma > \frac{1}{2}$*,*

*(2)* $\gamma = \frac{1}{2}$ *and* $\int_{[-1,1]} x^2 |\log|x|| \, \nu(dx) < \infty$*,*

*(3)* $\gamma \in (0, \frac{1}{2})$ *and* $\int_{[-1,1]} |x|^{1/(1-\gamma)} \, \nu(dx) < \infty$*.*

Note that $1/(1-\gamma) \in (1,2)$ when $\gamma \in (0, \frac{1}{2})$. Let us in particular consider

$$X_t = \int_0^t (t-s)^{H-1/\alpha} \, dZ_s, \qquad t \geq 0,$$

where $(Z_t)_{t\geq 0}$ is a symmetric $\alpha$-stable Lévy process with $\alpha \in (0,2]$ and $H > 0$ (note that $(X_t)_{t\geq 0}$ is well-defined). To avoid trivialities assume $H \neq 1/\alpha$. As a consequence of Corollary 3.5 ($\alpha \in (0,2)$) and Theorem 3.1 ($\alpha = 2$) it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if and only if $H > 1$ when $\alpha \in [1,2]$ or $H > 1/\alpha$ when $\alpha \in (0,1)$.

# 4 Proofs

Throughout this section $(X_t)_{t\geq 0}$ is given by (3.1). We extend $\phi$ to a function from $\mathbb{R}$ into $\mathbb{R}$ by setting $\phi(s) = 0$ for $s \in (-\infty, 0)$. For any function $f\colon \mathbb{R} \to \mathbb{R}$, let $\Delta_n f$ denote the function $s \mapsto n(f(1/n+s) - f(s))$ for all $n \geq 1$. We start by the following extension of Hardy and Littlewood (1928, Theorem 24).

**Lemma 4.1.** *Let $I$ be either $\mathbb{R}_+$ or $\mathbb{R}$, $f\colon I \to \mathbb{R}$ be locally integrable and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing convex function satisfying $g(x)/x \to \infty$ as $x \to \infty$. Then $f$ is absolutely continuous with a density having locally $g$-moment if and only if $(g(|\Delta_n f|))_{n\geq 1}$ is bounded in $L^1([a,b],\lambda)$ for all $a, b \in I$ with $a < b$. In this case $(g(\frac{f(t+\cdot)-f(\cdot)}{t}))_{t\in(0,b)}$ is bounded in $L^1([a,b],\lambda)$ for all $a, b \in I$ with $a < b$.*

If $(Z_t)_{t\geq 0}$ is of unbounded variation the above lemma can be applied with $\xi$ playing the role of $g$ ($\xi$ is given by (2.6)), since in this case $\xi$ satisfies all the conditions imposed on $g$ except $\xi$ is not convex. But $h$, defined by $h(x) = x^2 1_{\{x\leq 1\}} + (2x-1)1_{\{x>1\}})$ for all $x \geq 0$, is convex and if we let

$$g(x) = \int_{[-1,1]} h(|xs|)\,\nu(ds), \qquad x \geq 0, \tag{4.1}$$

then $g$ satisfies all the conditions in the lemma and $g/2 \leq \xi \leq g$. Thus, if $f\colon I \to \mathbb{R}$ is locally integrable then $f$ is absolutely continuous with a density having locally $\xi$-moment if and only if $(\xi(\Delta_n f))_{n\geq 1}$ is bounded in $L^1([a,b],\lambda)$ for all $a, b \in I$ with $a < b$.

*Proof.* Note that $g$ is continuous and $x \mapsto g(|x|)$ is a convex function from $\mathbb{R}$ into $\mathbb{R}$, since $g$ is increasing and convex. Let $a, b \in I$ satisfying $a < b$ be given and assume $(g(|\Delta_n f|))_{n\geq 1}$ is bounded in $L^1([a,b],\lambda)$. Since $g(x)/x \to \infty$ as $x \to \infty$, $\{\Delta_n f : n \geq 1\}$ is uniformly integrable and hence weakly sequentially compact in $L^1([a,b],\lambda)$ (see e.g. Dunford and Schwartz (1957, Chapter IV.8, Corollary 11)). Choose a subsequence $(n_k)_{k\geq 1}$ and an $h \in L^1([a,b],\lambda)$ such that $\Delta_{n_k} f \to h$ in the weak $L^1([a,b],\lambda)$-topology. For all $c, d \in [a,b]$ with $c < d$ we have

$$\int_c^d \Delta_{n_k} f\, d\lambda \to \int_c^d h\, d\lambda, \qquad \text{as } k \to \infty.$$

Moreover,

$$\begin{aligned}\int_c^d \Delta_{n_k} f\, d\lambda &= n_k\Big(\int_{c+1/n_k}^{d+1/n_k} f\, d\lambda - \int_c^d f\, d\lambda\Big)\\ &= n_k \int_d^{d+1/n_k} f\, d\lambda - n_k \int_c^{c+1/n_k} f\, d\lambda \to f(d) - f(c), \qquad \text{as } k \to \infty,\end{aligned}$$

for $\lambda \times \lambda$-a.a. $c < d$. Thus, we conclude that $f$ is absolutely continuous with density $h$. Choose a sequence $(\kappa_n)_{n\geq 1}$ of convex combinations of $(\Delta_{n_k} f)_{k\geq 1}$ such that $\kappa_n \to h$

in $L^1([a,b],\lambda)$ and a subsequence $(b_n)_{n\geq 1}$ such that $\kappa_{b_n} \to h$ $\lambda$-a.s. By Fatou's lemma and convexity and continuity of $g$ we have

$$\int_a^b g(|h|)\,d\lambda \leq \liminf_{n\to\infty} \int_a^b g(|\kappa_{b_n}|)\,d\lambda \leq \sup_{k\geq 1} \int_a^b g(|\Delta_{n_k} f|)\,d\lambda < \infty,$$

which shows that $h$ has $g$-moment on $[a,b]$. This completes the proof of the *if*-part.

Assume conversely that $f$ is absolutely continuous with a density, $h$, having locally $g$-moment. For all $t \in (0,b)$, we have by Jensen's inequality that

$$\begin{aligned}\int_a^b g\Big(\Big|t^{-1}\int_s^{s+t} h(u)\,du\Big|\Big)\,ds &\leq \int_a^b \Big(t^{-1}\int_0^t g(|h(u+s)|)\,du\Big)\,ds \\ &= t^{-1}\int_0^t \int_a^b g(|h(u+s|)\,ds\,du \leq \int_a^{2b} g(|h(s|)\,ds < \infty,\end{aligned}$$

which shows that $(g(\frac{f(t+\cdot)-f(\cdot)}{t}))_{t\in(0,b)}$ is bounded in $L^1([a,b],\lambda)$ and completes the proof. □

In following we are going to use two Lévy-Itô decompositions of $(Z_t)_{t\geq 0}$ (see e.g. Sato (1999, Theorem 19.2)).

