CENTRE FOR **STOCHASTIC GEOMETRY**
AND ADVANCED **BIOIMAGING**

Markus Kiderlen and Mario Hörig

Matérn's hard core models of types I and II with arbitrary compact grains

No. 05, July 2013

Matérn's hard core models of types I and II with arbitrary compact grains

Markus Kiderlen[1] and Mario Hörig[2]

[1]Aarhus University
[2]Karlsruhe Institute of Technology

Abstract

Matérn's classical hard core models can be interpreted as models obtained from a stationary marked Poisson process by dependent thinning. The marks are balls of fixed radius, and a point is retained when its associated ball does not hit any other balls (type I) or when its random birth time is strictly smaller than the birth times of all balls hitting it (type II). Extending ideas of [M. Månsson and M. Rudemo. Random patterns of nonoverlapping convex grains. Adv. in Appl. Probab., 34:718–738, 2002.], who considered grains that are isotropic rotations or random scalings of a fixed convex set, we discuss these two models in d-dimensional space when the marks are arbitrary random compact grains. We determine the intensity and the mark distribution after thinning, and find the second order factorial moment density of the ground process for model II under weak additional assumptions. By Brunn-Minkowski's inequality, the volume density associated to model II turns out to be bounded by 2^{-d}. This bound is sharp. It is attained asymptotically (when the proposal intensity tends to infinity) only when all grains coincide with one deterministic origin-symmetric convex set. We also discuss how known connections of this model with the process of intact grains of the dead leaves model and the Stienen model leads to analogous results for the latter.

Keywords: Matern hard core models of types I and II, Palm distribution, dependently thinned Poisson point process, germ grain model, soft core particle process, dead leaves model, Stienen model, volume density

1 Introduction

In his D. Sc. thesis [17] Matérn introduced a number of models for random collections of repulsive points in the plane that cannot be closer to one another than a certain prescribed distance $D > 0$. These models are obtained from dependent thinning of a stationary Poisson point process X^0 and are now known as Matérn hard core point processes of types I, II and III.

The type I process $X^0_{\mathbf{I}}$, illustrated in Figure 1, consists of all points of X^0 which have no D-neighbors in X^0, that is, no other point in X^0 of distance D or less.

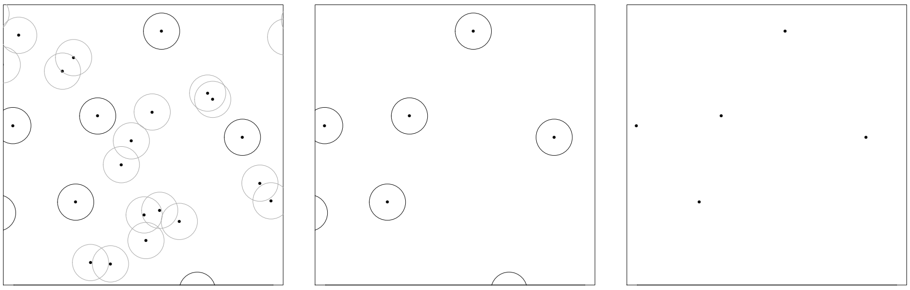

Figure 1: The classical Matérn hard core process of type I in a unit window: The initial Poisson point process with intensity $\gamma = 20$ and disks of radius 0.07 with the overlapping particles in gray (left), the particle process $X_{\mathbf{I}}$ after thinning (middle), and the hard core process $X_{\mathbf{I}}^0$ (right).

To construct the process $X_{\mathbf{II}}^0$ of type II the points of X^0 are assigned independent marks (uniformly distributed in the interval $(0, 1)$, usually interpreted as birth times) and only those points survive that are older than all their D-neighbors. Figure 2 depicts a realization of $X_{\mathbf{II}}$ and its ground process. The intensity of $X_{\mathbf{II}}^0$ is higher than that of $X_{\mathbf{I}}^0$. However, it could be further increased, as the procedure leading to $X_{\mathbf{II}}^0$ also deletes points whose D-neighbors all have been deleted by other points in X^0. Retaining such points would increase the intensity further. Matérn therefore mentions a third point process $X_{\mathbf{III}}^0$ obtained by an iterative procedure, where a point is deleted only if its D-neighborhood contains an older point that was retained. Due to the resulting long range dependence, even the intensity of $X_{\mathbf{III}}^0$ cannot be given in closed form, but recent work revealed that its likelihood function can be written down. This opens the door to likelihood-based inference [7] and perfect simulation [20] in bounded windows in $\mathbb{R}^d$. In the present paper we will only work with $X_{\mathbf{I}}$, $X_{\mathbf{II}}$ and generalizations of those.

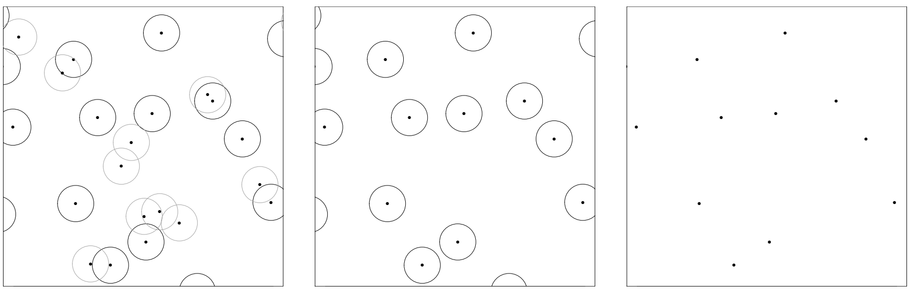

Figure 2: The classical Matérn hard core process of type II in a unit window: The initial Poisson point process with intensity $\gamma = 20$ and disks of radius 0.07 with particles that are 'younger' than a competing neighbor in gray (left), the particle process $X_{\mathbf{II}}$ after thinning (middle), and the hard core process $X_{\mathbf{II}}^0$ (right). Note that the same realization of X as in Figure 1 is used.

If every point in X^0 is a marked with a disk of diameter D, we obtain a special germ grain model X, and the elements of $X_{\mathbf{I}}$ (i.e., the points of $X_{\mathbf{I}}^0$ together with their marks) correspond to the non-overlapping particles of X. Similarly, the elements of $X_{\mathbf{II}}$ form the process of particles that are older than all other particles they overlap with. Inspired by problems of fatigue in materials science, Månsson and Rudemo [16] used Matérn's construction to obtain non-overlapping particle systems with shapes other than spheres. More precisely, they allowed for particles that are either random scalings or isotropic rotations of a given convex shape. For instance, applying Mathérn's construction with a distance other than the Euclidean one, the corresponding germ grain models after thinning would be non-overlapping translates of the unit ball of this distance. Closely following [16] we define generalizations of $X_{\mathbf{I}}$ and $X_{\mathbf{II}}$, but now allow for arbitrary compact particles (which may be different in shape for different germs). As all points of X^0 not only have an associated particle, but also a birth time, one works most conveniently with marked point processes. Assume that X is a stationary marked point process in $\mathbb{R}^d$ with two marks for each of its points, the first is a compact particle, the second is a birth time. The birth time may now have an arbitrary distribution on $\mathbb{R}$, and it may even depend on the particle. The marked point process X_M after global thinning is obtained by deleting all marked points of X whose particles hit another particle with later or equal birth time. This corresponds to a 'survival of the youngest'-rule, which is adopted here in order to ease comparison with [16] and is in contrast to Matérn's original 'survival of the oldest'. Of course, mathematically this choice does not change the class of the model, as one can always reflect the birth time at the origin in order to switch from one rule to the other. Even stronger, under very weak assumptions the two point processes produced with the two rules are equal in distribution; see Lemma 2.1 below. We will omit the auxiliary birth-times in X_M as they were only necessary for thinning, and thus consider X_M as a marked point process where each point has only *one* mark, namely the particle associated to this point. The process X_M will be referred to as *generalized Matérn particle process* (with arbitrary compact grains). It contains Matérn's original constructions as special cases. Choosing all particles equal to a fixed sphere of diameter D, and independent birth-times that are uniform in $(0, 1)$ yields $X_M = X_{\mathbf{II}}$, and only considering the ground process of points gives $X_M^0 = X_{\mathbf{II}}^0$. Whenever all competing particles have the same (deterministic) birth time, all overlapping particles are deleted. Hence, X_M coincides with $X_{\mathbf{I}}$ in this case. In general, however, X_M^0 is not hard-core. As the points of X_M^0 exhibit repulsion, X_M^0 can be seen as a model of a soft core process. In [5] X_M^0 is therefore called *generalized Stoyan soft core process* and shown to be Brillinger-mixing.

Figure 3 illustrates a realization based on a stationary Poisson process with intensity $\gamma = 30$ and a mark distribution $\mathbb{Q}$ that is concentrated on axis-parallel rectangles in $\mathcal{C}_0$. A typical rectangle with distribution $\mathbb{Q}$ has i.i.d. side lengths that are uniformly distributed in the interval $(0.02, 0.2)$. The birth times were chosen stochastically independent of the particles and uniform in $(0, 1)$. The process X_M after thinning is depicted in Figure 3, middle.

