Kähler Quantization and Hitchin Connections

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Preface

This thesis presents the outcome of my four years as a PhD student at the Department of Mathematical Sciences, Aarhus University. The subject of the thesis is quantization and lies at the boundary of mathematical physics. The main body of work is based on three papers [AGL], [AG] and [Gam], but the exposition also includes some results and ideas not presented elsewhere.

I would like to take this opportunity to thank some of the people who have had an important and positive impact on my time as a PhD student.

Above all, I would like to thank my advisor Jørgen Ellegaard Andersen. His insight and everlasting enthusiasm has been an inspiration, and I am grateful for all his support and his belief in my abilities.

During the spring of 2009, I had the pleasure of visiting the Mathematical Institute, University of Oxford, and I would like to thank Nigel Hitchin for his great hospitality and many interesting discussions. Also, in the fall of 2009, I spent an exciting semester at the Harvard Mathematics Department, and I am grateful to Peter Kronheimer for making this possible and for his time and interest in my work.

Finally, I would like to thank my friends and family for their support and for keeping me sane throughout this mathematical endeavor.

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Introduction

The topic of this thesis relates to the fundamental problem in physics of giving a sensible quantum-theoretical description of a physical system which explains the classical perception of that system. A general procedure for producing such a quantum description from its classical description is called a quantization scheme.

The need for quantization stems from the well-known fact that the laws of classical physics break down on atomic and subatomic scales. Strangely, measurements on this scale tend to be non-deterministic and allow only a discrete set of possible outcomes, and consequently a different 'quantum' description of a system is needed to explain and predict its behaviour.

Unfortunately, quantization is no straightforward procedure. There is no such thing as *the* corresponding quantum system of a classical system, and some quantum systems have no meaningful classical counterpart. In fact, it can be proved that in a certain sense no general procedure for quantization can exist. This has lead to a wealth of different quantization methods, each with their individual flaws and favors, and has spawned an entire field in mathematics, engaged in the study of the mathematical structure of these quantization schemes.

In this thesis, we shall be working with two different quantization methods called geometric quantization and deformation quantization. One of the characteristic features of geometric quantization is the need for a choice of an auxiliary structure — a complex structure on the classical phase space — to define it. From a physical perspective, the result of the quantization should, however, not depend on this structure, and this is where the notion of a Hitchin connection comes into the picture as a gadget for relating the quantizations produced from different choices of auxiliary structure. The construction and study of Hitchin connections forms a major part of this thesis.

Although not as ubiquitous in deformation quantization, the choice of an auxiliary complex structure can also be helpful when constructing this type of quantization. In fact, we shall present a general and explicit constuction of deformation quantization from a choice of complex structure. Once again, the final outcome of quantization should not depend on this choice, and we shall investigate this problem by similar techniques and define the notion of a Hitchin connection in the context of deformation quantization.

Scientific Context

Let us increase the level of mathematical detail and give a brief overview of the origins and development of deformation quantization on Kähler manifolds as well as the Hitchin connection.

Deformation Quantization on Kähler Manifolds

One of the first to study deformation quantization on Kähler manifolds was Berezin [Ber], who wrote down integral formulas for a star product in this setting, but had to make severe assumptions on the Kähler manifold.

Years later, the fundamental questions of existence and classification of formal deformation quantizations on a general Kähler manifold were solved by Karabegov [Kar1], who proved that deformation quantizations with separation of variables are in bijective correspondence with closed, formal $(1,1)$-forms.

A geometric but implicit construction of a deformation quantization on any compact Kähler manifold was given by Schlichenmaier [Sch], who used the asymptotic expansion of products of Toeplitz operators in geometric quantization to construct the star product.

The first explicit construction of a deformation quantization on a general Kähler manifold was given by Reshetikhin and Takhtajan [RT1], who interpreted Berezin's integral formulas formally and studied their asymptotic behavior. This yielded an explicit formula for a star product in terms of Feynman graphs interpreted as differential operators. However, the graphs produced by the expansion of Berezin's integrals have relations among them, expressing fundamental identities on the Kähler manifold. Moreover, the expansion produces disconnected graphs which prevent the star product from being normalized.

Inspired by the work of Reshetikhin and Takhtajan, and using Karabegov's classification, we shall give an explicit, combinatorial formula, in terms of Feynman graphs, for any deformation quantization with separation of variables on a Kähler manifold.

The Hitchin Connection

In [Wit], it was proposed that quantum Chern-Simons theory should form the two-dimensional part of a topological quantum field theory (TQFT) in 2+1 dimensions. This leads to the study of geometric quantization of the moduli space $\mathcal{M}$ of flat $\mathrm{SU}(n)$-connections on a surface Σ. This moduli space has a natural symplectic structure ω and admits a prequantum line bundle, which is a Hermitian line bundle $\mathcal{L}$ with a compatible connection whose curvature is given by the symplectic form. The Teichmüller space $\mathcal{T}$ of the surface Σ parametrizes complex structures on the moduli space, so for each point $\sigma \in \mathcal{T}$ and each natural number k, called the level of quantization, we have the quantum state space of geometric quantization, which is the space $\mathcal{Q}_k(\sigma) = H^0(\mathcal{M}_\sigma, \mathcal{L}^k)$ of holomorphic sections of the k'th tensor power of the prequantum line bundle. These form the fibers of a vector bundle $\hat{\mathcal{Q}}_k$ over $\mathcal{T}$, called the Verlinde bundle, and it was shown independently by Hitchin [Hit] and Axelrod, Della Pietra and Witten [ADW] that this bundle admits a natural projectively flat connection, which we shall call the Hitchin connection. Consequently, the quantum spaces associated with different complex structures are identified, as projective spaces, through the parallel transport of this connection.

The mapping class group Γ of the surface acts by symplectomorphisms on the moduli space, and this action lifts to an action on the prequantum line bundle. This gives an action of Γ on the Verlinde bundle, covering the action of Γ on the Teichmüller space. The Hitchin connection is equivariant with respect to the action of Γ, and consequently one gets a family of projective representations, called the quantum representations, of the mapping class group on the covariantly constant sections of the projectivized Verlinde bundle.

By studying the asymptotic relationship between Toeplitz operators and the Hitchin connection, Andersen [And3] has proved that the quantum representations are asymptotically faithful. Moreover, he has applied similar techniques to prove that the mapping class group does not have Kazhdan's property (T).

Andersen [And1] also extended his asymptotic analysis of the relationship between the Hitchin connection and the Toeplitz operators to higher orders, which lead him to define the notion of a formal Hitchin connection. The asymptotics of products of Toeplitz operators give rise to the Berezin-Toeplitz formal deformation quantization, and Andersen noticed that the formal Hitchin connection can be used to identify these deformation quantizations and obtain a mapping class group equivariant deformation quantization on the moduli space, provided that certain cohomology groups of the mapping class group vanish.

As part of his work, Andersen [And1] constructed the Hitchin connection in the bundle of quantum spaces arising by geometric quantization on a general symplectic manifold sharing certain properties with the moduli space. We shall generalize his construction to metaplectic quantization and calculate the curvature of the Hitchin connection. Moreover, we shall study the question of unitarity of the Hitchin connection and give some asymptotic results in this direction. Finally, we shall develop the formal Hitchin connection from a meteplectic perspective and give explicit expressions for it.

Summary

Let us give an outline of the structure and results of the thesis.

Chapter 1

This first chapter serves as an easy start by recalling some of the basic concepts from complex differential geometry. The purpose is to fix conventions and familiarize the reader with notation.

Chapter 2

The second chapter discusses the concept of quantization in general terms and introduces the notions of geometric quantization and deformation quantization.

Geometric quantization is presented using the three steps of prequantization, complex structure and metaplectic correction. Afterwards, the notion of a Hitchin connection is loosely discussed. The concept of metaplectic quantization is central to the thesis as one of the main results is the construction of a Hitchin connection in this setting.

The chapter then moves on to define the notion of formal deformation quantizations and discuss their classification on symplectic manifolds. The significance of a complex structure to support construction of deformation quantization is briefly touched upon, as is the notion of a formal Hitchin connection.

Chapter 3

In this chapter, we study deformation quantization on Kähler manifolds. We start by recalling Karabegov's classification of star products with separation of variables on a Kähler manifold. Then we prove the first main result of the thesis (Theorem 3.10) which gives an explicit combinatorial formula for any deformation quantization with separation of variables from its classifying Karabegov form.

Chapter 4

This chapter gives the definition and some af the main properties of Toeplitz operators in geometric quantization. In particular, we recall how the Berezin-Toeplitz star product arises through the asymptotic expansion of products of Toeplitz operators, and we discuss its properties and classification.

Furthermore, we recall the basics of differential operators on the quantum spaces of geometric quantization. We prove that a differential operator acts as a Toeplitz operator when followed by the projection back onto the quantum space, and we give an explicit formula for this Toeplitz operator.

Chapter 5

The fifth chapter lays the foundation for our construction of the Hitchin connection by investigating families of Kähler structures on a symplectic manifold. We show how a number of structures vary with the Kähler structure and discuss the important implications of two conditions, called holomorphicity and rigidity, on a family of Kähler structures.

Chapter 6

In Chapter 6, we finally introduce the Hitchin connection. We start by reviewing the explicit differential-geometric construction by Andersen [And1] in standard geometric quantization. Then we give an analogous construction of a Hitchin connection in metaplectic quantization (Theorem 6.10), which is one of the main results of the thesis. The assumptions of this theorem are fewer than needed in standard geometric quantization, but it does not give a completely explicit formula for the connection.

Whenever the conditions for the existence of the Hitchin connection in standard geometric quantization are met, we can also give an explicit formula for the Hitchin connection in metaplectic quantization (Theorem 6.12). Furthermore, we prove that the two variants of geometric quantization can be related and that the Hitchin connections agree (Theorem 6.14).

Towards the end of the chapter, we calculate the curvature of the Hitchin connection in metaplectic quantization, and we prove that it is projectively flat whenever the symplectic manifold has no holomorphic vector fields (Theorem 6.22).

Finally, we investigate the question of unitarity of the Hitchin connection, and we prove that if the connection is projectively flat, then it is asymptotically projectively unitary to any order (Theorem 6.25).

Chapter 7

In this chapter, we apply the ideas of the Hitchin connection to deformation quantization. We define the notion of a formal Hitchin connection for a family of deformation quantizations, and discuss how the parallel transport of such a connection can be used to relate the various deformation quantizations in the family.

Then we show how to construct a formal Hitchin connection for the Berezin-Toeplitz family of deformation quantizations by studying the asymptotics of covariant derivatives of Toeplitz operators with respect to the Hitchin connection in geometric quantization. We consider the Toeplitz operators in standard geometric quantization first and do the same analysis, with fewer details, for the metaplectic case, which is simpler in many ways.

Chapter 8

The final chapter applies the results of the thesis to the moduli space of flat connections on a surface. This is not only the most prominent example of application but also the setting in which many of the ideas were originally developed.

We give the definition of the moduli space and the main structures it carries, and then we apply our results on the Hitchin connection to the quantization of the moduli space. In particular, we prove that the Hitchin connection is projectively flat in this setting (Theorem 8.1). Some comments on the quantum representations of the mapping class groups are also made.

CHAPTER **1**

Complex Differential Geometry

In this first chapter, we shall briefly review some basic definitions and results from complex differential geometry. The material covered is fairly standard, and we include it with the purpose of fixing conventions and establishing notation for later chapters.

1.1 Almost Complex Structures

Consider a smooth manifold M of even dimension $2m$. An *almost complex structure* on M is a smooth section J of the endomorphism bundle $\mathrm{End}(TM)$ of the tangent bundle satisfying $J^2 = -\mathrm{Id}$. This turns the tangent bundle into a complex vector bundle TM_J, where multiplication by i is given by J. Therefore, any almost complex manifold must be even-dimensional and has a canonical orientation.

If we consider the complexified tangent bundle $TM_{\mathbb{C}}$, the complex linear extension of J induces a decomposition,

$$TM_{\mathbb{C}} = T'M_J \oplus T''M_J, \tag{1.1}$$

into the bundles of fiberwise eigenspaces for J, corresponding to the eigenvalues i and $-i$, respectively. The decomposition is explicitly given by the projections onto each summand,

$$\pi_J^{1,0} = \tfrac{1}{2}(\mathrm{Id} - iJ) \qquad \text{and} \qquad \pi_J^{0,1} = \tfrac{1}{2}(\mathrm{Id} + iJ).$$

We use the notation $X = X'_J + X''_J$ for the decomposition of a vector field on M.

Conjugation on $TM_{\mathbb{C}}$ identifies $T'M_J$ and $T''M_J$ as real vector bundles. Also, the projection $\pi_J^{1,0}$ gives an isomophism between the complex vector bundles TM_J and $T'M_J$.

The almost complex structure J acts on the cotangent bundle TM^* by $(J\alpha)X = \alpha(JX)$, and as above we get a decomposition, $TM^*_{\mathbb{C}} = T'M^*_J \oplus T''M^*_J$, into the bundles of eigenspaces. It is easily seen that $T'M^*_J$ is the subbundle of $TM^*_{\mathbb{C}}$ consisting of forms that vanish on $T''M_J$. Likewise, $T''M^*_J$ is the subbundle of forms that vanish on $T'M_J$.

The splittings of $TM_{\mathbb{C}}$ and $TM^*_{\mathbb{C}}$ induce splittings of the tensor bundles of $TM_{\mathbb{C}}$ into direct sums of tensorproducts of the eigensubbundles of $TM_{\mathbb{C}}$ and $TM^*_{\mathbb{C}}$. In particular, if we let $\bigwedge^{p,q} TM_J^* = \bigwedge^p T'M_J^* \otimes \bigwedge^q T''M_J^*$, then we get a decomposition,

$$\bigwedge^k TM_{\mathbb{C}}^* = \bigoplus_{p+q=k} \bigwedge^{p,q} TM_J^*,$$

which induces a splitting of the complex-valued differential forms,

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega_J^{p,q}(M),$$

into the spaces $\Omega_J^{p,q}(M) = C^\infty(M, \bigwedge^{p,q} TM_J^*)$ of complex valued differential forms of type (p,q). If α is a k-form on M, we denote by $\alpha^{p,q}$ its component of type (p,q).

Of course the bundle $\bigwedge^k TM_{\mathbb{C}}$ splits in a similar fashion, as does the symmetric powers $S^k(TM_{\mathbb{C}}^*)$ and $S^k(TM_{\mathbb{C}})$, and any tensor bundle in general.

The canonical line bundle K_J is defined by

$$K_J = \bigwedge^m T'M_J^*,$$

and will be of particular interest to us. At this point, it is just a complex line bundle, but its structure will become richer as we put more structure on M.

Using the projections

$$\pi_J^{p,q} \colon \Omega^{p+q}(M) \to \Omega_J^{p,q}(M)$$

and the exterior differential d, we can form the operators

$$\begin{aligned} \partial_J &\colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), & \partial_J &= \pi_J^{p+1,q} \circ d \\ \bar\partial_J &\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M), & \bar\partial_J &= \pi_J^{p,q+1} \circ d. \end{aligned}$$

These operators will determine the exterior derivative if the almost complex structure does in fact define a complex structure.

1.2 Complex Structures

A *complex structure* on M is a maximal atlas of smooth charts, $\varphi_j \colon U_j \to U_j' \subset \mathbb{C}^m$, such that every transition function,

$$\varphi_{kj} = \varphi_k \circ \varphi_j^{-1} \colon \varphi_j(U_k \cap U_j) \subset \mathbb{C}^m \to \varphi_k(U_k \cap U_j) \subset \mathbb{C}^m,$$

is holomorphic, in the sense that each coordinate function is holomorphic in each of its variables.

By the Cauchy-Riemann equations, the differentials of the transition functions $d\varphi_{ij} \colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ are complex-linear, viewed as transformations of $\mathbb{C}^m$.

Any complex manifold has a naturally induced almost complex structure on its tangent bundle. For local holomorphic coordinates $z^k = x^k + iy^k$, with corresponding coordinate vector fields X^k and Y^k, this is given by

$$JX^k = Y^k \qquad \text{and} \qquad JY^k = -X^k.$$

Since the transition functions on M are holomorphic, this definition is easily seen to be independent of the coordinates chosen. In this way, the tangent bundle becomes a complex vector bundle. Moreover, the usual coordinates for the tangent bundle have holomorphic transition maps, so TM_J has the structure of a holomorphic vector bundle.

An almost complex structure which is induced by a complex structure is called *integrable*. Amazingly, it turns out that integrability is characterized by a tensorial property. As one can easily verify, the expression

$$N_J(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY], \tag{1.2}$$

defines an anti-symmetric tensor on M, called the *torsion* or the *Nijenhuis tensor* of J. It is straightforward to show that an integrable almost complex structure is torsion-free in the sense that N_J vanishes. The converse statement is the famous theorem of Newlander and Nirenberg [NN].

Theorem 1.1 (Newlander-Nirenberg)**.** *Any torsion-free almost complex structure is induced by a unique complex structure.*

There are several equivalent formulations of integrability. We recall a few of them in the following proposition.

Proposition 1.2. *Let J be an almost complex structure on M. Then the following statements are equivalent*

1. *The Nijenhuis tensor N_J vanishes.*
2. *The bundle $T'M_J$ is preserved by the Lie-bracket.*
3. *The exterior differential decomposes as $d = \partial_J + \bar{\partial}_J$*

As usual, the third of these properties implies the identities

$$\partial_J^2 = 0, \qquad \bar{\partial}_J^2 = 0, \qquad \partial_J\bar{\partial}_J = -\bar{\partial}_J\partial_J. \tag{1.3}$$

In particular, we get a cochain complex

$$\Omega^{p,0}(M) \xrightarrow{\bar{\partial}_J} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}_J} \Omega^{p,2}(M) \xrightarrow{\bar{\partial}_J} \cdots,$$

for each non-negative integer p. The cohomology of this complex is denoted by $H_J^{p,q}(M,\mathbb{C})$ and called the Dolbeault cohomology of M.

Throughout this section, we have been careful to decorate anything depending on the almost complex structure by a subscripted J. In the future, we shall not follow this convention as meticulously, especially when the almost complex structure is clear from the context.

1.3 Symplectic and Poisson Manifolds

A *symplectic structure* on M is a non-degenerate and closed two-form $\omega \in \Omega^2(M)$. As with almost complex manifolds, a symplectic manifold must be of even dimension and has a canonical orientation given by ω^m. This also gives a canonical notion of volume, although the volume form is usually normalized as $\frac{\omega^m}{m!}$.

Since the symplectic form is non-degenerate, it defines an isomorphism $i_\omega \colon TM \to TM^*$ by contraction in the first entry. Using this isomorphism, we define the anti-symmetric bivector field $\tilde{\omega} = -(i_\omega^{-1} \otimes i_\omega^{-1})(\omega)$, which satisfies $\omega{\cdot}\tilde{\omega} = \tilde{\omega}{\cdot}\omega = \mathrm{Id}$. The Hamiltonian vector field of a function f is then defined by $X_f = i_\omega^{-1}(df) = df{\cdot}\tilde{\omega}$.

Notice the way we use a dot to denote contraction of tensors. For instance, the expression $\omega{\cdot}\tilde{\omega}$ denotes contraction of the right-most entry of ω with the left-most vector of $\tilde{\omega}$. This can be useful to keep track of contractions involving anti-symmetric tensors such as the symplectic form.

A symplectic structure gives rise to the Poisson bracket on functions by the formula

$$\{f, g\} = df{\cdot}\tilde{\omega}{\cdot}dg = -\omega(X_f, X_g),$$

which satisfies the Jacobi identity and the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

For this reason, the tensor $\tilde{\omega}$ is also called the *Poisson tensor*.

In general, a *Poisson structure* on a manifold is an anti-symmetric, bilinear map $\{\cdot,\cdot\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying the Jacobi identity and the Leibniz rule. The latter implies that such a Poisson bracket is given, as above, by an anti-symmetric bivector field, called the Poisson tensor. On a symplectic manifold, the Poisson tensor is non-degenerate, but this is not required of a general Poisson manifold. In fact, a Poisson structure is induced from a symplectic structure if and only if the Poisson tensor is non-degenerate.

On a Poisson manifold, we have the identity

$$[X_f, X_g] = X_{\{f,g\}}.$$

This shows that the association $f \mapsto X_f$ defines a homomorphism from the Lie algebra of smooth functions, equipped with the Poisson bracket, to the Lie algebra of Hamiltonian vector fields.

If X is a vector field on M, the *divergence* of X is the unique function δX such that

$$\mathcal{L}_X \omega^m = \delta X \omega^m.$$

Clearly, symplectic vector fields, and in particular Hamiltonian vector fields, are divergence free.

1.3.1 Compatible Almost Complex Structure

If M is a manifold equipped with a symplectic structure ω and an almost complex structure J, then ω and J are said to be *compatible* if the expression

$$g(X, Y) = \omega(X, JY) \tag{1.4}$$

defines a Riemannian metric on M. In other words, the bilinear form g must be symmetric and positive definite.

If J is an almost complex structure, g is a Riemannian metric and ω is a symplectic form, then the triple (J, g, ω) is called *compatible* if the three structures are related by (1.4). Clearly, each of the structures in a compatible triple is determined by the other two.

It is easily shown that symmetry of g is equivalent to J-invariance of ω, and therefore also of g. Consequently, both g and ω have type $(1, 1)$.

As usual, the metric g induces an isomorphism $i_g\colon TM \to TM^*$. This is related to the isomorphism i_ω by $i_\omega = i_g \circ J$. Since the metric and the symplectic form have type $(1,1)$, these isomorphisms interchange types. The inverse metric tensor is defined by $\tilde{g} = (i_g^{-1} \otimes i_g^{-1})(g)$, and it is the unique symmetric bivector field which satisfies $g{\cdot}\tilde{g} = \tilde{g}{\cdot}g = \mathrm{Id}$. The relation between the Poisson tensor $\tilde{\omega}$ and the inverse metric tensor $\tilde{g}$ is of course $\tilde{\omega} = J{\cdot}\tilde{g}$.

The Riemannian metric g induces a Hermitian structure $h^{T'M}$, or simply h, on the eigensubbundle $T'M$ by

$$h(X,Y) = g(X,\bar{Y}),$$

which in turn gives rise to a Hermitian structure h^K on the canonical line bundle K.

In general, we shall denote the Hermitian structure of a Hermitian vector bundle by h with the name of the bundle as a superscript. However, the relevant bundle will typically be clear from the context, and we shall therefore often drop the superscript, as above.

1.3.2 First Chern Class

The expression (1.4) defines an injective map from the space of compatible almost complex structures to the space of Riemannian metrics on M. In fact, one can define a retraction of this map from the space of Riemannian metrics to the space of compatible almost complex structures. Since the space of metrics is convex, the space of compatible almost complex structures is contractible.

Using this, we can define the *first Chern class* of a symplectic manifold by

$$c_1(M,\omega) = c_1(M,J) = -c_1(K_J),$$

where J is any almost complex structure compatible with ω. By integrality of the first Chern class, this definition is independent of J, since the space of almost complex structure is contractible.

The first Chern class is an element of the cohomology group $H^2(M,\mathbb{Z})$. Its reduction modulo 2 is a class $w_2(M) \in H^2(M,\mathbb{Z}_2)$, called the *second Stiefel-Whitney class*, which is independent of the symplectic structure. In other words, the second Stiefel-Whitney class is a topological invariant of M.

In general, if L is any complex line bundle on M, the image of its first Chern class $c_1(L)$ under the homomorphism $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})$ is denoted by $\tilde{c}_1(L)$ and called the *real first Chern class*. Recall that if ∇ is any connection on L, then the real first Chern class is given by

$$\tilde{c}_1(L) = \frac{i}{2\pi}[F_\nabla],$$

where F_∇ is the curvature of ∇.

1.3.3 Metaplectic Structure

Consider the positive Lagrangian Grassmannian L^+M, which is just the space of pairs (p, J_p) where $p \in M$ and J_p is a compatible almost complex structure on the tangent space T_pM. The space L^+M has the structure of a smooth bundle over M, with the obvious projection. Sections $J\colon M \to L^+M$ of this bundle correspond precisely to compatible almost complex structures on M. Therefore, the space of sections is contractible, and the projection $L^+M \to M$ is a homotopy equivalence with any section as homotopy inverse.

At each point $(p, J_p) \in L^+M$, we can consider the one dimensional space $K_{J_p} = \bigwedge^m T'M^*_{J_p}$. These form a smooth bundle K over L^+M, and the pullback by an almost complex structure, in the form of a section $J\colon M \to L^+M$, yields the canonical line bundle K_J associated with the almost complex structure.

A metaplectic structure is a square root of the bundle $K \to L^+M$. More precisely, we define

Definition 1.3. A *metaplectic structure* on the symplectic manifold (M, ω) is a line bundle $\delta \to L^+M$ and a map

$$\psi^\delta\colon \delta^2 \to K,$$

which is an isomorphism of line bundles over L^+M.

Clearly, a metaplectic structure exists if and only if the first Chern class $c_1(K) \in H^2(L^+M, \mathbb{Z})$ is even. But $H^2(L^+M, \mathbb{Z})$ is canonically isomorphic to $H^2(M, \mathbb{Z})$ since the projection $L^+M \to M$ is a homotopy equivalence. Moreover, as noted above, any compatible almost complex structure $J\colon M \to L^+M$ provides a homotopy inverse, inducing an isomorphism $J^*\colon H^2(L^+M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ which is independent of J. By naturality of the first Chern class, we have $J^*c_1(K) = c_1(K_J) = -c_1(M, \omega)$. It follows that $c_1(K)$ is even if and only if the second Stiefel-Whitney class of M vanishes. Thus we have proved

Proposition 1.4. *A symplectic manifold (M, ω) admits a metaplectic structure if and only if the second Stiefel-Whitney class of M vanishes.*

Clearly, a metaplectic structure defines an element $c_1(\delta) \in H^2(M, \mathbb{Z})$, satisfying $2c_1(\delta) = -c_1(M, \omega)$. Moreover, inequivalent choices of metaplectic structures are parametrized by $H^1(M, \mathbb{Z}_2)$.

The important feature of a metaplectic structure is that it provides a canonical choice of a square root of the canonical line bundle, for any compatible almost complex structure on a symplectic manifold. We shall have use for this later when we construct a Hitchin connection in metaplectic quantization.

1.4 Kähler Manifolds

A *Kähler manifold* is a complex, symplectic, and Riemannian manifold such that all three structures are compatible. More precisely, a Kähler manifold is a smooth manifold M equipped with a compatible triple (J, g, ω), where J is an integrable almost complex structure.

The Hermitian metric g on a Kähler manifold is called the *Kähler metric*, and the symplectic form ω is called the *Kähler form*.

Notice that choosing a Kähler structure on a symplectic manifold amounts to choosing a compatible and integrable almost complex structure. This will be used later when we consider families of Kähler structures on a symplectic manifold.

The Levi-Civita connection on a Kähler manifold is the unique connection ∇ on the tangent bundle which is both torsion-free,

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

and compatible with the metric, in the sense that

$$\nabla g = 0 \qquad \text{or} \qquad X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for any vector fields X, Y and Z on M. A crucial fact about Kähler manifolds is that the almost complex structure is parallel with respect to the Levi-Civita connection,

$$\nabla J = 0 \qquad \text{or} \qquad \nabla_X(JY) = J\nabla_X Y. \tag{1.5}$$

This implies that the Kähler form ω is parallel with respect to the Levi-Civita connection, since it is related to g and J, which are both parallel, by (1.4).

Furthermore, (1.5) implies that the Levi-Civita connection preserves types, in the sense that it preserves the subbundles $T'M_J$ and $T''M_J$ of $TM_{\mathbb{C}}$. Consequently, the Levi-Civita connection restricts to a connection $\nabla^{T'M}$ on $T'M$, which is compatible with the Hermitian and holomorphic structure of this bundle. As with Hermitian structures, we shall in general avoid the superscript on connections when the bundle in question is clear from the context.

On a Kähler manifold, we also have the canonical line bundle K. The Hermitian structure and compatible connection on $T'M$ induce a Hermitian structure h^K and a compatible connection ∇^K on K. The canonical line bundle inherits a holomorphic structure from $T'M$, which is of course compatible with the connection.

One of the characteristic properties of a Kähler manifold is the existence of certain special coordinates (see [Wel]).

Proposition 1.5. *Around any point p of a Kähler manifold, there exist complex coordinates $z^1, \dots, z^m$ such that the corresponding coordinate vector fields $Z^1, \dots, Z^m$ satisfy*

$$g(Z^j, \bar{Z}^k) = \delta_{jk} \qquad \text{and} \qquad \nabla Z^j = 0, \tag{1.6}$$

at the point p. Such coordinates are called geodesic coordinates.

1.4.1 Curvature

The *Kähler curvature* is simply the curvature of the Levi-Civita connection

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

It is a two-form with values in the endomorphism bundle $\mathrm{End}(TM)$. A simple consequence of (1.5) is that the curvature endomorphism commutes with J

$$R(X,Y)JZ = JR(X,Y)Z.$$

Thus the endomorphism-part of the curvature preserves types. By the usual symmetries of the Riemannian curvature, we see that $R(JX,JY) = R(X,Y)$, so that R is a (1,1)-form with values in $\mathrm{End}(T'M) \oplus \mathrm{End}(T''M)$.

As usual, the metric can be used to raise or lower indices. In particular, by lowering an index, the curvature can be viewed as a symmetric section of $\bigwedge^{1,1} TM_J^* \otimes \bigwedge^{1,1} TM_J^*$, known as the *curvature tensor*. Alternatively, by raising an index, we get a endomorphism of $\bigwedge^{1,1} TM_J^*$, known as the *curvature operator*.

The *Ricci tensor* r is the symmetric, J-invariant, bilinear form defined by

$$r(X,Y) = \mathrm{Tr}[Z \mapsto R(Z,X)Y],$$

and the associated skew-symmetric (1,1)-form ρ, given by

$$\rho(X,Y) = r(JX,Y),$$

is called the *Ricci form*. Using the symmetries of the Kähler curvature, it can be shown that the Ricci form is minus the image of the Kähler form under the curvature operator $\rho = -R(\omega)$.

The curvature of the canonical line bundle is easily calculated using the general fact that the curvature of the top exterior power of a vector bundle with connection is the trace of the endomorphism-part of the curvature. This tells us that

$$F_{\nabla^K} = \operatorname{Tr} F_{\nabla^{T^*}} = -iR(\omega) = i\rho, \tag{1.7}$$

where the second equality is easily verified using the relationship between the Kähler form and the Kähler metric. It follows that the Ricci form is closed and that the real first Chern class of a Kähler manifold is represented by $\frac{\rho}{2\pi}$.

1.4.2 Divergence

Even though the divergence of a vector field X on a Kähler manifold only depends on the symplectic structure, a simple computation shows that the divergence can be calculated using the Levi-Civita connection by the formula

$$\delta X = \operatorname{Tr} \nabla X,$$

in which the independence of the complex and Riemmanian structures is perhaps not so evident. This formula generalizes to tensors of higher order. For vector fields $X_1, \ldots, X_n$, we define

$$\delta(X_1 \otimes \cdots \otimes X_n) = \delta(X_1) X_2 \otimes \cdots \otimes X_n + \sum_j X_2 \otimes \cdots \otimes \nabla_{X_1} X_j \otimes \cdots \otimes X_n.$$

This defines a map $\delta \colon C^\infty(M, TM^n) \to C^\infty(M, TM^{n-1})$, also called the divergence, which does depend the Riemannian and complex structure. Repeated application of the divergence defines a map $\delta^k \colon C^\infty(M, TM^n) \to C^\infty(M, TM^{n-k})$.

The generalization of divergence to sections of the endomorphism bundle of the tangent bundle will also be convenient. If $\alpha \in \Omega^1(M)$ is a one-form and X is a vector field, we define

$$\delta(X \otimes \alpha) = \delta(X)\alpha + \nabla_X \alpha,$$

which gives a map $\delta \colon C^\infty(M, \operatorname{End}(TM)) \to \Omega^1(M)$.

1.4.3 Hodge Theory and the Ricci Potential

On any complex manifold M, closed forms are locally exact with respect to the $\partial\bar{\partial}$-operator. More precisely, if $\alpha \in \Omega^{p,q}(M)$ is a closed form and $U \subset M$ is some contractible open subset, then there exists a form $\beta \in \Omega^{p-1,q-1}(U)$ such that

$$\alpha|_U = \partial\bar{\partial}\beta.$$

On compact Kähler manifolds, a global version of this statement can be proved using Hodge theory.

Assume for the rest of this section that M is a compact Kähler manifold. Define an inner product on forms by

$$\langle \alpha, \beta \rangle = \int_M g(\alpha, \beta) \frac{\omega^m}{m!}, \tag{1.8}$$

where $g(\alpha, \beta)$ denotes the pointwise inner product on forms induced by the Kähler metric. Let d^* and $\bar{\partial}^*$ be the adjoints of d and $\bar{\partial}$, with respect to the inner product (1.8). These can be given explicitly in terms of the *Hodge star operator*, which is the unique bundle isomorphism

$$*\colon \bigwedge^k TM^* \to \bigwedge^{2m-k} TM^*$$

satisfying

$$\alpha \wedge *\beta = g(\alpha, \beta)\frac{\omega^m}{m!}.$$

In fact, using this operator, the adjoints are given by

$$d^* = - * d * \qquad \text{and} \qquad \bar{\partial}^* = - * \bar{\partial} * . \tag{1.9}$$

The Laplacians Δ and $\bar{\Box}$ are then defined by

$$\Delta = dd^* + d^*d \qquad \text{and} \qquad \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{1.10}$$

Forms in the kernel of Δ are called Δ-*harmonic*, and similarly for $\bar{\Box}$.

The following theorem says that Δ-harmonicity is the same as $\bar{\Box}$-harmonicity on a Kähler manifold (see [Wel]).