(a) Decompose $(Z_t)_{t\geq 0}$ as $Z_t = Z_t^1 + Z_t^2$, where $(Z_t^1)_{t\geq 0}$ and $(Z_t^2)_{t\geq 0}$ are two independent Lévy processes with characteristic triplets $(0,\sigma^2,\nu_1)$ respectively $(\gamma,0,\nu_2)$, where $\nu_1 = \nu|_{[-1,1]}$ and $\nu_2 = \nu|_{[-1,1]^c}$. $(Z_t^1)_{t\geq 0}$ and $(Z_t^2)_{t\geq 0}$ are $(\mathcal{F}_t^Z)_{t\geq 0}$-adapted. Moreover, when $\phi$ is locally bounded we let

$$X_t^1 = \int_0^t \phi(t-s)\,dZ_s^1, \quad \text{and} \quad X_t^2 = \int_0^t \phi(t-s)\,dZ_s^2, \qquad t\geq 0.$$

(b) Decompose $(Z_t)_{t\geq 0}$ as $Z_t = W_t + Y_t$, where $(W_t)_{t\geq 0}$ is a Wiener process with variance parameter $\sigma^2$ and $(Y_t)_{t\geq 0}$ is a Lévy process with characteristic triplet $(\gamma,0,\nu)$. $(W_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent and $(\mathcal{F}_t^Z)_{t\geq 0}$-adapted. Moreover, let

$$X_t^W = \int_0^t \phi(t-s)\,dW_s, \quad \text{and} \quad X_t^Y = \int_0^t \phi(t-s)\,dY_s, \qquad t\geq 0.$$

If $\sigma^2 = 0$ and $(X_t)_{t\geq 0}$ is càdlàg it follows by Rosiński (1989, Theorem 4) and a symmetrization argument that by modification on a set of Lebesgue measure 0, we may and do chose $\phi$ càdlàg.

The following lemma is closely related to Knight (1992, Theorem 6.5).

**Lemma 4.2.** *We have the following:*

*(i) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale if $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density.*

*(ii) Assume $(Z_t)_{t\geq 0}$ is a Wiener process. Then $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density if $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale.*

*Proof.* (i): Decompose $(Z_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ as in (a) above. Since both $\phi$ and $(Z^2_t)_{t\geq 0}$ are càdlàg and of bounded variation, $(X^2_t)_{t\geq 0}$ is càdlàg and of bounded variation as well. Hence, it is enough to show $(X^1_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale. Since

$$X^1_t = \int_0^t (\phi(t-s) - \phi(0))\, dZ^1_s + \phi(0) Z^1_t, \qquad t \geq 0,$$

we may and do assume $\phi(0) = 0$. Then, $\phi$ is absolutely continuous on $\mathbb{R}$ with locally square integrable density and hence for all $T > 0$, $\|\frac{\phi(t+\cdot)-\phi(\cdot)}{t}\|_{L^2([-T,T],\lambda)} \leq K$ for some constant $K > 0$ and all $t \in (0, T]$ by Lemma 4.1 with $g(x) = x^2$. By letting $c = E[|Z^1_1|^2]$ we have

$$E[(X^1_t - X^1_u)^2] = c\|\phi(t-\cdot) - \phi(u-\cdot)\|^2_{L^2([0,t],\lambda)} \leq cK^2(t-u)^2, \qquad \forall\, u, t \in [0, T],$$

which by the Kolmogorov-Čentsov Theorem (see Karatzas and Shreve (1991, Chapter 2, Theorem 2.8)) shows that $(X^1_t)_{t\geq 0}$ has a continuous modification (also to be denoted $(X^1_t)_{t\geq 0}$). Moreover, for all $0 = t_0 < \cdots < t_n = T$ we have

$$E[\sum_{i=1}^n |X^1_{t_i} - X^1_{t_{i-1}}|] \leq \sum_{i=1}^n \|X^1_{t_i} - X^1_{t_{i-1}}\|_{L^2(P)} \leq KT,$$

which shows that $(X^1_t)_{t\geq 0}$ is of integrable variation and hence an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale.

To show (ii) assume $(Z_t)_{t\geq 0}$ is a standard Wiener process and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale. Since $(X_t)_{t\geq 0}$ is a Gaussian process, Stricker (1983, Proposition 4+5) shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-quasimartingale on each compact interval $[0, N]$. Hence for $0 \leq u \leq t$ we have

$$\begin{aligned} E[|E[X_t - X_u|\mathcal{F}^Z_u]|] &= E[|\int_0^u \big(\phi(t-s) - \phi(u-s)\big)\, dZ_s|] \\ &= \sqrt{\frac{2}{\pi}} \|\int_0^u \big(\phi(t-s) - \phi(u-s)\big)\, dZ_s\|_{L^2(P)} \\ &= \sqrt{\frac{2}{\pi}} \Big(\int_0^u \big(\phi(t-s) - \phi(u-s)\big)^2\, ds\Big)^{1/2} \\ &= \sqrt{\frac{2}{\pi}} \Big(\int_0^u \big(\phi(t-u+s) - \phi(s)\big)^2\, ds\Big)^{1/2}, \end{aligned}$$

where the second equality follows by Gaussianity. Hence

$$\sum_{i=1}^{nN} E[|E[X_{i/n} - X_{(i-1)/n}|\mathcal{F}^Z_{(i-1)/n}]|] \geq \frac{Nn}{\sqrt{\pi 2}} \Big(\int_0^{N/2} \big(\phi(1/n+s) - \phi(s)\big)^2\, ds\Big)^{1/2}, \quad (4.2)$$

and since $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-quasimartingale on $[0, N]$ the left-hand side of (4.2) is bounded in $n$, which shows that $(\Delta_n \phi)_{n\geq 1}$ is bounded in $L^2([0, N/2], \lambda)$. By Lemma 4.1 with $g(x) = x^2$ this shows that $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density. $\square$

**Lemma 4.3.** *If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale then $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z^1})_{t\geq 0}$-semimartingale.*

*Proof.* Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale, fix $T > 0$ and let

$$A := \{\Delta Z_t^2 = 0 \ \forall t \in [0, T]\}.$$

Note that $P(A) > 0$ and $(Z_t^1)_{t\geq 0}$ is $P$-independent of $A$. Let $Q^A$ denote the probability measure given by $Q^A(B) := P(B \cap A)/P(A)$. $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale under $Q^A$, since $Q^A$ is absolutely continuous with respect to $P$. Moreover, since $(Z_t)_{t\geq 0}$ and $(Z_t^1)_{t\geq 0}$ are $Q^A$-indistinguishable it follows that $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z^1})_{t\geq 0}$-semimartingale under $Q^A$ and since $A$ is independent of $(Z_t^1)_{t\geq 0}$ this is also true under $P$. □

In the next lemma we study the jump structure of $(X_t)_{t\geq 0}$.

**Lemma 4.4.** *Assume $\sigma^2 = 0$ and $(X_t)_{t\geq 0}$ is càdlàg. Then $(\Delta X_t 1_{\{\Delta Z_t \neq 0\}})_{t\geq 0}$ and $(\phi(0)\Delta Z_t)_{t\geq 0}$ are indistinguishable.*

Before proving the lemma we note the following:

*Remark* 4.5.

(a) Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ denote two independent càdlàg processes such that $P(\Delta X_t = 0) = P(\Delta Y_t = 0) = 1$ for all $t \geq 0$. Then as a consequence of Tonelli's Theorem we have $P(\Delta X_t \Delta Y_t = 0,\ \forall t \geq 0) = 1$.

(b) If $\nu$ is concentrated on $[-1, 1]$ then the mapping $t \mapsto \int_0^t \phi(t-s)\, dZ_s$ is continuous from $\mathbb{R}_+$ into $L^1(P)$. This follows by approximating $\phi$ with continuous functions.

*Proof of Lemma 4.4.* Since $X_t = \int_0^t (\phi(t-s) - 1)\, dZ_s + Z_t$ we may and do assume $\phi(0) \neq 0$. Recall from page 9 that $\phi$ is chosen càdlàg; moreover $\Delta\phi(0) = \phi(0)$.

