Semi-classical properties of the quantum representations of mapping class groups

PhD Dissertation

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July 2013

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Contents

Preface

This dissertation is the outcome of my PhD studies undertaken at the Centre for Quantum Geometry of Moduli Spaces (QGM) at Aarhus University, during the years 2009–2013, extending work previously presented in my combined progress report and Master's thesis [Jør11] and including results contained in the research paper [AJ12] and summarized in [Jør12].

I would like to take this chance to thank the QGM and the Department of Mathematics, Aarhus University for creating an atmosphere suited for learning at a higher level, and the University of California, Berkeley for hosting me during the autumn of 2010.

In particular, I would like to thank my PhD supervisor Jørgen Ellegaard Andersen for introducing me to the world of quantum topology and guiding me in the maze of modern mathematics, and for his patience in helping me out with the many questions that have arised in the process. Indeed, all the results that are to be described in this thesis should be attributed as much, if not more, to him as to myself.

Moreover, I want to thank several of my coworkers at the QGM for stimulating conversations, many of which have inspired the contents of this dissertation. Among these, I want to mention Johan Martens, Brendan McLellan, Niels Leth Gammelgaard, Shehryar Sikander, and Jens Kristian Egsgaard in particular for discussions leading me to many insights in the mathematical fields concerned with my study.

Finally, I want to thank Jens Kristian Egsgaard, Jens-Jakob Kratmann Nissen, and Stefan Hansen for carefully proofreading a draft of this dissertation on a very short notice[1].

Aarhus, July 2013 Søren Fuglede Jørgensen

[1] As, inevitably, some typos will have gone unnoticed, an up to date version of the manuscript will be available at `http://maths.fuglede.dk`. You are currently reading the version compiled July 31, 2013.

Resume

Teorien diskuteret i denne afhandling går tilbage til Edward Wittens studium fra 1989 af Chern-Simons-teori som en $(2+1)$-dimensional kvantefeltteori [Wit89]: Lad $G = \mathrm{SU}(N)$, lad M være en kompakt orienteret 3-mangfoldighed indeholdende en orienteret lænke L og lad $P \to M$ være et principalbundt med strukturgruppe G. Lad $\mathcal{A}_P$ betegne rummet af konnektioner i $P \to M$, og lad $\mathcal{G}_P$ betegne rummet af gaugetransformationer. Virkningsfunktionalet i Chern-Simons-teori definerer da en afbildning $\mathrm{CS} : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}/\mathbb{Z}$.

Lad ydermere $k \in \mathbb{Z}_{>0}$ og vælg for hver komponent L_i af L en endelig-dimensional repræsentation R_i af G. Lad $\mathrm{hol}_A(L_i) \in G$ betegne holonomien af en konnektion $A \in \mathcal{A}_P$ langs L_i. Witten argumenterede da for, at stiintegralet

$$Z^{\mathrm{phys}}_{k,G}(M, L, R) = \int_{\mathcal{A}_P/\mathcal{G}_P} \prod_i \mathrm{tr}(R_i(\mathrm{hol}_{[A]}(L_i))) \exp(2\pi i k \mathrm{CS}([A]))\, \mathcal{D}A,$$

også kendt som partitionsfunktionen for Wittens Chern-Simons-teori, definerer en topologisk invariant af parret (M, L), og at denne invariant ydermere udvider til en topologisk kvantefeltteori (TQFT), hvilket kort sagt vil sige, at hvis $M = M_1 \cup_\Sigma M_2$ er en opdeling af M i mangfoldigheder M_1 og M_2 langs en flade Σ, da kan $Z_k(M, L)$ forstås ud fra data $Z_k(M_1, L \cap M_1, R)$, $Z_k(M_2, L \cap M_2, R)$ og data $V_k(\Sigma, L \cap \Sigma, R)$ associeret til fladen Σ.

Stiintegralet $Z^{\mathrm{phys}}_{k,G}(M, L, R)$ er imidlertid ikke defineret: Der eksisterer på nuværende tidspunkt ingen metode til på naturlig vis at knytte et mål $\mathcal{D}A$ til det uendelig-dimensionale rum $\mathcal{A}_P/\mathcal{G}_P$. Ikke desto mindre lykkedes det i begyndelsen af 1990'erne Resjetikhin og Turaev [RT90], [RT91] (se også [TW93], [Tur10]) ved brug af repræsentationsteorien for kvantegrupper at definere topologiske kvantefeltteorier (Z^G_k, V^G_k) med de ønskede egenskaber som foreskrevet af Wittens kvanteteori. Dette faktum står i stærk kontrast til de fleste andre andre gaugeteorier og gør Chern-Simons-teori særligt interessant fra et matematisk fysisk synspunkt.

Nærværende afhandling omfatter et studium af den to-dimensionale del af denne matematiske teori; aksiomerne for en $(2+1)$-dimensional TQFT sikrer for hvert k eksistensen af en projektiv repræsentation – kaldet kvanterepræsentationen – af afbildningsklassegruppen af enhver kompakt flade med passende randbetingelser. Størstedelen af afhandlingens matematiske substans vedrører to specifikke problemer. På den ene side undersøger vi asymptotic expansion-formodningen (formodningen om asymptotiske ekspansioner) [And02, Sect. 7.2] [AH12], der kan ses som et realitetstjek for kvantetopologi: Til trods for at stiintegralet $Z^{\mathrm{phys}}_{k,G}$ ikke er defineret, er det muligt heuristisk at deducere en matematisk formodning for dets opførsel for høje værdier af k; en af grundene til dette er, at rummet af klassiske løsninger i Chern-Simons-teori udgør et endelig-dimensionalt rum: Modulirummet af flade G-konnektioner på M. Dette faktum vil ydermere betyde, at geometrisk kvantisering spiller en central rolle i studiet af TQFT. Den resulterende asymptotiske opførsel kan da sammenholdes med den tilsvarende opførsel for de matematisk veldefinerede invarianter, hvilket giver anledning til asymptotic expansion-formodningen. En præcis formulering kan findes i Conjecture 5.2.

På den anden side diskuterer vi AMU-formodningen [AMU06], der siger, at kvanterepræsentationerne indeholder information om dynamikken af afbildningsklassegruppens virkning på fladen. Idet den matematiske definition af TQFT er baseret på Jonespolynomiet fra knudeteori, kan denne formodning ses som en pendant til spørgsmålet om, hvilken geometrisk information Jonespolynomiet indeholder (sammenlign dette med Volumenformodningen). Se Conjecture 6.1 for den præcise formulering.

En kapiteloversigt, samt en oversigt over de primære nye resultater indeholdt i denne afhandling, er givet i den engelsksprogede introduktion, der følger.

