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Cyclic constructions in mod two cohomology

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CYCLIC CONSTRUCTIONS IN MOD TWO COHOMOLOGY

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Contents

1. Introduction.

In this paper the coefficient ring of the cohomology groups is $\mathbb{F}_2$ when no other ring is presented in the notation. The mod two Steenrod algebra is denoted $\mathcal{A}$. Let $n = 2^r$ where r is a positive integer and let X be a space with finitely generated cohomology in each degree. The cyclic construction of order n on X is the orbit space

$$E_nX := EC_n \times_{C_n} X^n$$

where C_n is the cyclic group on n elements. The mod two cohomology of the cyclic construction can be computed using the following classical theorem:

Theorem 1.1. *There is a natural isomorphism*

$$\phi_X : H^*(E_nX) \longrightarrow H^*(C_n; H^*(X)^{\otimes n})$$

which takes the cup-product in $H^*(E_nX)$ to the group cohomology product in $H^*(C_n; H^*(X)^{\otimes n})$.

The group cohomology on the right hand side can be computed using the standard resolution of $\mathbb{F}_2$ with free $\mathbb{F}_2[C_n]$-modules. It is the homology of the complex

$$0 \longrightarrow H^*(X)^{\otimes n} \xrightarrow{1-T} H^*(X)^{\otimes n} \xrightarrow{N} H^*(X)^{\otimes n} \xrightarrow{1-T} \dots$$

where T denotes a generator of C_n and $N = 1 + T + T^2 + \cdots + T^{n-1}$. An invariant element $a \in (H^*(X)^{\otimes n})^{C_n}$ gives an element in the quotient $H^{2i}(C_n; H^*(X)^{\otimes n})$ (possibly zero) which we denote $e^{2i} \otimes a$. An element $b \in kerN$ gives an element in the quotient $H^{2i+1}(C_n; H^*(X)^{\otimes n})$ which we denote $e^{2i+1} \otimes b$. With this notation one obtains the following result for the cyclic construction of order two:

Proposition 1.2. *Let B consist of a basis for H^qX for every q. Then H^*E_2X has a basis of the following form*

$$\{e^0 \otimes (x \otimes y + y \otimes x), e^k \otimes z^{\otimes 2} | x, y, z \in B, x \neq y, k \geq 0\}$$

The most important fact about E_2X is that it can be used to define the Steenrod squares:

Theorem 1.3. *(Steenrod) Let $\Delta : X \longrightarrow X \times X$ denote the diagonal map. The map*

$$(1 \times \Delta)^* : H^*(E_2X) \longrightarrow H^*(BC_2 \times X)$$

satisfies

$$(1 \times \Delta)^*(e^k \otimes x^{\otimes 2}) = \sum_{j=0}^{|x|} e^{k+|x|-j} \otimes Sq^j x$$

Another important fact about the cyclic construction of order two is that one knows the mod two Steenrod algebra structure of H^*E_2X given the one for H^*X:

Theorem 1.4. *The mod two Steenrod algebra structure of H^*E_2X is given by the following formulas*

$$Sq^i(e^k \otimes z^{\otimes 2}) = \sum_{j \geq 0} \binom{k + |z| - j}{i - 2j} e^{k+i-2j} \otimes (Sq^j z)^{\otimes 2} + \delta_{k,0} \sum_{r=0}^{[\frac{i-1}{2}]} e^0 \otimes (Sq^r z \otimes Sq^{i-r} z + Sq^{i-r} z \otimes Sq^r z) \tag{1}$$

$$Sq^i(e^0 \otimes (x \otimes y + y \otimes x)) = \sum_{j=0}^{i} e^0 \otimes (Sq^j x \otimes Sq^{i-j} y + Sq^{i-j} y \otimes Sq^j x) \tag{2}$$

where δ denotes the Kronecker delta.

In this paper, these results are generalized to $E_n X$ where n is any power of two. First of all Theorem 1.1 gives the following extension of Prop 1.2

Proposition 1.5. *Pick a basis for $H^q X$ for every q, and let $B(n)$ denote the product basis for $H^*(X)^{\otimes n}$. C_n acts on $B(n)$ by cyclic permutation. Let $\mathcal{O}$ denote the orbit space of this action. Then one has*

$$H^* E_n X \cong \bigoplus_{\beta \in \mathcal{O}} H^*(C_n; V(\beta))$$

where $V(\beta)$ denotes the $\mathbb{F}_2$-span of the elements in β. Let $|\beta|$ denote the length of the orbit β. Let $s\beta$ denote the sum of the elements in β and let $\bar{\beta}$ denote a representative of β. For $|\beta| < n$ one has

$$\begin{aligned} H^{2q}(C_n; V(\beta)) &= \mathrm{Span}\{e^{2q} \otimes s\beta\} \\ H^{2q+1}(C_n; V(\beta)) &= \mathrm{Span}\{e^{2q+1} \otimes \bar{\beta}\} \end{aligned}$$

For $|\beta| = n$ one has

$$\begin{aligned} H^0(C_n; V(\beta)) &= \mathrm{Span}\{e^0 \otimes s\beta\} \\ H^q(C_n; V(\beta)) &= 0 \textit{ for } q \geq 1 \end{aligned}$$

One main result of this paper is the following generalization of Theorem 1.3

Theorem 1.6. *Let $m = 2^r$, $r \geq 0$ and let $\Delta : X \longrightarrow X^{4m}$ denote the diagonal map. Then the map $(1 \times \Delta)^* : H^*(E_{4m} X) \longrightarrow H^*(BC_{4m} \times X)$ satisfies*

$$\begin{aligned} (1 \times \Delta)^*(e^{2k} \otimes x^{\otimes 4m}) = &\sum_{i=0}^{|x|} e^{2k+2m(|x|-i)} \otimes (Sq^i x)^{2m} + \\ &\sum_{i=0}^{m|x|} e^{2k+1+2m|x|-2i} \otimes (Sq^{2i}(x^{2(m-1)} Sq^{|x|-1} x) + Q_{2m}^{2i-1}(x)) \end{aligned}$$

and

$$(1 \times \Delta)^*(e^{2k+1} \otimes x^{\otimes 4m}) = \sum_{i=0}^{|x|} e^{2k+1+2m(|x|-i)} \otimes (Sq^i x)^{2m}$$

where the Q-operation is defined by

$$Q_2^n(x) := \sum_{j=0}^{[\frac{n}{2}]} Sq^j(x)Sq^{n-j}(x)$$

$$Q_{2^k}^n(x) := \sum_{j=0}^{n} Sq^j(x^{2^k-2})Q_2^{n-j}(x)\ ,\ k \geq 2$$

Another main result is the generalization of Theorem 1.4.

Theorem 1.7. *The mod two Steenrod algebra structure of $H^*E_{4m}X$ is given by the following formulas.*

$$Sq^{2s}(e^{2k} \otimes x^{\otimes 4m}) = \sum_{j\geq 0} \binom{2k+2m|x|-j}{2s-2j} e^{2(k+s-j)} \otimes Sq^{2j}(x^{\otimes 4m}) \tag{3}$$

$$Sq^1(e^{2k} \otimes x^{\otimes 4m}) = \binom{|x|}{1} e^{2k+1}x^{\otimes 4m} + \delta_{k,0}e^0 \otimes Sq^1(x^{\otimes 4m}) \tag{4}$$

$$Sq^{2s}(e^{2k+1} \otimes x^{\otimes 4m}) = \sum_{j\geq 0} \binom{k+m|x|-mj}{s-2mj} e^{2(k+s-j)+1} \otimes (Sq^j x)^{\otimes 4m} \tag{5}$$

$$Sq^{2s+1}(e^{2k+1} \otimes x^{\otimes 4m}) = 0 \tag{6}$$

This gives a complete description of the structure since all elements of lower symmetry equals the transfer of an element of highest symmetry.

One should expect the above to work for a general unstable $\mathcal{A}$-algebra, not necessarily of the form H^*X for some space X. The following result from this paper confirms this and one gets some grip of the underlying algebra making things work.

Theorem 1.8. *Let M be an unstable $\mathcal{A}$-algebra. The formulas in Theorem 1.7 define an $\mathcal{A}$-module structure on the group cohomology modules $H^*(C_{4m}; M^{\otimes 4m})$ i.e. the Adem relations are respected. The formulas in Theorem 1.6 define a map*

$$\delta : H^*(C_{4m}; M^{\otimes 4m}) \longrightarrow H^*(C_{4m}) \otimes M$$

which is $\mathcal{A}$-linear with respect to this action.

The thing that makes the $\mathcal{A}$-linearity work is that some other operations closely related to the Q-operations above satisfy Adem-like relations (see Definition 6.3 for the definition of the operations and Theorem 6.8 for the Adem-like relations).

The last four chapters are about getting information of the free loop space using these results. These chapters do not have the character of a final theory. Let ΛX denote the free loop space on X. The S^1 homotopy orbits of ΛX is the space

$$(\Lambda X)_{hS^1} := ES^1 \times_{S^1} \Lambda X$$

This construction has interesting connections to Waldhausen's A-theory, see [K6].

We give an approximation to the mod two cohomology of $(\Lambda X)_{hS^1}$ which is a functor of the mod two cohomology of X. To be more precise let $ev : \Lambda X \longrightarrow X^n$ denote the map which evaluates a loop in the n'th roots of unity. It is a C_n-equivariant map thus it induces a map

$$(1 \times ev)^* : H^*(E_n X) \longrightarrow H^*(ES^1 \times_{C_n} \Lambda X)$$

We only use the maps where n is a power of two. For two different values of n we have a transfer map between the two corresponding right hand sides. Passing to the direct limit we get the reduced cohomology of $\Sigma((\Lambda X)_{hS^1})_+$. There is a limit system on the left compatible with the transfer limit system on the right. The approximation $\ell(X)$ is a quotient of the direct limit of the left hand side. Neglecting the suspension factor we get a map of degree -1.

$$\phi : \ell(X) \longrightarrow H^*((\Lambda X)_{hS^1})$$

It is shown that this map is an isomorphism when $X = \mathbb{C}P^\infty$ and $X = \mathbb{R}P^\infty$. There are however also examples where it is not an isomorphism.

The approximation can be used together with the Serre spectral sequence associated to the fibration $\Lambda X \longrightarrow ES^1 \times_{S^1} \Lambda X \longrightarrow BS^1$. We use it in the case $X = K(\mathbb{F}_2, n)$ to show that all differentials d_r with $r \geq 3$ are zero, and in this way we compute the mod two cohomology of $(\Lambda K(\mathbb{F}_2, n))_{hS^1}$ for all $n \geq 2$.

2. The cyclic construction.

Let $C_n = \{1, T, T^2, \dots T^{n-1}\}$ denote the cyclic group on n elements. Let X be a topological space. C_n acts on $X^n = X \times \cdots \times X$ by permuting the factors:

$$C_n \times X^n \to X^n;\ T\ (x_1, \dots, x_n) = (x_n, x_1, \dots, x_{n-1})$$

Let $\pi : EC_n \to BC_n$ denote the universal principal C_n-bundle. We get an associated fiber bundle

$$X^n \to EC_n \times_{C_n} X^n \xrightarrow{p} BC_n$$

from the equivariant map $EC_n \times X^n \to EC_n$. Since $EC_n \to BC_n$ is locally trivial p is a locally trivial bundle and thus a fibration. The space

$$E_n X := EC_n \times_{C_n} X^n = (EC_n \times X^n)/C_n$$

is called the cyclic construction of order n on X. A map $f : X \to Y$ induces a bundle map $E_n X \to E_n X$ over the identity thus the cyclic construction gives a functor from spaces to bundles over BC_n. The cohomology of the cyclic construction can be computed using the following theorem based on ideas of Dold [D].

Theorem 2.1. *Let $\mathbb{F}$ be a field. There is a natural isomorphism*

$$\phi_X : H^*(E_n X; \mathbb{F}) \to H^*(C_n; H^*(X; \mathbb{F})^{\otimes n})$$

which maps the cup-product of two classes in $H^*(E_nX;\mathbb{F})$ to the group cohomology product of their images in $H^*(C_n, H^*(X;\mathbb{F})^{\otimes n})$.

Proof. The essential parts are written in [May] and we use his notation. Let X be a simplicial set and let Λ be a commutative ring. The standard resolution of Λ with ΛC_n-modules is denoted $W_* \xrightarrow{\epsilon} \Lambda$. Let C_* denote the singular complex with Λ coefficients. According to [May] p. 193 there is an equivariant chain homotopy equivalence

$$W_* \otimes C_*(X)^{\otimes n} \xrightarrow{\Psi} W_* \otimes C_*(X^n)$$

There is also an equivariant equivalence

$$C_*(EC_n) \otimes C_*(X^n) \to C_*(EC_n \times X^n)$$

Combining these with a shift of resolution $f : W_* \to C_*(EC_n)$ one gets an equivariant equivalence

$$W_* \otimes C_*(X)^{\otimes n} \to C_*(EC_n \times X^n)$$

Since

$$W_* \otimes_{\Lambda C_n} C_*(X)^{\otimes n} \cong W_* \otimes C_*(X)^{\otimes n} \otimes_{\Lambda C_n} \Lambda$$

and

$$C_*(EC_n \times_{C_n} X^n) \cong C_*(EC_n \times X^n) \otimes_{\Lambda C_n} \Lambda$$

we actually have an equivalence

$$W_* \otimes_{\Lambda C_n} C_*(X)^{\otimes n} \to C_*(E_nX)$$

Suppose that $\Lambda = \mathbb{F}$ is a field. According to [May] p. 156 there is an equivalence

$$W_* \otimes_{\mathbb{F}C_n} H_*(X)^{\otimes n} \to W_* \otimes_{\mathbb{F}C_n} C_*(X)^{\otimes n}$$

We get a natural isomorphism

$$H_*(E_nX) \cong H_*(C_n; H_*(X)^{\otimes n})$$

Dually

$$H^*(E_nX) \cong H^*(C_n; H^*(X)^{\otimes n})$$

The claim about the product follows from a homotopy commutative diagram describing the two products by coproducts on the chain level. $\square$

The covering $q : EC_{mn} \times_{C_n} X^{nm} \to EC_{mn} \times_{C_{mn}} X^{mn}$ induces a transfer map

$$Tr : H^*(E_n(X^m)) \to H^*(E_{mn}X)$$

and we have a restriction map

$$Res = q^* : H^*(E_{mn}X) \to H^*(E_n(X^m))$$

Under the isomorphism of Theorem 2.1 they correspond to the transfer and restriction in group cohomology since we have the following result:

Proposition 2.2. *There are commutative diagrams*

$$\begin{array}{ccc} H^*(E_n(X^m)) & \xrightarrow{Tr} & H^*(E_{mn}X) \\ {\scriptstyle \phi_{X^m}}\downarrow & & {\scriptstyle \phi_X}\downarrow \\ H^*(C_n; H^*(X^m)^{\otimes n}) & \xrightarrow{tr} & H^*(C_{mn}, H^*(X)^{\otimes mn}) \end{array}$$

and

$$\begin{array}{ccc} H^*(E_n(X^m)) & \xleftarrow{Res} & H^*(E_{mn}X) \\ {\scriptstyle \phi_{X^m}}\downarrow & & {\scriptstyle \phi_X}\downarrow \\ H^*(C_n; H^*(X^m)^{\otimes n}) & \xleftarrow{res} & H^*(C_{mn}, H^*(X)^{\otimes mn}) \end{array}$$

where res and tr are the restriction and the transfer in group cohomology corresponding to the inclusion $C_n \subseteq C_{mn}$

Proof. We prove the result about the transfer maps. The proof for the restriction maps is similar. First we recall the definitions of the two transfer maps. Let H be a subgroup of a finite group G. Choose a projective resolution $P_* \to \Lambda$ of Λ with ΛG modules. Let M be a ΛG-module. The group cohomology transfer map is given by the following on the chain level

$$tr : P_* \otimes_{\Lambda G} M \to P_* \otimes_{\Lambda H} M \quad ; \quad tr(p \otimes m) = \sum_{r \in G/H} pr \otimes r^{-1}m$$

If G acts properly discontinuously on X then the transfer associated to the covering $X \to X/G$ is given by

$$Tr : C_*(X/G) \to C_*(X) \quad ; \quad Tr[t] = \sum_{g \in G} [gt']$$

where t' is a lifting of t. One checks that the following diagram is commutative:

$$\begin{array}{ccc}
W_* \otimes_{\mathbb{F}C_{nm}} H_*(X)^{\otimes nm} & \xrightarrow{tr} & W_* \otimes_{\mathbb{F}C_n} H_*(X)^{\otimes nm} \\
\downarrow & & \downarrow \\
W_* \otimes_{\mathbb{F}C_{nm}} C_*(X)^{\otimes nm} & \xrightarrow{tr} & W_* \otimes_{\mathbb{F}C_n} C_*(X)^{\otimes nm} \\
\downarrow & & \downarrow \\
W_* \otimes_{\mathbb{F}C_{nm}} C_*(X^{nm}) & \xrightarrow{tr} & W_* \otimes_{\mathbb{F}C_n} C_*(X^{nm}) \\
\downarrow & & \downarrow \\
C_*(EC_{nm}) \otimes_{\mathbb{F}C_{nm}} C_*(X^{nm}) & \xrightarrow{tr} & C_*(EC_{nm}) \otimes_{\mathbb{F}C_n} C_*(X^{nm}) \\
\downarrow & & \downarrow \\
C_*(EC_{nm} \times X^{nm}) \otimes_{\mathbb{F}C_{nm}} \mathbb{F} & \xrightarrow{tr} & C_*(EC_{nm} \times X^{nm}) \otimes_{\mathbb{F}C_n} \mathbb{F} \\
\downarrow & & \downarrow \\
C_*(EC_{nm} \times_{C_{nm}} X^{nm}) & \xrightarrow{Tr} & C_*(EC_{nm} \times_{C_n} X^{nm})
\end{array}$$

where the vertical arrows are the ones described in Theorem 2.1 $\square$

3. Group cohomology computations.

Let $\mathbb{F}_2$ be the field on two elements. We shall from now on use $\mathbb{F}_2$-coefficients everywhere unless otherwise is specified. Let $r \geq 1$ be an integer. We will calculate the group cohomology

$$H^*(C_{2^r}; H^*(X)^{\otimes 2^r}) \tag{7}$$

Put $n = 2^r$. We use the standard resolution

$$\mathbb{F}_2 \xleftarrow{\epsilon} W_0(n) \xleftarrow{d} W_1(n) \xleftarrow{d} \dots$$

of the trivial $\mathbb{F}_2[C_n]$-module $\mathbb{F}_2$ with free $\mathbb{F}_2[C_n]$-modules. That is $W_i(n) = \mathbb{F}_2[C_n]e_i$ with differential given by

$$de_{2i+1} = (1+T)e_{2i} \ , \ de_{2i} = Ne_{2i-1}$$

where $N = 1 + T + T^2 + \cdots + T^n$ is the norm element. The augmentation is defined by $\epsilon(e_0) = 1$.

For a field k and a finite group G there is an isomorphism of $k[G]$-modules

$$k[G] \xrightarrow{\cong} \mathrm{Hom}_k(k[G], k) = (k[G])^*$$

mapping an element $\alpha = \sum_{g\in G} \alpha_g g$ to $\hat{\alpha}$ given by $\hat{\alpha}(g) = \alpha_g$. The action on $\mathrm{Hom}_k(k[G], k)$ is given by $(g \cdot f)(h) = f(g^{-1}h)$. For any $k[G]$-module M we get an isomorphism

$$\mathrm{Hom}_{k[G]}(k[G], M) \cong M \cong k[G] \otimes_{k[G]} M \cong (k[G])^* \otimes_{k[G]} M$$

determined by $f \mapsto \hat{1} \otimes f(1)$. We will use this to get a convenient notation:

$$\mathrm{Hom}_{\mathbb{F}_2[C_n]}(W_i(n), H^*(X)^{\otimes n}) \xrightarrow{\cong} W_i^*(n) \otimes_{\mathbb{F}_2[C_n]} H^*(X)^{\otimes n}; \quad (e_i \mapsto x) \mapsto e^i \otimes x$$

The differential in the last complex is given by

$$\delta(e^{2i+1} \otimes x) = e^{2i+2} \otimes Nx$$
$$\delta(e^{2i} \otimes x) = e^{2i+1} \otimes (1+T)x$$

An invariant element $x \in (H^*(X)^{\otimes n})^{C_n}$ gives a class in the quotient

$$e^{2i} \otimes x \in H^{2i}(C_n; H^*(X)^{\otimes n})$$

and an element $y \in ker(H^*(X)^{\otimes n} \xrightarrow{N} H^*(X)^{\otimes n})$ gives a class in the quotient

$$e^{2i+1} \otimes y \in H^{2i+1}(C_n; H^*(X)^{\otimes n})$$

Observe that $e^{2i} \otimes x = 0$ when $x \in ImN$ and $i > 0$ and that $e^{2i+1} \otimes y = e^{2i+1} \otimes Ty$ for all $i \geq 0$, considered as elements in $H^*(C_n; H^*(X)^{\otimes n})$.

Assume that $H^*(X)$ is of finite type i.e. $H^i(X)$ is finite dimensional for each i. Let $a_1^i, \ldots, a_{n_i}^i$ denote a basis for $H^i(X)$. Then we have the following basis for $H^*(X)^{\otimes n}$

$$B(n) = \{a_{j_1}^{i_1} \otimes \cdots \otimes a_{j_n}^{i_n} | 1 \leq j_1 \leq n_{i_1}, \ldots, 1 \leq j_n \leq n_{i_n}\}$$

Let $\mathcal{O}$ be the set of orbits under the permutation action of C_n on $B(n)$. For an orbit β let $V(\beta)$ denote the $\mathbb{F}_2$-span of the elements in β and let $|\beta|$ denote the length of the orbit. We have a C_n-stable splitting

$$H^*(X)^{\otimes n} = \bigoplus_{\beta \in \mathcal{O}} V(\beta)$$

and the group cohomology splits

$$H^*(C_n; H^*(X)^{\otimes n}) = \bigoplus_{\beta \in \mathcal{O}} H^*(C_n; V(\beta))$$

The possible lengths of the orbits are 2^j where $j = 0, 1, \ldots, r$. Consider an element

$$\underline{a} = (a_1 \otimes \cdots \otimes a_m)^{\otimes \frac{n}{m}}$$

in an orbit β of length $|\beta| = m$. For $|\beta| < n$ one finds

$$H^{2i}(C_n; V(\beta)) = \mathrm{Span}_{\mathbb{F}_2}\{e^{2i} \otimes (1 + T + \cdots + T^{m-1})\underline{a}\}$$
$$H^{2i+1}(C_n, V(\beta)) = \mathrm{Span}_{\mathbb{F}_2}\{e^{2i+1} \otimes \underline{a}\}$$

and when $|\beta| = n$ one finds

$$H^0(C_n; V(\beta)) = \mathrm{Span}_{\mathbb{F}_2}\{e^0 \otimes N\underline{a}\}$$
$$H^k(C_n; V(\beta)) = 0\,,\ k > 0$$

The additive structure of the cohomology groups (7) is now determined.

Next let us determine transfer and restriction maps. Let T denote a generator of C_{nm} and t a generator of C_m where $m = 2^s$ and $n = 2^r$. C_m acts on $B(nm)$ via the inclusion $C_m \subseteq C_{nm}$, $t \mapsto T^n$. For an element $a \in B(nm)$ we let β_a^m denote the orbit of a under the action of C_m. For a subset $\beta \subseteq B(nm)$ we let $s\beta$ denote the sum of all elements in β.

Lemma 3.1. *Let $a = a_1 \otimes \cdots \otimes a_n$ where a_k is a basis element for $1 \le k \le n$. The transfer map*

$$tr_m^{nm} : H^*(C_m; (H^*(X)^{\otimes n})^{\otimes m}) \longrightarrow H^*(C_{nm}; H^*(X)^{\otimes nm})$$

is given by the following formulas when applied to highest symmetry elements

$$tr_m^{nm}(e^{2i} \otimes a^{\otimes m}) = e^{2i} \otimes \sum_{j=1}^{n} T^j a^{\otimes m}$$
$$tr_m^{nm}(e^{2i+1} \otimes a^{\otimes m}) = e^{2i+1} \otimes a^{\otimes m}$$

Proof. Define a $\mathbb{F}_2[C_m]$-linear chain map over the identity $s_* : W_*(nm) \longrightarrow W_*(m)$ by $T^k \mapsto t^{[\frac{k}{n}]}$ in even degrees and in odd degrees by $T^k \mapsto t^{[\frac{k}{n}]}$ for $k = -1 \bmod n$, $T^k \mapsto 0$ for $k \neq -1 \bmod n$. Then tr_m^{2m} is induced by the composite

$$\mathrm{Hom}_{\mathbb{F}_2[C_m]}(W_*(m), H^*(X)^{\otimes nm}) \xrightarrow{s^*} \mathrm{Hom}_{\mathbb{F}_2[C_m]}(W_*(nm), H^*(X)^{\otimes nm}) \xrightarrow{tr}$$
$$\mathrm{Hom}_{\mathbb{F}_2[C_{nm}]}(W_*(nm), H^*(X)^{\otimes nm})$$

where $tr(f)(x) = \sum_{i=0}^{n-1} T^i f(T^{-i}x)$. The result follows. $\square$

Note that by this lemma all elements in (7) which are not of highest symmetry equals the transfer on an element of highest symmetry. In this way the lemma gives a complete description of the transfer maps since $tr_{nm}^{knm} \circ tr_m^{nm} = tr_m^{knm}$.

