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CLOSED FORMS AND MULTI-MOMENT MAPS

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Closed forms and multi-moment maps

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Abstract

We extend the notion of multi-moment map to geometries defined by closed forms of arbitrary degree. We give fundamental existence and uniqueness results and discuss a number of essential examples, including geometries related to special holonomy. For forms of degree four, multi-moment maps are guaranteed to exist and are unique when the symmetry group is $(3, 4)$ -trivial, meaning that the group is connected and the third and fourth Lie algebra Betti numbers vanish. We give a structural description of some classes of $(3, 4)$ -trivial algebras and provide a number of examples.

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1 Introduction

The rich and varied field of symplectic geometry is the study of closed non-degenerate two-forms. It has origins in the study of Hamiltonian dynamics and the geometry of phase space. From a mathematical point of view it is natural to try to see how much of this theory may be extended to closed forms of higher degree. A number of authors have already made attempts at generalising the Hamiltonian picture to higher-degree, or multi-, phase spaces, often motivated by the interest in various field theories [11, 12, 19, 1, 2]. Indeed string- and M -theories with fluxes give a number of geometries equipped with closed differential forms of varying degrees, see [20] for one such example.

The purpose of this article is to study the geometry of closed differential forms in general, with particular emphasis on new techniques that are available in the presence of symmetry. One main tool in the construction of various symplectic manifolds is the Marsden-Weinstein quotient formed by taking quotients of the level sets of a moment map. One important feature of the moment map in symplectic geometry, is that it takes values in a finite-dimensional vector space depending only on the symmetry group and not on the underlying manifold. Previous attempts to extend moment maps to forms of higher degrees, have produced maps taking values in infinite-dimensional spaces of forms over the manifold, see the references above, though [32] provides an interesting alternative. In [29] we introduced a new notion of multi-moment map for geometries with a closed three-form, which shares the above basic property of symplectic moment maps. A thorough study of these new maps was made in [30]. In this paper we will show how this theory extends to forms of arbitrary degree, in large part based on ideas developed in the thesis [28]. Not only do these multi-moment maps take values in a finite-dimensional vector space, but there are existence results based on easily satisfied properties of the manifold or its symmetry group. We will thus describe the general theory, give examples of multi-moment reduction of various geometries, particular ones with a closed four-form, and study an algebraic condition on Lie groups that guarantees existence and uniqueness of multi-moment maps for four-form geometries.

One salient feature of symplectic geometry is that the two-form is non-degenerate. What this means for a form of higher degree is less clear and we start the paper in Section 2 by discussing a number of different possibilities. These distinguish a number of geometries that have importance in their own right, for example geometries with exceptional holonomy, but do not lead to any one good constraint, so for the general theory we do not impose such assumptions.

In Section 3 we then introduce the notion of multi-moment map for symmetries of closed geometries of arbitrary degree. In order to facilitate the proofs we develop some theory of multi-vectors on manifolds and in particular give an extension of the classical Cartan formula which expresses the Lie derivative of forms in terms of exterior derivatives and contractions. Multi-moment maps are then defined, and existence and uniqueness theorems proved under topological and under algebraic assumptions. Section 4 then gives a number of examples of closed geometries, computes multi-moment maps in a number of cases and discusses the geometries of quotients. Finally, in Section 5 we study the algebraic condition found in Section 3.2 for the existence and uniqueness of multi-moment maps for geometries with a closed four-

form. These conditions are expressed as the vanishing of the third and fourth Lie algebra cohomology groups. We show how to exploit the Hochschild-Serre spectral sequence to determine the algebraic structure of a wide class of such Lie algebras and give a number of examples.

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2 Distinguished differential forms

We will be considering geometries defined by closed differential forms. So as a first question we address the issue of whether there are any algebraically distinguished forms on a vector space. Recall that in symplectic geometry one makes repeated use of the ‘non-degeneracy’ of the symplectic two-form ω . Algebraically this leads to the fact that a symplectic manifold is of even dimension and then the closure of ω is used for Darboux’s Theorem, that there are local coordinates so that $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. For higher degree forms, the situation is not simple and it is not clear which definition is appropriate. Let us discuss some of the possibilities.

Let V be an n -dimensional vector space over \mathbb{R} . Write Λ^*V^* for the algebra of forms on V .

Definition 2.1. A form $\alpha \in \Lambda^rV^*$ is said to be fully non-degenerate if

$$\alpha(v_1, v_2, \dots, v_{r-1}, \cdot)$$

is non-zero whenever $v_1, \dots, v_{r-1} \in V$ are linearly independent.

For $r = 2$, this is the usual non-degeneracy of a two-form. For any two-form α there is a basis of V^* such that

$$\alpha = e_1 \wedge e_2 + \cdots + e_{2k-1} \wedge e_{2k}, \tag{2.1}$$

for some $k \leq \frac{1}{2} \dim V$. To see this start with a non-zero vector $X \in V$ and put $e_2 = \alpha(X, \cdot)$. Choose a vector Y such that $e_2(Y) = 1$ then choose a one-form e_1 with $e_1(X) = 1$ and $e_2(Y) = 0$. We now have $\alpha' = \alpha - e_1 \wedge e_2$ is zero on X and Y , and the result follows by induction. We see that α is non-degenerate if and only if $\dim V = 2k$.

For forms of degree 3, full non-degeneracy already gives much stronger restrictions.

Theorem 2.2. A vector space of dimension n admits a fully non-degenerate form of degree $r \geq 3$ if and only if $r = n$ or the pair (r, n) is either $(3, 7)$ or $(4, 8)$.

Proof. Choose an inner product $\langle \cdot, \cdot \rangle$ on V . A form $\alpha \in \Lambda^r V^*$ defines a cross-product like operation $V^{r-1} \rightarrow V$ via

$$\langle v_1 \times v_2 \times \cdots \times v_{r-1}, w \rangle = \alpha(v_1, v_2, \dots, v_{r-1}, w).$$

This operation is continuous and has the property that the product $v_1 \times \cdots \times v_{r-1}$ is orthogonal to each of the v_i . When α is fully non-degenerate, this product on linearly independent vectors is non-zero. Let $V_{r,n}$ denote the Stiefel manifold consisting of all r -tuples (f_1, \dots, f_r) of orthonormal vectors in \mathbb{R}^n . The map

$$(f_1, \dots, f_{r-1}) \mapsto \left(f_1, \dots, f_{r-1}, \frac{f_1 \times \cdots \times f_{r-1}}{\|f_1 \times \cdots \times f_{r-1}\|} \right)$$

is a cross section of the projection $V_{r,n} \rightarrow V_{r-1,n}$. It is a topological result of Whitehead [38] that such a cross-section exists only in the given cases. An elementary proof for the case of two-fold cross-products, $r = 3$, may be found in [31]. \square

For $r = n$, a volume form on V provides a fully non-degenerate form. Examples for the other two cases of this result are given by the three-forms

$$\phi_0 = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{347} - e_{356} \quad (2.2)$$

on \mathbb{R}^7 and the four-form

$$\begin{aligned} \Phi_0 = & e_{1234} + e_{1256} + e_{3478} + e_{3456} + e_{1278} + e_{1357} - e_{1368} \\ & - e_{2457} + e_{2468} - e_{1458} - e_{1467} - e_{2358} - e_{2367} + e_{5678} \end{aligned} \quad (2.3)$$

on \mathbb{R}^8 . Here e_1, \dots, e_n is a basis for $(\mathbb{R}^n)^*$ and wedge products have been omitted from the notation, so $e_{123} = e_1 \wedge e_2 \wedge e_3$, etc.

The forms ϕ_0 and Φ_0 have interesting geometric properties. In particular, if we consider the action of $GL(n, \mathbb{R})$ then the isotropy groups $\{g \in GL(n, \mathbb{R}) : g \cdot \alpha = \alpha\}$ are the compact 14-dimensional exceptional Lie group G_2 for $\alpha = \phi_0$ and the compact 21-dimensional group $Spin(7)$, the simply-connected double cover of $SO(7)$, for $\alpha = \Phi_0$, see Bryant [8]. We now see that the dimensions of the orbits of these forms are

$$\begin{aligned} \dim(GL(7, \mathbb{R}) \cdot \phi_0) &= 49 - 14 = 35, \\ \dim(GL(8, \mathbb{R}) \cdot \Phi_0) &= 64 - 21 = 43. \end{aligned}$$

The first of these is notable since $\dim \Lambda^3 \mathbb{R}^7 = 35$, so the orbit of ϕ_0 in $\Lambda^3 \mathbb{R}^7$ is open.

Definition 2.3 (Hitchin [23]). *A form $\alpha \in \Lambda^r V^*$ is stable if the orbit $GL(V) \cdot \alpha$ is open in $\Lambda^r V^*$.*

For general forms the condition of stability provides restrictions on the dimension of V .

Proposition 2.4. *A vector space of dimension n admits a stable form of degree r if and only if either $r \in \{1, 2, n-2, n-1, n\}$ or $r \in \{3, n-3\}$ with $n \in \{6, 7, 8\}$.*

Proof. We give the basic arguments, following Hitchin [23].

The dimension of the orbit $GL(n, \mathbb{R}) \cdot \alpha$ is at most $\dim(GL(n, \mathbb{R})) = n^2$. To have a stable form we thus need $n^2 \geq \dim \Lambda^r \mathbb{R}^n = \binom{n}{r}$. The binomial coefficient $\binom{n}{r}$ is a polynomial of degree r in n , which for $3 \leq r \leq n-3$ grows quicker than n^2 . Now for $r < n/2$, we have $\dim \Lambda^r \mathbb{R}^n < \dim \Lambda^{r+1} \mathbb{R}^n$, so we start by considering the case $r = 3$. In this case, we see that

$$\begin{aligned} \dim \Lambda^3 \mathbb{R}^n - \dim(GL(n, \mathbb{R})) &= \frac{1}{6}n(n-1)(n-2) - n^2 \\ &= \frac{1}{6}n((n-9)n+2) \end{aligned}$$

so an orbit in $\Lambda^3 \mathbb{R}^n$ can not be open if $n \geq 9$. In dimension $n = 8$, we have $\dim \Lambda^3 \mathbb{R}^8 < 64 = \dim GL(8, \mathbb{R})$, but $\dim \Lambda^4 \mathbb{R}^8 = 70 > 64$, so orbits of four-forms on \mathbb{R}^8 are never open. This gives the list of possible r and n in the statement.

It remains to show that each case can be realised. For $r = 1, n-1, n$, we take α to be any non-zero form of the given degree. For $r = 2$, open orbits are realised by forms as in (2.1) with $k = \lfloor n/2 \rfloor$. Taking the Hodge star of such a two-form gives a stable form of degree $n-2$.

Finally, we need to give appropriate three-forms in dimensions 6, 7 and 8; the case for $r = n-3$ will then follow by taking Hodge stars. For dimension $n = 6$, one can take α to be the real part of a complex volume form on $\mathbb{R}^6 = \mathbb{C}^3$. We have already seen ϕ_0 (2.2) is stable on \mathbb{R}^7 . Finally for $n = 8$, one identifies \mathbb{R}^8 with the Lie algebra $\mathfrak{su}(3)$. This carries an ad-invariant three-form $\alpha(X, Y, Z) = \langle [X, Y], Z \rangle$, which in an appropriate basis is

$$\begin{aligned} \rho_0 &= e_{123} + \frac{1}{2}e_1(e_{47} - e_{56}) + \frac{1}{2}e_2(e_{46} + e_{57}) \\ &\quad + \frac{1}{2}e_3(e_{45} - e_{67}) + \sqrt{3}/2e_8(e_{45} + e_{67}). \end{aligned} \tag{2.4}$$

The infinitesimal stabiliser of this form is $\mathfrak{su}(3)$ and so the orbit of ρ_0 has dimension $64 - 8 = 56 = \dim \Lambda^3 \mathbb{R}^8$ and is open. \square

Note that the connected subgroup of $GL(8, \mathbb{R})$ preserving ρ_0 is $PSU(3)$: the quotient of $SU(3)$ by its centre $\mathbb{Z}/3$.

So far we have considered two strong conditions on forms and found them to be rather restrictive. There is another condition that is useful more generally.

Definition 2.5. *A form α on V is (weakly) non-degenerate if*

$$v \lrcorner \alpha = \alpha(v, \cdot, \dots, \cdot)$$

is non-zero for each non-zero v .

Any non-zero form α gives rise to a non-degenerate form on the quotient $V/\ker \alpha$ where $\ker \alpha = \{v \in V : v \lrcorner \alpha = 0\}$. Conversely a volume form always provides a non-degenerate form on any vector space. For a particular degree of form there can be restrictions on the dimension. For two-forms weak and full non-degeneracy are the same and the space must be even-dimensional. In higher degree we have far fewer restrictions.

Proposition 2.6. *A vector space of dimension n admits a non-degenerate form of degree r with $r \geq 3$ if and only if $n \geq r$ and $n \neq r + 1$.*

Proof. For $n < r$, we have $\Lambda^r V^* = \{0\}$, so all r -forms are zero and thus degenerate. For $n = r + 1$, any form of degree r is the Hodge dual of a one-form and so has the form $\alpha = e_2 \wedge \cdots \wedge e_n$, which is degenerate.

To demonstrate existence of non-degenerate forms in the remaining cases, first consider $r = 3$. If $n \geq 3$ is odd, let ω be a non-degenerate two-form on \mathbb{R}^{n-1} , then $\alpha = \omega \wedge e_n$ is a non-degenerate three-form on \mathbb{R}^n . If $n \geq 6$ is even, then writing $\mathbb{R}^n = \mathbb{R}^3 \oplus \mathbb{R}^{n-3}$ we have a non-degenerate three-form given by $\alpha = e_{123} + \alpha'$, where e_{123} is a volume form on \mathbb{R}^3 and α' is non-degenerate on \mathbb{R}^{n-3} .