Introduction

The theory discussed in this dissertation may be traced back to Edward Witten's 1989 study of Chern–Simons theory as a $(2+1)$-dimensional quantum field theory, [Wit89]. Let $G = \mathrm{SU}(N)$, let M be a compact oriented 3-manifold containing an oriented link L, and let $P \to M$ be a principal G-bundle. Let $\mathcal{A}_P$ denote the space of connections in $P \to M$, and let $\mathcal{G}_P$ denote the space of gauge transformations. The action functional of Chern–Simons theory then defines a map $\mathrm{CS} : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}/\mathbb{Z}$.

Let moreover $k \in \mathbb{Z}_{>0}$ and choose for every component L_i of L a finite-dimensional representation R_i of G. Let $\mathrm{hol}_A(L_i) \in G$ denote the holonomy of a connection $A \in \mathcal{A}_P$ along L_i. Witten argued that the path integral

$$Z^{\mathrm{phys}}_{k,G}(M,L,R) = \int_{\mathcal{A}_P/\mathcal{G}_P} \prod_i \mathrm{tr}(R_i(\mathrm{hol}_{[A]}(L_i))) \exp(2\pi i k \mathrm{CS}([A]))\, \mathcal{D}A,$$

known as the partition function of quantum Chern–Simons theory, defines a topological invariant of the pair (M,L) and that, moreover, this invariant extends to a topological quantum field theory (TQFT), which roughly means that if $M = M_1 \cup_\Sigma M_2$ is a union of two 3-manifolds along a surface Σ, then $Z_k(M,L)$ is described in terms of $Z_k(M, L \cap M_1)$, $Z_k(M_2, L \cap M_2)$ and boundary data $V_k(\Sigma, L \cap \Sigma)$ associated to the surface Σ.

The path integral however is not defined: at the time of writing there is no method of associating in a natural way a measure $\mathcal{D}A$ to the infinite-dimensional space $\mathcal{A}_P/\mathcal{G}_P$. Nonetheless, after a few years, Reshetikhin and Turaev [RT90], [RT91] (see also [TW93], [Tur10]) defined, using the representation theory of quantum groups, topological quantum field theories (Z_k^G, V_k^G) with the properties prescribed by Witten's quantum theory. This fact contrasts what is the case for most other gauge theories and makes Chern–Simons theory particularly interesting from the point of view of mathematical physics.

The dissertation at hand consists of a study of the two-dimensional part of this mathematical theory; the axioms of a $(2+1)$-dimensional TQFT ensures for each k the existence of a projective representation – known as the quantum representation – of the mapping class group of any compact surface with suitable boundary conditions. The main part of the dissertation will revolve around two particular problems. On one hand we will investigate the Asymptotic Expansion Conjecture (AEC) [And02, Sect. 7.2] [AH12] which may be viewed as a reality check for quantum topology: even though the path integral $Z^{\mathrm{phys}}_{k,G}$ is not defined, it is possible to deduce through heuristic means a mathematical conjecture for its behaviour for large values of k; one of the reasons for this is that the space of classical solutions of Chern–Simons theory forms a finite dimensional space: the moduli space of flat G-connections on M. This fact furthermore means that geometric quantization plays a key role in the study of TQFT. The asymptotic behaviour resulting from this study may then be compared with that of the mathematically well-defined invariants; the AEC makes this precise. See Conjecture 5.2 for the precise statement.

On the other hand, we discuss the AMU conjecture [AMU06] which states that the quantum representations contain information about the dynamics of the mapping class group action on the underlying surface. As the mathematical definition of TQFT is based on the Jones polynomial of knot theory, this can be viewed as a counterpart to the problem of describing the geometrical meaning of the Jones polynomial (compare with the Volume Conjecture). See Conjecture 6.1 for the precise statement.

The dissertation is structured as follows: it is split into six chapters, the first three of which contain all relevant background material. The mathematical contents of these should be well-known to most experts of the field but is included to set up notation, to ease reference, and also to provide a more gentle introduction for newcomers to the field (but knowledge about symplectic geometry, the geometry of vector bundles, and knot theory will be assumed). In particular, the level of references is kept high through-out, hopefully allowing anyone interested to obtain a better overview of the currently available literature. The last three chapters contain the results obtained during my study. The following is a summary of the contents of the individual chapters.

In **Chapter 1**, we start off gently, introducing the fundamentals about mapping class groups of compact surfaces. The Dehn–Lickorish Theorem tells us that these are generated by particular mapping classes known as Dehn twists; these will be of particular importance in Chapter 5, and so we will describe these and relations between them in some detail. The chapter ends with the trichotomy of mapping classes given by the Nielsen–Thurston classification which will play a central role in our discussion of the AMU conjecture in Chapter 6.

We begin **Chapter 2** by giving a brief outline of gauge theory and the study of connections in principal bundles over manifolds, mainly in order to fix the notation used in later sections. We then turn to the concept of geometric quantization and discuss in particular the part of the process known as prequantization. Whereas prequantization can be discussed in the general framework of symplectic geometry, we will be focussing primarily on the example of the moduli space of flat $\mathrm{SU}(N)$-connections in a principal bundle over a surface and explain how this space is prequantizable by bridging its construction to Chern–Simons theory.

In **Chapter 3**, we first recall the setup of quantum Chern–Simons theory, thus placing our study in the appropriate framework of mathematical physics. We go on to describe mathematically the construction of a topological quantum field theory in the language of modular categories and modular functors. Whereas Reshetikhin and Turaev, as mentioned before, gave the first construction hereof, for our purposes it will be beneficial to also have at our disposal the skein theoretical construction of TQFT due to Blanchet, Habegger, Masbaum and Vogel [BHMV95]. Essentially equivalent to the construction of Reshetikhin and Turaev with gauge group $G = \mathrm{SU}(2)$, this construction has several computational advantages. Finally, we turn to a construction of a modular category by Blanchet [Bla00], using the skein theory of the HOMFLYPT polynomial and equivalent to the quantum group construction for $G = \mathrm{SU}(N)$.

In **Chapter 4**, we study the quantum representations of mapping class groups, building on the constructions of Chapter 3 and relating them to similar representations arising from geometric quantization. The case of the mapping class group of the torus is discussed in detail and related to representations of the modular group arising in conformal field theory, see Sections 4.3.2–4.3.3. Moreover, we discuss the algebraic properties of the quantum representations, their kernels, and their images, largely guided by computer experimentation made possible by the algorithmic approach to calculation ensured by the skein theoretical construction. As an example of the power of the computational approach, we extend, ever so slightly, a result by Masbaum, and show that for $g \geq 2$, there are mapping classes of the genus g surface, whose images under the level k quantum representations have infinite order, when $k \neq 1, 2, 4$ (see Proposition 4.16).