In the language of [16], the above thinning is based on a *global assignment rule*, as each point has exactly one birth time that is used 'globally' in all comparisons. An alternative to global assignment is the *pairwise assignment rule*, also suggested in [16]. Again X is an independently marked point process in $\mathbb{R}^d$ where each of its points x_i has two marks, namely a compact particle and a *sequence of birth times* $(t_k^{(i)})$. For each pair of points (x_i, x_j), $i \neq j$, with overlapping particles, x_i dies in the competition with x_j if $t_j^{(i)} \leq t_i^{(j)}$. Equivalently, one can think of this rule as a

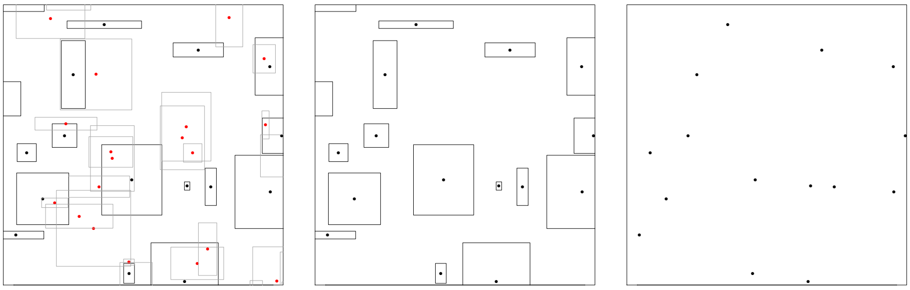

Figure 3: The generalized Matérn hard core process in a unit window: The initial Poisson point process with rectangular marks, where particles that are 'older' than a competing neighbor are gray (left), the particle process X_M after thinning (middle), and the soft core process X_M^0 (right).

pairwise competition of overlapping particles, where new birth-times are generated for each pair, and only the strictly younger of the two particles survives. Pairwise assignment does typically lead to a smaller intensity than global assignment [16, Corollary 2.3], and the corresponding intensity converges to zero when the initial intensity converges to ∞. We therefore only consider the global assignment rule, but emphasize that the theory of marked point process also allows to treat the pairwise assignment rule, generalizing the results in [16]. Quite recently, Teichmann et. al [28] have presented generalizations of $X_{\mathbf{I}}^0$ and $X_{\mathbf{II}}^0$, where the deterministic dependent thinning of the classical Matérn processes is replaced by a probabilistic thinning method. As their thinning probabilities depend on the Euclidean distance of competing point pairs, their approach can be interpreted as thinning of Euclidean balls, whereas the main focus of the present work is to consider arbitrarily shaped particles.

The paper is organized as follows. Section 2 presents the main results and their interpretations, while Section 3 is devoted to the proofs. In Section 2.1 we determine the first and certain second order properties of the stationary marked point process X_M, that is, its intensity, its mark distribution and the reduced second order moment density of X_M^0. These results are essentially taken from the doctoral thesis [6, Section 2.3.5] of the second author. If the birth-time distribution is independent of the corresponding particle and atom-free, we show in Lemma 2.1 that the *distribution* of X_M does not depend on the birth-time distribution. In this case, the volume density of the particle union of X_M is shown in Theorem 2.6 to be bounded by 2^{-d}, where this bound is attained asymptotically if and only if all particles are coinciding with a fixed origin-symmetric convex set. Using and extending a known connection of X_M to the process of intact grains of the dead leaves model, we show in Section 2.2 that the same bound also holds for the volume density of the intact grains. In Section 2.3, we show that also the volume fraction associated to the Stienen model (with arbitrary random compact grains) has the same bound with the same characterization of the equality case.

2 Main results

2.1 Matérn's construction with compact grains

Throughout the paper we assume that X is a stationary marked point process in $\mathbb{R}^d$ with intensity $\gamma > 0$ and mark space $\mathcal{C}_0 \times \mathbb{R}$; see e.g., [22] for standard notions in stochastic geometry. The first marginal $\mathbb{Q}$ of the mark distribution is called *(proposal) shape distribution* and is concentrated on the family $\mathcal{C}_0$ of all non-empty compact subsets of $\mathbb{R}^d$ with circumsphere at the origin. The second component of the mark space corresponds to the birth time. The distribution of the time mark T may depend on the particle $C \in \mathcal{C}_0$ and is therefore written as conditional probability $P_T(\cdot|C)$. For details, see Section 3.2. The marks C and T are independent, if and only if $P_T = P_T(\cdot|C)$ is (almost surely) independent of C. If C and T are independent, and if P_T is atom-free ($P_T(\{t\}) = 0$ for all $t \in \mathbb{R}$), it has been shown in special cases ([16] and [27]) that first and certain second order properties of the generalized Matérn process X_M, derived from X by global thinning, are independent of P_T. This es even true for the distribution of X_M.

Lemma 2.1. *If $P_T = P_T(\cdot|C)$ is independent of C and atom-free, then the distribution of X_M is independent of P_T.*

This Lemma and all other results stated in Section 2 will be proven in Section 3. In the following, λ stands for the Lebesgue measure in $\mathbb{R}^d$. For sets $A, B \subset \mathbb{R}^d$, $A + B = \{a + b : a \in A, b \in B\}$ is the Minkowski sum. For $\alpha \in \mathbb{R}$ we write $\alpha A = \{\alpha a : a \in A\}$. When α is negative, this can be thought of as a reflection $-A = (-1)A$ of A at the origin followed by a scaling with $|\alpha|$. We will often abbreviate $A + (-B)$ by $A - B$. Let $C \in \mathcal{C}_0$ and $t \in \mathbb{R}$ be given. We will show in Section 3.2 that

$$g_M(C,t) = \exp\left(-\gamma \int_{\mathcal{C}_0} P_T\left([t,\infty)\middle|C'\right) \lambda(C - C')\, \mathbb{Q}(dC')\right) \tag{2.1}$$

can be interpreted as the probability that a typical point survives, given its associated mark is (C,t). Hence,

$$g_M(C) = \int_{-\infty}^{\infty} g_M(C,t) P_T(dt|C),$$

is the retaining probability for a typical point, given its particle is C. The shape distribution $\mathbb{Q}$ is called *non-degenerate*, if

$$\int_{\mathcal{C}_0} \lambda(C - C')\mathbb{Q}(dC') > 0 \tag{2.2}$$

for $\mathbb{Q}$-almost all $C \in \mathcal{C}_0$. The shape distribution $\mathbb{Q}$ is clearly non-degenerate if

$$\mathbb{Q}(\{C \in \mathcal{C}_0 : \lambda(C) > 0\}) > 0.$$

The last condition is trivially true when $\mathbb{Q}$-almost all particles have positive volume, but we do not want to impose so strong a condition in order to allow for such

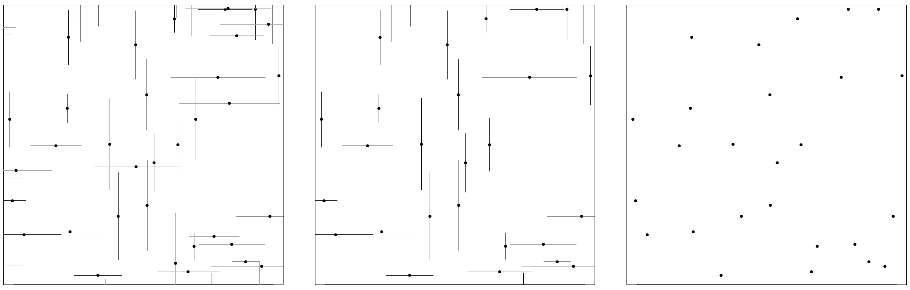

Figure 4: A model of non-overlapping line segments in a unit window: The initial Poisson point process with axis-parallel line segments as marks, where particles that are 'older' than a competing neighbor are gray (left), the particle process X_M after thinning (middle), and the ground process X_M^0 (right). Due to the requirement of disjointness the line segments appear to have locally preferred directions.

interesting models as the one depicted in Figure 4. In the illustrated model all particles are axis-parallel line segments with uniform random lengths in the interval $[.1, .4]$. For each of these line segments one of the two possible orientations is chosen with equal probability and stochastically independent of its length.

If $\mathbb{Q}$ is non-degenerate, then $g_M(C) < 1$ for $\mathbb{Q}$-almost all $C \in \mathcal{C}_0$. In other words, if $\mathbb{Q}$ is non-degenerate, the typical particle has a positive risk of being deleted.

Proposition 2.2. *Assume that $\mathbb{Q}$ is non-degenerate, and that $P_T = P_T(\cdot|C)$ is independent of C. Then*

$$g_M(C) \le \frac{1 - \exp\left[-\gamma \int_{\mathcal{C}_0} \lambda(C - C')d\mathbb{Q}(C')\right]}{\gamma \int_{\mathcal{C}_0} \lambda(C - C')d\mathbb{Q}(C')} \tag{2.3}$$

holds for $\mathbb{Q}$-almost all C. Equality holds in (2.3) *for $\mathbb{Q}$-almost all C if and only if P_T is atom free.*

The intensity and the shape distribution of the thinned process X_M is given in the next theorem.