Lemma 3.2. *Let $a \in B(nm)$ and put $r(a, n) = min\{|\beta_a^{nm}|, n\} - 1$. Then the restriction map*

$$res_m^{nm} : H^*(C_{nm}; H^*(X)^{\otimes nm}) \longrightarrow H^*(C_m; (H^*(X)^{\otimes n})^{\otimes m})$$

is given by the formulas

$$res_m^{nm}(e^{2i} \otimes s\beta_a^{nm}) = \sum_{j=0}^{r(a,n)} e^{2i} \otimes s\beta_{T^j a}^m$$

$$res_m^{nm}(e^{2i+1} \otimes a) = \sum_{j=0}^{n-1} e^{2i+1} \otimes T^j a$$

Especially $res_m^{nm}(e^{2i+1} \otimes a) = 0$ *when* $n > |\beta_a^{nm}|$.

Proof. $W_*(nm)$ is a $\mathbb{F}_2[C_m]$-module via the inclusion $C_m \subseteq C_{nm}$; $t \mapsto T^n$. Define a $\mathbb{F}_2[C_m]$-linear chain map over the identity

$$i_* : W_*(m) \longrightarrow W_*(nm)$$

by $e_{2i} \mapsto e_{2i}$ and $e_{2i+1} \mapsto \sum_{j=0}^{n-1} T^j e_{2i-1}$. This map induces res_m^{nm}. □

Corollary 3.3. *The restriction map*

$$res_m^{2m} : H^*(C_{2m}; H^*(X)^{\otimes 2m}) \longrightarrow H^*(C_m; (H^*(X)^{\otimes 2})^{\otimes m})$$

has the following kernel

$$ker(res_m^{2m}) = \mathrm{Span}_{\mathbb{F}_2}\{e^{2i+1} \otimes x^{\otimes 2m} | i \geq 0, x \in H^*(X) \text{ basis element}\}$$

The algebra structure of the group cohomology $H^*(C_{2^k})$ is as follows. For $k \geq 2$ one has

$$H^*(C_{2^k}) = \Lambda(v) \otimes \mathbb{F}_2[u]$$

that is an exterior algebra on v of degree one tensor a polynomial algebra on u of degree two. For $k = 1$ one gets a polynomial algebra $H^*(C_2) = \mathbb{F}_2[a]$ on a one dimensional generator a.

Lemma 3.4. *Let* $k \geq 1$ *and* $m = 2^k$ *then*

(1) $tr_m^{2m} : H^j(C_m) \longrightarrow H^j(C_{2m})$ *is zero for* j *even and the identity for* j *odd.*
(2) $res_m^{2m} : H^j(C_{2m}) \longrightarrow H^j(C_m)$ *is zero for* j *odd and the identity for* j *even.*

Proof. Use the standard resolution and the chain maps from Lemma 3.1 and Lemma 3.2 □

Our next step is to describe the multiplicative structure of (7). It is induced by the coproduct $\psi : W_*(n) \longrightarrow W_*(n) \otimes W_*(n)$ given by

$$\psi(e_{2i+1}) = \sum_{j+l=i} e_{2j} \otimes e_{2l+1} + \sum_{j+l=i} e_{2j+1} \otimes Te_{2l}$$

$$\psi(e_{2i}) = \sum_{j+l=i} e_{2j} \otimes e_{2l} + \sum_{j+l=i-1} \sum_{0 \leq r < s < n} T^r e_{2j+1} \otimes T^s e_{2l+1}$$

Let $f_1 \in \mathrm{Hom}_{\mathbb{F}_2[C_n]}(W_r(n), H^*(X)^{\otimes n})$ and $f_2 \in \mathrm{Hom}_{\mathbb{F}_2[C_n]}(W_s(n), H^*(X)^{\otimes n})$ and let ϕ be the composite

$$W_*(n) \xrightarrow{\psi} W_*(n) \otimes W_*(n) \xrightarrow{f_1 \otimes f_2} H^*(X)^{\otimes n} \otimes H^*(X)^{\otimes n} \xrightarrow{\mu} H^*(X)^{\otimes n}$$

Then $[f_1] \cup [f_2] = [\phi]$. The following table gives the relevant parts of ψ

r	s	$[\psi(e_{r+s})]_{r,s} \in W_r \otimes W_s$
$2i$	$2j+1$	$e_{2i} \otimes e_{2j+1}$
$2i+1$	$2j$	$e_{2i+1} \otimes Te_{2j}$
$2i$	$2j$	$e_{2i} \otimes e_{2j}$
$2i+1$	$2j+1$	$\sum_{0 \le u < v < n} T^u e_{2i+1} \otimes T^v e_{2j+1}$

From this it is easy to get the cup product of two given elements e.g.

$$(e^{2i} \otimes x^{\otimes n})(e^{2j} \otimes y^{\otimes n}) = e^{2(i+j)} \otimes (xy)^{\otimes n}$$

Finally we describe most of the action of the mod 2 Steenrod algebra $\mathcal{A}$ on $H^*(E_n X)$.

Theorem 3.5. *Let $m = 2^r$ for $r \geq 0$. Then*

$$Sq^{2s}(u^b \otimes x^{\otimes 4m}) = \sum_{j=0}^{s} \binom{2b + 2m|x| - j}{2(s-j)} u^{b+s-j} \otimes Sq^{2j}(x^{\otimes 4m})$$

Proof. It is enough to prove the formula when $X = K(\mathbb{F}_2, n)$ with fundamental class ι_n. The $\mathcal{A}$-action on $H^*(E_2 X)$ is known. By mapping both sides one step down by Res^{4m}_{2m} one gets by induction that the formula is right modulo some terms in the kernel of Res^{4m}_{2m}. But these elements have odd degrees and the degree of $Sq^{2s}(u^b \otimes \iota_n^{\otimes 4m})$ is even thus there are no such terms. $\square$

Remark that this also determines the action of the even squares on elements of the form $vu^b \otimes x^{\otimes n}$ since we can use the Cartan formula on

$$vu^b \otimes x^{\otimes n} = (v \otimes 1)(u^b \otimes x^{\otimes n})$$

Using that all other types of elements are hit by the transfer on elements of the highest symmetry, we have a complete description of the action of the even squares on $H^*(E_n X)$. (Recall that there is a description of the transfer map using geometrical maps, thus it commutes with the squares.)

Lemma 3.6.

$$Sq^1(e^0 \otimes x^{\otimes 4m}) = \alpha(|x|)e^1 \otimes x^{\otimes 4m} + e^0 \otimes Sq^1(x^{\otimes 4m})$$

where $\alpha(|x|) \in \mathbb{F}_2$.

Proof. It is enough to show the result for $X = K(\mathbb{F}_2, n)$. Mapping both sides one step down by Res_{2m}^{4m} one gets the following by induction

$$Sq^1(e^0 \otimes \iota_n^{\otimes 4m}) = \alpha(n)e^1 \otimes \iota_n^{\otimes 4m} + \beta(n)e^{4mn+1} \otimes 1 + e^0 \otimes Sq^1(\iota_n^{\otimes 4m})$$

By using the map $K(\mathbb{F}_2, n) \longrightarrow *$ where $*$ is a point one finds that $\beta(n) = 0$ for all n. □

We shall show later that $\alpha(n) = n$. One can already see that $\alpha(2n)$ is zero by choosing a map $f : K(\mathbb{F}_2, n) \longrightarrow K(\mathbb{F}_2, 2n)$ with $f^*(\iota_{2n}) = \iota_n^2$.

4. The Q_n^m and Sq_1 operation.

In this chapter we define some operations Q_n^m and Sq_1 and prove some purely algebraic results about them. The operations occurs in the description of the diagonal map

$$(1 \times \Delta)^* : H^*(E_{2^k} X) \longrightarrow H^*(BC_{2^k}) \otimes H^*(X)$$

as we shall see in the next chapter. The algebraic results in this chapter are all needed to study this diagonal map. Let K be an unstable $\mathcal{A}$-algebra.

Definition 4.1. Define the operations $Sq_i : K \longrightarrow K$, $k \geq 0$ by

$$Sq_i(x) = Sq^{|x|-i}(x)$$

Lemma 4.2.

$$Sq_n(xy) = \sum_{i+j=n} Sq_i(x) Sq_j(y)$$

Proof. Follows by the Cartan formula □

Lemma 4.3.

$$Sq^{2m}(Sq_1 x) = Sq_1(Sq^m x)$$
$$Sq^{2m+1}(Sq_1 x) = (m + |x|)(Sq^m x)^2$$

Proof. To prove the first relation we must show

$$Sq^{2m} Sq^{|x|-1}(x) = Sq^{|x|+m-1} Sq^m(x)$$

Both sides of this equation are zero when $m \geq |x|$ and when $m = |x|-1$ the equation is obvious. Assume that $m \leq |x|-2$. Then one can use the Adem relation on the left hand side

$$Sq^{2m}Sq^{|x|-1}(x) = \sum_{j=0}^{m} \binom{|x|-2-j}{2(m-j)} Sq^{2m+|x|-1-j}Sq^{j}(x)$$

but because of the degree the only term in the sum different from zero is the one corresponding to $j = m$ and the result follows. The proof of the second relation is similar. $\square$

Deviation from linearity will play a central role. For a set map $F : K \longrightarrow K$ we define the deviation from linearity $\Delta F : K \times K \longrightarrow K$ by

$$\Delta F(x,y) := F(x+y) - F(x) - F(y)$$

Note that for $G(x) = l(x)F(x)$ where l is $\mathbb{F}_2$-linear we have

$$\Delta G(x,y) = (l(x)+l(y))\Delta F(x,y) + l(x)F(y) + l(y)F(x) \tag{8}$$

Definition 4.4. Define $\Lambda_{2^k} : K \longrightarrow K$ for $k \geq 1$ by

$$\Lambda_{2^k}(x) := x^{2^k-2}Sq_1(x)$$

Lemma 4.5. *There is a product formula*

$$\Lambda_{2^k}(xy) = x^{2^k}\Lambda_{2^k}(y) + y^{2^k}\Lambda_{2^k}(x)$$

and the deviation of linearity is given by

$$\Delta\Lambda_{2^k}(x,y) = Sq_1(xy) \sum_{r+s=2^{k-1}-2} x^{2r}y^{2s} = \sum_{i=1}^{k-1} \sum_{r+s=2^{k-1-i}-1} \Lambda_{2^i}(x^{2r+1}y^{2s+1})$$

where $|x| = |y|$. The sums should be read as zero when $k = 1$.

Proof. By Lemma 4.2 we have

$$Sq_1(xy) = x^2Sq_1y + y^2Sq_1x \tag{9}$$

and the product formula follows directly from this. Put $m = 2^{k-1}$ and let $F(x) := x^{2m-2}$. By the binomial formula we get

$$\Delta F(x,y) = \sum_{r+s=m-1} x^{2r}y^{2s} - x^{2m-2} - y^{2m-2} = (xy \sum_{r+s=m-3} x^r y^s)^2$$

and using (8) this implies

$$\Delta\Lambda_{2m}(x,y) = y^2Sq_1(x) \sum_{r+s=m-3} x^{2r+2}y^{2s} + x^2Sq_1(x) \sum_{r+s=m-3} x^{2r}y^{2s+2} +$$
$$x^{2m-2}Sq_1(y) + y^{2m-2}Sq_1(x) = Sq_1(xy) \sum_{r+s=m-2} x^{2r}y^{2s}$$

and the first relation is proved. Note that (9) implies $Sq_1(ab^2) = Sq_1(a)b^4$. It follows that $Sq_1(x^{2r+1}y^{2s+1}) = Sq_1(xy)x^{4r}y^{4s}$ hence

$$\Lambda_{2^i}(x^{2r+1}y^{2s+1}) = x^{2(2^i r+2^{i-1}-1)}y^{2(2^i s+2^{i-1}-1)}Sq_1(xy)$$

It remains to show that

$$\sum_{r+s=2^{k-1}-2} x^{2r}y^{2s} = \sum_{i=1}^{k-1} \sum_{r+s=2^{k-1-i}-1} x^{2(2^i r+2^{i-1}-1)}y^{2(2^i s+2^{i-1}-1)}$$

This follows since $2^i r + 2^{i-1} - 1 + 2^i s + 2^{i-1} - 1 = 2^k - 1 - 2$ and

$$\coprod_{i=1}^{k-1}\{2^i r + 2^{i-1} - 1 | r = 0, 1, 2, 3, \dots, 2^{k-1-i} - 1\} = \{0, 1, 2, 3, \dots, 2^{k-1} - 2\}$$

The last is easily seen by using the binary expansion of the numbers. $\square$

Definition 4.6. Define operations $Q^n_{2^k} : K \longrightarrow K$ for $k \geq 0$ by

$$Q^n_1(x) := Sq^n(x)$$

$$Q^n_2(x) := \sum_{j=0}^{[\frac{n}{2}]} Sq^j(x)Sq^{n-j}(x)$$

$$Q^n_{2^k}(x) := \sum_{j=0}^{n} Sq^j(x^{2^k-2})Q^{n-j}_2(x)\ ,\ k \geq 2$$

and define the total operations $Q_{2^k} : K \longrightarrow K[t]$ by

$$Q_{2^k}(x) := \sum_{n\geq 0} Q^n_{2^k}(x)\, t^n$$

There is a recursive formula for these operations

$$Q^n_{2^{k+1}}(x) = \sum_{j=0}^{n} Sq^j(x^{2^k})Q^{n-j}_{2^k}(x)$$

This is seen by using total operations. Let Sq denote the total square i.e. $Sq(x) := \sum_{n\geq 0} Sq^n(x)\, t^n$. For $k \geq 1$ the following holds

$$Q_{2^{k+1}}(x) = Sq(x^{2^{k+1}-2})Q_2(x) = Sq(x^{2^k})Sq(x^{2^k-2})Q_2(x) = Sq(x^{2^k})Q_{2^k}(x)$$

giving the recursive formula above.

Theorem 4.7. *For $k \geq 1$ and $n \geq 0$ odd we have*

$$Q^n_{2^k}(xy) = \sum_{j=0}^{n}(Sq^j(x^{2^k})Q^{n-j}_{2^k}(y) + Sq^j(y^{2^k})Q^{n-j}_{2^k}(x))$$

Proof. We prove the formula by induction on k. We start by the initial case $k=1$ which is the hardest. Let $F(x,y)=Q_2^n(xy)$ and $G(x,y)=\sum_{j=0}^n(Sq^j(x^2)Q_2^{n-j}(y)+Sq^j(y^2)Q_2^{n-j}(x))$. We find

$$\Delta Q_2^n(x,y)=Sq^n(xy)$$

Since $\Delta_1 F(x_1,x_2,y)=\Delta Q_2^n(x_1y,x_2y)$ and $\Delta_2 F(x,y_1,y_2)=\Delta Q_2^n(xy_1,xy_2)$ this implies

$$\Delta_1 F(x_1,x_2,y)=Sq^n(x_1x_2y^2)\ ,\ \Delta_2 F(x_1,x_2,y)=Sq^n(x^2y_1y_2)$$

We also find

$$\begin{aligned}\Delta_1 G(x_1,x_2,y)&=\sum_{j=0}^n Sq^j(y^2)\Delta Q_2^{n-j}(x_1,x_2)\\&=\sum_{j=0}^n Sq^j(y^2)Sq^{n-j}(x_1x_2)=Sq^n(x_1x_2y^2)\end{aligned}$$

and $\Delta_2 G(x,y_1,y_2)=Sq^n(xy_1y_2)$. That is $\Delta_\nu(F-G)=0\ ,\ \nu=1,2$ thus $F-G$ is bilinear. We have that

$$(F-G)(x,y)=\sum Sq^{a_i}(x)Sq^{b_i}(x)Sq^{c_i}(y)Sq^{d_i}(y)$$

and since this is bilinear the terms $Sq^{a_i}(x)Sq^{b_i}(x)$ and $Sq^{c_i}(y)Sq^{d_i}(y)$ are linear. Let $f(x)=Sq^a(x)Sq^b(x)$. We find $\Delta f(x,y)=Sq^a(x)Sq^b(y)+Sq^a(y)Sq^b(x)$. We see that f is linear if and only if $a=b$. Thus $F-G=0$ since the degree n is odd and the initial case is done. Define the odd total Q-operations from K to $K[t]$ by

$$\overline{Q}_{2^k}(x)=\sum_{m=0}^\infty Q_{2^k}^{2m+1}(x)\,t^{2m+1}$$

We have

$$Sq(x^{2^k})\overline{Q}_{2^k}(x)=\overline{Q}_{2^{k+1}}(x)$$

for $k\geq 1$ and we would like to prove

$$\overline{Q}_{2^k}(xy)=Sq(x^{2^k})\overline{Q}_{2^k}(y)+Sq(y^{2^k})\overline{Q}_{2^k}(x)$$

Assume that this is OK for k. Then

$$\begin{aligned}\overline{Q}_{2^{k+1}}(xy)&=Sq((xy)^{2^k})\overline{Q}_{2^k}(xy)\\&=Sq(x^{2^k})Sq(y^{2^k})(Sq(x^{2^k})\overline{Q}_{2^k}(y)+Sq(y^{2^k})\overline{Q}_{2^k}(x))\\&=(Sq(x^{2^k}))^2Sq(y^{2^k})\overline{Q}_{2^k}(y)+(Sq(y^{2^k}))^2Sq(x^{2^k})\overline{Q}_{2^k}(x)\\&=Sq(x^{2^{k+1}})\overline{Q}_{2^{k+1}}(y)+Sq(y^{2^{k+1}})\overline{Q}_{2^{k+1}}(x)\end{aligned}$$

and the formula is OK for $k+1$. $\square$

Note that for n odd we have $Q_{2^k}^n(x^2)=0$.

Theorem 4.8. *For n odd, $|x| = |y|$ and for $k \geq 1$ we have*

$$\Delta Q^n_{2^k}(x,y) = \sum_{i=0}^{k-1} \sum_{r+s=2^{k-1-i}-1} Q^n_{2^i}(x^{2r+1}y^{2s+1})$$

Proof. The formula is proved by induction on k. The initial case is OK since

$$\Delta Q^n_2(x,y) = Sq^n(xy)$$

Assume that the formula is OK for k. Using (8) on the definition we get

$$\Delta Q^n_{2^{k+1}}(x,y) =$$
$$\sum_{j=0}^{n}[(Sq^j(x^{2^k}) + Sq^j(y^{2^k}))\Delta Q^{n-j}_{2^k}(x,y) + Sq^j(x^{2^k})Q^{n-j}_{2^k}(y) + Sq^j(y^{2^k})Q^{n-j}_{2^k}(x)]$$

By Theorem 4.7 this equals

$$\Delta Q^n_{2^{k+1}}(x,y) = Q^n_{2^k}(xy) + \sum_{j=0}^{n}(Sq^j(x^{2^k}) + Sq^j(y^{2^k}))\Delta Q^{n-j}_{2^k}(x,y)$$

where $\Delta Q^{n-j}_{2^k}(x,y)$ is known by the induction hypothesis. Using Theorem 4.7 and the fact $x^{2^k} = ((x^{2^{k-i-1}})^2)^{2^i}$ we find

$$\sum_{j=0}^{n} Sq^j(x^{2^k})Q^{n-j}_{2^i}(x^{2r+1}y^{2s+1}) = Q^n_{2^i}(x^{2^{k-i}+2r+1}y^{2s+1})$$

thus

$$\Delta Q^n_{2^{k+1}}(x,y) =$$
$$Q^n_{2^k}(xy) + \sum_{i=0}^{k-1} \sum_{r+s=2^{k-i-1}-1} (Q^n_{2^i}(x^{2^{k-i}+2r+1}y^{2s+1}) + Q^n_{2^i}(y^{2^{k-i}+2r+1}x^{2s+1})) =$$
$$Q^n_{2^k}(xy) + \sum_{i=0}^{k-1} \sum_{r+s=2^{k-i}-1} Q^n_{2^i}(x^{2r+1}y^{2s+1}) = \sum_{i=0}^{k} \sum_{r+s=2^{k-i}-1} Q^n_{2^i}(x^{2r+1}y^{2s+1})$$

□

Lemma 4.9. *Assume that $|x| = 1$ and $k \geq 1$. Then*

(1) $Q^{2i-1}_{2^k}(x) = x^{2^k+2i-1}$ *for* $1 \leq i \leq 2^{k-1}$ *and* 0 *otherwise.*
(2) $Sq^{2i}\Lambda_{2^k}(x) = x^{2^k+2i-1}$ *for* $0 \leq i \leq 2^{k-1}-1$ *and* 0 *otherwise.*

Proof. (1) $k=1$ is OK since $\overline{Q}_2(x)=x^3t$. Assume that the formula holds for k. We have

$$\overline{Q}_{2^{k+1}}(x)=Sq(x^{2^k})\overline{Q}_{2^k}(x)=(x^{2^k}+x^{2^{k+1}}t^{2^k})\sum_{j=1}^{2^{k-1}}x^{2^k+2j-1}t^{2j-1}=$$

$$\sum_{j=1}^{2^{k-1}}x^{2^{k+1}+2j-1}t^{2j-1}+\sum_{j=1}^{2^{k-1}}x^{2^{k+1}+2^k+2j-1}t^{2^k+2j-1}=\sum_{j=1}^{2^k}x^{2^{k+1}+2j-1}t^{2j-1}$$

(2) $Sq_1(x)=x$ hence $\Lambda_{2^k}(x)=x^{2^k-1}$ and $Sq^{2i}\Lambda_{2^k}(x)=\binom{2^k-1}{2i}x^{2^k+2i-1}$. $\square$

Let $s_{\alpha_1,\dots,\alpha_n}(x_1,\dots x_n)$ denote the symmetric polynomial one obtains from symmetricising the monomial $x_1^{\alpha_1}\dots x_n^{\alpha_n}$.

Lemma 4.10. *Let $k\geq 1$ and put $m=2^k$. Assume that $x_1,\dots,x_n$ all have degree 1 and put $x=x_1\dots x_n$. Then*

1. *$Sq^{2i}\Lambda_{2m}(x)+Q_{2m}^{2i-1}(x)=0$ for $i\notin m\mathbb{Z}$*
2. *$Sq^{2rm}\Lambda_{2m}(x)=s_{4m,\dots,4m,2m,\dots,2m,2m-1}(x_1,\dots,x_n)$ where there are r times $4m$ and $n-r-1$ times $2m$*
3. *$Q_{2m}^{2rm-1}(x)=s_{4m,\dots,4m,2m,\dots,2m,4m-1}(x_1,\dots,x_n)$ where there are $r-1$ times $4m$ and $n-r$ times $2m$*

Proof. (1) This is OK for $n=1$ by lemma 4.9. Assume that it holds for $n-1$ and let $i\notin m\mathbb{Z}$. By lemma 4.5 and the Cartan formula we find

$$Sq^{2i}\Lambda_{2m}(xx_n)=\sum_{j=0}^{2i}(Sq^j(x^{2m})Sq^{2i-j}\Lambda_{2m}(x_n)+Sq^j(x_n^{2m})Sq^{2i-j}\Lambda_{2m}(x))$$

And by lemma 4.7 we have

$$Q_{2m}^{2i-1}(xx_n)=\sum_{j=0}^{2i-1}(Sq^j(x^{2m})Q_{2m}^{2i-1-j}(x_n)+Sq^j(x_n^{2m})Q_{2m}^{2i-1-j}(x))$$

In these sums only the terms corresponding to $j\in m\mathbb{Z}$ contribute and for these j we have $Sq^{2i-j}\Lambda_{2m}(x)+Q_{2m}^{2i-1-j}(x)=0$ and $Sq^{2i-j}\Lambda_{2m}(x_n)+Q_{2m}^{2i-1-j}(x_n)=0$ and the result follows.

(2) $Sq_1(x)=s_{2,2,\dots,2,1}$ hence $\Lambda_{2m}(x)=s_{2m,2m,\dots,2m,2m-1}$ and we get $Sq^{2rm}\Lambda_{2m}(x)=s_{4m,\dots,4m,2m,\dots,2m,2m-1}$ with r times $4m$.