Chapter 5 is concerned with the Asymptotic Expansion Conjecture and contains the majority of the new results of this thesis. After first setting the scene with a derivation of the Semiclassical Approximation Conjecture underlying the AEC, we state the AEC and mention its applications for distinguishing 3-manifolds. We then set out to prove the AEC for mapping tori; these fit naturally with the 2-dimensional theme of the dissertation, as their quantum invariants are described by the characters of the quantum representations. The main results in this direction are computations of the quantum representations of mapping tori of all reducible torus homeomorphisms for $G = \mathrm{SU}(2)$ (see Theorem 5.15), and those of trace 2 for $G = \mathrm{SU}(3)$ (see Theorem 5.24). For the latter manifolds, we are able to determine all possible values of the $\mathrm{SU}(N)$-Chern–Simons action functional (see Proposition 5.28), thereby proving the AEC for the before-mentioned manifolds (Corollary 5.20 and Corollary 5.30) for $G = \mathrm{SU}(2)$, and $G = \mathrm{SU}(3)$. In the case of $G = \mathrm{SU}(2)$, this extends an earlier result by Jeffrey, and we collect known results on the AEC for torus bundles in Section 5.3.4. Moreover, in the case $G = \mathrm{SU}(2)$, we further analyze the moduli spaces of flat connections on the 3-manifolds and show that the growth rates of the quantum invariants are in accordance with that expected from the path integral point of view (see Theorem 5.22). These results are all contained in a joint paper with Andersen, [AJ12]. In Section 5.5 we discuss how one might apply the methods of the previous sections to mapping tori of Dehn twist and mention a few partial results in this direction, leaving the filling in of details as future work. Finally, in Section 5.6 we discuss how to generalize our results in a different direction by giving an extension of the AEC to the case where the 3-manifolds are allowed to contain links; part of the material in this section is joint work with – besides Jørgen Ellegaard Andersen – also Benjamin Himpel, Johan Martens, and Brendan McLellan.

Finally, **Chapter 6** deals with the AMU conjecture. This chapter has a slightly different nature than the previous chapters and revolves largely around conjecture building: motivated by a number of observations surrounding the quantum invariants of torus mapping tori, we pose different possible approaches to the general AMU conjecture. In particular, we pose a conjecture (see Conjecture 6.6) for the relation between quantum invariants of 3-manifolds and stretch factors of pseudo-Anosov homeomorphisms. These approaches are then discussed in Sections 6.2 and 6.3, the main upshot being Theorem 6.8 which determines the leading order asymptotics of the quantum representation characters for those mapping classes whose induced moduli space action has isolated fixed points. These results will appear in a paper joint with Andersen. The dissertation is rounded off in Section 6.4 with a discussion – which came about through conversation with Jens Kristian Egsgaard – of the relation between the quantum representations and well-known Jones representations of braid groups with a particular focus on possibly using these relations to generalize the original argument of [AMU06] to spheres with several punctures; as an illutration, we prove the AMU conjecture for a large number of mapping classes of a six-punctured sphere (see Theorem 6.32).

Chapter 1
Mapping class groups

1.1 Definition and preliminary remarks

The main algebraic object under scrutiny in this dissertation is the mapping class group of a surface. Very roughly, one thinks of the mapping class group as the group of symmetries of a given surface. Let $\Sigma = \Sigma_{g,n}$ be a compact surface of genus $g \geq 0$ with $n \geq 0$ boundary components. Let $\mathrm{Homeo}(\Sigma, \partial\Sigma)$ be the group of orientation-preserving homeomorphisms restricting to the identity on $\partial\Sigma$, and let $\mathrm{Homeo}_0(\Sigma, \partial\Sigma)$ denote the normal subgroup of those homeomorphisms that are isotopic (i.e. homotopic through homeomorphisms relative to the boundary) to the identity. The *mapping class group* of Σ is the quotient

$$\Gamma(\Sigma) = \mathrm{Homeo}(\Sigma, \partial\Sigma)/\mathrm{Homeo}_0(\Sigma, \partial\Sigma),$$

or, equivalently, $\Gamma(\Sigma) = \pi_0(\mathrm{Homeo}(\Sigma, \partial\Sigma))$. The class of a homeomorphism in $\Gamma(\Sigma)$ is called its *mapping class*. Obviously, homeomorphic surfaces have isomorphic mapping class groups, and we will often simply write $\Gamma_{g,n} = \Gamma(\Sigma)$. Also, we will write $\Gamma_g = \Gamma_{g,0}$.

Several variations on this theme exist. It is common to define the mapping class group of a surface as the group $\pi_0(\mathrm{Diff}(\Sigma, \partial\Sigma))$ of orientation-preserving diffeomorphisms of Σ rather than homeomorphisms. It is a non-trivial fact (see [FM11, Sect. 1.4]) that any homeomorphism is isotopic to a diffeomorphism, and that isotopy can be replaced by smooth isotopy, so we obtain an isomorphic group, and we will use the two interchangeably.

Occasionally, we will be considering surfaces $\Sigma_{g,n}^m$ with $m \geq 0$ punctures, i.e. m points removed from the interior of the surface, and consider homeomorphisms of the resulting non-compact surface. Equivalently, one could consider the surface with a set of $m \geq 0$ marked points, and require that homeomorphisms and isotopies fix this set. The resulting mapping class group will be denoted $\Gamma_{g,n}^m$. Similarly, some definitions ease the condition on the behaviour on the boundary and consider instead homeomorphisms and isotopies preserving the boundary setwise rather than pointwise.

1.2 Examples and generators

A guiding example in what follows will be the closed torus Σ_1. Homeomorphisms of the torus act by determinant 1 automorphisms on the first homology $H_1(\Sigma_1, \mathbb{Z}) \cong \mathbb{Z}^2$ of the torus. In fact, any element M of $\mathrm{SL}(2, \mathbb{Z})$ defines a homeomorphism of the torus, viewed as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, whose action on homology is exactly M. Likewise, it follows from $K(G, 1)$ theory, that any such homomorphism arises from a (based) map on the torus, unique up to homotopy. We thus obtain the following (and refer to [FM11, Thm. 2.5] for the details).

Theorem 1.1. *The homomorphism $\Gamma_1 \to \mathrm{SL}(2,\mathbb{Z})$ given by the action on homology is an isomorphism.*

Remark 1.2. It is an interesting open question whether or not general mapping class groups are linear, i.e. admit injective representations, as in the torus case. The case of Γ_2 was handled in [BB01], where an explicit 64-dimensional representation is constructed, but the same question for higher genus closed surfaces is still open.