Now for $r > 3$, given a non-degenerate form α' of degree $r - 1$ on \mathbb{R}^{n-1} we have that $\alpha = \alpha' \wedge e_n$ is non-degenerate of degree r on \mathbb{R}^n . \square

3 Multi-moment maps in general degree

The general situation we wish to study is where a symmetry group G acts on a manifold M preserving some closed form.

Definition 3.1. *For $r \geq 2$, a closed geometry of degree r on a manifold M is choice of a closed differential form $\alpha \in \Omega^r(M)$.*

Here α closed means $d\alpha = 0$ in the exterior algebra. This implies that if $\ker \alpha$ has constant dimension then $\mathcal{D} = \ker \alpha$ is integrable as a distribution. It follows that α induces a weakly non-degenerate closed form on M/\mathcal{D} when this quotient is a manifold. In general we will not make any non-degeneracy assumptions on α . However, when needed, the following terminology will be useful.

Definition 3.2 (Baez, Hoffnung and Rogers [1]). *A k -plectic structure is a closed geometry (M, α) of degree $r = k + 1$ with α (weakly) non-degenerate.*

It is easy to give a couple of elementary examples. Firstly, if $M = \mathbb{R}^n$ is a vector space, then any constant coefficient form α on M is closed, and the discussion of the previous section gives many k -plectic examples. Of particular importance are the forms ϕ_0 (2.2), on \mathbb{R}^7 , Φ_0 (2.3) and ρ_0 (2.4), both on \mathbb{R}^8 .

Example 3.3 (Multi-phase space). Given any manifold N we may consider $M = \Lambda^k T^*N$. This carries a tautological form $\beta \in \Omega^k(M)$ given by

$$\beta_b(X_1, \dots, X_k) = b(\pi_* X_1, \dots, \pi_* X_k),$$

where $\pi: M = \Lambda^k T^*N \rightarrow N$ is the bundle projection. Defining

$$\alpha = d\beta,$$

we get a closed $(k + 1)$ -form on M which turns out to be non-degenerate so (M, α) is k -plectic. To see this, choose local coordinates q^1, \dots, q^n on $U \subseteq N$ and note that

$dq^I = dq^{i_1} \wedge \cdots \wedge dq^{i_{r-1}}$ gives a basis for each fibre of $\Lambda^k T^*U \subset M$. Let p_I be the corresponding fibre coordinates, then

$$\beta = \sum_I p_I dq^I, \quad \alpha = \sum_I dp_I \wedge dq^I,$$

and non-degeneracy of α is manifest. This is the higher degree generalisation of the usual symplectic structure on phase space T^*N . It is the central example in mechanics and field theory based approaches to the geometry of closed forms. \diamond

In order to build multi-moment maps we need to construct closed one-forms out of the group action and the closed r -form α . This involves contracting α with linear combinations of $r - 1$ vector fields. Here it is convenient to use the notion of multi-vectors.

3.1 Calculus of multi-vectors

Recall that a multi-vector p of degree s on M is a sum $p = \sum_{i=1}^k q_i$ of simple multi-vectors of the form

$$q = X_1 \wedge X_2 \wedge \cdots \wedge X_s \tag{3.1}$$

with X_j smooth vector fields on M . We will use

$$\mathfrak{X}^s(M) = \Gamma(\Lambda^s TM)$$

to denote the space of degree s multi-vectors on M . This is dual to the space $\Omega^s(M)$ of differential forms of the same degree. We write \lrcorner for the partial evaluation map $\mathfrak{X}^s(M) \times \Omega^r(M) \rightarrow \Omega^{r-s}(M)$,

$$(q \lrcorner \beta)(Y_1, \dots, Y_{r-s}) = \beta(X_1, X_2, \dots, X_s, Y_1, \dots, Y_{r-s}).$$

When we consider symmetries, we will have use for a generalisation of Cartan's formula $\mathcal{L}_X \alpha = d(X \lrcorner \alpha) + X \lrcorner d\alpha$ for the Lie derivative. To this end note that one may regard $\mathfrak{X}(M)$ as a vector space over \mathbb{R} and form the exterior powers $\Lambda_{\mathbb{R}}^s \mathfrak{X}(M)$. These spaces are larger than $\mathfrak{X}^s(M)$ which is equal to the exterior product of $\mathfrak{X}(M)$ over $C^\infty(M)$. There is a natural \mathbb{R} -linear projection $\Lambda_{\mathbb{R}}^s \mathfrak{X}(M) \rightarrow \mathfrak{X}^s(M)$ given on decomposable elements by

$$Q = X_1 \wedge \cdots \wedge X_s \mapsto q = X_1 \wedge \cdots \wedge X_s, \tag{3.2}$$

where \wedge denotes the wedge product over \mathbb{R} . For a such a Q , we write

$$Q_{\wedge i} = (-1)^{i-1} X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_s$$

and $Q_{\wedge ij} = (Q_{\wedge i})_{\wedge j}$. We define

$$(\lrcorner \mathcal{L})_Q \alpha = \sum_{i=1}^s Q_{\wedge i} \lrcorner \mathcal{L}_{X_i} \alpha \quad \text{and} \tag{3.3}$$

$$L(Q) = \sum_{1 \leq i < j \leq s} [X_i, X_j] \wedge Q_{\wedge ij}, \tag{3.4}$$

and extend \mathbb{R} -linearly to $\Lambda_{\mathbb{R}}^s \mathfrak{X}(M)$.

Lemma 3.4 (Extended Cartan Formula). For $\alpha \in \Omega^r(M)$ and $p \in \mathfrak{X}^s(M)$, we have

$$p \lrcorner d\alpha - (-1)^s d(p \lrcorner \alpha) = (\lrcorner \mathcal{L})_p \alpha - L(P) \lrcorner \alpha$$

for any $P \in \Lambda_{\mathbb{R}}^s \mathfrak{X}(M)$ projecting to p .

Proof. The left-hand side is independent of the presentation of p and both sides are \mathbb{R} -linear, so it is enough to prove the corresponding formula for a decomposable Q projecting to q as in (3.2). Note that when $s = r + 1$ we have one of the standard formulae for the exterior derivative:

$$(d\alpha)(q) = (\lrcorner \mathcal{L})_Q \alpha - \alpha(L(Q)). \quad (3.5)$$

For general $s \leq r + 1$, write $Q' = Y_1 \wedge \cdots \wedge Y_t$ with $s + t = r + 1$. Note that we always have $Q \lrcorner \beta = q \lrcorner \beta$ for any form β . Now we compute, using (3.5), (3.3) and (3.4),

$$\begin{aligned} (q \lrcorner d\alpha)(q') &= d\alpha(q \wedge q') = (\lrcorner \mathcal{L})_{Q \wedge Q'} \alpha - \alpha(L(Q \wedge Q')) \\ &= \sum_{i=1}^s (\mathcal{L}_{X_i} \alpha)(Q \wedge \wedge_i Q') + (-1)^s \sum_{j=1}^t (\mathcal{L}_{Y_j} \alpha)(Q \wedge Q' \wedge \wedge_j) \\ &\quad - \alpha(L(Q) \wedge Q') - (-1)^s \alpha(Q \wedge L(Q')) \\ &\quad - (-1)^s \sum_{i=1}^s \sum_{j=1}^t \alpha([X_i, Y_j] \wedge Q \wedge \wedge_i Q' \wedge \wedge_j) \\ &= ((\lrcorner \mathcal{L})_Q \alpha)(Q') + (-1)^s \sum_{j=1}^t \mathcal{L}_{Y_j} (Q \lrcorner \alpha)(Q' \wedge \wedge_j) - (-1)^s \sum_{j=1}^t \alpha(\mathcal{L}_{Y_j} Q \wedge Q' \wedge \wedge_j) \\ &\quad - (L(Q) \lrcorner \alpha)(Q') - (-1)^s (Q \lrcorner \alpha)(L(Q')) + (-1)^s \sum_{j=1}^t \alpha(\mathcal{L}_{Y_j} Q \wedge Q' \wedge \wedge_j) \\ &= ((\lrcorner \mathcal{L})_Q \alpha + (-1)^s d(q \lrcorner \alpha) - L(Q) \lrcorner \alpha)(q'), \end{aligned}$$

which gives the claimed result. \square

3.2 Symmetries and multi-moment maps

Let (M, α) be a manifold with an r -form α not necessarily closed.

Definition 3.5. A group of symmetries of (M, α) is a connected Lie group G acting on M preserving α .

Infinitesimally this means that

$$\mathcal{L}_X \alpha = 0 \quad \text{for all } X \in \mathfrak{g},$$

where \mathfrak{g} is the Lie algebra of G and we write X for the vector field generated by X .

Example 3.6 (Multi-phase space). Suppose $M = \Lambda^k T^*N$ with the canonical k -plectic form α of Example 3.3. Then any diffeomorphism ϕ of the base N induces a symmetry ψ of (M, α) covering ϕ , namely take $\psi = (\phi^*)^{-1}$. In this way, any group G of diffeomorphisms of N lifts a group of symmetries of (M, α) . \diamond

The map sending an element \mathbf{X} of \mathfrak{g} to the vector field X on M generated by \mathbf{X} is \mathbb{R} -linear. So we may extend this to associate to each $\mathfrak{p} \in \Lambda^s \mathfrak{g}$ a unique multi-vector $p \in \mathfrak{X}^s(M)$. For a decomposable $\mathfrak{q} = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \cdots \wedge \mathbf{X}_s$, the corresponding multi-vector is exactly the q given in equation (3.1). When G preserves α , we have $(\lrcorner \mathcal{L})_p \alpha = 0$ for each $\mathfrak{p} \in \Lambda^s \mathfrak{g}$, so the extended Cartan formula reads

$$p \lrcorner d\alpha - (-1)^s d(p \lrcorner \alpha) = -L(p) \lrcorner \alpha \quad \text{for all } \mathfrak{p} \in \Lambda^s \mathfrak{g}, \quad (3.6)$$

where $L(p)$ is understood to be the multi-vector corresponding to $L(\mathfrak{p}) \in \Lambda^{s-1} \mathfrak{g}$ which is defined as in (3.4) but using the Lie bracket of \mathfrak{g} . In particular, when α is closed, we see that $p \lrcorner \alpha$ is closed whenever $L(\mathfrak{p}) = 0$.

Definition 3.7. *The k th Lie kernel of \mathfrak{g} is the \mathfrak{g} -module*

$$\mathcal{P}_{\mathfrak{g},k} = \ker(L: \Lambda^k \mathfrak{g} \rightarrow \Lambda^{k-1} \mathfrak{g}).$$

If G acts a group of symmetries for a closed geometry of degree r we may write

$$\mathcal{P}_{\mathfrak{g}} = \mathcal{P}_{\mathfrak{g},r-1}$$

for the corresponding Lie kernel of \mathfrak{g} .

Since each Ad_g , $g \in G$, is a Lie algebra automorphism of \mathfrak{g} , we see that $\mathcal{P}_{\mathfrak{g}}$ is a G -module.

If G is a Abelian, then $\mathcal{P}_{\mathfrak{g},k} = \Lambda^k \mathfrak{g}$. For any \mathfrak{g} , we have $\mathcal{P}_{\mathfrak{g},1} = \mathfrak{g}$.

Example 3.8. For $G = SU(2)$, let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ be a basis of $\mathfrak{su}(2)$ which satisfies $[\mathbf{X}_1, \mathbf{X}_2] = -2\mathbf{X}_3$, etc. The map $L: \Lambda^3 \mathfrak{su}(2) \rightarrow \Lambda^2 \mathfrak{su}(2)$ is given by the cyclic sum $L(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3) = \mathfrak{G}_{1,2,3}[\mathbf{X}_1, \mathbf{X}_2] \wedge \mathbf{X}_3 = -2 \sum_{i=1}^3 \mathbf{X}_i \wedge \mathbf{X}_i = 0$, so this L is identically zero. Thus $\mathcal{P}_{\mathfrak{su}(2),2} = \Lambda^3 \mathfrak{su}(2)^* \cong \mathbb{R}$. On the other hand, $L: \Lambda^2 \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$ is an isomorphism and $\mathcal{P}_{\mathfrak{su}(2),1} = \{0\}$.

More generally, if \mathfrak{g} is a simple Lie group of compact type, and $\mathbf{X}_1, \dots, \mathbf{X}_n$ is a basis for \mathfrak{g} consisting of unit length vectors for the Killing form, then one has that the element $\sum_{i,j=1}^n \mathbf{X}_i \wedge \mathbf{X}_j \wedge [\mathbf{X}_i, \mathbf{X}_j]$ lies in $\mathcal{P}_{\mathfrak{g},2}$. This element corresponds to the standard representative $\gamma(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \langle [\mathbf{X}, \mathbf{Y}], \mathbf{Z} \rangle$ of the third cohomology group $H^3(\mathfrak{g}) \cong \mathbb{R}$. For \mathfrak{g} simple, $L: \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$ is onto so we have $\mathcal{P}_{\mathfrak{g},1} \cong \Lambda^2 \mathfrak{g} / \mathfrak{g}$, which is a non-zero irreducible G -module when $\dim G > 3$, cf. Wolf [41]. \diamond

Now suppose that we have a closed geometry (M, α) and that G is a group of symmetries. Then for \mathfrak{p} in $\mathcal{P}_{\mathfrak{g}} = \mathcal{P}_{\mathfrak{g},r-1} \subseteq \Lambda^{r-1} \mathfrak{g}$ we have $p \lrcorner \alpha \in \Omega^1(M)$ and

$$d(p \lrcorner \alpha) = 0,$$

by Lemma 3.4. Thus $p \lrcorner \alpha$ is a closed one-form and locally the derivative of a function ν_p . Letting \mathfrak{p} vary over $\mathcal{P}_{\mathfrak{g}}$, we obtain a family of functions that may be combined into a local map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$ by setting $\langle \nu, \mathfrak{p} \rangle = \nu_p$. This motivates the following definition:

Definition 3.9. Let G be a symmetry group for a closed geometry (M, α) . A multi-moment map for this action is an equivariant map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$ satisfying

$$d\langle \nu, \mathfrak{p} \rangle = p_{\lrcorner} \alpha \quad (3.7)$$

for all $\mathfrak{p} \in \mathcal{P}_{\mathfrak{g}}$.