Theorem 2.3. *Assume that the stationary marked Poisson point process X has intensity $\gamma > 0$, and shape distribution $\mathbb{Q}$ on $\mathcal{C}_0$, and that the conditional distribution of the time mark given the shape C is $P_T(\cdot|C)$. Then X_M is a stationary marked point process with intensity*

$$\gamma_M = \gamma \int_{\mathcal{C}_0} g_M(C)\mathbb{Q}(dC).$$

If, in addition $\gamma_M > 0$, then X_M has shape distribution

$$\mathbb{Q}_M(\mathcal{A}) = \frac{\gamma}{\gamma_M} \int_{\mathcal{A}} g_M(C)\,\mathbb{Q}(dC)$$

for any measurable set $\mathcal{A} \subset \mathcal{C}_0$.

The condition

$$\int_{\mathcal{C}_0} \lambda(C + B^d)\, \mathbb{Q}(dC) < \infty, \tag{2.4}$$

is sufficient for γ_M being positive, but it is not necessary when $\mathbb{Q}$ is concentrated on the family of compact sets without interior points. We will use this criterion, as it is natural; see e.g., [22, Theorem 4.1.2]. If, for instance, $\mathbb{Q}$ is concentrated on a family $\{\alpha C_0 : \alpha > 0\}$ of scaled versions of a fixed $C_0 \in \mathcal{C}_0$ with interior points, then (2.4) is equivalent to the requirement that the image measure of $\mathbb{Q}$ under $\alpha C_0 \mapsto \alpha$ has finite dth moment. In view of Proposition 2.2, we obtain the most important special case of Theorem 2.3.

Corrolary 2.4. *Assume that the stationary marked Poisson point process X has intensity $\gamma > 0$, and non-degenerate shape distribution $\mathbb{Q}$ on $\mathcal{C}_0$, and that the conditional distribution $P_T(\cdot|C)$ of the time mark given the shape C is independent of C and atom-free. Then*

$$\gamma_M = \gamma \int_{\mathcal{C}_0} \frac{1 - \exp\left[-\gamma \int_{\mathcal{C}_0} \lambda(C - C') d\mathbb{Q}(C') \right]}{\gamma \int_{\mathcal{C}_0} \lambda(C - C') d\mathbb{Q}(C')} \mathbb{Q}(dC). \tag{2.5}$$

If, in addition, $\gamma_M > 0$,

$$\mathbb{Q}_M(\mathcal{A}) = \frac{\gamma}{\gamma_M} \int_{\mathcal{A}} \frac{1 - \exp\left[-\gamma \int_{\mathcal{C}_0} \lambda(C - C') d\mathbb{Q}(C') \right]}{\gamma \int_{\mathcal{C}_0} \lambda(C - C') d\mathbb{Q}(C')} \mathbb{Q}(dC) \tag{2.6}$$

for any measurable set $\mathcal{A} \subset \mathcal{C}_0$.

It should be noted that (2.3), (2.5), and (2.6) can be simplified using mixed volumes, when $\mathbb{Q}$ is concentrated on *convex* particles. If, in addition, $\mathbb{Q}$ is isotropic, (2.3), (2.5), and (2.6) can be expressed with the help of Minkowski functionals of C and C', thus avoiding the complicated volume $\lambda(C - C')$; see [6] for details. Also the second order properties of X_M after removing the marks can be calculated when the distribution of the birth time is independent of the particle and atom-free. For $C, C' \in \mathcal{C}_0$ and $x \in \mathbb{R}^d$ we use the abbreviations

$$\alpha = \int_{\mathcal{C}_0} \lambda\left((x + C - \overline{C}) \setminus (C' - \overline{C}) \right) \mathbb{Q}(d\overline{C}), \tag{2.7}$$

$$\beta = \int_{\mathcal{C}_0} \lambda\left((C' - \overline{C}) \setminus (x + C - \overline{C}) \right) \mathbb{Q}(d\overline{C}), \tag{2.8}$$

$$\kappa = \int_{\mathcal{C}_0} \lambda\left((x + C - \overline{C}) \cap (C' - \overline{C}) \right) \mathbb{Q}(d\overline{C}). \tag{2.9}$$

Theorem 2.5. *Let the assumptions of Theorem 2.3 be satisfied. If $P_T(\cdot|C)$ is independent of the particle and atom-free then the reduced second order factorial moment density of the ground process $X_M^0 = \{x : (x, C) \in X_M\}$ is*

$$\begin{aligned} \rho^{(2)}(x) = \int_{\{(C,C') \in \mathcal{C}_0^2 : (x+C) \cap C' = \emptyset\}} \Bigg[& \frac{1}{\alpha + \beta + \kappa} \left(\frac{1}{\alpha + \kappa} + \frac{1}{\beta + \kappa} \right) - \frac{e^{-\gamma(\alpha+\kappa)}}{\beta(\alpha + \kappa)} \\ & - \frac{e^{-\gamma(\beta+\kappa)}}{\alpha(\beta + \kappa)} + \frac{\alpha + \beta}{\alpha\beta(\alpha + \beta + \kappa)} e^{-\gamma(\alpha+\beta+\kappa)} \Bigg] \mathbb{Q}^2\big(d(C, C')\big). \end{aligned} \tag{2.10}$$

In the doctoral thesis [6, Satz 2.3.9] the second order moment density of the marked point process X_M, even with birth-mark retained, is given in full generality. This result is though less explicit than Theorem 2.5 as the birth-time distribution is not integrated out. Note that $\alpha + \kappa$ and $\beta + \kappa$ are independent of x. When $\mathbb{Q}$ is concentrated on one shape C_0, then

$$\rho^{(2)}(x) = 0 \tag{2.11}$$

for all $x \in C_0 - C_0$ and $\rho^{(2)}$ depends on C_0 only through $C_0 - C_0$. Actually, (2.11) holds if and only if $x \in C_0 - C_0$, which can be seen directly from (3.13) and (3.14), below. Hence, when $\mathbb{Q}(\{C_0\}) = 1$, the second order properties of the ground process determine C_0 if and only if C_0 is origin-symmetric. If C_0 is a ball with fixed radius, the formula for $\rho^{(2)}$ in Theorem 2.5 coincides with [27, (3.2)], if the first summand $1/(a+b)$ in that formula is replaced by the correct $1/(a(a+b))$; see also [25, p. 164], where the result is stated correctly. In [27, Theorem 2] the second order moment density is calculated in the special case, where all particles are random balls.

If (2.4) is satisfied, then $\{x + C : (x, C) \in X_M\}$ is a stationary particle process, cf. [22, Theorem 4.1.2]. Its union set

$$Z_M = \bigcup_{(x,C) \in X_M} (x + C)$$

is a stationary random set. Its volume density $\overline{V}_M(\gamma)$ will be considered as a function of γ. As the particles associated to X_M do not overlap, we have

$$\overline{V}_M(\gamma) = \gamma_M \int_{\mathcal{C}_0} \lambda(C)\mathbb{Q}_M(dC),$$

and Theorem 2.3 gives

$$\overline{V}_M(\gamma) = \gamma \int_{\mathcal{C}_0} \lambda(C) g_M(C)\mathbb{Q}(dC). \tag{2.12}$$

If the distribution of T is atom-free and independent of the particle C, Proposition 2.2 shows that $\overline{V}_M(\gamma)$ is increasing in γ. We consider therefore the asymptotic volume density

$$\overline{V}_M(\infty) := \lim_{\gamma \to \infty} \overline{V}_M(\gamma).$$

It will be shown in Subsection 3.2 that an application of the Brunn-Minkowski inequality gives a sharp upper bound for $\overline{V}_M(\infty)$.

Theorem 2.6. *Assume that $\mathbb{Q}$ is non-degenerate and satisfies* (2.4). *If $P_T(\cdot|C)$ is independent of C and atom free, then the asymptotic volume density satisfies*

$$\overline{V}_M(\infty) = \int_{\mathcal{C}_0} \frac{\lambda(C)}{\int_{\mathcal{C}_0} \lambda(C - C')\mathbb{Q}(dC')} \mathbb{Q}(dC) \leq 2^{-d}, \tag{2.13}$$

with equality on the right if and only if all particles are coinciding with a fixed origin symmetric convex set of positive volume (i.e., $\mathbb{Q}(\{C_0\}) = 1$ for some origin-symmetric convex $C_0 \in \mathcal{C}_0$, $\lambda(C_0) > 0$).

It should be noted that the first equality in (2.13) still holds for some models, where $P_T = P_T(\cdot|C)$ is independent of C but has atoms. This is for instance true when the supremum of all atoms of P_T is strictly smaller than the right endpoint of its support, as can be seen from (3.4) in Lemma 3.1, below.

If one considers only isotropic typical particles in Theorem 2.6, then equality in (2.13) can only hold if $\mathbb{Q}(\{C_0\}) = 1$ for an origin-symmetric C_0, which is invariant under all rotations of $\mathbb{R}^d$ fixing 0. In other words, in the isotropic case, the volume density is bounded by 2^{-d}, and the bound is attained if and only if all particles are Euclidean balls with the the same fixed radius. This confirms and generalizes [2, Conjecture 3.3]. The special case of this statement, where $d \in \{2, 3\}$ and $\mathbb{Q}$ is a Dirac measure, was already shown in [16, Theorem 4.4].

Under the assumptions of Proposition 2.2 we know that $\overline{V}_M(\gamma)$ is bounded by the middle term in (2.13), and we obtain directly the following corollary.