1.2.1 Dehn twists

The most important class of examples of mapping classes for our purposes are the Dehn twists about simple closed curves, which we intuitively think of as obtained by cutting the surface along a curve, giving one of the resulting boundary components a 2π left twist, and gluing the boundary components back together (see Figure 1.1 for a Dehn twist on the closed torus). More precisely, consider the annulus $A = S^1 \times [0,1]$ considered as an oriented surface in $\mathbb{R}^2$ via the map $(\theta, r) \mapsto (\theta, r+1)$ with the orientation induced by the orientation of $\mathbb{R}^2$. Define a map $t : A \to A$ by

$$t(\theta, r) = (\theta + 2\pi r, r),$$

as illustrated in Figure 1.2. Now let γ be a simple closed curve in an oriented surface Σ, and let N be a regular neighbourhood of γ. Choose an orientation-preserving homeomorphism $\varphi : A \to N$, and define the *Dehn twist about* γ, denoted $t_\gamma : \Sigma \to \Sigma$, by $t_\gamma = \varphi \circ t \circ \varphi^{-1}$ on N, and $t_\gamma = \mathrm{id}$ on $\Sigma \setminus N$. This defines an orientation-preserving homeomorphism on Σ. The mapping class of t_γ depends neither on the choice of regular neighbourhood, nor the homeomorphism φ. Furthermore, the mapping class is determined by the isotopy class of γ. If a is the isotopy class of γ, we write t_a for the resulting mapping class. We will often make a slight abuse of notation, writing t_γ for the mapping class as well.

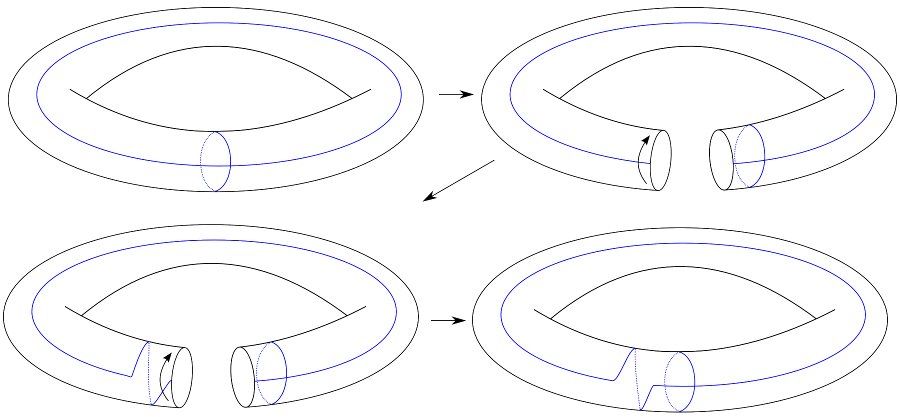


Figure 1.1: The action on two simple closed curves in a torus of the Dehn twist about a meridian.

Remark 1.3. What we defined above is really a *left Dehn twist*. Similarly, one could have used the map $t : A \to A$ given by $t(\theta, r) = (\theta - 2\pi r, r)$ to obtain instead a *right Dehn twist*. The mapping class of the resulting homeomorphism would be the inverse to the one obtained above.

The importance of Dehn twists stems from the fact that they generate the mapping class groups. Before discussing exactly how, we note some of their algebraic properties. In the following, let Σ be any surface. The *intersection number*, denoted $i(a, b)$, between two isotopy classes of curves a and b in Σ, is the minimal number of intersections between representative curves. A simple closed curve curve in Σ is called *essential*, if it is not

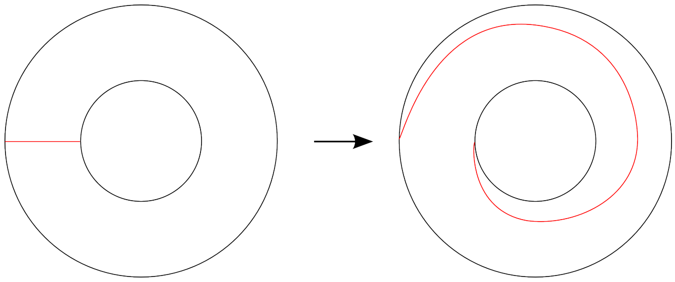


Figure 1.2: The action of the twist map $t : A \to A$ on a horizontal line in the annulus.

homotopic to a point, a boundary component or a marked point. We will need the following non-trivial fact. For a proof, see [FM11, Prop. 3.2].

Proposition 1.4. *Let a and b be isotopy classes of essential closed curves, and let $k \in \mathbb{Z}$. Then $i(t_a^k(b), b) = |k| i(a, b)^2$.*

Note that a Dehn twist about a simple closed curve homotopic to a point is trivial in the mapping class group. In general, however, Dehn twists are non-trivial:

Corollary 1.5. *Let a be the isotopy class of a simple closed curve α not homotopic to a point or a puncture in Σ. Then t_a has infinite order.*

Proof. By Proposition 1.4, it is enough to find an isotopy class b, such that $i(a, b) > 0$. Assume first that Σ has no boundary components. Then this is possible by the so-called change of coordinates principle: it follows from the classification of surfaces that there is an orientation-preserving homeomorphism of Σ taking one simple closed curve to another if and only if the two results of cutting the surface along the two curves will be homeomorphic surfaces. In other words, up to homeomorphism there is only one non-separating curve and finitely many separating ones, and we may assume that α is one of the curves in Fig. 1.3 (the separating curve might of course enclose more holes, punctures or boundary components). In both cases, the existence of the isotopy class b is obvious. In the case where Σ has boundary, using the same method as above, it remains to prove that Dehn twists about boundary components have infinite order – this is proven by a similar argument. □

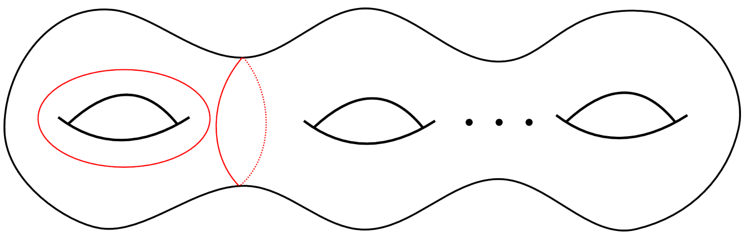


Figure 1.3: Using the change of coordinate principle to simplify α.

In the following, let Γ be the mapping class group of the surface Σ, and let a and b denote isotopy classes of simple closed curves in Σ.