First we show the lemma in the case where $\nu$ is a finite measure. Let $\tau_n$ denote the time of the $n$th jump of $(Z_t)_{t\geq 0}$ ($(\tau_{n+1} - \tau_n)_{n\geq 1}$ is thus an i.i.d. sequence of exponential distributions) and let $(\sigma_n)_{n\geq 1} \subseteq [0, \infty)$ denote the jump times of $\phi$. Note that the event

$$B := \{\exists\, (j, k) \neq (j', k') : \tau_j + \sigma_k = \tau_{j'} + \sigma_{k'}\},$$

has probability zero. Since $(Z_t)_{t\geq 0}$ only has finitely many jumps on each compact interval we may regard $(X_t)_{t\geq 0}$ as a pathwise Lebesgue-Stieltjes integral and hence it follows that

$$(\Delta X_t)_{t\geq 0} = \Big(\sum_{k\geq 1} \Delta Z_{t-\sigma_k} \Delta\phi(\sigma_k)\Big)_{t\geq 0}.$$

Let us show that on $B^c$ the series $\sum_{k\geq 1} \Delta Z_{t-\sigma_k}\Delta\phi(\sigma_k)$ has at most one term which differs from zero for all $t \geq 0$. Indeed, to see this assume that $\Delta Z_{t-\sigma_k}\Delta\phi(\sigma_k)$ and $\Delta Z_{t-\sigma_{k'}}\Delta\phi(\sigma_{k'})$ both differ from zero, where $k \neq k'$. Then there exist $n, n' \geq 1$ such that $\tau_n = t - \sigma_k$ and $\tau_{n'} = t - \sigma_{k'}$ which implies $\tau_n + \sigma_k = \tau_{n'} + \sigma_{k'}$, and hence we have a contradiction. In particular, if $\Delta Z_t \neq 0$ then $\Delta Z_t \Delta\phi(0) \neq 0$ and thus $\Delta X_t = \Delta Z_t \Delta\phi(0) = \phi(0)\Delta Z_t$.

Now let $(Z_t)_{t\geq 0}$ be a general Lévy process for which $\sigma^2 = 0$. For each $n \geq 1$, decompose $(Z_t)_{t\geq 0}$ as $Z_t = Y_t^n + U_t^n$, where $(Y_t^n)_{t\geq 0}$ and $(U_t^n)_{t\geq 0}$ are two independent Lévy processes with characteristic triplets $(0, 0, \nu|_{[-1/n,1/n]})$ respectively $(0, 0, \nu|_{[-1/n,1/n]^c})$. Moreover, set

$$X_t^{Y^n} = \int_0^t \phi(t-s)\, dY_s^n \quad \text{and} \quad X_t^{U^n} = \int_0^t \phi(t-s)\, dU_s^n.$$

Since $(U_t^n)_{t\geq 0}$ has piecewise constant sample paths the second integral is a pathwise Lebesgue-Stieltjes integral. Hence $(X_t^{U^n})_{t\geq 0}$ is càdlàg and it follows that $(X_t^{Y^n})_{t\geq 0}$ is càdlàg as well. Set

$$\begin{aligned} C &:= \bigcap_{n\geq 1} \{\Delta X_t^{Y^n} \Delta U_t^n = 0,\ \forall t \geq 0\}, \\ D &:= \bigcap_{n\geq 1} \{\Delta X_t^{U^n} 1_{\{\Delta U_t^n \neq 0\}} = \phi(0)\Delta U_t^n,\ \forall t \geq 0\}. \end{aligned}$$

From Remark 4.5 (b) it follows that $P(\Delta X_t^{Y^n} = 0) = 1$ for all $t \geq 0$ which together with Remark 4.5 (a) shows that $C$ has probability one. Moreover, from the first part of the proof it follows that $D$ has probability one. When $\Delta Z_t \neq 0$, choose $n \geq 1$ such that $|\Delta Z_t| > 1/n$. Thus, $\Delta U_t^n \neq 0$, and hence $\Delta X_t^{Y^n} = 0$ on $C$, which shows $\Delta X_t = \Delta X_t^{U^n} = \phi(0)\Delta U_t^n = \phi(0)\Delta Z_t$ on $C \cap D$ and completes the proof. □

**Lemma 4.6.** *Assume $\sigma^2 = \gamma = 0$, $\nu$ is concentrated on $[-1, 1]$ and $(X_t)_{t\geq 0}$ is a special $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Then $(\phi(0)Z_t)_{t\geq 0}$ is the martingale component of $(X_t)_{t\geq 0}$.*

*Proof.* Let $X_t = M_t + A_t$ denote the canonical decomposition of $(X_t)_{t\geq 0}$. Since $(Z_t)_{t\geq 0}$ is a Lévy process, it is quasi-left-continuous (see Jacod and Shiryaev (2003, Chapter II, Corollary 4.18)) and thus there exists a sequence of totally inaccessible stopping times $(\tau_n)_{n\geq 1}$ which exhausts the jumps of $(Z_t)_{t\geq 0}$. On the other hand, since $(A_t)_{t\geq 0}$ is predictable there exists a sequence of predictable times $(\sigma_n)_{n\geq 1}$ which exhausts the jumps of $(A_t)_{t\geq 0}$. From the martingale representation theorem for Lévy processes (see Jacod and Shiryaev (2003, Chapter III, Theorem 4.34)) it follows that $(M_t)_{t\geq 0}$ is a purely discontinuous martingale which jumps only when $(Z_t)_{t\geq 0}$ does. Furthermore, since

$$P(\exists\, n, k \geq 1 : \tau_n = \sigma_k < \infty) = 0,$$

Lemma 4.4 shows

$$\phi(0)\Delta Z_{\tau_n} = \Delta X_{\tau_n} = \Delta M_{\tau_n} + \Delta A_{\tau_n} = \Delta M_{\tau_n}, \quad P\text{-a.s. on } \{\tau_n < \infty\}\ \forall\, n \geq 1.$$

Hence $(\Delta M_t)_{t\geq 0}$ and $(\phi(0)\Delta Z_t)_{t\geq 0}$ are indistinguishable which implies that $(M_t)_{t\geq 0}$ and $(\phi(0)Z_t)_{t\geq 0}$ are indistinguishable since they both are purely discontinuous martingales (see Jacod and Shiryaev (2003, Chapter I, Corollary 4.19)). This completes the proof. □

The following lemma is concerned with the bounded variation case and it relies on an inequality by Marcus and Rosiński (2001).

**Lemma 4.7.** *Assume $\gamma = \sigma^2 = 0$, $\nu$ is concentrated on $[-1,1]$ and $(Z_t)_{t\geq 0}$ is of unbounded variation. Then $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation if and only if $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density having locally $\xi$-moment and $\phi(0) = 0$.*

Recall the definition of $\Delta_n\phi$ on page 8 and of $\mathrm{V}_{0,t}(f)$ in (2.4).

*Proof.* We start by showing the following (i) and (ii) under the assumptions stated in the lemma:

(i) If $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation, then $(X_t)_{t\geq 0}$ is of integrable variation.