Corrolary 2.7. *Assume that $\mathbb{Q}$ is non-degenerate and satisfies* (2.4). *If $P_T(\cdot|C)$ is independent of C, then the volume fraction after thinning satisfies*

$$V_\gamma(Z_M) < 2^{-d}.$$

This does not mean, however, that volume fractions larger than 2^{-d} cannot be obtained by global thinning. In fact, it was shown in [2, Theorem 4.1] that if the birth times T may depend on the particles C, a volume density arbitrarily close to 1 can be obtained even with finite intensity.

Note that the sharp *lower* bound of $V_\gamma(Z_M)$ without additional assumptions is zero. This is even true when the typical particle is isotropic. For instance, let $\mathbb{Q}$ be rotation invariant and concentrated on the family $\{B^d \cap x^\perp : \; 0 \neq x \in \mathbb{R}^d\}$ of all $(d-1)$-dimensional unit discs that are centered at the origin. However, if $\mathbb{Q}$ is known to be concentrated on *one full-dimensional convex* shape (i.e., $\mathbb{Q}(\{C_0\}) = 1$ for a convex set $C_0 \in \mathcal{C}_0$ with interior points), a non-trivial lower bound can be found when $P_T(\cdot|C)$ is independent of C and atom free. Under these assumptions

$$\overline{V}_M(\infty) \geq \binom{2d}{d}^{-1}$$

with equality if and only if C_0 is a simplex. This was shown in [16, Theorem 4.3] based on a difference-body inequality in [21].

2.2 Intact grains of the dead leaves model

In this section, we outline and exploit the connection between Matérn's hard core processes and the *dead leaves model* introduced by Matéron [18]; see also Serra's monograph [23] and the works by Jeulin [8, 10, 12]. This connection was first described in [26] for ball-shaped particles, and extended in [15]. The process of the *intact grains of the dead leaves model* (with arbitrary compact grains) can be defined as follows; see e.g., [9], [15]. Consider a stationary, independently marked Poisson point process on $\mathbb{R}^d \times \mathbb{R}$ with intensity one and mark distribution $\mathbb{Q}$ on $\mathcal{C}_0$. Let Y be its restriction to $\mathbb{R}^d \times (-\infty, 0] \times \mathcal{C}_0$. For a point $(x, t, C) \in Y$, t will be interpreted as birth time of the particle $x + C$ (a 'leave'). The process Y_{ig} of intact grains of the dead leaves model (at time zero) are all those triples (x, t, C) of Y for which the particle does not overlap with any other particle of Y that arrived later. In order to assure that Y_{ig} is locally finite, we assume again that $\mathbb{Q}$ is non-degenerate. After thinning we obtain a marked point process Y. Omitting the birth times in Y leads to a marked point process Y_{ig} with marks in $\mathcal{C}_0$, which is usually called the *process of intact grains of the dead leaves model.* Its ground point process is denoted by Y_{ig}^0. Figure 5 shows an example.

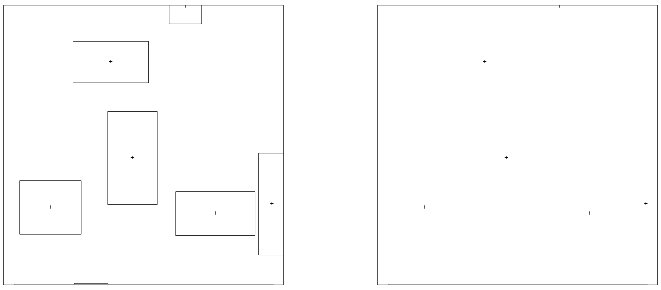

Figure 5: A realization of the intact grains of the dead leaves model Y_{ig} (left), where the proposal shape distribution $\mathbb{Q}$ is the same as in Figure 3 and the corresponding ground process Y_{ig}^0.

For $\gamma > 0$ let $Y(\gamma)$ be the point process $Y \cap (\mathbb{R}^d \times [-\gamma, 0] \times \mathcal{C}_0)$. Then $Y(\gamma)$ can be interpreted as a stationary Poisson point process on $\mathbb{R}^d$ with independent marks in $\mathbb{R} \times \mathcal{C}_0$, intensity γ and mark distribution $\mathrm{unif}[-\gamma, 0] \otimes \mathbb{Q}$, where $\mathrm{unif}[a, b]$ is the uniform distribution on the interval $[a, b] \subset \mathbb{R}$. Thinning of this subprocess by only retaining those particles that do not overlap with any other particle of $Y(\gamma)$ that arrived later, and subsequently ignoring the birth time, corresponds to the global thinning procedure described in Section 1. Hence the resulting point process $Y_M(\gamma)$ is a generalized Matérn process with initial shape distribution $\mathbb{Q}$ and a distribution of the birth-mark that is uniform in $[-\gamma, 0]$. Clearly $\bigcup_{\gamma>0} Y_M(\gamma) = Y_{ig}$, and one observes that the process of intact grains is the limit of globally thinned Matérn hard core processes as the intensity tends to infinity. A simple coupling argument makes this independent of the particular construction.

Proposition 2.8. *Let $\mathbb{Q}$ be a non-degenerate distribution on the mark space $\mathcal{C}_0$. Let Y_{ig} be the stationary process of intact grains of the dead leaves model with initial shape distribution $\mathbb{Q}$. For every $\gamma > 0$ let X be a stationary marked Poisson point process with intensity γ and mark distribution $\mathbb{Q} \times P_T$, where P_T is atom-free (and may vary with γ) and let $X_M(\gamma)$ be the derived generalized Matérn particle process. Then*

1. *Y_{ig} stochastically dominates $X_M(\gamma)$, that is, there is a process $X'_M(\gamma)$ with the same distribution as $X_M(\gamma)$ and $X'_M(\gamma) \subset Y_{ig}$,*
2. *as $\gamma \to \infty$, we have the following convergence in distribution:*

$$X_M(\gamma) \xrightarrow{d} Y_{ig}. \tag{2.14}$$

As Y_{ig} is a stationary marked point process with marks in $\mathcal{C}_0$, its intensity γ_{ig} (which is of course also the intensity of Y_{ig}^0) and mark distribution $\mathbb{Q}_{ig}$ are well-defined. They are obtained by taking the limit $\gamma \to \infty$ in Corollary 2.4.

Corrolary 2.9. *With the notation from above, we have*

$$\gamma_{ig} = \int_{\mathcal{C}_0} \left(\int_{\mathcal{C}_0} \lambda(C - C')\mathbb{Q}(dC') \right)^{-1} \mathbb{Q}(dC). \tag{2.15}$$

If $\gamma_{ig} > 0$, we have

$$\mathbb{Q}_{ig} = \gamma_{ig}^{-1} \int_{(\cdot)} \left(\int_{\mathcal{C}_0} \lambda(C - C')\mathbb{Q}(dC') \right)^{-1} \mathbb{Q}(dC). \tag{2.16}$$

The union of the particles in Y_{ig} thus has volume density $\overline{V}_M(\infty)$ given by (2.13).

We remark two consequences of this corollary. Firstly, Theorem 2.6 also holds for Y_{ig}: The volume density of Y_{ig} is bounded by 2^{-d} and this bound is attained if and only if $\mathbb{Q}$ is concentrated on one origin-symmetric convex particle. Under the assumptions $d \in \{2, 3\}$ and $\mathbb{Q}(\{C_0\}) = 1$ for some *convex* $C_0 \in \mathcal{C}_0$, this result was already loosely stated in [13, p. 128]. More generally, for shape distributions that are concentrated on scaled versions of a fixed convex particle, or isotropic shape distributions concentrated on rotations of a fixed convex particle the exact statement was shown in [15, Theorems 5.1 and 5.2] for $d \in \{2, 3\}$.

Another consequence of (2.16) are moment relations when all leaves are randomly scaled versions of a fixed, origin-symmetric particle $C_0 \in \mathcal{C}_0$ of positive volume. Then, also the intact grains are random scalings of this particle. Let R and R_{ig} be the random scaling factors of C_0 before and after thinning, respectively. Hence, R has distribution

$$\mathbb{Q}(\{\alpha C_0 : \alpha \in (\cdot)\}) = \int_{\mathcal{C}_0} \mathbf{1}_{\left(\frac{\lambda(C)}{\lambda(C_0)}\right)^{1/d} \in (\cdot)} \, \mathbb{Q}(dC), \tag{2.17}$$

and similarly for R_{ig} with $\mathbb{Q}$ replaced by $\mathbb{Q}_{ig}$. From (2.16) we obtain

$$ER^k = \gamma_{ig}\lambda(C_0) \sum_{j=0}^{d} \binom{d}{j} (ER_{ig}^{d+k-j})(ER^j), \tag{2.18}$$

$k = 0, 1, 2, \ldots$. We will show in Section 3.3 that (2.18) allows to determine all moments of R from the quantities γ_{ig}, and ER_{ig}^k, $k = 0, 1, 2, \ldots$ when $d = 2$. Whenever the moments ER^k of R do not grow too fast with k, for instance when Carleman's condition

$$\sum_{k=1}^{\infty} (ER^k)^{-1/(2k)} = \infty \tag{2.19}$$

holds, even the distribution of R, and hence $\mathbb{Q}$, is determined by Y_{ig}. See e.g., [1] or [24] for details on uniqueness in the Stieltjes and other moment problems.