For α a symplectic form, this is exactly the usual notion of moment map, since $\mathcal{P}_{\mathfrak{g}} = \mathcal{P}_{\mathfrak{g},1} = \mathfrak{g}$. Concrete examples of multi-moment maps will be given in Section 4.

The obstructions to constructing a multi-moment map are quite weak. To start with if $b_1(M) = 0$ then we can always find global functions $\nu_{\mathfrak{p}}$ with $d\nu_{\mathfrak{p}} = p_{\lrcorner} \alpha$. Using averaging arguments we have the following topological existence result.

Theorem 3.10. Let (M, α) be a closed geometry with G acting as a group of symmetries. Suppose $b_1(M) = 0$. Then there exists a multi-moment map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$ if either

- (i) G is compact, or
- (ii) M is compact and orientable, and G preserves a volume form on M .

Proof. The proofs given in [30] for the case of α a three-form carry directly over to this general situation. In the first case, one averages over G ; in the second, one averages over M . \square

A second useful existence result occurs when α is exact in a good way.

Proposition 3.11. Suppose G is a group of symmetries of a closed geometry (M, α) of degree r . If there exists a G -invariant form β with $\alpha = d\beta$, then

$$\langle \nu, \mathfrak{p} \rangle = (-1)^{r-1} \beta(p) \quad \text{for } \mathfrak{p} \in \mathcal{P}_{\mathfrak{g}}$$

defines a multi-moment map for the action of G .

Proof. Invariance of β implies that ν is equivariant, so we just need to verify equation (3.7). However, Lemma 3.4 gives

$$d\langle \nu, \mathfrak{p} \rangle = (-1)^{r-1} d(p_{\lrcorner} \beta) = p_{\lrcorner} d\beta = p_{\lrcorner} \alpha,$$

since $L(\mathfrak{p}) = 0$ and β is invariant. \square

Finally there is a purely algebraic existence and uniqueness result depending only on the Betti numbers of \mathfrak{g} . The dual of the map $L: \Lambda^{k+1} \mathfrak{g} \rightarrow \Lambda^k \mathfrak{g}$ is essentially the differential

$$\begin{aligned} d: \Lambda^k \mathfrak{g}^* &\rightarrow \Lambda^{k+1} \mathfrak{g}^* \\ (d\gamma)(X_1, X_2, \dots, X_k) &= -\gamma(L(X_1 \wedge X_2 \wedge \dots \wedge X_k)). \end{aligned} \quad (3.8)$$

The Jacobi identity implies $L \circ L = 0$ and so $d \circ d = 0$. Thus we have the Lie algebra homology $H_*(\mathfrak{g})$ of \mathfrak{g} defined by the complex $(\Lambda^* \mathfrak{g}, L)$ and the Lie algebra cohomology $H^*(\mathfrak{g})$ defined by $(\Lambda^* \mathfrak{g}^*, d)$. In particular,

$$H^k(\mathfrak{g}) = \frac{\ker(d: \Lambda^k \mathfrak{g}^* \rightarrow \Lambda^{k+1} \mathfrak{g}^*)}{\text{im}(d: \Lambda^{k-1} \mathfrak{g}^* \rightarrow \Lambda^k \mathfrak{g}^*)} = \frac{Z^k(\mathfrak{g})}{B^k(\mathfrak{g})}, \quad (3.9)$$

the quotient of the space $Z^k(\mathfrak{g}) = \ker d$ of cycles by the space of boundaries $B^k(\mathfrak{g}) = \text{im } d$, and we write

$$b_k(\mathfrak{g}) = \dim H^k(\mathfrak{g})$$

for the k th Betti number of \mathfrak{g} .

Our algebraic existence and uniqueness criteria are expressed in terms of vanishing of certain Betti numbers. It is therefore useful to introduce the following terminology.

Definition 3.12. *A connected Lie group G or its Lie algebra \mathfrak{g} is (cohomologically) $(k_1, k_2, \dots, k_\ell)$ -trivial if the Betti numbers $b_k(\mathfrak{g})$ vanish for $k = k_1, k_2, \dots, k_\ell$.*

We will discuss these type of conditions in some detail in Section 5, however let us note that a simple Lie algebra is always $(1, 2)$ -trivial, but has b_3 non-zero. Indeed looking up the Poincaré polynomials of the compact simple Lie algebras reveals the following:

Proposition 3.13 (Madsen [28]). *Every compact simple Lie algebra not isomorphic to $\mathfrak{su}(n)$, $n \geq 3$, is $(1, 2, 4, 5, 6)$ -trivial. \square*

Returning to multi-moment maps the algebraic existence and uniqueness result is:

Theorem 3.14. *Suppose (M, α) is a closed geometry of degree r and G is a group of symmetries. If G is $(r-1, r)$ -trivial then there exists a unique multi-moment map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$. If G is just $(r-1)$ -trivial, then ν is unique whenever it exists.*

Proof. The proof builds on the following observation. Taking the dual of the exact sequence

$$0 \longrightarrow \mathcal{P}_{\mathfrak{g},k} \xrightarrow{\iota} \Lambda^k \mathfrak{g} \xrightarrow{L} \Lambda^{k-1} \mathfrak{g}$$

we obtain the sequence

$$\Lambda^{k-1} \mathfrak{g}^* \xrightarrow{d} \Lambda^k \mathfrak{g}^* \xrightarrow{\pi} \mathcal{P}_{\mathfrak{g},k}^* \longrightarrow 0.$$

From this one sees that $\mathcal{P}_{\mathfrak{g},k}^* \cong \Lambda^k \mathfrak{g}^* / B^k(\mathfrak{g})$ and so the exterior derivative $d: \Lambda^k \mathfrak{g}^* \rightarrow \Lambda^{k+1} \mathfrak{g}^*$ induces a well-defined linear map $d_{\mathcal{P}}: \mathcal{P}_{\mathfrak{g}}^* \rightarrow \Lambda^{k+1}(\mathfrak{g})$ via $d_{\mathcal{P}}a = db$ where $a = b + B^k(\mathfrak{g}) \in \mathcal{P}_{\mathfrak{g},k}^* \cong \Lambda^k \mathfrak{g}^* / B^k(\mathfrak{g})$. We now see that $d_{\mathcal{P}}$ is injective if and only if $b_k(\mathfrak{g}) = 0$ and that the image of $d_{\mathcal{P}}$ is $B^{k+1}(\mathfrak{g})$.

Now let us consider the situation of the Theorem. The action of G on M defines a map

$$\begin{aligned} \Psi: M &\rightarrow Z^r(\mathfrak{g}), \\ \langle \Psi(x), \mathfrak{p} \rangle &= (-1)^r \alpha(p)_x, \end{aligned} \tag{3.10}$$

for all $\mathfrak{p} \in \Lambda^r \mathfrak{g}$ and $x \in M$. To see that the image lies in $Z^r(\mathfrak{g}) \leq \Lambda^r \mathfrak{g}^*$, use (3.6) for the invariant closed form α to get

$$\langle d(\Psi(x)), \mathfrak{q} \rangle = \langle \Psi(x), L(\mathfrak{q}) \rangle = (-1)^r \alpha(L(q))_x = (-1)^{r+1} (q \lrcorner d\alpha)_x = 0,$$

for each $\mathfrak{q} \in \Lambda^{r+1} \mathfrak{g}$.

Now if $b_r(\mathfrak{g}) = 0$, then $Z^r(\mathfrak{g}) = B^r(\mathfrak{g}) = \text{im } d_{\mathcal{P}}$, so we may find for each $x \in M$ a $\nu_x \in \mathcal{P}_{\mathfrak{g}}^*$ with $d_{\mathcal{P}}(\nu_x) = \Psi(x)$. If $b_{r-1}(\mathfrak{g}) = 0$, the map $d_{\mathcal{P}}$ is injective, so there is a unique choice of ν_x for each x . It follows that ν is equivariant.

Suppose we have an equivariant map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$ with $d_{\mathcal{P}}\nu = \Psi$ of equation (3.10) and that $b_{r-1}(\mathfrak{g}) = 0$. We claim that ν is a multi-moment map. The important fact here is that $b_{r-1}(\mathfrak{g}) = 0$ says $\ker d = \text{im } d$ in $\Lambda^{r-1}\mathfrak{g}^*$ which dually means that $\text{im } L = \ker L$ in $\Lambda^{r-1}\mathfrak{g}$. However, $\ker L = \mathcal{P}_{\mathfrak{g}}$ so $L: \Lambda^r\mathfrak{g} \rightarrow \Lambda^{r-1}\mathfrak{g}$ maps on to the Lie kernel $\mathcal{P}_{\mathfrak{g}}$. We may now compute, for $\mathfrak{p} = -L(\mathfrak{q}) \in \mathcal{P}_{\mathfrak{g}}$,

$$\begin{aligned} d\langle \nu, \mathfrak{p} \rangle &= -d\langle \nu, L(\mathfrak{q}) \rangle = d\langle d_{\mathcal{P}}(\nu), \mathfrak{q} \rangle \\ &= d\langle \Psi, \mathfrak{q} \rangle = (-1)^r d(q \lrcorner \alpha) \\ &= L(q) \lrcorner \alpha = p \lrcorner \alpha, \end{aligned}$$

by (3.6). Thus ν is indeed a multi-moment map. \square

4 Example geometries and their multi-moment maps

Having introduced the general theory of multi-moment maps we will now look at a number of concrete examples. For many examples the main focus will be on closed geometries of degree 4, but we will also consider other cases. When relevant we will also discuss the use of multi-moment maps to describe reductions of certain geometries.

Definition 4.1. *Suppose $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$ is a multi-moment map. Then for each $t \in \mathcal{P}_{\mathfrak{g}}^*$ fixed by the G -action, the reduction of M at level t is*

$$M \parallel_{\nu, t} G = \nu^{-1}(t)/G.$$

We set $M \parallel_{\nu} G = M \parallel_{\nu, 0} G$ for the reduction at level 0.

This makes sense, since ν is G -equivariant so the G -action preserves $\nu^{-1}(t)$ whenever t is fixed by G . This notion of reduction corresponds to the usual Marsden-Weinstein quotient in symplectic geometry. However because the structure of forms of higher degree is so varied, the type of geometry obtained on the quotient is often of a different character to the geometry on M . Also, one usually has to impose assumptions, such as freeness of the action of G and regularity of the value t , in order to obtain smooth quotients.

4.1 Multi-phase space

This is $M = \Lambda^{r-1}T^*N$ with the canonical r -form α of Example 3.3. If G is any group of diffeomorphisms of N , then as noted in Example 3.6, this induces an action of G on M preserving α . However, in this case we have α is equivariantly exact: the canonical form β is also G -invariant and satisfies $d\beta = \alpha$. By Proposition 3.11, there is a multi-moment map ν given by $\langle \nu, \mathfrak{p} \rangle = (-1)^{r-1}\beta(p)$.

A concrete example is provided by taking $N = \mathbb{R}^4$. If we consider the closed geometry of degree 4 on $M = \Lambda^3 T^* N \cong TN \cong \mathbb{R}^8$, we have α and β given by the cyclic sums

$$\alpha = \mathfrak{S}_{1,2,3,4} dp^1 \wedge dq^2 \wedge dq^3 \wedge dq^4, \quad \beta = \mathfrak{S}_{1,2,3,4} p^1 dq^2 \wedge dq^3 \wedge dq^4.$$

If $G = \mathbb{R}^4$ acts by translations on $N = \mathbb{R}^4$ then

$$\nu \left(\frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial q^3} \right) = p^4, \quad \text{etc.}$$

and ν is simply projection on to the fibres of $T\mathbb{R}^4 \rightarrow \mathbb{R}^4$.

4.2 Product manifolds

Let (N, α') be a $(k-1)$ -plectic manifold. Consider $M = S^1 \times N$ and write θ for the standard one-form on the S^1 -factor. Then as in the proof of Proposition 2.6, we have that $\alpha = \theta \wedge \alpha'$ is a k -plectic form on M .

If H is a group of symmetries of (N, α') , then $G = S^1 \times H$ is a group of symmetries of (M, α) , where the S^1 -factor of G acts non-trivially on just the S^1 -factor of M preserving θ .

Suppose $\nu': N \rightarrow \mathcal{P}_{\mathfrak{h}, k-1}^*$ is a multi-moment map for the action of H on N . Writing $\mathfrak{g} = \mathbb{R}\mathbb{T} \oplus \mathfrak{h}$, we have that

$$\Lambda^m \mathfrak{g} = (\mathbb{T} \wedge \Lambda^{m-1} \mathfrak{h}) \oplus \Lambda^m \mathfrak{h}. \quad (4.1)$$

Since \mathbb{T} commutes with \mathfrak{h} , the map $L: \Lambda^k \mathfrak{g} \rightarrow \Lambda^{k-1} \mathfrak{g}$ preserves the splittings (4.1) and we conclude that

$$\mathcal{P}_{\mathfrak{g}, k} \cong (\mathbb{T} \wedge \mathcal{P}_{\mathfrak{h}, k-1}) \oplus \mathcal{P}_{\mathfrak{h}, k}.$$

As $0 = \mathcal{L}_T \theta = d(T \lrcorner \theta)$, we may scale \mathbb{T} by a constant so that $\theta(T) = 1$. Let ϑ denote the element of \mathfrak{g}^* that annihilates \mathfrak{h} and has $\vartheta(\mathbb{T}) = 1$. We claim that

$$\nu = \vartheta \wedge \nu'$$

is a multi-moment map for the action of G on M . Firstly, ν is a map to $\mathbb{R}\vartheta \wedge \mathcal{P}_{\mathfrak{h}, k-1}^* \subset \mathcal{P}_{\mathfrak{g}, k}^*$ and it is equivariant for the action of $G = S^1 \times H$. Secondly, for $\mathfrak{p} \in \mathcal{P}_{\mathfrak{g}, k}$, we have $\mathfrak{p} = \mathbb{T} \wedge \mathfrak{p}' + \mathfrak{q}$ with $\mathfrak{p}' \in \mathcal{P}_{\mathfrak{h}, k-1}$ and $\mathfrak{q} \in \mathcal{P}_{\mathfrak{h}, k}$. Now ν is zero on \mathfrak{q} and

$$d\langle \nu, \mathfrak{p} \rangle = d\langle \vartheta \wedge \nu', \mathbb{T} \wedge \mathfrak{p}' \rangle = d\langle \nu', \mathfrak{p}' \rangle = \mathfrak{p}' \lrcorner \alpha' = (T \wedge \mathfrak{p}') \lrcorner (\theta \wedge \alpha') = \mathfrak{p} \lrcorner \alpha.$$

So ν satisfies (3.7) and is a multi-moment map.