Theorem 1.6. *The Laplacians satisfy $\Delta = 2\bar{\Box}$ on any Kähler manifold.*

Using this theorem, it is easily seen that the Laplacian of a function on a Kähler manifold can be calculated by

$$\Delta f = -2i\, \delta X'_f. \tag{1.11}$$

Indeed, the divergence of a vector field is related to the adjoint of the exterior derivative by

$$\delta X = -d^* i_g(X), \tag{1.12}$$

for any vector field X on M, which gives

$$\delta X'_f = -i\delta J X'_f = id^*\bar{\partial} f = i(\partial^* + \bar{\partial}^*)\bar{\partial} f = i\bar{\partial}^*\bar{\partial} f = i\bar{\Box} f = \frac{i}{2}\Delta f,$$

since $\partial^*\bar{\partial} f = -\bar{\partial}\partial^* f = 0$ on a Kähler manifold.

As can be seen by an elementary calculation, using the above definition of the star operator and the formulas (1.9) for the adjoints, the Kähler form is an example of a harmonic form.

It is a classical result (see e.g. [Wel]) that each cohomology class in $H^k(M, \mathbb{C})$ is represented uniquely by a harmonic form. Likewise, each cohomology class in $H^{p,q}(M, \mathbb{C})$ is represented uniquely by a harmonic form. The *harmonic part* α^H of a closed form $\alpha \in \Omega^k(M)$ is the unique harmonic representative of the cohomology class $[\alpha] \in H^k(M, \mathbb{C})$.

By using harmonic representatives, the Dolbeault spaces $H^{p,q}(M, \mathbb{C})$ sit as subspaces of $H^{p+q}(M, \mathbb{C})$, and we get the following Hodge decomposition of the cohomology

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M, \mathbb{C}). \tag{1.13}$$

Further Hodge techniques can be applied to prove (see [Bes])

Proposition 1.7. *For any exact form* $\alpha \in \Omega^{p,q}(M)$*, there is a* $\beta \in \Omega^{p-1,q-1}(M)$ *such that* $\alpha = 2i\partial\bar{\partial}\beta$*.*

We can apply this proposition to the Ricci form ρ, which is a real, closed (1,1)-form on M and consequently differs from its harmonic part ρ^H by a real, exact (1,1)-form. Therefore, we can write

$$\rho = \rho^H + 2i\partial\bar{\partial}F,$$

where $F \in C^\infty(M)$ is a real function, called a *Ricci potential*. Clearly, a Ricci potential is determined up to a constant, by the compactness of M, and consequently it is uniquely determined if we require that its average over M is zero.

Chapter **2**

Quantization

Quantization is concerned with the transition from classical to quantum mechanics. Starting from a classical theory, the aim is to produce a quantum theory which encodes its classical origin in the sense that the classical system can be recovered through a certain limit. Quantization is, however, not a straighforward procedure as there are quantum systems which do not have a meaningful classical limit, and also, different quantum systems might have the same classical limit.

Let us be a bit more precise about the goals of quantization. In the Hamiltonian viewpoint of classical mechanics, the phase space of states of a mechanical system can be described by a symplectic manifold, with observables being the smooth functions on the phase space. The dynamics evolve along the flow of the Hamiltonian vector field of a special function called the Hamiltonian.

On the other hand, the state space of quantum mechanics is a (projective) Hilbert space $\mathcal{Q}$, and the observables are self-adjoint operators on $\mathcal{Q}$.

Ideally, a quantization scheme associates a Hilbert space $\mathcal{Q}$ to a symplectic manifold (M, ω), and a self-adjoint operator $Q(f)$ on $\mathcal{Q}$ to a smooth function f on M, such that the assignment $f \mapsto Q(f)$ is linear and sends the constant function 1 to the identity operator, and such that

$$[Q(f), Q(g)] = i\hbar Q(\{f, g\}). \tag{2.1}$$

Furthermore, the quantization scheme should reproduce the so-called canonical quantization when applied to $\mathbb{R}^{2m}$ with the standard symplectic form. This means that if we give $\mathbb{R}^{2m}$ the standard coordinates of position $q_1, \ldots, q_m$ and momentum $p_1, \ldots, p_m$, then $\mathcal{Q}$ must be $L^2(\mathbb{R}^m, dq)$ and the quantum observables corresponding to position and momentum must be given by

$$Q(q_i)\psi = q_i\psi \qquad \text{and} \qquad Q(p_i)\psi = i\hbar\frac{\partial\psi}{\partial q_i}.$$

Finally, the quantization should also incorporate the symmetries of the classical theory. If there is a natural group of symmetries Γ, acting on M by symplectomorphisms, there should be an unitary action of Γ on $\mathcal{Q}$ so that the quantization of observables is equivariant with respect to the two actions.

In summary, a quantization should define a functor, from the category of symplectic manifolds to the category of Hilbert spaces, which satisfies (2.1) and reproduces canonical quantization on $\mathbb{R}^{2m}$.

Unfortunately, it can be proved that a quantization scheme satisfying all of these properties does not exist (see [AE]). There are several ways of handling this inconvenient fact. One is to restrict the set of quantizable observables to a subset of the smooth functions. However, in the quantization schemes that we will consider, this approach will severely limit the number of quantizable functions. Another approach is based on the principle that (2.1) should only hold asymptotically as h goes to zero and therefore be replaced by

$$\big[Q(f), Q(g)\big] = ihQ(\{f, g\}) + O(h^2) \quad \text{as} \quad h \to 0. \tag{2.2}$$

We shall be working with two quantization schemes based on this idea. The first is called *geometric quantization*, in which the quantum Hilbert space is constructed geometrically as an appropriate space of sections of a line bundle on the classical phase space. The second is *deformation quantization*, where the inherent non-commutative nature of the quantum observables is achieved through a deformation of the algebra of classical observables. In this way, deformation quantization can be viewed as a way of avoiding the need for a Hilbert space of quantum states.

In the following, we give a more detailed overview of these quantization methods. For further information on quantization, we refer to [AE] and [Woo].

2.1 Geometric Quantization

In geometric quantization, the Hilbert space of quantum states arises as sections of a certain Hermitian line bundle over the phase space. We shall present the construction by first introducing the notion of prequantization, which is a clean geometric construction carrying most of the properties required of a quantization.

2.1.1 Prequantization

Consider a classical phase space in the form of a symplectic manifold (M, ω) of dimension $2m$. We shall denote by Γ a group of symmetries, which acts on M by symplectomorphisms.

Definition 2.1. A *prequantum line bundle* over the symplectic manifold (M, ω) is a complex line bundle $\mathcal{L}$ endowed with a Hermitian metric $h^{\mathcal{L}}$ and a compatible connection $\nabla^{\mathcal{L}}$ of curvature

$$F_{\nabla^{\mathcal{L}}} = -i\omega.$$

Even though we shall be considering several different line bundles with connection, we shall almost always consider only one connection on each bundle, and conseqently we just denote the connection by ∇ when there is little or no chance of confusion.

With the symmetry group Γ acting on M, it is often natural to require that this action lifts to an action on the prequantum line bundle $\mathcal{L}$ by bundle maps which preserve the Hermitian structure and connection.

A symplectic manifold admitting a prequantum line bundle is called *prequantizable*. Evidently this is not the case for every symplectic manifold. Indeed, the real first Chern class of a prequantum line bundle is given by $\tilde{c}_1(\mathcal{L}) = \big[\frac{\omega}{2\pi}\big]$, leading us to the following necessary condition for prequantizability, called the *prequantum condition*,

$$\big[\tfrac{\omega}{2\pi}\big] \in \operatorname{Im}\big(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\big). \tag{2.3}$$

This is, in fact, also sufficient to ensure the existence of a prequantum line bundle. When they exist, the inequivalent prequantum line bundles over M are parametrized by $H^1(M, \mathrm{U}(1))$.

For any natural number k, we consider the Hilbert space $\mathcal{P}_k = C^\infty(M, \mathcal{L}^k)$ of quantum states (or rather its completion), with the inner product

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2) \frac{\omega^m}{m!}. \tag{2.4}$$

If $f \in C^\infty(M)$ is a classical observable, the corresponding quantum observable is given by the *prequantum operator*,

$$P_k(f) = \frac{i}{k} \nabla_{X_f} + f,$$

which acts on $\mathcal{P}_k$. It is easily checked that the prequantum operator of a real function is self-adjoint, and that they satisfy the relation

$$\big[P_k(f), P_k(g)\big] = \frac{i}{k} P_k(\{f, g\}). \tag{2.5}$$

The integer k is called the *level* of the quantization. By comparing (2.5) and (2.1), we see that k^{-1} acts as a descretized substitute for $\hbar$, which can only attain a discrete set of values by the prequantum condition.

Prequantization satisfies all the properties required of a quantization, except that it fails to reproduce canonical quantization when applied to $\mathbb{R}^{2m}$. In a sense, it produces a Hilbert space of wave functions which depend on twice as many variables as they should. A standard way around this is to pick an auxiliary polarization on M and consider the space of polarized sections of the line bundle. Several types of polarizations can be applied, but we shall only be working with Kähler polarizations.

2.1.2 Kähler Structure

From now on, we assume that the symplectic manifold admits a Kähler sturcture. If we choose a complex structure J on M which is compatible with the symplectic structure, then this gives M the structure of a Kähler manifold, which we denote by M_J. Since the Kähler form ω has type $(1,1)$, it follows that the connection on the prequantum line bundle $\mathcal{L}$ gives it a holomorphic structure. Therefore, we can define the space of quantum states to be the space of holomorphic sections,

$$\mathcal{Q}_k(J) = H^0(M_J, \mathcal{L}^k) = \{ s \in \mathcal{P}_k \mid \nabla_X s = 0, \ \forall X \in T'' M_J \},$$

which is a subspace of the prequantum space $\mathcal{P}_k$ of smooth sections. If the manifold M is compact, then $\mathcal{Q}_k(J)$ is a finite dimensional space by standard theory of elliptic operators.

This approach resolves the issue with the size of the quantum space when applied to $\mathbb{R}^{2m}$, but unfortunalely the prequantum operators do not in general preserve the space of holomorphic sections. One solution is to reduce the set of quantizable functions. Indeed, the operator P_f preserves $\mathcal{Q}_k(J)$ if and only the function f satisfies

$$[X_f, T'' M_J] \subset T'' M_J,$$

and only such functions will therefore be quantizable. For a real function f, this condition implies that X_f is a Killing vector field of the Kähler metric, and consequently the space of quantizable observables is at most finite-dimensional and often trivial (see [Woo]).

To get more quantizable observables, we shall take a different approach in the spirit of (2.2). The space $\mathcal{Q}_k(J)$ is in fact a closed subspace of $\mathcal{P}_k$ (see [Woo]), and therefore we have the orthogonal projection $\pi_k(J)\colon \mathcal{P}_k \to \mathcal{Q}_k(J)$. If $f \in C^\infty(M)$ is a classical observable, we define the corresponding quantum observable $Q_k(f)_J$ by

$$Q_k(f)_J = \pi_k(J) \circ P_k(f).$$

These operators do not form an algebra, but they satisfy (2.2) (at least if M is compact) in the sense that

$$\left\| [Q_k(f), Q_k(g)] - \frac{i}{k} Q_k(\{f,g\}) \right\| = O(k^{-2}) \quad \text{as} \quad k \to \infty, \tag{2.6}$$

with respect to the operator norm on $\mathcal{Q}_k(J)$. The proof of (2.6) relies on the fact that these operators are Toeplitz operators, which we shall have much more to say about later.

Although this quantization scheme gives a Hilbert space of the right size, it still fails to produce the right answers on basic examples from quantum mechanics. In the end, what really matters is the spectrum of the operators, and if the above procedure is applied to the one-dimensional harmonic oscillator, the quantization yields a spectrum which differs from the correct one by a shift. To deal with this problem, the so-called metaplectic correction is introduced.

2.1.3 Metaplectic Quantization

Recall that the canonical line bundle K_J on the Kähler manifold M_J is a Hermitian holomorphic line bundle, with Hermitian metric h^K_J and compatible connection ∇^K_J of curvature $i\rho_J$, where ρ_J is the Ricci form.

Suppose that M has vanishing second Stiefel-Whitney class and choose a square root δ_J of the canonical line bundle K_J. Then δ_J inherits the structure of a Hermitian holomorphic line bundle, with metric h^δ_J and connection ∇^δ_J of curvature $\frac{i}{2}\rho_J$.

In analogy with the previous section, we consider the infinite-dimensional vector space $\mathcal{P}^\delta_k(J) = C^\infty(M, \mathcal{L}^k \otimes \delta_J)$, and the quantum space of metaplectic quantization is then defined to be the subspace of holomorphic sections,

$$\mathcal{Q}^\delta_k(J) = H^0(M_J, \mathcal{L}^k \otimes \delta_J) = \{s \in \mathcal{P}^\delta_k(J) \mid \nabla_X s = 0,\ \forall X \in T''M_J\}.$$

Again, this is a closed subspace of $\mathcal{P}^\delta_k(J)$, and if M is compact, it is of finite dimension.

The prequantum operator $P^\delta_k(f)_J$, which now obviously depends on the complex structure through the bundle δ_J, and the quantum operator $Q^\delta_k(f)_J$ associated with a classical observable $f \in C^\infty(M)$ are defined by exactly the same formulas in terms of the covariant derivative.

Although metaplectic quantization is the right thing to consider from a physical perspective, a lot of the mathematical work on quantization is done in the simpler case of standard geometric quantization, that is, without the metaplectic correction. On many occasions, we shall work through both approaches and discuss some of the differences and similarities.

2.1.4 The Hitchin Connection

Obviously, the spaces $\mathcal{Q}_k(J)$ and $\mathcal{Q}_k^\delta(J)$ depend on the choice of auxiliary Kähler structure J, which from a physical perspective they should not. Moreover, the action of the symmetry group is not likely to be by automorphisms of the Kähler structure. Rather the group acts on the space of complex structures and permutes the quantum spaces associated with different complex structures.

Suppose that the spaces $\mathcal{Q}_k(J)$, for different choices of J, constitute the fibers of a bundle over the space of Kähler structures. Then one could try to relate the different fibers through parallel transport of some unitary connection. This is indeed the idea behind the Hitchin connection, which will be one of our main objects of study. If the Hitchin connection is flat, then the identification is essentially canonical. In addition, it is natural to require that the Hitchin connection is equivariant with respect to the action the symmetry group Γ. In particular, this would yield a representation of Γ on the space of covariantly constant sections with respect to the Hitchin connection.

Similar ideas apply to the spaces $\mathcal{Q}_k^\delta(J)$, but since these spaces also depend on the choice of square root of the canonical line bundle, the approach calls for a consistent way of choosing such a square root. This is achieved through the use of a metaplectic structure on the symplectic manifold.

Since the final outcome of quantization should not depend on the Kähler structure, we have so far been careful to indicate such depence in the notation throughout the construction. In order to simplify notation and avoid unnecessary clutter, we shall not be as meticulous in the future.

2.2 Deformation Quantization

Instead of constructing a Hilbert space of quantum states, the idea of deformation quantization is to construct a non-commutative deformation of the algebra of classical observables. More precisely, one looks for a family of non-commutative star products $*_h$ on $C^\infty(M)$, parametrized by a real parameter h, which reproduces the usual product of functions when the parameter is zero. By expansion in h, such a family of products would have the form

$$f *_h g = \sum_j C_j(f,g) h^j, \tag{2.7}$$

where we require that $C_0(f,g) = fg$.

In the description of quantization through Hilbert spaces and operators, one can think of the star product as representing the product of quantum operators, so that $Q_f Q_g = Q_{f *_h g}$. Therefore, to comply with (2.2), we must require that

$$C_1(f,g) - C_1(f,g) = i\{f,g\}.$$

There are approaches to quantization which sidestep the Hilbert space completely and define a spectrum of observables using the star product (see [BFF$^+$]).

When considering general schemes for constructing star products, the question of convergence in (2.7) is usually deferred by regarding h as a formal parameter. In this case, the star product becomes a product on the algebra $C_h^\infty(M) = C^\infty(M)[[h]]$ of formal functions, that is, of formal power series with coefficients in smooth functions.

Definition 2.2. A *formal deformation quantization* on a symplectic manifold (M, ω), is an associative and $\mathbb{C}[[h]]$-bilinear product on the space of formal functions $C_h^\infty(M)$,

$$f * g = \sum_j C_j(f, g) h^j,$$

which satisfies

$$C_0(f, g) = fg \qquad \text{and} \qquad C_1(f, g) - C_1(f, g) = i\{f, g\},$$

for all functions $f, g \in C^\infty(M)$.

It is often natural to impose further conditions on a deformation quantization. For instance, the star product is said to be *differential* if the operators C_k are bidifferential operators. Moreover, a star product is *normalized* if $1 * f = f * 1 = f$, for any function f. Equivalently a star product is normalized if $C_j(1, f) = C_j(f, 1) = 0$, for $j \geq 1$, and consequently the phrase *null on constants* is also used.

If the manifold M is equipped with a complex structure, a star product is said to be with *separation of variables* if $f * g = fg$ whenever f is holomorphic or g is anti-holomorphic.

Finally, in the presence of a symmetry group Γ acting on M, it is natural to seek a star product which is equivariant with respect to this action. More generally, the quantization should be functorial, in the sense that the star product should be equivariant with respect to any symplectomorphism.

2.2.1 Equivalence and Classification

An equivalence of two star products is an automorphism of $C_h^\infty(M)$ which sends one star product to the other. It is natural to require an equivalence to be h-linear, which implies that it has the form $A = \sum_j A_j h^j$, where each A_j is an endomorphism of $C^\infty(M)$. Moreover, since star products are deformations of the product on $C^\infty(M)$, the equivalence should restrict to the identity on this space.

Definition 2.3. An *equivalence* of two star products $*$ and $*'$ is a formal operator $A = \mathrm{Id} + \sum_{j\geq 1} A_j h^j$ on $C_h^\infty(M)$ which satisfies

$$A(f * g) = Af *' Ag,$$

for all functions $f, g \in C^\infty(M)$.

Since an equivalence A starts with the identity, the operators A_j do not have to be invertible to ensure invertibility of A.

If the operators A_k are differential operators, the equivalence is said to be *differential*. As proved in [GR], an equivalence of differential star products must in fact be a differential equivalence.

Clearly, the notion of deformation quantization makes sense for a general Poisson manifold. By the famous result of Kontsevich [Kon], any Poisson manifold admits a deformation quantization. Moreover, he gives a classification of equivalence classes of star products on Poisson manifolds by certain formal deformations of the Poisson tensor.

The existence of deformation quantizations on symplectic manifolds was first proved by De Wilde and Lecomte [DWL] through purely cohomological considerations. A more geometric proof of existence was given by Fedosov [Fed]. Using a symplectic connection, which is a torsion-free connection preserving the symplectic form, he constructs a flat (Fedosov) connection on the bundle of formal Weyl algebras on the tangent bundle. The covariantly constant sections are identified with formal functions on M, and the Weyl product defines a star product on this space.

Associated with a Fedosov connection is its Weyl curvature, which is a closed formal two form

$$-i\omega\frac{1}{h} + \omega_0 + \omega_1 h + \cdots ,$$

where ω in the first term is the symplectic form of the manifold.

The *characteristic class* of a Fedosov star product $*$ is the formal cohomology class,

$$\mathrm{cl}(*) \in \frac{[\omega]}{2\pi h} + H^2(M, \mathbb{C})[[h]],$$

represented by the Weyl curvature, multiplied by $\frac{i}{2\pi}$, of the Fedosov connection.

Fedosov showed that the star products defined by different Fedosov connections are equivalent if and only if their characteristic classes agree. Moreover, Nest and Tsygan [NT1, NT2] have showed that any differential star product is equivalent to a Fedosov star product, and consequently the notion of characteristic class applies to any differential star product.

2.2.2 Kähler Structure and the Formal Hitchin Connection

In order to construct deformation quantizations, additional structure is often helpful. As with geometric quantization, we shall call upon a Kähler structure to aid us in the construction of star products. On a Kähler manifold, Karabegov [Kar1] has proved that deformation quantizations with separation of variables are completely classified, not only up to equivalence, by formal deformations of the Kähler form. Using this classification, we shall prove a general explicit formula for any deformation quantization with separation of variables in Chapter 3.

On compact Kähler manifolds, a particular deformation quantization can be constructed using geometric quantization and the theory of Toeplitz operators. We shall look further into this so-called Berezin-Toeplitz deformation quantization in Chapter 4. It turns out that this star product fits into Karabegov's classification, and we can therefore give an explicit formula for it.

Once again, a quantization should not depend on the complex structure, and consequently it is natural to seek a way of identifying star products constructed using different complex structures. For the Berezin-Teoplitz star product on a compact Kähler manifold, Andersen [And1] proposed that the star products obtained from different complex structures should be identified by parallel transport of a formal analog of the Hitchin connection. The construction of a star product from a Kähler structure is not equivariant with respect to the action of a symmetry group Γ on the space of Kähler structures, but permutes the different star products. If the formal Hitchin connection is flat and equivariant with respect to the action, then this could be used to define a symmetry equivariant star product on the space of covariantly constant sections.

Chapter 3

Deformation Quantization on Kähler Manifolds

In this chapter, we take a closer look at deformation quantization on general Kähler manifolds. More precisely, we study star products with separation of variables, and we shall give a local formula for any such star product, specified in terms of it classifying Karabegov form.

The formula that we give is described in terms of combinatorial graphs in a way which is greatly inspired by the work of Reshetikhin and Takhtajan [RT1]. Whereas they derived their formula through asymptotic expansion of integrals, our presentation will be purely combinatorial.

We start with a review of Karabegov's classification of star products with separation of variables.

3.1 Karabegov's Classification

Throughout this chapter, we will consider an arbitrary Kähler manifold M. As usual, the Kähler metric is denoted by g, and, for reasons which will become obvious shortly, the Kähler form is denoted by ω_{-1}.

As with the classification of star products on Poisson and symplectic manifolds, the notion of a formal deformation of the structure on the manifold forms a cornerstone in Karabegov's classification.

Definition 3.1. A *formal deformation* of the Kähler form ω_{-1} is a formal two-form

$$\omega = \omega_{-1}\frac{1}{h} + \omega_0 + \omega_1 h + \omega_2 h^2 + \cdots ,$$

where each ω_k is a closed form of type $(1,1)$.

Following [Kar1], we assign a formal deformation of the Kähler form to any star product with separation of variables in the following way. For any set of local holomorphic coordinates $z^1, \ldots, z^m$ on a contractible subset $U \subset M$, there exists a set of formal functions $\Psi^1, \ldots, \Psi^m$ on U,

$$\Psi^k = \Psi^k_{-1}\frac{1}{h} + \Psi^k_0 + \Psi^k_1 h + \Psi^k_2 h^2 + \cdots ,$$

which satisfy the following system of equations

$$\Psi^k * z^l - z^l * \Psi^k = \delta^{kl}. \tag{3.1}$$

Using these functions, a formal two-form on U is defined by

$$\omega|_U = -i\bar{\partial}\Big(\sum_{k=1}^{m} \Psi^k dz^k\Big),$$

and as the notation suggests, this two-form is independent of the solution to (3.1) and the choice of local coordinates. The resulting global two-form ω is a formal deformation of the Kähler form, called the *Karabegov form* of the star product, and is denoted by $\mathrm{Kar}(*)$.

Theorem 3.2 (Karabegov)**.** *Any deformation quantization with separation of variables is completely determined by its Karabegov form.*

We stress that the Karabegov form determines the star product completely and not only up to equivalence. If ω is a formal deformation of the Kähler form, we denote by $*^\omega$ the unique star product such that

$$\mathrm{Kar}(*^\omega) = \omega.$$

Now, a deformation quantization with separation of variables is in particular a differential star product, and, as shown in [Kar2], the characteristic class is given in terms of the Karabegov form by

$$\mathrm{cl}(*) = \frac{[\mathrm{Kar}(*)]}{2\pi} - \frac{\tilde{c}_1(M)}{2}, \tag{3.2}$$

where $\tilde{c}_1(M)$ denotes the real first Chern class of the Kähler manifold. It follows immediately that two star products with separation of variables are equivalent if and only if their Karabegov forms are cohomologous.

3.1.1 Change of Parameter

Later, we will need to consider star products which arise from another star product by a change of parameter. Given a star product $*$, with coefficients C_k, and a formal constant

$$\varphi(h) = h + \varphi_2 h^2 + \varphi_3 h^3 + \cdots \in \mathbb{C}[[h]],$$

we can consider a new star product $*_{\varphi(h)}$ given by the following change of parameter

$$\begin{aligned} f *_{\varphi(h)} g = (f * g)[\varphi(h)] &= \sum_k C_k(f,g)(\varphi(h))^k \\ &= fg + C_1(f,g)h + (C_2(f,g) + \varphi_2 C_1(f,g))h^2 + \cdots . \end{aligned}$$

As in $(f*g)[\varphi(h)]$ above, we use square brackets to denote substitution in the formal parameter. If $*$ is with separation of variables, then so is $*_{\varphi(h)}$, and if $\Psi^1, \dots, \Psi^k$ are solutions to (3.1), then by substituting $\varphi(h)$ for h, we get that

$$\Psi^k[\varphi(h)] *_{\varphi(h)} z^l - z^l *_{\varphi(h)} \Psi^k[\varphi(h)] = \delta^{kl},$$

where the inverse of $\varphi(h)$, which is needed when substituting in the expression for Φ^k, is the inverse in the field of formal Laurent series.

This shows that the Karabegov form is equivariant with respect to parameter change,

$$\mathrm{Kar}(*_{\varphi(h)}) = \mathrm{Kar}(*)[\varphi(h)]. \tag{3.3}$$

It also proves that the form $\mathrm{cl}(*)$, characterizing the equivalence class of the star product, is equivariant with respect to change of parameter.

3.2 A Combinatorial Formula for Star Products

Having recalled the classification of deformation quantizations on a Kähler manifold, we set out to give an explicit formula for any such star product.

More precisely, we fix a formal deformation ω of the Kähler form ω_{-1}, and we shall ultimately present a local formula for the unique star product $*^\omega$ such that $\mathrm{Kar}(*^\omega) = \omega$.

The formula is given by an interpretation of certain combinatorial graphs as bi-differential operators, acting on pairs of functions on M, and we start by introducing the graphs that will constitute the main ingredient in the construction.

3.2.1 Graphs

A directed graph consists of vertices connected by directed edges. If G is a graph, then the set of vertices is denoted by V_G and the set of edges by E_G. The vertex on which an edge is incoming is called the *head*, and the vertex at the other end is called the *tail*.

An edge is a *loop* if it has the same head and tail, and a *cycle* is a path that starts and ends at the same vertex. If two edges connect the same vertices, they are said to be *parallel*.

We shall allow parallel edges in our graphs, but not cycles. In particular, we do not allow any loops. A graph without cycles is said to be *acyclic*, and if it is finite, it must have at least one vertex, called a *source*, with only outgoing edges and at least one *sink* with only incoming edges.

Any graph that we shall consider must also have a distinguished set of numbered vertices, which we shall call *external*. The rest of the vertices are called *internal*. The set of external vertices is denoted by $\mathrm{Ext}(G)$, and the internals by $\mathrm{Int}(G)$. The first external vertex must be a source and the last must be a sink. In general, only external vertices are allowed to be either a source or a sink.

A *weighting* of a graph is an assignment of a weight, which is an integer from the set $\{-1, 0, 1, 2, \ldots\}$, to every internal vertex. All graphs that we shall consider must be weighted, and furthermore, we require the vertices of weight -1 to have degree at least three.

If G is a graph, the weight of a vertex $v \in \mathrm{Int}(G)$ is denoted by $w(v)$. The total weight of G is defined by

$$W(G) = |E_G| + \sum_{v \in \mathrm{Int}(G)} w(v).$$

An isomorphism of two graphs is a bijective mapping of vertices to vertices and edges to edges, preserving the way vertices are connected by edges, and preserving

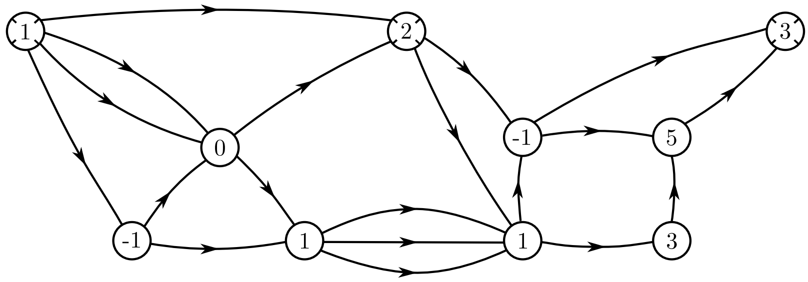

Figure 3.1: A weighted acyclic graph of total weight 27.

the external edges and their numbering. Moreover, an isomorphism should preserve the weights on internal vertices. The set of automorphisms of a graph G is denoted by $\mathrm{Aut}(G)$.

The set of isomorphism classes of finite, acyclic and weighted graphs with n external vertices is denoted by $\mathcal{A}_n$. The subset of graphs with total weight k is denoted by $\mathcal{A}_n(k)$.

3.2.2 Labelled Graphs and Partition Functions

In this section, we define the partition function of a graph. Let us fix a set of holomorphic coordinates $z^1, \dots, z^m$ on an open and contractible subset U of M. On the subset U, we choose a formal potential of the form ω, that is, we choose a formal function

$$\Phi = \Phi_{-1}\frac{1}{h} + \Phi_0 + \Phi_1 h + \Phi_2 h^2 + \cdots ,$$

such that $\omega|_U = i\partial\bar\partial\Phi$. The existence of such a potential is guaranteed by the fact that ω is closed and of type $(1,1)$.

On U, the Kähler metric is given by the matrix with entries

$$g_{p\bar q} = g(Z^p, \bar Z^q) = \frac{\partial^2 \Phi_{-1}}{\partial z^p \partial \bar z^q},$$

where Z^p denotes the coordinate vector field of z^p, as usual. Of course this matrix is invertible, and we denote the entries of the inverse by $g^{\bar q p}$. This is the matrix for the inverse metric tensor $\tilde g$ in local coordinates. With this notation, the Poisson bracket is given by

$$\{f_1, f_2\} = i \sum_{pq} g^{\bar q p}\Big(\frac{\partial f_1}{\partial z^p}\frac{\partial f_2}{\partial \bar z^q} - \frac{\partial f_1}{\partial \bar z^q}\frac{\partial f_2}{\partial z^p}\Big).$$

Having fixed notation, we now define the partition function of a graph. We start by considering graphs with additional structure.

A *labelling* l of a graph $G \in \mathcal{A}_n$ is an assignment of indices to the incoming and outgoing edges at each vertex of the graph. If v is a vertex and e is an incident edge, then the index specified by the labelling is an integer in the set $\{1, \dots, m\}$ and is denoted by $l(v, e)$.

An isomorphism of labelled graphs is of course an isomorphism preserving the labels. The set of labellings of a graph G is denoted by $\mathcal{L}(G)$, and the set of isomorphism classes of labelled graphs with n external edges is denoted by $\mathcal{L}_n$.

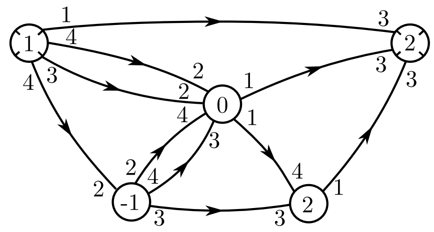

Figure 3.2: A labelled graph.

Consider smooth functions $f_1, \ldots, f_n$ on M, and let us introduce a partition function $\Lambda^l_G(f_1, \ldots, f_n)$ of a labelled graph G with labelling l. For notational convenience, we first define two maps $V^l_{f_1,\ldots,f_n} \colon V_G \to C^\infty(U)$ and $E^l \colon E_G \to C^\infty(U)$, which assign a function to each vertex and edge of the graph, respectively.

Let v be a vertex of G with p incoming and q outgoing edges, and suppose that the incoming edges are labelled with indices $i_1, \ldots, i_p$, and the outgoing edges are labelled with indices $j_1, \ldots, j_q$. If v is the k'th external vertex, then we define

$$V^l_{f_1,\ldots,f_n}(v) = \frac{\partial^{p+q} f_k}{\partial z^{i_1} \cdots \partial z^{i_p} \partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_q}}.$$

If v is an internal vertex of weight w, then we define

$$V^l_{f_1,\ldots,f_n}(v) = -\frac{\partial^{p+q} \Phi_w}{\partial z^{i_1} \cdots \partial z^{i_p} \partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_q}}.$$

Notice that this does not depend on the choice of potential Φ, since internal vertices have at least one incoming and outgoing edge.

Now, suppose that e is an edge from u to v, and let $s = l(u, e)$ and $r = l(v, e)$. Then the function E^l is defined by

$$E^l(e) = g^{\bar{s}r}.$$

Using this, we define the partition function of a labelled graph by

$$\Lambda^l_G(f_1, \ldots, f_n) = \Big(\prod_{v \in V_G} V^l_{f_1,\ldots,f_n}(v) \Big) \Big(\prod_{e \in E_G} E^l(e) \Big).$$

Finally, we can define the partition function of a graph without labelling.

Definition 3.3. For any graph $G \in \mathcal{A}_n$ and smooth functions $f_1, \ldots, f_n$, the *partition function* $\Gamma_G(f_1, \ldots, f_n) \in C^\infty(U)$ is given by

$$\Gamma_G(f_1, \ldots, f_n) = \sum_{l \in \mathcal{L}(G)} \Lambda^l_G(f_1, \ldots, f_n),$$

where the sum is taken over all labellings of G.

Let us give a more concise description of this partition function. First, we introduce the following notation. If $f \in C^\infty(U)$ is a function, we define, for each pair of non-negative integers p and q, a covariant symmetric tensor $f^{(p,q)}$ on U of type (p, q) by

$$f^{(p,q)}(Z^{i_1}, \ldots, Z^{i_p}, \bar{Z}^{j_1}, \ldots, \bar{Z}^{j_q}) = \frac{\partial^{p+q} f}{\partial z^{i_1} \cdots \partial z^{i_p} \partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_q}}.$$

Assign to each vertex $v \in V_G$, with p incoming and q outgoing edges, a symmetric tensor by the following rule. If v is the k-th external vertex, we associate the tensor $f_k^{(p,q)}$, and if v is an internal vertex of weight w, we associate the tensor $-\Phi_w^{(p,q)}$. Then the partition function $\Gamma_G(f_1, \ldots, f_n)$ is given by contracting the tensors associated with each vertex, using the Kähler metric, as prescribed by the edges of the graph. Since the tensors are completely symmetric, this contraction is well-defined.