Proposition 2.10. *Consider the process of intact grains Y_{ig} of a planar ($d = 2$) dead leaves model, where the shape distribution $\mathbb{Q}$ is concentrated on scaled versions of one single origin-symmetric shape C_0 with interior points. Assume that the distribution* (2.17) *of the random scaling R satisfies* (2.19).

Then γ_{ig} and $\mathbb{Q}_{ig}$ determine $\mathbb{Q}$.

Jeulin [11, p. 17] derived (2.18) in the plane with $C_0 = B^2$ and explains an estimation procedure of low order moments of R from those of R_{ig}. He also derives similar relationships for planar thickened fibre particles and explains stereological applications to the morphological analysis of powders. Generalizing this, (2.18) could naturally be extended to the case where $\mathbb{Q}$ is concentrated on convex sets (and not necessarily a Dirac measure) when mixed volumes are used. We omit explicit equations, though.

2.3 The generalized Stienen model

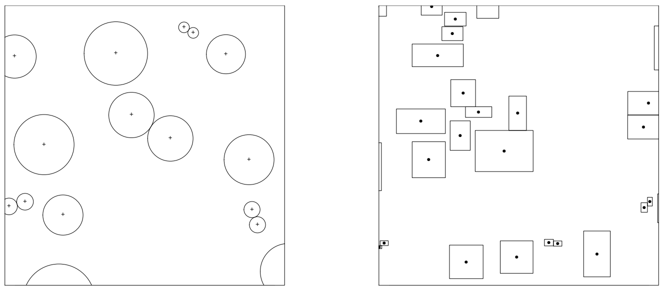

Figure 6: A realization of the classical Stienen model with intensity $\gamma = 15$ in a unit window (left) and a realization of the generalized Stienen model with the same shape distribution $\mathbb{Q}$ as in Figure 3.

The classical Stienen model is obtained by attaching random balls to the points of a stationary Poisson point process. The radius of a ball at a point of the process is half the distance to its closest neighbor; see Figure 6, left. Alternatively, one can think of this process in a dynamic way, by letting balls grow with constant speed. Then an (infinitesimal) ball attached to a given point grows until it first meets one of the other growing balls, if the latter would grow forever. This dynamic point of view was adapted in [15] to generalize the model by replacing the ball by a non-spherical convex particle (possibly randomized by an isotropic rotation) and allowing for different – random – constant growing speeds. The latter is equivalent to saying that each particle is scaled by a random factor before the growth starts. In this model all particles have the same shape, possibly up to rotation. In the following we drop this restriction and consider a Stienen model, where each point is independently marked by a random shape, not necessarily convex, thus allowing realizations where no two particles have the same shape. The proposal process is now an independently marked Poisson point process X with intensity $\gamma > 0$ and mark distribution $\mathbb{Q}$ on $\mathcal{C}_0$. We let Y_S be the marked point process of particles in the Stienen model derived from X; see Figure 6, right. More formally,

$$Y_S = \bigcup_{(x,C)\in X} \Big\{\Big(x, \tau\big(X; x, C\big)C\Big)\Big\},$$

where

$$\tau(F; x, C) = \inf_{(x',C')\in F\setminus\{(x,C)\}} \inf\{\alpha \geq 0 : (x + \alpha C) \cap (x' + \alpha C') \neq \emptyset\}$$

is defined for arbitrary $x \in \mathbb{R}^d$, $C \in \mathcal{C}_0$ and a locally finite set $F \subset \mathbb{R}^d \times \mathcal{C}_0$. Due to the lack of convexity, if all particles would grow indefinite, a given particle might hit another particle at a given time and be disjoint to all other particles at a later time point. Our definition of $\tau(F; x, C)$ implies that the particle stops growing when first hitting another one. If X is a marked stationary Poisson process as above, the cumulative distribution function of $\tau = \tau\big(X; x, C\big)$ is

$$F_\tau(t) = 1 - \exp\left(-\gamma \int_{\mathcal{C}_0} \lambda\left(t \operatorname{star}(C - C')\right) \mathbb{Q}(dC')\right) \tag{2.20}$$

$t \geq 0$. Here,

$$\operatorname{star} A = \{\alpha a : 0 \leq \alpha \leq 1, a \in A\}$$

is the *star hull* of $A \subset \mathbb{R}^d$, i.e., the smallest set containing A that is star shaped with respect to the origin. Equation (2.20) is essentially a consequence of the Poisson property, and shown in Section 3.4. If $\mathbb{Q}$ is concentrated on star-shaped sets the star-operator in (2.20) can be omitted. This is in particular true, if almost all particles are convex. Again, we assume that $\mathbb{Q}$ is non-degenerate, as this assures that almost all particles grow with a finite stopping time (and hence are compact).

Proposition 2.11. *Let X be a stationary marked Poisson process intensity $\gamma > 0$ and non-degenerate mark distribution $\mathbb{Q}$ on $\mathcal{C}_0$. Then the derived Stienen model Y_S is a stationary marked point process with intensity γ and mark distribution*

$$\mathbb{Q}_S = \int_{\mathcal{C}_0} P\big(\tau(X; 0, C)C \in (\cdot)\big)\mathbb{Q}(dC) \tag{2.21}$$

on $\mathcal{C}_0$. The volume density of the union of all its particles is

$$\overline{V}_S = \int_{\mathcal{C}_0} \frac{\lambda(C)}{\int_{\mathcal{C}_0} \lambda(\operatorname{star}(C - C'))\mathbb{Q}(dC')} \mathbb{Q}(dC). \tag{2.22}$$

A comparison of (2.22) with (2.13) reveals that the Stienen model with proposal shape distribution $\mathbb{Q}$ has the same volume fraction as the dead leaves model with shape distribution $\mathbb{Q}$ when $\lambda(\operatorname{star}(C - C')) = \lambda(C - C')$ for $\mathbb{Q}^2$-almost all (C, C'). In the case where $\mathbb{Q}$ is concentrated on convex sets in $\mathcal{C}_0$, this was shown in [15]. It is interesting to observe that this appears to be the only instance in the present theory, where convexity yields a qualitatively different result compared to the general (compact) case. As we have trivially $C - C' \subset \operatorname{star}(C - C')$, the upper bound for the asymptotic volume fraction in the Matérn case in Theorem 2.6 can be transferred to the Stienen model.

Theorem 2.12. *Let X be a stationary marked Poisson process in $\mathbb{R}^d$ with intensity $\gamma > 0$ and non-degenerate mark distribution $\mathbb{Q}$. Then the volume density of the union of all particles of the derived Stienen model satisfies*

$$\overline{V}_S \leq 2^{-d}$$

with equality if and only if $\mathbb{Q}$ is concentrated on one origin-symmetric convex particle $C_0 \in \mathcal{C}_0$ with positive volume.

Finally, we remark that Proposition 2.10 has an analogue for the Stienen model: assume that the proposal shape distribution $\mathbb{Q}$ of a planar Stienen model Y_S is concentrated on scaled versions of one single origin-symmetric convex particle $C_0 \in \mathcal{C}_0$. Then its intensity γ and shape distribution $\mathbb{Q}_S$ determine the distribution of the scaling. In the language of [15], when all particles of a planar Stienen model Y_S have the same origin-symmetric convex shape but possibly different growth speeds, then Y_S determines the growth speed of the typical particle.

3 Proofs

3.1 Some auxiliary results for random variables

Here and in the following, the notation $\mathcal{B}(S)$ always denotes the Borel-σ algebra of a topological space S. We start with some general remarks on distributions P on $\mathcal{B}(\mathbb{R})$. We denote by

$$\omega_P = \sup(\operatorname{supp} P) \in (-\infty, \infty]$$

the upper endpoint of the support of P, and by

$$a_P = \sup\{x \in \mathbb{R} : P(\{x\}) > 0\} \in [-\infty, \infty]$$

the supremum of all atoms of P. Let F be the cumulative distribution function of P and let $F^{\leftarrow}$ be the generalized inverse of F defined by

$$F^{\leftarrow}(s) = \inf\{x \in \mathbb{R} : F(x) \geq s\},$$

$s \in (0,1)$; see for instance [3, pp. 37–38]. If S is uniform in $(0,1)$ then $F^{\leftarrow}(S)$ has distribution P. For $0 < s, s' < 1$ we have

$$\mathbf{1}_{F^{\leftarrow}(s) \leq F^{\leftarrow}(s')} = \mathbf{1}_{s<s'} + \mathbf{1}_{F(F^{\leftarrow}(s)-) \leq s' \leq F(F^{\leftarrow}(s))}, \tag{3.1}$$

where $F(x-) = \lim_{y \nearrow x} F(y)$, as usual. Hence,

$$\mathbf{1}_{F^{\leftarrow}(s) \leq F^{\leftarrow}(s')} = \mathbf{1}_{s<s'} \tag{3.2}$$

holds for λ-almost all $s, s' \in (0,1)$ if and only if P is atom free.