Lemma 1.6. *If $t_a = t_b$, then $a = b$.*

Proof. Assume that $a \neq b$. Using the change of coordinates principle as above, considering all the various cases, it is possible to find an isotopy class c, such that $i(a, c) = 0$, $i(b, c) \neq 0$. By Proposition 1.4,

$$i(t_a(c), c) = i(a, c)^2 = 0 \neq i(b, c)^2 = i(t_b(c), c),$$

so $t_a(c) \neq t_b(c)$, and $t_a \neq t_b$. □

Lemma 1.7. *For $f \in \Gamma(\Sigma)$, we have $t_{f(a)} = f t_a f^{-1}$.*

Note here that when writing a product of mapping classes, we always apply them from right to left.

Proof. Let φ be a representative of f, and let γ a representative of a. Then φ^{-1} takes a regular neighbourhood of $\varphi(\gamma)$ to a regular neighbourhood of γ. Using this neighbourhood to define t_γ, we obtain $t_{\varphi(\gamma)} = \varphi t_\gamma \varphi^{-1}$. □

Lemma 1.8. *Dehn twists about two simple closed curves commute if and only if the isotopy classes of the curves have zero intersection number.*

Proof. That Dehn twists about non-intersecting curves commute is obvious. It follows from Lemma 1.6 and Lemma 1.7 that a given mapping class f commutes with a Dehn twist t_a, if and only if f fixes a. Thus, if $t_a t_b = t_b t_a$ for isotopy classes a and b of simple closed curves, we obtain $t_a(b) = b$, and by Proposition 1.4 $i(a,b)^2 = i(t_a(b), b) = 0$. □

Lemma 1.9 (Braid relation)**.** *If $i(a,b) = 1$ for isotopy classes of simple closed curves a and b, then $t_a t_b t_a = t_b t_a t_b$.*

Proof. We prove first that $t_a t_b(a) = b$. By using the change of coordinates principle, we assume that a and b are represented by curves α and β as in Figure 1.4 for which the equation is seen to hold by the sequence of mappings in the figure. It follows that $t_{t_a t_b(a)} = t_b$, and from Lemma 1.7, we obtain $(t_a t_b) t_a (t_a t_b)^{-1} = t_b$. □

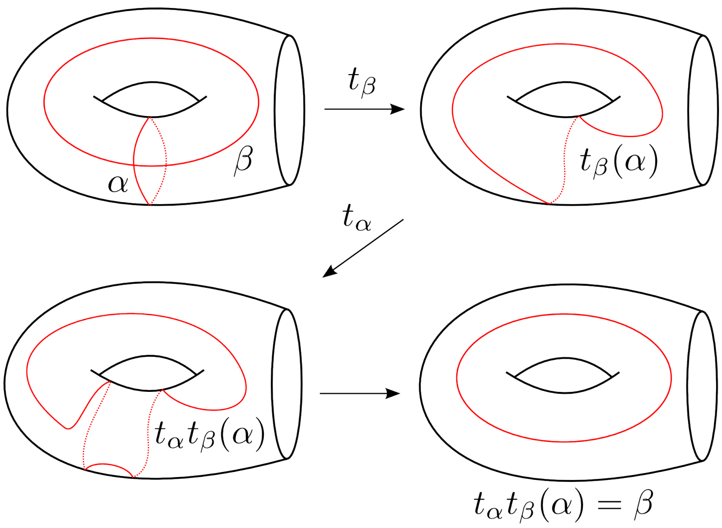


Figure 1.4: The curves α and β, and the equation $t_\alpha t_\beta(\alpha) = \beta$. The last arrow is a simple isotopy.

We consider now the effect on the mapping class group of a surface when adding boundary components to it (see [FM11, Sect. 3.6.2]). When Σ is a topologically closed subsurface of Σ', we define a homomorphism $\Gamma(\Sigma) \to \Gamma(\Sigma')$ as follows: given $\varphi \in \mathrm{Homeo}(\Sigma, \partial\Sigma)$ representing a mapping class $f \in \Gamma(\Sigma)$, we extend φ to a homeomorphism $\varphi \in \mathrm{Homeo}(\Sigma', \partial\Sigma')$ by letting it act identically on $\Sigma' \setminus \Sigma$. The induced map $\Gamma(\Sigma) \to \Gamma(\Sigma')$ is well-defined.

In the special case where Σ' is obtained from Σ by *capping* a boundary component, that is, $\Sigma' \setminus \Sigma$ is a once punctured disk with a boundary curve β, the resulting homomorphism $\Gamma(\Sigma) \to \Gamma(\Sigma')$ fits into a short exact sequence

$$1 \to \langle t_\beta \rangle \to \Gamma(\Sigma) \to \Gamma(\Sigma') \to 1. \tag{1.1}$$

As an example, we consider again the torus. Using the so-called Alexander trick, one can prove that the mapping class group of the once punctured torus $\Sigma^1_{1,0}$ is once again given by its action on homology, so $\Gamma(\Sigma^1_{1,0}) \cong \mathrm{SL}(2,\mathbb{Z})$. As above, the mapping class group of the torus with one boundary component $\Sigma_{1,1}$ thus fits into the exact sequence

$$1 \to \mathbb{Z} \to \Gamma(\Sigma_{1,1}) \to \Gamma(\Sigma^1_{1,0}) \to 1.$$

We can describe this mapping class group as follows: recall that $\mathrm{SL}(2,\mathbb{Z})$ has a presentation $\mathrm{SL}(2,\mathbb{Z}) \cong \langle a, b \mid aba = bab, (ab)^6 = \mathrm{id}\rangle$, explicitly given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto a, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mapsto b.$$

Consider the braid group on 3 strands, $B_3 \cong \langle a, b \mid aba = bab\rangle$. From the presentations, we get a homomorphism $B_3 \to \mathrm{SL}(2,\mathbb{Z})$ with kernel $\langle (ab)^6 \rangle \cong \mathbb{Z}$, and maps $\mathrm{SL}(2,\mathbb{Z}) \to \Gamma(\Sigma_{1,0})$, $\mathrm{SL}(2,\mathbb{Z}) \to \Gamma(\Sigma^1_{1,0})$, and $B_3 \to \Gamma(\Sigma_{1,1})$ given by mapping the generators a, b to Dehn twists about meridian and longitude curves respectively. These fit into the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & B_3 \arrow[r] \arrow[d] & \mathrm{SL}(2,\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & \mathbb{Z} \arrow[r] & \Gamma(\Sigma_{1,1}) \arrow[r] & \Gamma(\Sigma^1_{1,0}) \arrow[r] & 1. \end{tikzcd}$$

Applying the five-lemma, we obtain the following result:

Proposition 1.10. *The mapping class group of a torus with one boundary component is* $\Gamma(\Sigma_{1,1}) \cong B_3$.