(3) Let σ_i denote the elementary symmetric polynomial $s_{1,\dots,1,0,\dots,0}$ with i times 1. We put $\sigma_i=0$ for $i>n$. We have

$$Sq^t(x_1\dots x_n)=\sigma_t\sigma_n$$

Define $f_{2^k,t} := s_{2^k,\dots,2^k,2^k-1,0,\dots,0}$ with k times 2^k. For $m = 1$ we must show

$$f_{2,r-1} = \sum_{j=0}^{r-1} \sigma_j \sigma_{2r-1-j}$$

We prove this by taking partial derivatives. For a polynomial $p(x_1, \dots, x_n)$ we let $(p)_i$ denote the polynomial $(p)_i(x_1, \dots, x_n) = p(x_1, \dots, \hat{x}_i, \dots, x_n)$. We have

$$\frac{\partial}{\partial x_i} f_{2,r-1} = (\sigma_{r-1})_i^2$$

and this equals

$$\frac{\partial}{\partial x_i} \sum_{j=0}^{r-1} \sigma_j \sigma_{2r-1-j} = \sum_{j=0}^{r-1} ((\sigma_{j-1})_i \sigma_{2r-1-j} + \sigma_j (\sigma_{2r-2-j})_i) =$$

$$x_i \sum_{j=0}^{r-1} ((\sigma_{j-1})_i (\sigma_{2r-2-j})_i + (\sigma_{j-1})_i (\sigma_{2r-2-j})_i) + \sum_{j=0}^{r-1} ((\sigma_{j-1})_i (\sigma_{2r-1-j})_i + (\sigma_j)_i (\sigma_{2r-2-j})_i) =$$

$$(\sigma_{r-1})_i^2$$

We now have that the difference of the two sides is a symmetric polynomial q with all partial derivatives equal to zero. Since the degree of q is odd Euler's formula

$$\sum_i x_i \frac{\partial q}{\partial x_i} = deg(q) q$$

gives that $q = 0$.

for $m \geq 2$ we must show

$$f_{2m,r-1} = \sum_{j=0}^{2r-1} \sigma_j^m f_{m,2r-1-j}$$

Again we take partial derivatives:

$$\frac{\partial}{\partial x_i} f_{2m,r-1} = x_i^{2m-2} (s_{2m,\dots,2m,0,\dots,0})_i = x_i^{2m-2} (\sigma_{r-1})_i^{2m}$$

This equals

$$\frac{\partial}{\partial x_i} \sum_{j=0}^{2r-1} \sigma_j^m f_{m,2r-1-j} = \sum_{j=0}^{2r-1} \sigma_j^m x_i^{m-2} (\sigma_{2r-1-j})_i^m =$$

$$x_i^{2m-2} \sum_{j=1}^{2r-1} (\sigma_{j-1})_i^m (\sigma_{2r-1-j})_i^m + x^{m-2} \sum_{j=0}^{2r-1} (\sigma_j)_i^m (\sigma_{2r-1-j})_i^m$$

since $\sum_{j=0}^{2r-1} (\sigma_j)_i (\sigma_{2r-1-j})_i = 0$ and $\sum_{j=1}^{2r-1} (\sigma_{j-1})_i (\sigma_{2r-1-j})_i = (\sigma_{r-1})_i^2$. The result follows using Euler's formula. $\square$

5. The diagonal map.

Let $n = 2^r$ and let $\Delta : X \longrightarrow X^n$; $x \mapsto (x, \dots, x)$ be the diagonal map. This is a C_n-equivariant map hence it gives rise to a map

$$BC_n \times X \xrightarrow{1\times\Delta} E_n X$$

In this section we will determine the induced map

$$\Delta^* = H^*(1 \times \Delta) : H^*(E_n X) \longrightarrow H^*(C_n) \otimes H^*(X)$$

It is enough to find $\Delta^*(e^k \otimes x^{\otimes n})$ since the other classes in $H^*(E_n X)$ are hit by the transfer map applied to this type of elements.

Lemma 5.1. *Let $n = 2^k$ and $m = 2^l$ and let*

$$\Delta : X \longrightarrow X^{nm} \, , \; \Delta_1 : X^m \longrightarrow (X^m)^n \, , \; \Delta_2 : X \longrightarrow X^m$$

be the corresponding diagonal maps. There is a commutative diagram

$$\begin{tikzcd}
& H^*(BC_n) \otimes H^*(X)^{\otimes m} \arrow[dr, "1\otimes\Delta_2^*"] & \\
H^*(E_n(X^m)) \arrow[ur, "(1\times\Delta_1)^*"] \arrow[rr, "\Delta^*"'] \arrow[d, "tr_n^{nm}"'] & & H^*(BC_n) \otimes H^*(X) \arrow[d, "tr_n^{nm}\otimes 1"] \\
H^*(E_{nm}X) \arrow[rr, "\Delta^*"'] & & H^*(BC_{nm}) \otimes H^*(X)
\end{tikzcd}$$

Proof. The upper part of the diagram commutes since Δ equals the composite

$$EC_{nm}/C_n \times X \xrightarrow{\Delta_2} EC_{nm}/C_n \times X^m \xrightarrow{\Delta_1} EC_{nm} \times_{C_n} X^{nm}$$

The lower part of the diagram commutes since the following diagram commutes

$$\begin{tikzcd}
C_*(EC_{nm} \times_{C_n} X^{nm}) & C_*(EC_{nm}/C_n \times X) \arrow[l, "\Delta_*"'] \\
C_*(EC_{nm} \times_{C_{nm}} X^{nm}) \arrow[u, "tr"] & C_*(EC_{nm}/C_{nm} \times X) \arrow[l, "\Delta_*"'] \arrow[u, "tr"']
\end{tikzcd}$$

The commutativity of the last diagram is easily seen: Let Δ^n denote the geometrical n-simplex. A map $\alpha : \Delta^n \longrightarrow EC_{nm}/C_n \times X$ has a lifting $\alpha' : \Delta^n :\longrightarrow EC_{nm}/C_n \times X$ that is $q \circ \alpha' = \alpha$ where $q : EC_{nm}/C_n \times X \longrightarrow EC_{nm}/C_n \times X$ is the quotient map. We have

$$\Delta_* \circ tr([\alpha]) = \Delta_*(\sum_{g\in C_{nm}/C_n} g[\alpha']) = \sum_{g\in C_{nm}/C_n} g[\Delta \circ \alpha']$$

and this equals

$$tr \circ \Delta_*([\alpha]) = tr([\Delta \circ \alpha]) = \sum_{g \in C_{nm}/C_n} g[\Delta \circ \alpha']$$

□

Lemma 5.2. *Let $n = 2^k$. The composite*

$$H^*(EC_n \times X^n) \xrightarrow{Tr} H^*(E_n X) \xrightarrow{\Delta^*} H^*(BC_n \times X)$$

is zero. That is $\Delta^*(e^0 \otimes Na_1 \otimes \ldots \otimes a_n) = 0$.

Proof. See [St-Ep]. The map $q^* : H^*(BC_4 \times X) \to H^*(EC_4 \times X)$ is surjective and we have the commutative diagram by lemma 5.1

$$\begin{tikzcd}
 & H^*(EC_n \times X^n) \arrow[r, "Tr"] \arrow[d, "\Delta^*"] & H^*(EC_n \times_{C_n} X^n) \arrow[d, "\Delta^*"] \\
H^*(EC_n \times_{C_n} X) \arrow[r, "q^*"] & H^*(EC_n \times X) \arrow[r, "Tr"] & H^*(EC_n \times_{C_n} X)
\end{tikzcd}$$

The composite on the bottom line is multiplication by n thus the 0-map. □

Lemma 5.3.

$$\Delta^*(e^i \otimes 1^{\otimes 2^k}) = e^i \otimes 1$$

Proof. Put $n = 2^k$ and let $* \in X$. We have maps $r : X \longrightarrow *$ and $i : * \longrightarrow X$ with $r \circ i = id$. There is a commutative diagram

$$\begin{tikzcd}
E_n X \arrow[d, "E_n(r)"'] & BC_n \times X \arrow[l, "1\times\Delta"'] \arrow[d, "1\times r"'] \\
E_n(*) & BC_n \times * \arrow[l, "1\times\Delta"']
\end{tikzcd}$$

Note that $E_n(*) = BC_n \times *^n$ and that $1 \times \Delta$ induces an isomorphism in cohomology on the bottom line. We have $e^i \otimes 1^{\otimes n} = H^* E_n(r)(e^i \otimes 1^{\otimes n})$ and $(1 \otimes r^*) \circ \Delta^*(e^i \otimes 1^{\otimes n}) = 1 \otimes r^*(e^i \otimes 1) = e^i \otimes 1$ and the result follows. □

Corollary 5.4. $\Delta^* : H^* E_{2^k}(X) \to H^*(C_{2^k}) \otimes H^*(X)$ *is* $H^*(C_{2^k})$*-linear*

Theorem 5.5. *Let $k \geq 0$ and let $m = 2^k$. Define the maps*

$$R_{4m}, \Gamma_{4m} : H^*(X) \to H^*(C_{4m}) \otimes H^*(X)$$

by

$$R_{4m}(x) = \sum_{i=0}^{|x|} u^{m(|x|-i)} \otimes (Sq^i x)^{2m}$$

$$\Gamma_{4m}(x) = \sum_{i=0}^{m|x|} vu^{m|x|-i} \otimes (Sq^{2i}(x^{2(m-1)}Sq_1 x) + Q_{2m}^{2i-1}(x))$$

Then

$$\Delta^*(e^0 \otimes x^{\otimes 4m}) = R_{4m}(x) + \Gamma_{4m}(x)$$

Proof. Let $pr_e, pr_o : H^*(C_{4m}) \longrightarrow H^*(C_{4m})$ be the maps defined by letting pr_e be the identity in even degrees and zero in odd degrees and pr_0 zero in even degrees and the identity in odd degrees.

Step1: $(pr_e \otimes 1) \circ \Delta^*(e^0 \otimes x^{\otimes 4m}) = R_{4m}(x)$. We prove this by induction on k. We have a commutative diagram induced from a commutative diagrams of spaces

$$\begin{array}{ccccc}
H^*(E_{4m}X) & \xrightarrow{\Delta^*} & H^*(C_{4m}) \otimes H^*(X) & \xrightarrow{res_{4m}^{2m}} & H^*(C_{2m}) \otimes H^*(X) \\
{\scriptstyle Res_{4m}^{2m}}\downarrow & & & & \| \\
H^*(E_{2m}(X^2)) & \xrightarrow{\Delta^*} & H^*(C_{2m}) \otimes H^*(X^2) & \xrightarrow{1\otimes\Delta^*} & H^*(C_{2m}) \otimes H^*(X)
\end{array}$$

For $k = 0$ this gives

$$(res_4^2 \otimes 1) \circ \Delta^*(e^0 \otimes x^{\otimes 4}) =$$

$$(1 \otimes \Delta^*) \circ \Delta^*(e^0 \otimes (x \otimes x)^{\otimes 2}) = 1 \otimes \Delta^* \sum_{j=0}^{2|x|} a^{2|x|-j} \otimes Sq^j(x \otimes x) =$$

$$\sum_{j=0}^{2|x|} a^{2|x|-j} \otimes Sq^j(x^2) = \sum_{i=0}^{|x|} a^{2(|x|-i)} \otimes (Sq^i x)^2$$

and the initial case is OK. Assume that Step 1 is OK for k then

$$(res_{8m}^{4m} \otimes 1) \circ \Delta^*(e^0 \otimes x^{\otimes 8m}) = (1 \otimes \Delta^*)R_{4m}(x \otimes x) = R_{4m}(x^2) = R_{8m}(x)$$

thus Step 1 is OK for $k+1$ and we are done. The proof of

$$(pr_0 \otimes 1) \circ \Delta^*(e^0 \otimes x^{4m}) = \Gamma_{4m}(x)$$

is divided into two steps. First we show that the formula have the right deviation from linearity. Since R_{4m} is linear this means

Step 2: For $|x| = |y|$ we have

$$\Delta^*(e^0 \otimes (x+y)^{\otimes 4m} - e^0 \otimes x^{\otimes 4m} - e^0 \otimes y^{\otimes 4m}) = \Delta\Gamma_{4m}(x, y)$$

We prove this by induction on k. Let $k = 0$ and define

$$\Gamma_2(z) := \sum_{i\geq 0} a^{|z|-2i+1} \otimes Sq^{2i-1}(z)$$

for elements z with $|z|$ even. We have

$$e^0 \otimes (x+y)^{\otimes 4} - e^0 \otimes x^{\otimes 4} - e^0 \otimes y^{\otimes 4} = e^0 \otimes (1+T)x \otimes y \otimes x \otimes y = Tr_2^4(e^0 \otimes (x \otimes y)^{\otimes 2})$$

and since $\Delta^* \circ Tr_2^4 = (Tr_2^4 \otimes \Delta^*) \circ \Delta^*$ by lemma 5.1 we get

$$\Delta^*(e^0 \otimes (1+T)x \otimes y \otimes x \otimes y) = \Gamma_2(xy) = \Delta\Gamma_4(x, y)$$

and the initial case is OK. As before define

$$\Gamma_2(x) = \sum_{i\geq 0} vu^{2|z|-i} \otimes Sq^{2i-1}(z)$$

for $|z|$ even. Assume that Step 2 is proved for all $l \leq k-1$. Let

$$M = \{a_1 \otimes \ldots \otimes a_{4m} | \forall i : a_i \in \{x, y\}\}$$

C_{4m} acts on M by cyclic permutation. For $a \in M$ we let β_a denote the orbit of a and $s\beta_a = \sum_{\gamma\in\beta_a} \gamma$ the sum of the elements in the orbit. We have a splitting

$$(x+y)^{\otimes 4m} = \sum_{i=0}^{k+2} \sum_{|\beta|=2^i} s\beta$$

Let $\tau_{2^i} = \sum_{|\beta|=2^i} e^0 \otimes s\beta$. We claim that

$$\Delta^*(\tau_{2i}) = \sum_{r+s=2^{i-1}-1} \Gamma_{2^{k+2-i}}(x^{2r+1}y^{2s+1})$$

for $i = 1, 2, \ldots, k+1$. Let $i \in \{1, 2, \ldots k+1\}$ and put $n = 2^i$, $r = \frac{4m}{n}$. Let a be an element with $|\beta| = n$. We can write $a = (a_1 \otimes \ldots \otimes a_n)^{\otimes r}$ where $a_l \in \{x, y\}$ for all l. Using lemma 5.1 we get

$$\Delta^*(e^0 \otimes s\beta_a) = \Delta^* \circ Tr_r^{4m}(e^0 \otimes (a_1 \otimes \ldots \otimes a_n)^r) = \Gamma_r(a_1 \ldots a_n)$$

We define the type of β_a as the number $t(\beta_a)$ of x‘is in $a_1 \otimes \ldots \otimes a_n$. We see that orbits α and β with the same length and type have $\Delta^*(e^0 \otimes s\alpha) = \Delta^*(e^0 \otimes s\beta)$. We have already observed that $Q_{2^r}^{2s+1}(z^2) = 0$ and since $Sq_1(z^2) = 0$ we have $\Gamma_{2^l}(z^2) = 0$ for all z and all l. Thus orbits of even type maps to 0 under Δ^*. The number of orbits with length n and type $2s+1$ equals

$$\frac{1}{n}\binom{n}{2s+1}$$

since a string $a_1 \otimes \ldots \otimes a_n$ with an odd number of x'is satisfy $a_1 \otimes \ldots \otimes a_{\frac{n}{2}} \neq a_{\frac{n}{2}+1} \otimes \ldots \otimes a_n$. The 2-valuation of this number is

$$v_2(\frac{1}{n}\binom{n}{2s+1}) = v_2(\frac{(2^i-2)(2^i-4)(2^i-6)\ldots(2^i-2s)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2s}) = 0$$

thus there is a odd number of these orbits and the claim follows. We have

$$\Delta^*(e^0 \otimes (x+y)^{\otimes 4m} - e^0 \otimes x^{4m} - e^0 \otimes y^{4m}) = \Delta^*(\sum_{i=1}^{k+1} \tau_{2^i}) =$$
$$\sum_{i=1}^{k+1} \sum_{r+s=2^{i-1}-1} \Gamma_{2^{k+2-i}}(x^{2r+1}y^{2s+1})$$

and since $|x^{2r+1}y^{2s+1}| = 2^i|x|$ for $r+s = 2^{i-1}-1$ this equals

$$\sum_{l=0}^{2^k|x|} vu^{2^k|x|-l} \otimes \sum_{i=1}^{k+1} \sum_{r+s=2^{i-1}-1} [Sq^{2l}\Lambda_{2^{k+1-i}} + Q^{2l-1}_{2^{k+1-i}}](x^{2s+1}y^{2r+1})$$

By lemma 4.5 and lemma 50 this is the same as

$$\sum_{l=0}^{2^k|x|} vu^{2^k|x|-l} \otimes \Delta[Sq^{2l}\Lambda_{2^{k+1}} + Q^{2l-1}_{2^{k+1}}](x,y)$$

and Step 2 is proved. Now we have that

$$\Delta^*(e^0 \otimes x^{\otimes 4m}) = R_{4m}(x) + \Gamma_{4m}(x) + L_{4m}(x)$$

where L_{4m} is linear. It remains to show that $L_{4m} = 0$. This follows from Step 3 below.

Step 3: Let $X = \prod_{i=0}^n \mathbb{R}P^\infty$. We have $H^*(X) = \mathbb{F}_2[x_1, \ldots, x_n]$ where $|x_l| = 1$ for $1 \le l \le n$. Put $x = x_1 \ldots x_n$ then

$$\Delta^*(e^0 \otimes x^{\otimes 4m}) = R_{4m}(x) + \Gamma_{4m}(x)$$

We start by proving the case $n = 1$ where $H^*(X) = \mathbb{F}_2[x]$ and $|x| = 1$. We can write

$$\Delta^*(e^0 \otimes x^{\otimes 4m}) = R_{4m}(x) + \sum_{i=1}^{2m} \lambda_i vu^{2m-i} \otimes x^{2i-1}$$

for some $\lambda_i \in \mathbb{F}_2$. We can determine $\lambda_2, \lambda_3, \ldots, \lambda_{2m}$ by Step 2. Put $Y = \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ Since $\mathbb{R}P^\infty = K(\mathbb{F}_2, 1)$ we can find maps $f, g, h : Y \longrightarrow \mathbb{R}P^\infty$ such that $f^*(x) = x_1$,

$g^*(x) = x_2$, $h^*(x) = x_1 + x_2$. We find

$$\Delta^*(e^0 \otimes (x_1+x_2)^{\otimes 4m} - e^0 \otimes x_1^{\otimes 4m} - e^0 \otimes x_2^{\otimes 4m}) =$$
$$\Delta^* \circ (H^*E_{4m}(f) + H^*E_{4m}(g) +^* E_{4m}(h))(e^0 \otimes x^{\otimes 4m}) =$$
$$(1 \otimes f^* + 1 \otimes g^* + 1 \otimes h^*) \circ \Delta^*(e^0 \otimes x^{\otimes 4m}) =$$
$$\sum_{i=1}^{2m} \lambda_i v u^{2m-i} \otimes ((x_1+x_2)^{2i-1} - x_1^{2i-1} - x_2^{2i-1})$$

and by Step 2 this equals $\Delta\Gamma_{4m}(x_1, x_2)$. Note that $(x_1+x_2)^{2i-1} - x_1^{2i-1} - x_2^{2i-1}$ is nonzero for $i \geq 2$. By lemma 4.9 we find

$$\Delta\Gamma_{4m}(x_1, x_2) =$$
$$v \otimes [(x_1+x_2)^{4m-1} - x_1^{4m-1} - x_2^{4m-1}] + vu^m \otimes [(x_1+x_2)^{2m-1} - x_1^{2m-1} - x_2^{2m-1}]$$

For $m = 1$ this gives $\lambda_2 = 1$ and for $m \geq 1$ this gives $\lambda_{2m} = \lambda_m = 1$ and all other λ's are zero. That is

$$\Delta^*(e^0 \otimes x^{\otimes 4}) = 1 \otimes x^4 + u \otimes x^2 + v \otimes x^3 + \lambda_1 vu \otimes x$$

and

$$\Delta^*(e^0 \otimes x^{\otimes 4m}) = 1 \otimes x^{4m} + u^m \otimes x^{2m} + v \otimes x^{4m-1} + vu^m \otimes x^{2m-1} + \lambda_1 vu^{2m-1} \otimes x$$

for $m \geq 2$. The $\mathcal{A}$-action on H^*E_2Y is known. Especially

$$Sq^1(e^0 \otimes y^{\otimes 2}) = \binom{|y|}{1} e^1 \otimes y^{\otimes 2} + e^0 \otimes (1+t)y \otimes Sq^1(y)$$

This implies

$$Sq^1 \circ Res_2^{4m}(e^0 \otimes x^{\otimes 4m}) = Sq^1(e^0 \otimes (x^{\otimes 2m})^{\otimes 2}) = e^0 \otimes (1+t)x^{\otimes 2m} \otimes Sq^1(x^{\otimes 2m})$$

This equals $Res_2^{4m} \circ Sq^1(e^0 \otimes x^{\otimes 4m})$. Since the degree of $Sq^1(e^0 \otimes x^{\otimes 4m})$ is $4m+1$ we can write

$$Sq^1(e^0 \otimes x^{\otimes 4m}) = \sum_{j=0}^{2m} \sum_{i=1}^{n_j} e^{2j+1} \otimes z_{2m-j,i}^{\otimes 2} + \sum_i e^0 \otimes Ny_i$$

where $z_{k,i} = a_1 \otimes \ldots \otimes a_{2m}$ with $a_l \in \{1, x\}$ for all l and the number of x's equals k. Using lemma 3.2 we see that

$$Sq^1(e^0 \otimes x^{\otimes 4m}) = \alpha e^{4m+1} \otimes 1^{\otimes 4m} + \beta e^1 \otimes x^{\otimes 4m} + \sum_i e^0 \otimes Ny_i$$

for some $\alpha, \beta \in \mathbb{F}_2$. Now we have

$$\Delta^* \circ Sq^1(e^0 \otimes x^{\otimes 4m}) =$$
$$\alpha vu^{2m} \otimes 1 + \beta v\Delta^*(e^0 \otimes x^{\otimes 4m}) = \alpha vu^{2m} \otimes 1 + \beta(v \otimes x^{\otimes 4m} + vu^m \otimes x^{2m})$$

This equals

$$Sq^1 \circ \Delta^*(e^0 \otimes x^{\otimes 4m}) = \lambda_1 vu^{2m-1} \otimes x^2 + v \otimes x^{4m} + vu^m \otimes x^{2m}$$

for $m \geq 2$ and $\lambda_1 vu \otimes x^2 + v \otimes x^4$ for $m = 1$. Thus $\alpha = 0$ and $\beta = 1$ and $\lambda_1 = 1$ for $m = 1$ and $\lambda_1 = 0$ for $m \geq 2$. We have shown

$$\Delta^*(e^0 \otimes x^{\otimes 4m}) = 1 \otimes x^{4m} + u^m \otimes x^{2m} + v \otimes x^{4m-1} + vu^m \otimes x^{2m-1}$$

for all $m \geq 1$. By lemma 4.9 this equals $R_{4m}(x) + \Gamma_{4m}(x)$ and step 3 is proved for $n = 1$. Now consider the general case $x = x_1 \dots x_n$. Since $e^0 \otimes x^{\otimes 4m} = (e^0 \otimes x_1^{\otimes 4m}) \dots (e^0 \otimes x_n^{\otimes 4m})$ we have

$$\Delta^*(e^0 \otimes x^{\otimes 4m}) = \prod_{j=1}^{n} \Delta^*(e^0 \otimes x_j^{\otimes 4m}) = \prod_{j=1}^{n} (1 \otimes x_j^{4m} + u^m \otimes x_j^{2m} + v \otimes x_j^{4m-1} + vu^m \otimes x_j^{2m-1})$$

Since $v^2 = 0$ this equals

$$\prod_{j=1}^{n} (1 \otimes x_j^{4m} + u^m \otimes x_j^{2m}) + \sum_{s=0}^{n} (vu^m \otimes x_s^{2m-1} + v \otimes x_s^{4m-1}) \prod_{j \neq s} (1 \otimes x_j^{4m} + u^m \otimes x_j^{2m})$$

The first term clearly equals $R_{4m}(x)$ and the second term equals $\Gamma_{4m}(x)$ by lemma 4.10. $\square$

Note that the product formulas in lemma 4.5 and lemma 4.7 now follows from the fact that Δ^* is a ring homomorphism:

$$\Delta^*(e^0 \otimes x^{\otimes 2^k}) \Delta^*(e^0 \otimes y^{\otimes 2^k}) = \Delta^*(e^0 \otimes (xy)^{\otimes 2^k})$$

We can now find the action of Sq^1 on $H^* E_{4m} X$.

Proposition 5.6.

$$Sq^1(e^0 \otimes x^{\otimes 4m}) = \binom{|x|}{1} e^1 \otimes x^{\otimes 4m} + e^0 \otimes Sq^1(x^{\otimes 4m})$$

Proof. We know from Lemma 3.6 that

$$Sq^1(e^0 \otimes x^{\otimes 4m}) = \alpha(|x|) e^1 \otimes x^{\otimes 4m} + e^0 \otimes Sq^1(x^{\otimes 4m})$$

We can now find $\alpha(|x|)$ by using the $\mathcal{A}$-linearity of Δ^*. We have

$$\Delta^*(Sq^1(e^0 \otimes x^{\otimes 4m})) = Sq^1 \Delta^*(e^0 \otimes x^{\otimes 4m}) = Sq^1(\Gamma_{4m}(x)) =$$
$$\sum_{i=0}^{m|x|} vu^{m|x|-i} \otimes (Sq^{2i+1}(x^{2(m-1)} Sq_1 x) + Sq^1 Q_{2m}^{2i-1}(x))$$

This equals

$$\Delta^*(\alpha(|x|)e^1 \otimes x^{\otimes 4m} + e^0 \otimes Sq^1(x^{\otimes 4m})) =$$

$$\alpha(|x|)v\Delta^*(e^0 \otimes x^{\otimes 4m}) = \alpha(|x|)vS_{4m}(x) = \alpha(|x|)\sum_{i=0}^{|x|} vu^{m(|x|-i)} \otimes (Sq^i x)^{2m}$$

Especially for $i = 0$ we get

$$Sq^1(x^{2(m-1)}Sq_1 x) = \alpha(|x|)x^{2m}$$

By Lemma 4.3 we have $Sq^1 Sq_1 x = \binom{|x|}{1}x^2$ and using eg. $X = K(\mathbb{F}_2, n)$ we see that $\alpha(|x|) = \binom{|x|}{1}$. $\square$

6. The case of a general unstable $\mathcal{A}$-algebra.

Let M be an unstable $\mathcal{A}$-algebra. The formulas in Theorem 3.5 and Proposition 5.6 define an $\mathcal{A}$-module structure on $H^*(C_{4m}; M^{\otimes 4m})$ that is the Adem relations are respected. We also have a map

$$\delta : H^*(C_{4m}; M^{\otimes 4m}) \longrightarrow H^*(C_{4m}) \otimes M$$

defined by the formulas in Theorem 5.5. We now show directly that δ is $\mathcal{A}$-linear.