3.2.3 A Local Star Product

Using the partition functions of graphs, we can define the following formal multi-differential operator

$$D(f_1, \ldots, f_n) = \sum_{G \in \mathcal{A}_n} \frac{1}{|\mathrm{Aut}(G)|} \Gamma_G(f_1, \ldots, f_n) h^{W(G)}.$$

If we define the operators

$$D_k(f_1, \ldots, f_n) = \sum_{G \in \mathcal{A}_n(k)} \frac{1}{|\mathrm{Aut}(G)|} \Gamma_G(f_1, \ldots, f_n),$$

then D is given by the formal power series $D = \sum_k D_k h^k$.

The following important theorem will take a few sections to prove.

Theorem 3.4. *The product*

$$f_1 \bullet f_2 = D(f_1, f_2) = \sum_k D_k(f_1, f_2) h^k \tag{3.4}$$

defines a normalized formal deformation quantization with separation of variables on the coordinate neighborhood U.

Since the only graph with two external vertices and total weight zero is the graph with no edges and no internal vertices, we clearly have

$$D_0(f_1, f_2) = f_1 f_2.$$

Moreover, there is only one graph of total weight one, namely the graph with no internal vertices and only one edge connecting the two external vertices. Therefore,

$$D_1(f_1, f_2) = \sum_{pq} g^{\bar{q}p} \frac{\partial f_1}{\partial \bar{z}^q} \frac{\partial f_2}{\partial z^p},$$

and we get that

$$D_1(f_1, f_2) - D_1(f_2, f_1) = i\{f_1, f_2\},$$

as required of a deformation quantization. It follows that $\bullet$ defines a star product if it is associative, and this is indeed thar hardest part to prove.

If we assume associativity for moment, then we note that the expression for the star product $\bullet$ is with separation of variables since the first external vertex has no incoming edges and the second has no outgoing. Also, note that the star product is normalized since any graph of total weight higher than zero must have edges, and therefore the external vertices must have degree at least one.

Thus the only part of Theorem 3.4 that remains to be proved is associativity of the star product. We will prove this by combinatorial arguments involving certain modifications on graphs. Since the size of the automorphism group of a graph does not behave well under these modifications, the expression for the star product given above is not suitable to work with. Therefore, we need to find a different expression which behaves better when modifying the graphs.

3.2.4 Circuit Graphs

If G is a graph in $\mathcal{A}_n$, a *circuit structure* on G is a total ordering of the incoming as well of the outgoing edges at each vertex of G. This gives rise to a numbering of the incoming as well as the outgoing edges at each vertex. More precisely, if v is a vertex of G with an incident edge e, then the circuit structure specifies a natural number $c(v, e)$. An isomorphism of circuit graphs is an isomorphism which preserves the ordering on the incoming and outgoing edges at each vertex.

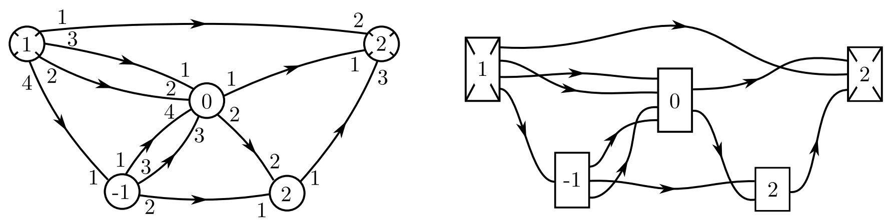

Figure 3.3: Different representations of a circuit graph.

Figure 3.3 shows two ways of representing a circuit structure graphically. The latter, with rectangular vertices, is usually preferred. This also motivates the name circuit structure as it resembles a diagram of an electrical circuit, where a number of chips, with input and output pins, are connected by wires. This analogy is also supported by the fact that our graphs are acyclic.

The set of circuit structures on G is denoted by $\mathcal{C}(G)$, and the set of isomorphism classes of circuit graphs with n external vertices is denoted by $\mathcal{C}_n$.

Very often, we shall be working with graphs equipped with both a labelling and a circuit structure, and we will need to enforce a certain compatibility between the two structures.

If $G \in \mathcal{A}_n$ is a graph equipped with a labelling l and a circuit structure c, we say that l and c are *compatible* if for any vertex v and any two edges e and e' incident to v, with the same orientation, we have that $c(v, e) \leq c(v, e')$ implies $l(v, e) \leq l(v, e')$. In other words, the incoming edges of a vertex should be labelled ascendingly with respect to the ordering given by the circuit structure, and likewise for the outgoing edges.

If G is a graph with labelling l, the set of compatible circuit structures is denoted by $\mathcal{C}(G, l)$. The set of isomorphism classes of labelled graphs with a compatible circuit structure is denoted by $\mathcal{L}_n^C$.

Given a labelled graph, the number of compatible circuit structures will be important to us. To calculate this, we will need some notation.

Recall that a multi-index is an m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$. The length of α is defined to be $|\alpha| = \alpha_1 + \ldots + \alpha_m$, and we define $\alpha! = \alpha_1! \cdots \alpha_m!$. A labelling of a graph assigns two multi-indices to each vertex in a canonical way.

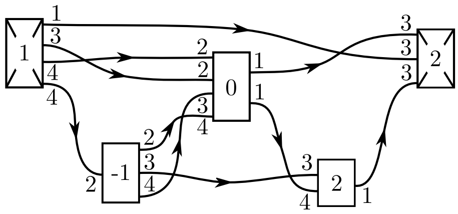

Figure 3.4: A labelled circuit graph.

More precisely, if G is a graph with labelling l, then we have two canonically defined maps $\alpha_l, \beta_l \colon V_G \to \mathbb{N}_0^m$. If v is a vertex of G, then the multi-index $\alpha_l(v)$ counts the number of occurrences of each label among the incoming edges of v. Similarly, the multi-index $\beta_l(v)$ counts the occurrences of each label among the outgoing edges.

Now, given the graph G with labelling l, the number of compatible circuit structures is given by

$$C(G,l) = \prod_{v \in V_G} \alpha_l(v)!\beta_l(v)!.$$

Using this, we can rewrite the formula for the operator D as

$$D(f_1, \dots, f_n) = \sum_{G \in \mathcal{A}_n} \sum_{l \in \mathcal{L}(G)} \sum_{c \in \mathcal{C}(G,l)} \frac{1}{|\mathrm{Aut}(G)|C(G,l)} \Lambda_G^l(f_1, \dots, f_n) h^{W(G)},$$

since the circuit structure does not influence on the value of the partition function.

Suppose that $G \in \mathcal{A}_n$ is any graph with n external edges. If we pick a labelling l and a compatible circuit structure c, then (G, l, c) represents an element of $\mathcal{L}_n^C$. If we choose a different labelling l' and circuit structure c' on G, then (G, l', d') represents the same isomorphism class in $\mathcal{L}_n^C$ if and only if there exists an automorphism of G which sends the labelling l to l' and the circuit structure c to c'. Such an automorphism is uniquely determined, since circuit structures do not have any automorphisms, and thus we have proved the following proposition

Proposition 3.5. *The operator D is given by*

$$D(f_1, \dots, f_n) = \sum_{G \in \mathcal{L}_n^C} \frac{1}{C(G)} \Lambda_G(f_1, \dots, f_n) h^{W(G)},$$

for any functions $f_1, \dots, f_n$.

As we shall often do when the additional structure is clear from the context, we have omitted the labelling from the notation in this proposition.

3.2.5 Associativity of the Star Product

With the alternative expression for the operator D given in Proposition 3.5, we are ready to prove associativity of the local product defined by (3.4). This is an immediate corollary of the following theorem.

Theorem 3.6. *We have*

$$D(f_1, D(f_2, f_3)) = D(f_1, f_2, f_3) = D(D(f_1, f_2), f_3),$$

for any functions f_1, f_2 and f_3.

We shall only prove the first equality of this theorem as the second equality will follow by analagous arguments. To prove Theorem 3.6, we must have a better understanding of the expression $D(f_1, D(f_2, f_3))$. Writing this out in further detail, we have

$$D(f_1, D(f_2, f_3)) = \sum_{G_1 \in \mathcal{L}_2^C} \sum_{G_2 \in \mathcal{L}_2^C} \frac{1}{C(G_1)C(G_2)} \Lambda_{G_1}(f_1, \Lambda_{G_2}(f_2, f_3)) h^{W(G_1)} h^{W(G_2)},$$

and consequently $\Lambda_{G_1}(f_1, \Lambda_{G_2}(f_2, f_3))$ is the crucial part to understand.

Before we prove Theorem 3.6, let us illustrate, with an example, how graphs in the expression for $D(f_1, f_2, f_3)$ arise from $D(f_1, D(f_2, f_3))$.

Example 3.7. Suppose that we have two graphs G_1 and G_2 in $\mathcal{L}_2^C$ as depicted in Figure 3.5.

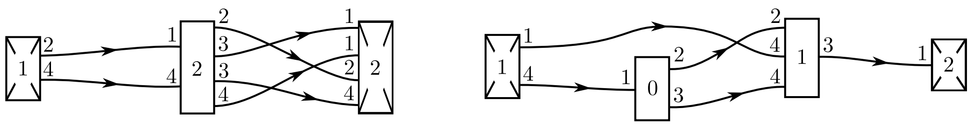

Figure 3.5: The graphs G_1 and G_2.

We think of G_2 as representing a term of the inner D in $D(f_1, D(f_2, f_3))$, and G_1 as representing a term of the outer D. More precisely, we let

$$\hat{f} = \Lambda_{G_2}(f_2, f_3) = \frac{\partial^2 f_1}{\partial \bar{z}^1 \partial \bar{z}^4} \frac{\partial^3 \Phi_0}{\partial z^1 \partial \bar{z}^2 \partial \bar{z}^3} \frac{\partial^4 \Phi_1}{\partial z^2 \partial z^4 \partial z^4 \partial \bar{z}^3} \frac{\partial f_2}{\partial z^1} g^{\bar{1}4} g^{\bar{4}1} g^{\bar{2}2} g^{\bar{3}4} g^{\bar{3}1},$$

and we want to calculate the partition function

$$\Lambda_{G_1}(f_1, \hat{f}) = -\frac{\partial^2 f_1}{\partial \bar{z}^2 \partial \bar{z}^4} \frac{\partial^6 \Phi_2}{\partial z^1 \partial z^4 \partial \bar{z}^2 \partial \bar{z}^3 \partial \bar{z}^3 \partial \bar{z}^4} \frac{\partial^4 \hat{f}}{\partial z^1 \partial z^1 \partial z^2 \partial z^4} g^{\bar{2}1} g^{\bar{4}4} g^{\bar{2}2} g^{\bar{3}1} g^{\bar{3}4} g^{\bar{4}1}.$$

Informally, we have the picture in Figure 3.6 in mind as a graphical representation of this expression.

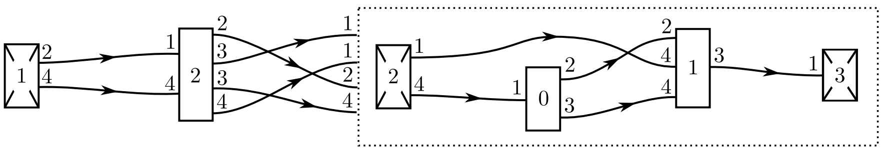

Figure 3.6: Calculating $\Lambda_{G_1}(f_1, \Lambda_{G_2}(f_2, f_3))$.

Since $\hat{f}$ is given by a product, the Leibniz rule says that $\frac{\partial^4 \hat{f}}{\partial z^1 \partial z^1 \partial z^2 \partial z^4}$ is given by a sum where each term represents a certain way of distributing the partial derivatives among the factors.

Let us focus on one such term, say the one where the first and the third partial derivative from the left hit the factor $\frac{\partial^3 \Phi_0}{\partial z^1 \partial \bar{z}^2 \partial \bar{z}^3}$, the second derivative hits the factor $\frac{\partial^2 f_1}{\partial \bar{z}^1 \partial \bar{z}^4}$, and the fourth hits the factor $g^{\bar{3}4}$. That term is then given by

$$\frac{\partial^3 f_1}{\partial z^1 \partial \bar{z}^1 \partial \bar{z}^4} \frac{\partial^5 \Phi_0}{\partial z^1 \partial z^1 \partial z^2 \partial \bar{z}^2 \partial \bar{z}^3} \frac{\partial^4 \Phi_1}{\partial z^2 \partial z^4 \partial z^4 \partial \bar{z}^3} \frac{\partial f_2}{\partial z^1} g^{\bar{1}4} g^{\bar{4}1} g^{\bar{2}2} \frac{\partial g^{\bar{3}4}}{\partial z^4} g^{\bar{3}1}.$$

But partial derivatives of the inverse metric can be easily expressed in terms of partial derivatives of the metric, as in

$$\frac{\partial g^{\bar{3}4}}{\partial z^4} = -\sum_{pq} g^{\bar{3}p}\frac{\partial g_{p\bar{q}}}{\partial z^4}g^{\bar{q}4} = -\sum_{pq} g^{\bar{3}p}\frac{\partial^3\Phi_{-1}}{\partial z^4\partial z^p\partial\bar{z}^q}g^{\bar{q}4}.$$

If we choose particular values, say $p = 1$ and $q = 2$, for the summation variables, then we arrive at

$$-\frac{\partial^3 f_1}{\partial z^1\partial\bar{z}^1\partial\bar{z}^4}\frac{\partial^5\Phi_0}{\partial z^1\partial z^1\partial z^2\partial\bar{z}^2\partial\bar{z}^3}\frac{\partial^4\Phi_1}{\partial z^2\partial z^4\partial z^4\partial\bar{z}^3}\frac{\partial^3\Phi_{-1}}{\partial z^1\partial z^4\partial\bar{z}^2}\frac{\partial f_2}{\partial z^1}g^{\bar{1}4}g^{\bar{4}1}g^{\bar{2}2}g^{\bar{3}1}g^{\bar{2}4}g^{\bar{3}1}$$

as an example of what terms in the expression for $\frac{\partial^4\hat{f}}{\partial z^1\partial z^1\partial z^2\partial z^4}$ look like.

If we insert this into the expression for $\Lambda_{G_1}(f_1,\hat{f})$ above, we get an example of what terms in the expression for $D(f_1, D(f_2, f_3))$ look like. But this particular example can be represented graphically by $\Lambda_G(f_1, f_2, f_3)$, where $G \in \mathcal{L}_3^C$ is the graph shown in Figure 3.7.

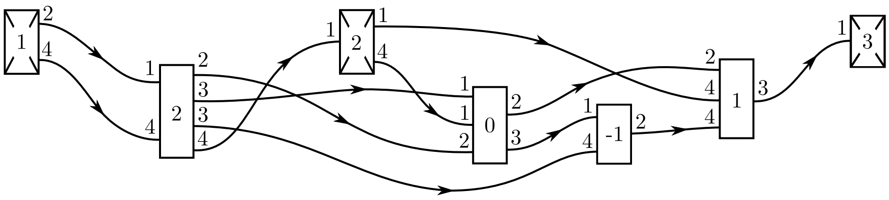

Figure 3.7: A fusion G of the two graphs G_1 and G_2.

With Example 3.7 in mind, let us turn to more general considerations. The graph in Figure 3.7 is an example of a *fusion* of the graphs G_1 and G_2. Let us define this notion more carefully.

Let G_1 and G_2 be two graphs in $\mathcal{L}_2^C$. A fusion of G_1 onto G_2 is a graph $G \in \mathcal{L}_3^C$, with three external vertices, obtained through the following procedure. Cut out the second external vertex of G_1, leaving a collection of labelled edges with loose ends. Connect each of these loose ends, one at a time, to the graph G_2 in one of two possible ways. The first is to connect a loose end to one of the vertices of G_2 and extend the circuit structure at the vertex, in any way compatible with the labelling, to include the newly attached edge. The second possibility os to attach a loose end to one of the edges of G_2. This is done by adding a vertex of weight -1 on the edge, choosing any labelling of the two edges incident to the new vertex, attaching the loose end to the new vertex, and choosing a circuit structure at the vertex. Finally, the first and second external vertices of G_2 will be the second and third external vertex of the fusion, respectively.

Clearly, a fusion of two graphs results in a labelled circuit graph with three external vertices. The set of isomorphism classes of such graphs, which can be obtained from two graphs G_1 and G_2 through a fusion procedure, is denoted by $\mathcal{F}(G_1, G_2)$.

Given a labelled circuit graph $G \in \mathcal{L}_3^C$, it is natural to ask if G can be obtained as a fusion of two graphs G_1 and G_2 in $\mathcal{L}_2^C$. Moreover, it is natural to ask how much information about the graphs G_1 and G_2 is encoded in a fusion.

Given two vertices u and v of a graph, we say that v is a *successor* of u if there exists a directed path from u to v. A crucial observation is that when G_1 is fused to G_2, any vertex in G_2 which is a successor of the first external vertex in G_2 will be a successor of the second external vertex in the fusion. Moreover, vertices that arose by attaching a loose end to an edge of G_2 will also be successors of the second external vertex. On the other hand, none of the vertices of G_1 will succeed the second external vertex in the fusion.

These observations can be used to reconstruct nearly all the information about the structure of the graphs G_1 and G_2 from a fusion of these. Moreover, as we shall see, any labelled circuit graph with three external vertices arises as a fusion.

Suppose that $G \in \mathcal{L}_3^C$ is any labelled circuit graph. We seek two labelled circuit graphs G_1 and G_2 such that $G \in \mathcal{F}(G_1, G_2)$. We can completely determine the isomorphism class of G_2 in $\mathcal{L}_2^C$ by the following procedure. Delete all vertices from G which are not successors of the second external vertex, as well as all edges incident to at least one such vertex. The result may contain vertices of weight -1 and degree 2. These are the remnants of vertices arising during the fusion when a loose edge end is connected to an edge of G_2. Every such vertex is deleted and the resulting two loose ends are spliced, forgetting their labelling. Finally, the second and third external vertices are the only external vertices left, and they will be the first and second external vertices in G_2, respectively.

In a similar way, we can almost determine the isomorphism class of the labelled circuit graph G_1 by deleting all successors of the second external vertex in G, and all edges between two such successors, and then connect all the remaining loose edge ends to a new vertex, which will be the second external vertex of G_1. There is however no canonical way of telling what the circuit structure at the second external vertex should be.

To deal with this ambiguity, we define an equivalence relation on the set $\mathcal{L}_2^C$ of labelled circuit graphs with two external vertices. Consider two graphs G and G' in $\mathcal{L}_2^C$, with labellings l and l' and circuit structures c and c'. We say that these graphs are equivalent, and we write $G \sim G'$, if there exists an isomorphism between G and G' which preserves the labelling at all vertices, and which preserves the circuit structure, except possibly at the second external vertex. In the discussion above, the equivalence class of the graph G_1 is then completely determined.

We summarize our findings in the following proposition

Proposition 3.8. *For any labelled circuit graph $G \in \mathcal{L}_3^C$, there exist two labelled circuit graphs $G_1, G_2 \in \mathcal{L}_2^C$ such that $G \in \mathcal{F}(G_1, G_2)$. Moreover, the equivalence class of G_1 is uniquely determined by G, and so is the isomorphism class of G_2.*

When calculating $D(f_1, D(f_2, f_3))$, we are basically faced with the task of calculating $\Lambda_{G_1}(f_1, \Lambda_{G_2}(f_2, f_3))$ for any two labelled circuit graphs G_1 and G_2. As illustrated in Example 3.7, this is given by a sum where each term can be represented by a fusion of G_1 and G_2.

Now, suppose that during a fusion an edge incident to the second external vertex in G_1 and with label j is attached to a vertex v in G_2, and suppose that v already has k incoming edges with label j. Then, when extending the circuit structure at v to include the newly attached edge, there are $k+1$ ways of placing the new edge in the ordering of the incoming edges. Moreover, suppose that l is the labelling of G_1, and let u be the second external vertex. Then the size of the equivalence class $[G_1]$ is given by $\alpha_l(u)!$.

These observations suffice to realize that

$$\sum_{G\in[G_1]} \frac{1}{C(G)C(G_2)}\Lambda_G(f_1,\Lambda_{G_2}(f_2,f_3)) = \sum_{G\in\mathcal{F}(G_1,G_2)} \frac{1}{C(G)}\Lambda_G(f_1,f_2,f_3).$$

Since $W(G_1)+W(G_2) = W(G)$ if $G\in\mathcal{F}(G_1,G_2)$, we can multiply the left-hand side by $h^{W(G_1)}h^{W(G_2)}$ and the right-hand side by $h^{W(G)}$ and sum over all graphs $G_2\in\mathcal{L}_2^C$ and all equivalence classes $[G_1]$ in $\mathcal{L}_2^C/\sim$ to get

$$\begin{aligned} D(f_1,D(f_2,f_3)) &= \sum_{G_1\in\mathcal{L}_2^C}\sum_{G_2\in\mathcal{L}_2^C} \frac{1}{C(G_1)C(G_2)}\Lambda_{G_1}(f_1,\Lambda_{G_2}(f_2,f_3))h^{W(G_1)}h^{W(G_2)} \\ &= \sum_{[G_1]\in\mathcal{L}_2^C/\sim}\ \sum_{G_2\in\mathcal{L}_2^C}\ \sum_{G\in\mathcal{F}(G_1,G_2)} \frac{1}{C(G)}\Lambda_G(f_1,f_2,f_3)h^{W(G)}. \end{aligned}$$

But as $[G_1]$ runs through all equivalence classes of $\mathcal{L}_2^C/\sim$, and G_2 runs through $\mathcal{L}_2^C$, then Proposition 3.8 tells us that the sets $\mathcal{F}(G_1,G_2)$ partition the set $\mathcal{L}_3^C$, that is, they form a collection of disjoint sets whose union is all of $\mathcal{L}_3^C$. Thus we conclude that

$$D(f_1,D(f_2,f_3)) = \sum_{G\in\mathcal{L}_3^C} \frac{1}{C(G)}\Lambda_G(f_1,f_2,f_3)h^{W(G)} = D(f_1,f_2,f_3).$$

This proves the first equality of Theorem 3.6. The other equality is proved by similar methods, and therefore the theorem is proved. Furthermore, this also proves Theorem 3.4, which follows as an immediate corollary.

3.3 Coordinate Invariance and Classification

In this section, we prove that the local star product of Theorem 3.4 is independent of the coordinates used in its definition. This implies that it defines a global star product on M, and, as we shall see, the Karabegov form of this global star product is given by the formal deformation ω of the Kähler form which was used in its construction. These claims will be simple consequences of the following theorem.

Theorem 3.9. *The local star product $\bullet$ on U has Karabegov form $\omega|_U$.*

Proof. We shall prove that the formal functions $\Psi^r = \partial\Phi/\partial z^r$ satisfy the relation

$$\Psi^r \bullet z^s - z^s \bullet \Psi^r = \delta^{rs}. \tag{3.5}$$

With reference to Section 3.1, this will prove the theorem since

$$\omega|_U = i\partial\bar\partial\Phi = -i\bar\partial(\sum_k \Psi^k dz^k).$$

Clearly, we have $D_0(\Psi^r_{-1},z^s) - D_0(z^s,\Psi^r_{-1}) = 0$ and

$$D_1(\Psi^r_{-1},z^s) - D_1(z^s,\Psi^r_{-1}) = -i\{\Psi^r_{-1},z^s\} = \delta^{rs},$$

so the identity (3.5) is equivalent to the system of identities

$$\sum_{l=-1}^{k-1} D_{k-l}(\Psi_l^r, z^s) = 0, \qquad k \geq 1.$$

To prove these, we define a modification on graphs called a *budding*. If $l > -1$ and $G \in \mathcal{A}_2(k-l)$ is a graph, we define the budded graph $B(G) \in \mathcal{A}_2(k+1)$ by the following procedure. Let u denote the first external vertex of G and convert this to an internal vertex of weight l. Then add a new first external vertex and connect it to u by a single edge.

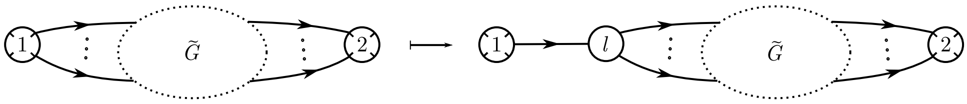

Figure 3.8: A budding of a graph.

We had to exclude the case $l = -1$ because the first external vertex of G might have degree one, in which case the budded graph would not satisfy the rule that internal vertices of weight -1 must have degree at least three. However, if we let $\mathcal{A}_2^1(k+1)$ be the set of graphs with degree one on the first external vertex, and $\mathcal{A}_2^{>1}(k+1)$ be the set of graphs with degree more than one on the first external vertex, then the budding construction defines a map $B\colon \mathcal{A}_2^{>1}(k+1) \to \mathcal{A}_2^1(k+1)$.

We conclude that the budding construction gives a map

$$B\colon \mathcal{A}_2^{>1}(k+1) \cup \bigcup_{l=0}^{k-1} \mathcal{A}_2(k-l) \to \mathcal{A}_2^1(k+1).$$

The inverse map can easily be constructed, so clearly this map is a bijection. Furthermore, it is clear that the budding map preserves the size of the automorphism group.

Now, the crucial property of the budding map is that

$$\Gamma_{B(G)}(\Psi_{-1}^r, z^s) = -\,\Gamma_G(\Psi_l^r, z^s),$$

for any graph G in the domain of B. Since B is a bijection which preserves the size of the automorphism group, this implies that

$$\begin{aligned}
\sum_{l=-1}^{k-1} D_{k-l}(\Psi_l^r, z^s) &= \sum_{l=-1}^{k-1} \sum_{G\in\mathcal{A}(k-l)} \frac{1}{|\mathrm{Aut}(G)|}\Gamma_G(\Psi_l^r, z^s) \\
&= \sum_{G\in\mathcal{A}_2^1(k+1)} \frac{\Gamma_G(\Psi_{-1}^r, z^s)}{|\mathrm{Aut}(G)|} + \sum_{G\in\mathcal{A}_2^{>1}(k+1)} \frac{\Gamma_G(\Psi_{-1}^r, z^s)}{|\mathrm{Aut}(G)|} + \sum_{l=0}^{k-1} \sum_{G\in\mathcal{A}_2(k-l)} \frac{\Gamma_G(\Psi_l^r, z^s)}{|\mathrm{Aut}(G)|} \\
&= \quad 0.
\end{aligned}$$

This proves the theorem. □

Karabegov's classification has the obvious property that restriction of a star product to an open subset corresponds to restriction of the Karabegov form. Therefore, it follows immediately that $\bullet$ is the restriction of the unique star product $*^\omega$ on M with Karabegov form ω. In particular, the explicit expression (3.4) for $\bullet$ must be independent of the local coordinates used on U.

Let us write the partition functions as Γ_G^ω to emphasize the dependence on our initial choice of formal deformation of the Kähler structure ω. With this notation, we can summarize our findings of this chapter in the following theorem.

Theorem 3.10. *The unique formal deformation quantization on M with Karabegov form ω is given by the local formula*

$$f_1 *^\omega f_2 = \sum_{G \in \mathcal{A}_2} \frac{1}{|\mathrm{Aut}(G)|} \Gamma_G^\omega(f_1, f_2) h^{W(G)},$$

for any functions f_1 and f_2 on M.

This theorem implies that the operator D is coordinate independent when applied to two functions, and hence also when applied to three by Theorem 3.6. We conjecture that the general formula for D is coordinate invariant and that there are relations analagous to Theorem 3.6 when applied to a larger collection of functions.

Finally, it would be interesting to use the local combinatorial formula for the star product to give a global formula for the star product in terms of covariant derivatives and the global form ω. Possibly this could be achieved by applying a similar graphical tensor notation.

Chapter 4

Differential and Toeplitz Operators

In this chapter, we return to geometric quantization. As discussed in Chapter 2, the observables in quantization are self-adjoint operators on the space of quantum states. In geometric quantization, this quantum space is formed by sections of a line bundle, and we shall investigate the properties of two particular types of operators on this space, namely differential and Toeplitz operators, and the relationship between the two.

As usual, we consider a Kähler manifold M with a prequantum line bundle $\mathcal{L}$. All results of this chapter are valid for both metaplectic and standard geometric quantization, so let us work in a slightly more general setting which includes both. Let L be any Hermitian holomorphic line bundle on M and consider the space $\mathcal{P}_k^L = C^\infty(M, \mathcal{L}^k \otimes L)$ of smooth sections and the quantum space $\mathcal{Q}_k^L = H^0(M, \mathcal{L}^k \otimes L)$ of holomorphic sections. Metaplectic and standard geometric quantization correspond to the cases where L is a square root of the canonical line bundle and the trivial bundle, respectively. The prequantum and quantum operators are defined by the usual expressions and denoted by $P_k^L(f)$ and $Q_k^L(f)$, respectively.

4.1 Differential Operators

A differential operator acting on $\mathcal{P}_k^L$ is an operator which, with a choice of local coordinates, can be written as a polynomial in the partial derivatives with coefficients in smooth functions. A differential operator has order at most n if, around every point in M, it has a local representation with no terms of degree higher than n in the partial derivatives. The space of differential operators on $\mathcal{P}_k^L$ of finite order is denoted by $\mathcal{D}(M, \mathcal{L}^k \otimes L)$.

The Hermitian holomorphic line bundle $\mathcal{L}^k \otimes L$ is equipped with the Chern connection ∇, and for any number of vector fields $X_1, \dots, X_d$ on M, we consider the inductively defined differential operator on $\mathcal{P}_k^L$,

$$\nabla^n_{X_1,\dots,X_n} s = \nabla_{X_1} \nabla^{n-1}_{X_2,\dots,X_n} s - \sum_j \nabla^{n-1}_{X_1,\dots,\nabla_{X_1} X_j,\dots,X_n} s, \tag{4.1}$$

with the obvious induction start given by the covariant derivative. It is easily verified that this expression is tensorial in the vector fields, so we get a map

$$\nabla^n \colon C^\infty(M, TM^n) \to \mathcal{D}(M, \mathcal{L}^k \otimes L).$$

To any differential operator $D \in \mathcal{D}(M, \mathcal{L}^k \otimes L)$ of order at most n, we can assign the principal symbol $\sigma_P(D) \in C^\infty(M, S^n(TM))$, which is a symmetric section of the n'th tensor power of the tangent bundle. If the principal symbol vanishes, then D is of order at most $n-1$. It is easily shown that for any tensor field $T_n \in C^\infty(M, TM^n)$, the symbol of $\nabla^n_{T_n}$ is given by the symmetrization $\mathcal{S}(T_n) \in C^\infty(M, S^n(TM))$ of T_n.

In general, there is no good notion of lower order symbols of differential operators, but in our case we can use the covariant derivative and the Levi-Civita connection to define symbols of all orders. If $D \in \mathcal{D}(M, \mathcal{L}^k \otimes L)$ is an operator, of order at most n, with principal symbol $\sigma_P(D) = S_n \in C^\infty(M, S^n(TM))$, then the operator $D - \nabla^n_{S_n}$ is of order at most $n-1$, since its principal symbol vanishes. It follows that the operator D can be written uniquely in the form

$$D = \nabla^n_{S_n} + \nabla^{n-1}_{S_{n-1}} + \cdots + \nabla_{S_1} + S_0,$$

where $S_d \in C^\infty(M, S^d(TM))$ is called the *symbol of order* d and gives rise to a map

$$\sigma_d \colon \mathcal{D}(M, \mathcal{L}^k \otimes L) \to C^\infty(M, S^d(TM)).$$

Any finite order differential operator is uniquely determined by the values of these symbol maps.

4.1.1 Adjoints

In case the manifold M is compact, we have the Hermitian inner product

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2) \frac{\omega^m}{m!} \tag{4.2}$$

on the space $\mathcal{P}^L_k$ of smooth sections. We will need to know the adjoints of several differential operators with respect to this inner product. Building on the idea of the covariant derivative as the basic differential operator, we have the following fundamental lemma

Lemma 4.1. *The adjoint of* ∇_X *is given by*

$$(\nabla_X)^* = -\nabla_{\bar{X}} - \delta\bar{X},$$

for any complex vector field $X \in C^\infty(M, TM_{\mathbb{C}})$.

Proof. Recalling that the divergence $\delta\bar{X}$ is the unique function satisfying $\mathcal{L}_{\bar{X}}\omega^n = \delta\bar{X}\omega^n$, we get that

$$\mathcal{L}_{\bar{X}} h(s_1, s_2)\omega^m = h(\nabla_{\bar{X}} s_1, s_2)\omega^m + h(s_1, \nabla_X s_2)\omega^m + h(s_1, s_2)\delta\bar{X}\omega^m,$$

for any smooth sections $s_1, s_2 \in C^\infty(M, \mathcal{L}^k \otimes L)$. By the Cartan formula for the Lie-derivative, this is an exact expression, and therefore integration and Stokes theorem yields

$$\langle (\nabla_X)^* s_1, s_2 \rangle = -\langle \nabla_{\bar{X}} s_1, s_2 \rangle - \langle \delta\bar{X} s_1, s_2 \rangle,$$

which is the desired statement. □

The adjoints of the higher-order operators $\nabla^n_{S_n}$ are not easily calculated in general. However, if $B \in C^\infty(M, TM^2)$ is a bivector field, we introduce the following second-order differential operator

$$\Delta_B = \nabla^2_B + \nabla_{\delta B},$$

and the adjoint of this operator is easily calculated and given by the following lemma.

Lemma 4.2. *The adjoint of Δ_B is given by*

$$\Delta_B^* = \Delta_{\bar{B}},$$

for any complex, symmetric bivector field $B \in C^\infty(M, S^2(TM_{\mathbb{C}}))$.