1.2.2 Generators of mapping class groups

In the examples above, we saw that the mapping class groups could be generated by particular Dehn twists in the case of the closed torus, the once punctured torus and the torus with one boundary component. The case of the closed torus is a special case of the Dehn–Lickorish theorem (sometimes also called the Lickorish twist theorem).

Theorem 1.11 (Dehn–Lickorish)**.** *For $g \geq 1$, the group Γ_g is generated by $3g-1$ Dehn twists about non-separating simple closed curves (see Figure 1.5).*

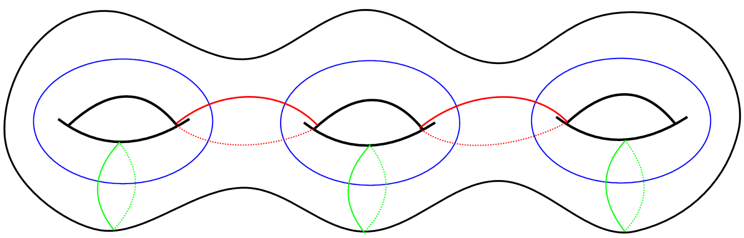


Figure 1.5: The $3g-1$ curves appearing in the Dehn–Lickorish theorem in the case $g = 3$.

In fact, Humphries [Hum79] has proved that the minimal (and realizable) number of Dehn twists required to generate Γ_g, $g > 1$, is $2g+1$. In the case where the surface has boundary components, the picture changes but we remark that Dehn twists still generate the mapping class group (see [FM11, Sect. 4.4.4]).

Since we will need it later, we note that Γ_2 has the following presentation, due to Birman and Hilden, [BH73]:

$$\begin{aligned}\Gamma_2 \cong \langle a_1, \dots, a_5 \mid a_i a_j &= a_j a_i, \;\; |i-j| \geq 2, \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1}, \\ (a_1 a_2 a_3 a_4 a_5)^6 &= 1, \\ (a_1 \dots a_5 a_5 \dots a_1)^2 &= 1, \\ [a_1 \dots a_5 a_5 \dots a_1, a_1] &= 1\rangle.\end{aligned} \tag{1.2}$$

As in the torus case, we can realize the generators $a_1, \dots, a_5$ as Dehn twists about the five curves shown in Figure 1.6.

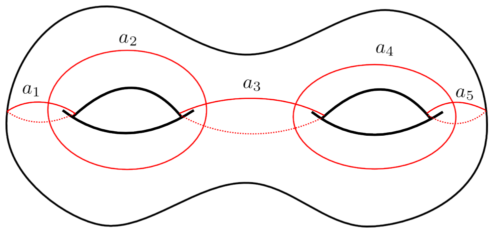


Figure 1.6: The Dehn twists generating Γ_2.

1.2.3 Finite order elements

Whereas non-trivial Dehn twists have infinite order in the mapping class group, we will also need to discuss finite order mapping classes. In the case of the torus, $\Gamma_1 \cong \mathrm{SL}(2,\mathbb{Z})$, and examples of order 2, 3, 4, and 6 are

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

respectively. An example of an order 2 mapping class in Γ_2 is given by the element $a_1 \dots a_5 a_5 \dots a_1$ described above. More generally, the mapping class of the homeomorphism obtained by an angle π rotation about the axis shown in Figure 1.7 has order 2 and is called *hyperelliptic involution* $\iota = \iota_g$. In fact, hyperelliptic involutions are the only possible central

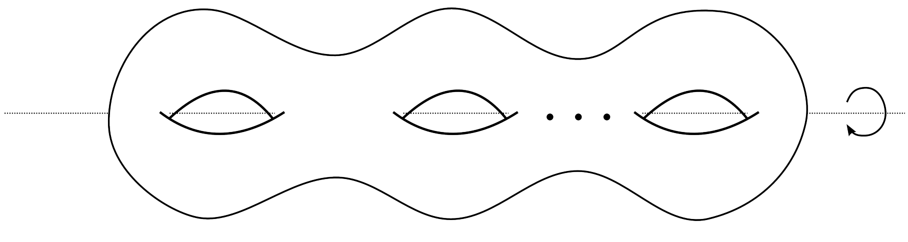


Figure 1.7: The hyperelliptic involution as a rotation of a surface.

elements of mapping class groups of closed surfaces (see [FM11, Sect. 3.4]).

Theorem 1.12. *The center $Z(\Gamma_g)$ of Γ_g is isomorphic to $\mathbb{Z}_2$ for $g = 1, 2$ and trivial otherwise.*

While we will not go into all details of the proof of this, one way of understanding this is as follows: as in the proof of Lemma 1.8, any central element will fix the isotopy class of every simple closed curve. A combinatorial argument using the so-called Alexander method shows that no non-trivial elements can do this when $g \geq 3$ and leaves ι_1 and ι_2 as the only possible central elements in Γ_1 and Γ_2. To prove that these two elements are in fact central, it suffices to check that they fix the isotopy classes of the curves giving rise to the Dehn twists used to generate the respective mapping class groups in Theorem 1.11.

This theorem is no longer true when the surface is not closed. For example, the Dehn twist about a boundary component will always be central.

1.3 The Nielsen–Thurston classification

The examples considered above are in some sense the simplest. Finite order mapping classes can be realized by homeomorphisms of finite order, and while Dehn twists have infinite order, they will still have a simple action on certain curves (namely they fix the isotopy class of the curve used to define them). We end this chapter with a brief discussion of the Nielsen–Thurston classification and in particular we discuss the notion of a pseudo-Anosov homeomorphism. One of the main goals of the project at hand will be to analyze the behaviour of quantum representations under the trichotomy of the classification.

To define a pseudo-Anosov homeomorphism we need the notion of a transverse measure in a singular foliation.

Definition 1.13. A *singular foliation* on a surface Σ is a decomposition of the surface into a disjoint union of leaves such that all but finitely many singular points in Σ will have smooth charts $U \to \mathbb{R}^2$ taking leaves to horizontal lines. The singular points have smooth charts taking leaves to k-prong singularities as in Figure 1.8. Punctures are allowed to have 1-prong singularities as in Figure 1.9. We require also that every boundary component has at least one singularity, and that boundary components are unions of leaves connecting the singularities. Two singular foliations are called *transverse* if they have the same singular points and have transverse leaves at all other points (see Figure 1.10).