Lemma 3.1. *Let P be a distribution on $\mathcal{B}(\mathbb{R})$ and $\beta > 0$. Then*

$$\beta \int_{\mathbb{R}} e^{-\beta P([x,\infty))} P(dx) \leq 1 - e^{-\beta}. \tag{3.3}$$

Equality holds in (3.3) *if and only if P is atom free. Furthermore, if $a_P < \omega_P$, then*

$$\lim_{\beta \to \infty} \beta \int_{\mathbb{R}} e^{-\beta P([x,\infty))} P(dx) = 1. \tag{3.4}$$

Proof. Let F be the cumulative distribution function of P. Using the fact that $F^{\leftarrow}(S)$ has distribution P when S is uniform in $(0,1)$ we get

$$\begin{aligned} f(\beta) &= \beta \int_{\mathbb{R}} e^{-\beta P([t,\infty))} P(dt) \\ &= \beta \int_0^1 \exp\left(-\beta \int_0^1 \mathbf{1}_{F^{\leftarrow}(s) \le F^{\leftarrow}(s')} ds'\right) ds. \end{aligned}$$

Hence, by (3.1),

$$\begin{aligned} f(\beta) &\le \beta \int_0^1 \exp\left(-\beta \int_0^1 \mathbf{1}_{s \le s'} ds'\right) ds \\ &= 1 - e^{-\beta}, \end{aligned}$$

using elementary integration. Equality holds here if and only if P is atom free. To show (3.4), we set $\alpha_p = P((a_P, \omega_P)) > 0$ and define the atom-free probability measure

$$\tilde{P} = \alpha_P^{-1} P(\cdot \cap (a_P, \omega_P)).$$

Then,

$$f(\beta) \ge \beta \int_{a_P}^{\omega_P} e^{-\beta P([x,\infty))} P(dx) = \alpha_P \beta \int_{a_P}^{\omega_P} e^{-\alpha_P \beta \tilde{P}([x,\infty))} \tilde{P}(dx) = 1 - e^{-\alpha_P \beta},$$

using the equality case in (3.3) where β and P are replaced by $\alpha_P \beta$ and the atom-free distribution $\tilde{P}$, respectively. Together with (3.3) this shows the assertion (3.4). $\square$

Proposition 3.2. *Let R and R' two i.i.d. non-negative stochastic variables, and let $d \in \mathbb{N}$ be given. Then*

$$E_R \frac{R^d}{E_{R'}(R+R')^d} \le 2^{-d} \tag{3.5}$$

with equality if and only if $R = R'$ is a fixed positive constant.

Proof. If $E_R R = E_{R'} R' = 0$, the left hand side of (3.5) is understood to be zero, so (3.5) holds with strict inequality. Assume now that $E_R R = E_{R'} R' > 0$. The convexity of $f(x) = 1/x$ on $(0, \infty)$ implies

$$f\Big(\sum_{i=0}^d \alpha_i x_i\Big) \le \sum_{i=0}^d \alpha_i f(x_i) \tag{3.6}$$

for all positive $\alpha_0, \ldots, \alpha_d$ summing up to 1, and all $x_0, \ldots, x_d > 0$. As f is strictly convex, equality holds here if and only if $x_0 = \ldots = x_d$. For $R > 0$ we set

$$\alpha_i = 2^{-d} \binom{d}{i} \qquad \text{and} \qquad x_i = \frac{E_{R'}(R')^i}{R^i},$$

$i = 0, \ldots, d$. Then (3.6) gives

$$2^d \frac{R^d}{E_{R'}(R+R')^d} \le 2^{-d} \sum_{i=0}^d \binom{d}{i} \frac{R^i}{E_{R'}(R')^i},$$

which obviously also holds for $R = 0$. Taking expectations gives (3.5) and the characterization for the equality case. $\square$

3.2 Proofs for the generalized Matérn particle process

We use standard notions from stochastic geometry; see [22]. Let $\mathcal{C}'$ be the family of all non-empty compact subsets of $\mathbb{R}^d$, and let $\mathcal{C}_0$ be the set of all $C \in \mathcal{C}'$ with center of their circumsphere $z(C)$ at the origin. As $z(\cdot)$ is translation covariant, $\mathcal{C}_0$ contains exactly one member of the translation class of any $C \in \mathcal{C}'$. Let X be an independently marked stationary point process in $\mathbb{R}^d$ with mark space $M = \mathcal{C}_0 \times \mathbb{R}$, where the first component of the mark is the grain C at a given location and the second component is the birth time T. If the unmarked point process X^0 has intensity $\gamma > 0$, and the marks have distribution Λ on M, then $\gamma(\lambda \otimes \Lambda)$ is the intensity measure of X. The first marginal $\mathbb{Q} = \Lambda(\cdot \times \mathbb{R})$ is the shape distribution, and represents the distribution of a typical grain before thinning. As $\mathcal{C}_0$ is a measurable subset of a Polish space [22, p. 101 and Theorem 12.2.1] there exists a regular version $P_T(\cdot|C)$ of the conditional distribution of T given the shape C; see [14, Theorems 6.3 and A1.2]. It satisfies

$$\Lambda(\mathcal{A} \times A) = \int_{\mathcal{A}} P_T(A|C)\, \mathbb{Q}(dC), \tag{3.7}$$

for all $\mathcal{A} \in \mathcal{B}(\mathcal{C}_0)$ and $A \in \mathcal{B}(\mathbb{R})$.

The generalized Matérn particle process X_M is now formally given by

$$X_M = \{(x, C) : (x, C, t) \in X \text{ with } f_M(X; x, C, t) = 1\}, \tag{3.8}$$

where the thinning function f_M is defined by

$$f_M(X; x, C, t) = \prod_{(x', C', t') \in X \setminus \{(x, C, t)\}} (1 - \mathbf{1}_{(x+C)\cap(x'+C') \neq \emptyset} \mathbf{1}_{t' \geq t}). \tag{3.9}$$

This function is measurable, and as $X_M^0 \subset X^0$, X_M is a marked point process on $\mathbb{R}^d$ with mark space $\mathcal{C}_0$. The stationarity of X implies the stationarity of X_M.

If $P_T = P_T(\cdot|C)$ does not depend on C, *Lemma 2.1* is now a direct consequence of the properties of $F^{\leftarrow}$ for the cumulative distribution function F of P_T. Indeed, X has the same distribution as a marked point process, where the distribution of T is the image of the uniform distribution on $(0, 1)$ under $F^{\leftarrow}$. But the definition of X_M in (3.8) and (3.9) only depends on the birth times through comparison of pairs. Hence, (3.2) shows that one can instead directly compare marks that are uniform in $(0, 1)$ if P_T is atom free. Summarizing, if $P_T = P_T(\cdot|C)$ is atom-free, it can be replaced by the uniform distribution on $(0, 1)$ without changing the distribution of X_M.

We now show that (2.1) *is the retaining probability of a point given its marks are* (C, t). The marked point process X is a random element with values in the family $\mathcal{F}_{\mathrm{lf}}$ of all locally finite subsets of $\mathbb{R}^d \times \mathcal{C}_0 \times \mathbb{R}$. (Note that all point processes occurring here are simple, and we therefore identify these random counting measures with there supports.) We let $P^{0,C,t}$ be the Palm distribution of X. It can be interpreted as the distribution of X under the condition that $(0, C, t) \in \mathbb{R}^d \times \mathcal{C}_0 \times \mathbb{R}$ is a point of X. This typical point survives the thinning if $f_M(X; 0, C, t) = 1$, and thus its retaining probability is

$$g_M(C, t) = \int_{\mathcal{F}_{\mathrm{lf}}} f_M(F; 0, C, t) P^{0,C,t}(dF). \tag{3.10}$$

To show that (3.10) and (2.1) coincide, one can use Slivnyak's theorem [22, Theorem 3.5.9], stating that $P^{0,C,t}$ coincides with the distribution of $X \cup \{(0, C, t)\}$, [22, Theorem 3.2.4] expressing the generating functional $G_X(f) = \prod_{x\in X} f(x)$ of the Poisson process X at a measurable function f with values in $(0, 1)$ by

$$G_X(f) = \exp\left(-\gamma \int_{\mathbb{R}^d}\int_{\mathcal{C}_0}\int_{\mathbb{R}} 1 - f(x, C, t)P_T(dt|C)\mathbb{Q}(dC)\lambda(dx)\right), \tag{3.11}$$

and the finally the fact that

$$\{x' \in \mathbb{R}^d : C \cap (x' + C') \neq \emptyset\} = C - C'.$$

To prove *Proposition 2.2*, note that the definition of $g_M(C)$ simplifies to

$$g_M(C) = \int_{-\infty}^{\infty} \exp\left(-\gamma \int_{\mathcal{C}_0} \lambda(C - C')\mathbb{Q}(dC')P_T([t, \infty))\right) P_T(dt),$$

when $P_T(\cdot|C) = P_T$ is independent of C. Lemma 3.1 with $\beta = \gamma \int_{\mathcal{C}_0} \lambda(C-C')\mathbb{Q}(dC')$ and $P = P_T$ now implies the claim, as $\beta > 0$ for $\mathbb{Q}$-almost all $C \in \mathcal{C}_0$.