Starting with (N, ω) a symplectic manifold with a Hamiltonian action of H , iteration of the above construction produces a k -plectic structure on $M = T^{k-1} \times N$ together with a multi-moment map for the action of $G = T^{k-1} \times H$.

4.3 Symplectic manifolds

If $\omega \in \Omega^2(M)$ is an ordinary symplectic form, then each power $\omega^k \in \Omega^{2k}(M)$ with $2k \leq \dim M$ is $(2k-1)$ -plectic. In particular, we may consider the four-form

$\alpha = \omega \wedge \omega = \omega^2$ as a 3-plectic form on M . Let us take $\dim M \geq 6$ and assume that M is simply-connected.

If X is a vector field preserving α , then we have $0 = \mathcal{L}_X \alpha = 2\omega \wedge \mathcal{L}_X \omega$. But the map $\omega \wedge \cdot : \Lambda^2 T^* M \rightarrow \Lambda^4 T^* M$ is injective when $\dim M \geq 6$, so we have $\mathcal{L}_X \omega = 0$ and X also preserves ω . Thus symmetries of (M, ω^2) are nothing but symplectomorphisms of (M, ω) .

Let us first consider actions of Abelian groups. Suppose $G = \mathbb{R}^3$ acts generated by vector fields X_1, X_2 and X_3 . Then by the extended Cartan formula (Lemma 3.4), we have $d(\omega(X_i, X_j)) = d((X_i \wedge X_j) \lrcorner \omega) = -[X_i, X_j] \lrcorner \omega = 0$, showing that $\omega(X_i, X_j)$ is constant. Taking constant linear combinations of our vector fields we may therefore assume that $\omega(X_i, X_3) = 0$, for $i = 1, 2$, and that $\omega(X_1, X_2) = \delta \in \{0, 1\}$.

Now a multi-moment map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^* = \Lambda^3 \mathfrak{g}^* \cong \mathbb{R}$ has differential

$$\begin{aligned} d\nu &= (X_1 \wedge X_2 \wedge X_3) \lrcorner \alpha = (X_1 \wedge X_2 \wedge X_3) \lrcorner \omega^2 \\ &= \mathfrak{S}_{1,2,3} \omega(X_1, X_2) \omega(X_3, \cdot) = \delta X_3 \lrcorner \omega. \end{aligned} \tag{4.2}$$

Thus if $\delta = 0$, i.e. the orbits of the G -action are isotropic, then ν is constant. Fixing now $\delta = 1$, we note that if μ_i is a symplectic moment map for X_i , then $X_2 \mu_1 = d\mu_1(X_2) = \omega(X_1, X_2) = \delta = 1$. Thus in this case X_1 and X_2 do not Poisson commute and there is no symplectic moment map for the action of the whole of G . However, $\nu = \mu_3$ is a multi-moment map for the action of G .

A second case is given by considering $G = SU(2)$ with generators $X_i \in \mathfrak{su}(2)$ as in Example 3.8. Since G is compact and $b_1(M) = 0$, there is a symplectic moment map $\mu = (\mu_1, \mu_2, \mu_3): M \rightarrow \mathfrak{su}(2)^* \cong \mathbb{R}^3$ satisfying $L_{X_1} \mu_2 = -2\mu_3$, etc. As above we find that $L_{X_1} \mu_2 = d\mu_2(X_1) = -\omega(X_1, X_2)$. For a multi-moment map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{su}(2),2} \cong \mathbb{R}$, equation (4.2) gives

$$d\nu = (X_1 \wedge X_2 \wedge X_3) \lrcorner \alpha = 2 \sum_{i=1}^3 \mu_i d\mu_i = d\|\mu\|^2.$$

Thus a multi-moment map is $\nu = \|\mu\|^2$. It is unique up to the addition of a constant.

The quotient $M //_{\nu} SU(2) = \nu^{-1}(0) / SU(2)$ is nothing other than the symplectic quotient of M by $SU(2)$, and thus inherits both a symplectic form ω' and a closed four-form $\alpha' = (\omega')^2$.

For $t > 0$, the geometry of the reduction $M //_{\nu,t} SU(2) = \nu^{-1}(t) / SU(2)$ is more complicated. Part of the reason for this is that even in the good case when $SU(2)$ acts freely on M_t , there is no canonical choice of connection form for the $SU(2)$ -bundle $M_t \rightarrow M //_{\nu,t} SU(2)$. This problem is remedied in geometries that come equipped with a metric.

4.4 HyperKähler manifolds

A variant of the construction considered above arises in the setting of quaternionic geometry. A quaternion-Hermitian manifold Q is a $4n$ -dimensional Riemannian manifold with a rank three subbundle $\mathcal{G} \subset \text{End}(TQ)$ which is locally trivialised by anti-commuting almost complex structures I, J and K that satisfy $K = IJ$. In

addition the Riemannian metric g must be compatible with \mathcal{G} , meaning $g(\mathcal{I}X, \mathcal{I}Y)$ for each $X, Y \in T_x Q$ and $\mathcal{I} \in \mathcal{G}_x$; in particular $\omega_I = g(I \cdot, \cdot)$, etc., are locally defined non-degenerate two-forms. A quaternion-Hermitian manifold carries a non-degenerate four-form Ω which may locally be expressed as

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

In dimension eight and above one says that Q is *quaternionic Kähler* if the fundamental form is parallel, $\nabla^{\text{LC}}\Omega = 0$. This implies that Ω is closed. In dimensions 12 and higher $d\Omega = 0$ is actually equivalent to the quaternionic Kähler condition [36]. Gray [21] showed that the stabiliser of Ω under the action of $GL(4n, \mathbb{R})$, $n > 1$, is the compact group $Sp(n) Sp(1)$ and so Ω determines the metric g . In dimension four, these considerations no longer hold and a quaternionic Kähler manifold is instead defined to be an oriented Riemannian manifold which is Einstein and self-dual.

If the subbundle \mathcal{G} can be globally trivialised by I, J, K and these almost complex structures are integrable then we have a hyper-Hermitian manifold. This will then be hyperKähler provided that the two-forms ω_I , etc., are closed.

In [37] it was shown that to any quaternionic Kähler manifold Q^{4n} of positive scalar curvature one may associate a hyperKähler manifold $M^{4n+4} = \mathcal{U}(Q)$ which acts as a hyperKähler generalisation of the twistor space; this is known as the *Swann bundle* and may be written as $\mathcal{U}(Q) = \mathbb{R}_{>0} \times \mathcal{S}$, where \mathcal{S} is the bundle of triples (I, J, K) . Conversely given a $(4n+4)$ -dimensional hyperKähler manifold M admitting a special type of $SU(2)$ -action then a version of the Marsden-Weinstein reduction produces a quaternionic-Kähler manifold of positive scalar curvature; this latter reduction process can be realised in terms of multi-moment maps.

The relevant type of $SU(2)$ -symmetry often arises due to the presence of a vector field X on (M^{4n+4}, g, I, J, K) , a *special homothety* (cf. [33]), with the following properties:

$$\mathcal{L}_X g = g, \quad \mathcal{L}_{IX} g = 0, \quad \mathcal{L}_{IX} I = 0, \quad \mathcal{L}_{IX} J = -K, \quad \mathcal{L}_{IX} K = J, \quad \text{etc.}$$

Special homotheties generate a local action of \mathbb{H}^* and in good cases the vector fields IX, JX, KX integrate to give an action of $SU(2)$ which is necessarily locally free.

Proposition 4.2 (Madsen [28]). *Let (M^{4n+4}, g, I, J, K) be a hyperKähler manifold, and X a special homothety. If IX, JX, KX generate a locally free action of $SU(2)$ then this action preserves Ω and a multi-moment map $\nu: M \rightarrow \mathbb{R} \cong \mathcal{P}_{\text{su}(2),2}^*$ is given by*

$$\nu = -3\|X\|^4. \quad \square$$

Any non-zero $t \in \nu(M)$ is a regular value. The level sets correspond to $\|X\|$ is constant and the results of [37] show that $M \parallel_{\nu,t} SU(2)$ is a quaternionic Kähler orbifold of positive scalar curvature.

4.5 Holonomy $Spin(7)$

A $Spin(7)$ structure is a geometry modelled on the form Φ_0 of equation (2.3).

Definition 4.3. *An eight-manifold M has a $Spin(7)$ -structure if there is a form $\Phi \in \Omega^4(M)$ such that $(T_x M, \Phi_x)$ is linearly isomorphic to (\mathbb{R}^8, Φ_0) for each $x \in M$.*

Now Φ determines a volume form and Riemannian metric on M via the relations

$$\Phi^2 = 14 \text{ vol}, \quad ((X \wedge Y) \lrcorner \Phi)^2 \wedge \Phi = 6 \|X \wedge Y\|_g^2 \text{ vol}.$$

Comparing with Φ_0 we see that vol and g correspond to the standard volume $\text{vol}_0 = e_{12345678}$ and metric $g_0 = \sum_{i=1}^8 e_i^2$ on \mathbb{R}^8 . We also see that Φ is a self-dual four-form $*\Phi = \Phi$. In particular, a closed $Spin(7)$ -structure has Φ harmonic and it follows that the holonomy group of g is contained in $Spin(7)$. This is one of the two exceptional holonomies in the Berger classification [5, 6]. Examples of metrics with holonomy exactly $Spin(7)$ are not easy to find. Local existence was proved by Bryant [8], the first complete examples were produced by Bryant and Salamon [9] and the first compact examples were found by Joyce [25]. The complete examples produced by Bryant & Salamon have many symmetries, in fact the symmetry group acts with cohomogeneity one, so the principal orbit is of codimension one. Further systematic study of cohomogeneity one examples with compact symmetry group has been made by Reidegeld [34, 35]. One sees that many of the examples and candidates have compact symmetry groups of rank 3, so an interesting class of $Spin(7)$ -manifolds are those with T^3 -symmetry.

Given a closed $Spin(7)$ -structure (M, Φ) with free T^3 -symmetry, fix a basis U_1, U_2, U_3 for $\mathfrak{t} \cong \mathbb{R}^3$. Then we have the following two-forms on M :

$$\omega_1 = U_2 \lrcorner U_3 \lrcorner \Phi, \quad \omega_2 = U_3 \lrcorner U_1 \lrcorner \Phi, \quad \omega_3 = U_1 \lrcorner U_2 \lrcorner \Phi. \quad (4.3)$$

These forms are all closed, by the Lemma 3.4.

To see the structure of these forms, we consider the geometry of (\mathbb{R}^7, Φ_0) . Isolating e_1 in the expression (2.3) for Φ_0 we have

$$\begin{aligned} \Phi_0 &= e_1 \wedge (e_{234} + e_{256} + e_{278} + e_{357} - e_{368} - e_{458} - e_{467}) \\ &\quad + e_{5678} + e_{3478} + e_{3456} + e_{2468} - e_{2457} - e_{2367} - e_{2358} \\ &= e_1 \wedge \varphi_0 + *_7 \varphi_0, \end{aligned} \quad (4.4)$$

where we recognise φ_0 on $V_7 = \langle E_2, \dots, E_8 \rangle$ as a rewritten version of ϕ_0 in (2.2) and $*_7$ is the Hodge star operator on V_7 with respect to the induced metric and volume. In particular, we see that the stabiliser of e_1 under the action of $Spin(7)$ is the stabiliser of φ_0 which is the exceptional group G_2 . The orbit of e_1 under the action of the compact group $Spin(7)$ is thus of dimension $\dim Spin(7) - \dim G_2 = 21 - 14 = 7$. As the $Spin(7)$ -action preserves the metric g_0 , we conclude that $Spin(7)$ acts transitively on the unit sphere $S^7 \subset \mathbb{R}^8$. Thus for any unit vector $v \in \mathbb{R}^8$, we have that $v \lrcorner \Phi_0$ is a G_2 -form on v^\perp .

We may now repeat this argument, isolating e_2 in the expression for φ_0 to get

$$\begin{aligned} \varphi_0 &= e_2 \wedge (e_{34} + e_{56} + e_{78}) \\ &\quad + e_{357} - e_{368} - e_{458} - e_{468} \\ &= e_2 \wedge \omega + \psi_+. \end{aligned}$$

This time ω is a symplectic form on $V_6 = \langle E_3, \dots, E_8 \rangle$ and ψ_+ is the real part of the complex volume form $(e_3 + ie_4) \wedge (e_5 + ie_6) \wedge (e_7 + ie_8)$ on $V_6 \cong \mathbb{R}^6 \cong \mathbb{C}^3$. The stabiliser of e_2 under the action of G_2 is the stabiliser of the pair (ω, ψ_+) which is $SU(3)$. The orbit of $e_2 \in V_7$ has dimension $\dim G_2 - \dim SU(3) = 14 - 8 = 6$ and is just the unit sphere $S^6 \subset V_7$. Finally, the orbit of $e_3 \in V_6$ under the action of $SU(3)$ is $S^5 = SU(3)/SU(2)$. This demonstrates the following well-known result, cf. Bryant [8].