Proof. If we write $B = \sum_j X_j \otimes Y_j$, then

$$\Delta_B = \sum_j \nabla_{X_j} \nabla_{Y_j} + \nabla_{\delta(X_j) Y_j}.$$

For any complex vector fields X and Y, Lemma 4.1 yields

$$\begin{aligned}(\nabla_X \nabla_Y)^* &= (\nabla_Y)^* (\nabla_X)^* \\ &= (\nabla_{\bar{Y}} + \delta(\bar{Y})(\nabla_{\bar{X}} + \delta(\bar{X})) \\ &= \nabla_{\bar{Y}} \nabla_{\bar{X}} + \nabla_{\bar{Y}} \delta(\bar{X}) + \delta(\bar{Y}) \nabla_{\bar{X}} + \delta(\bar{Y}) \delta(\bar{X}),\end{aligned}$$

and

$$\begin{aligned}(\nabla_{\delta(X)Y})^* &= -\nabla_{\delta(\bar{X})\bar{Y}} - \delta(\delta(\bar{X})\bar{Y}) \\ &= -\delta(\bar{X})\nabla_{\bar{Y}} - \bar{Y}[\delta(\bar{X})] - \delta(\bar{X})\delta(\bar{Y}) \\ &= -\nabla_{\bar{Y}} \delta(\bar{X}) - \delta(\bar{X})\delta(\bar{Y}),\end{aligned}$$

so we conclude that

$$\Delta_B^* = \sum_j \nabla_{\bar{Y}_j} \nabla_{\bar{X}_j} + \delta(\bar{Y}_j) \nabla_{\bar{X}_j} = \Delta_{\bar{B}},$$

since B is symmetric. □

Operators of the type Δ_B will play a central role in our construction of the Hitchin connection in Chapter 6.

Although this entire section on differential operators was formulated for sections of the line bundle $\mathcal{L}^k \otimes L$, the statements are also true when k is equal to zero, and hence for the general line bundle L. In particular, for the trivial line bundle with the trivial connection, the sections are just functions on M, and the operator $\Delta_{\tilde{g}}$ is the Laplace-Beltrami operator, which is equal to minus the Laplace-de Rham operator Δ defined in (1.10).

4.2 Toeplitz Operators

Another important class of operators is the Toeplitz operators. If $f \in C^\infty(M)$ is a smooth function, we define the associated Toeplitz operator $T_k^L(f) \colon \mathcal{P}_k^L \to \mathcal{Q}_k^L$ at level k by

$$T_k^L(f)s = \pi_k^L f s,$$

where $s \in \mathcal{P}_k^L$ is any smooth section, and $\pi_k^L : \mathcal{P}_k^L \to \mathcal{Q}_k^L$ is the orthogonal projection onto the closed subspace $\mathcal{Q}_k^L$. Although this is not indicated in the notation, the Toeplitz operators clearly depend on the complex structure of the manifold.

The definition of the Toeplitz operators does not require the manifold M to be compact, but the work of Bordemann, Meinrenken and Schlichenmaier [BMS, Sch] shows that the Toeplitz operators have powerful asymptotic properties in this case. Henceforth, we shall therefore assume the manifold M to be compact.

The Toeplitz operators restrict to endomorphisms of the finite-dimensional space $\mathcal{Q}_k^L$, which is equipped with the operator norm associated with the Hermitian inner product (4.2). The following theorem shows that the collection of all Toeplitz operators associated with a function represent the function faithfully.

Theorem 4.3. *For any function $f \in C^\infty(M)$, the Toeplitz operators satisfy*

$$\lim_{k\to\infty} \|T_k^L(f)\| = \|f\|_\infty,$$

and this limit is approached from below.

This theorem is proved by Bordemann, Meinrenken and Schlichenmaier [BMS] in the case of trivial L. The general case is considered by Hawkins in [Haw].

The Toeplitz operators do not form an algebra since the product of two Toeplitz operators does not yield a Toeplitz operator in general. Asymptotically, however, the product of two Toeplitz operators can be approximated by Toeplitz operators. We shall have more to say about this when we discuss the Berezin-Teoplitz deformation quantization in Chapter 4.4, but for the moment, we just state the following theorem on the commutator of two Toeplitz operators.

Theorem 4.4. *The Toeplitz operators satisfy*

$$\Big\| \big[T_k^L(f), T_k^L(g)\big] - \frac{i}{k} T_k^L(\{f,g\}) \Big\| = O(k^{-2}) \quad \text{as} \quad k \to \infty,$$

for any smooth functions $f, g \in C^\infty(M)$.

It follows that the Toeplitz operators satisfy (2.2) in the precise meaning of the theorem. The quantization scheme that uses the Toeplitz operators to quantize observables is called Berezin-Toeplitz quantization.

Now, the Toeplitz operators are related to the quantum operators Q_k^L by the following theorem, which we shall prove in the next section.

Theorem 4.5 (Tuynman)**.** *The quantum operators are Toeplitz operators and satisfy*

$$Q_k^L(f) = T_k^L\Big(\frac{1}{2k}\Delta f + f\Big),$$

for any smooth function $f \in C^\infty(M)$.

As an easy consequence of Tyanman's theorem and Theorem 4.4, the quantum operators also satisfy the condition (2.2)

Theorem 4.6. *The quantum operators satisfy*

$$\Big\| \big[Q_k^L(f), Q_k^L(g)\big] - \frac{i}{k} Q_k^L(\{f,g\}) \Big\| = O(k^{-2}) \quad \text{as} \quad k \to \infty,$$

for any smooth functions $f, g \in C^\infty(M)$.

Tuynman's theorem does not require compactness of the manifold M, and in fact, it is a special case of a general relationship between differential operators and Toeplitz operators.

4.3 Toeplitz Operators from Differential Operators

The quantum operators of geometric quantization are given by the prequantum operators, which are differential operators, followed by the projection onto holomorphic sections. By Tuynman's theorem, this yields Toeplitz operators, and in this section, we shall see that in general it is also the case that a differential operator followed by the projection is a Toeplitz operator of a function.

Proposition 4.7. *If $X \in C^\infty(M, T'M)$ is a smooth section of the holomorphic tangent bundle on M, then we have*

$$\pi_k^L \nabla_X s = -T_k^L(\delta X)s,$$

for any smooth section $s \in \mathcal{P}_k^L$.

Proof. The result follows by partial integration. If $s' \in \mathcal{Q}_k^L$ is any holomorphic section, then

$$X[h(s,s')] = h(\nabla_X s, s') + h(s, \nabla_{\bar{X}} s') = h(\nabla_X s, s'),$$

so by taking the Lie-derivative along X of $h(s,s')\omega^n$, we obtain

$$d(h(s,s') i_X \omega^n) = h(\nabla_X s, s')\omega^n + h(s,s')\delta X \omega^n.$$

Finally, integration over M yields

$$0 = \langle \nabla_X s, s' \rangle + \langle \delta X s, s' \rangle,$$

and the result follows. □

Tuynman's theorem follows easily from Proposition 4.7 and the formula (1.11) for the Laplacian. Indeed, for any holomorphic section $s \in \mathcal{Q}_k^L$, we have

$$Q_k^L(f) = \pi_k^L\Big(\frac{i}{k}\nabla_{X'_f} + f\Big)s = \pi_k^L\Big(-\frac{i}{k}(\delta X'_f) + f\Big)s = \pi_k^L\Big(\frac{1}{2k}\Delta f + f\Big)s,$$

which is the statement of Tuynman's theorem.

A generalization of Proposition 4.7 to higher-order differential operators is given by the following theorem.

Theorem 4.8. *If $S_n \in C^\infty(M, S^n(T'M))$ is a smooth tensor field, then we have*

$$\pi_k^L \nabla_{S_n}^n s = (-1)^n T_k^L(\delta^n S_n)s,$$

for any smooth section $s \in \mathcal{P}_k^L$.

Proof. If $X_1, \dots, X_n$ are smooth sections of the holomorphic tangent bundle $T'M$, then the expression (4.1) can also be written as

$$\nabla^n_{X_1 \otimes \cdots \otimes X_n} s = \nabla_{X_1} \nabla^{n-1}_{X_2 \otimes \cdots \otimes X_n} s - \nabla^{n-1}_{\delta(X_1 \otimes \cdots \otimes X_n)} s + \delta X_1 \nabla^{n-1}_{X_2 \otimes \cdots \otimes X_n} s.$$

Applying the projection π_k^L to both sides, Proposition 4.7 implies that

$$\pi_k^L \nabla^n_{X_1\otimes\cdots\otimes X_n} s = -\pi_k^L \nabla^{n-1}_{\delta(X_1\otimes\cdots\otimes X_n)} s,$$

and by induction, we get that

$$\pi_k^L \nabla^n_{X_1\otimes\cdots\otimes X_n} s = (-1)^n \pi_k^L \delta^n (X_1\otimes\cdots\otimes X_n) s,$$

which implies the desired statement. □

The following is an immediate corollary of this theorem.

Corollary 4.9. *If $B \in C^\infty(M, S^2(T'M))$ is a symmetric bivector field, then the operator Δ_B satisfies*

$$\pi_k^L \Delta_B s = 0,$$

for any smooth section $s \in \mathcal{P}_k^L$.

4.4 The Berezin-Toeplitz Star Product

The asymptotic property of Toeplitz operators in Theorem 4.4 is a consequence of a more general statement about the asymptotics of the product of two Toeplitz operators. As previously mentioned, the product of two Toeplitz operators is in general not a Toeplitz operator, but it can be approximated by Toeplitz operators in the sense of the following theorem.

Theorem 4.10. *For any smooth functions $f, g \in C^\infty(M)$, there exists a sequence of uniquely determined functions $C_j^L(f,g)$ such that*

$$\Big\| T_k^L(f) T_k^L(g) - \sum_{j=0}^{N} T_k^L(C_j^L(f,g)) \Big(\frac{1}{k}\Big)^j \Big\| = O\Big(\frac{1}{k^{N+1}}\Big) \quad as \quad k\to\infty, \tag{4.3}$$

for any natural number N. Moreover, the product $\star^L$ on $C_h^\infty(M)$ given by

$$f \star^L g = \sum_{j=0}^{\infty} C_j^L(f,g) h^j$$

defines a formal deformation quantization on M.

The theorem gives an asymptotic expansion of the product of Toeplitz operators and we shall write

$$T_k^L(f) T_k^L(g) \sim \sum_{j=0}^{\infty} T_k^L(C_j^L(f,g)) \Big(\frac{1}{k}\Big)^j, \tag{4.4}$$

or even

$$T_k^L(f) T_k^L(g) \sim T_k(f \star^L g) \Big[\frac{1}{k}\Big]$$

with the precise meaning of (4.3).

The theorem is proved by Schlichenmaier [Sch] in the case where L is the trivial line bundle, but his proof should extend to the general case with minimal modifications. The general case is also studied by Hawkins in [Haw], where he proves that the classifying characteristic class of the star product is given by

$$\mathrm{cl}(\star^L) = \frac{[\omega]}{2\pi h} - \frac{\tilde{c}_1(M)}{2} - \tilde{c}_1(L). \tag{4.5}$$

We shall refer to the star product $\star^L$ as the *twisted Berezin-Toeplitz star product.* In case L is trivial, the star product is just denoted by $\star$, and is commonly known as the *Berezin-Toeplitz star product.*

As proved by Karabegov and Schlichenmaier [KS], the Berezin-Toeplitz star product is with separation of variables, albeit with the roles of holomorphic and antiholomorphic switched. For a general star product $*$, we can define its *opposite* by $f *_o g = g * f$, which defines a star product on the opposite symplectic manifold. As shown in [KS], the Karabegov form of the opposite to the Berezin-Toeplitz star product is given by

$$\mathrm{Kar}(\star_o) = -\omega\frac{1}{h} + \rho, \tag{4.6}$$

where ρ denotes the Ricci form on M. By Theorem 3.10, we can therefore give an explicit formula to all orders of the Berezin-Toeplitz star product.

Although this has not been done, we suspect that the methods of [KS] could be used to prove that the opposite to the twisted Berezin-Toeplitz star product has separation of variables and that

$$\mathrm{Kar}(\star_o^L) = -\omega\frac{1}{h} + \rho + iF_{\nabla^L}, \tag{4.7}$$

where F_{∇^L} is the curvature of the line bundle L. Since the characteristic class satisfies $\mathrm{cl}(*_o) = -\mathrm{cl}(*)$, this would comply with the formula (4.5), by using (3.2) and the fact that the Ricci form represents the first Chern class of the manifold.

Finally, we remark that the characteristic class of the twisted Berezin-Toeplitz star product arising through metaplectic quantization is trivial,

$$\mathrm{cl}(\star^\delta) = \frac{[\omega]}{2\pi h}, \tag{4.8}$$

which follows immediately from (4.5) since δ is a square root of the canonical line bundle.

CHAPTER 5

Families of Kähler Structures

In this chapter, we study families of Kähler structures on a symplectic manifold. Such families lie at the heart of our construction of the Hitchin connection, which relates the quantum spaces associated with different choices of Kähler structures through parallel transport in a bundle which has the quantum spaces a fibers. The results obtained will therefore play an important role in subsequent chapters.

5.1 Families of Kähler Structures

If $\mathcal{T}$ is a smooth manifold and (M, ω) is a symplectic manifold, we define

Definition 5.1. A *family of Kähler structures* on (M, ω) parametrized by $\mathcal{T}$ is a map

$$J \colon \mathcal{T} \to C^\infty(M, \operatorname{End}(TM)),$$

which associates an integrable and ω-compatible almost complex structure to every point $\sigma \in \mathcal{T}$.

A family of Kähler structures is called smooth if J is a smooth map, in the sense that it defines a smooth section of the pullback bundle $\pi_M^* \operatorname{End}(TM)$ over $\mathcal{T} \times M$, where $\pi_M \colon \mathcal{T} \times M \to M$ denotes the projection.

In the presence of a symmetry group Γ acting on M, we shall require that Γ also acts on $\mathcal{T}$ and that the map J is equivariant with respect to this action.

For any point $\sigma \in \mathcal{T}$, the manifold endowed with the Kähler structure defined by ω and $J(\sigma)$ is denoted by M_σ. The corresponding Kähler metric is denoted by g_σ and similar notation will be used for other structures which depend on the complex structure. We shall, however, often omit the σ from the notation, indicating that the formula in question is valid for any point in $\mathcal{T}$.

5.2 Infinitesimal Deformations

For a smooth family of Kähler structures, we can take its derivative along a vector field V on $\mathcal{T}$ to obtain a map

$$V[J] \colon \mathcal{T} \to C^\infty(M, \operatorname{End}(TM)).$$

Differentiating the identity $J^2 = -\operatorname{Id}$, we see that $V[J]$ satisfies

$$V[J]J + JV[J] = 0, \tag{5.1}$$

which implies that $V[J]_\sigma$ interchanges types on M_σ. Thus we get a decomposition

$$V[J] = V[J]' + V[J]'', \tag{5.2}$$

where $V[J]'_\sigma \in C^\infty(M, T'M_\sigma \otimes T''M^*_\sigma)$, and $V[J]''_\sigma \in C^\infty(M, T''M_\sigma \otimes T'M^*_\sigma)$ is its conjugate.

Notice that this splitting occurs for any infinitesimal deformation of a complex structure on M and defines an almost complex structure on the space of almost complex structures on M.

Differentiating the integrability condition on J, stating that the Nijenhuis tensor (1.2) vanishes, reveals that $V[J]' \in \Omega^{0,1}(M, T'M)$ satisfies $\bar\partial V[J]' = 0$, and the associated cohomology class in $H^1(M, T'M)$ is the Kodaira-Spencer class of the deformation (see [Kod]).

Define a bivector field $\tilde G(V) \in C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$ by the relation

$$V[J] = \tilde G(V)\cdot\omega,$$

for any vector field V on $\mathcal{T}$. Differentiating the identity $\tilde g = -J\cdot\tilde\omega$ along V, we get

$$V[\tilde g] = -V[J]\cdot\tilde\omega = -\tilde G(V),$$

and since $\tilde g$ is symmetric, this implies that $\tilde G(V)$ is a symmetric bivector field. Furthermore, the combined types of $V[J]$ and ω yield a decomposition,

$$\tilde G(V) = G(V) + \bar G(V),$$

where $G(V)_\sigma \in C^\infty(M, S^2(T'M_\sigma))$ and $\bar G(V)_\sigma \in C^\infty(M, S^2(T''M_\sigma))$.

Similarly, the variation of the Kähler metric is obtained by differentiating the identity $g = \omega\cdot J$, which yields

$$V[g] = \omega\cdot V[J] = \omega\cdot\tilde G(V)\cdot\omega.$$

By the types of ω and $\tilde G(V)$, it follows that the $(1,1)$-part of $V[g]$ vanishes.

Finally, we shall need the variation of the Levi-Civita connection, which is the tensor field

$$V[\nabla] \in C^\infty(M, S^2(TM^*) \otimes TM)$$

given by (see [Bes] Theorem 1.174)

$$\begin{aligned} 2g(V[\nabla]_X Y, Z) \;=\; & \nabla_X(V[g])(Y,Z) \\ & + \nabla_Y(V[g])(X,Z) \\ & - \nabla_Z(V[g])(X,Y), \end{aligned} \tag{5.3}$$

for any vector fields X, Y and Z on M.

5.3 The Canonical Line Bundle of a Family

For a family of complex structures, we can consider the vector bundle

$$\hat{T}'M \to \mathcal{T} \times M$$

with fibers $\hat{T}'M_{(\sigma,p)} = T'_p M_\sigma$ given by the holomorphic tangent spaces of M. As we have done here, we shall in general use a hat in our notation to indicate that we are working over the product $\mathcal{T} \times M$. Following this convention, the exterior differential on $\mathcal{T} \times M$ is denoted by $\hat{d}$, whereas the differential on $\mathcal{T}$ is denoted by $d_{\mathcal{T}}$ and by d on M.

The Kähler metric induces a Hermitian structure $\hat{h}^{T'M}$ on $\hat{T}'M$, and the Levi-Civita connection gives a compatible partial connection along the directions of M. We can extend this partial connection to a full connection $\hat{\nabla}^{T'M}$ on $\hat{T}'M$ in the following way.

If $Z \in C^\infty(\mathcal{T} \times M, \hat{T}'M)$ is a smooth family of sections of the holomorphic tangent bundle, and V is a vector field on $\mathcal{T}$, then we define

$$\hat{\nabla}_V Z = \pi^{1,0} V[Z].$$

In other words, we regard Z as a smooth family of sections of the complexified tangent bundle $TM_{\mathbb{C}}$, and then we simply differentiate Z along V in this bundle, which does not depend on the point in $\mathcal{T}$, and project the result back onto the holomorphic tangent bundle.

Of course, the connection $\hat{\nabla}^{T'M}$ preserves the Hermitian structure in the directions of M since it is induced by the Levi-Civita connection. Moreover, if V is a vector field on $\mathcal{T}$, and X and Y are sections of $\hat{T}'M$, we get that

$$\begin{aligned} V[h(X,Y)] = V[g(X,\bar{Y})] &= V[g](X,\bar{Y}) + g(V[X],\bar{Y}) + g(X,\overline{V[Y]}) \\ &= h(\hat{\nabla}_V X, Y) + h(X, \hat{\nabla}_V Y), \end{aligned}$$

since the $(1,1)$-part of $V[g]$ vanishes. It follows that $\hat{\nabla}^{T'M}$ preserves the Hermitian structure on $\hat{T}'M$.

Now consider the line bundle

$$\hat{K} = \bigwedge^m \hat{T}'M^* \to \mathcal{T} \times M,$$

which we shall call the *canonical line bundle of the family* of complex structures. As usual, the Hermitian structure and connection on $\hat{T}'M$ induce a Hermitian structure $\hat{h}^K$ and a compatible connection $\hat{\nabla}^K$ on $\hat{K}$.

5.3.1 Curvature of the Canonical Line Bundle

The next proposition gives formulas for the curvature of $\hat{\nabla}^K$, but before stating it, we introduce the following notation. For any vector fields V and W on $\mathcal{T}$, we define

$$\theta(V,W) = -\frac{i}{4} \operatorname{Tr} \pi^{1,0} \big[V[J], W[J]\big]. \tag{5.4}$$

Clearly, this defines a two-form $\theta \in \Omega^2(\mathcal{T}, C^\infty(M))$ on $\mathcal{T}$ with values in smooth functions on M.

Proposition 5.2. *The curvature of $\hat{\nabla}^K$ is given by*

$$F_{\hat{\nabla}^K}(X,Y) = i\rho(X,Y)$$
$$F_{\hat{\nabla}^K}(V,X) = \frac{i}{2}\delta(V[J])X$$
$$F_{\hat{\nabla}^K}(V,W) = i\theta(V,W),$$

for any vector fields X, Y on M and V, W on $\mathcal{T}$.

Proof. Since the curvature is a tensor, we can assume the vector fields X, Y, V and W to be commuting. The curvature in the direction of M is clearly given by the Ricci form since $\hat{\nabla}^K$ extends the Chern connection in the canonical line bundle on M.

To calculate the curvature of $\hat{\nabla}^K$ in the mixed directions, we first calculate the curvature of $\hat{\nabla}^{T'M}$ in these directions. For any smooth section $Z \in C^\infty(M, \hat{T}'M)$, we get that

$$\begin{aligned} F_{\hat{\nabla}^{T'M}}(V,X)Z &= \hat{\nabla}_V\hat{\nabla}_X Z - \hat{\nabla}_X\hat{\nabla}_V Z \\ &= \pi^{1,0}V[\nabla_X Z] - \nabla_X\pi^{1,0}V[Z] \\ &= \pi^{1,0}V[\nabla_X Z] - \pi^{1,0}\nabla_X V[Z] \\ &= \pi^{1,0}V[\nabla]_X Z. \end{aligned}$$

Now fix a point $\sigma \in \mathcal{T}$ and a point $p \in M$, and let $e_1, \dots, e_m$ be a basis of the fiber $\hat{T}'M_{(\sigma,p)} = T'_pM_\sigma$ which satisfies the orthogonality condition $g_\sigma(e_j, \bar{e}_k) = \delta_{jk}$. Then we get that

$$F_{\hat{\nabla}^K}(V,X) = -\operatorname{Tr} F_{\hat{\nabla}^{T'M}}(V,X) = -\operatorname{Tr}\pi^{1,0}V[\nabla]_X = -\sum_j g(V[\nabla]_X e_j, \bar{e}_j).$$

Taking into account the type of $V[g]$, the formula (5.3) yields

$$\begin{aligned} g(V[\nabla]_X e_j, \bar{e}_j) &= \frac{1}{2}\nabla_{e_j}(V[g])(\bar{e}_j, X) - \frac{1}{2}\nabla_{\bar{e}_j}(V[g])(e_j, X) \\ &= \frac{1}{2}\omega(\bar{e}_j, \nabla_{e_j}(V[J])X) - \frac{1}{2}\omega(e_j, \nabla_{\bar{e}_j}(V[J])X) \\ &= -\frac{i}{2}g(\bar{e}_j, \nabla_{e_j}(V[J])X) - \frac{i}{2}g(e_j, \nabla_{\bar{e}_j}(V[J])X). \end{aligned}$$

After summing over j, we get that

$$F_{\hat{\nabla}^K}(V,X) = \frac{i}{2}\delta(V[J])X,$$

which is the desired expression of the curvature in the mixed directions.

Finally, we calculate the curvature in the directions of $\mathcal{T}$. For any section $Z \in C^\infty(\mathcal{T} \times M, \hat{T}'M)$, we have

$$\hat{\nabla}_V Z = \pi^{1,0}V[Z] = V[\pi^{1,0}Z] - V[\pi^{1,0}]Z = V[Z] + \frac{i}{2}V[J]Z,$$

and this can be used to calculate

$$\begin{aligned} &\hat{\nabla}_V\hat{\nabla}_W Z \\ &\quad= \hat{\nabla}_V(W[Z] + \frac{i}{2}W[J]Z) \\ &\quad= VW[Z] + \frac{i}{2}VW[J]Z + \frac{i}{2}W[J]V[Z] + \frac{i}{2}V[J]W[Z] - \frac{1}{4}V[J]W[J]Z. \end{aligned}$$

Since V and W commute, this implies that

$$\begin{aligned} F_{\hat{\nabla}^{T'M}}(V,W)Z &= \hat{\nabla}_V \hat{\nabla}_W Z - \hat{\nabla}_W \hat{\nabla}_V Z \\ &= \frac{1}{4}(W[J]V[J] - V[J]W[J])Z \\ &= -\frac{1}{4}\big[V[J], W[J]\big] Z, \end{aligned}$$

so the curvature of $\hat{\nabla}^K$ is given by

$$F_{\hat{\nabla}^K}(V,W) = -\operatorname{Tr} F_{\hat{\nabla}^{T'M}}(V,W) = \frac{1}{4}\operatorname{Tr}\pi^{1,0}\big[V[J], W[J]\big] = i\theta(V,W).$$

This proves the proposition. □

5.3.2 The Bianchi Identity

By applying the Bianchi identity to the connection $\hat{\nabla}^K$, and using the formulas of Proposition 5.2, we get three useful results.

The first is a trivial reformulation of the Bianchi identity on three vector fields on $\mathcal{T}$.

Proposition 5.3. *The two-form $\theta \in \Omega^2(\mathcal{T}, C^\infty(M))$ is closed.*

The second result is a formula for the variation of the Ricci form.

Proposition 5.4. *The variation of the Ricci form is given by*

$$V[\rho] = \frac{1}{2} d\delta(V[J]),$$

for any vector field V on $\mathcal{T}$.

Proof. This is a statement about tensors, so we can let X and Y be commuting vector fields on M. The Bianchi identity for the connection $\hat{\nabla}^K$ then gives that

$$0 = V[F_{\hat{\nabla}^K}(X,Y)] + X[F_{\hat{\nabla}^K}(Y,V)] + Y[F_{\hat{\nabla}^K}(V,X)].$$

Inserting the appropriate expressions for the curvature, which are given in Proposition 5.2, we get that

$$V[i\rho(X,Y)] = \frac{i}{2}X[\delta(V[J])Y] - \frac{i}{2}Y[\delta(V[J])X],$$

and since the vector fields commute, this is equivalent to

$$2V[\rho](X,Y) = d\delta(V[J])(X,Y),$$

which proves the proposition. □

The third result follows by applying the Bianchi identity to two vector fields on $\mathcal{T}$ and one vector field on M.

Proposition 5.5. *We have the identity*

$$d\theta(V,W) = \frac{1}{2}W[\delta](V[J]) - \frac{1}{2}V[\delta](W[J])$$

for all vector fields V and W on $\mathcal{T}$.

Proof. Choose commuting vector fields V and W on $\mathcal{T}$ and X on M. Then the Bianchi identity yields

$$0 = X[F_{\hat{\nabla}^K}(V,W)] + V[F_{\hat{\nabla}^K}(W,X)] + W[F_{\hat{\nabla}^K}(X,V)].$$

Applying Proposition 5.2, and using the fact that the vector fields commute, we get

$$\begin{aligned} 0 &= 2X[\theta(V,W)] + V[\delta(W[J])X] - W[\delta(V[J])X] \\ &= 2X[\theta(V,W)] + V[\delta](W[J])X - W[\delta](V[J])X + \delta(VW[J] - WV[J])X \\ &= 2X[\theta(V,W)] + V[\delta](W[J])X - W[\delta](V[J])X. \end{aligned}$$

This proves the proposition. □

5.4 Holomorphic Families of Kähler Structures

In case the manifold $\mathcal{T}$ is itself a complex manifold, we can require the family J to be a holomorphic map from $\mathcal{T}$ to the space of complex structures. This is made precise by the following definition, which uses the splitting (5.2) of $V[J]$.

Definition 5.6. Suppose that $\mathcal{T}$ is a complex manifold, and that J is a family of complex structures on M, parametrized by $\mathcal{T}$. Then J is *holomorphic* if

$$V'[J] = V[J]'' \qquad \text{and} \qquad V''[J] = V[J]'',$$

for any vector field V on $\mathcal{T}$.

Let us give an alternative characterization of holomorphic families of Kähler structures. If I denotes the integrable almost complex structure on $\mathcal{T}$ induced by its complex structure, then we get an almost complex structure $\hat{J}$ on $\mathcal{T} \times M$ defined by

$$\hat{J}(V \oplus X) = IV \oplus J_\sigma X, \qquad V \oplus X \in T_{(\sigma,p)}(\mathcal{T} \times M).$$

The following proposition gives another characterization of holomorphic families.

Proposition 5.7. *The family J is holomorphic if and only if $\hat{J}$ is integrable.*

Proof. We show that J is holomorphic if and only if the Nijenhuis tensor for $\hat{J}$ vanishes. Clearly, the Nijenhuis tensor vanishes when evaluated only on vectors tangent to $\mathcal{T}$ since I is integrable. Likewise, it will vanish when evaluated only on vectors tangent to M since J is a family of integrable almost complex structures. Thus we are left with the case of mixed directions.

Let X and V be vector fields on M and $\mathcal{T}$, respectively. Then $[V, JX] = V[J]X$, and we get that

$$\begin{aligned} N_{\hat{J}}(V', X) &= [IV', JX] - [V', X] - \hat{J}[IV', X] - \hat{J}[V', JX] \\ &= i[V', JX] - \hat{J}[V', JX] \\ &= iV'[J]X - JV'[J]X \\ &= 2i\pi^{0,1}V'[J]X. \end{aligned}$$

Similarly, one shows that $N_{\hat{J}}(V'', X) = -2i\pi^{1,0}V''[J]X$, and we see that $N_{\hat{J}}(V, X)$ vanishes if and only if

$$\pi^{0,1}V'[J]X = 0 \quad \text{and} \quad \pi^{1,0}V''[J]X = 0.$$

This proves the proposition. □

By Proposition 5.7, a holomorphic family induces a complex structure on the product manifold $\mathcal{T} \times M$. Clearly, the projection $\pi_{\mathcal{T}} \colon \mathcal{T} \times M \to \mathcal{T}$ is a holomorphic map, and its differential is the projection $d\pi_{\mathcal{T}} \colon \hat{T}'\mathcal{T} \oplus \hat{T}'M \to T'\mathcal{T}$, where $\hat{T}'\mathcal{T}$ is the pullback of $T'\mathcal{T}$ by $\pi_{\mathcal{T}}$. Since the bundle $\hat{T}'M$ over $\mathcal{T} \times M$ is the kernel of this map, it has the structure of a holomorphic vector bundle, and it is easily verified that the connection $\hat{\nabla}^{T'M}$ is compatible with this holomorphic structure. Since the connection also preserves the Hermitian structure, it must be the Chern connection. Of course the same holds for the induced connection $\hat{\nabla}^K$ on $\hat{K}$, so we have proved

Proposition 5.8. *The Chern connection on the canonical line bundle $\hat{K}$ of a holomorphic family is given by $\hat{\nabla}^K$.*

Whenever we have a holomorphic family of complex structures, we can prove the following useful lemma on the second-order variation.

Lemma 5.9. *If J is a holomorphic family of Kähler structures, then*

$$W''V'[J] = \frac{i}{2}\big[V'[J], W''[J]\big] \tag{5.5}$$

for any vector fields fields V and W on $\mathcal{T}$ such that V' and W'' commute.

Proof. The holomorphicity of J implies that $V'[J]\pi^{1,0} = 0$, which yields

$$W''V'[J]\pi^{1,0} = \frac{i}{2}V'[J]W''[J]$$

by differentiation along W''. Similarly, by differentiation of $W''[J]\pi^{0,1} = 0$, we obtain

$$V'W''[J]\pi^{0,1} = -\frac{i}{2}W''[J]V'[J].$$

By adding these identities, and using the fact that V' and W'' commute, the lemma is proved. $\square$

Holomorphicity has other useful consequences. For instance, it implies that

$$\tilde{G}(V') = V'[J]\cdot\tilde{\omega} = V[J]'\cdot\tilde{\omega} = G(V),$$

and a similarly $\tilde{G}(V'') = \bar{G}(V)$. This means that (5.5) can also be written in the form

$$W''[G(V)] = \frac{i}{2}G(V)\cdot\omega\cdot\bar{G}(W) - \frac{i}{2}\bar{G}(W)\cdot\omega\cdot G(V). \tag{5.6}$$

Finally, we remark that the two-form θ, defined in (5.4), has type (1,1) on $\mathcal{T}$ since the composition $V[J]W[J]$ is zero whenever V and W have the same type.

5.5 Rigid Families of Kähler Structures

The following rather serious assumption on a family of Kähler structures will be crucial to our construction of the Hitchin connection.

Definition 5.10. A family of Kähler structures J is called *rigid* if

$$\nabla_{X''}G(V) = 0, \tag{5.7}$$

for all vector fields V on $\mathcal{T}$ and X on M.

In other words, the family J is rigid if $G(V)$ is a holomorphic section of $S^2(T'M)$, for any vector field V on $\mathcal{T}$. Let us give a simple example to illustrate that rigid deformations of a complex structure exist.

Example 5.11. Let (M, ω) be $\mathbb{R}^2$ with the standard symplectic form $\omega = dx \wedge dy$, and let $\mathcal{T} = \mathbb{R}^n$. Consider the following family of complex structures,

$$J_\sigma \left(\frac{\partial}{\partial x} \right) = A(\sigma, x, y) \frac{\partial}{\partial x} + B(\sigma, x, y) \frac{\partial}{\partial y},$$

given by functions $A, B \in C^\infty(\mathcal{T} \times M)$. The identity $J^2 = -\operatorname{Id}$ yields

$$J \left(\frac{\partial}{\partial y} \right) = -\left(\frac{1}{B} + \frac{A^2}{B} \right) \frac{\partial}{\partial x} - A \frac{\partial}{\partial y},$$

and it is easily verified that ω is J-invariant, and that $g = \omega \cdot J$ is positive definite when $B > 0$.