The following proof of *Theorem 2.3* follows closely the proofs of the known special cases, and is thus kept concise. The refined Campbell theorem states that

$$\begin{aligned} &E \sum_{(x,C,t)\in X} f(X; x, C, t) \\ &\quad = \gamma \int_{\mathbb{R}^d}\int_{\mathcal{C}_0}\int_{\mathbb{R}}\int_{\mathcal{F}_{\mathrm{lf}}} f(F + x; x, C, t)P^{0,C,t}(dF)P_T(dt|C)\mathbb{Q}(dC)\lambda(dx) \end{aligned}$$

for any measurable $f \geq 0$ on $\mathcal{F}_{\mathrm{lf}} \times \mathbb{R}^d \times \mathcal{C}_0 \times \mathbb{R}$. The intensity measure of X_M at $B \times \mathcal{A} \in \mathcal{B}(\mathbb{R}^d \times \mathcal{C}_0)$ is therefore given by

$$\begin{aligned} E\#\left(X_M \cap (B \times \mathcal{A})\right) &= E \sum_{(x,C,t)\in X} f_M(X; x, C, t)\mathbf{1}_{B\times\mathcal{A}}(x, C) \\ &= \gamma\lambda(B) \int_{\mathcal{A}}\int_{\mathbb{R}} g_M(C, t)\, P_T(dt|C)\mathbb{Q}(dC), \end{aligned}$$

where we also used the fact that f_M is invariant under simultaneous translations of its first two arguments. Choosing a set B with $\lambda(B) = 1$ and using the fact that the intensity measure of the stationary marked point process X_M is $\gamma_M(\lambda \otimes \mathbb{Q}_M)$, we get

$$\gamma_M\mathbb{Q}_M(\mathcal{A}) = \gamma \int_{\mathcal{A}}\int_{\mathbb{R}} g_M(C, t)P_T(dt|C)\mathbb{Q}(dC). \tag{3.12}$$

Note that (3.12) holds even without the Poisson assumption for X, if $g_M(C, t)$ is defined by (3.10). Relation (3.12) now implies Theorem 2.3 as the two definitions of $g_M(C, t)$ in (3.10) and (2.1) are equivalent.

We show *Theorem 2.5*. By Lemma 2.1 we may assume that the birth times are distributed uniformly in $(0, 1)$ and independent of the particles. The second factorial moment measure [22, p. 55] of X^0_M is given by

$$\Lambda^{(2)}(A \times B) = E \sum_{((x,C,t),(x',C',t'))\in X^2_{\neq}} \mathbf{1}_{(x,x')\in A\times B} f_M(X; x, C, t) f_M(X; x', C', t'),$$

$A, B \in \mathcal{B}(\mathbb{R}^d)$, where $X^2_{\neq}$ is the point process of all different marked pairs of X. The Slivnyak-Mecke formula [22, Corollary 3.2.3] gives

$$\Lambda^{(2)}(A \times B) = \gamma^2 \int_{A\times B} \int_{\mathcal{C}_0^2} \int_{[0,1]^2} Ef_M(X \cup \{(x', C', t')\}; x, C, t) \\ \times f_M(X \cup \{(x, C, t)\}; x', C', t') d(t, t') \mathbb{Q}\big(d(C, C')\big) \lambda^2(d(x, x')),$$

where we have used that

$$f_M(X \cup \{(x', C', t'), (x, C, t)\}; x, C, t) = f_M(X \cup \{(x', C', t')\}; x, C, t)$$

(and a similar identity with (x, C, t) and (x', C', t') interchanged) holds. From the above equality the density $\tilde{\rho}^{(2)}(x, x')$ of $\Lambda^{(2)}$ can be read off. As usual, the stationarity of X, and the fact that thinning only depends on the relative positions of points, imply that $\tilde{\rho}^{(2)}(x, x')$ only depends on $x - x'$ and the reduced density $\rho^{(2)}(x) = \tilde{\rho}^{(2)}(x, 0)$ is thus given by

$$\begin{aligned} \rho^{(2)}(x) &= \gamma^2 \int_{\mathcal{C}_0^2} \int_{[0,1]^2} Ef_M(X \cup \{(0, C', t')\}; x, C, t) \\ &\qquad \times f_M(X \cup \{(x, C, t)\}; 0, C', t') d(t, t') \mathbb{Q}^2\big(d(C, C')\big) \\ &= \gamma^2 \int_{\mathcal{C}_0^2} \mathbf{1}_{(x+C)\cap C' = \emptyset} \int_{[0,1]^2} Ef_M(X; x, C, t) \\ &\qquad \times f_M(X; 0, C', t') d(t, t') \mathbb{Q}^2\big(d(C, C')\big). \end{aligned} \tag{3.13}$$

By (3.11) the expectation in the last expression satisfies

$$Ef_M(X; x, C, t) f_M(X; 0, C', t') = e^{-\gamma(\alpha(1-t) + \beta(1-t') + \kappa(1 - \min\{t, t'\}))}, \tag{3.14}$$

with α, β, and κ given by (2.7), (2.8), and (2.9). Inserting (3.14) in (3.13), the integrations w.r.t. t and t' can be carried out, leading to (2.10).

We now show *Theorem 2.6* assuming that the distribution of the birth-time is atom-free and independent of the associated particle. Then Corollary 2.4 and (2.12) imply by monotone convergence

$$\begin{aligned} \overline{V}_M(\infty) &= \lim_{\gamma\to\infty} \int_{\mathcal{C}_0} \lambda(C) \frac{1 - \exp\big[-\gamma \int_{\mathcal{C}_0} \lambda(C - C') d\mathbb{Q}(C')\big]}{\int_{\mathcal{C}_0} \lambda(C - C') d\mathbb{Q}(C')} \mathbb{Q}(dC) \\ &= \int_{\mathcal{C}_0} \frac{\lambda(C)}{\int_{\mathcal{C}_0} \lambda(C - C') d\mathbb{Q}(C')} \mathbb{Q}(dC), \end{aligned} \tag{3.15}$$

which is the equality in (2.13). Using Brunn-Minkowski's inequality for compact sets (see e.g., [4, 8.1.1. Theorem]), we get

$$\int_{\mathcal{C}_0} \lambda(C - C') \mathbb{Q}(dC') \geq \int_{\mathcal{C}_0} [\lambda(C)^{1/d} + \lambda(C')^{1/d}]^d \mathbb{Q}(dC'), \tag{3.16}$$

with equality if and only if almost all C' are convex and homothetic to $-C$. Let $R \geq 0$ and $R' \geq 0$ be i.i.d. copies of the random variable $\lambda(C)^{1/d}$, where C is the typical particle (so P_R is the image measure of $\mathbb{Q}$ under $C \mapsto \lambda(C)^{1/d}$). Using this notation, we get from (3.15) and (3.16) that

$$\overline{V}_M(\infty) \leq E_R \frac{R^d}{E_{R'}(R+R')^d}.$$

In view of Proposition 3.2, the right hand side of this inequality is bounded by 2^{-d} with equality if and only if $R = R'$ is a positive constant. We conclude that $\overline{V}_M(\infty) \leq 2^{-d}$ with equality if and only if $\mathbb{Q}(\{C_0\}) = 1$ for some origin-symmetric convex set C_0 with positive volume. Theorem 2.6 is shown.

3.3 Proofs for the dead leaves model

To show *Proposition 2.8*, we use the construction described in the paragraph before this proposition and make use of the processes $Y_M(\gamma)$ and Y_{ig}. With the notations of the Proposition, $X_M(\gamma)$ is a generalized Matérn process with initial intensity γ, shape distribution $\mathbb{Q}$ and birth a time distribution that is independent and atom-free. By Lemma 2.1 its distribution is thus independent of P_T, and we may take $P_T = \mathrm{unif}[-\gamma, 0]$. This shows that $X_M(\gamma)$ has the same distribution as $Y_M(\gamma)$. Assertion 1. thus follows by setting $X'_M(\gamma) = Y_M(\gamma) \subset Y_{ig}$, and (2.14) follows if we can show

$$Y_M(\gamma) \xrightarrow{d} Y_{ig}, \tag{3.17}$$

as $\gamma \to \infty$. By [19, Theorem 6.5] this is equivalent to the pointwise convergence of the capacity functionals (or the void functionals) on all compact sets in the continuity family of Y_{ig}. Now if $\mathcal{A} \subset \mathbb{R}^d \times \mathcal{C}_0$ is compact, then, for all realizations, the events $[Y_M(\gamma) \cap \mathcal{A} = \emptyset] \searrow [Y_{ig} \cap \mathcal{A} = \emptyset]$, as $\gamma \to \infty$, and hence

$$\lim_{\gamma \to \infty} P(Y_M(\gamma) \cap \mathcal{A} = \emptyset) = P(Y_{ig} \cap \mathcal{A} = \emptyset),$$

which implies (3.17). To prove *Corollary 2.9*, it is enough to notice that $X_M(\gamma)$ and $Y_M(\gamma)$ have the same intensity and mark distribution (which is given by Corollary 2.4), and that (3.17) together with $Y_M(\gamma) \nearrow Y_{ig}$ implies convergence of these to intensity and mark distribution of Y_{ig}, respectively.