Lemma 4.4. *Spin(7) acts transitively on orthonormal pairs (v_1, v_2) and orthonormal triples (v_1, v_2, v_3) in \mathbb{R}^8 .* \square

In particular, $Spin(7)$ acts transitively on the sets of unit length simple bivectors $u_1 \wedge u_2$ and unit length simple trivectors $u_1 \wedge u_2 \wedge u_3$ on \mathbb{R}^8 . Using the first of these statements, we describe the form ω_3 of (4.3) as pointwise corresponding to a multiple of $(E_1 \wedge E_2) \lrcorner \Phi_0 = e_{34} + e_{56} + e_{78}$. Thus each ω_i is a two-form of rank 6.

Furthermore, suppose that $\nu: M \rightarrow \mathbb{R}$ is a multi-moment map for the action of T^3 . Then $d\nu = (U_1 \wedge U_2 \wedge U_3) \lrcorner \Phi$ which corresponds to a multiple of $(E_1 \wedge E_2 \wedge E_3) \lrcorner \Phi_0 = e_4$. This implies that on a level set $\nu^{-1}(t) \subset M$, the pull-back $i^*\omega_3$ of ω_3 under the inclusion map i corresponds to a multiple of $e_{56} + e_{78}$. One may prove that the T^3 -invariant forms $i^*\omega_1, i^*\omega_2, i^*\omega_3$ each vanishes on U_1, U_2 and U_3 and thus they descend to two-forms on the four-manifold $N_t = M \parallel_{\nu, t} T^3$. The following terminology will be used:

Definition 4.5. *A triple of $\sigma_1, \sigma_2, \sigma_3$ of symplectic structures on a manifold N of dimension four is weakly coherent if the forms are pointwise linearly independent, define the same orientation and the pairing $\sigma_i \wedge \sigma_j$ has definite sign.*

A more detailed analysis of our situation gives the following description of the quotients N_t .

Proposition 4.6 (Madsen [27]). *Let (M, Φ) be a closed Spin(7)-structure. Suppose T^3 acts freely on M preserving Φ and with a multi-moment map ν . Then for each $t \in \nu(M)$, the four-manifold $N_t = M \parallel_{\nu, t} T^3$ admits a real-analytic weakly coherent triple of symplectic structures $\sigma_1, \sigma_2, \sigma_3$.* \square

The fact that the quotient geometry is real-analytic follows from the remark that T^3 acts by isometries and that closed Spin(7) structures are Ricci-flat. It follows that the vector fields U_i are real-analytic.

To fully describe the relationship between the geometries of M and N_t , one may construct a connection one-form $\theta \in \Omega^1(M, \mathfrak{t})$ as follows. Let $G = (g_{ij})$ with $g_{ij} = g(U_i, U_j)$ and let $U^b = (U_1^b, U_2^b, U_3^b)$, where $U_i^b = g(U_i, \cdot)$. Then θ is given by

$$\theta = U^b G^{-1}.$$

This satisfies $\theta_i(U_j) = \delta_{ij}$ as required. The matrix G here turns out to be determined the geometry on N_t .

Lemma 4.7 (Madsen [27]). $G^{-1} = h^2 Q$, where $h = 1/\sqrt{\det G}$ and $Q = (q_{ij})$ is the matrix given by

$$\sigma_i \wedge \sigma_j = 2q_{ij} \text{ vol}$$

on N_t . □

Here vol denotes the volume form on N_t induced from the quotient.

As $\nu^{-1}(t)$ is a T^3 -bundle over N_t its curvature $F = d\theta$ is a closed form with integral periods. We write $F \in \Omega_{\mathbb{Z}}^2(N, \mathfrak{t})$ to indicate this. Conversely, given such an F , one may construct a principal T^3 -bundle over N with curvature F . Madsen shows that F satisfies the following symmetry condition

$$F_i \wedge \sigma_j = F_j \wedge \sigma_i, \quad \text{for all } i, j. \quad (4.5)$$

It turns out that this data is sufficient to invert the construction.

Theorem 4.8 (Madsen [27]). *Suppose N is a connected four-manifold with a real-analytic weakly coherent symplectic triple $(\sigma_1, \sigma_2, \sigma_3)$ and volume form vol , with the same orientation as σ_i^2 . For each real-analytic $F \in \Omega_{\mathbb{Z}}^2(N, \mathfrak{t})$ satisfying (4.5) there is a unique maximal connected closed $\text{Spin}(7)$ -structure (M, Φ) with T^3 -symmetry and multi-moment map ν , such that $(N, \sigma_i, \text{vol}, F)$ is the reduction of M at level 0. □*

The idea of the proof is to construct a G_2 -geometry on the T^3 -bundle $P \rightarrow N$ determined by F . One then uses a modified variant of the Hitchin flow to extend this to a closed $\text{Spin}(7)$ -geometry on a maximal open subset of $P \times \mathbb{R}$. Explicit examples of this construction are given in [27]. Note that even the reduction of \mathbb{R}^8 by the maximal torus of $\text{Spin}(7)$ gives non-trivial tri-symplectic geometries on \mathbb{R}^4 that are not hyperKähler.

4.6 G_2 -manifolds

A G_2 -structure on a seven-manifold M is a choice of three-form $\phi \in \Omega^3(M)$ which on each tangent space $T_x M$ is linearly equivalent to the form ϕ_0 of (2.2) on \mathbb{R}^7 . As in the $\text{Spin}(7)$ case, the form ϕ determines a volume form and metric on M via the relation

$$(X \lrcorner \phi) \wedge (Y \lrcorner \phi) \wedge \phi = 6g(X, Y) \text{ vol}. \quad (4.6)$$

To interpret this formula, note that g is required to be positive definite, which determines the sign of vol and the conformal class of g . Now the fact that vol is required to be of unit length with respect to g , fixes the conformal factor: scaling g by $f^2 > 0$ scales vol by f^7 and the right-hand side of (4.6) scales by f^9 (cf. [8, 26]).

The form ϕ_0 is stable in the sense of Definition 2.3 and so is its Hodge dual, the four-form $*_7 \phi_0$. However, the stabiliser of $*_7 \phi_0$ is $G_2 \times \mathbb{Z}_2$ rather than just G_2 , as may be seen by noting that $-\phi_0$ defines the opposite orientation on \mathbb{R}^7 .

There are a number of classes of G_2 -structures with closed four-form that have particular interest for us.

Definition 4.9. Let (M, ϕ) be a G_2 -structure. The structure is

- (i) cosymplectic if $d*_7\phi = 0$,
- (ii) parallel if $d\phi = 0$ and $d*_7\phi = 0$,
- (iii) nearly parallel if $d\phi = 4*_7\phi$.

Any oriented hypersurface M of a closed $Spin(7)$ -manifold Y carries a cosymplectic G_2 -structure. Indeed write $i: M \rightarrow Y$ for the inclusion and let \mathbf{N} be a unit normal to the hypersurface, then formula (4.4) gives

$$\Phi|_Y = \mathbf{N}^\flat \wedge \phi + *_7\phi$$

with $\phi = i^*(\mathbf{N} \lrcorner \Phi)$ defining the G_2 -structure. This gives $d*_7\phi = di^*\Phi = i^*d\Phi = 0$, showing that the G_2 -structure is cosymplectic.

Parallel G_2 -structures are so-called because the equations $d\phi = 0$ and $d*_7\phi = 0$ imply that ϕ is parallel for the Levi-Civita connection of g , as shown by Fernández and Gray [15]. This implies that g is Ricci-flat, cf. Bonan [7], and that the holonomy is contained in G_2 .

Given a parallel G_2 -structure (M, ϕ) we may consider $Y = S^1 \times M$ with the four-form $\Phi = \theta \wedge \phi + *_7\phi$, and see that Φ gives a closed $Spin(7)$ -structure. If T^2 acts on M preserving ϕ , then it also preserves $*_7\phi$. Now much as in Section 4.2, a multi-moment map ν for T^2 on (M, ϕ) gives a multi-moment map $\tilde{\nu}$ for $T^3 = S^1 \times T^2$ acting on $(Y = S^1 \times M, \Phi)$.

The theory of reductions of T^3 -invariant closed $Spin(7)$ -structures, discussed in the previous section, may now be applied to T^2 -invariant parallel G_2 -structures. This gives that $N_t = M //_{\nu, t} T^2$ carries *coherent* triple of symplectic structures $\sigma_0, \sigma_1, \sigma_2$, meaning that they are weakly coherent and $\sigma_0 \wedge \sigma_i = 0$ for $i = 1, 2$. Also the curvature associated to the S^1 -factor of Y is trivial, so $\nu^{-1}(t) \rightarrow N_t$ is a T^2 -bundle with curvature form $F \in \Omega_{\mathbb{Z}}^2(N_t, \mathfrak{t})$, whose self-dual part has no σ_0 -component, and which satisfies the condition (4.5). A direct description of this situation is given in [30].

The third class of Definition 4.9 is the manifolds of nearly parallel G_2 -structures. These also go under the name of weak holonomy G_2 in the terminology of Gray [22], who showed that the associated metric g is Einstein with positive scalar curvature. The simplest example of such a G_2 -structure is the unit sphere $S^7 \subset \mathbb{R}^8$, with the geometry induced from the flat $Spin(7)$ -structure. The discussion above shows that $S^7 = Spin(7)/G_2$ so the maximal torus T^3 acts preserving this geometry. The $Spin(7)$ -geometry on \mathbb{R}^8 can be recovered as a warped product.

Indeed, suppose (M, ϕ) is a nearly parallel G_2 -structure. Put $C(M) = \mathbb{R}_{>0} \times M$ with the form

$$\Phi_C = s^3 ds \wedge \phi + s^4 *_7\phi,$$

where s is the parameter on $\mathbb{R}_{>0}$. At level s , the induced structure on $\{s\} \times M$ is given by $s^3\phi$, which is a G_2 -structure with metric s^2g . This shows that at each point Φ_C is linearly isomorphic to the $Spin(7)$ -form Φ_0 of (2.3); so Φ_C defines a $Spin(7)$ -structure on $C(M)$. As $d\phi = 4*_7\phi$, the four-form $*_7\phi$ is closed and we find that

$$d\Phi_C = -s^3 ds \wedge d\phi + 4s^3 ds \wedge *_7\phi + s^4 d*_7\phi = 0.$$

Thus $(C(M), \Phi_C)$ is a closed $Spin(7)$ -structure. Its metric is the warped product $g_C = ds^2 + s^2g$. This construction was used by Bär [3] to relate the Killing spinors of (M, g) to parallel spinors of $(C(M), g_C)$. For more on the Killing spinor approach to these G_2 -structures see [4, 18].

Now any symmetry of (M, ϕ) induces a symmetry of $(C(M), \Phi_C)$ that preserves s . Thus a nearly parallel G_2 -structure with T^3 -symmetry corresponds to a certain class of T^3 -invariant closed $Spin(7)$ -structures. As in the previous section, let U_1, U_2, U_3 be vector fields generating the T^3 -action. For $u = U_1 \wedge U_2 \wedge U_3$, the nearly parallel condition and the extended Cartan formula (Lemma 3.4) give

$$u \lrcorner * \phi = \frac{1}{4} u \lrcorner d\phi = -\frac{1}{4} d(\phi(u)).$$

Thus $\nu = -\frac{1}{4}\phi(u)$ is a multi-moment map for the T^3 -action on $(M, * \phi)$. Also we have that

$$\begin{aligned} u \lrcorner \Phi_C &= -s^3 ds \phi(u) + s^4 u \lrcorner * \phi \\ &= -\frac{1}{4} d(s^4 \phi(u)). \end{aligned}$$

So $\nu_C = s^4 \nu$ is a multi-moment map for the action on the cone $(C(M), \Phi_C)$.

Now for general t , the level set $\nu_C^{-1}(t)$ consists of the (s, m) such that $s^4 \nu(m) = t$. Since $s \in \mathbb{R}_{>0}$, this relation simplifies when $t = 0$ and we have that

$$\nu_C^{-1}(0) = C(\nu^{-1}(0)).$$

We will therefore only consider the reductions at level 0.

The reduction $N_C = \nu_C^{-1}(0)/T^3$ of the cone carries a weakly coherent triple of symplectic forms $\sigma_1, \sigma_2, \sigma_3$ with for example $\pi_C^* \sigma_3 = i_C^*(U_1 \lrcorner U_2 \lrcorner \Phi_C)$, where $i_C: C(M)_0 = \nu_C^{-1}(0) \hookrightarrow C(M)$ is the inclusion and $\pi_C: C(M)_0 \rightarrow N_C$ is the projection. However,

$$\begin{aligned} \pi_C^* \sigma_3 &= i_C^*(U_1 \lrcorner U_2 \lrcorner \Phi_C) \\ &= i_C^*(s^3 ds \wedge (U_1 \lrcorner U_2 \lrcorner \phi) + \frac{1}{4} U_1 \lrcorner U_2 \lrcorner d\phi) \\ &= i_C^* d(\frac{1}{4} s^4 \pi^* \eta_3), \end{aligned}$$

where $\pi^* \eta_3 = i^*(U_1 \lrcorner U_2 \lrcorner \phi)$, with $i: M_0 = \nu^{-1}(0) \hookrightarrow M$ the inclusion and $\pi: M_0 \rightarrow N = M //_{\nu} T^3$ the projection. The fact that σ_3 is non-degenerate on $N_C = C(N)$ then corresponds to η_3 being a contact structure on N .

In this way, we see that the reduction $N = M //_{\nu} T^3$ carries a pointwise linearly independent triple (η_1, η_2, η_3) of contact structures. The condition that the σ_i are weakly coherent corresponds to the requirements that the forms $\eta_i \wedge d\eta_i$ define the same orientation and that the symmetric matrix with entries corresponding to $\eta_i \wedge d\eta_j + \eta_j \wedge d\eta_i$ is positive definite. Thus one example is provided by taking $N = S^3 = SU(2)$, with $d\eta_1 = -2\eta_2 \wedge \eta_3$ etc. Conversely the standard basis of one forms for $SL(2, \mathbb{R})$ does not give a weakly coherent triple.