For simplicity, suppose that B is constant along $\mathcal{T}$. Given a vector field V on $\mathcal{T}$, the variation of J is then given by

$$V[J] = V[A] \frac{\partial}{\partial x} dx - \left(\frac{2AV[A]}{B} \frac{\partial}{\partial x} + V[A] \frac{\partial}{\partial y} \right) dy,$$

and the relation $V[J] = \tilde{G}(V) \cdot \omega$ gives the formula

$$\tilde{G}(V) = -2V[A] \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{2AV[A]}{B} \frac{\partial^2}{\partial x^2}.$$

From this, we calculate that

$$G(V) = \frac{-2iV[A]}{B} \frac{\partial^2}{\partial z^2},$$

which implies that J is rigid if

$$0 = -V[A] \frac{\partial B}{\partial y} + B \frac{\partial V[A]}{\partial y} = V[A] \frac{\partial B}{\partial x} - B \frac{\partial V[A]}{\partial x}.$$

These equations have solutions $B(x, y) = B_0(x, y)$ and $A(\sigma, x, y) = A_0(x, y) + \sum_{i=1}^{l} \sigma_i B_0(x, y)$, where A_0 and B_0 are arbitrary functions on M. This means that given any initial complex structure,

$$J_0 \left(\frac{\partial}{\partial x} \right) = A_0(x, y) \frac{\partial}{\partial x} + B_0(x, y) \frac{\partial}{\partial y},$$

we have obtained a rigid family of deformations parametrized by $\mathbb{R}^n$.

If we differentiate the rigidity condition (5.7) along $\mathcal{T}$, we get the following important result

Proposition 5.12. *Any rigid and holomorphic family of Kähler structures satisfies*

$$\mathcal{S}(G(V) \cdot \nabla G(W)) = \mathcal{S}(G(W) \cdot \nabla G(V)), \tag{5.8}$$

for any vector fields V and W on $\mathcal{T}$.

Proof. Throughout the proof, let V and W be vector fields on $\mathcal{T}$ such that V' and W' commute, and let U and Z be any vector fields on M. Furthermore, fix a $\sigma \in \mathcal{T}$, and let X and Y be any vector fields of type $(1,0)$ on M_σ.

We shall first prove a fact about the variation of the Levi-Civita connection. By (5.3), we get that

$$2g(V'[\nabla]_{Z''}X, U) = \nabla_{Z''}(V'[g])(X,U) + \nabla_X(V'[g])(Z'',U) - \nabla_U(V'[g])(Z'',X).$$

The first term vanishes because the family is holomorphic and rigid, and the last term vanishes because $V'[g]$ has no $(1,1)$-part, so we get

$$V'[\nabla]_{Z''}X\cdot g\cdot U = \frac{1}{2}\nabla_X(V'[g])(Z'',U) = \frac{i}{2}Z\cdot\omega\cdot\nabla_X(G(V))\cdot g\cdot U,$$

and it follows that

$$V'[\nabla]_{Z''}X = \frac{i}{2}Z\cdot\omega\cdot\nabla_X(G(V)),$$

which is the fact we need.

Locally on M_σ, we can write $G(W) = \sum_j X_j \otimes Y_j$, and we get that

$$\begin{aligned} V'[\nabla]_{Z''}(G(W)) &= \sum_j V'[\nabla]_{Z''}X_j \otimes Y_j + \sum_j X_j \otimes V'[\nabla]_{Z''}Y_j \\ &= \frac{i}{2}\sum_j Z\cdot\omega\cdot\nabla_{X_j}G(V)\otimes Y_j + \frac{i}{2}\sum_j X_j\otimes Z\cdot\omega\cdot\nabla_{Y_j}(G(V)) \\ &= \frac{3i}{2}Z\cdot\omega\cdot\mathcal{S}(G(W)\cdot\nabla G(V)) - \frac{i}{2}Z\cdot\omega\cdot G(W)\cdot\nabla G(V), \end{aligned} \tag{5.9}$$

where the last equality can be verified by writing out the symmetrization and using the symmetry of $G(W)$.

Now the family is rigid, so $\nabla_{Z''}G(W)$ vanishes, and by differentiating along V', we get that

$$0 = V'[\nabla]_{Z''}(G(W)) - \frac{i}{2}Z\cdot\omega\cdot G(V)\cdot\nabla G(W) + \nabla_{Z''}(V'[G(W)]).$$

Combining this with (5.9), we get that

$$\begin{aligned} &3Z\cdot\omega\cdot\mathcal{S}(G(W)\cdot\nabla G(V)) \\ &\quad = Z\cdot\omega\cdot G(V)\cdot\nabla G(W) + Z\cdot\omega\cdot G(W)\cdot\nabla G(V) + 2i\nabla_{Z''}(V'[G(W)]), \end{aligned}$$

and clearly this is symmetric in V and W because

$$W'[G(V)] = -W'V'[\tilde{g}] = -V'W'[\tilde{g}] = V'[G(W)].$$

This proves the proposition. □

By taking the divergence of (5.8), we get the following result.

Proposition 5.13. *Any rigid and holomorphic family of Kähler structures satisfies*

$$\begin{aligned} &\nabla^2_{G(V)}G(W) + \nabla_{\delta G(V)}G(W) + 2\,\mathcal{S}(G(V)\cdot\nabla\delta G(W)) \\ &\quad = \nabla^2_{G(W)}G(V) + \nabla_{\delta G(W)}G(V) + 2\,\mathcal{S}(G(W)\cdot\nabla\delta G(V)). \end{aligned}$$

for any vector fields V and W on $\mathcal{T}$.

These propositions will prove very useful when we calculate the curvature of the Hitchin connection.

5.6 Families of Ricci Potentials

Throughout this section, we will assume that M is compact, which enables us to apply Hodge theory to a family of Kähler structures and ensure the existence of Ricci potentials.

For each $\sigma \in \mathcal{T}$, we have a Ricci potential on the Kähler manifold M_σ. Recall that this is a real function $F_\sigma \in C^\infty(M)$ satisfying

$$\rho_\sigma = \rho_\sigma^H + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma, \tag{5.10}$$

where ρ_σ^H is the unique harmonic part of the Ricci form. By compactness of M, such a function is uniquely determined up to a constant, so we can fix the Ricci potential by demanding that it has zero average on M,

$$\int_M F_\sigma \omega^m = 0. \tag{5.11}$$

Clearly, with this normalization, the Ricci potentials define a smooth function $\hat{F} \in C^\infty(\mathcal{T} \times M)$, which we interpret as a smooth map $F \colon \mathcal{T} \to C^\infty(M)$. In general, we say that a smooth map $F \colon \mathcal{T} \to C^\infty(M)$ is a *smooth family of Ricci potentials* if it satisfies (5.10) for any $\sigma \in \mathcal{T}$.

The family af Ricci potentials defined by the normalization (5.11) is equivariant with respect to the action of a symmetry group Γ since this acts by symplectomorphisms on M and hence preserves the volume form.

Now, we will make the additional assumption that there exists an $n \in \mathbb{Z}$ such that the real first Chern class of (M, ω) is given by

$$\tilde{c}_1(M, \omega) = n[\tfrac{\omega}{2\pi}].$$

But the real first Chern class is also represented by $\frac{\rho}{2\pi}$, since $c_1(M, \omega) = -c_1(K_\sigma)$ for any $\sigma \in \mathcal{T}$, and consequently the identity (5.10) becomes

$$\rho = n\omega + 2i\partial\bar{\partial}F, \tag{5.12}$$

since the Kähler form ω is harmonic. The following proposition gives a useful identity, involving the variation of the Ricci potential, under this assumption.

Proposition 5.14. *Suppose that M is a compact, symplectic manifold which satisfies $H^1(M, \mathbb{R}) = 0$ and $\tilde{c}_1(M, \omega) = n[\frac{\omega}{2\pi}]$, and let J be a holomorphic family of Kähler structures on M. Then any family of Ricci potentials satisfies*

$$4i\bar{\partial}V'[F] = \delta(V'[J]) + 2dF \cdot V'[J], \tag{5.13}$$

for any vector field V on $\mathcal{T}$.

Proof. By differentiating the identity (5.12) in the direction of V', we get

$$V'[\rho] = -d(dF \cdot V'[J]) + 2i\partial\bar{\partial}V'[F],$$

and by using Proposition 5.4 on the left-hand side, this yields

$$d\delta(V'[J]) + 2d(dF \cdot V'[J]) - 4id\bar{\partial}V'[F] = 0.$$

On one hand, it follows that the one-form

$$\delta(V'[J]) + 2dF{\cdot}V'[J] - 4i\bar{\partial}V'[F]$$

is closed and hence exact by the assumption $H^1(M,\mathbb{R}) = 0$. On the other hand, it is of type (0,1) since J is holomorphic, so by the compactness of M, it cannot be exact unless it is zero. This proves the lemma. $\square$

On several occasions, we will use (5.13) on the form

$$\bar{\partial}V'[F] = \frac{i}{4}\omega{\cdot}\delta G(V) + \frac{i}{2}\omega{\cdot}G(V){\cdot}dF, \tag{5.14}$$

so we state it here for easy reference.

Using Proposition 5.14, we can express the divergence of $V[J]$ in terms of the Ricci potential.

Lemma 5.15. *With the assumptions of Proposition 5.14, we have that*

$$\delta(V[J])X = 4iV'X''[F] - 4iV''X'[F],$$

for any vector fields X on M and V on $\mathcal{T}$.

Proof. By Proposition 5.14, we get that

$$4iV'X''[F] = -2(V'[J]X)[F] + 4iX''V'[F] = \delta(V'[J])X,$$

and by conjugation, this implies that

$$-4iV''X'[F] = \delta(V''[J])X.$$

The result follows by adding these identities. $\square$

5.6.1 Curvature and Ricci Potentials

As we shall see in this section, the curvature of the canonical line bundle of a family of Kähler structures can be expressed in terms of Ricci potentials.

Proposition 5.16. *Given the assumptions of Proposition 5.14, we have that any family of Ricci potentials satisfies*

$$\theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F \in \Omega^{1,1}(\mathcal{T}).$$

In other words, the form takes values in constant functions on M, and hence it defines an ordinary two-form on $\mathcal{T}$.

Proof. Choose commuting vector fields V' and W'' on $\mathcal{T}$ and X on M. Then, by Proposition 5.5 and Lemma 5.15, we get that

$$\begin{aligned} 2X[\theta(V',W'')] &= W''[\delta](V'[J])X - V'[\delta](W''[J])X \\ &= W''[\delta(V'[J])X] - V'[\delta(W''[J])X] \\ &= 4iW''V'X''[F] + 4iV'W''X'[F] \\ &= 4iV'W''X[F] \\ &= 4iXV'W''[F] \\ &= 4iX[\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F(V',W'')]. \end{aligned} \tag{5.15}$$

In other words, the expression

$$\theta(V', W'') - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F(V', W''),$$

is constant along M, which proves the proposition. □

Proposition 5.17. *Suppose that the assumptions of Proposition 5.14 are satisfied and that the manifold $\mathcal{T}$ is contractible. Then there exists a family of Ricci potentials $\tilde{F}$ such that*

$$\theta = 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}\tilde{F}. \tag{5.16}$$

Proof. Fix a smooth family F of Ricci potentials, say the one satisfying (5.11). From Proposition 5.16, we know that

$$\alpha = \theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F$$

defines a two-form $\alpha \in \Omega^{1,1}(\mathcal{T})$, and by Proposition 5.3 this two-form is closed. Since $\mathcal{T}$ is contractible, it follows that we can find a real function $A \in C^\infty(\mathcal{T})$ such that $\alpha = 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}A$. But then the expression

$$\tilde{F} = F + A,$$

defines another smooth family of Ricci potentials, and we see that

$$\theta = 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}\tilde{F},$$

as desired. □

This proposition shows that the curvature of $\hat{\nabla}^K$ in the directions of $\mathcal{T}$ can be expressed in terms of an appropriately chosen family of Ricci potentials, although this is likely to break the equivariance of the Ricci potential with respect to a symmetry action.

Clearly, the assumption that $\mathcal{T}$ is contractible is not the minimal assumption needed to prove Proposition 5.17, but is chosen for simplicity. We shall usually apply this assumption to avoid restrictions caused by the topology of $\mathcal{T}$, and in the important example discussed in Chapter 8 it is actually satisfied.

The next proposition shows that the curvature in the remaining directions can also be expressed in terms of this particular family of Ricci potentials.

Proposition 5.18. *Suppose that the assumptions of Proposition 5.14 are satisfied and that manifold $\mathcal{T}$ is contractible. Then there exists a family of Ricci potentials $\tilde{F}$ such that*

$$F_{\hat{\nabla}^K} = ni\pi_M^*\omega - 2\hat{\partial}\hat{\bar{\partial}}\tilde{F}. \tag{5.17}$$

Proof. Let X and Y be vector fields on M, and let V and W be vector fields on $\mathcal{T}$. By Proposition 5.17, we can find a family of Ricci potentials satisfying (5.16), and by Proposition 5.2, this equation is exactly the desired statement in the directions along $\mathcal{T}$. For the curvature in the directions of M, the identity (5.12) yields

$$R_{\hat{\nabla}^K}(X, Y) = i\rho(X, Y) = ni\omega(X, Y) - 2\partial\bar{\partial}\tilde{F}(X, Y).$$

In the mixed directions, a direct calculation yields

$$\begin{aligned}
\bar{\hat{\partial}}\hat{\partial}F(X'',V') &= \hat{d}\hat{\partial}F(X'',V') \\
&= X''(\hat{\partial}F(V')) - V'(\hat{\partial}F(X'')) - \hat{\partial}F([X'',V']) \\
&= X''V'[F] + \frac{i}{2}dF{\cdot}V'[J]X'' \\
&= -\frac{i}{4}\delta(V'[J])X'' \\
&= -\frac{1}{2}F_{\hat{\nabla}^K}(V',X''),
\end{aligned}$$

where we used Proposition 5.14 and Proposition 5.2 for the last two equalities. The case of X' and V'' is similar, by conjugation of the identity in Proposition 5.14, and the proposition is proved. □

Chapter 6

The Hitchin Connection

In this chapter, we give a differential geometric construction of a Hitchin connection in the bundle of quantum spaces associated with a family of Kähler structures on a symplectic manifold.

We start by reviewing the construction by Andersen [And1] of a Hitchin connection in the bundle of quantum spaces arising from standard geometric quantization. Then we give an analogous construction in the metaplectic setting and show how the two constructions can be related. After calculating the curvature of the connection, we give sufficient conditions for flatness and discuss the issue of unitarity.

6.1 The Hitchin Connection in Standard Geometric Quantization

In this section, we recall the differential geometric construction of a Hitchin connection in the setting of standard geometric quantization. Although the results are all proved in [And1], we include the proofs to illustrate the similarities with the metaplectic setting, which we shall be dealing with afterwards.

Consider a compact, symplectic manifold (M, ω), equipped with a prequantum line bundle $\mathcal{L}$, and assume that $H^1(M, \mathbb{R}) = 0$ and that the real first Chern class of (M, ω) is given by

$$\tilde{c}_1(M, \omega) = n\big[\tfrac{\omega}{2\pi}\big], \tag{6.1}$$

for some integer $n \in \mathbb{Z}$. Further, assume that M is of Kähler type, and let J be a rigid and holomorphic family of Kähler structures on (M, ω), parametrized by some complex manifold $\mathcal{T}$.

The prequantum space $\mathcal{P}_k = C^\infty(M, \mathcal{L}^k)$ forms the fiber of a trivial, infinite-rank vector bundle over $\mathcal{T}$,

$$\hat{\mathcal{P}}_k = \mathcal{T} \times \mathcal{P}_k.$$

If ∇^t denotes the trivial connection on this bundle, we consider a connection on $\hat{\mathcal{P}}_k$ of the form

$$\boldsymbol{\nabla} = \nabla^t + a, \tag{6.2}$$

where $a \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ is a one-form on $\mathcal{T}$ with values in the space of differential operators on sections of $\mathcal{L}^k$, and we seek an a which makes the connection $\boldsymbol{\nabla}$ preserve the subspaces $\mathcal{Q}_k(\sigma) = H^0(M_\sigma, \mathcal{L}^k)$ of holomorphic sections, inside each fiber of $\hat{\mathcal{P}}_k$.

Definition 6.1. A *Hitchin connection* on the bundle $\hat{\mathcal{P}}_k$ is a connection of the form (6.2) which preserves the fiberwise subspaces $\mathcal{Q}_k$.

To prove the existence of such a connection, we first look at the bundle $\hat{\mathcal{P}}_k$ from a slightly different perspective. Consider the pullback $\hat{\mathcal{L}} = \pi_M^* \mathcal{L}$ of the line bundle $\mathcal{L}$ to the product $\mathcal{T} \times M$. Sections of $\hat{\mathcal{P}}_k$ are in one-to-one correspondence with sections of $\hat{\mathcal{L}}^k$, and the prequantum connection on $\mathcal{L}^k$ defines a partial connection on $\hat{\mathcal{L}}^k$ along the directions of M. We can easily extend this to a full connection $\hat{\nabla}^{\mathcal{L}^k}$ on $\hat{\mathcal{L}}^k$. Indeed, for any section s of $\hat{\mathcal{L}}^k$ and any vector field V on $\mathcal{T}$, we define

$$\hat{\nabla}_V s = \nabla_V^t s = V[s],$$

simply expressing differentiation in the direction of V, in which $\hat{\mathcal{L}}^k$ is trivial.

The bundle $\hat{\mathcal{L}}^k$ has a Hermitian structure, which is induced from $\mathcal{L}$, and $\hat{\nabla}$ is easily seen to be compatible with this. Moreover, it is a simple task to calculate the curvature.

Proposition 6.2. *The curvature of the connection $\hat{\nabla}$ on $\hat{\mathcal{L}}^k$ is given by*

$$F_{\hat{\nabla}} = -ik\pi_M^* \omega,$$

where $\pi_M \colon \mathcal{T} \times M \to M$ denotes the projection.

It follows that the curvature of $\hat{\nabla}$ has type $(1,1)$ on $\mathcal{T} \times M$, and consequently it defines holomorphic structure on $\hat{\mathcal{L}}^k$.

The fact that $\hat{\nabla}$ only has curvature in directions along M can be used to prove

Proposition 6.3. *The connection $\boldsymbol{\nabla}$ preserves the fiberwise subspaces $\mathcal{Q}_k$ of $\hat{\mathcal{P}}_k$ if and only if the one-form a satisfies*

$$\nabla^{0,1} a(V)s + \frac{i}{2}\omega \cdot G(V) \cdot \nabla s = 0, \tag{6.3}$$

for any vector field V on $\mathcal{T}$, any point $\sigma \in \mathcal{T}$ and any section $s \in \mathcal{Q}_k(\sigma)$.

Proof. Let V and X be vector fields on $\mathcal{T}$ and M, respectively. It is easily calculated that

$$[X'', V] = -\frac{i}{2} V[J]X = \frac{i}{2} X \cdot \omega \cdot \tilde{G}(V). \tag{6.4}$$

Consider a point $\sigma \in \mathcal{T}$, and suppose that $s \in \mathcal{Q}_k(\sigma)$. Let $\hat{s}$ be any extension of s to a smooth section of $\hat{\mathcal{P}}_k \to \mathcal{T}$. Then we have that $\hat{\nabla}_{X''} \hat{s} = 0$ at the point σ, and we get that

$$\begin{aligned} \hat{\nabla}_{X''} \boldsymbol{\nabla}_V \hat{s} &= \hat{\nabla}_{X''} \hat{\nabla}_V \hat{s} + \hat{\nabla}_{X''} a(V)\hat{s} \\ &= R_{\hat{\nabla}}(X'', V)\hat{s} + \hat{\nabla}_V \hat{\nabla}_{X''} \hat{s} + \hat{\nabla}_{[X'',V]} \hat{s} + \hat{\nabla}_{X''} a(V)\hat{s} \\ &= \frac{i}{2} X \cdot \omega \cdot G(V) \cdot \hat{\nabla} \hat{s} + \hat{\nabla}_{X''} a(V)\hat{s}, \end{aligned} \tag{6.5}$$

at the point σ, where the curvature term vanishes by Proposition 6.2. Finally, it is clear that $(\boldsymbol{\nabla}_V \hat{s})_\sigma \in \mathcal{Q}_k(\sigma)$ if and only if the left-hand side of (6.5) vanishes at σ, and the proposition follows. $\square$

If we can find a one-form a satisfying (6.3), then it follows that $\boldsymbol{\nabla}$ preserves the fiberwise subspaces $\mathcal{Q}_k$ of the bundle $\hat{\mathcal{P}}_k$. In this case, these subspaces must form a smooth subbundle $\hat{\mathcal{Q}}_k$ since it can be trivialized locally through parallel transport by $\boldsymbol{\nabla}$. Futhermore, $\boldsymbol{\nabla}$ induces a connection in this subbundle.

To solve the equation (6.3), we need the following proposition.

Proposition 6.4. *The operator $\Delta_{G(V)}$ satisfies*

$$\nabla^{0,1}\Delta_{G(V)}s = -2ik\omega{\cdot}G(V){\cdot}\nabla s - i\rho{\cdot}G(V){\cdot}\nabla s - ik\omega{\cdot}\delta(G(V))s, \tag{6.6}$$

for and any (local) holomorphic section s of $\mathcal{L}^k$.

Proof. Fix a vector field V and a point $\sigma \in \mathcal{T}$. The statement of the proposition is local on M, and since the family of complex structures is rigid, the bivector field $G(V)$ is holomorphic and can therefore be expressed locally as $G(V) = \sum_j X_j \otimes Y_j$, where X_j and Y_j are local holomorphic vector fields on M_σ. With this notation, the operator $\Delta_{G(V)}$ has the expression

$$\Delta_{G(V)} = \sum_j \nabla_{X_j}\nabla_{Y_j} + \nabla_{\delta(X_j)Y_j}.$$

For any local holomorphic section s of $\mathcal{L}^k$ and any local anti-holomorphic vector field $\bar{Z}$, we get that

$$\nabla_{\bar{Z}}\nabla_{X_j}\nabla_{Y_j}s = -ik\omega(\bar{Z}, X_j)\nabla_{Y_j}s - ik\omega(\bar{Z}, Y_j)\nabla_{X_j}s - ik\omega(\bar{Z}, \nabla_{X_j}Y_j)s,$$

since the Kähler form is parallel. On the other hand, we have

$$\bar{Z}[\delta(X_j)] = \bar{Z}[\operatorname{Tr}\nabla X_j] = \operatorname{Tr}\nabla_{\bar{Z}}\nabla X_j = \operatorname{Tr} R(\bar{Z},\cdot)X_j = -i\rho(\bar{Z}, X_j),$$

so we get that

$$\nabla_{\bar{Z}}\nabla_{\delta(X_j)Y_j}s = -ik\omega(\bar{Z}, \delta(X_j)Y_j)s - i\rho(\bar{Z}, X_j)\nabla_{Y_j}s.$$

Finally, using the expressions above, it follows that

$$\nabla_{\bar{Z}}\Delta_{G(V)} = -\sum_j ik\omega(\bar{Z}, X_j)\nabla_{Y_j}s + i\rho(\bar{Z}, X_j)\nabla_{Y_j}s + ik\omega(\bar{Z}, \delta(X_j \otimes Y_j))s,$$

which is the statement of the proposition. □

We emphasize the importance of the rigidity condition on the family of Kähler structures in this proposition.

If not for the last two terms of (6.6), we could use $\frac{1}{4k}\Delta_{G(V)}$ as our $a(V)$, so in the following, we shall try to get rid of these terms. As we shall see, the Ricci potential will play a centrol role in this.

Recall that the assumption (6.1) on the first Chern class implies that $\rho^H = n\omega$ since the Käher form is harmonic. In particular, for any smooth family of Ricci potentials F, we have the identity

$$\rho = n\omega + 2i\partial\bar{\partial}F. \tag{6.7}$$

Inserting this in (6.6), and using the identity

$$\nabla^{0,1}\nabla_{G(V)\cdot dF}s = -\partial\bar{\partial}F{\cdot}G(V){\cdot}\nabla s - ik\omega{\cdot}G(V){\cdot}dFs,$$

which is valid at any $\sigma \in \mathcal{T}$ and for any (local) holomorphic section s of $\mathcal{L}^k$, we get that

$$\begin{aligned}\nabla^{0,1}\big(\Delta_{G(V)}s + 2\nabla_{G(V)\cdot dF}s\big) \\ = -(2k+n)i\omega\cdot G(V)\cdot\nabla s - ik\omega\cdot\delta(G(V))s - 2ik\omega\cdot G(V)\cdot dFs.\end{aligned} \tag{6.8}$$

The obvious improvement over (6.6) is that we have replaced a first-order term by a zero-order term. Moreover, we can get rid of the last two terms using (5.14), and it follows that the one-form $a \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ defined by

$$a(V) = \frac{1}{4k+2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]) \tag{6.9}$$

satisfies the condition (6.3). Later, it will be convenient to write the operator $a(V)$ in the form

$$a(V) = \frac{1}{k+n/2}b(V) + V'[F] \quad \text{with} \quad b(V) = \frac{1}{4}\big(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} - 2nV'[F]\big),$$

which splits the operator into orders of k. In conclusion, we have proved the following theorem.

Theorem 6.5 (Andersen)**.** *Let (M, ω) be a compact, prequantizable symplectic manifold with $H^1(M, \mathbb{R}) = 0$ and $\tilde{c}_1(M, \omega) = n\big[\frac{\omega}{2\pi}\big]$. Further, let J be a rigid, holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$. Then the expression*

$$\boldsymbol{\nabla}_V = \nabla^t_V + \frac{1}{4k+2n}(\Delta_{G(V)} + 2dF\cdot G(V)\cdot\nabla + 4kV'[F])$$

defines a Hitchin connection in the bundle $\hat{\mathcal{Q}}_k$ over $\mathcal{T}$.

Clearly, with an equivariant family of Ricci potentials, the Hitchin connection given in this theorem is equivariant with respect to the action of the symmetry group Γ.

Having reviewed the construction by Andersen of a Hitchin connection in standard geometric quantization, we turn to the metaplectically corrected setting.

6.2 The Hitchin Connection in Metaplectic Quantization

It turns out that a Hitchin connection in metaplectic quantization can be constructed with fewer assumptions than in standard geometric quantization. We start by describing the setup that we shall consider.

Let (M, ω) be a symplectic manifold equipped with a prequantum line bundle $\mathcal{L}$. Further, let J be a rigid family of Kähler structures, parametrized by a smooth manifold $\mathcal{T}$ and satisfying that the Dolbeault cohomology group $H^{0,1}(M_\sigma, \mathbb{C})$ vanishes for every point $\sigma \in \mathcal{T}$.

Assume that the second Stiefel-Whitney class of M vanishes, and pick a metaplectic structure δ on M. The family J can be viewed as a smooth map $J\colon \mathcal{T} \times M \to L^+M$, and if we pull back the metaplectic structure $\delta \to L^+M$ by this map, we get a line bundle $\hat{\delta} \to \mathcal{T} \times M$.

The isomorphism ψ^δ in Definition 1.3 induces an isomorphism between the canonical line bundle $\hat{K} \to \mathcal{T} \times M$ and the bundle $\hat{\delta}^2$. The Hermitian structure and compatible connection $\hat{\nabla}^K$ on $\hat{K}$ induce a Hermitian structure and compatible connection $\hat{\nabla}^\delta$ on δ through this isomorphism. The restriction of $\hat{\delta}$ to a point $\sigma \in \mathcal{T}$ yields a square root δ_σ of the canonical line bundle K_σ over the Kähler manifold M_σ.

The spaces $\mathcal{P}_k^\delta(\sigma) = C^\infty(M, \mathcal{L}^k \otimes \delta_\sigma)$ form the fibers of a vector bundle

$$\hat{\mathcal{P}}_k^\delta \to \mathcal{T}, \tag{6.10}$$

and we wish to define a connection in $\hat{\mathcal{P}}_k^\delta$ which preserves the fiberwise subspaces $\mathcal{Q}_k^\delta(\sigma) = H^0(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$. We shall approach this in a way which is similar to what we did for standard goeometric quantization, where we used the trivial connection as a reference point in the space of connections and added an appropriate one-form to make the connection preserve the subspaces of holomorphic sections over M.

Since the trivial connection is not available to us in the bundle $\hat{\mathcal{P}}_k^\delta$, we must first choose a good connection to use as a reference point. The space of sections of $\hat{\mathcal{P}}_k^\delta \to \mathcal{T}$ is equal to the space of sections of the line bundle $\hat{\mathcal{L}}^k \otimes \hat{\delta} \to \mathcal{T} \times M$. The bundle $\hat{\mathcal{L}}^k$ is equipped with the connection $\hat{\nabla}^{\mathcal{L}^k}$, which we used in the previous section, and $\hat{\delta}$ is equipped with the connection $\hat{\nabla}^\delta$, as discussed above. Together, these define a connection $\hat{\nabla}^{\mathcal{L}^k \otimes \delta}$ on the line bundle $\hat{\mathcal{L}}^k \otimes \hat{\delta}$, and in particular this defines a connection on $\hat{\mathcal{P}}_k^\delta \to \mathcal{T}$.

The curvature of the line bundle $\hat{\mathcal{L}}^k \otimes \hat{\delta}$, which we will need shortly, is easily calculated from Proposition 5.2, and we state it here for convenience.

Proposition 6.6. *The curvature of the connection $\hat{\nabla}$ on $\hat{\mathcal{L}}^k \otimes \hat{\delta}$ is given by*

$$\begin{aligned} F_{\hat{\nabla}}(X, Y) &= -ik\omega(X, Y) + \frac{i}{2}\rho(X, Y) \\ F_{\hat{\nabla}}(V, X) &= \frac{i}{4}\delta(V[J])X \\ F_{\hat{\nabla}}(V, W) &= \frac{i}{2}\theta(V, W), \end{aligned}$$

for any vector fields X, Y on M and V, W on $\mathcal{T}$.

The spaces $\mathcal{D}(M, \mathcal{L}^k \otimes \delta_\sigma)$ of differential operators on $\mathcal{L}^k \otimes \delta_\sigma$ form a bundle $\hat{\mathcal{D}}(M, \mathcal{L}^k \otimes \delta)$ over $\mathcal{T}$, and we seek a one-form $a^\delta \in \Omega^1(\mathcal{T}, \hat{\mathcal{D}}(M, \mathcal{L}^k \otimes \delta))$ such that the connection

$$\boldsymbol{\nabla}^\delta = \hat{\nabla} + a^\delta \tag{6.11}$$

preserves the subspaces $\mathcal{Q}_k^\delta$ inside each fiber of $\hat{\mathcal{P}}_k^\delta$. In complete analogy with Definition 6.1, such a connection is called a Hitchin connection.

As in the case of standard geometric quantization, the property of being a Hitchin connection reduces to a condition on the one-form a^δ.

Lemma 6.7. *The connection $\boldsymbol{\nabla}^\delta$ preserves the fiberwise subspaces $\mathcal{Q}_k^\delta$ of $\hat{\mathcal{P}}_k^\delta$ if and only if the one-form a^δ satisfies*

$$\nabla^{0,1} a^\delta(V)s + \frac{i}{2}\omega \cdot G(V) \cdot \nabla s + \frac{i}{4}\omega \cdot \delta(G(V))s = 0, \tag{6.12}$$

for any vector field V on $\mathcal{T}$, any point $\sigma \in \mathcal{T}$ and any section $s \in \mathcal{Q}_k^\delta(\sigma)$.

Proof. Let V and X be vector fields on $\mathcal{T}$ and M, respectively, and consider a point $\sigma \in \mathcal{T}$ and a section $s \in \mathcal{Q}_k^\delta(\sigma)$. Let $\hat{s}$ be any extension of s to a smooth section of $\hat{\mathcal{P}}_k^\delta \to \mathcal{T}$. Then we have that $\hat{\nabla}_{X''}\hat{s} = 0$ at the point σ, and we find that

$$\begin{aligned}\hat{\nabla}_{X''}\boldsymbol{\nabla}_V^\delta \hat{s} &= \hat{\nabla}_{X''}\hat{\nabla}_V\hat{s} + \hat{\nabla}_{X''}a^\delta(V)\hat{s} \\ &= \hat{\nabla}_V\hat{\nabla}_{X''}\hat{s} + F_{\hat{\nabla}}(X'', V)\hat{s} + \hat{\nabla}_{[X'',V]}\hat{s} + \hat{\nabla}_{X''}a^\delta(V)\hat{s} \\ &= \frac{i}{4}X\cdot\omega\cdot\delta(G(V))\hat{s} + \frac{i}{2}X\cdot\omega\cdot G(V)\cdot\hat{\nabla}\hat{s} + \hat{\nabla}_{X''}a^\delta(V)\hat{s},\end{aligned}$$

at the point $\sigma \in \mathcal{T}$, where we used (6.4) and Proposition 6.6 for the last equality. The proposition follows since $(\boldsymbol{\nabla}_V^\delta\hat{s})_\sigma \in \mathcal{Q}_k^\delta(\sigma)$ if and only if the left-hand side vanishes. $\square$

To solve the equation (6.12), we once again turn to the operator $\Delta_{G(V)}$. The following proposition is similar to Proposition 6.4 and its proof is completely analogous.

Proposition 6.8. *The operator $\Delta_{G(V)}$ satisfies*

$$\nabla^{0,1}\Delta_{G(V)}s = -2ik\omega\cdot G(V)\cdot\nabla s - ik\omega\cdot\delta(G(V))s + \frac{i}{2}\delta(\rho\cdot G(V))s, \tag{6.13}$$

for any $\sigma \in \mathcal{T}$ and any (local) holomorphic section s of $\mathcal{L}^k \otimes \delta_\sigma$.