(ii) For all $N \geq 1$ we have

$$\begin{aligned}\frac{N}{8}\sup_{n\geq 1}\Big\{\Big(\int_{-N/2}^{N/2}\xi(|\Delta_n\phi(s)|)\,ds\Big)\wedge\Big(\int_{-N/2}^{N/2}\xi(|\Delta_n\phi(s)|)\,ds\Big)^{1/2}\Big\} \qquad (4.3)\\ \leq E[\mathrm{V}_{0,N}(X)] \leq 3N\sup_{n\geq 1}\Big\{\int_{-N}^{N}\xi(|\Delta_n\phi(s)|)\,ds + 1\Big\}.\end{aligned}$$

To show (i) assume $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation. According to Rosiński (1989, Theorem 4) we may and do assume that $\phi$ is of bounded variation. Fix $t_0 > 0$ and let $T := [0,t_0]\cap\mathbb{Q}$. Moreover, let $\underline{X}\colon\Omega\to\mathbb{R}^T$ denote the canonical random element induced by $(X_t)_{t\in T}$ and let $\mu$ be given by

$$\mu(A) = (\lambda\times\nu)\left((s,x)\in[0,t_0]\times\mathbb{R} : x\phi(\cdot - s)\in A\setminus\{\underline{0}\}\right), \qquad A\in\mathcal{B}(\mathbb{R}^T). \quad (4.4)$$

For all $t_1,\dots,t_n\in T$, $(X_{t_1},\dots,X_{t_n})$ is infinitely divisible with Lévy measure $\mu\circ p^{-1}_{t_1,\dots,t_n}$, where $p_{t_1,\dots,t_n}(f) = (f(t_1),\dots,f(t_n))$ for all $f\in\mathbb{R}^T$. For $f\in\mathbb{R}^T$ let $q(f)$ denote the total variation of $f$ on $T$. Then $q\colon\mathbb{R}^T\to[0,\infty]$ is clearly a lower-semicontinuous pseudonorm on $\mathbb{R}^T$. Since $\nu$ has compact support and $\phi$ is of bounded variation there exists an $r_0 > 0$ such that $\mu(f\in\mathbb{R}^T : q(f) > r_0) = 0$ and hence by Lemma 2.2 in Rosiński and Samorodnitsky (1993), $E[e^{\epsilon q(\underline{X})}] < \infty$ for some $\epsilon > 0$. In particular $(X_t)_{t\geq 0}$ is of integrable variation.

(ii): From Marcus and Rosiński (2001, Corollary 1.1) we have

$$\tfrac{1}{4}\min(a_{i,n}, a_{i,n}^{1/2}) \leq E[|n(X_{i/n} - X_{(i-1)/n})|] \leq 3\max(a_{i,n}, a_{i,n}^{1/2}), \qquad (4.5)$$

where

$$a_{i,n} := \int_0^\infty\int_{[-1,1]}\left(|xf_{i,n}(s)|^2\wedge|xf_{i,n}(s)|\right)\nu(dx)\,ds = \int_{-1/n}^{(i-1)/n}\xi(|\Delta_n\phi(s)|)\,ds,$$

and $f_{i,n}(s) = n(\phi(i/n - s) - \phi((i-1)/n - s))$. Since

$$E[\mathrm{V}_{0,N}(X)] = \sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{nN}E[|n(X_{i/n} - X_{(i-1)/n})|],$$

it follows that

$$\frac{N}{2}\sup_{n\geq 1}\inf_{nN/2<i\leq Nn} E[|n(X_{i/n}-X_{(i-1)/n})|]\leq E[\mathrm{V}_{0,N}(X)]$$
$$\leq N\sup_{n\geq 1}\sup_{1\leq i\leq Nn} E[|n(X_{i/n}-X_{(i-1)/n})|],$$

which by (4.5) shows (4.3).

By combining (i) and (ii) it follows that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation if and only if $(\xi(\Delta_n\phi))_{n\geq 1}$ is bounded in $L^1([-a,a],\lambda)$ for all $a>0$. Hence the discussion just below Lemma 4.1 completes the proof, since $(Z_t)_{t\geq 0}$ is of unbounded variation. □

The following remark is a consequence of the Bichteler-Dellacherie Theorem (see Dellacherie and Meyer (1982, Theorem 80)).

*Remark* 4.8. Let $(Y_t)_{t\geq 0},(U_t)_{t\geq 0},(\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{U}_t)_{t\geq 0}$ denote four processes such that $(U_t)_{t\geq 0},(\tilde{U}_t)_{t\geq 0}$ are càdlàg, $(Y_t)_{t\geq 0}$ is $(\mathcal{F}^U_t)_{t\geq 0}$-adapted, $(\tilde{Y}_t)_{t\geq 0}$ is $(\mathcal{F}^{\tilde{U}}_t)_{t\geq 0}$-adapted and $(Y_\cdot,U_\cdot)\overset{\mathcal{D}}{=}(\tilde{Y}_\cdot,\tilde{U}_\cdot)$. If $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}^U_t)_{t\geq 0}$-semimartingale then $(\tilde{Y}_t)_{t\geq 0}$ has a modification which is an $(\mathcal{F}^{\tilde{U}}_t)_{t\geq 0}$-semimartingale.

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* We prove the result in the following three steps (1)-(3). Recall (a) and (b) on page 9.

(1) Let $\sigma^2>0$.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale. Let $\tilde{Z}_t=Y_t-W_t$ and $\tilde{X}_t=\int_0^t\phi(t-s)\,d\tilde{Z}_s$. We have $\mathcal{F}^Z_t=\mathcal{F}^W_t\vee\mathcal{F}^Y_t=\mathcal{F}^{-W}_t\vee\mathcal{F}^Y_t=\mathcal{F}^{\tilde{Z}}_t$ and since $(X_\cdot,Z_\cdot)\overset{\mathcal{D}}{=}(\tilde{X}_\cdot,\tilde{Z}_\cdot)$, Remark 4.8 shows that $(\tilde{X}_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale. Hence, $(X^W_t)_{t\geq 0}=((X_t-\tilde{X}_t)/2)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale and thus an $(\mathcal{F}^W_t)_{t\geq 0}$-semimartingale, and by Lemma 4.2 (ii) we conclude that $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a locally square integrable density.

On the other hand, if $\phi$ is absolutely continuous with a locally square integrable density it follows by Lemma 4.2 (i) that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale.

(2) Let $\sigma^2=0$ and $(Z_t)_{t\geq 0}$ be of unbounded variation.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^Z_t)_{t\geq 0}$-semimartingale. By Lemma 4.3 it follows that $(X^1_t)_{t\geq 0}$ is an $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-semimartingale. Let $T=\mathbb{Q}\cap[0,t]$, $q(f)=\sup_{s\in T}|f(s)|$ for all $f\in\mathbb{R}^T$ and $\mu$ be given by (4.4) with $\nu$ replaced by $\nu_1$. Since $\nu_1$ has compact support and $\phi$ is locally bounded (recall from page 9 that $\phi$ is chosen càdlàg) there exists an $r_0>0$ such that $\mu(f\in\mathbb{R}^T:q(f)\geq r_0)=0$ and hence, according to Rosiński and Samorodnitsky (1993, Lemma 2.2), $E[\sup_{s\in[0,t]}|X^1_s|]<\infty$. This shows that $(X^1_t)_{t\geq 0}$ is a special $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-semimartingale. Let $X^1_t=M_t+A_t$ denote the canonical $(\mathcal{F}^{Z^1}_t)_{t\geq 0}$-decomposition of $(X^1_t)_{t\geq 0}$. Then Lemma 4.6 yields $(M_t)_{t\geq 0}=(\phi(0)Z^1_t)_{t\geq 0}$ and hence $(A_t)_{t\geq 0}$, given by

$$A_t=\int_0^t\psi(t-s)\,dZ^1_s,\qquad t\geq 0,\tag{4.6}$$

where $\psi(t)=\phi(t)-\phi(0)$ for $t\geq 0$, is of bounded variation. Thus, by Lemma 4.7 we conclude that $\psi$, and hence also $\phi$, is absolutely continuous on $\mathbb{R}_+$ with a density having locally $\xi$-moment.