The remaining data for the $Spin(7)$ -geometry are the curvature forms F_1, F_2, F_3 satisfying (4.5). These forms are invariant under the action of the Euler vector field on N_C , so they have the form $F = d \log s \wedge a_i + b_i$, with $a_i \in \Omega^1(N)$ closed and

$b_i \in \Omega_{\mathbb{Z}}^2(N)$. Equation (4.5) becomes

$$a_i \wedge d\eta_j + 4b_i \wedge \eta_j = a_j \wedge d\eta_i + 4b_j \wedge \eta_i, \quad \text{for all } i, j.$$

Thus N carries three-contact forms and the closed forms a_1, \dots, b_3 .

4.7 PSU(3)-structures

A $PSU(3)$ -structure on an oriented eight-manifold M is a three-form $\rho \in \Omega^3(M)$ pointwise modelled on ρ_0 of equation (2.4). These geometries were studied by Witt [40, 39]. Such a form ρ determines a metric g and the five-form $*\rho$. A $PSU(3)$ -structure is said to be *harmonic* if $d\rho = 0$ and $d*\rho = 0$.

The case when a harmonic $PSU(3)$ -structure admits a free two-torus symmetry with multi-moment map ν was discussed in [28] reinterpreting some results of Witt. In this case one has four two-forms given by

$$\begin{aligned} \omega_0 &= -(d\nu)^\sharp \lrcorner U_1 \lrcorner U_2 \lrcorner *\rho, & \omega_1 &= U_1 \lrcorner \rho, \\ \omega_2 &= U_2 \lrcorner \rho, & \omega_3 &= U_1 \lrcorner U_2 \lrcorner \alpha^\sharp \lrcorner *\rho, \end{aligned}$$

where α is contraction of ρ by ω_0 . At a regular value t of ν , these forms induce two-forms σ_i on the reduction $N = M //_{\nu, t} T^2$ and α induces a one-form a . Together $(a, \sigma_1, \sigma_2, \sigma_3)$ give N the structure of an $SU(2)$ -manifold, see Conti and Salamon [13], and a conformal scaling of $a \wedge \sigma_0$ can be 2-plectic.

4.8 Homogeneous k-plectic manifolds

Suppose (M, α) is a closed geometry of degree $r = k+1$ with a group G of symmetries that acts transitively on M . Then the equivariant map $\Psi: M \rightarrow Z^r(\mathfrak{g})$ given by (3.10) has image a single G -orbit in $Z^r(\mathfrak{g})$. Conversely, we may use equation (3.10) to define closed geometries that map to a given orbit $G \cdot \Psi \subset Z^r(\mathfrak{g})$, as follows. Let K_Ψ be the connected subgroup of G with Lie algebra $\ker \Psi = \{X \in \mathfrak{g} : X \lrcorner \Psi = 0\}$. Then for each closed subgroup H of G containing K_Ψ , equation (3.10) gives a well-defined closed r -form α on $M = G/H$.

Now suppose that $\Psi = d_{\mathcal{P}}\beta$ for some $\beta \in \mathcal{P}_{\mathfrak{g}}^*$. If the map $d_{\mathcal{P}}$ is injective, then the orbits $G \cdot \Psi$ and $G \cdot \beta$ are identified and the map $\Psi: M \rightarrow Z^r(\mathfrak{g})$ may now be interpreted as a map $\nu: M \rightarrow \mathcal{P}_{\mathfrak{g}}^*$. Injectivity of $d_{\mathcal{P}}$ is equivalent to the condition $b_{r-1}(\mathfrak{g}) = 0$ and the proof of Theorem 3.14 shows that ν is a multi-moment map for the action of G .

Theorem 4.10. *Suppose G is a connected Lie group with $b_k(\mathfrak{g}) = 0$. Let $\mathcal{O} = G \cdot \beta \subset \mathcal{P}_{\mathfrak{g}}^*$ be an orbit of G acting on the dual of the k th Lie kernel. Then there are homogeneous closed geometries $(G/H, \alpha)$, with $\alpha \in \Omega^{k+1}(G/H)$ corresponding to $\Psi = d_{\mathcal{P}}\beta$, such that \mathcal{O} is the image of G/H under the (unique) multi-moment map ν .*

The closed geometry may be realised on the orbit \mathcal{O} itself if and only if

$$\text{stab}_{\mathfrak{g}} \beta = \ker(d_{\mathcal{P}}\beta). \quad (4.7)$$

In this situation, the orbit is k -plectic and ν is simply the inclusion $\mathcal{O} \hookrightarrow \mathcal{P}_{\mathfrak{g}}^$.*

Proof. It only remains to prove the assertions of the last paragraph of the theorem. We have $\mathcal{O} = G/K$ with $K = \text{stab}_G \beta$, a closed subgroup of G . Now equation (4.7), shows that K has Lie algebra $\ker(d_{\mathcal{P}}\beta)$, so the component of the identity K^0 of K is $K^0 = K_{\Psi}$ for $\Psi = d_{\mathcal{P}}\beta$. In particular, Ψ vanishes on elements of \mathfrak{k} and induces a well-defined form on $T_{\beta}\mathcal{O} = \mathfrak{g}/\mathfrak{k}$. The result now follows. \square

Remark 4.11. In the case when $r = 2$, condition (4.7) is automatic and we get the result of Kirillov-Kostant-Souriau that each orbit of \mathfrak{g}^* is symplectic. \triangle

Example 4.12. Suppose G is a $(k, k+1)$ -trivial Lie group. Then, taking $H = \{e\}$, we see that every $\Psi \in Z^{k+1}(\mathfrak{g})$ gives rise to a closed geometry on G with multi-moment map whose image is diffeomorphic to the G -orbit of Ψ . \diamond

5 Cohomology of Lie algebras

Recall from Section 3.2 that for a closed geometry of degree r , multi-moment maps exist and are unique for any symmetry group G which is $(r-1, r)$ -trivial. This condition means that the cohomology groups $H^k(\mathfrak{g})$ of \mathfrak{g} are trivial in degrees $k = r-1$ and r . While this is a concise statement, it is not clear which, if any algebras, satisfy these conditions. In this section, we will discuss some techniques to gain more information and show that there are in fact many such algebras.

Note first that the definitions (3.8) and (3.9) give $H^1(\mathfrak{g}) = \ker d \leq \mathfrak{g}^*$. As d is dual to the Lie bracket $L = [\cdot, \cdot]: \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$, the vanishing of $b_1(\mathfrak{g}) = \dim H^1(\mathfrak{g})$ is equivalent to the surjectivity of L . This says that $b_1(\mathfrak{g}) = 0$ if and only if \mathfrak{g} is equal to its derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Indeed $b_1(\mathfrak{g})$ is exactly the codimension of \mathfrak{g}' in \mathfrak{g} . Lie algebras with $\mathfrak{g} = \mathfrak{g}'$ are called *perfect*. Any semi-simple Lie algebra is perfect, but other examples may be constructed as the semi-direct product $\mathfrak{h} \ltimes V$ of a semi-simple algebra \mathfrak{h} with a faithful representation V . For example, the group of isometries of \mathbb{R}^n with the standard flat metric is perfect for each $n \geq 3$.

Interpretation of the vanishing of higher Betti numbers is more complicated. We gave some of the vanishing properties satisfied by compact simple Lie groups in Section 3.2. In particular, these are $(1, 2)$ -trivial, leading to the usual existence and uniqueness results for symplectic moment maps. Furthermore any $(1, 2)$ -trivial algebra is semi-simple. The structure of the $(2, 3)$ -trivial groups was described in [30] and classification results in small dimensions given in the same paper and in [29]. In particular, we found that $(2, 3)$ -trivial Lie algebras are always *solvable*, meaning that $\mathfrak{g}^m = \{0\}$ for some $m > 0$, where $\mathfrak{g}^m = [\mathfrak{g}^{m-1}, \mathfrak{g}^{m-1}]$ is the m th derived algebra of \mathfrak{g} .

The general structure theory of Lie algebras says that any \mathfrak{g} has a maximal solvable ideal \mathfrak{r} , the *solvable radical*, and that the quotient $\mathfrak{g}/\mathfrak{r}$ is semi-simple. In addition, for any solvable Lie algebra \mathfrak{h} the derived algebra $\mathfrak{k} = \mathfrak{h}'$ is always *nilpotent*, meaning that $\mathfrak{k}_m = \{0\}$ for some $m > 0$, where $\mathfrak{k}_m = [\mathfrak{k}, \mathfrak{k}_{m-1}]$ and $\mathfrak{k}_1 = \mathfrak{k}'$. In [30], a result of Hochschild and Serre [24] relating the cohomologies of \mathfrak{g} , \mathfrak{r} and $\mathfrak{g}/\mathfrak{r}$ was used in [30] to prove:

Lemma 5.1. *Any non-zero Lie algebra with $b_3(\mathfrak{g}) = 0$ is solvable and thus has $b_1(\mathfrak{g}) \neq 0$. Such a \mathfrak{g} is not nilpotent unless $\mathfrak{g} = \mathbb{R}$ or $\mathfrak{g} = \mathbb{R}^2$.* \square

The essential point is that the semi-simple algebra $\mathfrak{g}/\mathfrak{r}$ has b_3 non-zero, and this feeds through to $b_3(\mathfrak{g})$ if $\mathfrak{r} \neq \mathfrak{g}$. For a solvable algebra \mathfrak{g}' is strictly smaller than \mathfrak{g} , so $b_1(\mathfrak{g}) \neq 0$.

The (2,3)-trivial algebras were then found to be exactly those solvable \mathfrak{g} such that the derived algebra $\mathfrak{k} = \mathfrak{g}'$ has codimension 1 in \mathfrak{g} and such that $\mathfrak{g}/\mathfrak{k}$ acts invertibly on the cohomology groups $H^i(\mathfrak{k})$, for $i = 1, 2, 3$.

5.1 (3,4)-trivial Lie algebras

Let us work towards a description of (3,4)-trivial Lie algebras. As a first result we consider direct sums of algebras.

Proposition 5.2. *A non-trivial direct sum $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2$ has $b_3(\mathfrak{g}) = 0$ if and only if each summand \mathfrak{h}_i is (2,3)-trivial. Consequently, \mathfrak{g} is product of at most two summands.*

The direct sum $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2$ is (3,4)-trivial if and only if each summand is (2,3,4)-trivial.

Proof. This is almost direct from the Künneth formula, which gives the following sum of positive terms:

$$b_3(\mathfrak{g}) = b_3(\mathfrak{h}_1) + b_3(\mathfrak{h}_2) + b_2(\mathfrak{h}_1)b_1(\mathfrak{h}_2) + b_1(\mathfrak{h}_1)b_2(\mathfrak{h}_2). \quad (5.1)$$

Thus $b_3(\mathfrak{g}) = 0$ immediately gives $b_3(\mathfrak{h}_i) = 0$. However, by Lemma 5.1 we know that $b_1(\mathfrak{h}_i) \neq 0$, so the vanishing of $b_3(\mathfrak{g})$ also gives $b_2(\mathfrak{h}_i) = 0$.

Now a similar argument, given in [30], shows that (2,3)-trivial algebras are not direct sums of smaller ideals. Thus the summands \mathfrak{h}_i are not direct sums and \mathfrak{g} has at most two summands.

In the second case, the Künneth formula gives

$$b_4(\mathfrak{g}) = b_4(\mathfrak{h}_1) + b_4(\mathfrak{h}_2) + b_2(\mathfrak{h}_1)b_2(\mathfrak{h}_2) + b_3(\mathfrak{h}_1)b_1(\mathfrak{h}_2) + b_1(\mathfrak{h}_1)b_3(\mathfrak{h}_2), \quad (5.2)$$

so the extra condition $b_4(\mathfrak{g}) = 0$ forces $b_4(\mathfrak{h}_i) = 0$ too.

The converse statements are immediate from (5.1) and (5.2). \square

To go further and study other cases we need to use stronger techniques. Our main tool will be the Hochschild-Serre spectral sequence of a Lie algebra \mathfrak{g} with respect to an ideal \mathfrak{k} . We will consider the case when the \mathfrak{k} contains the derived algebra \mathfrak{g}' . Then the quotient algebra $\mathfrak{a} = \mathfrak{g}/\mathfrak{k}$ is Abelian of rank at most $b_1(\mathfrak{g})$. This algebra acts on the cohomology of \mathfrak{k} via

$$A \cdot [\alpha] = [A \lrcorner d\alpha], \quad \text{for } A \in \mathfrak{a} = \mathfrak{g}/\mathfrak{k}, [\alpha] \in H^q(\mathfrak{k}). \quad (5.3)$$

Indeed the above formula defines an action of \mathfrak{g} for which \mathfrak{k} acts trivially. Note that here d is the differential in \mathfrak{g} ; we will write d_0 for the differential in \mathfrak{k} , so $\alpha \in \ker d_0$ in (5.3). Moreover, in our case with \mathfrak{a} Abelian, this action induces the coboundary map d_1 on the cochains

$$C^p(\mathfrak{a}, H^q(\mathfrak{k})) = \Lambda^p \mathfrak{a}^* \otimes H^q(\mathfrak{k})$$

via

$$(d_1 f)(\mathbf{a}) = \sum_{i=1}^{p+1} f(\mathbf{a}_{\wedge i}),$$

for $\mathbf{a} \in \Lambda^{p+1} \mathfrak{a}$.

The Hochschild-Serre spectral sequence [24] has E_2 -page given by the cohomology of the operator d_1 above:

$$E_2^{p,q} = H^p(\mathfrak{a}, H^q(\mathfrak{k})).$$

Note that we have

$$H^0(\mathfrak{a}, H^q(\mathfrak{k})) = \{ b \in H^q(\mathfrak{k}) : \mathbf{A} \cdot b = 0 \text{ for all } \mathbf{A} \in \mathfrak{k} \} = H^q(\mathfrak{k})^{\mathfrak{g}}$$

the fixed-point set of the action of G on $H^q(\mathfrak{k})$. Also note that $E_2^{p,q} = 0$ for $p > \dim \mathfrak{a}$.