We see that if not for the last term we could use $\frac{1}{4k}\Delta_{G(V)}$ as our $a^\delta(V)$. Notice, however, that (6.13) is valid for any integer k. In the particular case of $k = 0$, the proposition yields

$$\nabla^{0,1}\Delta_{G(V)}s = \frac{i}{2}\delta(\rho\cdot G(V))s.$$

By applying the $\bar{\partial}$-operator on both sides, we see that

$$\bar{\partial}\delta(\rho\cdot G(V)) = 0,$$

at every point $\sigma \in \mathcal{T}$. Since we assumed that the family J satisfies $H^{0,1}(M_\sigma, \mathbb{C}) = 0$ for every $\sigma \in \mathcal{T}$, we have proved

Proposition 6.9. *There exists a one-form $b_0^\delta \in \Omega^1(\mathcal{T}, C^\infty(M))$ on $\mathcal{T}$ such that*

$$\bar{\partial}b_0^\delta(V) = -\frac{i}{8}\delta(\rho\cdot G(V)),$$

for any vector field V on $\mathcal{T}$.

It follows that the one-form $a^\delta \in \Omega^1(\mathcal{T}, \hat{\mathcal{D}}(M, \mathcal{L}^k \otimes \delta))$ defined by

$$a^\delta(V) = \frac{1}{4k}(\Delta_{G(V)} + 4b_0^\delta(V)) \tag{6.14}$$

satisfies the condition (6.12). The peculiar name for b_0^δ is explained if we write

$$a^\delta(V) = \frac{1}{k}b^\delta(V) \qquad \text{with} \qquad b^\delta(V) = \frac{1}{4}\Delta_{G(V)} + b_0^\delta(V),$$

as we did in the case of standard geometric quantization, so that $b_0^\delta(V)$ is the zero-order term of $b^\delta(V)$. This way of writing a^δ will be convenient later.

In conclusion, we have proved the following theorem.

Theorem 6.10. *Let (M,ω) be a prequantizable symplectic manifold with vanishing second Stiefel-Whitney class. Further, let J be a rigid family of Kähler structures on M, parametrized by a smooth manifold $\mathcal{T}$, all satifying $H^{0,1}(M,\mathbb{C})=0$. Then there exists a one-form $b_0^\delta \in \Omega^1(\mathcal{T}, C^\infty(M))$ satisfying $\bar\partial b_0^\delta(V) = -\frac{i}{8}(\rho\cdot G(V))$, and the connection*

$$\boldsymbol{\nabla}_V^\delta = \hat\nabla_V + \frac{1}{4k}(\Delta_{G(V)} + 4b_0^\delta(V))$$

is a Hitchin connection on the bundle $\hat{\mathcal{Q}}_k^\delta$ over $\mathcal{T}$.

Let us highlight some of the differences between this result and Theorem 6.5. The most important difference is that the assumption on the first Chern class of the symplectic manifold is no longer needed. This comes at the cost of an explicit expression for the Hitchin connection, however. Furthermore, we do not need the family of Kähler structures to be holomorphic, and in particular, the manifold $\mathcal{T}$ does not have to be complex. Finally, the symplectic manifold is not required to be compact, but we remark that in the compact case, the Hodge decomposition theorem ensures that the assumption $H^{0,1}(M,\mathbb{C})=0$ on each of the Kähler structures can be replaced by $H^1(M,\mathbb{R})=0$.

6.3 Relating the Hitchin Connections

Let us impose the combined assumptions of Theorem 6.5 and Theorem 6.10, which ensures the existence of a Hitchin connection in metaplectic as well as standard geometric quantization. This will allow us to give an explicit formula for the Hitchin connection in the metaplectic setting and eventually compare it with the connection in standard geometric quantization.

To rephrase our assumptions, the manifold M is assumed to be compact with $H^1(M,\mathbb{R})=0$ and $w_2(M)=0$. Moreover, we assume that the real first Chern class satisfies

$$\tilde c_1(M,\omega) = n[\tfrac{\omega}{2\pi}], \tag{6.15}$$

for some integer $n\in\mathbb{Z}$. In fact, we shall strengthen this assumption a little by requiring that the first Chern class $c_1(M,\omega)$ is divisible by n in $H^2(M,\mathbb{Z})$. Since first Chern class must be even, we shall also assume that the integer n is even.

6.3.1 Explicit Formula for the Hitchin Connection

As a first consequence of our additional assumptions, we are able to give an explicit formula for the one-form b_0^δ in Theorem 6.10.

Proposition 6.11. *If F is a smooth family of Ricci potentials, then the one-form $b_0^\delta\in\Omega^1(\mathcal{T}, C^\infty(M))$ given by*

$$b_0^\delta(V) = -\frac{1}{4}\big(\Delta_{G(V)}F + dF\cdot G(V)\cdot dF + 2nV'[F]\big) \tag{6.16}$$

satisfies $\bar\partial b_0^\delta(V) = -\frac{i}{8}\delta(\rho\cdot G(V))$, for any vector field V on $\mathcal{T}$.

Proof. Since ω is parallel with respect to the Levi-Civita connection, we get

$$-\frac{i}{2}\delta(\rho\cdot G(V)) = -\frac{i}{2}n\omega\cdot\delta(G(V)) + \delta(\partial\bar{\partial}F\cdot G(V)).$$

Moreover, it is easily verified that

$$\begin{aligned}\delta(\partial\bar{\partial}F\cdot G(V)) &= -i\rho\cdot G(V)\cdot dF - \bar{\partial}\delta(dF\cdot G(V))\\ &= -in\omega\cdot G(V)\cdot dF + 2\partial\bar{\partial}F\cdot G(V)\cdot dF - \bar{\partial}\delta(dF\cdot G(V)).\end{aligned}$$

But $\Delta_{G(V)}F = \delta(dF\cdot G(V))$, so the lemma follows by (5.14) and the identity

$$\bar{\partial}(dF\cdot G(V)\cdot dF) = -2\partial\bar{\partial}F\cdot G(V)dF,$$

which is easily verified using the symmetry of $G(V)$. □

Under the assumptions of this section, we have thus given a completely explicit formula for the Hitchin connection. Let us restate Theorem 6.10 with this formula.

Theorem 6.12. *Let (M,ω) be a compact, prequantizable symplectic manifold with $H^1(M,\mathbb{R}) = 0$, vanishing second Stiefel-Whitney class and $\tilde{c}_1(M,\omega) = n\big[\frac{\omega}{2\pi}\big]$. Further, let J be a rigid, holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$. Then the expression*

$$\boldsymbol{\nabla}_V = \hat{\nabla}_V + \frac{1}{4k}(\Delta_{G(V)} - \Delta_{G(V)}(F) - dF\cdot G(V)\cdot dF - 2nV'[F])$$

defines a Hitchin connection in the bundle $\hat{\mathcal{Q}}_k^\delta$ over $\mathcal{T}$.

6.3.2 Relating the Quantum Spaces and Hitchin Connections

We wish to relate the quantum spaces of metaplectic quantization to the spaces of standard geometric quantization, with the intent to describe the relation between the Hitchin connections in the two settings.

To this end, we must choose our prequantum line bundle wisely. More precisely, it must by chosen to be compatible with the choice of metaplectic structure, and this is possible by the assumption on the first Chern class of M.

Lemma 6.13. *There exists a prequantum line bundle $\mathcal{L}$ on M such that*

$$\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta), \tag{6.17}$$

where $c_1(\delta)$ is the class specified by the metaplectic structure on M.

Proof. Let $\mathcal{L}_0$ be any prequantum line bundle, and let J be any Kähler structure on M. Further, let F be a Ricci potential on M, and consider the Hermitian line bundles $(\mathcal{L}_0^{-n/2}, e^F h^{\mathcal{L}_0})$ and (δ, h^δ) over M. It is easily calculated that these line bundles have the same curvature, and therefore the tensor product of the former with the dual of the latter yields a flat Hermitian line bundle L_1. Since $2c_1(\delta) = c_1(K) = -c_1(M,\omega)$, which is divisible by n, there exists a flat Hermitian line bundle L_2 such that $L_2^{n/2} \cong L_1$. Finally, the line bundle $\mathcal{L} = \mathcal{L}_0 \otimes L_2$ has the structure of a prequantum line bundle, and $\frac{n}{2}c_1(\mathcal{L}) = c_1(\mathcal{L}^{n/2}) = -c_1(\delta)$. It follows that $\mathcal{L}$ is the desired prequantum line bundle. □

From now on, we will assume that our prequantum line bundle satisfies (6.17). Suppose that $\mathcal{T}$ is contractible, and let $\tilde{F}$ be a family of Ricci potentials satisfying (5.16). We wish to find an isomorphism $\hat{\varphi}$ of Hermitian holomorphic line bundles over $\mathcal{T} \times M$,

$$\hat{\varphi}\colon (\hat{\mathcal{L}}^{k-n/2}, e^{\tilde{F}}\hat{h}^{\mathcal{L}}) \to (\hat{\mathcal{L}}^k \otimes \hat{\delta}, \hat{h}). \tag{6.18}$$

Since $\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta)$, the line bundles are isomorphic as complex line bundles. The curvature of $\hat{\mathcal{L}}^{k-n/2}$, with the given Hermitian structure, is easily calculated and seen to agree with the curvature of the bundle $\hat{\mathcal{L}}^k \otimes \hat{\delta}$, which can be found using Proposition 5.18.

Thus the obstruction to finding the structure preserving isomorphism $\hat{\varphi}$ lies in the first cohomology of $\mathcal{T} \times M$. But this is trivial by the Künneth formula since $\mathcal{T}$ is contractible and $H^1(M, \mathbb{R}) = 0$, by assumption. The isomorphism (6.18) gives an isomorphism of the bundles of quantum spaces $\hat{\mathcal{Q}}_{k-n/2}$ and $\hat{\mathcal{Q}}_k^\delta$.

It is easily seen that the pullback under $\hat{\varphi}$ of the Chern connection $\hat{\nabla}^{\mathcal{L}^k \otimes \delta}$ is given by

$$\hat{\varphi}^* \hat{\nabla}^{\mathcal{L}^k \otimes \delta} = \hat{\nabla}^{\mathcal{L}^{k-n/2}} + \hat{\partial}\tilde{F} \tag{6.19}$$

since the right hand side is the unique Hermitian connection compatible with the holomorphic structure on $\hat{\mathcal{L}}^{k-n/2}$.

Recall that the Hitchin connection in the bundle $\hat{\mathcal{Q}}_k$ is given by $\boldsymbol{\nabla}_V = \nabla^t_V + a(V)$, where

$$a(V) = \frac{1}{4k+2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot d\tilde{F}} - 2nV'[\tilde{F}]) + V'[\tilde{F}]. \tag{6.20}$$

Using (6.19), it is straightforward to verify that the pullback by $\hat{\varphi}$ of the operator $\Delta_{G(V)}$, acting on sections of $\hat{\mathcal{L}}^k \otimes \hat{\delta}$, is given by

$$\hat{\varphi}^* \Delta_{G(V)} = \Delta_{G(V)} + 2\nabla_{G(V)\cdot d\tilde{F}} - 4b_0^\delta(V) - 2nV'[\tilde{F}], \tag{6.21}$$

where $b_0^\delta(V)$ is given by the expression in Proposition 6.11, but in terms of $\tilde{F}$. Furthermore, in the bundle $\hat{\mathcal{L}}^{k-n/2}$, the formula (6.20) becomes

$$\begin{aligned} a(V) &= \frac{1}{4k}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot d\tilde{F}} - 2nV'[\tilde{F}]) + V'[\tilde{F}] \\ &= \frac{1}{4k}(\hat{\varphi}^*\Delta_{G(V)} + 4b_0^\delta(V)) + V'[\tilde{F}] \\ &= \hat{\varphi}^* a^\delta(V) + V'[\tilde{F}], \end{aligned} \tag{6.22}$$

and this means that the pullback by $\hat{\varphi}$ of the Hitchin connection in the metaplectic setting is given by

$$\begin{aligned} \hat{\varphi}^* \boldsymbol{\nabla}_V^\delta &= \hat{\varphi}^* \hat{\nabla}_V + \hat{\varphi}^* a^\delta(V) \\ &= \nabla^t_V + V'[\tilde{F}] + \hat{\varphi}^* a^\delta(V) \\ &= \nabla^t_V + a(V) \\ &= \boldsymbol{\nabla}_V. \end{aligned} \tag{6.23}$$

Thus the two connections agree, and we have proved

Theorem 6.14. *Let (M,ω) be a compact, prequantizable symplectic manifold with vanishing second Stiefel-Whitney class and $H^1(M,\mathbb{R})=0$. Further, let J be a rigid, holomorphic family of Kähler structures on M, parametrized by a contractible complex manifold $\mathcal{T}$. Assume that the first Chern class of (M,ω) is divisible by an even integer n and that its image in $H^2(M,\mathbb{R})$ satisfies*

$$c_1(M,\omega) = n[\tfrac{\omega}{2\pi}].$$

Then there exists a smooth family $\tilde{F}$ of Ricci potentials on M and an isomorphism of vector bundles

$$\varphi\colon \hat{\mathcal{Q}}_{k-n/2} \to \hat{\mathcal{Q}}_k^\delta$$

such that

$$\varphi^* \boldsymbol{\nabla}^\delta = \boldsymbol{\nabla},$$

where $\varphi^*\boldsymbol{\nabla}^\delta$ is the pullback of the Hitchin connection on $\hat{\mathcal{Q}}_k^\delta$, and $\boldsymbol{\nabla}$ is the Hitchin connection on $\hat{\mathcal{Q}}_{k-n/2}$, both of which are expressed in terms of $\tilde{F}$.

6.4 Curvature of the Hitchin Connection

In this section, we calculate the curvature of the Hitchin connection and prove that it is projectively flat under certain conditions. We will focus on calculating the curvature in the metaplectic case, and therefore we shall work under the assumptions of Theorem 6.12, which guarantee that the Hitchin connection exists and that we have an explicit expression for it.

Recall that the Hitchin connection is given by

$$\boldsymbol{\nabla}_V^\delta = \hat{\nabla}_V + \frac{1}{k} b^\delta(V) \qquad \text{with} \qquad b^\delta(V) = \frac{1}{4}\Delta_{G(V)} + b_0^\delta(V),$$

where $b_0^\delta(V)$ is given by (6.16). Since the family of Kähler structures is holomorphic, the one-form b^δ has type $(1,0)$ on the complex manifold $\mathcal{T}$, as can be seen from its explicit expression. But then

$$F_{\boldsymbol{\nabla}^\delta}(V'',W'') = F_{\hat{\nabla}}(V'',W'') = \frac{i}{2}\theta(V'',W'') = 0, \tag{6.24}$$

for any vector fields V and W on $\mathcal{T}$, where the second equality follows by Proposition 6.6, and the last equality is due to the fact that θ has type $(1,1)$ when the family is holomorphic. In other words, the $(2,0)$-part of the curvature of the Hitchin connection vanishes. In particular, this means that the Hitchin connection induces a holomorphic structure on the bundle $\hat{\mathcal{Q}}_k^\delta$ over $\mathcal{T}$.

The $(1,1)$-part of the curvature is somewhat more laborious to calculate, so we split the calculation into a few lemmas.

Lemma 6.15. *On sections of $\hat{\mathcal{Q}}_k^\delta$, the commutator $\big[\hat{\nabla}_{W''}, \Delta_{G(V)}\big]$ is given by*

$$\big[\hat{\nabla}_{W''}, \Delta_{G(V)}\big] = 2ik\theta(V',W'') - \frac{1}{4}\operatorname{Tr}\bar{G}(W)\cdot\omega\cdot G(V)\cdot\rho + \frac{i}{4}\delta(\delta\bar{G}(W)\cdot\omega\cdot G(V)),$$

for any vector fields V and W on $\mathcal{T}$ such that V' and W'' commute.

Proof. Let s be any section of $\hat{\mathcal{Q}}^\delta_k$. Using Proposition 6.6, which gives the curvature of the connection $\hat\nabla$ on $\hat{\mathcal{L}}^k \otimes \hat\delta$, one can calculate that

$$\big[\hat\nabla_{W''}, \Delta_{G(V)}\big]s = \Delta_{W''[G(V)]}s + \frac{i}{2}\nabla_{\delta\bar G(W)\cdot\omega\cdot G(V)}s + \frac{i}{4}\delta(\delta\bar G(W)\cdot\omega\cdot G(V))s,$$

even if s was a section of $\hat{\mathcal{P}}^\delta_k$. Now the family is rigid, so the formula (5.6) tells us that

$$\delta(W''[G(V)]) = \frac{i}{2}\delta G(V)\cdot\omega\cdot\bar G(V) - \frac{i}{2}\delta\bar G(W)\cdot\omega\cdot G(V),$$

and therefore we get

$$\Delta_{W''[G(V)]}s = \nabla^2_{W''[G(V)]}s + \nabla_{\delta(W''[G(V)])}s = \nabla^2_{W''[G(V)]} - \frac{i}{2}\nabla_{\delta\bar G(W)\cdot\omega\cdot G(V)}s$$

since s is a section of $\hat{\mathcal{Q}}^\delta_k$, so we conclude that

$$\big[\hat\nabla_{W''}, \Delta_{G(V)}\big]s = \nabla^2_{W''[G(V)]}s + \frac{i}{4}\delta(\delta\bar G(W)\cdot\omega\cdot G(V))s.$$

For any pair of vector fields X and Y on M, we clearly have

$$\nabla^2_{X',Y''}s = \nabla_{X'}\nabla_{Y''}s - \nabla_{\nabla_{X'}Y''}s = 0,$$

and furthermore

$$\nabla^2_{Y'',X'}s = \nabla_{Y''}\nabla_{X'}s - \nabla_{\nabla_{Y''}X'}s = -ik\omega(Y'',X')s + \frac{i}{2}\rho(Y'',X')s,$$

so

$$\begin{aligned}\nabla^2_{W''[G(V)]}s &= -\frac{i}{2}\nabla^2_{\bar G(W)\cdot\omega\cdot G(V)}s \\ &= \frac{k}{2}\operatorname{Tr}\big(\bar G(W)\cdot\omega\cdot G(V)\cdot\omega\big)s - \frac{1}{4}\operatorname{Tr}\big(\bar G(W)\cdot\omega\cdot G(V)\cdot\rho\big)s \\ &= 2ik\theta(V',W'')s - \frac{1}{4}\operatorname{Tr}\big(\bar G(W)\cdot\omega\cdot G(V)\cdot\rho\big)s,\end{aligned}$$

which proves the lemma. □

Lemma 6.16. *The one-form b^δ_0 satisfies*

$$\begin{aligned}W''[b^\delta_0(V)] = &-\frac{in}{4}\theta(V',W'') + \frac{1}{16}\operatorname{Tr}\bar G(W)\cdot\omega\cdot G(V)\cdot\rho \\ &- \frac{i}{16}\delta(\delta\bar G(W)\cdot\omega\cdot G(V)) - \frac{n}{2}W''V'[F].\end{aligned}$$

for any vector fiels V and W on $\mathcal{T}$, such that V' and W'' commute.

Proof. Since $\Delta_{G(V)}F = \delta(G(V)\cdot dF)$, the identities (5.6) and (5.14) give us that

$$\begin{aligned}W''[\Delta_{G(V)}(F)] &= \delta(W''[G(V)]\cdot dF) + \delta(G(V)\cdot W''[F]) \\ &= \frac{i}{4}\delta(\delta\bar G(W)\cdot\omega\cdot G(V)) - \frac{i}{2}\delta(\bar G(W)\cdot\omega\cdot G(V)\cdot dF) \\ &= \frac{i}{4}\delta(\delta\bar G(W)\cdot\omega\cdot G(V)) - \frac{i}{2}\delta\bar G(W)\cdot\omega\cdot G(V)\cdot dF + \frac{i}{2}\partial\bar\partial F(\bar G(W)\cdot\omega\cdot G(V)),\end{aligned}$$

where rigidity of the family of Kähler structures was used for the last equality. The Ricci potential satisfies the equation $\rho = n\omega + 2i\partial\bar{\partial}F$, and so we get that

$$\begin{aligned}\frac{i}{2}\partial\bar{\partial}F(\bar{G}(W)\cdot\omega\cdot G(V)) &= \frac{n}{4}\operatorname{Tr}\bar{G}(W)\cdot\omega\cdot G(V)\cdot\omega - \frac{1}{4}\operatorname{Tr}\bar{G}(W)\cdot\omega\cdot G(V)\cdot\rho \\ &= in\theta(V', W'') - \frac{1}{4}\operatorname{Tr}\bar{G}(W)\cdot\omega\cdot G(V)\cdot\rho.\end{aligned}$$

Finally, the identities (5.6) and (5.14) can be used to verify that

$$W''[dF\cdot G(V)\cdot dF] = \frac{i}{2}\delta\bar{G}(W)\cdot\omega\cdot G(V)\cdot dF,$$

and the proposition follows by combining the identities above. □

Proposition 6.17. *The* $(1,1)$*-part of the curvature of the Hitchin connection is given by*

$$F^{1,1}_{\boldsymbol{\nabla}^{\delta}} = \frac{in}{4k}(\theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F)$$

on the bundle $\hat{\mathcal{Q}}^{\delta}_k$.

Proof. Let V and W be vector fields on $\mathcal{T}$ such that V' and W'' commute. Then the curvature of the Hitchin connection is given by

$$\begin{aligned}F_{\boldsymbol{\nabla}^{\delta}}(V', W'') &= \big[\boldsymbol{\nabla}^{\delta}_{V'}, \boldsymbol{\nabla}^{\delta}_{W''}\big] \\ &= \big[\hat{\nabla}_{V'}, \hat{\nabla}_{W''}\big] + \frac{1}{k}\big[b^{\delta}(V), \hat{\nabla}_{W''}\big].\end{aligned}$$

By Proposition 6.6, the first term is given by

$$\big[\hat{\nabla}_{V'}, \hat{\nabla}_{W''}\big] = \frac{i}{2}\theta(V', W''),$$

and by Lemma 6.15 and Proposition 6.17, the last term is given by

$$\begin{aligned}\frac{1}{k}\big[b^{\delta}(V), \hat{\nabla}_{W''}\big] &= -\frac{1}{k}\big[\hat{\nabla}_{W''}, \frac{1}{4}\Delta_{G(V)} + b^{\delta}_0(V)\big] \\ &= -\frac{i}{2}\theta(V', W'') + \frac{in}{4k}(\theta(V', W'') - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F(V', W'')).\end{aligned}$$

This proves the proposition. □

Notice that Proposition 5.16 implies that the $(1,1)$-part of the curvature of the Hitchin connection is just a projective factor, in the sense that its endomorphism-part is just multiplication by a scalar.

Finally, we shall calculate the $(2,0)$-part of the curvature of the Hitchin connnection, and once again, we shall aid the computation with a few lemmas. First, we have

Lemma 6.18. *For any Kähler structure on* M*, any vector field* $X \in C^{\infty}(M, T'M)$ *and any bivector field* $B \in C^{\infty}(M, S^2(T'M))$*, we have the symbols*

$$\begin{aligned}\sigma_2(\big[\nabla_X, \nabla^2_B\big]) &= \nabla_X B - 2\,\mathcal{S}(B\cdot\nabla X) \\ \sigma_1(\big[\nabla_X, \nabla^2_B\big]) &= -\nabla^2_B X \\ \sigma_0(\big[\nabla_X, \nabla^2_B\big]) &= 0,\end{aligned}$$

for the commutator of the operators ∇_X *and* ∇^2_B *on* $\mathcal{P}^{\delta}_k$.

Proof. Locally, we can write $B = \sum_j Y_j \otimes Z_j$, where Y_j and Z_j are local vector fields of type $(1,0)$. By writing out the commutator in terms of these vector fields, and using that the $(2,0)$-part of the curvature of $\mathcal{L}^k \otimes \delta$ vanishes, it is easily verified that

$$\big[\nabla_X, \nabla^2_B\big] = \nabla^2_{\nabla_X B} - \nabla^2_{B\cdot\nabla X} - \nabla^2_{\nabla(X)\cdot B} - \nabla_{\nabla^2_B X} + \sum_j R(Y_j, X) Z_j.$$

But the Kähler curvature has type $(1,1)$, so the last term vanishes, and the operator ∇^2 only depends on the symmetric part of a bivector field of type $(2,0)$, so the lemma follows. □

The next lemma is proved in a completely analogous way, although the computations are lengthier.

Lemma 6.19. *For any Kähler structure on M and any bivector fields $A, B \in C^\infty(M, S^2(T'M))$, we have the symbols*

$$\begin{aligned}
\sigma_3\big(\big[\nabla^2_A, \nabla^2_B\big]\big) &= 2\,\mathcal{S}(A\cdot\nabla B) - 2\,\mathcal{S}(B\cdot\nabla A)\\
\sigma_2\big(\big[\nabla^2_A, \nabla^2_B\big]\big) &= \nabla^2_A B - \nabla^2_B A\\
\sigma_1\big(\big[\nabla^2_A, \nabla^2_B\big]\big) &= 0\\
\sigma_0\big(\big[\nabla^2_A, \nabla^2_B\big]\big) &= 0,
\end{aligned}$$

for the commutator of the operators ∇^2_A and ∇^2_B on $\mathcal{P}^\delta_k$.

Using these two lemmas, we can prove the following important proposition.

Proposition 6.20. *The commutator of $b^\delta(V)$ and $b^\delta(W)$ is a first-order operator on $\mathcal{P}^\delta_k$ with symbols given by*

$$\begin{aligned}
\sigma_1\big(\big[b^\delta(V), b^\delta(W)\big]\big) = {}& \frac{1}{16}\nabla^2_{G(V)}\delta G(W) - \frac{1}{16}\nabla^2_{G(W)}\delta G(V) + \frac{1}{16}\big[\delta G(V), \delta G(W)\big]\\
&+ \frac{1}{2} db^\delta_0(W)\cdot G(V) - \frac{1}{2} db^\delta_0(V)\cdot G(W)
\end{aligned}$$

and

$$\sigma_0\big(\big[b^\delta(V), b^\delta(W)\big]\big) = \frac{1}{4}\Delta_{G(V)} b^\delta_0(W) - \frac{1}{4}\Delta_{G(W)} b^\delta_0(V),$$

for any vector fields V and W on $\mathcal{T}$.

Proof. First of all, we get

$$\big[b^\delta(V), b^\delta(W)\big] = \frac{1}{16}\big[\Delta_{G(V)}, \Delta_{G(W)}\big] + \frac{1}{4}\big[\Delta_{G(V)}, b^\delta_0(W)\big] - \frac{1}{4}\big[\Delta_{G(W)}, b^\delta_0(V)\big].$$

By applying Lemma 6.19 and Proposition 5.12, the third-order symbol of the first term vanishes,

$$\begin{aligned}
\sigma_3\big(\big[\Delta_{G(V)}, \Delta_{G(W)}\big]\big) &= \sigma_3\big(\big[\nabla^2_{G(V)}, \nabla^2_{G(W)}\big]\big)\\
&= 2\,\mathcal{S}(G(V)\cdot\nabla G(W)) - 2\,\mathcal{S}(G(W)\cdot\nabla G(V))\\
&= 0.
\end{aligned}$$

Likewise, by Lemma 6.18, Lemma 6.19 and Proposition 5.13, the second-order symbol vanishes,

$$
\begin{aligned}
&\sigma_2([\Delta_{G(V)}, \Delta_{G(W)}]) \\
&\quad = \sigma_2([\nabla^2_{G(V)}, \nabla^2_{G(W)}]) + \sigma_2([\nabla_{\delta G(V)}, \nabla^2_{G(W)}]) - \sigma_2([\nabla_{\delta G(W)}, \nabla^2_{G(V)}]) \\
&\quad = \nabla^2_{G(V)} G(W) + \nabla_{\delta G(V)} G(W) + 2\,\mathcal{S}(G(V) \cdot \nabla\delta G(W)) \\
&\qquad - \nabla^2_{G(W)} G(V) - \nabla_{\delta G(W)} G(V) - 2\,\mathcal{S}(G(W) \cdot \nabla\delta G(V)) \\
&\quad = 0.
\end{aligned}
$$

It follows that the commutator is a first-order operator, with the claimed symbols, as is easily verified using Lemma 6.18 and Lemma 6.19. □

With the above results at hand, we can calculate the $(2,0)$-part of the curvature of the Hitchin connection.

Proposition 6.21. *The $(2,0)$-part of the curvature of the Hitchin connection is the first-order operator on $\mathcal{P}^\delta_k$ given by*

$$
F^{2,0}_{\boldsymbol{\nabla}^\delta}(V, W) = \frac{1}{k} \partial_{\mathcal{T}} b^\delta_0(V, W) + \frac{1}{k^2} [b^\delta(V), b^\delta(W)], \tag{6.25}
$$

for any vector fields V and W on $\mathcal{T}$.

Proof. Let V and W be vector fields on $\mathcal{T}$, such that V' and W' commute. Then the curvature of the Hitchin connection is given by

$$
\begin{aligned}
F_{\boldsymbol{\nabla}^\delta}(V', W') &= [\boldsymbol{\nabla}^\delta_{V'}, \boldsymbol{\nabla}^\delta_{W'}] \\
&= [\hat{\nabla}_{V'}, \hat{\nabla}_{W'}] + \frac{1}{k}[\hat{\nabla}_{V'}, b^\delta(W)] - \frac{1}{k}[\hat{\nabla}_{W'}, b^\delta(V)] + \frac{1}{k^2}[b^\delta(V), b^\delta(W)].
\end{aligned}
$$

The curvature of the bundle $\hat{\mathcal{L}}^k \otimes \hat{\delta}$ has type $(1,1)$, which implies that the first term vanishes. Furthermore, it implies that

$$
[\hat{\nabla}_{V'}, \Delta_{G(W)}] = \Delta_{V'[G(W)]},
$$

as is easily verified, and therefore

$$
\begin{aligned}
&[\hat{\nabla}_{V'}, b^\delta(W)] - [\hat{\nabla}_{W'}, b^\delta(V)] \\
&\quad = \frac{1}{4}\Delta_{V'[G(W)]} - \frac{1}{4}\Delta_{W'[G(V)]} + V'[b^\delta_0(W)] - W'[b^\delta_0(V)] \\
&\quad = d_{\mathcal{T}} b^\delta_0(V', W'),
\end{aligned}
$$

where we used that $V'[G(W)] = W'[G(V)]$ for the last equality. □

The symbols of the curvature are easily obtained from Proposition 6.20. Furthermore, we can use Proposition 6.21 to prove projective flatness of the Hitchin connection under certain conditions.

Theorem 6.22. *If none of the Kähler structures in the family admit a holomorphic vector field on M, then the Hitchin connection is projectively flat.*

Proof. We must prove that the curvature of the Hitchin connection at a point $\sigma \in \mathcal{T}$ is a projective factor, that is, its endomorphism-part is given by multiplication with a scalar. By Proposition 6.17 and (6.24), this is the case for the $(1,1)$ and $(0,2)$-part, and by Proposition 6.21, the $(2,0)$-part is at most a first-order operator.

Since the Hitchin connection preserves the subbundle $\hat{\mathcal{Q}}_k^\delta$ of $\hat{\mathcal{P}}_k^\delta$, the endomorphism-part of the curvature must preserve the subspace $\mathcal{Q}_k^\delta(\sigma)$ of $\mathcal{P}_k^\delta(\sigma)$, and consequently the symbol must be a holomorphic vector field on M_σ. By assumption, such a vector field cannot exist, and therefore the endomorphism-part of the curvature must be an operator of order zero on $\mathcal{Q}_k^\delta(\sigma)$. In other words, it is given by multiplication by a function on M, which must also be holomorphic and hence constant by compactness. □

If the Hitchin connection is projectively flat, then each of the terms in (6.25),

$$\partial_{\mathcal{T}} b_0^\delta(V, W) \qquad \text{and} \qquad \big[b^\delta(V), b^\delta(W)\big],$$

must be a projective factor since the curvature must be a projective factor for each value of k. In particular, this means that the first-order symbol of $\big[b^\delta(V), b^\delta(W)\big]$ must vanish, and the symbol of order zero must be constant on M. But for any function f on M, the function $\Delta_{G(V)} f = \delta(G(V) \cdot df)$ has average zero since

$$\int_M \delta(X)\omega^m = \int_M \mathcal{L}_X \omega^m = \int_M d i_X \omega^m = 0,$$

for any vector field X on M, and conseqently we have

$$\big[b^\delta(V), b^\delta(W)\big] = 0,$$

for all vector fiels V and W on $\mathcal{T}$. Moreover, if we define the one-form β on $\mathcal{T}$ by

$$\beta(V) = \frac{1}{\mathrm{Vol}(M)} \int_M b_0^\delta(V)\omega^m = -\frac{1}{4\,\mathrm{Vol}(M)} \int_M dF \cdot G(V) \cdot dF \frac{\omega^m}{m!},$$

then $\partial_{\mathcal{T}} \beta = \partial_{\mathcal{T}} b_0^\delta$ since the latter takes values in constant functions on M, by projective flatness of the Hitchin connection. In conclusion, we have proved

Proposition 6.23. *If the Hitchin connection is projectively flat, its curvature is given by*

$$F^{2,0}_{\boldsymbol{\nabla}^\delta} = \frac{1}{k}\partial_{\mathcal{T}}\beta, \qquad F^{1,1}_{\boldsymbol{\nabla}^\delta} = \frac{in}{4k}(\theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}} F), \qquad F^{0,2}_{\boldsymbol{\nabla}^\delta} = 0,$$

on the bundle $\hat{\mathcal{Q}}_k^\delta$.

If the Hitchin connection is projectively flat, the parallel translation maps along homotopic curves are equal up to scale. Thus, if the parameter space $\mathcal{T}$ is simply connected, the Hitchin connection gives a canonical identification of the projectivized quantum spaces associated with different complex structures. In this sense, the quantization is independent of the complex structure. Furthermore, the action of a symmetry group Γ permutes the various quantum spaces, so if they are canonically identified as projective vector spaces, then we get a projective representations of Γ, for each level k of the quantization.

But quantization is not only about producing the projective vector space. The Hilbert space structure on the quantum space is an essential part of the theory, as is the quantization of observables. Ultimately, the interest is in the spectrum of the quantized observables, since the spectrum represents the possible outcome of measurements. The identification of the quantum spaces, provided by the Hitchin connection, should therefore preserve the Hermitian structure and the quantization of observables.