Assume conversely that $\phi$ is absolutely continuous with a density having locally $\xi$-moment. Since $\phi$ and $(Z_t^2)_{t\geq 0}$ are càdlàg and of bounded variation it follows that $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation as well. Let $(A_t)_{t\geq 0}$ is given by (4.6). By Lemma 4.7 it follows that $(A_t)_{t\geq 0}$ is càdlàg and of bounded and hence $(X_t^1)_{t\geq 0} = (\phi(0)Z_t^1 + A_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale and we have shown that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale.

(3) Let $(Z_t)_{t\geq 0}$ be of bounded variation.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. By arguing as in (2) it follows that $(A_t)_{t\geq 0}$ given by (4.6) is of bounded variation. Hence Rosiński (1989, Theorem 4) and a symmetrization argument shows that $\psi$, and hence also $\phi$, is of bounded variation.

Assume conversely that $\phi$ is of bounded variation. Since $(Z_t)_{t\geq 0}$ is càdlàg and of bounded variation it follows that $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation and hence an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. □

To show Proposition 3.2 we need the following Fubini type result.

**Lemma 4.9.** *Let $T > 0$, $\mu$ denote a finite measure on $\mathbb{R}_+$ and let $f\colon \mathbb{R}_+^2 \to \mathbb{R}$ be a measurable function such that either* (i) *or* (ii) *are satisfied, where*

*(i)* $\sigma^2 = 0$, $\xi(|f(t,\cdot)|) \in L^1([0,T],\lambda)$ *for all* $t \geq 0$ *and* $\xi(|f|) \in L^1(\mathbb{R}_+ \times [0,T], \mu \times \lambda)$.

*(ii)* $\sigma^2 > 0$, $f(t,\cdot) \in L^2([0,T],\lambda)$ *for all* $t \geq 0$, *and* $f \in L^2(\mathbb{R}_+ \times [0,T], \mu \times \lambda)$.

*Then* $(\int_0^T f(t,s)\,dZ_s)_{t\geq 0}$ *can be chosen measurable and in this case*

$$\int \Big(\int_0^T f(t,s)\,dZ_s\Big)\mu(dt) = \int_0^T \Big(\int f(t,s)\,\mu(dt)\Big)dZ_s \qquad P\text{-a.s.} \tag{4.7}$$

*Proof.* Assume (i) is satisfied. To show (4.7) we may and do assume that $(Z_t)_{t\geq 0}$ has characteristic triplet $(0,0,\nu)$ where $\nu$ is concentrated on $[-1,1]$. Let $g$ be given by (4.1). Since $g$ is 0 at 0, symmetric, increasing, convex, $\lim_{x\to\infty} g(x) = \infty$ and $g(2x) \leq 4g(x)$ for all $x \geq 0$, $g$ is a Young function satisfying the $\Delta_2$-condition (see Rao and Ren (1991, page 5+22)). Let $L^g([0,T],\lambda)$ denote the Orlicz space of measurable functions with finite $g$-moment on $[0,T]$ equipped with the norm

$$\|h\|_g = \inf\{c > 0 : \int_0^T g(c^{-1}h(s))\,ds \leq 1\}.$$

According to Chapter 3.3, Theorem 10, and Chapter 3.5, Theorem 1, in Rao and Ren (1991), $L^g([0,T],\lambda)$ is a separable Banach space. Let $f_t := f(t,\cdot)$ for all $t \geq 0$. Since $\xi(|f_t|) \in L^1([0,T],\lambda)$ for all $t \geq 0$, it is easy to check that $f_t$ satisfies (2.1)–(2.3) and hence $Y_t := \int_0^T f_t(s)\,dZ_s$ is well-defined for all $t \geq 0$. We show that $(Y_t)_{t\geq 0}$ has a measurable modification. Since $L^g([0,T],\lambda)$ is separable and $t \mapsto \|f_t - h\|_g$ is measurable for all $h \in L^g([0,T],\lambda)$ it follows that $t \mapsto f_t$ is a measurable mapping from $\mathbb{R}_+$ into $L^g([0,T],\lambda)$. Furthermore, since $L^g([0,T],\lambda)$ is separable there exists

$(h_k^n)_{n,k\geq 1} \subseteq L^g([0,T],\lambda)$ and disjoint measurable sets $(A_k^n)_{k\geq 1}$ for all $n\geq 1$ such that with

$$f_t^n(s) = \sum_{k\geq 1} h_k^n(s) 1_{A_k^n}(t),$$

we have $\|f_t - f_t^n\|_g \leq 2^{-n}$ for all $t \geq 0$. Set $Y_t^n = \sum_{k\geq 1}\int_0^T h_k^n(s)\,dZ_s 1_{A_k^n}(t)$ for all $t\geq 0$ and $n\geq 1$. Then $(Y_t^n)_{t\geq 0}$ is a measurable process and by Marcus and Rosiński (2001, Theorem 2.1) it follows that

$$\|Y_t^n - Y_t\|_{L^1(P)} \leq 3\|f_t^n - f_t\|_g \leq 3\times 2^{-n}, \qquad \forall t\geq 0,\, \forall n\geq 1. \tag{4.8}$$

For all $t\geq 0$ and $\omega\in\Omega$ let $\tilde{Y}_t(\omega) = \lim_n Y_t^n(\omega)$ when the limit exists in $\mathbb{R}$ and zero otherwise. Then $(\tilde{Y}_t)_{t\geq 0}$ is measurable and for all $t\in\mathbb{R}$, $\tilde{Y}_t = Y_t$ $P$-a.s. by (4.8). Thus we have constructed a measurable modification of $(Y_t)_{t\geq 0}$.

Let us show that both sides of (4.7) are well-defined. Since $g/2\leq \xi\leq g$ and $\xi(ax)\leq (a+1)^2\xi(x)$ for all $x,a>0$, it follows by Jensen's inequality that

$$\int_0^T \xi\Big(\int |f(t,s)|\,\mu(dt)\Big)\,ds \leq \frac{2(\mu(\mathbb{R})+1)^2}{\mu(\mathbb{R})}\int_0^T\int \xi(|f(t,s)|)\,\mu(dt)\,ds < \infty.$$

Thus, the right-hand side of (4.7) is well-defined. The left-hand side is well-defined as well since

$$\begin{aligned} &E\Big[\int\Big|\int_0^T f(t,s)\,dZ_s\Big|\mu(dt)\Big] \\ &\qquad \leq 3\int\Big(\int_0^T \xi(|f_t(s)|)\,ds\Big)\vee\Big(\int_0^T \xi(|f_t(s)|)\,ds\Big)^{1/2}\mu(dt) < \infty, \end{aligned}$$

where the first inequality follows by Marcus and Rosiński (2001, Corollary 1.1). Furthermore, (4.7) is obviously true for simple $f$ on the form

$$f(t,s) = \sum_{i=1}^n \alpha_i 1_{(s_{i-1},s_i]}(t) 1_{(t_{i-1},t_i]}(s).$$

If $f$ is a given function satisfying (i) we can choose a sequence of simple $(f_n)_{n\geq 1}$ converging to $f$ and satisfying $|f_n|\leq |f|$. We have

$$\int\Big(\int_0^T f_n(u,s)\,dZ_u\Big)\mu(ds) = \int_0^T\Big(\int f_n(u,s)\,\mu(ds)\Big)dZ_u, \tag{4.9}$$

and by estimates as above it follows that we can go to the limit in $L^1(P)$ in (4.9), which shows (4.7).