Proposition 6.1.

$$Sq^1 \circ \delta = \delta \circ Sq^1$$

Proof. By the proof of Lemma 5.6 we must show

$$Sq^{2i+1}(x^{2(m-1)}Sq_1 x) + Sq^1 Q_{2m}^{2i-1}(x) = \binom{|x|}{1} Sq^{2i}(x^{2m})$$

This is done by induction. For $m = 1$ we have

$$\begin{aligned}&Sq^1 Q_2^{2i-1}(x) = \\ &\sum_{j=0}^{i-1} Sq^1 Sq^j(x) Sq^{2i-1-j}(x) + \sum_{j=0}^{i-1} Sq^j(x) Sq^1 Sq^{2i-1-j}(x) = \\ &\sum_{j=0}^{i-1} \delta^2_{j,0} Sq^{j+1}(x) Sq^{2i-1-j}(x) + \sum_{j=0}^{i-1} \delta^2_{j,1} Sq^j(x) Sq^{2i-j}(x) = \\ &\delta^2_{i,1}(Sq^i x)^2\end{aligned}$$

and by Lemma 4.3 we have

$$Sq^{2i+1}(Sq_1 x) = \binom{i+|x|}{1}(Sq^i x)^2$$

Thus the formula holds when $m = 1$. Assume that the formula is OK for m. Writing $x^{2(2m-1)}Sq_1x$ as $x^{2m}x^{2(m-1)}Sq_1x$ and using the Cartan formula we get

$$Sq^{2i+1}(x^{2(2m-1)}Sq_1x) = \sum_{j=0}^{i} Sq^{2(i-j)}(x^{2m})Sq^{2j+1}(x^{2(m-1)}Sq_1x)$$

The other term in the formula can be written as

$$Sq^1Q_{4m}^{2i-1}(x) = \sum_{l=0}^{2i-1} Sq^l(x^{2m})Sq^1Q_{2m}^{2i-1-l}(x) = \sum_{l=0}^{2i-1} Sq^{2i-1-l}(x^{2m})Sq^1Q_{2m}^{l}(x) =$$
$$\sum_{j=0}^{i} Sq^{2(i-j)}(x^{2m})Sq^1Q_{2m}^{2j-1}(x)$$

Using induction and these two equations we get

$$Sq^{2i+1}(x^{2(2m-1)}Sq_1x) + Sq^1Q_{4m}^{2i-1}(x) = \sum_{j=0}^{i} Sq^{2(i-j)}(x^{2m})\binom{|x|}{1}Sq^{2j}(x^{2m})$$

and by the Cartan formula we are done. □

We will now prove that

$$Sq^{2s} \circ \delta = \delta \circ Sq^{2s}$$

First some notation. Given a sequence $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we write $l(\alpha) := m$ for the length of α and $|\alpha| := \alpha_1 + \cdots + \alpha_m$ for the degree of α. Put

$$A(m,i) := \{\alpha | l(\alpha) = m\ ,\ |\alpha| = i\}$$

C_m acts on $A(m,i)$ by cyclic permutation: $T(\alpha_1, \dots, \alpha_m) = (\alpha_m, \alpha_1, \dots, \alpha_{m-1})$. Let $A(m,i,r)$ be the set of multiindices in $A(m,i)$ lying in an orbit of length r. We use the following notation for the quotients: $B(m,i,r) := A(m,i,r)/C_m$ and $B(m,i) := A(m,i)/C_m$. Finally for $\alpha \in A(m,i)$ we define

$$Sq^{\alpha}_{\otimes}(x) := Sq^{\alpha_1}(x) \otimes \ldots \otimes Sq^{\alpha_m}(x)$$
$$Sq^{\alpha}_{\mu}(x) := Sq^{\alpha_1}(x) \ldots Sq^{\alpha_m}(x)$$

The following description of the $Q_{2^k}^n$-operations is needed.

Theorem 6.2.

$$Q_{2^k}^n(x) = \sum_{\alpha \in B(2^k,n)} Sq^{\alpha}_{\mu}(x)$$

Proof. Let $D_{2^k}^n(x)$ denote the right hand side of the equation in the theorem. It is clear that $D_1^n(x) = Q_1^n(x)$ and that $D_2^n(x) = Q_2^n(x)$. Thus it is enough to show

$$D_{2^k}^n(x) = \sum_{j=0}^{n} Sq^j(x^{2^{k-1}})D_{2^{k-1}}^{n-j}(x)$$

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ we define the type of α as $t(\alpha) = (i_0, \dots, i_r)$ where $r = max\{\alpha_1, \dots, \alpha_n\}$ and i_l is the number of l's in the string $\alpha_1, \dots, \alpha_n$. Remark that an α with $t(\alpha) = (i_0, \dots, i_r)$ has $|\alpha| = 0 \cdot i_0 + 1 \cdot i_1 + \cdots + r i_r$ and $l(\alpha) = i_0 + \cdots + i_r$. Given a sequence of non negative integers $(i_0, \dots i_r)$ with $i_0 + \cdots + i_r = 2^k$ and $0 \cdot i_0 + 1 \cdot i_1 + \cdots + r i_r = n$. The number of multiindices of type $(i_0, \dots, i_r)$ is

$$\binom{2^k}{i_0, \dots, i_r} = \frac{2^k!}{i_0! \dots i_r!}$$

and the number of orbits of length 2^l and type $(i_0, \dots, i_r)$ is

$$\frac{1}{2^l}[\binom{2^l}{2^{-k+l} i_0, \dots, 2^{-k+l} i_r} - \binom{2^{l-1}}{2^{-k+l+1} i_0, \dots, 2^{-k+l+1} i_r}]$$

where a multinomial coefficient should be read as zero if the indices are not all nonnegative integers. The number of orbits of type $(i_0, \dots, i_r)$ is thus

$$\sigma(i_0, \dots, i_r) = \sum_{l=1}^{k} \frac{1}{2^l}[\binom{2^l}{2^{-k+l} i_0, \dots, 2^{-k+l} i_r} - \binom{2^{l-1}}{2^{-k+l+1} i_0, \dots, 2^{-k+l+1} i_r}]$$

We now have the formula

$$D^n_{2^k}(x) = \sum_{\substack{i_0 + \cdots + i_r = 2^k \\ 0 \cdot i_0 + 1 \cdot i_1 + \cdots + r i_r = n}} \sigma(i_0, \dots, i_r)(Sq^0 x)^{i_0} \dots (Sq^r x)^{i_r}$$

and we must show

$$\sum_{t=0}^{r} \sigma(i_0, \dots, i_t - 2^{k-1}, \dots, i_r) \equiv \sigma(i_0, \dots, i_r)\, mod\, 2$$

We start by proving this in the case where there is an odd i_l. Since $i_1 + \cdots + i_r = 2^k$ there is at least two odd i_l's. Assume that this is i_0 and i_1. We have

$$\sigma(i_0, \dots, i_r) = \frac{2^k - 1}{i_0 i_1} \frac{(2^k - 2)!}{(i_0 - 1)!(i_1 - 1)! i_2! \dots i_r!} \equiv \binom{2^k - 2}{i_0 - 1, i_1 - 1, i - 2, \dots, i_r}$$

From the definition of the multinomial coefficient one sees the following relation

$$\binom{m}{j_0, \dots, j_r} = \binom{m}{j_0} \binom{m - j_0}{j_1} \binom{m - j_0 - j_1}{j_2} \dots \binom{m - j_0 - \cdots - j_{r-1}}{j_r}$$

From this we see that the multinomial coefficient modulo 2 can be determined as follows. Write the binary expansion of the numbers $j_0, \dots j_r$ under each other. If there is more than one 1 in a column we get 0 otherwise we get 1. Using this and the fact that $i_0 + \cdots + i_r = 2^k$ we see that $\sigma(i_0, \dots, i_r) = 1$ if and only if there are exactly two ones in the column corresponding to 2^0 and one one in each of the columns corresponding to $2^1, 2^2, \dots, 2^{k-1}$. The result follows. The general case follows from the case above if we can prove

$$\sigma(2j_0, \dots, 2j_r) \equiv \sigma(j_0, \dots, j_r) \, mod\, 2$$

This relation is easy to verify. Assume that $j_0 + \cdots + j_r = 2^k$. We have

$$\sigma(2j_0, \dots, 2j_r) - \sigma(j_0, \dots, j_r) = \frac{1}{2^{k+1}}[\binom{2^{k+1}}{2j_0, \dots, 2j_r} - \binom{2^k}{j_0, \dots, j_r}] \in \mathbb{Z}$$

Let ν_2 denote two valuation i.e. $\nu_2(a)$ equals the number of times 2 divides a. Then it is easy to see that

$$\nu_2(\binom{2^l}{s_0, \dots, s_r}) = \sum_{l=0}^{r} \#(s_l) - 1$$

where $\#(a)$ is the number of one's in the binary expansion of a. We see that

$$\nu_2(\binom{2^{k+1}}{2j_0, \dots, 2j_r}) = \nu_2(\binom{2^k}{j_0, \dots, j_r})$$

and the result follows. $\square$

Definition 6.3. Define operations $S^{a,b}_{2^k} : M \longrightarrow M$ for $k \geq 0$ by

$$\begin{aligned} S^{a,b}_1(x) &:= Sq^a Sq^b(x) \\ S^{a,b}_2(x) &:= Sq^a Q^b_2(x) + \delta^2_{b,0} Q^a_2(Sq^{\frac{b}{2}} x) \\ S^{a,b}_{2^k}(x) &:= \sum_{r=0}^{a} \sum_{s=0}^{b} S^{r,s}_1(x^{2^k-2}) S^{a-r,b-s}_2(x) \, , \; k \geq 2 \end{aligned}$$

and define a total operation

$$S_{2^k} : M \longrightarrow M[s,t] \; ; \; S_{2^k}(x) := \sum_{a,b \geq 0} S^{a,b}_{2^k}(x) t^a s^b$$

Note that $S_{2^{k+1}}(x) = S_1(x^{2^k}) S_{2^k}(x)$ for $k \geq 1$ giving the recursive formula

$$S^{a,b}_{2^{k+1}}(x) = \sum_{r=0}^{a} \sum_{s=0}^{b} S^{r,s}_1(x^{2^k}) S^{a-r,b-s}_{2^k}(x)$$

Theorem 6.4. *Let $k \geq 1$. For j odd we have*

$$\delta(e^0 \otimes Sq^{2j}(x^{\otimes 2^{k+1}})) = \sum_{i \geq 0} vu^{[\frac{j}{2}]+2^{k-1}|x|-i} \otimes Sq^{2i} Q^j_{2^k}(x)$$

and for j even we have

$$\delta(e^0 \otimes Sq^{2j}(x^{\otimes 2^{k+1}})) = \delta_{j,0}^{2^k} R_{2^{k+1}}(Sq^{2^{-k}j}(x))$$
$$+\sum_{i\geq 0} vu^{[\frac{j}{2}]+2^{k-1}|x|-i} \otimes (Sq^{2i}Sq^j\Lambda_{2^k}(x) + S_{2^k}^{2i-1,j}(x))$$

Proof. We have

$$Sq^{2j}(x^{\otimes 2^{k+1}}) = \sum_{l=0}^{k+1} \sum_{\alpha\in B(2^{k+1},2j,2^l)} (1+T+\cdots+T^{2^l-1})Sq_{\otimes}^{\alpha}(x)$$

From this we get

$$e^0 \otimes Sq^{2j}(x^{\otimes 2^{k+1}}) = \sum_{l=0}^{k+1} \sum_{\beta\in B(2^l,2^{l-k}j,2^l)} tr_{2^{k+1-l}}^{2^{k+1}}(e^0 \otimes (Sq_{\otimes}^{\beta}(x))^{\otimes 2^{k+1-l}}) \tag{10}$$

Assume that j is odd. Then this reduces to the terms $l = k$ and $l = k+1$ since $B(m,n,r) = \emptyset$ when n is not an integer. By definition of δ we get

$$\delta(e^0 \otimes Sq^{2j}(x^{\otimes 2^{k+1}})) = \sum_{i\geq 0} vu^{[\frac{j}{2}]+2^{k-1}|x|-i} \otimes Sq^{2i}(\sum_{\beta\in B(2^k,j,2^k)} Sq_{\mu}^{\beta}(x))$$

Since j is odd every orbit in $A(2^k,j)/C_{2^k}$ is of maximal length. Using this and Theorem 6.2 we get

$$\delta(e^0 \otimes Sq^{2j}(x^{\otimes 2^{k+1}})) = \sum_{i\geq 0} vu^{[\frac{j}{2}]+2^{k-1}|x|-i} \otimes Sq^{2i}Q_{2^k}^{j}(x)$$

and the result follows from the lemma below.

Assume that j is even From equation 10 we get

$$\delta(e^0 \otimes Sq^{2j}(x^{\otimes 2^{k+1}})) - \delta_{j,0}^{2^k} R_{2^{k+1}}(Sq^{2^{-k}j}(x)) = \sum_{i\geq 0} vu^{\frac{j}{2}+2^{k-1}|x|-i} \otimes$$

$$[Sq^{2i}[\sum_{l=0}^{k-1} \sum_{\alpha\in B(2^l,2^{l-k}j,2^l)} \Lambda_{2^{k-l}}(Sq_{\mu}^{\alpha}(x))] + \sum_{l=0}^{k} \sum_{\alpha\in B(2^l,2^{l-k}j,2^l)} Q_{2^{k-l}}^{2i-1}(Sq_{\mu}^{\alpha}(x))]$$

the result follows from the lemma below. $\square$

Lemma 6.5. *For $k \geq 1$ we have*

(1)
$$S_{2^k}^{2n,2m+1}(x) = Sq^{2n}Q_{2^k}^{2m+1}(x)$$

(2)
$$Sq^{2m}(\Lambda_{2^k}(x)) = \sum_{l=0}^{k-1} \sum_{\alpha\in B(2^l,2^{l-k+1}m,2^l)} \Lambda_{2^{k-l}}(Sq_{\mu}^{\alpha}(x))$$

(3)

$$S^{2n+1,2m}_{2^k}(x) = \sum_{l=0}^{k} \sum_{\alpha\in B(2^l,2^{l-k+1}m,2^l)} Q^{2n+1}_{2^{k-l}}(Sq^{\alpha}_{\mu}(x))$$

Proof. 1) By definition of the Q-operations and the Cartan formula

$$Sq^{2n}Q^{2m+1}_{2^k}(x) = \sum_{r=0}^{2m+1}\sum_{t=0}^{2n} Sq^tSq^r(x^{2^k-2})Sq^{2n-t}Q^{2m+1-r}_2(x)$$

Since the terms with r or t odd are zero and $Sq^{2i}Q^{2j+1}_2(x) = S^{2i,2j+1}_2(x)$ we are done.

2) From the product formulas for the Λ-operations we find

$$\Lambda_{2^k}(x_1\dots x_n) = \sum_{i=1}^{n}\Lambda_{2^k}(x_i)(x_1\dots\hat{x}_i\dots x_n)^{2^k}$$

Using this we get

$$\sum_{\alpha\in B(2^l,2^{l-k+1}m,2^l)} \Lambda_{2^{k-l}}(Sq^{\alpha}_{\mu}(x)) =$$

$$\sum_{\alpha\in A(2^l,2^{l-k+1}m,2^l)} \Lambda_{2^{k-l}}(Sq^{\alpha_1}(x))(Sq^{\alpha_2}(x)\dots Sq^{\alpha_{2^l}}(x))^{2^{k-l}}$$

Using $\Lambda_{2^r}(y) = y^{2^r-2}Sq_1(y)$ we see that this equals

$$\sum_{\beta\in A(2^{k-1},m,2^l)} Sq_1(Sq^{\beta_1}x)(Sq^{\beta_2}(x)\dots Sq^{\beta_{2^{k-1}}}(x))^2$$

hence the right hand side of the formula (2) in this lemma equals

$$\sum_{\beta\in A(2^{k-1},m)} Sq_1(Sq^{\beta_1}x)(Sq^{\beta_2}(x)\dots Sq^{\beta_{2^{k-1}}}(x))^2$$

Since $Sq_1(Sq^r(x)) = Sq^{2r}(Sq_1(x))$ and $(Sq^r(x))^2 = Sq^{2r}(x)$ we can rewrite this as $Sq^{2m}(x^{2^k-2}Sq_1x)$ by the Cartan formula.

3) From the product formulas for the Q-operations we get

$$\begin{aligned}\overline{Q}_{2^r}(x_1\dots x_s) &= \sum_{t=1}^{s} Sq((x_1\dots\hat{x}_t\dots x_s)^{2^r})\overline{Q}_{2^r}(x_t)\\ &= \sum_{t=1}^{s} Sq((x_1\dots\hat{x}_t\dots x_s)^{2^r}x_t^{2^r-2})\overline{Q}_2(x_t)\end{aligned}$$

Using this we find for $l < k$

$$\sum_{\alpha\in B(2^l,2^{l-k+1}m,2^l)} Q^{2n-1}_{2^{k-l}}(Sq^{\alpha}_{\mu}(x)) =$$

$$\sum_{A(2^l,2^{l-k+1}m,2^l)} \sum_{j=0}^{2n+1} Sq^j[(\prod_{i=1}^{2^l-1} Sq^{\alpha_i}(x))^{2^{k-l}}(Sq^{\alpha_{2^l}}x)^{2^{k-l}-2}]Q_2^{2n+1-j}(Sq^{\alpha_{2^l}}x) =$$

$$\sum_{\alpha\in A(2^{k-1},m,2^l)} \sum_{j=0}^{2n+1} Sq^j[(\prod_{i=1}^{2^{k-l}-1} Sq^{\alpha_i}(x))^2]Q_2^{2n+1-j}(Sq^{\alpha_{2^{k-1}}}x)$$

Using this and Theorem 6.2 together with the fact that $Sq^{2n+1}(y^2) = 0$ we see that the right hand side of the formula (3) in this lemma equals

$$\sum_{\alpha\in A(2^{k-1},m)} \sum_{j=0}^{2n+1} Sq^j[(\prod_{i=1}^{2^{k-1}-1} Sq^{\alpha_i}(x))^2]Q_2^{2n+1-j}(Sq^{\alpha_{2^{k-1}}}x) + Sq^{2n+1}Q^{2m}_{2^k}(x) =$$

$$\sum_{r=0}^{m}\sum_{j=0}^{2n+1} Sq^j[(Sq^r(x^{2^{k-1}-1}))^2]Q_2^{2n+1-j}(Sq^{m-r}x) + Sq^{2n+1}(\sum_{r=0}^{2m} Sq^r(x^{2^k-2})Q_2^{2m-r}(x)) =$$

$$\sum_{r=0}^{2m}\sum_{j=0}^{2n+1} Sq^jSq^r(x^{2^k-2})Q_2^{2n+1-j}(Sq^{m-\frac{r}{2}}x) + \sum_{r=0}^{2m}\sum_{j=0}^{2n+1} Sq^jSq^r(x^{2^k-2})Sq^{2n+1-j}Q_2^{2m-r}(x)$$

and this equals $S^{2n+1,m}_{2^k}(x)$ $\square$

Theorem 6.6.

$$\delta\circ Sq^{2s} = Sq^{2s}\circ\delta$$

Proof. It is enough to show that the equation holds when evaluated on $u^b\otimes x^{\otimes 2^{k+1}}$ for $k\geq 1$ and $b\geq 0$ that is we must show

$$Sq^{2s}\circ\delta(u^b\otimes x^{\otimes 2^{k+1}}) = \sum_{t=0}^{s}\binom{2c-t}{2s-2t}\delta(u^{b+s-t}\otimes Sq^{2t}(x^{\otimes 2^{k+1}})) \tag{11}$$

where $c = b + 2^{k-1}|x|$. As before define degree preserving maps $p_e, p_0 : H^*(C_{2^{k+1}}) \longrightarrow H^*(C_{2^{k+1}})$ by letting p_e be zero in odd degrees and the identity in even degrees and p_o zero in even degrees and the identity in odd degrees. If we apply $p_e\otimes 1$ on both sides of equation 11 we get

$$Sq^{2s}(u^bR_{2^{k+1}}(x)) = \sum_{t=0}^{s}\binom{2c-t}{2s-2t}\delta^{2^k}_{t,0}R_{2^{k+1}}(Sq^{2^{-k}t}x)$$

and this straight forward to verify. Thus it is enough to show that if we apply $p_o\otimes 1$ on both sides of equation 11 then the resulting equation is true. From the left hand side we get

$$(p_o \otimes 1) \circ Sq^{2s} \circ \delta(u^b \otimes x^{\otimes 2^{k+1}}) =$$
$$Sq^{2s}(\sum_{j=0}^{2^k|x|} vu^{c-j} \otimes (Sq^{2j}\Lambda_{2^k}(x) + Q_{2^k}^{2i-1}(x))) =$$
$$\sum_{j=0}^{2^k|x|}\sum_{i=0}^{s}\binom{c-j}{s-i} vu^{c+s-i-j} \otimes (Sq^{2i}Sq^{2j}\Lambda_{2^k}(x) + S_{2^k}^{2i,2j-1}(x))$$

and from the right hand side we get using Theorem 6.4

$$\sum_{t=0}^{s}\binom{2c-t}{2s-2t}(p_o \otimes 1) \circ \delta(u^{b+s-t} \otimes Sq^{2t}(x^{2^{k+1}})) =$$
$$\sum_{t=0}^{s}\binom{2c-t}{2s-2t}\sum_{i\geq 0} vu^{c+s-i-[\frac{t+1}{2}]} \otimes [\delta_{t,0}^2(Sq^{2i}Sq^{t}\Lambda_{2^k}(x) + S_{2^k}^{2i-1,t}(x)) + \delta_{t,1}^2 S_{2^k}^{2i,t}(x)] =$$
$$\sum_{r\geq 0} vu^{c+s-r} \otimes \sum_{i+j=r}[\binom{2c-2j}{2s-4j}(Sq^{2i}Sq^{2j}\Lambda_{2^k}(x) + S_{2^k}^{2i-1,2j}(x)) +$$
$$\binom{2c-2t+1}{2s-4j+2} S_{2^k}^{2i,2j-1}(x)]$$

Thus it is enough to show the two equations

$$\sum_{i+j=r}[\binom{c-j}{s-2j} + \binom{c-j}{s-i}]Sq^{2i}Sq^{2j}\Lambda_{2^k}(x) = 0 \tag{12}$$

and

$$\sum_{i+j=r}[\binom{c-j}{s-2j} S_{2^k}^{2i-1,2j}(x) + [\binom{c-j}{s+1-2j} + \binom{c-j}{s-i}]S_{2^k}^{2i,2j-1}(x)] = 0 \tag{13}$$

We can rewrite equation 12 by using the Adem relations when possible and Lemma 6.7

$$\sum_{3j\leq r}(\binom{c-j}{s-2j} + \binom{c-j}{s-r+j})Sq^{2(r-j)}Sq^{2j}\Lambda_{2^k}(x) =$$
$$\sum_{r<3j\leq 3r}(\binom{c-j}{s-2j} + \binom{c-j}{s-r+j})\sum_{t\geq 0}\binom{2j-1-2t}{2(r-j)-4t}Sq^{2(r-t)}Sq^{2t}\Lambda_{2^k}(x)$$

thus equation 12 follows if we can prove the following for $3j \leq r$

$$\sum_{r<3k\leq 3r}(\binom{c-k}{s-2k} + \binom{c-k}{s-r+k})\binom{k-1-j}{r-k-2j} = \binom{c-j}{s-2j} + \binom{c-j}{s-r+j} \tag{14}$$

The left hand side of equation 13 equals

$$\sum_{t\geq 0}[\binom{2c-t}{2s-2t}+\binom{2c-t}{2s-2r+1+t}]S_{2^k}^{2r-1-t,t}(x)$$

By Theorem 6.8 this equals

$$\sum_{2r-1<3t}[\binom{2c-t}{2s-2t}+\binom{2c-t}{2s-2r+1+t}]\sum_{l\geq 0}\binom{t-1-l}{2r-1-t-2l}S_{2^k}^{2r-1-l,l}(x)+$$
$$\sum_{2r-1\geq 3l}[\binom{2c-l}{2s-2l}+\binom{2c-l}{2s-2r+1+l}]S_{2^k}^{2r-1-l,l}(x)$$