Given the E_2 -page of the spectral sequence, the general theory defines maps $d_2: E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ induced by the exterior derivative d in \mathfrak{g} and sets $E_3^{p,q}$ to be the corresponding cohomology group. More generally, $d_r: E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$ and so the spectral sequence stabilises at level $r = \dim \mathfrak{a}$. So $E_\infty^{p,q} = E_{\dim \mathfrak{a}}^{p,q}$ and one then has

$$H^k(\mathfrak{g}) \cong \bigoplus_{p+q=k} E_\infty^{p,q}.$$

Note that if we choose a linear splitting of the exact sequence

$$0 \rightarrow \mathfrak{k} \rightarrow \mathfrak{g} \rightarrow \mathfrak{a} \rightarrow 0$$

then the image of $W = \mathfrak{k}^*$ in \mathfrak{g}^* has $dW \subset \Lambda^2 W + \mathfrak{a}^* \wedge W + \Lambda^2 \mathfrak{a}^*$. In particular, the differential of \mathfrak{g} on $E_0^{p,q}$ has components in $E_0^{p,q+1}$, $E_0^{p+1,q}$ and $E_0^{p+2,q-1}$. Thus this is more general than the spectral sequence of a bicomplex.

We are now ready to state our first characterisation result, the proof of which will be given after some discussion of consequences.

Theorem 5.3. *A Lie algebra \mathfrak{g} is (3,4)-trivial if and only if \mathfrak{g} is solvable and for any codimension one ideal \mathfrak{k} containing \mathfrak{g}' one has $H^i(\mathfrak{k})^{\mathfrak{g}} = 0$ for $i = 2, 3, 4$. Here $H^i(\mathfrak{k})^{\mathfrak{g}}$ is the part of the cohomology of \mathfrak{k} that is invariant under the action of \mathfrak{g} .*

This result already gives a number of examples of (3,4)-trivial algebras.

Example 5.4. Let \mathfrak{k} be an Abelian algebra \mathbb{R}^m of dimension m . The differential in \mathfrak{k} is zero and $H^k(\mathfrak{k}) = \Lambda^k \mathfrak{k}^*$ for each k . Let T be a diagonalisable operator acting by $T(\mathbf{K}_i) = \lambda_i \mathbf{K}_i$, where $\mathbf{K}_1, \dots, \mathbf{K}_m$ is a basis of eigenvectors. Write $\kappa_1, \dots, \kappa_m$ for the dual basis of \mathfrak{k}^* . Then T acts on $H^k(\mathfrak{k})$ with eigenvalue $\lambda_{i_1} + \dots + \lambda_{i_k}$ on $\kappa_{i_1} \wedge \dots \wedge \kappa_{i_k}$. We thus have that $H^i(\mathfrak{k})^T = 0$ for $i = 2, 3, 4$ if and only if $\lambda_i \neq -\lambda_j, -\lambda_j - \lambda_k, -\lambda_j - \lambda_k - \lambda_\ell$ whenever i, j, k, ℓ are distinct. This then gives a (3,4)-trivial algebra \mathfrak{g} defined by

$$\mathfrak{g} = \mathbb{R}\mathbf{A} + \mathfrak{k},$$

where $[\mathbf{A}, \mathbf{K}_i] = T\mathbf{K}_i = \lambda_i \mathbf{K}_i$. Note that if some λ_i is 0, then this occurs for only one index i and \mathfrak{g} splits as a product $\mathbb{R} + \mathfrak{h}$, with T acting invertibly on $\mathfrak{h}' \cong \mathbb{R}^{m-1}$.

Some concrete (3, 4)-trivial non-product examples are given as follows. First consider $\mathfrak{k} = \mathbb{R}^2$. Then $\mathfrak{g} = (0, 12, \mu.13)$, $\mu \neq 0, -1$, is (3, 4)-trivial. Here the notation means that \mathfrak{g}^* has a basis e_1, e_2, e_3 with $de_1 = 0$, $de_2 = e_1 \wedge e_2$ and $de_3 = \mu e_1 \wedge e_3$. For a higher-dimensional case with $\mathfrak{k} = \mathbb{R}^4$, another example is provided by the algebras $\mathfrak{g} = (0, 12, 13, 14, \mu.15)$ with $\mu \neq 0, -1, -2, -3$. For general m , by taking λ_i strictly positive for each i , we obtained (3, 4)-trivial algebras in all dimensions. \diamond

Example 5.5. Suppose \mathfrak{k} is *positively graded*, meaning that as a vector space $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \cdots \oplus \mathfrak{k}_r$ with $[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j}$ for all i, j . Such a \mathfrak{k} is necessarily nilpotent. Choosing a linear operator T on \mathfrak{k} , we may obtain a Lie algebra $\mathfrak{g} = \mathbb{R}A + \mathfrak{k}$ with $[A, K] = T(K)$ if only if $T[K_1, K_2] = [T(K_1), K_2] + [K_1, T(K_2)]$ for all $K_i \in \mathfrak{k}$. In the positively graded situation we may thus take $T(K) = jK$, for $K \in \mathfrak{k}_j$.

Every nilpotent algebra of dimension at most 6 admits a positive grading (see [29] for concrete gradings), so each of these nilpotent Lie algebras may be realised as the derived algebra of a (3, 4)-trivial algebra. As a concrete example, the seven-dimensional algebra

$$(0, 12, 3.13, 4.14 + 23, 5.15 + 24, 6.16 + 25, 7.17 + 34 + 26)$$

obtained from $(0, 0, 12, 13, 14, 23 + 15)$ via the grading given by the weights $(1, 3, 4, 5, 6, 7)$ is (3, 4)-trivial. \diamond

Theorem 5.3 has the following consequence, which completes the characterisation Proposition 5.2 of (3, 4)-trivial algebras that are direct sums.

Corollary 5.6. *Let \mathfrak{g} be a Lie algebra. Write $\mathfrak{k} = \mathfrak{g}'$ for the derived algebra and $\mathfrak{a} = \mathfrak{g} / \mathfrak{k}$. Then \mathfrak{g} is (2, 3, 4)-trivial if and only if \mathfrak{g} is solvable, \mathfrak{a} has dimension 1 and $H^i(\mathfrak{k})^{\mathfrak{g}} = \{0\}$, for $i = 1, 2, 3, 4$.*

Proof. By [30] any (2, 3)-trivial algebra has $\text{codim } \mathfrak{g}' = 1$ and $H^i(\mathfrak{g}')^{\mathfrak{g}} = \{0\}$, for $i = 1, 2, 3$. Combining this with Theorem 5.3 applied to $\mathfrak{k} = \mathfrak{g}'$ gives the result. \square

Proof (of Theorem 5.3). By Lemma 5.1 we know \mathfrak{g} is solvable, so \mathfrak{g}' has codimension at least one. Now with respect to a codimension one ideal \mathfrak{k} , the E_1 -page of the spectral sequence has row q given by

$$\begin{array}{c|cccccc} q+1 & \cdots & & & & & \\ q & H^q(\mathfrak{k}) & \xrightarrow{d_1} & \mathfrak{a}^* \otimes H^q(\mathfrak{k}) & 0 & 0 & \cdots \\ q-1 & \cdots & & & & & \\ \hline & 0 & 1 & 2 & 3 & \cdots & \end{array}$$

For the E_2 -terms we thus have $E_2^{0,q} = \ker d_1 = H^q(\mathfrak{k})^{\mathfrak{g}}$ and $E_2^{1,q} = \text{coker } d_1$. But $\dim \mathfrak{a} = 1$, so d_1 is a map between vector spaces of the same dimension. This implies $\text{coker } d_1$ and $\ker d_1$ have the same dimension and so we may identify $E_2^{1,q}$ with $H^q(\mathfrak{k})^{\mathfrak{g}}$, non canonically.

The E_2 -page of the spectral sequence is thus

$$\begin{array}{c|cccccc}
\vdots & & \dots & & & & \\
4 & H^4(\mathfrak{k})^{\mathfrak{g}} & H^4(\mathfrak{k})^{\mathfrak{g}} & 0 & 0 & \dots & \\
3 & H^3(\mathfrak{k})^{\mathfrak{g}} & H^3(\mathfrak{k})^{\mathfrak{g}} & 0 & 0 & \dots & \\
2 & H^2(\mathfrak{k})^{\mathfrak{g}} & H^2(\mathfrak{k})^{\mathfrak{g}} & 0 & 0 & \dots & \\
1 & H^1(\mathfrak{k})^{\mathfrak{g}} & H^1(\mathfrak{k})^{\mathfrak{g}} & 0 & 0 & \dots & \\
0 & \mathbb{R} & \mathbb{R} & 0 & 0 & \dots & \\
\hline
& & 0 & 1 & 2 & 3 & \dots
\end{array}$$

The spectral sequence degenerates at the E_2 -term and we conclude that

$$H^3(\mathfrak{g}) \cong H^3(\mathfrak{k})^{\mathfrak{g}} + H^2(\mathfrak{k})^{\mathfrak{g}}, \quad H^4(\mathfrak{g}) \cong H^4(\mathfrak{k})^{\mathfrak{g}} + H^3(\mathfrak{k})^{\mathfrak{g}},$$

from which the result follows. \square

In the case when \mathfrak{n} is a nilpotent Lie algebra, one may use the above spectral sequence to prove Dixmier's result [14] that $b_k(\mathfrak{n}) \geq 2$ for each $0 < k < \dim \mathfrak{n}$, since in this situation $\mathfrak{g} = \mathfrak{n}$ acts nilpotently on $H^q(\mathfrak{k})$ and so $H^q(\mathfrak{k})^{\mathfrak{g}}$ is non-zero if $H^q(\mathfrak{k})$ is non-zero. A refinement of the above argument was also used by Cairns and Jessup [10] to improve Dixmier's bounds.

Proposition 5.7. *A Lie algebra \mathfrak{g} with derived algebra \mathfrak{g}' of codimension at least two is $(3, 4)$ -trivial if and only if for each ideal \mathfrak{k} of \mathfrak{g} containing \mathfrak{g}' one has $H^i(\mathfrak{k})^{\mathfrak{g}} = 0$ for $i = 1, 2, 3, 4$.*

The reason for giving this result in addition to Theorem 5.3 is that we will see later that we can often assume that $\text{codim } \mathfrak{g}' \leq 2$ and therefore take $\mathfrak{k} = \mathfrak{g}'$ to be nilpotent.

Example 5.8. Suppose \mathfrak{k} is Abelian as in Example 5.4. We may now consider two commuting diagonalisable operators T_1 and T_2 acting on \mathfrak{k} . Then \mathfrak{k} splits as a direct sum of common eigenspaces of T_1 and T_2 . We wish to consider when the Lie algebra given by

$$\begin{aligned}
\mathfrak{g} &= \langle \mathbf{A}_1, \mathbf{A}_2 \rangle + \mathfrak{k}, \\
[s\mathbf{A}_1 + t\mathbf{A}_2, \mathbf{K}] &= (sT_1 + tT_2)(\mathbf{K}), \quad \text{for } s, t \in \mathbb{R}, \mathbf{K} \in \mathfrak{k},
\end{aligned} \tag{5.4}$$

is $(3, 4)$ -trivial.

The cohomology conditions of Proposition 5.7 are satisfied if T_1 and T_2 have no common 0-eigenspace on $\Lambda^i \mathfrak{k}^*$ for $i = 1, 2, 3, 4$. This is a vanishing condition on finitely many linear combinations of the eigenvalues of T_1 and of T_2 . When it is satisfied we may thus find a generic linear combination $T = sT_1 + tT_2$ such that the action of T on these $\Lambda^i \mathfrak{k}^*$ has trivial kernel. Taking $T_1 = T$, we are now free to have T_2 any linear transformation of \mathfrak{k} that commutes with T_1 , and need not assume that T_2 is diagonalisable.

Thus for example when $\mathfrak{k} \cong \mathbb{R}^2$, we see that the algebra

$$(0, 0, 13 + 24, 14)$$

is $(3, 4)$ -trivial. \diamond

Example 5.9. If \mathfrak{k} is positively graded, as in Example 5.5, we may take $T_1 = T$ as in that example and let A_2 be any derivation T_2 of \mathfrak{k} commuting with T_1 . Then (5.4) defines a $(3, 4)$ -trivial algebra. A simple example is obtained by taking \mathfrak{k} to be the Heisenberg algebra $(0, 0, 12)$. This has a positive grading with weights $(1, 1, 2)$. A second derivation T_2 acts on \mathfrak{k}^* by $e_1 \mapsto e_2$, $e_2 \mapsto -e_1$, $e_3 \mapsto 0$ and this commutes with T_1 . We thus conclude that

$$(0, 0, 13 + 24, 14 - 23, 2.15)$$

is $(3, 4)$ -trivial. Another choice for T_2 is the nilpotent transformation $e_1 \mapsto e_2$, $e_2 \mapsto 0$, $e_3 \mapsto 0$ of \mathfrak{k}^* , which gives the $(3, 4)$ -trivial algebra

$$(0, 0, 13 + 24, 14, 2.15). \quad \diamond$$

Proof (of Proposition 5.7). This time $\mathfrak{a}^* \cong \mathbb{R}^2$ and $\Lambda^2 \mathfrak{a}^* \cong \mathbb{R}$, so the E_1 -page of the spectral sequence has row q isomorphic to

$$\begin{array}{c|cccccc} q+1 & \cdots & & & & & \\ q & H^q(\mathfrak{k}) & \xrightarrow{d_1} & 2H^q(\mathfrak{k}) & \xrightarrow{d_1} & H^q(\mathfrak{k}) & 0 & \cdots \\ q-1 & \cdots & & & & & & \\ \hline & 0 & & 1 & & 2 & 3 & \cdots \end{array}$$