The case (ii) follows by a similar argument. In this case we have to work in $L^2([0,T],\lambda)$ instead of $L^g([0,T],\lambda)$. $\square$

Proposition 3.2 is an immediate consequence of Theorem 3.1 and Lemma 4.9, since

$$\phi(t-s) = \phi(0) + \int_0^{t-s}\phi'(u)\,du = \phi(0) + \int_0^t 1_{\{s\leq u\}}\phi'(u-s)\,du, \qquad s\in[0,t].$$

# 5 The two-sided case

Let $(X_t)_{t\geq 0}$ be given by

$$X_t = \int_{-\infty}^{t} (\phi(t-s) - \psi(-s))\, dZ_s, \qquad t \geq 0, \tag{5.1}$$

where $(Z_t)_{t\in\mathbb{R}}$ is a (two-sided) nondeterministic Lévy process with characteristic triplet $(\gamma, \sigma^2, \nu)$ and $\phi, \psi \colon \mathbb{R} \to \mathbb{R}$ are measurable functions for which the integral exists (still in the sense of Rajput and Rosiński (1989, page 460)). Also assume that $\phi$ and $\psi$ are 0 on $(-\infty, 0)$ and let $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$ denote the least filtration for which $\sigma(Z_s : -\infty < s \leq t) \subseteq \mathcal{F}_t^{Z,\infty}$ for all $t \geq 0$. Let $(X_t^1)_{t\geq 0}$ and $(X_t^2)_{t\geq 0}$ be given by

$$X_t^1 = \int_0^t \phi(t-s)\, dZ_s, \quad \text{and} \quad X_t^2 = \int_{-\infty}^{0} (\phi(t-s) - \psi(-s))\, dZ_s, \quad t \geq 0.$$

Similar to Remark 4.8 we have the following.

*Remark* 5.1. Let $(Y_t)_{t\geq 0}, (U_t)_{t\in\mathbb{R}}, (\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{U}_t)_{t\in\mathbb{R}}$ denote four processes such that $(U_t)_{t\in\mathbb{R}}, (\tilde{U}_t)_{t\in\mathbb{R}}$ are càdlàg, $(Y_t)_{t\geq 0}$ is $(\mathcal{F}_t^{U,\infty})_{t\geq 0}$-adapted, $(\tilde{Y}_t)_{t\geq 0}$ is $(\mathcal{F}_t^{\tilde{U},\infty})_{t\geq 0}$-adapted and $(Y_\cdot, U_\cdot) \overset{\mathcal{D}}{=} (\tilde{Y}_\cdot, \tilde{U}_\cdot)$. If $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{U,\infty})_{t\geq 0}$-semimartingale then $(\tilde{Y}_t)_{t\geq 0}$ has a modification which is an $(\mathcal{F}_t^{\tilde{U},\infty})_{t\geq 0}$-semimartingale.

**Lemma 5.2.** *Assume $(Z_t)_{t\in\mathbb{R}}$ is symmetric. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z})_{t\geq 0}$-semimartingale and $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation.*

*Proof.* The *if*-part is trivial. To show the *only if*-part assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. Let $\tilde{X}_t = X_t^1 - X_t^2$ and let $\tilde{Z}_t = Z_t$ for $t \geq 0$ and $\tilde{Z}_t = -Z_t$ when $t < 0$. Since $(Z_t)_{t\in\mathbb{R}}$ is symmetric $(X_\cdot, Z_\cdot) \overset{\mathcal{D}}{=} (\tilde{X}_\cdot, \tilde{Z}_\cdot)$ and from Remark 5.1 it follows that $(\tilde{X}_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0}$-semimartingale and hence an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale since $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0} = (\mathcal{F}_t^{Z,\infty})_{t\geq 0}$. Thus, $(X_t^1)_{t\geq 0} = ((X_t + \tilde{X}_t)/2)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale and hence an $(\mathcal{F}_t^{Z})_{t\geq 0}$-semimartingale. Moreover, $(X_t^2)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale and hence càdlàg and of bounded variation since $X_t^2$ is $\mathcal{F}_0^{Z,\infty}$-measurable for all $t \geq 0$. □

We have the following consequence of Lemma 5.2 and Theorem 3.1.

**Proposition 5.3.** *Let $(X_t)_{t\geq 0}$ be given by* (5.1) *and assume it is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale.*

*If $(Z_t)_{t\in\mathbb{R}}$ is of unbounded variation then $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density $\phi'$ satisfying* (1.2)-(1.3).

*If $(Z_t)_{t\in\mathbb{R}}$ is of bounded variation then $(X_t)_{t\geq 0}$ is of bounded variation and $\phi$ is of bounded variation as well.*

*Proof.* Let $\tilde{Z}_t = Z_t - Z_t'$ where $(Z_t')_{t\in\mathbb{R}}$ is an independent copy of $(Z_t)_{t\in\mathbb{R}}$ and let $(X_t')_{t\geq 0}$ be given by

$$X_t' = \int_{-\infty}^{t} (\phi(t-s) - \psi(-s))\, dZ_s', \qquad t \geq 0.$$

By Remark 5.1, $(X_t')_{t\geq 0}$ is an $(\mathcal{F}_t^{Z',\infty})_{t\geq 0}$-semimartingale, which by independence of filtrations shows that $(\tilde{X}_t)_{t\geq 0} := (X_t - X_t')_{t\geq 0}$ is a semimartingale in the $(\mathcal{F}_t^{Z,\infty} \vee \mathcal{F}_t^{Z',\infty})_{t\geq 0}$-filtration and hence in the $(\mathcal{F}_t^{\tilde{Z},\infty})_{t\geq 0}$-filtration. Since $(\tilde{Z}_t)_{t\in\mathbb{R}}$ is symmetric Lemma 5.2 shows that $(\tilde{X}_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^{\tilde{Z}})_{t\geq 0}$-semimartingale and since $(\tilde{Z}_t)_{t\geq 0}$ has characteristic triplet $(0, 2\sigma^2, \tilde{\nu})$ where $\tilde{\nu}(A) = \nu(A) + \nu(-A)$, the proposition follows by Theorem 3.1. □

Let $(X_t)_{t\geq 0}$ denote a fractional Lévy motion, that is

$$X_t = \int_{-\infty}^{t} ((t-s)^{\gamma} - (-s)_+^{\gamma})\, dZ_s, \qquad t \geq 0, \tag{5.2}$$

where $\gamma$ is such that the integral exists and $x_+ := x \vee 0$ for all $x \in \mathbb{R}$. In the following let us assume $(Z_t)_{t\in\mathbb{R}}$ has no Brownian component. It is necessary that $\gamma < \frac{1}{2}$, $\int_{[-1,1]^c} |x|^{1/(1-\gamma)}\, \nu(dx) < \infty$ and (a)-(c) on page 7 are satisfied for $(X_t)_{t\geq 0}$ to be well-defined and when $(Z_t)_{t\in\mathbb{R}}$ is symmetric these conditions are also sufficient. Marquardt (2006) studies processes on the form (5.2) under the assumptions that $\sigma^2 = 0$, $\int_{[-1,1]^c} |x|^2\, \nu(dx) < \infty$, $\gamma = -\int_{[-1,1]^c} x\, \nu(dx)$ and $0 < \gamma < \frac{1}{2}$.