By putting $c'=2c$, $s'=2s$ and $r'=2r-1$ we see that equation 13 also holds if we can prove equation 14. Equation 14 is proved in Lemma 7.4. □

Lemma 6.7.

$$Sq^{2r+1}Sq^{2s+1}\Lambda_{2^k}(x)=0$$

Proof. For $k=1$ the result follows from Lemma 4.3. Assume the formula holds for some $k\geq 1$. Then

$$Sq^{2s+1}\Lambda_{2^{k+1}}(x)=Sq^{2s+1}(x^{2^k}\Lambda_{2^k}(x))=\sum_{i=0}^{s}(Sq^i(x^{2^{k-1}}))^2Sq^{2(s-i)+1}\Lambda_{2^k}(x)$$

If we apply Sq^{2r+1} to this we get zero □

Theorem 6.8. *The operations $S_{2^k}^{a,b}$ satisfy odd Adem relations i. e. For $a<2b$ and $a+b$ odd we have*

$$S_{2^k}^{a,b}=\sum_{j=0}^{[\frac{a}{2}]}\binom{b-1-j}{a-2j}S_{2^k}^{a+b-j,j}$$

Proof. The case $k=0$ is the usual Adem relations for $a+b$ odd. The case $k=1$ is proved as follows. We may view $S_2^{a,b}$ as an element in $S^2(\mathcal{A})=\mathcal{A}\otimes\mathcal{A}/(a\otimes b-b\otimes a)$. There is a commutative diagram

$$\begin{tikzcd}
 & \Phi(\mathcal{A}) \arrow[d] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "p"] \arrow[d, "1+tw"'] & S^2(\mathcal{A}) \arrow[d, "q"] \\
\mathcal{A}\otimes\mathcal{A} & \Lambda^2(\mathcal{A}) \arrow[l, "\phi"']
\end{tikzcd}$$

where p and q are the quotient maps, $\Phi(\mathcal{A})$ is the kernel of q, tw the twisting map i.e. $tw(a \otimes b) = b \otimes a$ and ϕ is the inclusion $\phi(x \wedge y) = x \otimes y + y \otimes x$. Recall that $\mathcal{A}$ is a Hopf algebra with diagonal map

$$\psi : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}\,,\; \psi(Sq^r) = \sum_{i=0}^{r} Sq^i \otimes Sq^{r-i}$$

In $\mathcal{A} \otimes \mathcal{A}$ we have the element

$$C_2^{a,b} = \psi(Sq^a) \sum_{j=0}^{[\frac{b}{2}]} Sq^j \otimes Sq^{b-j} + \delta_{b,0}^2 \sum_{t=0}^{[\frac{a}{2}]} Sq^t Sq^{\frac{b}{2}} \otimes Sq^{a-t} Sq^{\frac{b}{2}}$$

satisfying $p(C_2^{a,b}) = S_2^{a,b}$. The following relations holds

$$(1 + tw)(C_2^{a,b}) = \psi(Sq^a Sq^b) + \delta_{b,0}^2 \delta_{a,0}^2 Sq^{\frac{a}{2}} Sq^{\frac{b}{2}} \otimes Sq^{\frac{a}{2}} Sq^{\frac{b}{2}}$$

When $b + a$ is odd the last term is zero. Defining

$$R_2^{a,b} := C_2^{a,b} + \sum_{j=0}^{[\frac{a}{2}]} \binom{b-1-j}{a-2j} C_2^{a+b-j,j}$$

for $a < 2b$ and $a + b$ odd we see that $(1 + tw)(R_2^{a,b}) = 0$. Thus $p(R_2^{a,b}) \in \Phi^2(\mathcal{A})$. But since $a + b$ is odd $p(R_2^{a,b}) \cap \Phi^2(\mathcal{A}) = 0$ and $p(R_2^{a,b}) = 0$ proving the case $k = 1$. The case $k \geq 2$ follows form the lemma below. $\square$

Lemma 6.9. *Let $A = \mathbb{F}_2[a^{i,j} | i, j \geq 0]$ and $B = \mathbb{F}_2[b^{i,j} | i, j \geq 0]$. Let $\bar{A}$ be A modulo the Adem relation and the relation $a^{p,q} = 0$ for p or q odd. Let $\bar{B}$ be B modulo odd Adem relations. Define*

$$c^{a,b} = \sum_{k,l} a^{k,l} \otimes b^{a-k,b-l} \in A \otimes B$$

Then $c^{a,b} \in \bar{A} \otimes \bar{B}$ satisfy odd Adem relations.

Proof. Define $\psi : B \longrightarrow A \otimes B$ by $\psi(b^{r,s}) = \sum_{k,l} a^{k,l} \otimes b^{r-k,s-l}$. We must show that there is a map $\bar{\psi}$ making the following diagram commutative

$$\begin{array}{ccc} B & \xrightarrow{\psi} & A \otimes B \\ \downarrow & & \downarrow \\ \bar{B} & \xrightarrow{\bar{\psi}} & \bar{A} \otimes \bar{B} \end{array}$$

Dually we must show that there is a map ρ making the following diagram commutative

$$\begin{tikzcd} B^* & A^* \otimes B^* \arrow[l, "\psi^*"'] \\ \bar{B}^* \arrow[u] & \bar{A}^* \otimes \bar{B}^* \arrow[l, "\rho"'] \arrow[u] \end{tikzcd}$$

Defining $\phi : A \longrightarrow A \otimes A$ by $\phi(a^{i,j}) = \sum_{k,l} a^{k,l} \otimes a^{i-k,j-l}$ we have a coproduct on A and the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[d, "\psi"'] & A \otimes B \arrow[d, "\phi\otimes 1"] \\ A \otimes B \arrow[r, "1\otimes\psi"] & A \otimes A \otimes B \end{tikzcd}$$

Thus ψ^* defines an A^* module structure on B^*. There is an isomorphism between $\bar{A}$ and $\mathcal{A}_{(2)} := \mathbb{F}_2[Sq^i Sq^j]$ given by $a^{p,q} \mapsto Sq^{\frac{a}{2}} Sq^{\frac{b}{2}}$. Dually we have an ring isomorphism of $\bar{A}^*$ with $\mathcal{A}_(2)^*$. The dual algebra of the mod 2 Steenrod algebra is a polynomial algebra $\mathcal{A}^* = \mathbb{F}_2[\xi_i | i \geq 0]$ where $deg(\xi_i) = 2^i - 1$. We find that $\mathcal{A}^*_{(2)} = \mathbb{F}_2[\xi_1, \xi_2]$. Thus it is enough to show that the image of $\xi_1 \cdot : \bar{B}^* \longrightarrow B^*$ is contained in $\bar{B}^*$ and that the image of $\xi_2 \cdot : \bar{B}^* \longrightarrow B^*$ is contained in $\bar{B}^*$. Dually we must show that the composite

$$\alpha_\nu : B \xrightarrow{\psi} A \otimes B \longrightarrow \bar{A} \otimes \bar{B} \longrightarrow \bar{B}$$

where the last map is projection on the $a^{2,0}$ coefficient when $\nu = 1$ and on the $a^{2,4}$ coefficient when $\nu = 2$ factors through $\bar{B}$ for $\nu = 1, 2$. This is OK for α_1 since $\alpha_1(b^{r,s}) = b^{r-2,s} + b^{r,s-2}$ and thus corresponds to $\kappa \circ \kappa$. We find that $\alpha_2(b^{r,s}) = b^{r-4,s-2}$. Assume that $r + s$ is odd and $r < 2s$. We have

$$\alpha_2\Big(b^{r,s} + \sum_{j=0}^{[\frac{r}{2}]} \binom{s-1-j}{r-2j} b^{r+s-j,j}\Big) = b^{r-4,s-2} + \sum_{j=0}^{[\frac{r}{2}]} \binom{s-1-j}{r-2j} b^{r+s-4-j,j-2}$$

Changing summation index $t = j - 2$ in the last sum we get the uneven Adem relation for $b^{r-4,s-2}$ and α_2 also factors through $\bar{B}$. $\square$

7. Binomial coefficients modulo 2.

For any real number z and any integer n the binomial coefficient $\binom{z}{n}$ is defined by

$$\binom{z}{n} := \begin{cases} \frac{z(z-1)\dots(z-n+1)}{n!} & , n > 0 \\ 1 & , n = 0 \\ 0 & , n < 0 \end{cases}$$

The following properties holds. If m is a positive integer and $n > m$ then $\binom{m}{n} = 0$. For all real numbers z and any integer n the Pascal triangle equality holds

$$\binom{z}{n} + \binom{z}{n+1} = \binom{z+1}{n+1} \tag{15}$$

For $z > 0$ and any integer n

$$\binom{-z}{n} = (-1)^n \binom{z+n-1}{n}$$

It follows that for any integers m and n we have

$$\binom{m}{n} = (-1)^n \binom{-m+n-1}{n} \tag{16}$$

In the rest of this section binomial coefficients is always taken modulo 2. The following summation formula can be found in [Ad]. Here we give an easier proof.

Theorem 7.1. *For $a, b, c \in \mathbb{Z}$ the following formula holds*

$$\sum_{k=0}^{c} \binom{a-k}{k}\binom{b+k}{c-k} = \binom{a+b+1}{c} \tag{17}$$

Proof. If $c < 0$ the theorem is trivial. Assume that $c \geq 0$. (17) is equivalent to showing the following for integers a, b, c where $c \geq 0$

$$\sum_{n+m=c} \binom{2m+a}{m}\binom{2n+b}{n} = \binom{2c+a+b}{c} \tag{18}$$

This is seen by using (16) on each of the three binomial coefficients in the formula. Define for all integers a

$$f_a(t) := \sum_{m=0}^{\infty} \binom{2m+a}{m} t^m \in \mathbb{F}_2[[t]]$$

Note that $f_0(t) = 1$. We see that (18) is equivalent to

$$f_a(t) f_b(t) = f_{a+b}(t)$$

i.e. we must show

$$f_a(t) = (f_1(t))^a \tag{19}$$

for all integers a. By (15) we get

$$f_a(t) + f_{a+1}(t) = t f_{a+2}(t) \tag{20}$$

Using the binary expansion $m = \sum_{l=0}^{t} m_l \cdot 2^l$, $m_l = 0, 1$ we have

$$\binom{2m+1}{m} = \binom{m_t}{0}\binom{m_{t-1}}{m_t} \cdots \binom{m_1}{m_2}\binom{m_0}{m_1}\binom{1}{m_0}$$

thus $\binom{2m+1}{m} = 1$ if and only if $m = 2^s - 1$ for some integer $s \geq 0$. That is

$$f_1(t) = \sum_{s=0}^{\infty} t^{2^s-1}$$

and from this $f_1(t) + t(f_1(t))^2 = 1$. Now (19) follows for $a \geq 0$ by induction. It is true for $a = 0, 1$. Assume it is true for a and $a+1$ where $a \geq 0$. Then

$$t(f_1(t))^{a+2} = (f_1(t))^a(1 + f_1(t)) = f_a(t) + f_{a+1}(t) = tf_{a+2}(t)$$

Finally we prove that $f_{-a}(t) = (f_1(t))^{-a}$ for $a \geq 1$ by induction. Assume it is OK for $-a+1$ and $-a+2$. Then

$$\begin{aligned}&(f_1(t))^a f_{-a}(t) = (f_1(t))^a(tf_{-a+2}(t) + f_{-a+1}(t)) =\\&(f_1(t))^a(t(f_1(t))^{-a+2} + (f_1(t))^{-a+1}) = t(f_1(t))^2 + f_1(t) = 1\end{aligned}$$

and we are done. □

For $a > 0$ and $c \geq a - 1$ we have

(21) $$\sum_{k=0}^{a-1} \binom{b-k}{k}\binom{k-a}{c-k} = \binom{b-c}{a-1}$$

This follows from by (17) and (16) since

$$\binom{k-a}{c-k} = \binom{a+c-1-2k}{c-k} = \binom{a+c-1-2k}{a-1-k} = \binom{-c+k-1}{a-1-k}$$

From (17) and (21) we get for $b \geq c \geq a \geq 0$

(22) $$\sum_{k=a}^{c} \binom{b-k}{k}\binom{k-a}{c-k} = \binom{b-a+1}{c} + \binom{b-c}{a-1}$$

Lemma 7.2. *For $x \geq 0$ and $y \in \mathbb{Z}$ the following relation holds*

$$\sum_{j=0}^{x} \binom{x-j}{y+j}\binom{-1+j}{x-j} = \binom{x}{|y|}$$

Proof. We first assume that $y \geq 0$. For $x = 0$ the relation is trivial and we assume that $x \geq 1$. In (22) we perform the substitution $a = y+1$, $b = c = x+y$, $k = y+j$. We obtain

$$\sum_{j=1}^{x} \binom{x-j}{y+j}\binom{j-1}{x-j} = \binom{x}{x+y} + \binom{0}{y} = 0$$

Next assume that $y \leq 0$. By the substitution $a = -y-1$, $b = c = x+y$ in (17) we get

$$\binom{x}{-y} = \binom{x}{x+y} = \sum_{k=0}^{x+y} \binom{x+y-k}{k}\binom{-y-1+k}{x+y-k} = \sum_{j=0}^{x} \binom{x-j}{y+j}\binom{-1-j}{x-j}$$

□

Lemma 7.3. *For $a \geq 0$ and $b, c \in \mathbb{Z}$ the following holds*

$$\sum_{l=0}^{a} [\binom{b+a-l}{c+a-2l} + \binom{b+2l}{c+l}]\binom{l-1}{a-l} = 0$$

Proof. Let $f(a,b,c)$ denote the left hand side. We have $f(a,b,c) = 0$ for $c \leq -a$ and $f(a,b,c) + f(a,b,c+1) = f(a,b+1,c+1)$. Because of this it is enough to show $f(a,0,c) = 0$ for all c. Using Lemma 7.2 we find

$$\sum_{k=0}^{a} \binom{a-k}{c+a-2k}\binom{k-1}{a-k} = \sum_{k=0}^{a} \binom{a-k}{-c+k}\binom{k-1}{a-k} = \binom{a}{|c|} \tag{23}$$

Assume that $c \leq 0$. We have

$$\sum_{k=0}^{a} \binom{2k}{c+k}\binom{k-1}{a-k} = \sum_{k=-c}^{a} \binom{2k}{c+k}\binom{k-1}{a-k} = \sum_{k=-c}^{a} \binom{c-k-1}{c+k}\binom{k-1}{a-k}$$

By the substitution $l = c+k$ and (17) this gives

$$\sum_{k=0}^{a} \binom{2k}{c+k}\binom{k-1}{a-k} = \binom{c-1}{a+c} = \binom{a}{a+c} = \binom{a}{-c} \tag{24}$$

Assume that $c \geq 0$. Since

$$\binom{2k}{c+k} = \binom{2k}{k-c} = \binom{-c-k-1}{k-c}$$

we have

$$\sum_{k=0}^{a} \binom{2k}{c+k}\binom{k-1}{a-k} = \sum_{k=c}^{a} \binom{-c-k-1}{-c+k}\binom{k-1}{a-k} = \binom{a}{c} \tag{25}$$

where we have used (24) to see the lase equality. Combining (24) and (25) we get for c any integer

$$\sum_{k=0}^{a} \binom{2k}{c+k}\binom{k-1}{a-k} = \binom{a}{|c|} \tag{26}$$

From (23) and (26) we see that $f(a,0,c) = 0$. □

Lemma 7.4. *For integers c, s, r, j with $3j \leq r$ we have*

$$\sum_{3k>r} (\binom{c-k}{s-2k} + \binom{c-k}{s-r+k})\binom{k-1-j}{r-k-2j} = \binom{c-j}{s-2j} + \binom{c-j}{s-r+j}$$

Proof. Let $f(c,s,r,j)$ be the sum of the left hand side and the right hand side. We have $f(c,s,r,j) + f(c,s+1,r,j) = f(c+1,s+1,r,j)$ and it is enough to show that $f(0,s,r,j) = 0$. Since $k > \frac{r}{3} \geq j$ we have $k-1-j \geq 0$ and $\binom{k-1-j}{r-k-2j} = 0$ unless $k-1-j \geq r-k-2j$. Using this and the fact that $j \leq \frac{r}{3}$ we see that we can extend the summation to $1+j \leq k < \infty$. Using this and (16) we obtain

$$f(0,s,r,j) = \sum_{k=j}^{\infty} [\binom{s-1-k}{s-2k} + \binom{s-1-r+2k}{s-r+k}]\binom{k-1-j}{r-k-2j}$$

By the substitution $l = k-j$, $a = r-3j \geq 0$, $b = s-1-r+2j$, $c = s-r+j$ the result follows from Lemma 7.3. $\square$

8. An approximation to the S^1-homotopy orbits of the free loop space.

The free loop space on X is defined as the set of all maps from the circle S^1 into X with the compact open topology. We denote this space ΛX. The circle S^1 is a compact lie group under complex multiplication. Let BS^1 denote the corresponding classifying space and ES^1 its universal cover. The free loop space has an obvious S^1-action induced by the multiplication in the circle group. Thus one can form the Borel construction

$$(\Lambda X)_{hS^1} := ES^1 \times_{S^1} \Lambda X$$

That is the S^1 homotopy orbits of ΛX.

The cohomology of BS^1 is a polynomial algebra on a two dimensional class u. For the cohomology of the classifying space of a cyclic group of two power order we still use the notation

$$H^*(BC_{2^m}) = \begin{cases} \Lambda(v) \otimes \mathbb{F}_2[u] & \text{when } m \geq 2 \\ \mathbb{F}_2[a] & \text{when } m = 1 \end{cases}$$

here a and v are one dimensional and u is two dimensional. The inclusion $j : C_{2^m} \subseteq S^1$ satisfies $(Bj)^*(u) = u$ when $m \geq 2$ and $(Bj)^*(u) = a^2$ when $m = 2$ as seen by the spectral sequence associated to $S^1 \longrightarrow BC_{2^m} \longrightarrow BS^1$. To construct the approximation functor we shall need the following result.

Theorem 8.1. *Let $m \geq 2$ be an integer and let Y be a connected S^1-space. Consider the following diagram*

$$\begin{tikzcd} ES^1 \times_{C_{2^m}} Y \arrow[r, "Q"] \arrow[d, "pr_1"'] & ES^1 \times_{S^1} Y \arrow[d, "pr_1"'] \\ BC_{2^m} \arrow[r, "Bj"] & BS^1 \end{tikzcd} \tag{27}$$

where Q denotes the quotient map. There is an isomorphism

$$\theta : H^*(BC_{2^m}) \otimes_{H^*(BS^1)} H^*(ES^1 \times_{S^1} Y) \xrightarrow{\cong} H^*(ES^1 \times_{C_{2^m}} Y)$$

defined by $x \otimes y \mapsto pr_1^*(x)Q^*(y)$. The transfer map

$$Tr_m^{m+1} : H^*(ES^1 \times_{C_{2^m}} Y) \longrightarrow H^*(ES^1 \times_{C_{2^{m+1}}} Y)$$

is zero on elements of the form $1 \otimes Q^*(y)$ and the identity on elements of the form $v \otimes Q^*(y)$ (here $v = a$ when $m = 1$). We get an isomorphism

$$\varinjlim H^*(ES^1 \times_{C_{2^m}} Y) \cong \widetilde{H}^*(\Sigma(ES^1 \times_{S^1} Y)_+)$$

Proof. Filling in the fibers of diagram (27) we get

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y \arrow[r, "id"] \arrow[d] & Y \arrow[d] \\ S^1 \arrow[r] \arrow[d] & ES^1 \times_{C_{2^m}} Y \arrow[r, "Q"] \arrow[d, "pr_1"'] & ES^1 \times_{S^1} Y \arrow[d, "pr_1"'] \\ S^1 \arrow[r] & BC_{2^m} \arrow[r, "Bj"] & BS^1 \end{tikzcd}$$

Since the fundamental group of BS^1 is zero, the Serre spectral sequence $\hat{E}$ of the lowest horizontal fibration has trivial coefficients, and since Q is the pull back of this, the Serre spectral sequence E associated to Q also has trivial coefficients. We get

$$E_2^{**} = H^*(ES^1 \times_{S^1} Y) \otimes H^*(S^1) \Rightarrow H^*(ES^1 \times_{C_{2^m}} Y)$$
$$\hat{E}_2^{**} = H^*(BS^1) \otimes H^*(S^1) \Rightarrow H^*(BC_{2^m})$$

and pr_1 gives a map of these two spectral sequences. Writing $H^*(S^1) = \Lambda(v)$ we know that $d_2(v) = 0$ in $\hat{E}$. Since $pr_1^*(v) = v$ we get that $d_2 v = 0$ in E hence $E_2 = E_\infty$. It is now obvious that the map θ is an isomorphism. Using the map pr_1^* we get that $Tr_m^{m+1}(v) = v$. By Frobenius reciprocity the description of the transfer map follows. $\square$

The above theorem is inspired by the following result of Tom Goodwillie, which can be found in [BHM].

Theorem 8.2. *(Goodwillie) For any based S^1-space Z, there is a map*

$$\tau : \widetilde{Q}(\Sigma ES^1_+ \wedge_{S^1} Z) \to \underleftarrow{holim}\, \widetilde{Q}(ES^1_+ \wedge_{C_{p^n}} Z)$$

which induces an isomorphism on homotopy groups with $\mathbb{F}_p$ coefficients.

Definition 8.3. Let L_m and P_m be the endofunctors on the category of spaces defined by

$$L_m(X) = ES^1 \times_{C_{2^m}} \Lambda X$$
$$P_m(X) = ES^1 \times_{C_{2^m}} X^{2^m}$$

for $m = 0, 1, 2, 3, \ldots$

By applying Theorem 8.1 to each component of the free loop space we obtain

Corollary 8.4. *For any space X there is an isomorphism*

$$H^* L_m X \cong H^*(ES^1 \times_{S^1} \Lambda X) \oplus v H^*(ES^1 \times_{S^1} \Lambda X)$$

where v is a one dimensional class. When $m \geq 2$ we have $v^2 = 0$ and when $m = 1$ we have $v^2 = pr_1^*(u)$. The transfer map

$$Tr_m^{m+1} : H^* L_m X \longrightarrow H^* L_{m+1} X$$

is the identity on the v-component and zero on the other component thus

$$\varinjlim H^* L_m X \cong \widetilde{H}^*(\Sigma (ES^1 \times_{S^1} \Lambda X)_+)$$

There are natural transformations, defined below, making the following diagram commute

$$\begin{array}{ccc} H^* P_{n+m} & \xrightarrow{ev_{n+m}^*} & H^* L_{n+m} \\ {\scriptstyle t_m^{m+n}} \uparrow & & \uparrow {\scriptstyle Tr_m^{m+n}} \\ H^* P_m & \xrightarrow{ev_m^*} & H^* L_m \end{array} \tag{28}$$

Let $\zeta_k = e^{2\pi i/k}$ and consider the evaluation map, which evaluates a loop in the k'th roots of unity:

$$ev : \Lambda X \longrightarrow X^k \; ; \; ev(f) = (f(1), f(\zeta_k), \ldots, f(\zeta_k^{k-1}))$$

We get a commutative diagram of spaces

$$
\begin{tikzcd}
ES^1 \times_{C_{rs}} X^{rs} & ES^1 \times_{C_{rs}} \Lambda X \arrow[l, "1\times ev"'] \\
ES^1 \times_{C_s} (X^r)^s \arrow[u, "q"] \arrow[d, "1\times(pr_1)^s"'] & \\
ES^1 \times_{C_s} X^s & ES^1 \times_{C_s} \Lambda X \arrow[l, "1\times ev"] \arrow[uu, "Q"'] \arrow[ul, "1\times ev"']
\end{tikzcd}
\tag{29}
$$

where q and Q are quotient maps and pr_1 projection on the first factor. Put $s = 2^m$ and $r = 2^n$. Then Tr_m^{m+n} is the transfer associated to the covering Q. The map t_m^{m+n} is defined as the composite $t_m^{m+n} = Tr \circ (1 \times (pr_1)^s)^*$ where Tr is the transfer associated to the covering q. Finally ev_m^* is the map induced by the evaluation $ev_m^* = (1 \times ev)^*$. The commutativity of (28) follows from (29). Passing to the direct limit in (28) one obtains a map

$$
\varinjlim H^* P_m X \xrightarrow{ev^*} \varinjlim H^* L_m X \tag{30}
$$

The approximation functor is a quotient of the left hand side. The relations arise from the following result.