Since the maps d_1 on the bottom row $q = 0$ are zero, the relevant part of the E_2 -page is

$$\begin{array}{c|cccccc} \vdots & \cdots & & & & & \\ 4 & H^4(\mathfrak{k})^{\mathfrak{g}} & \xrightarrow{E_2^{1,4}} & E_2^{2,4} & 0 & \cdots \\ 3 & H^3(\mathfrak{k})^{\mathfrak{g}} & \xrightarrow{E_2^{1,3}} & E_2^{2,3} & 0 & \cdots \\ 2 & H^2(\mathfrak{k})^{\mathfrak{g}} & \xrightarrow{E_2^{1,2}} & E_2^{2,2} & 0 & \cdots \\ 1 & H^1(\mathfrak{k})^{\mathfrak{g}} & \xrightarrow{E_2^{1,1}} & E_2^{2,1} & 0 & \cdots \\ 0 & \mathbb{R} & \mathbb{R}^2 & \mathbb{R} & 0 & \cdots \\ \hline & 0 & 1 & 2 & 3 & \cdots \end{array}$$

The maps d_2 from and to the middle column $E_2^{1,p}$ are zero, and the spectral sequence degenerates at the E_3 -level, so

$$\begin{aligned} H^3(\mathfrak{g}) &\cong E_3^{0,3} + E_2^{1,2} + E_3^{2,1}, \\ H^4(\mathfrak{g}) &\cong E_3^{0,4} + E_2^{1,3} + E_3^{2,2}. \end{aligned}$$

Thus $b_3(\mathfrak{g}) = 0 = b_4(\mathfrak{g})$ implies that $E_2^{1,2} = 0 = E_2^{1,3}$. This says that the middle cohomologies of the E_1 -page on rows $q = 2$ and $q = 3$ are zero. However, for a sequence $V \xrightarrow{\alpha} 2V \xrightarrow{\beta} V$, vanishing of the middle cohomology implies that α is injective and β is surjective. Thus the cohomologies at each end of these rows are also zero. In particular,

$$H^2(\mathfrak{k})^{\mathfrak{g}} = 0 \quad \text{and} \quad H^3(\mathfrak{k})^{\mathfrak{g}} = 0.$$

The E_2 -page is now

| | | | | | |
|----------|------------------------------------|----------------|--------------|---|---------|
| \vdots | \dots | | | | |
| 4 | $H^4(\mathfrak{k})^{\mathfrak{g}}$ | $E_2^{1,4}$ | $E_2^{2,4}$ | 0 | \dots |
| 3 | 0 | 0 | 0 | 0 | \dots |
| 2 | 0 | 0 | 0 | 0 | \dots |
| 1 | $H^1(\mathfrak{k})^{\mathfrak{g}}$ | $E_2^{1,1}$ | $E_2^{2,1}$ | 0 | \dots |
| 0 | \mathbb{R} | \mathbb{R}^2 | \mathbb{R} | 0 | \dots |
| | | 0 | 1 | 2 | 3 |
| | | | | | \dots |

all the d_2 maps marked in the first picture of the E_2 -page are zero and

$$H^3(\mathfrak{g}) \cong E_2^{2,1}, \quad H^4(\mathfrak{g}) \cong H^4(\mathfrak{k})^{\mathfrak{g}}.$$

Thus (3, 4)-trivial implies

$$H^4(\mathfrak{k})^{\mathfrak{g}} = 0.$$

To prove the necessity of the cohomology conditions it only remains to show that $E_2^{2,1} = 0$ if and only if $H^1(\mathfrak{k})^{\mathfrak{g}} = 0$. In fact to get sufficiency we will show that $H^q(\mathfrak{k})^{\mathfrak{g}} = 0$ is equivalent to $E_2^{2,q} = 0$.

Write $V = H^q(\mathfrak{k}) \otimes \mathbb{C} = H^q(\mathfrak{k})_{\mathbb{C}}$ and let \mathbf{A}, \mathbf{B} be a basis for \mathfrak{a} . The space $(E_2^{2,q})_{\mathbb{C}}$ is the cokernel of $d_1: V \times V \rightarrow V$ with $d_1(f_1, f_2) = Af_2 - Bf_1$, where A, B are the linear operators of the action of \mathbf{A} and \mathbf{B} on V . Thus $(E_2^{2,q})_{\mathbb{C}}$ is $V/(\text{im } A + \text{im } B)$, whereas $H^q(\mathfrak{k})_{\mathbb{C}}^{\mathfrak{g}}$ is $\ker A \cap \ker B$.

Now A and B commute, so each preserves the generalised eigenspaces of the other. Decompose V as a direct sum of the common generalised eigenspaces $V = \bigoplus E(\lambda)$, where $E(\lambda_1, \lambda_2) = E_A(\lambda_1) \cap E_B(\lambda_2) = \ker((A - \lambda_1)^n) \cap \ker((B - \lambda_2)^n)$, $n = \dim V$. We see that A or B is invertible on each space $E(\lambda)$ with $\lambda \neq (0, 0)$. Thus $\ker A \cap \ker B$ is a subspace of $E(0)$ and $V/(\text{im } A + \text{im } B)$ is a quotient space of $E(0)$.

On $E(0)$, the operators A and B are nilpotent and commute. They thus generate a nilpotent Lie algebra and by Engel's Theorem, there is a basis v_1, \dots, v_m for $E(0)$ such that A and B are upper triangular. In particular $\text{im } A + \text{im } B \leq \langle v_1, \dots, v_{m-1} \rangle$ and $v_1 \in \ker A \cap \ker B$. Thus $V/(\text{im } A + \text{im } B)$ or $\ker A \cap \ker B$ is zero, if and only if $E(0)$ is zero. This proves that $H^q(\mathfrak{k})^{\mathfrak{g}} = 0$ if and only if $E_2^{2,q} = 0$.

We thus have that the given cohomology conditions are necessary. To see that they are sufficient, note that the vanishing of $H^q(\mathfrak{k})^{\mathfrak{g}}$ implies that the sequence on the

q -row of the E_1 -page has the form $V \xrightarrow{\alpha} 2V \xrightarrow{\beta} V$ with α injective and β surjective. It follows that the middle cohomology is also zero and that $E_2^{p,q}$ is zero for all $1 \leq p \leq 4$. This implies that $b_3(\mathfrak{g}) = 0 = b_4(\mathfrak{g})$ as required. \square

Theorem 5.3 and Proposition 5.7 give a full structural description of the $(3, 4)$ -trivial algebras with \mathfrak{g}' of codimension at most two. The next result gives a couple of conditions under which this codimension assumption is guaranteed.

Proposition 5.10. *A Lie algebra \mathfrak{g} with $b_3(\mathfrak{g}) = 0$ has derived algebra $\mathfrak{k} = \mathfrak{g}'$ of codimension at most two if either*

- (i) \mathfrak{g} is split, so \mathfrak{g} is the semi-direct product of $\mathfrak{a} = \mathfrak{g}/\mathfrak{k}$ and \mathfrak{k} , or
- (ii) $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{k})^{\mathfrak{g}}$.

Proof. When \mathfrak{g} is split, we have $\mathfrak{g} = \mathfrak{a} + \mathfrak{k}$, $[\mathfrak{a}, \mathfrak{a}] = 0$, and so the differential d of \mathfrak{g} satisfies $d\mathfrak{a}^* = 0$ and $d\mathfrak{k}^* \subset \Lambda^2 \mathfrak{k}^* + \mathfrak{a}^* \wedge \mathfrak{k}^*$. In particular, no element of $\Lambda^k \mathfrak{a}^*$ is exact, and there is an injection $\Lambda^k \mathfrak{a}^* \rightarrow H^k(\mathfrak{g})$ for each k . Thus $b_3(\mathfrak{g}) = 0$ implies $\Lambda^3 \mathfrak{a}^* = 0$ and \mathfrak{a} has dimension at most two, as required.

For the second case of the Proposition, we consider the spectral sequence. Choose a basis A_1, \dots, A_r for \mathfrak{a} . The proof of Proposition 5.7 shows that $H^q(\mathfrak{k})^{\mathfrak{g}} = \{0\}$ is equivalent to common generalised eigenspace $E(0) = \bigcap_{i=1}^r \ker(A_i^n)$ of the induced operators A_i on $H^q(\mathfrak{k})$ being zero. This implies that some linear combination A of the A_i acts invertibly on $H^q(\mathfrak{k})$. We may thus choose our basis A_i so that A_1 acts invertibly on the given $H^q(\mathfrak{k})$.

Now consider the cohomology of

$$H^q(\mathfrak{k}) \xrightarrow{d_1} \mathfrak{a}^* \otimes H^q(\mathfrak{k}) \xrightarrow{d_1} \Lambda^2 \mathfrak{a}^* \otimes H^q(\mathfrak{k}).$$

The first map is given by $v \mapsto (A_1 v, \dots, A_r v)$, the second map by $(f_1, \dots, f_r) \mapsto (A_i f_j - A_j f_i)$. For $\mathbf{f} = (f_1, \dots, f_r) \in \ker d_1 \subset \mathfrak{a}^* \otimes H^q(\mathfrak{k})$, we may write $v = A_1^{-1} f_1$. Then $A_1 f_j - A_j f_1 = 0$, implies that $f_j = A_1^{-1} (A_j f_1) = A_j v$ and that $\mathbf{f} \in \text{im } d_1$. Thus the d_1 -cohomology vanishes at the second step.

Under the hypotheses of the Proposition, the E_2 -page of the spectral sequence for \mathfrak{g} with respect the derived algebra \mathfrak{k} is now

$$\begin{array}{cccccc}
 \vdots & & \dots & & & \\
 3 & & H^3(\mathfrak{k})^{\mathfrak{g}} & & E_2^{1,3} & & E_2^{2,3} & & E_2^{3,3} & & \dots \\
 2 & & 0 & \cdots & 0 & & E_2^{2,2} & & E_2^{3,2} & & \dots \\
 1 & & 0 & & 0 & & E_2^{2,1} & & E_2^{3,1} & & \dots \\
 0 & & \mathbb{R} & & \mathfrak{a}^* & & \Lambda^2 \mathfrak{a}^* & \xrightarrow{\quad} & \Lambda^3 \mathfrak{a}^* & & \dots \\
 & & 0 & & 1 & & 2 & & 3 & & \dots
 \end{array}$$

with the solid arrow to $\Lambda^3 \mathfrak{a}^*$ representing d_2 and the dashed arrow indicating d_3 . This gives that $\Lambda^3 \mathfrak{a}^* = E_2^{3,0} = E_3^{3,0} = E_\infty^{3,0}$ is a summand of $H^3(\mathfrak{g})$. Hence $b_3(\mathfrak{g}) \geq \dim \Lambda^3 \mathfrak{a}^*$ and $b_3(\mathfrak{g}) = 0$ implies that $\dim \mathfrak{a} \leq 2$, as required. \square

Remark 5.11. Note that the condition $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\}$ does not appear in Theorem 5.3. However, when $\mathfrak{k} = \mathfrak{g}'$ is of codimension one, this space is automatically zero. Indeed, suppose that $\mathfrak{g} = \mathbb{R}\mathbf{A} + \mathfrak{k}$ with \mathfrak{k} an ideal. A non-zero element of $H^1(\mathfrak{k})^{\mathfrak{g}}$ is an element $\gamma \in \mathfrak{k}^*$, such that $d_0\gamma = 0$ and $\mathbf{A} \cdot \gamma = 0$. The first condition implies that γ annihilates \mathfrak{k}' , the second says that $\gamma([A, \mathbf{K}]) = 0$ for each $\mathbf{K} \in \mathfrak{k}$. Thus γ annihilates \mathfrak{g}' , but so does \mathfrak{a}^* , so \mathfrak{g}' is at least codimension two. \triangle

The Betti numbers of Lie algebras of dimension at most 6 are given in Freibert and Schulte-Hengesbach [16, 17]. One may thus use their tables to read off which algebras of dimension 4, 5 or 6 are (3, 4)-trivial. Table 5.1 summarises the resulting sets of Betti numbers that occur. The tables confirm that \mathfrak{g}' is of codimension at most two in these cases. When the codimension equals two, i.e., $b_1(\mathfrak{g}) = 2$, one sees that $b_n(\mathfrak{g}) = 0$. In other words, these examples are not unimodular. The next example shows that this is no longer true in higher dimensions.

| $n = \dim \mathfrak{g}$ | (b_1, \dots, b_n) |
|-------------------------|--|
| 4 | (1, 0, 0, 0), (2, 1, 0, 0) |
| 5 | (1, 0, 0, 0, 0), (2, 1, 0, 0, 0) |
| 6 | (1, 0, 0, 0, 0, 0), (1, 0, 0, 0, 1, 1), (2, 1, 0, 0, 0, 0) |

Table 5.1. The sets of Betti numbers of the (3, 4)-trivial Lie algebras \mathfrak{g} appearing in the low-dimensional classifications of [16, 17].

Example 5.12. Consider an $(m + 2)$ -dimensional Lie algebra of the form as in Example 5.8 with T_1 and T_2 commuting and diagonalisable. The proof of Proposition 5.7 shows that we may assume that T_1 is invertible. The action of T_i on $H^m(\mathfrak{k})$ is simply the trace of the action \mathfrak{k}^* . When $m \geq 5$, we may ensure that the cohomology conditions of Proposition 5.7 are satisfied and that each T_i acts trivially on $H^m(\mathfrak{k})$, for example by giving both T_i strictly positive eigenvalues on some $(m - 1)$ -dimensional subspace of \mathfrak{k} and requiring T_i to be trace-free. Now this trace-free condition ensures that the maps $d_1: E_1^{p,m} \rightarrow E_1^{p+1,m}$ of the Hochschild-Serre spectral sequence for \mathfrak{g} with respect to the ideal \mathfrak{k} are identically zero, so $H^n(\mathfrak{g}) = E_2^{2,m} = E_1^{2,m} = H^m(\mathfrak{k}) = \mathbb{R}$, showing that \mathfrak{g} is unimodular. A concrete non-product unimodular (3, 4)-trivial algebra of dimension $n = m + 2 \geq 7$ with $b_1 = 2$ is given by

$$(0, 0, 13 + 23, 14, 15, \dots, (1 - m) \cdot 1n - 2n). \quad \diamond$$

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