To avoid trivialities assume $\gamma \neq 0$. As an application of Proposition 5.3 and Corollary 3.5 we have the following.

**Corollary 5.4.** *Assume $(Z_t)_{t\in\mathbb{R}}$ has no Brownian component and let $(X_t)_{t\geq 0}$ be given by (5.2). If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale then $\gamma \in (0, \frac{1}{2})$ and $\int_{[-1,1]} |x|^{1/(1-\gamma)}\, \nu(dx) < \infty$.*

In particular let $(X_t)_{t\geq 0}$ denote a (linear) fractional stable motion with indexes $\alpha \in (0, 2]$ and $H \in (0, 1)$, that is

$$X_t = \int_{-\infty}^{t} \left((t-s)^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\right) dZ_s, \qquad t \geq 0,$$

where $(Z_t)_{t\in\mathbb{R}}$ is a symmetric $\alpha$-stable Lévy process (see Samorodnitsky and Taqqu (1994, Definition 7.4.1)). For $\alpha = 2$, $(X_t)_{t\geq 0}$ is a fractional Brownian motion (fBm) with Hurst parameter $H$ (up to a scaling constant). From Corollary 5.4 it follows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $H = 1/\alpha$.

Let $(X_t)_{t\geq 0}$ be given by (5.1) and assume $(Z_t)_{t\in\mathbb{R}}$ is a symmetric $\alpha$-stable Lévy process with $\alpha \in (1, 2]$. If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale it follows by Proposition 5.3 and (1) on page 5 that $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density having locally $\alpha$-moment. The next result shows that this condition is actually necessary and sufficient for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if we delete "locally". Thus, extending Knight (1992, Theorem 6.5) from $\alpha = 2$ to $\alpha \in (1, 2]$ we have the following.

**Proposition 5.5.** *Let $(X_t)_{t\geq 0}$ be given by (5.1) and assume $(Z_t)_{t\in\mathbb{R}}$ is a symmetric $\alpha$-stable Lévy process with $\alpha \in (1, 2]$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale if and only if $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^{\alpha}(\mathbb{R}_+, \lambda)$.*

Let $B$ denote a Banach space (not necessarily separable) and assume there exists a countable subset $D$ of the unit ball of $B'$ (the dual space of $B$) such that

$$\|x\| = \sup_{F\in D} |F(x)|, \qquad \forall x \in B. \tag{5.3}$$

Following Ledoux and Talagrand (1991, page 133), a $B$-valued random element $X$ is called $\alpha$-stable if $\sum_{i=1}^n a_i F_i(X)$ is a real-valued $\alpha$-stable random variable for all $n \geq 1$, $F_1, \dots, F_n \in D$ and $a_1, \dots, a_n \in \mathbb{R}$.

Let $T$ denote an interval in $\mathbb{R}_+$ and let $B$ denote the subspace of $\mathbb{R}^T$ containing all functions which are càdlàg and of bounded variation. Then $B$ is a Banach space in the total variation norm (but not separable) and since the unit ball of $B'$ consists of $F$ on the form

$$F(f) = \sum_{i=1}^n \alpha_i (f(t_i) - f(t_{i-1})), \qquad f \in B,$$

where $(a_i)_{i=1}^n \subseteq [-1, 1]$ and $(t_i)_{i=0}^n$ is an increasing sequence in $T$, it follows that $B$ satisfies (5.3).

*Proof of Proposition 5.5.* For $\alpha = 2$ the result follows by Cherny (2001, Theorem 3.1); thus let us assume $\alpha \in (1, 2)$.

Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. According to Proposition 5.2 $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation. Consider $(X_t^2)_{t\geq 0}$ as an $\alpha$-stable random element with values in the Banach space consisting of functions which are càdlàg and of bounded variation equipped with the total variation norm. Hence from Ledoux and Talagrand (1991, Proposition 5.6) it follows that $(X_t^2)_{t\geq 0}$ is of integrable variation on each compact interval. Moreover, by Marcus and Rosiński (2001, Corollary 1.1) we have

$$E[|n(X_{i/n}^2 - X_{(i-1)/n}^2)|] \geq \tfrac{1}{4}\left(a_{i,n} \wedge \sqrt{a_{i,n}}\right), \qquad i, n \geq 1,$$

where

$$a_{i,n} := \int_{(i-1)/n}^{\infty} \tilde{\xi}(|\Delta_n \phi(s)|)\, ds, \quad \text{and} \quad \tilde{\xi}(x) := \int (|xs|^2 \wedge |xs|)\, \nu(ds).$$

Since $i \mapsto a_{i,n}$ is decreasing it follows that

$$E[\mathrm{V}_{0,1}(X^2)] = \sup_{n\geq 1} \sum_{i=1}^n E[|X_{i/n}^2 - X_{(i-1)/n}^2|] \geq \sup_{n\geq 1} \tfrac{1}{4}\left(a_{n,n} \wedge \sqrt{a_{n,n}}\right). \tag{5.4}$$

By (5.4) we conclude that $(a_{n,n})_{n\geq 1}$ is bounded and hence $(\tilde{\xi}(|\Delta_n \phi|))_{n\geq 1}$ is bounded in $L^1([1, \infty), \lambda)$. A straightforward calculation shows $\tilde{\xi}(x) = c_1 x^\alpha$ for all $x \geq 0$ for some constant $c_1 > 0$, which implies that $(\Delta_n \phi)_{n\geq 1}$ is bounded in $L^\alpha([1, \infty), \lambda)$. Since $\alpha > 1$, a sequence in $L^\alpha([1, \infty), \lambda)$ is bounded if and only if it is weakly sequentially compact (see Dunford and Schwartz (1957, Chapter IV.8, Corollary 4)). Thus, by arguing as in Lemma 4.1 it follows that $\phi$ is absolutely continuous with a density in $L^\alpha([1, \infty), \lambda)$. Furthermore, since $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale it follows by Corollary 3.4 that $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density locally in $L^\alpha(\mathbb{R}_+,\lambda)$. This shows the *only if*-part.

Assume conversely $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^\alpha(\mathbb{R}_+,\lambda)$. By Corollary 3.4 $(X_t^1)_{t\geq 0}$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$-semimartingale. Thus it is enough to show that $(X_t^2)_{t\geq 0}$ is càdlàg and of bounded variation. Since $\phi$ is absolutely continuous on $\mathbb{R}_+$ with a density in $L^\alpha(\mathbb{R}_+,\lambda)$ it follows by arguing as in Lemma 4.1 that $\|\phi(t-\cdot)-\phi(u-\cdot)\|_{L^\alpha((-\infty,0),\lambda)} \leq c(t-u)$ for some $c>0$ and all $0\leq u\leq t$. For all $p\in[1,\alpha)$ and all $u,t\geq 0$ we have

$$\|X_t^2 - X_u^2\|_{L^p(P)} = K_{p,\alpha}\|\phi(t-\cdot)-\phi(u-\cdot)\|_{L^\alpha((-\infty,0),\lambda)} \leq K_{p,\alpha}c|t-u|, \qquad (5.5)$$

for some constant $K_{p,\alpha}>0$ only depending on $p$ and $\alpha$. By letting $p\in(1,\alpha)$, (5.5) and the Kolmogorov-Čentsov Theorem show that $(X_t^2)_{t\geq 0}$ has a continuous modification. Moreover, by letting $p=1$ (5.5) shows that this modification is of integrable variation on each compact interval. This completes the proof. $\square$

Motivated by Lemma 5.2 we study in the following proposition infinitely divisible processes $(X_t)_{t\geq 0}$ of bounded variation, where $(X_t)_{t\geq 0}$ is on the form $X_t = \int_{\mathbb{R}} f(t,s)\,dZ_s$. Assume $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation. Rosiński (1989, Theorem 4) shows that $t\mapsto f(t,s)$ is of bounded variation for $\lambda$-a.a. $s\in\mathbb{R}$. Extending this we show that the total variation of $f(\cdot,s)$ must satisfy an integrability condition which is equivalent to the existence of $\int_{\mathbb{R}} \mathrm{V}_{0,t}(f(\cdot,s))\,dZ_s$ for all $t>0$ when $(Z_t)_{t\in\mathbb{R}}$ is symmetric and has no Brownian component.