Proposition 8.5. *Let $\Delta : X \to X \times X$ be the diagonal map. In the diagram below, the square with the quotient maps, the square with the transfer maps and the triangle are commutative.*

$$
\begin{tikzcd}
H^*P_m(X) \arrow[r, "ev_m^*"] \arrow[d, shift left, "q^*"] & H^*L_m(X) \arrow[d, shift left, "Q^*"] \\
H^*P_{m-1}(X^2) \arrow[r, "ev^*"] \arrow[u, shift left, "Tr"] \arrow[d, "\Delta^*"'] & H^*L_{m-1}(X) \arrow[u, shift left, "Tr_{m-1}^m"] \\
H^*P_{m-1}(X) \arrow[ur, "ev_{m-1}^*"'] &
\end{tikzcd}
\tag{31}
$$

Proof. We need only prove the commutativity of the triangle since the rest follows from (29). We do it by showing that the corresponding triangle of spaces is homotopy commutative. The evaluation $ev_m : \Lambda X \to X^{2^m}$ is homotopic to the composite

$$
\Lambda X \xrightarrow{ev_{m-1}} X^{2^{m-1}} \xrightarrow{\Delta^{2^{m-1}}} X^{2^m}
$$

via the homotopy $H : I \times \Lambda X \to X^{2^m}$ defined by

$$
H(t, f) = (f(1), f(\zeta_s^t), f(\zeta_s^2), f(\zeta_s^{t+2}), \ldots, f(\zeta_s^{s-2}), f(\zeta_s^{s-2+t}))
$$

where $s = 2^m$. This homotopy is actually $C_{2^{m-1}}$-equivariant when the action is trivial on I, by ζ_s^2 on ΛX and by cyclic permutation on X^s. Thus we obtain a homotopy

$$I \times L_{m-1}(X) \to ES^1 \times_{C_{2^{m-1}}} (I \times \Lambda X) \xrightarrow{1 \times H} P_{m-1}(X^2)$$

with $1 \times ev_m$ in one end and $(1 \times \Delta^{2^m}) \circ (1 \times ev_{m-1})$ in the other. This completes the proof. $\square$

We use the notation from chapter 3 for the cohomology classes in $H^* P_m X$.

Corollary 8.6. *Let $r = 2^n$, $s = 2^m$ and let $k \geq 0$. Suppose that $a_1 \otimes \ldots \otimes a_r$ is in an orbit of length r under cyclic permutation by C_r. Then the following holds*

$$ev^*_{n+m}(e^{2k} \otimes \sum_{j=1}^{r} T^j (a_1 \otimes \ldots \otimes a_r)^{\otimes s}) = Tr^{n+m}_m \circ ev^*_m (e^{2k} \otimes (\prod_{i=1}^{r} a_i)^{\otimes s}) \tag{32}$$

$$ev^*_{n+m}(e^{2k+1} \otimes (a_1 \otimes \ldots \otimes a_r)^{\otimes s}) = Tr^{n+m}_m \circ ev^*_m (e^{2k+1} \otimes (\prod_{i=1}^{r} a_i)^{\otimes s}) \tag{33}$$

Proof. The elements in $H^* P_{n+m} X$ above are hit by the transfer from $H^* P_n(X^s)$. By iteration of (31) the result follows. $\square$

Because of this corollary it is enough to consider elements of highest symmetry i.e. of the form $e^k \otimes x^{\otimes 2^k}$ when one wants to determine the evaluation map (30). Proposition 8.5 reduces this problem still further because of the following two corollaries.

Corollary 8.7. *The evaluation map (30) maps elements of the form $e^{2k} \otimes (x^2)^{\otimes 2^n}$ to zero.*

Proof. Put $s = 2^{m-1}$ and look at $e^{2k} \otimes (x^2)^{\otimes s}$ in the cohomology group $H^* P_{m-1}(X)$ placed at the bottom of diagram (31). We will show that the image of this class

$$z = ev^*_{m-1}(e^{2i} \otimes (x^2)^{\otimes s})$$

maps to zero under the transfer i.e. $Tr^m_{m-1}(z) = 0$. Since the diagonal map satisfy $\Delta^*(e^{2k} \otimes (x^{\otimes 2})^{\otimes s}) = e^{2k} \otimes (x^2)^{\otimes s}$ the commutativity of the triangle of (31) implies

$$z = ev^*_{m-1}(e^{2k} \otimes (x^{\otimes 2})^{\otimes s})$$

We also have $q^*(e^{2k} \otimes x^{\otimes 2s}) = e^{2k} \otimes (x^{\otimes 2})^{\otimes s}$. Thus the quotient map square of (31) implies

$$z = Q^* \circ ev^*_m (e^{2i} \otimes x^{\otimes 2r})$$

Since $Tr^m_{m-1} \circ Q^*$ is zero (multiplication by 2) z is mapped to zero by the transfer Tr^m_{m-1}. $\square$

Corollary 8.8. *The evaluation map* (30) *satisfy*

$$ev^*[e^{2k+1} \otimes (x^2)^{\otimes 2^n}] = ev^*[e^{2k+1} \otimes x^{\otimes 2^{n+1}}]$$

for all $k \geq 0$ *and* $n \geq 1$.

Proof. As above put $s = 2^{m-1}$ and look at $e^{2k+1} \otimes (x^2)^{\otimes s}$ in the cohomology group $H^*P_{m-1}(X)$ placed at the bottom of diagram (31). From the triangle of (31) we obtain

$$ev^*_{m-1}(e^{2k+1} \otimes (x^2)^{\otimes s}) = ev^*_{m-1}(e^{2k+1} \otimes (x^{\otimes 2})^{\otimes s})$$

Since $Tr(e^{2k+1} \otimes (x^{\otimes 2})^{\otimes s}) = e^{2k+1} \otimes x^{\otimes 2s}$ the transfer part of the square of (31) shows that

$$Tr^m_{m-1} \circ ev^*_{m-1}(e^{2k+1} \otimes (x^2)^{\otimes s}) = ev^*_m(e^{2k+1} \otimes x^{\otimes 2s})$$

and the result follows. □

Definition 8.9. Define a functor ℓ from the category of spaces to the category of unstable $\mathcal{A}$-modules by

$$\ell(X) = \varinjlim H^*P_m(X)/\sim$$

where $\sim$ is the equivalence relation generated by

$$[e^{2k} \otimes \sum_{j=1}^{r} T^j(a_1 \otimes \ldots \otimes a_r)^{\otimes s}] \sim [e^{2k} \otimes (\prod_{i=1}^{r} a_i)^{\otimes s}] \tag{34}$$

$$[e^{2k+1} \otimes (a_1 \otimes \ldots \otimes a_r)^{\otimes s}] \sim [e^{2k+1} \otimes (\prod_{i=1}^{r} a_i)^{\otimes s}] \tag{35}$$

$$[e^{2k} \otimes (x^2)^{\otimes s}] \sim 0 \tag{36}$$

$$[e^{2k+1} \otimes (x^2)^{\otimes s}] \sim [e^{2k+1} \otimes x^{\otimes 2s}] \tag{37}$$

where $r = 2^n$ and $s = 2^m$ for integers $n, m \geq 0$.

Note that the limit systems starts at $H^*P_0X = H^*X$ and in this particular group we only have classes of the form $e^0 \otimes x$. The classes $e^k \otimes x$ with $k \geq 1$ are zero.

The evaluation map (30) gives a map

$$\ell(X) \xrightarrow{ev^*} \varinjlim H^*L_mX \tag{38}$$

and by neglecting the suspension factor we get a map of degree -1

$$\ell(X) \xrightarrow{\phi} H^*(ES^1 \times_{S^1} \Lambda X) \tag{39}$$

We use $\ell(X)$ as an approximation to the cohomology of $(\Lambda X)_{hS^1}$ via this $\mathcal{A}$-linear map.

The results of chapter 5 give informations about the map ϕ. Let $i : X \longrightarrow \Lambda X$ denote the map which sends a point $x \in X$ to the constant loop with value x. We get a commutative diagram of spaces

$$\begin{array}{ccc} ES_1 \times_{C_r} \Lambda X & \xleftarrow{1\times i} & BC_r \times X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Bj\times 1} \\ ES^1 \times_{C_{rs}} \Lambda X & \xleftarrow{1\times i} & BC_{rs} \times X \end{array}$$

where $j : C_r \subseteq C_{rs}$ is the inclusion. From this and (28) we get a commutative diagram

$$\begin{array}{ccccc} H^*P_{n+m}X & \xrightarrow{ev^*_{n+m}} & H^*L_{n+m}X & \xrightarrow{i^*_{n+m}} & H^*(BC_{2^{n+m}}) \otimes H^*(X) \\ {\scriptstyle t^{m+n}_m}\uparrow & & {\scriptstyle Tr^{m+n}_m}\uparrow & & {\scriptstyle Tr^{m+n}_m\otimes 1}\uparrow \\ H^*P_mX & \xrightarrow{ev^*_m} & H^*L_mX & \xrightarrow{i^*_m} & H^*(BC_{2^m}) \otimes H^*(X) \end{array} \tag{40}$$

Since the composite $X \xrightarrow{i} \Lambda X \xrightarrow{ev} X^n$ is the diagonal map $\Delta : X \longrightarrow X^n$, the horizontal composites of (40) are computed in chapter 5. Passing to the limit we get

$$\Delta^* : \ell(X) \xrightarrow{ev^*} \varinjlim H^*L_mX \xrightarrow{i^*} \varinjlim H^*(BC_{2^m}) \otimes H^*(X)$$

We finish this chapter with a lemma which shall be used to do computations in the next two chapters. It is well known but since we were not able to find a reference for it we also give a proof.

Lemma 8.10. *Let Y be a connected S^1-space and let $\lambda : S^1 \times Y \longrightarrow Y$ denote the action map. Define a map $d : H^*(Y;\mathbb{Z}) \longrightarrow H^{*-1}(Y;\mathbb{Z})$ by*

$$\lambda^* : H^*(Y;\mathbb{Z}) \longrightarrow H^*(S^1;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z})$$
$$\lambda^*(y) = \sigma \otimes d(y) + 1 \otimes y$$

The fibration $Y \longrightarrow ES^1 \times_{S^1} Y \longrightarrow BS^1$ have the following Serre spectral sequence

$$E_2^{**} = H^*(BS^1;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \Rightarrow H^*(ES^1 \times_{S^1} Y;\mathbb{Z})$$

The differential in the E_2-term is connected with the action map in the following way

$$d_2 : H^*(Y;\mathbb{Z}) \longrightarrow uH^*(Y;\mathbb{Z})\,;\; d_2(y) = \pm u d(y)$$

where u is the two dimensional algebra generator of the cohomology of the base space i.e. $H^*(BS^1;\mathbb{Z}) = \mathbb{Z}[u]$.

Proof. The map λ^* is injective since $\lambda \circ \gamma = id$ where γ is the map defined by $\gamma(y) = (1, y)$. We also see that λ^* have the form stated in the lemma. We check that d is a differential i.e. that $d \circ d = 0$. There is a commutative diagram

$$\begin{array}{ccc} S^1 \times S^1 \times Y & \xrightarrow{1\times\lambda} & S^1 \times Y \\ {\scriptstyle \mu\times 1}\downarrow & & \downarrow{\scriptstyle \lambda} \\ S^1 \times Y & \xrightarrow{\lambda} & Y \end{array}$$

where $\mu : S^1 \times S^1 \longrightarrow S^1$ is the multiplication map. In cohomology it satisfy $\mu^*(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$. By pulling back a class $y \in H^*(Y;\mathbb{Z})$ the two ways in the diagram it follows that $d \circ d(y) = 0$.

Letting S^1 act on $S^1 \times Y$ by μ on S^1 and trivially on Y we have that λ is an equivariant map. We get a map of fibrations

$$\begin{array}{ccccc} S^1 \times Y & \longrightarrow & ES^1 \times_{S^1} (S^1 \times Y) & \longrightarrow & BS^1 \\ {\scriptstyle \lambda}\downarrow & & {\scriptstyle 1\times\lambda}\downarrow & & {\scriptstyle id}\downarrow \\ Y & \longrightarrow & ES^1 \times_{S^1} Y & \longrightarrow & BS^1 \end{array}$$

giving a map of the corresponding spectral sequences $\lambda^* : E \longrightarrow \hat{E}$ where $\hat{E}$ is the spectral sequence associated to the upper fibration. Since $ES^1 \times_{S^1} S^1 \times Y \simeq ES^1 \times Y$ it looks like

$$\hat{E}_2^{**} = H^*(BS^1;\mathbb{Z}) \otimes H^*(S^1;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \Rightarrow H^*(Y;\mathbb{Z})$$

In $\hat{E}_2$ the differential on the fiber elements are given by

$$d_2(\sigma \otimes z) = \pm u(1 \otimes z)\ ,\ d_2(1 \otimes z) = 0 \tag{41}$$

To see this, assume that Y is contractible. Then the E_∞ term has a $\mathbb{Z}$ placed at $(0,0)$ and zero at all other places, It follows that $d_2(\sigma) = \pm u$ and $d_2(1) = 0$. Since the action on Y was trivial this case imply (41). Let $y \in H^*(Y;\mathbb{Z})$ be a fiber class in E_2. By λ^* it maps to $\sigma \otimes dy + 1 \otimes y$ and applying the $\hat{E}_2$- differential to this we get $\pm u(1 \otimes dy)$. We see that $d_2(y) = ux$ where $x \in H^*(Y;\mathbb{Z})$ is a class with $\lambda^*(x) = \pm 1 \otimes dy$. Since $x = \pm dy$ satisfy this equation and λ^* were injective the result follows. □

9. The approximation for the infinite complex projective space.

In this section we compute the map (39) in the case where $X = \mathbb{C}P^\infty$. We shall need the following result

Proposition 9.1. *Let A be an Abelian group and $n \geq 2$ an integer. There is a homotopy equivalence*

$$\Lambda K(A,n) \simeq K(A,n-1) \times K(A,n)$$

where the product is the product in the category of CW-complexes.

Proof. $K(A,n)$ is an Abelian group since the functor B applied to an Abelian group gives an Abelian group. $\Lambda K(A,n)$ is an Abelian group by the pointwise operations. There is a commutative diagram

$$\begin{tikzcd}
\Omega K(A,n) \arrow[r, "i_1"] \arrow[d, "id"'] & \Omega K(A,n) \times K(A,n) \arrow[r, "pr_2"] \arrow[d, "m"'] & K(A,n) \arrow[d, "id"'] \\
\Omega K(A,n) \arrow[r, "i"] & \Lambda K(A,n) \arrow[r, "ev"] & K(A,n)
\end{tikzcd}$$

The map ev evaluates a loop in $1 \in S^1$ and the map i_1 maps a loop f to (f,e) where $e \in K(A,n)$ is the neutral element. There is a section $s : K(A,n) \longrightarrow \Lambda K(A,n)$ mapping a point to the constant loop at that point. The map m is defined by $m(f,p) = i(f)s(p)$. By the long exact sequences of homotopy groups for the two horizontal fibrations and the five lemma we get that m is a homotopy equivalence. $\square$

Since $\mathbb{C}P^\infty = K(\mathbb{Z},2)$ we get $\Lambda\mathbb{C}P^\infty \simeq S^1 \times \mathbb{C}P^\infty$. We now compute the cohomology of the space $(\Lambda\mathbb{C}P^\infty)_{hS^1}$. The fibration

$$\Lambda\mathbb{C}P^\infty \xrightarrow{j} ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty \xrightarrow{\pi} BS^1 \tag{42}$$

give the following Serre spectral sequence

$$E_2^{**} = H^*(BS^1;\mathbb{Z}) \otimes H^*(\Lambda\mathbb{C}P^\infty;\mathbb{Z}) \Rightarrow H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty;\mathbb{Z}) \tag{43}$$

The cohomology of the base space is a polynomial algebra on a two dimensional generator u i.e. $H^*(BS^1;\mathbb{Z}) = \mathbb{Z}[u]$ and the cohomology of the fiber is an exterior algebra on a one dimensional generator e tensor a polynomial algebra on a two dimensional generator c i.e.

$$H^*(\Lambda\mathbb{C}P^\infty;\mathbb{Z}) = \Lambda(e) \otimes \mathbb{Z}[c]$$

Lemma 9.2. *The differential in the E_2-term of the spectral sequence (43) is given by*

$$d_2(e) = 0\ ,\ d_2(c) = \pm eu$$

Proof. Any constant loop in $\Lambda\mathbb{C}P^\infty$ is a S^1-fixpoint, thus there is a section

$$s : BS^1 \longrightarrow ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty\ ,\ \pi \circ s = id \tag{44}$$

In this way $\mathbb{Z}[u]$ becomes a direct summand in the cohomology of the total space implying $d_2(e) = 0$.

By Lemma 8.10 one may describe the differential d_2 by

$$d_2 : H^*(\Lambda\mathbb{C}P^\infty) \longrightarrow uH^*(\Lambda\mathbb{C}P^\infty)\,;\ d_2(y) = \pm ud(y)$$

where d is determined by the action

$$\lambda : S^1 \times \Lambda\mathbb{C}P^\infty \longrightarrow \Lambda\mathbb{C}P^\infty$$
$$\lambda^*(y) = \sigma \otimes d(y) + 1 \otimes y$$

We will find $d(c)$. Since it has degree one we get $d(c) = ke$ for a constant $k \in \mathbb{Z}$. Consider the composite

$$\psi : S^1 \times \Omega\mathbb{C}P^\infty \xrightarrow{1\times i} S^1 \times \Lambda\mathbb{C}P^\infty \xrightarrow{\lambda} \Lambda\mathbb{C}P^\infty \xrightarrow{ev} \mathbb{C}P^\infty$$

satisfying $\psi(z, f) = f(z)$. Since $ev^*(c) = c$ and $\lambda^*(c) = \sigma \otimes ke + 1 \otimes c$ we obtain

$$\psi^*(c) = \sigma \otimes ke$$

The map ψ factors through the smash product

$$\begin{tikzcd} S^1 \times \Omega\mathbb{C}P^\infty \arrow[r, "\psi"] \arrow[d, "Q"'] & \mathbb{C}P^\infty \\ S^1 \wedge \Omega\mathbb{C}P^\infty \arrow[ur, "\hat{\psi}"'] & \end{tikzcd}$$

We first consider the map $\hat{\psi}$. There is a commutative diagram of homotopy groups

$$\begin{tikzcd} \pi_2(S^1 \wedge \Omega\mathbb{C}P^\infty) \arrow[r, "\hat{\psi}_*"] & \pi_2(\mathbb{C}P^\infty) \arrow[dl, "\cong", "adj"'] \\ \pi_1(\Omega\mathbb{C}P^\infty) \arrow[u, "\Sigma"] & \end{tikzcd}$$

where Σ is the suspension map and adj is the adjunction isomorphism. All three homotopy groups in the diagram equals $\mathbb{Z}$ and it follows that $\hat{\psi}_*$ is multiplication by plus or minus one. By Hurewicz theorem we get the same result in dimension two homology and by universal coefficients we find

$$H^2(\mathbb{C}P^\infty; \mathbb{Z}) \xrightarrow{\hat{\psi}^*} H^2(S^1 \wedge \Omega\mathbb{C}P^\infty; \mathbb{Z})\,,\ \hat{\psi}^*(c) = \pm\sigma \otimes e$$

To complete the proof we only need to show that Q^* is an isomorphism in degree 2. Consider the long exact cohomology sequence associated to the pair $i : S^1 \vee S^1 \subseteq S^1 \times S^1$:

$$\ldots \leftarrow H^2(S^1 \vee S^1) \xleftarrow{i^*} H^2(S^1 \times S^1) \xleftarrow{Q^*} H^n(S^1 \wedge S^1) \leftarrow \ldots$$

Since $H^2(S^1 \vee S^1) = 0$ we see that $Q^* : \mathbb{Z} \longrightarrow \mathbb{Z}$ is surjective thus it is multiplication by plus or minus one. $\square$

We go back to mod two coefficients, and use the same names for the algebra generators i.e. $H^*(BS^1) = \mathbb{F}_2[u]$ and $H^*(\Lambda\mathbb{C}P^\infty) = \Lambda(e) \otimes \mathbb{F}_2[c]$.

Theorem 9.3.

$$H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty) = \Lambda(e,\xi) \otimes \mathbb{F}_2[b,u]/I$$

where I is the ideal generated by the elements eu and $e\xi$. The degrees are $|e| = 1$, $|\xi| = 3$, $|b| = 4$ and $|u| = 2$.

Proof. From the lemma above we find the E_3 term of the spectral sequence associated to the fibration (42)

$$E_3^{**} = \Lambda(e,\xi) \otimes \mathbb{F}_2[b,u]/I$$

where $\xi = ec$ and $b = c^2$. For dimensional reasons all higher differentials on the classes e and b are zero, and the only possible non-zero higher differential on ξ are $d_4\xi = u^2$. But because of the section (44) the class u^2 survives to E_∞ hence $d_4\xi = 0$. We have shown $E_3^{**} = E_\infty^{**}$. Choose liftings of the generators $e, \xi, b, u \in H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty)$. By using the section (44) when necessarily one finds that the liftings satisfy the relations $e^2 = 0$, $eu = 0$, $e\xi = 0$ and $\xi^2 = \lambda bu$ where $\lambda \in \mathbb{F}_2$ is a constant. The constant loop inclusion $i : \mathbb{C}P^\infty \to \Lambda\mathbb{C}P^\infty$ induces a map from the trivial fibration

$$\mathbb{C}P^\infty \to BS^1 \times \mathbb{C}P^\infty \to BS^1$$

to the fibration (42). By the corresponding map of spectral sequences one finds that the map of total spaces

$$(1 \times i)^* : H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty) \to H^*(BS^1 \times \mathbb{C}P^\infty)$$

satisfy $\xi \mapsto 0$, $x \mapsto x$ and $b \mapsto c^2 + \gamma cu$ for a $\gamma \in \mathbb{F}_2$. It follows that $\lambda = 0$. Thus the liftings satisfy the all the relations in the E_∞-term. Because of this we can define a ring homomorphism

$$\Lambda(e,\xi) \otimes \mathbb{F}_2[b,u]/I \to H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty)$$

which is easily seen to be an isomorphism. □

We proceed by finding the $\mathcal{A}$-module structure of $H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty)$. To do so it is enough to find $Sq^1\xi$, $Sq^2\xi$, $Sq^1 b$ and $Sq^2 b$. The constant loop section and the inclusion of the fiber map

$$H^*(\Lambda\mathbb{C}P^\infty) \xleftarrow{j^*} H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty) \xrightarrow{s^*} H^*(BS^1)$$

gives the following information about these

$$Sq^1\xi = 0\ ,\ Sq^2\xi = eb + \lambda_1 u\xi\ ,\ Sq^1 b = \lambda_2 u\xi\ ,\ Sq^2 b = \lambda_3 ub \tag{45}$$

where $\lambda_\nu \in \mathbb{F}_2$ are constants to be determined. The map

$$H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty) \xrightarrow{i^*} H^*(BS^1 \times \mathbb{C}P^\infty)$$

maps e and ξ to zero, u to u and going to the fiber one finds that $i^*(b) = c^2 + \lambda_4 uc$ for a constant $\lambda_4 \in \mathbb{F}_2$. This together with the quotient map shows that

$$H^* L_2 \mathbb{C}P^\infty \xrightarrow{i_1} H^*(BC_2 \times \mathbb{C}P^\infty)$$

is given by $e, \xi \mapsto 0$, $a \mapsto a$ and $b \mapsto c^2 + \lambda_4 a^2 c$. Since the diagonal map

$$\Delta^* : H^* P_2 \mathbb{C}P^\infty \longrightarrow H^*(BC_2 \times \mathbb{C}P^\infty)$$

satisfy $\Delta^*(1 \otimes c^{\otimes 2}) = a^2 c + c^2$, and $\Delta^* = i_1^* \circ ev_1^*$ we see that $\lambda_4 = 1$ and $i_1(b) = c^2 + a^2 c$, $i^*(b) = c^2 + uc$. Further more we get that

$$ev^*(1 \otimes c^{\otimes 2}) = b + \lambda_5 a\xi$$

for a constant $\lambda_5 \in \mathbb{F}_2$. Now we can find $Sq^2 b$ using the $\mathcal{A}$-linearity of i^*. We have

$$Sq^2(i^*(b)) = Sq^2(c^2 + uc) = u^2 c + uc^2$$

and this equals

$$i^*(Sq^2 b) = i^*(\lambda_3 ub) = \lambda_3 u(c^2 + uc) = \lambda_3(uc^2 + u^2 c)$$

thus $\lambda_3 = 1$ and $Sq^2 b = ub$.

We can use Lemma 9.2 to find $Sq^1 b$. The E_3-term of the spectral sequence (43) is determine by Lemma 9.2. The only possible non-zero higher differential is $d_4(ec)$ but the section (44) shows that it is zero. One finds that $H^q(ES^1 \times_{S^1} \Lambda \mathbb{C}P^\infty; \mathbb{Z})$ is $\mathbb{Z} \oplus \mathbb{Z}$ for $q = 4$ and $\mathbb{Z} \oplus \mathbb{Z}/2$ for $q = 5$. By the long exact sequence associated to the coefficient sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

one sees that there is a non-trivial bockstein from dimension four to five. Dimension four of the mod two cohomology of the total space consists of the classes b and u^2. Since $Sq^1(u^2) = 0$ we get $Sq^1 b \neq 0$, thus $\lambda_2 = 1$ and $Sq^1 b = u\xi$.