In the following section, we discuss the relation between the Hermitian structure and the Hitchin connection. The relation between the Hitchin connection and the quantized observables will take us back to deformation quantization and lead us to define the notion of a formal Hitchin connection. This will be the topic of the next chapter.

6.5 Unitarity of the Hitchin Connection

Throughout this section, we shall work under the assumptions of Theorem 6.12 to ensure that the Hitchin connection exists and that we have an explicit formula for it. Moreover, we shall assume that the Hitchin connection is projectively flat.

By projective flatness, the parallel transport identifies the quantum spaces as projective vector spaces, and we would like to prove that this identification is isometric, which asserts that the Hitchin connection is compatible with the Hermitian structure

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2) \frac{\omega^m}{m!}, \tag{6.26}$$

on the bundle $\hat{\mathcal{Q}}_k^\delta$. But the Hitchin has no chance of preserving this Hermitian structure, or indeed any other, unless the $(2,0)$-part of its curvature vanishes, which is not generally the case by Proposition 6.23. This can be fixed, however, by considering $\tilde{\boldsymbol{\nabla}}^\delta = \boldsymbol{\nabla}^\delta - \frac{1}{k}\beta$, which is also a Hitchin connection. Indeed, this connection preserves the subbundle $\hat{\mathcal{Q}}_k^\delta$ since β is constant along M, and furthermore the curvature has type $(1,1)$, as one can easily verify.

Now, the restriction of the connection $\hat{\nabla}$ on $\hat{\mathcal{L}}^k \otimes \hat{\delta}$ to directions along $\mathcal{T}$ preserves the inner product (6.26). Indeed, we get that

$$V[\langle s_1, s_2 \rangle] = \int_M \big(h(\hat{\nabla}_V s_1, s_2) + h(s_1, \hat{\nabla}_{\bar{V}} s_2)\big) \frac{\omega^m}{m!} = \langle \hat{\nabla}_V s_1, s_2 \rangle + \langle s_1, \hat{\nabla}_{\bar{V}} s_2 \rangle,$$

for any vector field V on $\mathcal{T}$, and consequently the Hitchin connection does not generally preserve this inner product. It might, however, preserve a different Hermitian structure, so consider a general inner product on $\hat{\mathcal{Q}}_k^\delta$, given by

$$H_k(s_1, s_2) = \langle D_k s_1, s_2 \rangle = \int_M h(D_k s_1, s_2) \frac{\omega^m}{m!}, \tag{6.27}$$

where $D_k \colon \mathcal{T} \to C^\infty(M)$ is a smooth family of real positive functions on M, depending on the level k of the quantization. The Hitchin connection preserves this Hermitian structure if and only if

$$0 = \tilde{\boldsymbol{\nabla}}_V^\delta (H_k)(s_1, s_2) = V[H_k(s_1, s_2)] - H_k(\tilde{\boldsymbol{\nabla}}_V^\delta s_1, s_2) - H_k(s_1, \tilde{\boldsymbol{\nabla}}_{\bar{V}}^\delta s_2), \tag{6.28}$$

which is equivalent to

$$\langle V'[D_k] s_1, s_2 \rangle = \frac{1}{k} \langle D_k b^\delta(V) s_1, s_2 \rangle - \frac{1}{k} \langle D_k \beta(V) s_1, s_2 \rangle, \tag{6.29}$$

since D_k takes values in real functions.

Now for any smooth section s of $\hat{\mathcal{P}}_k^\delta$, the operator $\Delta_{G(V)}$ satisfies

$$\pi_k^\delta D_k \Delta_{G(V)} s = \pi_k^\delta (\Delta_{G(V)} D_k - 2\nabla_{G(V)\cdot dD_k} - \Delta_{G(V)}(D_k))s = \pi_k^\delta \Delta_{G(V)}(D_k)s,$$

where we used Corollary 4.9 and Proposition 4.7. This can be applied to (6.29), which is therefore equivalent to

$$\langle V'[D_k]s_1, s_2\rangle = \frac{1}{k}\langle b^\delta(V)(D_k)s_1, s_2\rangle - \frac{1}{k}\langle \beta(V)D_k s_1, s_2\rangle$$

whenever s_2 is holomorphic. It follows that the Hitchin connection preserves H_k on $\hat{\mathcal{Q}}_k^\delta$ if and only if

$$V'[D_k] = \frac{1}{k} b^\delta(V)D_k - \frac{1}{k}\beta(V)D_k. \tag{6.30}$$

If we introduce $\tilde{b}^\delta(V)$ and $\tilde{\beta}(V)$ which are given by replacing $G(V)$ with $\tilde{G}(V)$ in the expressions for $b^\delta(V)$ and $\beta(V)$, then the following proposition follows by conjugating the condition (6.30).

Proposition 6.24. *The Hitchin connection* $\tilde{\boldsymbol{\nabla}}^\delta$ *preserves the Hermitian structure* H_k *on* $\hat{\mathcal{Q}}_k^\delta$ *if and only if the family of functions* D_k *satisfies*

$$V[D_k] = \frac{1}{k}\tilde{b}^\delta(V)D_k - \frac{1}{k}\tilde{\beta}(V)D_k, \tag{6.31}$$

for all vector fields V *on* $\mathcal{T}$.

The question is therefore whether (6.31) has any solutions. This question becomes much simpler if we consider an expansion of D_k in powers of k and write

$$D_k = \tilde{D}_0 + \frac{1}{k}\tilde{D}_1 + \frac{1}{k^2}\tilde{D}_2 + \cdots,$$

where each $\tilde{D}_j : \mathcal{T} \to C^\infty(M)$ is a family of smooth real functions, which is independent of the level k of the quantization. The equation (6.30), which is equivalent to (6.31), then becomes the recursive system of equations

$$V'[\tilde{D}_0] = 0 \qquad \text{and} \qquad V'[\tilde{D}_{j+1}] = b^\delta(V)\tilde{D}_j - \beta(V)\tilde{D}_j. \tag{6.32}$$

Clearly, the constant function $\tilde{D}_0 = 1$ solves the first equation, so we can try to solve (6.32) inductively. Assume that solutions have already been found up to some natural number j. If V and W are vector fields on $\mathcal{T}$ such that V' and W' commute, then (6.32) can only have a solution if

$$\begin{aligned} 0 &= V'[b^\delta(W)\tilde{D}_j] - W'[b^\delta(V)\tilde{D}_j] - V'[\beta(W)\tilde{D}_j] + W'[\beta(V)\tilde{D}_j] \\ &= (b^\delta(W) - \beta(W))V'[\tilde{D}_j] - (b^\delta(V) - \beta(V))W'[\tilde{D}_j] + \partial_{\mathcal{T}}(b_0^\delta - \beta)(V,W)\tilde{D}_j, \end{aligned}$$

where we used that $V'[\Delta_{G(W)}] = \Delta_{V'[G(W)]} = \Delta_{W'[G(V)]} = W'[\Delta_{G(V)}]$ for the last equation. But the function $\tilde{D}_j$ is assumed to satisfy (6.32), so the equation above is equivalent to

$$0 = \big[b^\delta(W), b^\delta(V)\big]\tilde{D}_{j-1} + \partial_{\mathcal{T}}(b_0^\delta - \beta)(V,W)\tilde{D}_{j-1}.$$

As we proved in the previous section, both of these terms vanish since the Hitchin connection is projectively flat. Consequently, there is no local obstruction to solving the system of equations (6.32). Let us therefore assume that $\mathcal{T}$ is contractible, which implies that we can find functions $\tilde{D}_j$ satisfying the equations.

Using these solutions, we define the functions

$$D_k^N = \sum_{j=0}^{N} \frac{1}{k^j} \tilde{D}_j$$

and the Hermitian structures

$$H_k^N(s_1, s_2) = \langle D_k^N s_1, s_2 \rangle = \sum_{j=0}^{N} \frac{1}{k^j} \int_M h(D_j s_1, s_2) \frac{\omega^m}{m!}, \tag{6.33}$$

for any sections s_1 and s_2 of $\hat{\mathcal{Q}}_k^\delta$.

Since the functions $\tilde{D}_j$ satisfy the equations (6.32), it is easily verified that the covariant derivative of these Hermitian structures with respect to the Hitchin connection are given by

$$\tilde{\boldsymbol{\nabla}}_V^\delta(H_k^N)(s_1, s_2) = -\frac{1}{k^{N+1}} \Big(\langle b^\delta(V)(\tilde{D}_N) s_1, s_2 \rangle + \langle \beta(V) \tilde{D}_N s_1, s_2 \rangle \Big). \tag{6.34}$$

This shows that the Hitchin connection preserves the Hermitian structure H_k^N up to order N in k. In fact, if we use the inner product (6.26) to define an analog of the operator norm on bilinear forms by

$$\big\| \tilde{\boldsymbol{\nabla}}_V^\delta(H_k^N) \big\| = \sup_{s_1, s_2 \in \mathcal{Q}_k^\delta} \frac{\big| \tilde{\boldsymbol{\nabla}}_V^\delta(H_k^N)(s_1, s_2) \big|}{\|s_1\| \|s_2\|},$$

then we have the following asymptotic result regarding unitarity of the Hitchin connection.

Theorem 6.25. *Assume that $\mathcal{T}$ is contractible and that the Hitchin connection is projectively flat. Then, for any natural number N, there exist Hermitian structures H_k^N on $\hat{\mathcal{Q}}_k^\delta$ such that*

$$\big\| \tilde{\boldsymbol{\nabla}}_V^\delta(H_k^N) \big\| = O\Big(\frac{1}{k^{N+1}} \Big)$$

as k tends to infinity.

Proof. Let H_k^N be the Hermitian structures defined by (6.33), so that (6.34) is satisfied. But

$$\begin{aligned} \big| \langle b^\delta(V)(\tilde{D}_N) s_1, s_2 \rangle \big| &\leq \big\| T_k^\delta(b^\delta(V)(\tilde{D}_N)) s_1 \big\| \|s_2\| \\ &\leq \big\| T_k^\delta(b^\delta(V)(\tilde{D}_N)) \big\| \|s_1\| \|s_2\| \\ &\leq \big\| b^\delta(V)(\tilde{D}_N) \big\|_\infty \|s_1\| \|s_2\|, \end{aligned}$$

where we used Theorem 4.3 for the last inequality, and similarly,

$$\big| \langle \beta(V) \tilde{D}_N s_1, s_2 \rangle \big| \leq \big\| \beta(V)(\tilde{D}_N) \big\|_\infty \|s_1\| \|s_2\|,$$

so the theorem follows immediately from (6.34). □

In the presence of a symmetry group Γ, the Hermitian structure should also be preserved by the symmetries. This is clearly the case for the inner product (5.13), but not necessarily for the Hermitian structures H_k^N constructed above. Indeed, we must solve the equations (6.31) equivariantly to obtain an invariant Hermitian structure, and we can only guarantee that this is possible if the equivariant cohomology group $H^1_\Gamma(\mathcal{T}, C^\infty(M))$ vanishes. If the manifold $\mathcal{T}$ is contractible, this cohomology group is equal to the the group cohomology $H^1(\Gamma, C^\infty(M))$, which is therefore required to vanish.

Although the Hitchin connection $\boldsymbol{\nabla}^\delta$ has no chance of being unitary, due to the $(2,0)$-part of its curvature, it might be preserve some Hermitian structure projectively. By definition, this is the case if there exists a Hermitian structure H_k and a one-form $\kappa_k \in \Omega^1(\mathcal{T})$ such that

$$\boldsymbol{\nabla}^\delta_V(H_k)(s_1, s_2) = \kappa_k(V) H_k(s_1, s_2), \tag{6.35}$$

for all vector fields V on $\mathcal{T}$ and all sections s_1 and s_2 of $\hat{\mathcal{Q}}^\delta_k$. This condition implies that the parallel transport identifies the quantum spaces with the inner product H_k as projective Hilbert spaces. Ultimately, this is all we need, since the projective Hilbert space is the object of interest in quantization.

If we assume that $\tilde{\boldsymbol{\nabla}}^\delta$ preserves a Hermitian structure H_k, then $\boldsymbol{\nabla}^\delta$ is projectively unitary with respect to the same Hermitian structure. Indeed, it is easily verified that $\boldsymbol{\nabla}^\delta$ satisfies (6.35) with $\kappa_k(V) = -\frac{1}{k}(\beta(V) + \overline{\beta(V)})$. Consequently, the particular choice of Hitchin connection is not important from a metric point of view.

6.6 Quantization Revisited

Having discussed the Hitchin connection in detail, let us briefly recall the objective of geometric quantization and the role of the Hitchin connection.

Suppose that the manifold $\mathcal{T}$ is contractible and that the Hitchin connection $\boldsymbol{\nabla}^\delta$ is projectively flat on the bundle $\hat{\mathcal{Q}}^\delta_k$ of quantum spaces associated with different complex structures. Moreover, assume that the Hitchin connection is projectively unitary with respect to some Hermitian structure H_k on $\hat{\mathcal{Q}}^\delta_k$.

Then the various quantum spaces are canonically identified as projective Hilbert spaces through the parallel transport of the Hitchin connection, and in this sense, the quantization is independent of the choice of complex structure. More precisely, one can consider the space of covariantly constant sections of the projectivization of $\hat{\mathcal{Q}}^\delta_k$, and this projective Hilbert space should be the true space of quantum states.

The quantization of an observable $f \in C^\infty(M)$ should be a self-adjoint operator acting on the space of quantum states. The quantum operators of Chapter 2 are not really the right operators to consider as they are based on the Hermitian structure (2.4), and consequently they are not self-adjoint with respect to H_k. Neither are they covariantly constant with respect to the Hitchin connection, which is required in order to act on the space of covariantly constant sections.

However, as we saw in Section 4.2, the quantum operators are examples of Toeplitz operators, and it is natural to follow this idea and define the quantization of the observable f by

$$Q^\delta_k(f)s = \pi^H_k f s,$$

where $\pi^H_k \colon \hat{\mathcal{P}}^\delta_k \to \hat{\mathcal{Q}}^\delta_k$ denotes the orthogonal projection with respect to the Hermitian structure H_k. Clearly, these operators are self-adjoint with respect to H_k, and it is easily checked that they are covariantly constant with respect to the Hitchin connection, so they act on the space of covariantly constant sections of (the projectivization of) $\hat{\mathcal{Q}}_k^\delta$, which is the space of quantum states.

With our limited understanding of unitarity, these operators are not very practical to work with, however. In Section 7.4 of the next chapter, we shall briefly return to the discussion of quantum observables and show how Toeplitz operators, with respect to the Hermitian structure (6.26), can be used to construct quantum observables which are self-adjoint with respect to (6.26) and covariantly constant with respect to the Hitchin connection to arbitrary order in k.

Chapter **7**

Formal Hitchin Connections

In this chapter, we return to the setting of formal deformation quantization. Previously, we have discussed various ways of constructing a formal deformation quantization on a symplectic manifold whenever a Kähler structure has been chosen. Once again the fundamental problem is that a quantization should not depend on this choice of complex structure. In geometric quantization, we approached this problem through the Hitchin connection, and in this chapter, we shall pursue an anologous idea and define a formal Hitchin connection. The ideas applied are heavily influenced by Andersen [And1], and so is the exposition given.

The Toeplitz operators provide a link between geometric and deformation quantization. On one hand, the quantum operators of geometric quantization are Toeplitz operators, and on the other hand, the asymptotic expansion of products of Toeplitz operators gives rise to a formal deformation quantization. As we shall see, the Hitchin connection manifests itself in the world of formal deformation quantizations through this link, and parallel transport of this formal Hitchin connection can be used to relate the deformation quantizations arising from different Kähler structures.

7.1 Formal Hitchin Connections

Let (M, ω) be any symplectic manifold, and let $\mathcal{T}$ be a smooth manifold. Consider the trivial bundle $\hat{C}_h^\infty(M) = \mathcal{T} \times C_h^\infty(M)$ over $\mathcal{T}$ of formal functions on M. Sections of this bundle are in bijective correspondence with the space $C^\infty(\mathcal{T} \times M)[[h]]$ of formal functions on $\mathcal{T} \times M$. Furthermore, let $\mathcal{D}_h(M) = \mathcal{D}(M)[[h]]$ be the space of formal differential operators on M.

Suppose that we have a smooth family $\hat{*}$ of formal deformation quantizations on M, parametrized by $\mathcal{T}$. In other words, for each $\sigma \in \mathcal{T}$, we have a star product $\hat{*}_\sigma$ on the fiber $C_h^\infty(M)$ of the bundle $\hat{C}_h^\infty(M)$.

As we did in the setting of geometric quantization, we consider a connection $\boldsymbol{D}$ on $\hat{C}_h^\infty(M)$ of the form

$$\boldsymbol{D} = D^t + A, \tag{7.1}$$

where D^t is the trivial connection on $\hat{C}_h^\infty(M)$, and $A \in \Omega^1(\mathcal{T}, \mathcal{D}_h(M))$ is a one-form on $\mathcal{T}$ with values in formal differential operators on M. Such a connection will be called a *formal connection*, and we seek an A such that this connection preserves the quantization in an appropriate sense. More precisely, the parallel transport with respect to the connection should constitute an equivalence between the star products on different fibers, which leads us to the following infinitesimal definition of a formal Hitchin connection.

Definition 7.1. A *formal Hitchin connection* for a family of star products $\hat{*}$ is a formal connection $\boldsymbol{D}$ on $\hat{C}_h^\infty(M)$, of the form (7.1), which satisfies

$$\boldsymbol{D}_V(f \mathbin{\hat{*}} g) = \boldsymbol{D}_V(f) \mathbin{\hat{*}} g + f \mathbin{\hat{*}} \boldsymbol{D}_V(g),$$

for any sections f and g of $\hat{C}_h^\infty(M)$.

In other words, a formal Hitchin connection should be a derivation for the family of star products. Since the one-form A takes values in formal differential operators, we shall write it as

$$A(V) = \sum_j A_j(V) h^j$$

for any vector field V on $\mathcal{T}$, where $A_j(V)$ is a differential operator on M.

If we have a symmetry group Γ acting on M and $\mathcal{T}$ in such a way that the family $\hat{*}$ is equivariant, then it is natural to require that a formal Hitchin connection is also equivariant.

7.1.1 Curvature and Formal Trivializations

Flat connections admit covariantly constant sections over simply connected subsets of the base space. For a formal connection, we formulate this fact in terms of formal trivializations.

Definition 7.2. A *formal trivialization* of a formal connection $\boldsymbol{D}$ is a map $P\colon \mathcal{T} \to \mathcal{D}_h(M)$ satisfying

$$\boldsymbol{D}_V(P(f)) = 0,$$

for all vector fields V on $\mathcal{T}$ and all functions f on M.

A formal trivialization therefore identifies formal functions on M with covariantly constant sections of $\hat{C}_h^\infty(M)$. Since the map $P\colon \mathcal{T} \to \mathcal{D}_h(M)$ is a family of formal differential operators, we shall write

$$P = \sum_j P_j h^j.$$

The next proposition gives sufficient conditions on a formal connection for the existence of a formal trivialization.

Proposition 7.3 (Andersen)**.** *If $\mathcal{T}$ is contractible, any flat formal connection $\boldsymbol{D} = D^t + A$ with $A_0 = 0$ admits a formal trivialization.*

Proof. A family of formal differential operators $P\colon \mathcal{T} \to \mathcal{D}_h(M)$ is a formal trivialization for $\boldsymbol{D}$ if and only if

$$0 = \boldsymbol{D}_V P = \sum_{j=0}^{\infty} \Big(V[P_j] + \sum_{r=0}^{j} A_r(V) P_{j-r} \Big) h^j.$$

This formal equation is of course equivalent to the system of equations

$$V[P_j] = -\sum_{r=1}^{j} A_r(V) P_{j-r}, \tag{7.2}$$

for each natural number j. Notice how the assumption $A_0 = 0$ implies that the variation of P_j is determined by the terms of P of lower degree. This means that we can try to solve the equations inductively.

Clearly, $P_0 = \mathrm{Id}$ satisfies the first of the equations. To prove existence of the higher-degree terms of P, we assume that the equations (7.2) have been solved up to some degree $j-1$, and we must show that right-hand side of (7.2) defines a closed one-form on $\mathcal{T}$. More precisely, the term P_j only has a chance of satisfying (7.2) if

$$0 = \sum_{r=1} d_{\mathcal{T}} A_r(V,W) P_{j-r} + \sum_{r=1}^{j} \sum_{s=1}^{j-r} \big[A_r(V), A_s(W)\big] P_{j-r-s},$$

for any commuting vector fields V and W on $\mathcal{T}$. This is indeed the case if the formal connection $\boldsymbol{D}$ is flat, as can be seen by considering the term of degree j of the curvature, which is given by

$$0 = F_{\boldsymbol{D}}(V,W) = d_{\mathcal{T}} A(V,W) + \big[A(V), A(W)\big].$$

Consequently, there is no local obstruction to finding a local formal trivialization, and the proposition follows since $\mathcal{T}$ is contractible. □

If the space $\mathcal{T}$ is contractible and the cohomology group $H^1(\Gamma, \mathcal{D}(M))$ vanishes, then the equivariant cohomology $H^1_\Gamma(\mathcal{T}, \mathcal{D}(M))$ vanishes, and the proof above can be used to produce an equivariant formal trivialization.

7.1.2 The Invariant Star Product

A formal trivialization of a formal Hitchin connection for a family of star products can be used to identify the star products corresponding to different points in $\mathcal{T}$.

Proposition 7.4 (Andersen)**.** *Suppose that $\boldsymbol{D}$ is a formal Hitchin connection for a family of star products $\hat{*}$ and that P is a formal trivialization of $\boldsymbol{D}$. Then*

$$f * g = P^{-1}(P(f) \mathbin{\hat{*}} P(g)) \tag{7.3}$$

defines a star product which is independent of the point in $\mathcal{T}$.

Proof. Clearly, the expression (7.3) defines a new family of star products, and we must show that the variation of this family along any vector field V on $\mathcal{T}$ vanishes.

On one hand, we have that

$$V[f*g] = V[P^{-1}](P(f) \mathbin{\hat{*}} P(g)) + P^{-1} V[P(f) \mathbin{\hat{*}} P(g)],$$

and on the other hand, since P is a formal trivialization of $\boldsymbol{D} = D^t + A$, we get that

$$0 = P^{-1}\boldsymbol{D}_V(P(f) \mathbin{\hat{*}} P(g)) = P^{-1} V[P(f) \mathbin{\hat{*}} P(g)] + P^{-1} A(V)(P(f) \mathbin{\hat{*}} P(g))$$

But since P is a formal trivialization, we get that

$$0 = \boldsymbol{D}_V(P(f)) = V[P](f) + A(V)P(f),$$

which implies that

$$V[P^{-1}] = -P^{-1}V[P]P^{-1} = P^{-1}A(V),$$

and if we apply this to the above, we get that $V[f * g] = 0$, as desired. □

If one has a good way of constructing a deformation quantization from a choice of complex structure, then this proposition can be used to obtain a quantization which is independent of the complex structure, as a good quantization should be.

In the presence of a symmetry group Γ, the star product $*$ will clearly be equivariant if the family $\hat{*}$ and the formal trivialization are equivariant.

7.2 The Berezin-Toeplitz Formal Hitchin Connection

By the classification of deformation quantizations with separation of variables on a Kähler manifold, we have plenty of examples of families of deformation quantizations. For the purpose of constructing a formal Hitchin connection, however, the family of deformation quantizations given by the Berezin-Toeplitz star products has the advantage that it is defined in terms of the Toeplitz operators in geometric quantization, where we have already constructed a Hitchin connection. As we shall see, this will help us construct a formal Hitchin connection for the family of Berezin-Toeplitz star products. The key is to understand the interaction of the Toeplitz operators and the Hitchin connnection.

Throughout all of this section, we shall assume that the conditions of Theorem 6.5 are satisfied, so that the Hitchin connection exists and we have an explicit formula for it.

7.2.1 Covariant Derivatives of Toeplitz Operators

The Hitchin connection $\boldsymbol{\nabla}$ on $\hat{\mathcal{Q}}_k$ induces a connection on the endomorphism bundle $\operatorname{End}(\hat{\mathcal{Q}}_k) \to \mathcal{T}$, and the Toeplitz operators, which act on the quantum spaces, are sections of this bundle. More precisely, if $f\colon \mathcal{T} \to C^\infty(M)$ is a family of smooth functions on M, then $T_k(f)$ defines a smooth section of the bundle $\operatorname{End}(\hat{\mathcal{Q}}_k) \to \mathcal{T}$. The Toeplitz operators are not covariantly constant with respect to the Hitchin connection, but as we shall see, the covariant derivative of a Toeplitz operator, much like the product of two Toeplitz operators, has an asymptotic expansion in terms of Toeplitz operators.

First, we must find a formula for the covariant derivative of a Toeplitz operator with respect to the Hitchin connection. For any section s of $\hat{\mathcal{P}}_k$, this is given by

$$\boldsymbol{\nabla}_V(T_k(f))s = \nabla^t_V(T_k(f))s + \big[a(V), T_k(f)\big]s.$$

Since the Hitchin connection preserves the bundle $\hat{\mathcal{Q}}_k$, the covariant derivative of a Toeplitz operator will also take values in this bundle, so we might as well compose with the projection onto $\hat{\mathcal{Q}}_k$, which yields

$$\pi_k \boldsymbol{\nabla}_V(T_k(f)) = \pi_k V[\pi_k]f + \pi_k V[f] + \pi_k a(V)\pi_k f - \pi_k f a(V). \tag{7.4}$$

To develop this expression further, we need the following lemma from [And1].

Lemma 7.5. *The variation of the projection $\pi_k \colon \hat{\mathcal{P}}_k \to \hat{\mathcal{Q}}_k$ satisfies*

$$\pi_k V[\pi_k] = \pi_k a(V)^* \pi_k - \pi_k a(V)^*,$$

for any vector field V on $\mathcal{T}$.

We recall that the explicit formula for the Hitchin connection is given by

$$a(V) = \frac{1}{k+n/2} b(V) + V'[F] \quad \text{with} \quad b(V) = \frac{1}{4}\big(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} - 2nV'[F]\big).$$

For the sake of notation, we further introduce the operator-valued one-form

$$c(V) = \frac{1}{4}\Big(\Delta_{\tilde{G}(V)} - 2\nabla_{\tilde{G}(V)\cdot dF} - 2\Delta_{\tilde{G}(V)}(F) + 2nV[F]\Big), \tag{7.5}$$

with zero-order part

$$c_0(V) = \frac{1}{2}\big(- \Delta_{\tilde{G}(V)}(F) + nV[F])\big).$$

With this notation and Lemma 7.5, the expression (7.4) becomes

$$\begin{aligned}\pi_k \boldsymbol{\nabla}_V(T_k(f)) &= \pi_k V[f] + \pi_k(a(V) + a(V)^*)\pi_k f - \pi_k f a(V) - \pi_k a(V)^* f \\ &= \pi_k V[f] + \pi_k V[F]\pi_k f + \pi_k V[F] f \\ &\quad + \frac{1}{k+n/2}\Big(\pi_k\big(b(V) + b(V)^*\big)\pi_k f - \pi_k f b(V) - \pi_k b(V)^* f\Big).\end{aligned} \tag{7.6}$$

Using Lemma 4.1 and Lemma 4.2, the adjoint of $b(V)$ is easily calculated,

$$b(V)^* = \frac{1}{4}\Big(\Delta_{\bar{G}(V)} - 2\nabla_{\bar{G}(V)\cdot dF} - 2\Delta_{\bar{G}(V)}(F) + 2nV''[F]\Big) = c(V''),$$

so by Proposition 4.7 and Corollary 4.9, we get that

$$\pi_k\big(b(V) + b(V)^*\big)\pi_k = \frac{1}{2}\pi_k\big(- \Delta_{\tilde{G}(V)}(F) + nV[F]\big)\pi_k = \pi_k c_0(V)\pi_k.$$

Finally, we get

$$\begin{aligned}\pi_k f b(V) &= \pi_k\Big(\frac{1}{4}\Delta_{G(V)} f - \frac{1}{2}\nabla_{G(V)\cdot df} + \frac{1}{2}\nabla_{G(V)\cdot dF} f - b(V)(f)\Big) \\ &= \frac{1}{4}\pi_k\Big(\Delta_{G(V)}(f) - 2\Delta_{G(V)}(F) f - 2dF\cdot G(V)\cdot df + 2nV'[F] f\Big) \\ &= \pi_k c(V')(f),\end{aligned}$$

and

$$\pi_k b(V)^* f \pi_k = \pi_k b(V)^*(f)\pi_k = \pi_k c(V'')(f)\pi_k,$$

so by combining the identities above with (7.6), we have proved

Proposition 7.6. *The covariant derivative, with respect to the Hitchin connection, of a Toeplitz operator acts by*

$$\begin{aligned}\boldsymbol{\nabla}_V(T_k(f)) &= T_k(V[f]) + T_k(V[F])T_k(f) - T_k(V[F]f) \\ &\quad + \frac{1}{k+n/2}\Big(T_k(c_0(V))\, T_k(f) - T_k(c(V)(f))\Big),\end{aligned} \tag{7.7}$$

on sections of $\hat{\mathcal{Q}}_k$.

This proposition will give us an asymptotic expansion of the covariant derivative of a Toeplitz operator since we already have an expansion for products of Toeplitz operators by Theorem 4.10. This expansion is, however, in powers of k, which does not play well with the factor $k+n/2$ in (7.7). It would therefore be more convenient to work with an asymptotic expansion of products of Toeplitz operators in powers of $k+n/2$.

7.2.2 The Shifted Berezin-Toeplitz Star Product

Recall from Theorem 4.10 that the product of two Toeplitz operators has an asymptotic expansion

$$T_k(f)T_k(g) \sim \sum_{j=0}^{\infty} T_k(C_j(f,g))\Big(\frac{1}{k}\Big)^j, \tag{7.8}$$

where the functions $C_j(f,g)$ are uniqely determined and specify the coefficients of the Berezin-Toeplitz star product

$$f \star g = \sum_j C_j(f,g)h^j.$$

If we consider the real function $\varphi(t) = \frac{t}{1-tn/2}$, which satisfies $\varphi\big(\frac{1}{k+n/2}\big) = \frac{1}{k}$, then (7.8) becomes

$$T_k(f)T_k(g) \sim \sum_{j=0}^{\infty} T_k(C_j(f,g))\Big(\varphi\Big(\frac{1}{k+n/2}\Big)\Big)^j. \tag{7.9}$$

The asymptotic expansion of φ, in powers of t as t tends to zero, is just the Taylor series expansion of φ around zero, and the corresponding formal series, with the same coefficients, yields the formal constant

$$\varphi(h) = \sum_{j=1}^{\infty} \Big(\frac{n}{2}\Big)^{j-1} h^j = h + \frac{n}{2}h^2 + \frac{n^2}{4}h^3 + \cdots .$$

Then it is a simple task to verify that we get an asymptotic expansion

$$T_k(f)T_k(g) \sim \sum_{j=0}^{\infty} T_k(\tilde{C}_j(f,g))\Big(\frac{1}{k+n/2}\Big)^j,$$

where the functions $\tilde{C}_j(f,g)$ are determined by

$$\sum_j \tilde{C}_j(f,g)h^j = \sum_j C_j(f,g)(\varphi(h))^j = (f \star g)[\varphi(h)] = f \star_{\varphi(h)} g.$$

For convenience, let us denote the star product $\star_{\varphi(h)}$ by $\tilde{\star}$.

To summarize, the star product $\tilde{\star}$ gives the asymptotic expansion of the product of two Toeplitz operators, but in powers of $k+n/2$,

$$T_k(f)T_k(g) \sim T_k(f \,\tilde{\star}\, g)\Big[\frac{1}{k+n/2}\Big]. \tag{7.10}$$

It is not difficult to fit the star product $\tilde{\star}$ into the classification schemes. The multiplicative inverse of the formal constant $\varphi(h)$ is given by

$$\frac{1}{\varphi(h)} = \frac{1}{h} - \frac{n}{2},$$

and consequently, by (3.3) and (4.6), the Karabegov form of $\tilde{\star}$ is given by

$$\mathrm{Kar}(\tilde{\star}_o) = \mathrm{Kar}(\star_o)[\varphi(h)] = -\omega\frac{1}{\varphi(h)} + \rho = -\omega\frac{1}{h} - \frac{n}{2}\omega + \rho.$$

Therefore, Theorem 3.10 gives an explicit expression for this star product. Moreover, by (3.2), the characteristic class of $\tilde{\star}$ is easily found

$$\mathrm{cl}(\tilde{\star}) = -\,\mathrm{cl}(\tilde{\star}_o) = \frac{1}{2\pi}\Big([\omega]\frac{1}{h} + \frac{n}{2}[\omega] - \frac{1}{2}[\rho]\Big) = \frac{1}{2\pi}\Big([\omega]\frac{1}{h} - i[d\bar{\partial}F]\Big) = \frac{[\omega]}{2\pi h}. \quad (7.11)$$

By comparing this with (4.8), we see that $\tilde{\star}$ is equivalent to the twisted Berezin-Toeplitz star product $\star^\delta$ arising from metaplectic quantization.

7.2.3 The Formal Hitchin Connection

Having introduced the star product $\tilde{\star}$, we can derive an asymptotic expansion of the covariant derivative of a Toeplitz operator from Proposition 7.6 and (7.10).

Theorem 7.7. *The formal connection $\tilde{\boldsymbol{D}} = D^t + \tilde{A}$, where $\tilde{A}$ is given by*

$$\tilde{A}(V)(f) = V[F] \,\tilde{\star}\, f - V[F]f + \big(c_0(V) \,\tilde{\star}\, f - c(V)(f)\big)h,$$

is the unique formal connection such that

$$\boldsymbol{\nabla}_V(T_k(f)) \sim T_k(\tilde{\boldsymbol{D}}_V f)\Big[\frac{1}{k+n/2}\Big], \quad (7.12)$$

for any vector field V on $\mathcal{T}$ and any family of smooth functions f on M.