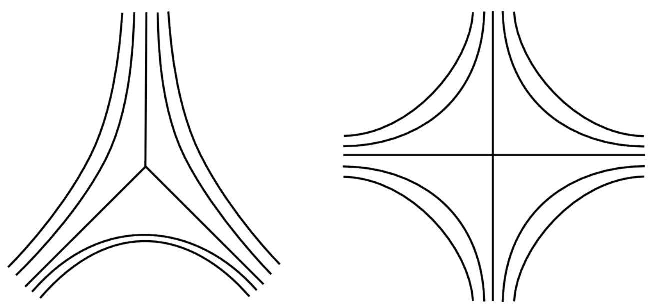


Figure 1.8: The k-prong singularities for $k = 3, 4$.

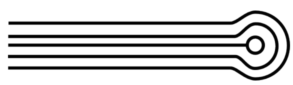


Figure 1.9: The 1-prong singularity for a punctured surface.

Definition 1.14. Let $\mathcal{F}$ be a singular foliation. A smooth arc α is called *transverse* to $\mathcal{F}$, if it misses all singular points and is transverse to the leaves. A *transverse measure* μ in a singular foliation $\mathcal{F}$ defines on every arc transverse to $\mathcal{F}$ a (non-negative) Borel measure $\mu(\alpha)$ such that:

1. If β is a subarc of α, then $\mu(\beta)$ is the restriction of $\mu(\alpha)$ to β.

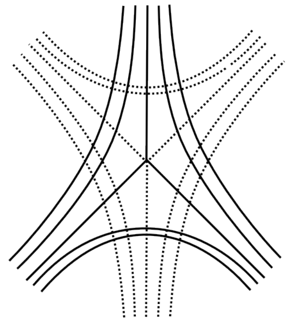


Figure 1.10: A pair of transverse singular foliations.

2. If two arcs $\alpha_0, \alpha_1 : I \to \Sigma$ are related by a homotopy $H : I \times I \to \Sigma$ such that $H(I \times \{0\}) = \alpha_0$, $H(I \times \{1\}) = \alpha_1$, and such that $H(\{a\} \times I)$ is contained in a single leaf for each $a \in I$, then $\mu(\alpha_0) = \mu(\alpha_1)$, identifying here the leaves using the homotopy.

A singular foliation together with a transverse measure is called a *measured foliation*.

Homeomorphisms of Σ act on measured foliations by $\varphi \cdot (\mathcal{F}, \mu) = (\varphi(\mathcal{F}), \varphi_* \mu)$, where $\varphi_* \mu(\alpha)$ is the push-forward of $\mu(\varphi^{-1}(\alpha))$ under φ. We can now state the following important result about mapping class groups (see [BC88, Thm. 6.3], [FLP79, Exposé 1, Thm. 5], [FM11, Sect. 13.4]).

Theorem 1.15 (The Nielsen–Thurston classification)**.** *A mapping class f in Γ_g, $g \geq 0$, has exactly one of the following three properties:*

1. *The class f has finite order in Γ_g.*

2. *The class f has infinite order but is* reducible*. That is, some power of f preserves the isotopy class of an essential simple closed curve.*

3. *The class f is* pseudo-Anosov *meaning that there exist transverse measured foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$, and $\lambda > 1$ real, such that f is represented by a homeomorphism φ (which we will also call pseudo-Anosov) satisfying*

$$\varphi \cdot (\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s), \quad \varphi \cdot (\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u).$$

In the pseudo-Anosov case, $\mathcal{F}^s$ and $\mathcal{F}^u$ are called the *stable* and *unstable* foliations respectively, and the number λ which turns out to depend only on f is called the *stretch factor* or *dilatation* of f. The term "pseudo-Anosov" is generally reserved for hyperbolic surfaces; it generalizes the notion of an *Anosov* map of a torus. For punctured surfaces, we define pseudo-Anosov mapping classes exactly as in the closed case.

1.3.1 Constructing pseudo-Anosov homeomorphisms

We give an explicit construction of pseudo-Anosov elements that will form the basis for later case studies. The construction is due to Penner, [Pen88], and generalizes earlier work by Thurston, [Thu88]. For an overview of this and other constructions, see [FM11, Sect. 14.1]. In short, Thurston's idea was the following: it turns out that every pseudo-Anosov homeomorphism carries a so-called train track on the surface (see below for details). Train tracks provide a way of translating the a priori non-linear problem of determining the action of pseudo-Anosovs on curves to a linear and combinatorial problem; associated to the pseudo-Anosov homeomorphism and the train track carried by it is a so-called incidence matrix, which is *Perron–Frobenius*; that is, it satisfies the conditions of the following theorem.

Theorem 1.16 (Perron–Frobenius)**.** *Let A be an $n \times n$ matrix with integer entries. If A has a power whose entries are positive, then A has a unique eigenvector v of unit length with non-negative entries. The eigenvalue λ corresponding to v is larger in absolute value than all other eigenvalues.*

The absolute value of the eigenvalue of the incidence matrix coming from this theorem turns out to be exactly the stretch factor of the pseudo-Anosov map being considered. Conversely, under certain conditions, if a given homeomorphism carries a particular train track, and the associated incidence matrix is Perron–Frobenius, then the given homeomorphism is pseudo-Anosov. By explicitly constructing train tracks for a certain class of homeomorphisms, Penner immediately constructs a large family of pseudo-Anosovs.

Definition 1.17. A *multicurve* in a surface Σ is a collection of disjoint simple closed curves in Σ. We say that two multicurves $A = \{\alpha_1, \dots, \alpha_n\}$, $B = \{\beta_1, \dots, \beta_m\}$ *fill* Σ, if the isotopy class of any essential simple closed curve has non-zero intersection with the isotopy class of one of the α_i or β_j.

Theorem 1.18 ([Pen88])**.** *Assume that $A = \{\alpha_1, \dots, \alpha_n\}$ and $B = \{\beta_1, \dots, \beta_m\}$ fill Σ. Then any product of positive powers of t_{α_i} and negative powers of t_{β_j}, with all curves appearing at least once, is pseudo-Anosov.*

Remark 1.19. Penner conjectures that all pseudo-Anosovs arise this way. More precisely, he conjectures that for any pseudo-Anosov ψ, there exist A and B as in the theorem, and $n > 0$, such that ψ^n is a word in positive powers of Dehn twists of curves from A and negative powers of those from B.

As already mentioned, Penner proves this by giving an explicit construction of which we give an example relevant in our later studies. Recall the presentation of Γ_2 given in Section 1.2.2. The two multicurves giving rise to the Dehn twists a_1, a_3, a_5 and a_2, a_4 respectively fill the surface. Let $\mathcal{W} \subseteq \Gamma_2$ be the semigroup consisting of words in positive powers of a_i, i odd, and negative powers of a_i, i even, with each a_i appearing at least once. By Theorem 1.18, all elements of $\mathcal{W}$ are pseudo-Anosov. As described above, we can determine the stretch factor of the mapping class using the theory of train tracks.