We can use this to determine λ_5. By the formulas for the $\mathcal{A}$ action on $H^* P_1 X$ we get that $Sq^1(1 \otimes c^{\otimes 2}) = 0$ and hence

$$0 = Sq^1\, ev_1^*(1 \otimes c^{\otimes 2}) = Sq^1(b + \lambda_5 \xi a) = a^2 \xi + \lambda_5 \xi a^2$$

giving $\lambda_5 = 1$ and

$$ev_1^*(1 \otimes c^{\otimes 2}) = b + a\xi$$

We get further information by applying Sq^2. We have

$$Sq^2(1 \otimes c^{\otimes 2}) = a^2 \otimes c^{\otimes 2} + 1 \otimes (c^2 \otimes c + c \otimes c^2)$$

and from this

$$ev_1^*(Sq^2(1 \otimes c^{\otimes 2})) = a^2 b + a^3 \xi + ev_1^*(1 \otimes (c^2 \otimes c + c \otimes c^2))$$

This equals

$$Sq^2\, ev_1^*(1 \otimes c^{\otimes 2}) = Sq^2(b + a\xi) = a^2 b + \lambda_1 a^3 \xi + aeb$$

Multiplication with a annihilates the terms aeb and $ev_1^*(1 \otimes (c^2 \otimes c + c \otimes c^2))$ giving $\lambda_1 = 1$ and $Sq^2(\xi) = eb + u\xi$. Further we get that

$$ev_1^*(1 \otimes (c^2 \otimes c + c \otimes c^2)) = aeb$$

Since $(1 \otimes c^{\otimes 2})(1 \otimes (c \otimes 1 + 1 \otimes c))$ it follows that

$$(b + a\xi)ev_1^*(1 \otimes (c \otimes 1 + 1 \otimes c)) = aeb$$

In degree two we have a^2 and ea and it follows that

$$ev_1^*(1 \otimes (c \otimes 1 + 1 \otimes c)) = ae$$

Since $t_0^2(1 \otimes c) = 1 \otimes (c \otimes 1 + 1 \otimes c)$ and $ev_0^*(1 \otimes c) = c$ it follows that $Tr_0^1(c) = ae$. By Frobenius reciprocity this determines Tr_0^1. We have show the following two propositions

Proposition 9.4. *The transfer maps*

$$Tr_m^{m+1} : H^* L_m \mathbb{C}P^\infty \longrightarrow H^* L_{m+1} \mathbb{C}P^\infty$$

are given by the following. When $m = 0$ the elements c^{2k} and ec^k maps to zero and c^{2k+1} maps to aeb^k. When $m = 1$ monomials with a in an even power maps to zero and monomials with a in an odd power $2k+1$ maps to the same monomial with a^{2k+1} replaced by vu^k. When $m \geq 2$ monomials without a v is mapped to zero and monomials with a v is mapped identically.

Let $Sq(x)$ denote the total square of the class x i.e. the formal power series

$$Sq(x) = \sum_{i \geq 0} (Sq^i x) t^i$$

Proposition 9.5. *The $\mathcal{A}$-module structure of $H^*(ES^1 \times_{S^1} \Lambda \mathbb{C}P^\infty)$ is given by the following. The total squares on the exterior generators are*

$$Sq(e) = e\ ,\ Sq(\xi) = \xi + (eb + u\xi)t^2$$

and the total squares on the polynomial generators are

$$Sq(u) = u + ut^2\ ,\ Sq(b) = b + u\xi t + ubt^2 + u^2\xi t^3 + b^2 t^4$$

The next step is to compute ev_m^* for each m. We shall need the following result

Lemma 9.6. *For $m \geq 2$ and $r = 0, 1, 2, \ldots, 2^{m-1} - 1$ one has*

$$Sq^2(\xi b^r u^{2^m - 2(r+1)}) = \begin{cases} \xi b^r u^{2^m - 2r - 1} & \text{, when } r \text{ is even} \\ eb^{2^{m-1}} & \text{, when } r = 2^{m-1} - 1 \\ 0 & \text{, otherwise} \end{cases}$$

$$Sq^4(\xi b^r u^{2^m - 2(r+1)}) = \xi b^r u^{2^m - 2(r+1)} \left(\binom{r}{1} b + \binom{2^m - 1 - r}{2} u^2 \right)$$

Proof. Since both e and ξ annihilates ξ we have

$$Sq(\xi b^r u^{2^m-2(r+1)}) = (\xi + (eb+u\xi)t^2)(b+ubt^2+b^2t^4)^r(u+u^2t^2)^{2^m-2(r+1)}$$

and by finding the coefficient to t^2 we get

$$Sq^2(\xi b^r u^{2^m-2(r+1)}) = (eb+u\xi)b^r u^{2^m-2(r+1)} + \binom{r}{1}\xi b^r u^{2^m-2r-1}$$

which equals the result stated above. Since $eu=0$ we map put $e=0$ in the expression of the total square when computing Sq^4. The expression then equals

$$\xi u^{2^m-2(r+1)} b^r (1+ut^2+bt^4)^r(1+ut^2)^{2^m-2r-1}$$

by the binomial formula we get

$$\begin{aligned}(1+ut^2+bt^4)^r(1+ut^2)^{2^m-2r-1} =&(1+ut^2)^{2^m-1-r} + \binom{r}{1}bt^r(1+ut^2)^{2^m-2-r}\\ &+\sum_{i=2}^{r}\binom{r}{i}b^i t^{4i}(1+ut^2)^{2^m-1-r-i}\end{aligned}$$

We may neglect the last sum since the powers of t are to big and the stated result follows by finding the coefficient to t^4. $\square$

Theorem 9.7. *For all integers $m \geq 1$ we have*

$$ev_m^*(1\otimes c^{2^m}) = b^{2^{m-1}} + v\xi b^{2^{m-1}-1}$$

Proof. We have already seen that this is correct when $m=1$. Since $Sq^1 c = 0$ the Q_2-operation from chapter 4 satisfy $Q_2^{2i+1}(c) = 0$ and it follows that $Q_r^{2i+1}(c) = 0$ for $r = 2^k$ with k any positive integer. We also have $Sq_1 c = Sq^1 c = 0$ And the formula for the diagonal map reduces to

$$\Delta^*(1\otimes c^{\otimes 2^m}) = (uc)^{2^{m-1}} + c^{2^m}$$

The map i_m^* is given by $i_m^*(v) = v$ and the same as the map i^* on the other algebra generators. Since $\Delta^* = i_m^* \circ ev_m^*$ we find that

$$ev_m^*(1\otimes c^{2^m}) = b^{2^{m-1}} + y$$

for some y in the kernel of i_m^*. The even dimensional elements in the kernel of i_m^* is spanned by the elements veb^l and $v\xi b^r u^s$ with $l,r,s \geq 0$. Since the degree of y is 2^{m+1} and the degree of veb^l is $4l+2$, y is in the span of $v\xi b^r u^s$ where r and s satisfy $2^m = 2+2r+s$. We conclude that

$$ev_m^*(1\otimes c^{\otimes 2^m}) = b^{2^{m-1}} + \sum_{r=0}^{2^{m-1}-1} k_r v\xi b^r u^{2^m-2(r+1)}$$

for some constants $k_i \in \mathbb{F}_2$. In the case $m = 2$ we have

$$Sq^2(1 \otimes c^{\otimes 4}) = 1 \otimes Nc^2 \otimes c \otimes c \otimes c$$

and this element is the transfer on $c^2 \otimes c \otimes c \otimes c$ in $H^*P_0((\mathbb{C}P^\infty)^4)$. By applying Proposition 8.5 we obtain

$$ev_2^*(Sq^2(1 \otimes c^{\otimes 4})) = Tr_0^4 \circ ev_0^*(1 \otimes c^5) = veb^2$$

and by the $\mathcal{A}$-linearity of ev_2^* this equals

$$Sq^2(b^2 + k_0 v\xi u^2 + k_1 v\xi b) = k_0 vu^3\xi + k_1 veb^2$$

Thus $k_0 = 0$ and $k_1 = 1$ and we have proved the stated formula when $m = 2$. Assume $m \geq 3$. By Lemma 9.6 the map

$$(Sq^2, Sq^4) : H^*L_m\mathbb{C}P^\infty \to H^*L_m\mathbb{C}P^\infty \oplus H^*L_m\mathbb{C}P^\infty$$

is injective when restricted to the subspace spanned by the elements $\xi b^r u^{2^m-2(r+1)}$, $r = 0, 1, \dots, 2^{m-1}-1$. Thus it is enough to show that

$$ev_m^*(Sq^i(1 \otimes c^{\otimes 2^m})) = Sq^i(b^{2^{m-1}} + v\xi b^{2^{m-1}-1})$$

for $i = 2$ and $i = 4$. We have

$$Sq^2(1 \otimes c^{\otimes 2^m}) = 1 \otimes Nc^2 \otimes c \otimes \cdots \otimes c$$

and

$$\begin{aligned} Sq^4(1 \otimes c^{\otimes 2^m}) =& 1 \otimes Sq^4(c^{2^m}) = 1 \otimes \sum_{k=1}^{2^{m-1}} T^k(c^2 \otimes c \otimes \cdots \otimes c)^{\otimes 2} \\ &+ \sum 1 \otimes Nc^2 \otimes c \otimes \cdots \otimes c \otimes c^2 \otimes c \otimes \cdots \otimes c \end{aligned}$$

By applying Proposition 8.5 as above we get

$$ev_m^*(Sq^2(1 \otimes c^{\otimes 2^m})) = veb^{2^{m-1}}$$

and by applying Proposition 8.5 to the equation for square four we get

$$ev_m^*(Sq^4(1 \otimes c^{2^m})) = Tr_1^m(1 \otimes (c^{2^{m-1}+1})^{\otimes 2}) + \sum Tr_0^m \circ ev_1^*(1 \otimes c^{2^m+2})$$

The proof of corollary 8.7 gives that the last sum is zero and we have

$$ev_m^*(Sq^4(1 \otimes c^{2^m})) = Tr_1^m((b + a\xi)^{2^{m-1}+1}) = v\xi b^{2^{m-1}}$$

By Lemma 9.6 we have

$$Sq^2(b^{2^{m-1}} + v\xi b^{2^{m-1}-1}) = veb^{2^{m-1}}\ ,\ Sq^4(b^{2^{m-1}} + v\xi b^{2^{m-1}-1}) = v\xi b^{2^{m-1}}$$

which completes the proof. $\square$

Using that ev_m^* is a ring homomorphism and the description of the transfer maps, we get the following main result.

Theorem 9.8. *The map*

$$\ell(\mathbb{C}P^\infty) \xrightarrow{\phi} H^*(ES^1 \times_{S^1} \Lambda\mathbb{C}P^\infty)$$

is an isomorphism given by

$$\begin{aligned}
&[1 \otimes c^{2k+1}] \mapsto eb^k \\
&[e^{2i+1} \otimes (c^k)^{\otimes 2}] \mapsto u^i b^k \\
&[e^{2i} \otimes (c^{2k+1})^{\otimes 2^m}] \mapsto u^i \xi b^{2^m k + 2^{m-1} - 1}
\end{aligned}$$

where $m \geq 1$ *and* $k \geq 0$

Note that any nonnegative integer r can be written as $r = 2^m k + 2^{m-1} - 1$ for a unique $m \geq 1$ and $k \geq 0$. If r is even m must be one and $k = r/2$. If r is odd $m-1$ must be the two valuation of $r+1$ since $r + 1 = 2^{m-1}(2k+1)$ and we can solve this to find k.

10. The S^1-homotopy orbits of the Eilenberg MacLane spaces $K(\mathbb{F}_2, n)$.

Let n be an positive integer and let $K_n = K(\mathbb{F}_2, n)$ denote an Eilenberg MacLane space of type $(\mathbb{F}_2, n)$. Assume that $n \geq 2$. The fibration

$$\Lambda K_n \longrightarrow ES^1 \times_{S^1} \Lambda K_n \xrightarrow{\pi} BS^1$$

has the following Serre spectral sequence

$$E_2^{**} = H^*(BS^1) \otimes H^*(\Lambda K_n) \Rightarrow H^*(ES^1 \times_{S^1} \Lambda K_n) \tag{46}$$

The cohomology of the fiber is $H^*(\Lambda K_n) = H^*(K_{n-1}) \otimes H^*(K_n)$ by Proposition 9.1. We shall denote the fundamental class of $H^*(K_r)$ by ι_r.

Proposition 10.1. *In the above spectral sequence the differential* d_2 *is given by* $d_2(\iota_{n-1}) = 0$ *and* $d_2(\iota_n) = u\iota_{n-1}$.

Proof. The map π has a section

$$BS^1 \xrightarrow{s} ES^1 \times_{S^1} \Lambda K_n \tag{47}$$

since any constant loop is a fixpoint in ΛK_n. Because of this section the class u survives to E^∞ and $d_2(\iota_{n-1}) = 0$. (At this place we only need this argument when $n = 2$). By lemma 8.10 one may describe the differential d_2 by

$$\begin{aligned}
&d_2 : H^*(\Lambda K_n) \longrightarrow uH^*(\Lambda K_n) \\
&d_2(y) = ud(y)
\end{aligned}$$

where d is determined by the action

$$\begin{aligned}
&\lambda : S^1 \times \Lambda K_n \longrightarrow \Lambda K_n \\
&\lambda^*(y) = \sigma \otimes d(y) + 1 \otimes y
\end{aligned}$$

We will find $d(\iota_n)$. Since it has degree $n-1$ we get

$$d(\iota_n) = k\iota_{n-1}\ ,\ k \in \mathbb{F}_2$$

Consider the composite

$$\phi : S^1 \times \Omega K_n \xrightarrow{1\times i} S^1 \times \Lambda K_n \xrightarrow{\lambda} \Lambda K_n \xrightarrow{ev_1} K_n$$

satisfying $\psi(z,f) = f(z)$. Since $ev_1^*(\iota_n) = \iota_n$ and $\lambda^*(\iota_n) = \sigma \otimes k\iota_{n-1} + 1 \otimes \iota_n$ we obtain

$$\psi^*(\iota_n) = \sigma \otimes k\iota_n$$

and it is enough to show that

$$\psi^* : H^n(K_n) \longrightarrow H^n(S^1 \times \Omega K_n)$$

is non trivial. The map ψ factors through the smash product

$$\begin{tikzcd} S^1 \times \Omega K_n \arrow[r, "\psi"] \arrow[d, "Q"'] & K_n \\ S^1 \wedge \Omega K_n \arrow[ur, "\hat{\psi}"'] & \end{tikzcd}$$

We first consider the map $\hat{\psi}$. There is a commutative diagram of homotopy groups

$$\begin{tikzcd} \pi_q(S^1 \wedge \Omega K_n) \arrow[r, "\hat{\psi}_*"] & \pi_q(K_n) \arrow[dl, "adj", "\cong"'] \\ \pi_{q-1}(\Omega K_n) \arrow[u, "\Sigma"] & \end{tikzcd}$$

where Σ is the suspension map and adj is the adjunction isomorphism.

Assume that $n \geq 3$. By Freudenthal's suspension theorem Σ is an isomorphism for $q < 2n-2$ especially for $q \leq n$. Thus $\hat{\phi}_*$ is an isomorphism for $q \leq n$. By Hurewicz theorem

$$H_n(S^1 \wedge \Omega K_n) \xrightarrow{\hat{\psi}_*} H_n(K_n)$$

is an isomorphism. To complete the $n \geq 3$ case we only need to show that Q^* is non zero in degree n. Consider the long exact cohomology sequence associated to the pair $i : S^1 \vee K_{n-1} \subseteq S^1 \times K_{n-1}$:

$$\ldots \leftarrow H^n(S^1 \vee K_{n-1}) \xleftarrow{i^*} H^n(S^1 \times K_{n-1}) \xleftarrow{Q^*} H^n(S^1 \wedge K_{n-1}) \leftarrow \ldots$$

Since $i^* : \mathbb{F}_2 \oplus \mathbb{F}_2 \longrightarrow \mathbb{F}_2$ has a non trivial kernel we see that Q^* is non trivial. When $n = 2$ the proposition follows from Lemma 9.2 by using the reduction map $K(\mathbb{Z},2) \longrightarrow K(\mathbb{F}_2, 2)$. $\square$

Recall that a multi index of non-negative integers $I = (i_1, i_2, \dots, i_k)$ is called admissible if $i_{s-1} \geq 2i_s$ for $k \geq s \geq 2$ and $i_k \geq 1$. We denote the set of admissible multiindeces by A. The excess of I is the number

$$e(I) = i_1 - i_2 - \cdots - i_k$$

Theorem 10.2. *Let $n \geq 2$ be an integer. The cohomology $H^*(ES^1 \times_{S^1} \Lambda K_n)$ is the quotient of the $\mathbb{F}_2$-polynomial algebra generated by the elements*

$$\begin{aligned}
&u \text{ of degree } 2\\
&\{b_J | J \in A\ ,\ e(J) \leq n-2\} \text{ with } |b_J| = 2(|J| + n)\\
&\{X_I | I \in A\ ,\ e(J) = n-1\} \text{ with } |X_I| = |I| + n\\
&\{D(J_1, \dots, J_r) | r \geq 1\ ;\ J_1, \dots, J_r \in A\ ;\ e(J_i) \leq n-2, i = 1, \dots, r\}\\
&\text{with } |D(J_1, \dots, J_r)| = |J_1| + \cdots + |J_r| + rn - 1
\end{aligned}$$

by the ideal generated by the elements

$$\begin{aligned}
&D(J_1, \dots, J_r) + D(J_{\sigma(1)}, \dots, J_{\sigma(r)})\\
&D(J, J, J_1, \dots, J_s) + b_J D(J_1, \dots, J_s)\\
&D(J_1, \dots, J_r) D(K_1, \dots, K_s) + \sum_{i=1}^{r} D(J_i) D(J_1, \dots, \hat{J}_i, \dots, J_r, K_1, \dots, K_s)\\
&uD(J_1, \dots, J_r)
\end{aligned}$$

here σ is any permutation of the numbers from 1 to r, and the "hat" means that the term is left out.

Proof. We start by computing the E_3-term of the spectral sequence (46). Again we use the notation

$$d_2 : H^*(\Lambda K_n) \longrightarrow uH^*(\Lambda K_n)\ ;\ d_2(y) = ud(y)$$

and we can write the E_3-term as follows

$$E_3^{**} = ker(d) \oplus uH(d) \oplus u^2 H(d) \oplus \dots$$

Because of (9.1) we get the cohomology of the fiber

$$H^*(\Lambda K_n) \cong \mathbb{F}_2[Sq^I \iota_n, Sq^J \iota_{n-1} | I, J \in A,\ e(I) \leq n-1\ ,\ e(J) \leq n-2] \tag{48}$$

Assume that I is admissible and $e(I) = n-1$. We will express $dSq^I \iota_n = Sq^I \iota_{n-1}$ as a polynomial in the variables from (48). Define $I(t)$ for $t = 1, 2, \dots, k$ by

$$I(t) = (i_t, i_{t+1}, \dots, i_k)$$

We have

$$e(I(t)) - e(I(t+1)) = i_t - 2i_{t+1} \geq 0$$

hence the excess is weakly decreasing with increasing t. Define the number $\sigma(I)$ by

$$\sigma(I) = max\{t | e(I(t)) = n-1\}$$

and put $L(I) = (i_1, \dots, i_{\sigma(I)-1})$ and $R(I) = (i_{\sigma(I)+1}, \dots, i_k)$. We claim that

$$dSq^I \iota_n = (Sq^{R(I)} \iota_{n-1})^{2^{\sigma(I)}}$$

where $R(I)$ is admissible with $e(R(I)) \leq n-2$. This follows since

$$i_1 = 2i_2, i_2 = 2i_3, \dots, i_{\sigma(I)-1} = 2i_{\sigma(I)}$$

and

$$i_{\sigma(I)} = i_{\sigma(I)+1} \cdots + i_k + n - 1 = |Sq^{R(I)} \iota_{n-1}|$$

By definition of $\sigma(I)$ we have $e(R(I)) \leq n-2$. We chose another set of polynomial generators for (48) as follows. Define X_I for I admissible with $e(I) = n-1$ by

$$X_I = Sq^I \iota_n + Sq^{L(I)}(Sq^{R(I)} \iota_n Sq^{R[I]} \iota_{n-1})$$

and we have

(49) $H^*(\Lambda K_n) = \mathbb{F}_2[Sq^J \iota_{n-1}, Sq^J \iota_n, X_I | I, J \in A\ ,\ e(J) \leq n-2\ ,\ e(I) = n-1]$

where

$$dSq^J \iota_{n-1} = 0\ ,\ dSq^J \iota_n = Sq^J \iota_{n-1}\ ,\ dX_I = 0$$

Applying the Künneth formula we find the cohomology of d:

$$H(d) = \mathbb{F}_2[b_J, X_J | I, J \in A\ ,\ e(I) = n-1\ ,\ e(J) \leq n-2]$$

where $b_J = (Sq^J \iota_n)^2$. Thus the E_3-term is determined except for the first column. Splitting the kernel of d, we may write it as

$$E_3^{**} = im(d) \oplus H(d) \oplus uH(d) \oplus u^2 H(d) \oplus \dots$$

where $uy = 0$ for all $y \in im(d)$.

We will now prove that all higher differentials vanish such that $E_3^{**} = E_\infty^{**}$. Because of the quotient map

$$Q : L_m X \longrightarrow ES^1 \times_{S^1} \Lambda X \qquad (50)$$

the Serre spectral sequence of the fibration

$$\Lambda X \longrightarrow L_m X \longrightarrow BC_{2^m}$$

also have E_2-differential d_2 given by Proposition 10.1 and the E_3-term looks as follows

$$\hat{E}_2^{**} = im(d) \oplus H(d) \oplus v(im(d) \oplus H(d)) \oplus uH(d) \oplus vuH(d) \oplus \dots$$

where the classes v and u comes from

$$H^*(BC_{2^m}) = \Lambda(v) \otimes \mathbb{F}_2[u]\ ,\ |v| = 1\ ,\ |u| = 2$$

when $m \geq 2$. For $m = 1$ we have $H^*(BC_2) = \mathbb{F}_2[a]$ where a is a one dimensional class, and the above should be interpretaded by $v = a$ and $u = a^2$. The quotient map (50) also gives that a non zero higher differential in $\hat{E}_*^{**}$ implies one in E_*^{**} and visa versa.

Assume that $d_s = 0$ for all $s \leq r - 1$. If r is odd, $d_r = 0$ since it begins or ends in a zero column in E_r^{**}. Assume r is even, $r = 2l$. For a class $y \in im(d)$ we can write $d_r y = u^l z$ for some $z \in H(d)$. But since $uy = 0$ we have

$$0 = d_r(uy) = ud_r y = u^{l+1} z$$

which implies $z = 0$ and we have shown $d_r y = 0$. Thus it is enough to show that $d_r b_J = 0$ and $d_r X_I = 0$. At this point we use (40).

Let us examine the map of spectral sequences induced by the constant loop inclusion i.

$$\begin{tikzcd}
\Lambda K_n \arrow[d] & K_n \arrow[l, "i"'] \arrow[d] \\
L_m K_n \arrow[d] & BC_{2^m} \times K_n \arrow[l, "1\times i"'] \arrow[d] \\
BC_{2^m} \arrow[r, equal] & BC_{2^m}
\end{tikzcd}$$

on the fibers we have

$$i^* : H^*(\Lambda K_n) \longrightarrow H^*(K_n)\ ;\ \iota_{n-1} \mapsto 0\ ,\ \iota_n \mapsto \iota_n$$

because of the degree and the fact that the composite

$$H^*(K_n) \xrightarrow{ev^*} H^*(\Lambda K_n) \xrightarrow{i^*} H^*(K_n)$$

is the identity. This implies that $i^*(y) = 0$ for all $y \in im(d)$ and

$$i^*(X_I) = Sq^I \iota_n\ ,\ i^*(b_J) = (Sq^J \iota_n)^2$$

as a map of spectral sequences.

Assume J is admissible with $e(J) \leq n - 2$. The map

$$\Delta^* : H^* P_1(X) \to H^*(BC_2) \otimes H^*(X)$$

satisfy

$$\Delta^*(a \otimes (Sq^J \iota_n)^{\otimes 2}) = a(Sq^J \iota_n)^2 + a^2 Sq^{|J|+n-1} Sq^J \iota_n + \dots$$

There is a class $z \in H^* L_1(X)$ which maps to this under the map i_1^* in (40). Thus the class $w = Tr_1^2(z)$ maps to

$$v(Sq^J \iota_n)^2 + vuSq^{|J|+n-2} Sq^J \iota_n + \dots$$

under the map i_2^*. We claim that the class corresponding to w in the spectral sequence has zero in the first column. This follows from a general fact. The composite

$$H^*L_1X \xrightarrow{Tr_1^2} H^*L_2X \xrightarrow{j^*} H^*\Lambda X \tag{51}$$

is zero, where j is the inclusion of the fiber. To prove this observe that we have a commutative diagram

$$\begin{array}{ccc} S^0 \times \Lambda X & \longrightarrow & L_1X \\ \downarrow & & \downarrow \\ \Lambda X & \xrightarrow{j} & L_2X \end{array} \tag{52}$$

from which we get the following commutative diagram in homology

$$\begin{array}{ccc} H_*(\Lambda X) \oplus H_*(\Lambda X) & \xrightarrow{j_{1*}+j_{2*}} & H_*L_1X \\ {\scriptstyle Tr}\uparrow & & {\scriptstyle Tr_2^1}\uparrow \\ H_*(\Lambda X) & \xrightarrow{j_*} & H_*L_2X \end{array}$$

where j_1 and j_2 are the two inclusions coming from the top horizontal arrow of (52). Clearly j_1 and j_2 are homotopic and $j_{1*} + j_{2*} = 2j_{1*}$ thus it is zero with mod two coefficients. Passing to cohomology, we see that (51) is the zero map.

we can now conclude that there is a class c in the E_∞-term such that $i^*(vc) = v(Sq^J\iota_n)^2$. This forces the class b_J to survive.