**Proposition 5.6.** *Let $f\colon \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ denote a measurable function such that $X_t = \int_{\mathbb{R}} f(t,s)\,dZ_s$ is well-defined for all $t\geq 0$. If $(X_t)_{t\geq 0}$ is càdlàg and of bounded variation then*

$$\iint \big(1\wedge |x\,\mathrm{V}_{0,t}(f(\cdot,s))|^2\big)\,\nu(dx)\,ds < \infty, \qquad \forall t>0. \qquad (5.6)$$

Let $(\epsilon_i)_{i\geq 1}$ denote a Rademacher sequence, i.e. $(\epsilon_i)_{i\geq 1}$ is an i.i.d. sequence such that $P(\epsilon_1=-1)=P(\epsilon_1=1)=\frac{1}{2}$. It is well-known that if $(\alpha_i)_{i\geq 1}\subseteq\mathbb{R}$ then $\sum_{i=1}^\infty \epsilon_i\alpha_i$ converges $P$-a.s. if and only if $\sum_{i=1}^\infty \alpha_i^2<\infty$. Let $B$ denote a Banach space satisfying (5.3). Following Ledoux and Talagrand (1991, page 99), a $B$-valued random element $X$ is called a vector-valued Rademacher series if there exists a sequence $(x_i)_{i\geq 1}$ in $B$ such that $\sum_{i=1}^\infty F^2(x_i)<\infty$ for all $F\in D$ and $(F_1(X),\dots,F_n(X))$ equals $(\sum_{i=1}^\infty \epsilon_i F_1(x_i),\dots,\sum_{i=1}^\infty \epsilon_i F_n(x_i))$ in distribution for all $n\geq 1$ and all $F_1,\dots,F_n\in D$.

*Proof of Proposition 5.6.* By a symmetrization argument we may and do assume that $\sigma^2=0$ and $(Z_t)_{t\in\mathbb{R}}$ is symmetric. Define

$$Y_t = \sum_{j=1}^\infty \epsilon_j C_j f(t,U_j), \qquad t\geq 0, \qquad (5.7)$$

where $(\epsilon_j)_{j\geq 1}$ is a Rademacher sequence, $(\tau_j)_{j\geq 1}$ are the partial sums of i.i.d. standard exponential random variables and $(U_j)_{j\geq 1}$ are i.i.d. standard normal random variables with density $\rho$, and $(\epsilon_j)_{j\geq 1}$, $(\tau_j)_{j\geq 1}$ and $(U_j)_{j\geq 1}$ are independent. Let

$\nu^{\leftarrow}\colon \mathbb{R}_+ \to \mathbb{R}_+$ denote the right-continuous inverse of the mapping $x \mapsto \nu((x,\infty))$, that is, $\nu^{\leftarrow}(s) = \inf\{x > 0 : \nu((x,\infty)) \le s\}$, and let $C_j := \nu^{\leftarrow}(\tau_j \rho(U_j))$ for all $j \ge 1$. By Rosiński (1989, Proposition 2), the series (5.7) converges $P$-a.s. and $(Y_t)_{t\ge0}$ has the same finite dimensional distributions as $(X_t)_{t\ge0}$. Thus, $(Y_t)_{t\ge0}$ has a càdlàg modification of locally bounded variation. Hence we may and do assume $(X_t)_{t\ge0}$ is given by (5.7). Moreover we may define $(\epsilon_j)_{j\ge1}$ on a probability space $(\Omega', \mathcal{F}', P')$, $(\tau_j)_{j\ge1}$ and $(U_j)_{j\ge1}$ on a probability space $(\Omega'', \mathcal{F}'', P'')$ and $(X_t)_{t\ge0}$ on the product space. Let $T = [0,t]$ denote a compact interval in $\mathbb{R}_+$ and let $B$ denote the subspace of $\mathbb{R}^T$ consisting of functions which are càdlàg and of bounded variation. Inspired by Marcus and Rosiński (2003) let us fix $\omega'' \in \Omega''$ and consider $\underline{X} = (X_t)_{t\in T}$ as a $B$-valued Rademacher series under $P'$. From Ledoux and Talagrand (1991, Theorem 4.8) it follows that $E'[e^{\alpha\|\underline{X}\|^2}] < \infty$ for all $\alpha > 0$, which in particular shows that $(X_t)_{t\in T}$ is of $P'$-integrable variation. By Khinchine's inequality there exists a constant $c > 0$ such that $E[|X_t - X_u|] \ge c\|X_t - X_u\|_{L^2(P)}$ for all $u, t \ge 0$. Together with the triangle inequality in $l^2$ this shows that

$$\begin{aligned} E'[\sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}|] &\ge c\sum_{i=1}^n \Big(\sum_{j=1}^\infty C_j^2 (f(t_i,U_j) - f(t_{i-1},U_j))^2\Big)^{1/2} \\ &\ge c\Big(\sum_{j=1}^\infty \Big(\sum_{i=1}^n |C_j(f(t_i,U_j) - f(t_{i-1},U_j))|\Big)^2\Big)^{1/2} \\ &= c\Big(\sum_{j=1}^\infty \Big(|C_j|\sum_{i=1}^n |f(t_i,U_j) - f(t_{i-1},U_j)|\Big)^2\Big)^{1/2}. \end{aligned}$$

Thus, by monotone convergence we conclude

$$E'[\mathrm{V}_{0,t}(X)] \ge c\Big(\sum_{j=1}^\infty \big(C_j \,\mathrm{V}_{0,t}(f(\cdot,U_j))\big)^2\Big)^{1/2},$$

and in particular $(C_j \,\mathrm{V}_{0,t}(f(\cdot,U_j)))_{j\ge1} \in l^2$. Thus, we have shown that the series $\sum_{j=1}^\infty \epsilon_j C_j \,\mathrm{V}_{0,t}(f(\cdot,U_j))$ converges $P$-a.s. and from Theorem 2.4 and Proposition 2.7 in Rosiński (1990) it follows that

$$\int_0^\infty \int \big(1 \wedge H(u,v)^2\big)\,\rho(v)\,dv\,du < \infty, \tag{5.8}$$

where $H(u,v) = \nu^{\leftarrow}(u\rho(v))\,\mathrm{V}_{0,t}(f(\cdot,v))$. Furthermore, (5.8) equals

$$\begin{aligned} &\iint \big(1 \wedge (\nu^{\leftarrow}(u)\,\mathrm{V}_{0,t}(f(\cdot,v)))^2\big)\frac{1}{\rho(v)}\,du\,\rho(v)\,dv \\ &\qquad = \iint \big(1 \wedge (u\,\mathrm{V}_{0,t}(f(\cdot,v))^2\big)\,\nu(du)\,dv, \end{aligned}$$

which shows (5.6). □

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