The precise meaning of (7.12) in the theorem is

$$\Big\|\boldsymbol{\nabla}^e_V(T_k(f)) - T_k(V[f]) - \sum_{j=0}^{N} T_k(\tilde{A}_j(V)(f))\Big(\frac{1}{k+n/2}\Big)^j\Big\| = O\Big(\frac{1}{k^{N+1}}\Big), \quad (7.13)$$

as k tends to infinity. Also, the formal connection is equivariant with respect the action of a symmetry group Γ, which is clear from the explicit expression.

We remark that

$$\tilde{A} = 0 \mod h \quad (7.14)$$

since the zero-degree term of a star product is just the usual product on functions. In particular, this means that the Toeplitz operators are covariantly constant to first order in k. More precisely, we have

$$\big\|\boldsymbol{\nabla}_V(T_k(f))\big\| = O\Big(\frac{1}{k}\Big), \quad (7.15)$$

for any smooth function f on M, which follows by (7.13) for $N = 0$.

We aim to prove that the formal connection of Theorem 7.7 is a formal Hitchin connection for the family of star products defined by the shifted Berezin-Toeplitz star product $\tilde{\star}$. To prove this, we will need the following proposition from [And1]

Proposition 7.8. *We have the asymptotic expansion*

$$\boldsymbol{\nabla}_V\big(T_k(f)T_k(g)\big) \sim \boldsymbol{\nabla}_V\big(T_k(f \,\tilde{\star}\, g)\big)\Big[\frac{1}{k+n/2}\Big],$$

for any families of smooth functions f and g on M.

Proof. In [And1], it is proved that

$$\boldsymbol{\nabla}_V\big(T_k(f)T_k(g)\big) \sim \boldsymbol{\nabla}_V\big(T_k(f \star g)\big)\Big[\frac{1}{k}\Big],$$

which implies the desired statement by the relation between $\tilde{\star}$ and $\star$. $\square$

With this proposition, we can finally prove

Theorem 7.9. *The formal connection of Theorem 7.7 is a formal Hitchin connection for the family of shifted Berezin-Toeplitz star products $\tilde{\star}$.*

Proof. From Proposition 7.8, we have that

$$\boldsymbol{\nabla}_V\big(T_k(f)T_k(g)\big) \sim \boldsymbol{\nabla}_V\big(T_k(f \,\tilde{\star}\, g)\big)\Big[\frac{1}{k+n/2}\Big], \tag{7.16}$$

and by using (7.12) on each of the terms $\boldsymbol{\nabla}_V(T_k(C_j(f,g)(k+n/2)^j))$ on the right-hand side of (7.16), it follows that

$$\boldsymbol{\nabla}_V\big(T_k(f)T_k(g)\big) \sim T_k\big(\tilde{\boldsymbol{D}}_V(f \,\tilde{\star}\, g)\big)\Big[\frac{1}{k+n/2}\Big].$$

On the other hand, the Leibniz rule gives

$$\boldsymbol{\nabla}_V\big(T_k(f)T_k(g)\big) = \boldsymbol{\nabla}_V(T_k(f))T_k(g) + T_k(f)\boldsymbol{\nabla}_V(T_k(g)),$$

and by applying the expansions (7.10) and (7.12), it follows that

$$\boldsymbol{\nabla}_V\big(T_k(f)T_k(g)\big) \sim T_k\big(\tilde{\boldsymbol{D}}_V(f) \,\tilde{\star}\, g + f \,\tilde{\star}\, \tilde{\boldsymbol{D}}_V(g)\big)\Big[\frac{1}{k+n/2}\Big].$$

Therefore, uniqueness of asymptotic expansions implies that

$$\tilde{\boldsymbol{D}}_V(f \,\tilde{\star}\, g) = \tilde{\boldsymbol{D}}_V(f) \,\tilde{\star}\, g + f \,\tilde{\star}\, \tilde{\boldsymbol{D}}_V(g),$$

which shows that $\tilde{\boldsymbol{D}}$ is a derivation for $\tilde{\star}$, as desired. $\square$

Recall that the shifted Berezin-Toeplitz star product $\tilde{\star}$ is given by

$$f \,\tilde{\star}\, g = (f \star g)[\varphi(h)].$$

The inverse function of φ is given by $\varphi^{-1}(t) = \frac{t}{1+tn/2}$, and its Taylor expansion around zero gives rise to the formal constant

$$\varphi^{-1}(h) = \sum_{j=1}^{\infty}\Big(-\frac{n}{2}\Big)^{j-1}h^j = h - \frac{n}{2}h^2 + \frac{n^2}{4}h^3 - \cdots .$$

If we define a formal connection $\boldsymbol{D} = \tilde{\boldsymbol{D}}[\varphi^{-1}(h)]$ by substituting this formal constant into $\tilde{\boldsymbol{D}}$, then the following is an immediate corollary of Theorem 7.9.

Theorem 7.10. *The formal connection $\boldsymbol{D}$ is a formal Hitchin connection of the Berezin-Toeplitz star product $\star$.*

The next proposition concerns the curvature of the formal Hitchin connection.

Proposition 7.11. *If the Hitchin connection on $\hat{\mathcal{Q}}_k$ is projectively flat, then the formal Hitchin connection $\tilde{\boldsymbol{D}}$ for the shifted Berezin-Toeplitz star product $\tilde{\star}$ is flat.*

Proof. From Proposition 7.6, we get that

$$\begin{aligned}\boldsymbol{\nabla}_W\big(\boldsymbol{\nabla}_V(T_k(f))\big) &= \boldsymbol{\nabla}_W\big(T_k(V[f])\big) + \boldsymbol{\nabla}_W\big(T_k(V[F])T_k(f)\big) - \boldsymbol{\nabla}_W\big(T_k(V[F]f)\big) \\ &\quad + \frac{1}{k+n/2}\Big(\boldsymbol{\nabla}_W\big(T_k(c_0(V))\,T_k(f)\big) - \boldsymbol{\nabla}_W\big(T_k(c(V)(f))\big)\Big),\end{aligned}$$

and by applying Theorem 7.7 and Proposition 7.8, we see that

$$\boldsymbol{\nabla}_W\big(\boldsymbol{\nabla}_V(T_k(f))\big) \sim T_k(\tilde{\boldsymbol{D}}_W\tilde{\boldsymbol{D}}_V f)\Big[\frac{1}{k+n/2}\Big].$$

This implies that the formal Hitchin connection is flat if the Hitchin connection on the endomorphism bundle of $\hat{\mathcal{Q}}_k$ is flat, and this is indeed the case if the Hitchin connection on $\hat{\mathcal{Q}}_k$ is projectively flat. □

Since the formal Hitchin connection $\boldsymbol{D}$ for the Berezin-Toeplitz star product $\star$ is flat if and only if $\tilde{\boldsymbol{D}}$ is flat, a statement analogous to Proposition 7.11 obviously holds for $\boldsymbol{D}$.

7.2.4 The Invariant Star Product

If we assume that $\mathcal{T}$ is contractible and that the formal Hitchin connection $\tilde{\boldsymbol{D}}$ is flat, then Proposition 7.3 and (7.14) guarantee the existence of a formal trivialization $\tilde{P}$ of the formal Hitchin connection $\tilde{\boldsymbol{D}}$ for the shifted Berezin-Toeplitz star product $\tilde{\star}$. By Proposition 7.4, we can then define a star product

$$f \mathbin{\tilde{\star}} g = \tilde{P}^{-1}(\tilde{P}(f) \mathbin{\tilde{\star}} \tilde{P}(g)),$$

which is independent on the complex structure, and which must have trivial characteristic class by (7.11).

It turns out that the formal Hitchin connection $\tilde{\boldsymbol{D}}$ is always flat to second order, which implies that it has a formal trivialization to first order. In fact, we can give an explicit formula for the first-order term.

Proposition 7.12. *Any family of formal operators of the form*

$$\tilde{P} = \mathrm{Id} + \Big(\frac{1}{4}\Delta - i\nabla_{X''_F}\Big)h + O(h^2) \tag{7.17}$$

is a formal trivialization to first order of the formal Hitchin connection $\tilde{\boldsymbol{D}}$ for the shifted Berezin-Toeplitz star product $\tilde{\star}$.

Proof. With reference to the proof of Proposition 7.3 and the formula for the formal Hitchin connection given in Theorem 7.7, we must show that $\tilde{P}_1 = \frac{1}{4}\Delta - i\nabla_{X''_F}$ satisfies

$$V[\tilde{P}_1] = -A_1(V) = \frac{1}{4}\Delta_{\tilde{G}(V)} - \frac{1}{2}\nabla_{\tilde{G}(V)\cdot dF} - \tilde{C}_1(V[F], \cdot),$$

where $\tilde{C}_1$ is the first-degree term of the shifted Berezin-Toeplitz star product, which is given by

$$\tilde{C}_1(f_1, f_2) = -\partial f_1 \cdot \tilde{g} \cdot \bar{\partial} f_2 = i \nabla_{X''_{f_1}} f_2,$$

according to Theorem 3.10. The variation of the Laplace operator is given by

$$V[\Delta] = -V[\Delta_{\tilde{g}}] = -\Delta_{V[\tilde{g}]} = \Delta_{\tilde{G}(V)},$$

and it is easily verified that

$$iV[\nabla_{X''_F} f] = V[\tilde{C}_1(F, f)] = \frac{1}{2} dF \cdot \tilde{G}(V) \cdot df + \tilde{C}_1(V[F], f),$$

for any function f on M, so the proposition follows. □

If we choose a formal trivialization of the form (7.17), then we can calculate the first-order term of the invariant star product $\tilde{\star}$. In fact, a simple calculation shows that

$$f \mathbin{\tilde{\star}} g = fg + \frac{i}{2}\{f, g\} h + O(h^2), \tag{7.18}$$

which is perhaps not too surprising.

Since the two formal Hitchin connections $\boldsymbol{D}$ and $\tilde{\boldsymbol{D}}$ agree on their first two terms, the discussion above can be carried through, word for word, for the connection $\boldsymbol{D}$. In particular, we obtain an invariant star product $\star$ from the Berezin-Toeplitz star product $\star$, which is also of the form (7.18).

As previously mentioned, if $\mathcal{T}$ is contractible and we have a group action by Γ for which the cohomology $H^1(\Gamma, \mathcal{D}(M))$ vanishes, then the invariant star products will be equivariant.

7.3 The Metaplectic Case

All of what we have done above for the Berezin-Toeplitz star product $\star$ can also be done for the twisted Berezin-Toeplitz star product $\star^\delta$ arising from metaplectic quantization. In this section, we shall briefly go through the analogous results in the metaplectic setting, and as we shall see, the simplicity of formulas suggest that this is in fact a more natural setting to consider.

Recall from Theorem 6.12 that the Hitchin connection in metaplectic quantization is given by $\boldsymbol{\nabla}^\delta = \hat{\nabla} + \frac{1}{k} b^\delta$, where

$$b^\delta(V) = \frac{1}{4}\Delta_{G(V)} + b_0^\delta(V),$$

with zero-order part $b_0^\delta(V)$ given by

$$b_0^\delta(V) = -\frac{1}{4}\big(\Delta_{G(V)}(F) + dF \cdot G(V) \cdot dF + 2nV'[F]\big).$$

The proof of Lemma 7.5 in [And1] can be used without modification to prove

$$\pi_k^\delta V[\pi_k^\delta] = \frac{1}{k}\pi_k^\delta b^\delta(V)^* \pi_k^\delta - \frac{1}{k}\pi_k^\delta b^\delta(V)^*,$$

and by an analysis similar to the proof of Proposition 7.6, one can deduce that

$$\boldsymbol{\nabla}^\delta_V\big(T^\delta_k(f)\big) = T^\delta_k(V[f]) + \frac{1}{k}\Big(T^\delta_k(\tilde{b}^\delta_0(V))T^\delta_k(f) - T^\delta_k\big(\tilde{b}^\delta(V)(f)\big)\Big), \tag{7.19}$$

on sections of $\hat{\mathcal{Q}}^\delta_k$, where $\tilde{b}^\delta_0(V)$ is the function given by

$$\tilde{b}^\delta_0(V) = -\frac{1}{4}(\Delta_{\tilde{G}(V)}(F) + dF\cdot\tilde{G}(V)\cdot dF + 2nV[F]),$$

and $\tilde{b}^\delta(V)$ is the operator given by

$$\tilde{b}^\delta(V) = \frac{1}{4}\Delta_{\tilde{G}(V)} + \tilde{b}^\delta_0(V).$$

In other words, $\tilde{b}^\delta(V)$ and $\tilde{b}^\delta_0(V)$ are obtained by replacing $G(V)$ with $\tilde{G}(V)$ in the expressions for $b^\delta(V)$ and $b^\delta_0(V)$.

By Theorem 4.10, the twisted Berezin-Toeplitz star product $\star^\delta$ encodes the asymptotic expansion of the product of two Toeplitz operators in metaplectic quantization. Applied to (7.19), this gives the asymptotic expansion

$$\boldsymbol{\nabla}^\delta_V(T^\delta_k(f)) \sim T^\delta_k(\boldsymbol{D}^\delta_V f)\Big[\frac{1}{k}\Big], \tag{7.20}$$

where $\boldsymbol{D}^\delta = D^t + A^\delta$ is the formal connection with

$$A^\delta(V)(f) = \frac{1}{4}\Delta_{\tilde{G}(V)}(f)h + (\tilde{b}^\delta_0(V)\star^\delta f - \tilde{b}^\delta_0(V)f)h.$$

Using the expansion

$$\boldsymbol{\nabla}^\delta_V\big(T^\delta_k(f)T^\delta_k(g)\big) \sim \boldsymbol{\nabla}^\delta_V\big(T^\delta_k(f\star^\delta g)\big)\Big[\frac{1}{k}\Big], \tag{7.21}$$

which is the analogous statement of Proposition 7.8 in the metaplectic setting, we get that

$$\boldsymbol{D}^\delta_V(f\star^\delta g) = \boldsymbol{D}^\delta_V(f)\star^\delta g + f\star^\delta \boldsymbol{D}^\delta_V(g).$$

It follows that $\boldsymbol{D}^\delta$ is a formal Hitchin connection for $\star^\delta$, and it is flat if the Hitchin on $\hat{\mathcal{Q}}^\delta_k$ is projectively flat. In this case, the existence of a formal trivialization P^δ is guaranteed by Proposition 7.3, and it is easily seen that it can be chosen of the form

$$P^\delta = \mathrm{Id} + \frac{1}{4}\Delta h + O(h^2). \tag{7.22}$$

Moreover, a simple calculation shows that the invariant star product has the form

$$f\star^\delta g = (P^\delta)^{-1}(P^\delta(f)\star^\delta P^\delta(g)) = fg + \frac{i}{2}\{f,g\}h + O(h^2).$$

Finally, the characteristic class of this star product is trivial, and since this was also the case for $\tilde{\star}$, the invariant star products $\tilde{\star}$ and $\star^\delta$ are equivalent.

7.4 Observables in Geometric Quantization

Formal trivializations of the formal Hitchin connection can be used to produce quantized observables in geometric quantization which are covariantly constant with respect to the Hitchin connection up to any given order in k.

Suppose that the Hitchin connection on $\hat{\mathcal{Q}}_k^\delta$ is projectively flat, so that the formal Hitchin connection $\boldsymbol{D}^\delta$ is flat, and let P^δ be a formal trivialization of the formal Hitchin connection of the form (7.22).

If $f \in C^\infty(M)$ is a classical observable and N is some natural number, we can use the formal trivialization to define

$$P_k^N(f) = \sum_{j=0}^{N} P_j^\delta(f)\frac{1}{k^j},$$

and by the expansion (7.20), this function satisfies

$$\big\|\boldsymbol{\nabla}_V^\delta\big(T_k^\delta(P_k^N(f))\big)\big\| = O\Big(\frac{1}{k^{N+1}}\Big),$$

which shows that the operator

$$Q_k^N(f) = T_k^\delta(P_k^N(f))$$

is covariantly constant to order N in k. In this way, the operator

$$Q_k^1(f) = T_k^\delta\Big(f + \frac{1}{4k}\Delta f\Big),$$

is a more natural quantization of f than the usual quantum operator, which is given by Tuynman's formula in Theorem 4.5, because it is covariantly constant to a higher order.

Furthermore, this quantization of observables gives a better approximation of the property (2.1) than (2.2) because it satisfies

$$\Big\|\big[Q_k^1(f), Q_k^1(g)\big] - \frac{i}{k}Q_k^1(\{f,g\})\Big\| = O(k^{-3}) \quad \text{as} \quad k \to \infty. \tag{7.23}$$

To see this, one must verify that the Laplace operator on the Poisson bracket of two functions is given by

$$\Delta\{f,g\} = \{\Delta f, g\} + \{f, \Delta g\} + 4C_2^\delta(f,g) - 4C_2^\delta(g,f), \tag{7.24}$$

where the operator C_2^δ is the second-order coefficient of the star product $*$ whose opposite star product $*_o$ has Karabegov form

$$\mathrm{Kar}(*_o) = -\omega\frac{1}{h} + \frac{1}{2}\rho.$$

Theorem 3.10 gives an explicit formula for the operator C_2^δ, and using this, it is a simple matter to verify the identity (7.24). Furthermore, if the formula (4.7) holds true, then the star product $*$ is exactly the twisted Berezin-Toeplitz star product $\star^\delta$ which gives the expansion

$$T_k^\delta(f)T_k^\delta(g) \sim T_k^\delta(f \star^\delta g)\Big[\frac{1}{k}\Big].$$

Using this and the identity (7.24), it takes a simple computation to verify (7.23).

We find it unlikely that the property (7.23) is coincidential, and we conjecture that the operators Q_k^N have a similar property to higher orders in k. This would make them even better candidates for the quantization of observables.

Chapter **8**

The Moduli Space of Flat Connections

The purpose of this chapter is to present an interesting example of a symplectic manifold to which our work on quantization can be applied. This example is the moduli space of flat $\mathrm{SU}(n)$-connections on a Riemann surface, and in fact, many of the ideas that we have discussed were first developed in this setting.

As mentioned in the introduction, the moduli space of flat connections appears naturally in classical Chern-Simons theory, where the flat connections are the critical points of the Chern-Simons action functional. The quantization of the moduli space should then form the two-dimensional part of a topological quantum field theory, as proposed by Witten in [Wit].

There is a vast amount of literature that studies the moduli space. Some of the primary references for us are [AB], [And1] and [ADW]. Also, we refer to [Vil] which gives a good exposition of the moduli space that is well adapted to our situation.

8.1 Definition of the Moduli Space

Let Σ be a compact, smooth and orientable surface of genus $g \geq 2$ and with one boundary component. Furthermore, let π_1 denote the fundamental group of Σ, with respect to some arbitrary but fixed base point $x \in \partial\Sigma$ on the boundary.

We consider the space

$$\mathcal{M} = \mathrm{Hom}(\pi_1, \mathrm{SU}(n)) / \mathrm{SU}(n) \tag{8.1}$$

of representations of the fundamental group, endowed with the compact-open topology, modulo conjugation. This is called the *moduli space of flat* $\mathrm{SU}(n)$*-connections* on Σ. The name comes from the standard identification, through the holonomy representation, of this space with the space of gauge equivalence classes of flat connections on a principal $\mathrm{SU}(n)$-bundle over the surface.

Let us say a few more words about this identification. Consider a principal $\mathrm{SU}(n)$-bundle P over Σ, and let $\mathcal{G}$ denote the corresponding group of gauge transformations, which is the group of bundle maps from P to itself. The gauge tranformations act by pullback on the space $\mathcal{A}$ of connections on P. The curvature of a pullback connection is the pullback of the curvature, and therefore the gauge tranformations preserve the subspace $\mathcal{F}$ of flat connections.

The quotient $\mathcal{F}/\mathcal{G}$ is the moduli space of flat connections. It is identified with (8.1) through the following map. For any flat connection, the parallel translation of a point in the fiber over x along a closed curve only depends on the homotopy class of the curve, and therefore it defines a homomorphism from π_1 to $\mathrm{SU}(n)$, called the holonomy representation of the connection. Had we chosen to transport a different point in the fiber, we would have obtained a conjugate homomorphism, and consequently we get a well-defined map $\mathcal{F} \to \mathcal{M}$. Moreover, gauge equivalent connections have conjugate holonomy representations, so in fact we get a map

$$\mathcal{F}/\mathcal{G} \to \mathcal{M},$$

which is surjective, since any representation can be realized as the holonomy of a flat connection, and injective, since the the holonomy determines the gauge equivalence class of a connection.

Since the surface Σ has one boundary component, the fundamental group is a free group on $2g$ generators, and consequently the space $\mathrm{Hom}(\pi_1, \mathrm{SU}(n))$ is a smooth manifold, which can be identified with the product of $2g$ copies of $\mathrm{SU}(n)$. The moduli space, which is the quotient by the conjugation action, is rarely a smooth manifold, however. This is due to the existence of reducible representations. In fact, the space $\mathrm{Hom}^{\mathrm{irr}}(\pi_1, \mathrm{SU}(n))$ of irreducible representations is a dense and open subset of $\mathrm{Hom}(\pi_1, \mathrm{SU}(n))$, and the quotient

$$\mathcal{M}^o = \mathrm{Hom}^{\mathrm{irr}}(\pi_1, \mathrm{SU}(n))/\,\mathrm{SU}(n)$$

is a smooth manifold.

Using the fact that Σ has boundary, we shall go further and require the representations to have a fixed central holonomy around the boundary. More precisely, let $\gamma \in \pi_1$ be the element represented by a curve going once around the boundary of Σ, and let $D = e^{2\pi i d} I$, where $d \in \mathbb{Z}_n$, be a generator of the center of $\mathrm{SU}(n)$. Then we define

$$\mathrm{Hom}_d(\pi_1, \mathrm{SU}(n)) = \{\rho \in \mathrm{Hom}(\pi_1, \mathrm{SU}(n)) \mid \rho(\gamma) = D\},$$

which is preserved by the conjugation action since D is central. It is not difficult to see that every representation in $\mathrm{Hom}_d(\pi_1, \mathrm{SU}(n))$ is irreducible, which leads to the the conclusion that the space

$$\mathcal{M}^d = \mathrm{Hom}_d(\pi_1, \mathrm{SU}(n))/\,\mathrm{SU}(n)$$

is a smooth, compact manifold.

The moduli space was studied by Atiyah and Bott in [AB], where they proved that it is simply connected with $H^2(\mathcal{M}^d, \mathbb{Z}) = \mathbb{Z}$. Furthermore, the second Stiefel-Whitney class of the moduli space vanishes.

8.2 Symplectic Structure

The moduli space has a natural symplectic structure, defined in the following way. The space $\mathcal{A}$ of connections on P is an affine space for the space $\Omega^1(\Sigma, \mathrm{Ad}\, P)$ of one-forms with values in the adjoint bundle, and we can define a non-degenerate, skew-symmetric pairing on such one-forms by

$$\omega(\alpha, \beta) = -\int_M \mathrm{Tr}(\alpha \wedge \beta).$$

This pairing constitutes a symplectic form on $\mathcal{A}$, being constant and hence closed on this space.

The gauge group acts by symplectomorphisms, and the moment map of the action by the subgroup $\mathcal{G}_0$ of gauge transformations that restrict to the identity on the boundary is given by the curvature map on connections. Consequently, the preimage of zero by this moment map is exactly the flat connections, and therefore the quotient $\mathcal{F}/\mathcal{G}_0$ is a symplectic space. The group $\mathcal{G}/\mathcal{G}_0$ acts on this space by symplectomorphisms, and the quotient of this action is the moduli space $\mathcal{M}^o$, which inherits a Poisson structure. The symplectic leaves are obtained by fixing the conjugacy class of the holonomy around the boundary (see [Aud]). In particular, the moduli space $\mathcal{M}^d$ is symplectic.

If A is a flat connection on P, the tangent space to $\mathcal{M}^o$ at $[A]$ is given by the first cohomology group $H^1_A(\Sigma, \operatorname{Ad} P)$ of the complex of $\operatorname{Ad} P$-valued forms with the covariant exterior derivate d_A, which squares to zero by the flatness of A. If $[\alpha]$ and $[\beta]$ are tangent vectors to $\mathcal{M}^d$ at $[A]$, then the symplectic form is given by

$$\omega([\alpha], [\beta]) = -\int_\Sigma \operatorname{Tr}(\alpha \wedge \beta),$$

which is easily seen to be independent of representatives.

The symplectic structure can also be described in purely algebraic terms. The tangent spaces of the moduli space are expressed as cohomology groups of π_1 with coefficients in the Lie-algebra of $\mathrm{SU}(n)$, and the symplectic form can be defined using the cup product on cohomology. For further details, we refer to [Gol] and [Vil].

8.3 The Mapping Class Group

The moduli space has a natural group of symmetries. Let $\operatorname{Diff}(\Sigma)$ be the group of diffeomorphisms of Σ which restrict to the identity on the boundary, and denote by $\operatorname{Diff}_0(\Sigma)$ the subgroup of diffeomorphisms isotopic to the identity. The quotient $\Gamma(\Sigma) = \operatorname{Diff}(\Sigma)/\operatorname{Diff}_0(\Sigma)$ is the *mapping class group* of the surface.

The mapping class group acts on the moduli space of flat connections through its action on the fundamental group. Indeed, since the base point for the fundamental group was chosen on the boundary, it is fixed by every diffeomorphism in $\operatorname{Diff}(\Sigma)$, and we get an action of this group on π_1. Clearly, the subgroup $\operatorname{Diff}_0(\Sigma)$ acts by the identity, and we get an action of the mapping class group on π_1 which induces an action on the moduli space $\mathcal{M}^d$.

One can easily verify that the action of the mapping class group preserves the symplectic structure on the moduli space.

8.4 Teichmüller Space and Kähler Structure

Another space related to the surface is the Teichmüller space $\mathcal{T}(\Sigma)$ of complex structures on Σ. Any Riemannian metric on Σ gives rise to a Hodge star operator $*$ on forms, and this restricts to an anti-involution $* : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$, which only depends on the conformal class of the metric. In particular, it defines an almost complex structure on the surface, which is integrable for dimensional reasons. This is the well-known correspondence between complex and conformal structures on a surface. Now, we consider the space $\mathcal{C}(\Sigma)$ of conformal equivalence classes of Riemannian metrics on Σ. The group of diffeomorphisms $\operatorname{Diff}_0(\Sigma)$ acts on this space by pullback, and the quotient $\mathcal{T}(\Sigma) = \mathcal{C}(\Sigma)/\operatorname{Diff}_0(\Sigma)$ is the *Teichmüller space* of the surface.

Clearly, the full group of diffeomorphisms also acts on $\mathcal{C}(\Sigma)$, so we get an action of the mapping class group on Teichmüller space, and the quotient by this action is the Riemann moduli space of the surface.

It is a classical result that the Teichmüller space is contractible. Moreover, it has a natural complex structure defined as follows. A point $\sigma \in \mathcal{T}(\Sigma)$ is represented by a complex structure on Σ, and the tangent space to $\mathcal{T}(\Sigma)$ at σ is given by the first cohomology group $H^1(\Sigma, T'\Sigma)$ with coefficients in the sheaf of sections of the holomorphic tangent bundle of Σ with respect to the complex structure specified by σ. This is clearly a complex vector space, and we get an almost complex structure on $\mathcal{T}(\Sigma)$, which is in fact integrable.

The Teichmüller space parametrizes Kähler structures on the moduli space in the following way. A conformal structure on Σ gives rise to a Hodge star operator $*$, and after extending this operator to $\operatorname{Ad} P$-valued forms, one defines the adjoint $d_A^* = - * d_A *$ and the Laplacian $\Delta_A = d_A d_A^* + d_A^* d_A$ in the usual way. By standard Hodge theory, the tangent space $T_{[A]}\mathcal{M}^o = H_A^1(\Sigma, \operatorname{Ad} P)$ is identified with the space of harmonic forms. But harmonicity is preserved by $*$, and therefore $J = -*$ defines an almost complex structure on $\mathcal{M}^o$. In other words, we have constructed a map $J \colon \mathcal{C}(\Sigma) \to C^\infty(\mathcal{M}^o, \operatorname{End}(T\mathcal{M}^o))$. Since this map is equivariant with respect to the action of $\operatorname{Diff}(\Sigma)$, and since the subgroup $\operatorname{Diff}_0(\Sigma)$ acts trivially on the moduli space, this induces a map $J \colon \mathcal{T}(\Sigma) \to C^\infty(\mathcal{M}^o, \operatorname{End}(T\mathcal{M}^o))$, which is equivariant with respect to action of the mapping class group.

The almost complex structures parametrized by the Teichmüller space are easily seen to be compatible with the symplectic structure. Moreover, they are integrable, and hence Kähler, by the result of Narasimhan and Seshadri [NS]. Finally, they preserve the subspace $\mathcal{M}^d$, which is therefore a compact Kähler manifold. In conclusion, the Teichmüller space parametrizes a family of Kähler structures on the moduli space $\mathcal{M}^d$, and this family is holomorphic, in the sense of Definition 5.6, with respect to the complex structure on $\mathcal{T}(\Sigma)$.

8.5 Quantization of the Moduli Space

The results of previous chapters can be applied to the symplectic manifold $(\mathcal{M}^d, \omega)$. Indeed, the first Chern class of the moduli space is given by $c_1(\mathcal{M}^d, \omega) = 2n[\frac{\omega}{2\pi}]$, and consequently the prequantum condition (2.3) is satisfied. A prequantum line bundle $\mathcal{L}$ is constructed by Freed in [Fre], and using this construction one can lift the action of the mapping class group to an action on $\mathcal{L}$ by Hermitian bundle maps. Moreover, as observed by Hitchin [Hit], the family of Kähler structures parametrized by the Teichmüller space is rigid in the sense of Definition 5.10.

In summary, all the assumptions of Theorem 6.5 and Theorem 6.12 are satisfied, and consequently a Hitchin connection exists in the bundles $\hat{\mathcal{Q}}_k$ and $\hat{\mathcal{Q}}_k^\delta$ over Teichmüller space.

Traditionally, the bundle $\hat{\mathcal{Q}}_k = H^0(\mathcal{M}, \mathcal{L}^k)$ over Teichmüller space is called the Verlinde bundle. This was the bundle originally studied by Hitchin in [Hit], where he proved that it has a natural projectively flat connection. The work of Narasimhan and Ramanan [NR] shows that the moduli space does not admit any holomorphic vector fields, and Hitchin relied on this fact to prove projective flatness, by using methods specific to the moduli space to show that the curvature must be a first-order holomorphic operator.

Using that no holomorphic vector fields exist on the moduli space, we can give an alternative, although similar, proof of projective flatness of the Hitchin connection. Indeed, Theorem 6.22 implies that the Hitchin connection in the metaplectic setting is projectively flat, and by Theorem 6.14 this is the case if and only if the Hitchin connection in standard geometric quantization is projectively flat. Thus we have proved

Theorem 8.1. *The Hitchin connection in the Verlinde bundle over the moduli space $\mathcal{M}^d$ is projectively flat.*

We emphasize that our proof uses rigidity of the family of Kähler structures to deduce that the curvature is a first-order operator.

We can also apply the theory developed in Chapter 7 to the moduli space. The Berezin-Toeplitz star products give rise to a family of star products on $\mathcal{M}^d$ parametrized by Teichmüller space, and this family is equivariant with respect to the action of the mapping class group. Furthermore, Theorem 7.9 ensures that we have a formal Hitchin connection for this family, which is flat by Theorem 8.1 and Proposition 7.11. Since the Teichmüller space is contractible, it follows from Proposition 7.3 that we can find a formal trivialization, which in turn gives us an invariant star product on the moduli space. If the cohomology $H^1(\Gamma(\Sigma), \mathcal{D}(\mathcal{M}^d))$ vanishes, then we can choose the formal trivialization to be equivariant with respect to the action of the mapping class group, and this will imply that the invariant star product is equivariant with respect to this action. Furthermore, if the cohomology group $H^1(\Gamma(\Sigma), C^\infty(\mathcal{M}^d))$ vanishes, then this is the unique equivariant star product on the moduli space (see [And1]).

It is, however, not known whether these cohomology groups of the mapping class group do indeed vanish, although some results in this direction, with various other coefficients, have been proved by Andersen and Villemoes [AV2, Vil, AV1].

8.5.1 Quantum Representations and Unitarity

By the explicit formulas for the Hitchin connection, it is clear that it is equivariant with respect to the action of the mapping class group, and consequently we get a projective representation of the mapping class group on the space of covariantly constant sections of the projectivized Verlinde bundle. In fact, we get such a representation for each level k of the quantization, and these representations are known as the quantum representations of the mapping class group. Using the fact that the Toeplitz operators are asymptotically flat, in the sense of (7.15), Andersen [And3] has proved that the quantum representations are asymptotically faithful, meaning that the intersection of their kernels is trivial. Furthermore, Andersen [And2] used the asymptotic relationship between the Toeplitz operators and the Hitchin connection to prove that the mapping class group does not have Kazhdan's property (T).

The question of unitarity of the Hitchin connection in the moduli space setting is not well understood within the framework of geometric quantization. However, through the relation with other fully established TQFT constructions, such as the algebraic construction by Reshetikhin and Turaev [RT2, RT3, Tur], using representations of quantum groups, or the combinatorial construction by Blanchet, Habegger, Masbaum and Vogel [BHMV1, BHMV2], using skein theory, it is known that there exists a Hermitian structure on the Verlinde bundle which is preserved by the Hitchin connection and invariant with respect to the action of the mapping class group. This follows by work of Andersen and Ueno, using also work of Laszlo, and we refer to [And2] for a more detailed outline and further references. In particular, the correspondence proves that the quantum represention are equal to the representations produced by the aforementioned TQFTs, which are known to by unitary.

At least if the cohomology $H^1(\Gamma(\Sigma), C^\infty(\mathcal{M}^d))$ vanishes, then Theorem 6.25 gives a mapping class group equivariant asymptotic approximation to the Hermitian structure preserved by the Hitchin connection.

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