Assume $I = (i_1, \ldots, i_k)$ is admissible with $e(I) = n-1$. Let K be the multi index $K = I(2) = (i_2, \ldots, i_k)$. We have

$$\Delta^*(1 \otimes (Sq^K\iota_n)^{\otimes 2}) = 1 \otimes (Sq^K\iota_n)^2 + a \otimes Sq^{|K|+n-1}Sq^K\iota_n + a^2 \otimes Sq^{|K|+n-2}Sq^K\iota_n + \ldots$$

and since $e(I) = n-1$ we have that $i_1 = n - 1 + i_2 + \cdots + i_k$ thus

$$Sq^I\iota_n = Sq^{|K|+n-1}Sq^K\iota_n$$

A similar argument to the one above forces the class X_I to survive to E_∞^{**}. We have proved $E_3^{**} = E_\infty^{**}$.

Next we give a precise description of the $im(d)$ part. For a sequence $J_1, \ldots, J_r$ of admissible multiindeces all with excess at most $n-2$ we define the class

$$D(J_1, \ldots, J_r) := d\left(\prod_{i=1}^r Sq^{J_i}\iota_n\right) = \sum_{i=1}^r Sq^{J_i}\iota_{n-1} \prod_{j \neq i} Sq^{J_j}\iota_n \tag{53}$$

The following relations are obvious

$$D(J_1, \ldots J_r) = D(J_{\sigma(1)}, \ldots, J_{\sigma(r)}) \text{ for all } \sigma \in \Sigma_r \tag{54}$$

$$D(J, J, J_1, \ldots, J_s) = b_J D(J_1, \ldots J_s) \tag{55}$$

Where Σ_r denotes the symmetric group. The following relation is easily verified

$$D(J_1,\dots,J_r)D(K_1,\dots,K_s)=\sum_{i=1}^{r}D(J_i)D(J_1,\dots,\hat{J}_i,\dots,J_r,K_1,\dots,K_s) \tag{56}$$

where the "hat" means that the element is left out. We see that $im(d)$ is a module over the ring

$$\mathbb{F}_2[D(J),b_J,X_I|I,J\in A\ ,\ e(J)\leq n-2\ ,\ e(I)=n-1]$$

and as such generated by the classes (53). In fact (54) , (55) and (56) generates all relations in $im(d)$ and we have

$$\begin{aligned} E_\infty^{**}=\mathbb{F}_2[u,D(J_1,\dots,J_r),b_{J_0},X_I|J_i\in A\ ,\ e(J_i)\leq n-1 \text{ for } j=0,\dots,r\\ \text{and } I\in A\ ,\ e(I)=n-1]/R\end{aligned} \tag{57}$$

where R is the ideal generated by (54), (55) and (56) and the relations

$$uD(J_1,\dots,J_r)=0 \tag{58}$$

Choose liftings of the algebra generators of (57) in $H^*(ES^1\times_{S^1}\Lambda X)$. We can do this such that the relations (58) still holds. With this choice the other relations (54) , (55) and (56) also holds. Using these liftings we can define an algebra morphism

$$Al^*\xrightarrow{f}H^*(ES^1\times_{S^1}\Lambda X)$$

where Al^* is the algebra in (57). Any element in E_∞^{**} is the quotient of an element of the form $f(a)$ where $a\in Al^*$, and from this it follows that f is surjective. It is equally easy to see that f is injective. □

In order to complete the description we consider the initial case $K_1=\mathbb{R}P^\infty=BC_2$. For a discrete group G one has a homotopy equivalence

$$\Lambda BG\simeq\coprod_{x\in<G>}BC_G(x)$$

where $C_G(x)$ is the centralizer of x and $<G>$ is the set of conjugacy classes in G. With $G=C_2$ we find

$$\Lambda BC_2\simeq BC_2\coprod BC_2 \tag{59}$$

Let $i_0: BC_2\longrightarrow\Lambda BC_2$ denote the constant loop inclusion i.e. $i_0(x)(z)=x$ for all $z\in S^1$. Since BC_2 is connected $i_0(BC_2)$ is contained in one of the two components in $\Lambda(BC_2)$. We denote this component $(\Lambda BC_2)_0$ and the other $(\Lambda(BC_2))_1$. Let $ev:\Lambda BC_2\longrightarrow BC_2$ be the map which evaluates a loop in $1\in S^1$. Since the composite

$$BC_2\xrightarrow{i_0}(\Lambda BC_2)_0\xrightarrow{ev}BC_2$$

is the identity $i_{0*}:\pi_q(BC_2)\longrightarrow\pi_q((\Lambda BC_2)_0)$ is an injective map. Since both homotopy groups equals $\mathbb{F}_2$ when $q=1$ and 0 when $q\geq 2$ we have that i_{0*} is an isomorphism. Since $(\Lambda BC_2)_0$ has the type of a CW-complex by (59) the Whitehead theorem shows that $i_0 : BC_2 \longrightarrow (\Lambda BC_2)_0$ is a homotopy equivalence. Let $(BC_2)_0$ denote BC_2 with the trivial S^1-action. Then $i_0 : (BC_2)_0 \longrightarrow (\Lambda BC_2)_0$ is a homotopy equivalence and an S^1-equivariant map. By the Whitehead theorem we get a homotopy equivalence

$$BS^1 \times BC_2 = ES^1 \times_{S^1} (BC_2)_0 \xrightarrow{1\times i_0} ES^1 \times_{S^1} (\Lambda BC_2)_0$$

There is an homeomorphism $S^1 \approx S^1/C_2$ given by $z \mapsto \pm\sqrt{z}$. Via this we get a free S^1-action on BC_2:

$$S^1/C_2 \times ES^1/C_2 \longrightarrow ES^1/C_2 \tag{60}$$

From this action we define a map $i_1 : BC_2 \longrightarrow \Lambda BC_2$ by $i_1([e])(z) = z \cdot [e] = [\sqrt{z}e]$ for $z \in S^1$ and $e \in ES^1$. $i_1(BC_2)$ lies in the component $(\Lambda BC_2)_\nu$ of ΛBC_2 and the composite $ev \circ i_1$ is the identity. By a similar argument as above we get that $i_1 : BC_2 \longrightarrow (\Lambda BC_2)_\nu$ is a homotopy equivalence. Letting $(BC_2)_1$ denote BC_2 with the S^1-action (60) we have that $i_1 : (BC_2)_1 \longrightarrow (\Lambda BC_2)_\nu$ is a homotopy equivalence and an S^1-equivariant map. We get a homotopy equivalence

$$ES^1 \times_{S^1} (BC_2)_1 \xrightarrow{1\times i_1} ES^1 \times_{S^1} (\Lambda BC_2)_\nu$$

Since the S^1-action on $(BC_2)_1$ was free projection on the second factor of the left hand side gives a homotopy equivalence to $(BC_2)_1/S^1 \simeq BS^1$ thus $\nu = 1$ and we have shown

$$(BS^1 \times BC_2) \coprod BS^1 \simeq ES^1 \times_{S^1} \Lambda BC_2$$

Hence the following cohomology

$$H^*(ES^1 \times_{S^1} \Lambda BC_2) \cong \mathbb{F}_2[u,t] \oplus \mathbb{F}_2[u]$$

where the degree of t is one. Of cause we also have a homotopy equivalence

$$ES^1 \times_{C_{2^m}} (BC_2)_0 \coprod ES^1 \times_{C_{2^m}} (BC_2)_1 \xrightarrow{i_0 \coprod i_1} L_m BC_2$$

and since the second factor on the left hand side is homotopic to $(BC_2)_1/C_{2^m} \simeq BC_{2^{m+1}}$ we get

$$(BC_{2^m} \times BC_2) \coprod BC_{2^{m+1}} \simeq L_m BC_2$$

In this initial case the approximation (39) has also been determined. It has the following form:

Theorem 10.3. *The map*

$$\ell(\mathbb{R}P^\infty) \xrightarrow{\phi} H^*(ES^1 \times_{S^1} \Lambda \mathbb{R}P^\infty) \cong \mathbb{F}_2[u,t] \oplus \mathbb{F}_2[u]$$

is given by

$$
\begin{aligned}
&[1\otimes t^{2r+1}] \mapsto (0\ ,\ u^r)\\
&[e^{2k+1}\otimes (t^r)^{\otimes 2}] \mapsto (\sum_{i=0}^{[r/2]}\binom{r}{2i}u^{k+i}\otimes t^{2(r-i)}\ ,\ \delta_{k,0}u^r)\\
&[e^{2k}\otimes (t^{2r+1})^{\otimes 2}] \mapsto (\sum_{i=0}^{r}\binom{r}{i}u^{k+i}\otimes t^{2(r-i)-1}\ ,\ 0)\\
&[e^{2k}\otimes (t^{2r+1})^{\otimes 2^m}] \mapsto (\sum_{i=0}^{2r+1}\binom{2r+1}{i}u^{k+2^{m-2}i}\otimes t^{2^m(2r+1)-1-2^{m-1}i}\ ,\ 0)
\end{aligned}
$$

where $k,r\geq 0$ *and* $m\geq 2$.

Proof. We first check that $pr_1\circ\phi$ is as stated. The composite

$$BC_{2^m}\times BC_2 \xrightarrow{1\times i_0} L_mBC_2 \xrightarrow{ev_m} P_mBC_2$$

equals the diagonal $1\times\Delta$ and from chapter 5 we get for $m\geq 2$:

$$\Delta^*(1\otimes t^{\otimes 2^m}) = 1\otimes t^{2^m} + u^{2^{m-2}}\otimes t^{2^{m-1}} + v\otimes t^{2^m-1} + vu^{2^{m-2}}\otimes t^{2^{m-1}-1}$$

Since Δ^* is a ring homomorphism $\Delta^*(1\otimes(t^r)^{\otimes 2^m})$ is the r'th power of the right hand side. We compute this by the binomial formula using that $v^2=0$ and obtain after some reductions

$$\Delta^*(1\otimes(t^r)^{\otimes 2^m}) = \sum_{j=0}^{r}\binom{r}{j}u^{2^{m-2}j}\otimes t^{2^mr-2^{m-1}j} + \binom{r}{1}\sum_{j=0}^{r}\binom{r}{j}vu^{2^{m-2}j}\otimes t^{2^mr-1-2^{m-1}j}$$

This implies the stated result. When $m=1$ we have by Steenrod's formula

$$\Delta^*(1\otimes(t^r)^{\otimes 2}) = \sum_{j=0}^{r}a^j\otimes Sq^{r-j}(t^r) = \sum_{j=0}^{r}\binom{r}{j}a^j\otimes t^{2r-j}$$

implying the stated results. Finally for $m=0$ we have that the composite

$$ES^1\times BC_2 \xrightarrow{1\times i_0} ES^1\times\Lambda BC_2 \xrightarrow{1\times ev} ES^1\times BC_2$$

is the identity thus $1\otimes t^{2s+1}\mapsto 1\otimes t^{2s+1}$ and in the limit we get zero.

Next we consider $pr_2\circ\phi$. The equivalence

$$ES^1\times_{C_{2^m}}(BC_2)_1 \xrightarrow{pr_2} (BC_2)_1/C_{2^m}$$

goes in the wrong direction thus we need an inverse to pr_2^*. There is a map $s: ES^1/C_2\longrightarrow ES^1$ satisfying

$$s([ze]) = z^2s([e])$$

for all $z \in S^1$ and $e \in ES^1$. We construct this simplicially. There is a simplicial map $ES^1 \xrightarrow{s} ES^1$ given by $(z_0, \ldots, z_n) \mapsto (z_0^2, \ldots, z_n^2)$ in degree n. Since S^1 is abelian we have

$$s(z(z_0, \ldots, z_n)) = z^2 s(z_0, \ldots, z_n)$$

The realization $s : ES^1 \longrightarrow ES^1$ is a map satisfying $s(ze) = z^2 s(e)$ and since $s(1e) = s((-1)e)$ it factors through the quotient ES^1/C_2. Define a map

$$f : ES^1/C_2 \longrightarrow ES^1 \times (ES^1/C_2)$$

by $f([e]) = (s([e]), [e])$. It is easy to verify that f is S^1-equivariant and we get a map

$$f : (ES^1/C_2)/C_{2^m} \longrightarrow ES^1 \times_{C_{2^m}} (ES^1/C_2)$$

Clearly $pr_2 \circ f = id$ hence $f^* \circ pr_2^* = id$ and since we know that pr_2^* is an isomorphism we have $f^* = f^* \circ pr_2^* \circ (pr_2^*)^{-1} = (pr_2^*)^{-1}$. We wish to compute the cohomology of the composite

$$g : BC_{2^{m+1}} \xrightarrow{f} ES^1 \times_{C_{2^m}} (BC_2)_1 \xrightarrow{1 \times i_1} L_m BC_2 \xrightarrow{1 \times ev_m} P_m BC_2$$

First consider

$$BC_{2^{m+1}} \xrightarrow{g} ES^1 \times_{C_{2^m}} (BC_2)_1 \xrightarrow{pr_1} BC_{2^m}$$

given by $[e] \mapsto [s([e])]$. It is the reduction map $B(-)^2 : BC_{2^{m+1}} \longrightarrow BC_{2^m}$. By lemma 10.4 we get $g^*(u \otimes 1^{\otimes 2^m}) = 0$ and $g^*(v \otimes 1^{\otimes 2^m}) = v$ To finish the computation of g^* we only need to find $g^*(1 \otimes t^{\otimes 2^m})$. There is a commutative diagram

$$\begin{array}{ccc}
BC_{2^{m+1}} & \xrightarrow{g} & ES^1 \times_{C_{2^m}} (BC_2)^{2^m} \\
{\scriptstyle Bi}\uparrow & & {\scriptstyle Q}\uparrow \\
BC_2 & \longrightarrow & ES^1 \times (BC_2)^{2^m}
\end{array}$$

where $i : C_2 \subseteq C_{2^{m+1}}$ is the inclusion and the bottom map is homotopic to the diagonal map. Since $\Delta^* \circ Q^*(1 \otimes t^{\otimes 2^m}) = a^{2^m}$ we get

$$g^*(1 \otimes t^{\otimes 2^m}) = u^{2^{m-1}}\ ,\ m \geq 1$$

When $m = 0$ we simply have the identity

$$id : ES^1 \times BC_2 \xrightarrow{1 \times i_1} L_0 BC_2 \xrightarrow{1 \times ev} P_0 BC_2$$

Since the limit system are by transfer maps, we get the stated result. $\square$

Lemma 10.4. *Let $s = 2^m$ and let $r : C_{2s} \longrightarrow C_s$ be the reduction map $T \mapsto t$ where T is a generator for C_{2s} and t is a generator for C_s. Then the induced map*

$$r^* : H^*(BC_s) \longrightarrow H^*(BC_{2s})$$

is given by $v \mapsto v$ and $u \mapsto 0$.

Proof. Since r is a group homomorphism, it induces a map in group cohomology

$$r^* : H^*(C_s;\mathbb{Z}) \longrightarrow H^*(C_{2s}; r^{\#}(\mathbb{Z})) \tag{61}$$

The $r^{\#}$ means that the action is via r. Here $\mathbb{Z}$ has trivial C_s action and then $r^{\#}(\mathbb{Z})$ is of cause $\mathbb{Z}$ with trivial C_{2s}-action. Let $W_*(C_k)$ denote the standard resolution of $\mathbb{Z}$ with free C_k-modules. There is a chain map over the identity $F_* : W_*(C_{2s}) \longrightarrow red^{\#}W_*(C_s)$ and it is uniquely determined up to homotopy. The map (61) is induced by

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{Z}[C_s]}(W_*(C_s);\mathbb{Z}) \xrightarrow{''id''} \mathrm{Hom}_{\mathbb{Z}[C_{2s}]}(r^{\#}W_*(C_s); r^{\#}\mathbb{Z})\\ &\xrightarrow{F^*} \mathrm{Hom}_{\mathbb{Z}[C_{2s}]}(W_*(C_{2s}), r^{\#}\mathbb{Z})\end{aligned}$$

We choose F_* as below

$$\begin{array}{ccccccccc}
\mathbb{Z} & \longleftarrow & \mathbb{Z}[C_{2r}] & \xleftarrow{1-T} & \mathbb{Z}[C_{2r}] & \xleftarrow{N} & \mathbb{Z}[C_{2r}] & \xleftarrow{1-T} & \dots \\
{\scriptstyle id}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle 2r}\downarrow & & \\
\mathbb{Z} & \longleftarrow & r^{\#}\mathbb{Z}[C_r] & \xleftarrow{1-t} & r^{\#}\mathbb{Z}[C_r] & \xleftarrow{n} & r^{\#}\mathbb{Z}[C_r] & \xleftarrow{1-t} & \dots
\end{array}$$

where $N = 1+T+\dots T^{2s-1}$ and $n = 1+t+\dots t^{s-1}$ are the norm elements. Passing to mod two coefficients the result follows. $\square$

Theorem 10.5. *The map*

$$\ell(\mathbb{R}P^\infty) \xrightarrow{\phi} H^*(ES^1 \times_{S^1} \Lambda\mathbb{R}P^\infty)$$

is an isomorphism.

Proof. We first prove that it is a surjective map. Since $[1\otimes t^{2r+1}] \mapsto (0,u^r)$ it is enough to show that $pr_1\circ\phi$ is surjective. We must hit $u^k\otimes t^q$ for all $k,q\geq 0$. Assume that q is even $q=2r$. We have $[e^{2k+1}\otimes 1^{\otimes 2}] \mapsto u^k$ hence the elements with $r=0$ are hit. Assume that $u^k\otimes t^{2r}$ are hit for all k and $0\leq r\leq s-1$. Then

$$[e^{2k+1}\otimes (t^s)^{\otimes 2}] \mapsto u^k\otimes t^{2s} + \sum_{j=1}^{[s/2]} \binom{s}{2j} u^{k+j}\otimes t^{2(s-j)}$$

and we can hit the first term since the last sum is hit by induction. Thus all elements with q even are hit. Assume q is odd $q=2r+1$. Since $[e^{2k}\otimes t^{\otimes 2}] \mapsto u^k\otimes t$ the elements with $r=0$ are hit. Assume that the elements $u^k\otimes t^{2r+1}$ are hit for all k and $0\leq r\leq s-1$. Let b be the two valuation of $s+1$. Then $1+s = 2^b(2i+1)$ for an integer i. If $b=0$ we have

$$[e^{2k}\otimes (t^{2i+1})^{\otimes 2}] \mapsto u^k\otimes t^{2s+1} + \sum_{j=1}^{i}\binom{i}{j} u^{k+j}\otimes t^{4i+1-2j}$$

and the last sum is hit by induction. If $b \geq 1$ we use

$$[e^{2k} \otimes (t^{2i+1})^{\otimes 2^{b+1}}] \mapsto u^k \otimes t^{2s+1} + \sum_{j=1}^{i} \binom{i}{j} u^{k+2^{b-1}j} \otimes t^{2^{b+1}(2i+1)-1-2^b j}$$

Again the last sum is hit by induction. We have proved that ϕ is surjective. To complete the proof we check that the dimensions in corresponding degrees are the same. In degree $2s-1$ on the right hand side, we have the span of the elements $(u^k \otimes t^{2(s-k)-1}, 0)$ where $k = 0, 1, \dots s-1$. Thus the dimension is s in this degree. In the corresponding degree $2s$ on the left hand side we have the span of the elements $e^{2k} \otimes (t^{2r+1})^{\otimes 2^m}$ with $2k + 2^m(2r+1) = 2s$. Given k there are exactly one choice for m and r. m must be the two valuation of $2(s-k)$ and $2r+1$ must equal $2^{1-m}(s-k)$. Since $k = 0, 1, \dots, s-1$ the dimension of the left hand side in degree $2s$ is also s. In degree $2s$ of the right hand side the dimension is $s+2$ and this equals the dimension on the left hand side in degree $2s+1$. $\square$

11. Problems with the approximating functor.

As seen in the last two chapters the map (39) is an isomorphism when the space X is $\mathbb{C}P^\infty$ or $\mathbb{R}P^\infty$. There are however also examples where the approximation does not work very well. When $X = K(\mathbb{F}_2, 2)$ one can count the dimensions in corresponding degrees and see that the map (39) is not surjective e.g. $dim(H^3((\Lambda X)_{hS^1})) = 2$ and $dim(\ell^4(X)) = 1$. Another example where we have computed the map (39) is when $X = BT$ where T denote the torus $T = S^1 \times S^1$. It is of course related to the $\mathbb{C}P^\infty$ case since $BT = BS^1 \times BS^1 = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. The free loop space $\Lambda BT = \Lambda\mathbb{C}P^\infty \times \Lambda\mathbb{C}P^\infty$ has the following cohomology:

$$H^*(\Lambda BT) = \Lambda(e, e') \otimes \mathbb{F}_2[c, c']$$

where the degrees of e and e' are one and the degrees of c and c' are two. The cohomology of $(\Lambda BT)_{hS^1}$ can be found using the Serre spectral sequence of the fibration

$$\Lambda BT \xrightarrow{j} ES^1 \times_{S^1} \Lambda BT \xrightarrow{\pi} BS^1$$

The E_2 term equals $H^*(BS^1) \otimes H^*(\Lambda BT)$ and using the two projections

$$BT = \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{pr_\nu} \mathbb{C}P^\infty \ , \ \nu = 1, 2 \tag{62}$$

one finds $d_2(e) = d_2(e') = 0$, $d_2(c) = eu$ and $d_2(c') = e'u$. Introducing the names $\xi = ec$, $\xi' = e'c'$, $b = c^2$, $b' = (c')^2$ and $\eta = ec' + e'c$ one gets the following E^3-term

$$E_3^{**} = \Lambda(e, e', \xi, \xi', \eta) \otimes \mathbb{F}_2[b, b', u]/I$$

Here I is the ideal generated by the elements $e\xi$, $e'\xi'$, $e\eta + e'\xi$, $e'\eta + e\xi'$, $bee' + \xi\eta$, $b'ee' + \xi'\eta$, eu, $e'u$ and ηu. By the constant loop section of π and the two projections (62) one sees that all higher differentials on the algebra generators are zero thus E_3=E_∞. Thus the additive structure of the cohomology of the total space is determined. One can apply the constant loop section and the two inclusion maps of $\mathbb{C}P^\infty$ in $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ together with this and obtain a description of the algebra structure of the total space.

$$H^*(ES^1 \times_{S^1} \Lambda BT) = \Lambda(e, e', \xi, \xi', \eta) \otimes \mathbb{F}_2[b, b', u]/I$$

where I is as above.

We need also determine the lowest transfer map Tr_0^1. From the $\mathbb{C}P^\infty$ case one obtains $Tr_0^1(c) = ae$ and $Tr_0^1(c') = ae'$. By Frobenius reciprocity we have $eTr_0^1(cc') = Tr_0^1(ecc') = \xi Tr_0^1(c') = ae'\xi$ especially $Tr_0^1(cc') \neq 0$. Because of the dimension we have

$$Tr_0^1(cc') = \lambda_1 a\xi + \lambda_2 a\xi' + \lambda_3 a\eta + \lambda_4 a^4$$

for some constants $\lambda_\nu \in \mathbb{F}_2$. Since $Tr_0^1(cc') = Tr_0^1 \circ ev_0^*(1 \otimes cc')$ we see that $aTr_0^1(cc') = 0$ and $\lambda_1 = \lambda_2 = \lambda_4 = 0$. Since $Tr_0^1(cc')$ was non-zero we have $\lambda_3 = 1$ and $Tr_0^1(cc') = a\eta$.

For each m the evaluation $ev_m^* : H^*P_mX \longrightarrow H^*L_mX$ is an algebra homomorphism. Thus the computation of these maps when $X = \mathbb{C}P^\infty$ gives these maps when $X = BT$. Passing to the limit we find that the map (39) is given by

$$[1 \otimes c^{2i+1}(c')^{2j}] \mapsto eb^i(b')^j \ , \ [1 \otimes c^{2i}(c')^{2j+1}] \mapsto e'b^i(b')^j$$

$$[1 \otimes c^{2i+1}(c')^{2j+1}] \mapsto \eta b^i(b')^j$$

$$[e^{2j+1} \otimes (c^r(c')^s)^{\otimes 2}] \mapsto u^j b^r(b')^s + u^{j+1}\xi\xi' b^{r-1}(b')^{s-1}$$

$$[e^{2j} \otimes (c^{2r+1}(c')^{2s})^{\otimes 2^m}] \mapsto u^j \xi b^{2^{m-1}(2r+1)-1}(b')^{2^m s}$$

$$[e^{2j} \otimes (c^{2r}(c')^{2s+1})^{\otimes 2^m}] \mapsto u^j \xi' b^{2^m r}(b')^{2^{m-1}(2s+1)-1}$$

$$[e^{2j} \otimes (c^{2r+1}(c')^{2s+1})^{\otimes 2^m}] \mapsto u^j b^{2^m r}(b')^{2^m s}(\xi' b^{2^{m-1}}(b')^{2^{m-1}-1} + \xi b^{2^{m-1}-1}(b')^{2^{m-1}})$$

We see that the map is not surjective since e.g. elements of the form $ee'b^r(b')^s$ are not hit.

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ADDENDUM TO THE PH.D. THESIS: CYCLIC CONSTRUCTIONS IN MOD TWO COHOMOLOGY

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