We finally show *Proposition 2.10*, and assume without loss of generality that $\lambda(C_0) = 1$. As (2.19) implies that all moments of R are finite, (2.15) implies $\gamma_{ig} > 0$. It is enough to show that the two equations in (2.18) with $k = 0$ and $k = 1$ determine ER and ER^2, as all higher order moments of R then can be determined recursively using (2.18) with $k = 2, 3, 4, \ldots$. The equations in question are, explicitly,

$$\begin{aligned} \gamma_{ig}^{-1} &= ER_{ig}^2 + 2ER_{ig}ER + ER^2, \\ \gamma_{ig}^{-1}ER &= ER_{ig}^3 + 2ER_{ig}^2 ER + ER_{ig}ER^2. \end{aligned}$$

These equations determine ER and ER^2 if and only if $2\,\mathrm{var}(R_{ig}) \neq \gamma_{ig}^{-1}$. But

$$2\,\mathrm{var}(R_{ig}) \leq 2ER_{ig}^2 = 2\gamma_{ig}^{-1} E_R \left(\frac{R^2}{E_{R'}\lambda(RC_0 + R'C_0)} \right) \leq 2\gamma_{ig}^{-1} E_R \left(\frac{R^2}{E_{R'}(R+R')^2} \right),$$

where we inserted the distribution of R_{ig} to obtain the first equality, and $RC_0 + R'C_0 \supset (R+R')C_0$ for the last inequality. Now (3.5) implies

$$2\,\mathrm{var}(R_{ig}) \le \gamma_{ig}^{-1}/2 < \gamma_{ig}^{-1}.$$

The assertion is shown.

3.4 Proofs for the generalized Stienen model

We first show (2.20). For $0 \neq x' \in \mathbb{R}^d$ and $C, C' \in \mathcal{C}_0$ we have

$$\begin{aligned} \tau(\{(x',C')\};0,C) \le t \iff& \inf\{\alpha \ge 0 : (\alpha C) \cap (x' + \alpha C') \neq \emptyset\} \le t \\ \iff& \inf\{\alpha \ge 0 : x' \in \alpha(C - C')\} \le t \\ \iff& x' \in t\,\mathrm{star}(C - C'). \end{aligned} \tag{3.18}$$

As X is stationary, and $(x,X) \mapsto \tau(X;x,C)$ is translation covariant, we have

$$P(\tau(X;x,C) > t) = P(\tau(X;0,C) > t)$$

for any fixed $x \in \mathbb{R}^d$ and $C \in \mathcal{C}_0$. As X is a Poisson process on $\mathbb{R}^d \times \mathcal{C}_0$ with intensity measure $\gamma(\lambda \otimes \mathbb{Q})$, its generating functional is explicitly known, so

$$\begin{aligned} 1 - F_\tau(t) &= E \prod_{(x',C') \in X} \left(1 - \mathbf{1}_{\tau(\{(x',C')\};0,C) \le t}\right) \\ &= \exp\left(-\gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\tau(\{(x',C')\};0,C) \le t} \lambda(dx') \mathbb{Q}(dC')\right). \end{aligned}$$

Inserting the last term of (3.18) here, gives (2.20).

We now show *Proposition 2.11.* As $\mathbb{Q}$ is non-degenerate, (2.20) implies that the marks in the definition of Y_S are almost surely compact. Clearly, Y_S is a stationary marked point process with intensity γ. For $\mathcal{A} \in \mathcal{B}(\mathcal{C}_0)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ with $\lambda(B) = 1$, its mark distribution is given by

$$\begin{aligned} \mathbb{Q}_S(\mathcal{A}) &= \frac{1}{\gamma} E \sum_{(x,C) \in Y_S, x \in B} \mathbf{1}_{\mathcal{A}}(C) \\ &= \frac{1}{\gamma} E \sum_{(x,C) \in X, x \in B} \mathbf{1}_{\mathcal{A}}\big(\tau(X;x,C)C\big) \\ &= \int_{\mathcal{C}_0} \int_B P(\tau(X;x,C)C \in \mathcal{A})\, \lambda(dx) \mathbb{Q}(dC), \end{aligned}$$

where we used the Slivnyak-Mecke formula [22, Corollary 3.2.3] in the last step. Again, by stationarity of X and the translation covariance of $(x,X) \mapsto \tau(X;x,C)$, (2.21) follows. As the particles in Y_S do not overlap, we have

$$\overline{V}_S = \gamma \int_{\mathcal{C}_0} \lambda(C) \mathbb{Q}_S(dC).$$

In view of (2.21) this implies

$$\overline{V}_S = \gamma \int_{\mathcal{C}_0} \lambda(C) E\tau^d(X;0,C)\,\mathbb{Q}(dC).$$

From (2.20), we get

$$E\tau^d(X;0,C) = d\int_0^\infty t^{d-1}(1-F_\tau(t))dt = \left(\gamma \int_{\mathcal{C}_0} \lambda(\mathrm{star}(C-C'))\mathbb{Q}(dC')\right)^{-1},$$

and substituting this in the above equality yields (2.22).

Acknowledgements

The first author is grateful for support from *The Villum Foundation* in the framework of the *Centre for Stochastic Geometry and Advanced Bioimaging*. The authors also want to thank the master student Heidi Lund Jensen for writing some of the R code for the figures.

References

[1] N.I. Akhiezer. *The Classical Moment Problem.* Hafner, New York, 1965.

[2] J. Andersson, O. Häggström, and M. Månsson. The volume fraction of a non–overlapping germ–grain model. *Electron. Commun. Probab.*, 11:78–88, 2006.

[3] S. Asmussen and P.W. Glynn. *Stochastic Simulation. Algorithms and Analysis.* Springer, New York, 2007.

[4] Yu. D. Burago and V. A. Zalgaller. *Geometric Inequalities.* Springer, Heidelberg, 1988.

[5] L. Heinrich and Z. Pawlas. Absolute regularity and Brillinger-mixing of stationary point processes. *To appear in: Lithuanian Mathematical Journal*, 2014.

[6] M.S. Hörig. *Zufällige harte Partikelsysteme.* PhD thesis, Karlsruhe Institute of Technology, 2010.

[7] M. L. Huber and R. L. Wolpert. Likelihood-based inference for Matérn type III repulsive point processes. *Advances in Applied Probability*, 41:958–977, 2009.

[8] D. Jeulin. Random structure analysis and modelling by mathematical morphology. In Spencer A.J.M., editor, *Proc. CMDS5*, pages 217–226. Balkema, Rotterdam, 1987.

[9] D. Jeulin. Morphological modeling of images by sequential random functions. *Signal Processing*, 16:403–431, 1989.

[10] D. Jeulin. *Modèles morphologiques de structures aléatoires et de changement d'échelle.* Thèse de Doctorat d'Etat, University of Caen, 1991.

[11] D. Jeulin. Random models for morphological analysis of powders. *Journal of Microscopy*, 172:13 – 21, 1993.

[12] D. Jeulin. Dead leaves models - from space tessellation to random functions. In Jeulin D, editor, *Proceedings of the Symposium on the Advances in the Theory and Applications of Random Sets, Fontainebleau, 9-11 October 1996*. World Scientific Publishing, Singapore, 1997.

[13] D. Jeulin. Random texture models for material structures. *Statistics and Computing*, 10:121–132, 2000.

[14] O. Kallenberg. *Foundations of Modern Probability*. Springer, New York, 2nd edition, 2002.

[15] M. Månsson. A connection between the volume fractions of the Stienen model and the dead leaves model. *Adv. in Appl. Probab.*, 39:41–52, 2007.

[16] M. Månsson and M. Rudemo. Random patterns of nonoverlapping convex grains. *Adv. in Appl. Probab.*, 34:718–738, 2002.

[17] B. Matérn. *Spatial Variation. Stochastic Models and their Application to some Problems in Forest Surveys and other Sampling Investigations*, volume 49. Medd. Statens Skogsforskningsinstitut, 1960.

[18] G. Mathéron. Schéma booléen séquentiel de partition aléatiore. Technical report, Res. Rep. N83 Centre de Morphologie Mathématique, École des Mines de Paris, 1968.

[19] I. Molchanov. *Theory of Random Sets*. Springer, New York, 2005.

[20] J. Møller, M. L. Huber, and R. L. Wolpert. Perfect simulation and moment properties for the Matérn type III process. *Stochastic Processes and their Applications*, 120(11):2142 – 2158, 2010.

[21] C.A. Rogers and G.C. Shephard. The difference body of a convex body. *Archiv der Mathematik*, 8:220–233, 1957.

[22] R. Schneider and W. Weil. *Stochastic Geometry and Integral Geometry*. Springer, Heidelberg, 2008.

[23] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, London, 1982.

[24] B. Simon. The classical moment problem as a self-adjoint finite difference operator. *Adv. Math.*, 137:82–203, 1998.

[25] D. Stoyan, W.S. Kendall, and J. Mecke. *Stochastic Geometry and its Applications*. Wiley, New York, 2nd edition, 1995.

[26] D. Stoyan and M. Schlather. Random sequential adsorption: Relationship to dead leaves and characterization of variability. *J. Stat. Phys.*, 100:969–979, 2000.

[27] D. Stoyan and H. Stoyan. On one of Matérn's hard-core point process models. *Mathematische Nachrichten*, 122:205–214, 1985.

[28] J. Teichmann, F. Ballani, and K.G. van den Boogaart. Generalizations of Matérn's hard-core point processes. *Spat. Stat.*, 3:33–53, 2013.