Definition 1.20. A *train track* on a surface Σ is a finite graph τ embedded in Σ satisfying the following:

1. Each edge is the smooth image of an interval. Edges are called *branches* of τ.
2. If b_1 and b_2 are branches meeting a vertex, their one-sided tangents in the vertex point either coincide or differ by a rotation by an angle π of the tangent plane. Vertices are called *switches*.
3. No component of $\Sigma \setminus \tau$ is an embedded nullgon, monogon, bigon, once punctured nullgon, or annulus.

Here, an *n-gon* is a disc embedded in Σ whose boundary tangents have n discontinuities. We say that a train track τ *carries* another track τ', if there is a C^1 map $\Phi : \Sigma \to \Sigma$ homotopic to the identity with $\Phi(\tau') \subseteq \tau$ and such that $d\Phi_p$ has non-zero restriction to tangents of τ', for all $p \in \tau'$. In this case we define the *incidence matrix* M of Φ as follows: for every branch b_i of τ, we choose a point x_i in the interior of b_i, and define $M_{ij} = \#\{\Phi^{-1}(x_i) \cap c_j\}$, where the c_j range over branches of τ'.

Let us return to the mapping classes $\mathcal{W} \subseteq \Gamma_2$ defined above. One defines a train track τ associated to the filling multicurves on Σ_2 by "smoothing" as in Figure 1.11. It is now easy to see that the train track is carried by each of the Dehn twists constituting the generators of $\mathcal{W}$, and the carrying property is clearly transitive, so any element $w \in \mathcal{W}$ will carry the train track. Penner shows that this procedure generalizes to those mapping classes satisfying the conditions of Theorem 1.18.

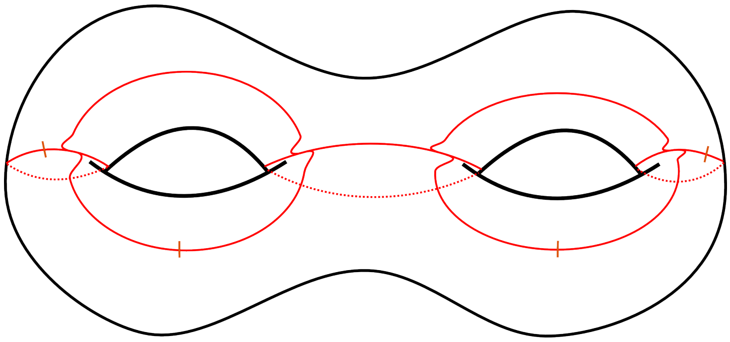


Figure 1.11: The train track carried by $w \in \mathcal{W}$.

The incidence matrix of this particular τ will be 12×12. In practice it will be easier to consider measured train tracks. A *measure* on a general train track τ is an assignment of non-negative integers called *weights* to branches of τ such that these satisfy the *switch condition*: the branches of τ are divided into two sets by the second condition of Definition 1.20. We require that the sums of weights of branches in the two sets agree. In our example, the weights of the branches in the 12-branch train track are determined by the weights of 4 of the branches, denoted in Figure 1.11 by dashes, and one defines a reduced incidence matrix using only these 4 branches. Thus, the study of the particular family of pseudo-Anosovs in $\mathcal{W}$ boils down to that considered in [BC88, pp. 77–79] to which we refer for the details left out here (note that in [BC88], Dehn twists twist to the right by convention). In particular, the reduced incidence matrix of an element $w \in \mathcal{W}$ is given by its action $h : \Gamma_2 \to \mathrm{Sp}(4, \mathbb{Z})$ on homology, which in its standard basis is given by

$$h(t_1) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h(t_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h(t_3) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h(t_4)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad h(t_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

correcting a few minor typos in [BC88]. We summarize this discussion as follows, recalling that the *spectral radius* of a bounded linear operator is the supremum among the absolute values of the elements in its spectrum.

Proposition 1.21. *The subset $\mathcal{W} \subseteq \Gamma_2$ consists of pseudo-Anosov homeomorphisms, and for $w \in \mathcal{W}$, the stretch factor of w is the spectral radius of the action $h(w)$ of w on $H_1(\Sigma_2, \mathbb{R})$.*

In general, there is no reason to expect that the action on homology carries as much information as was the case above; indeed, for any $g \geq 2$, the *Torelli subgroup* of Γ_g consisting of the mapping classes whose action on homology is trivial will contain pseudo-Anosov elements. This was first shown by Thurston (see [Thu88, p. 431]); for an explicit construction of such elements in Γ_2, see [Bro98, App. A].

Following e.g. [HK06], we will call a pseudo-Anosov $\varphi \in \Gamma_g$ *orientable*, if its stable and unstable foliations are orientable. In this case, we have the following theorem (see [Ryk99] and the proof of [FM11, Thm. 14.8]).

Theorem 1.22. *If $\varphi \in \Gamma_g$ is an orientable pseudo-Anosov, then the stretch factor of φ is the spectral radius of the action of φ on $H_1(\Sigma_g, \mathbb{R})$.*

Roughly, this is due to the fact that under the assumption of orientability, the foliations may be described as the kernels of null-cohomologous 1-forms, scaled by exactly the stretch factor under the action of φ. This also shows that stretch factors of orientable pseudo-Anosovs are algebraic integers; the same is true for non-orientable pseudo-Anosovs as can be seen by combining the above theorem with a covering space argument – see [FM11, Thm. 14.8].

Chapter 2

Geometric quantization and classical Chern–Simons theory

In this chapter, we recall the framework that will later result in particular representations of the mapping class group.

In general, geometric quantization is one of several schemes involving the passage from a classical physical theory to a quantum mechanical analogue. The phase space of a classical system is a symplectic manifold (M, ω), and observables correspond to smooth functions on M. Geometric quantization associates to M a complex line bundle $\mathcal{L} \to M$ and a Hilbert space $\mathcal{H}$ of states consisting of certain sections of $\mathcal{L}$. To a subset of the observables on M it associates self-adjoint operators on $\mathcal{H}$. The process of geometric quantization is typically divided into three parts; *prequantization* which is concerned with the associations above, *polarization* which restricts the collection of quantizable observables through a choice of a certain distribution on M, and finally *metaplectic correction* which involves repairs to the quantization which are necessary, for example in order to obtain the correct energy values for the harmonic oscillator. We will largely avoid discussion of the underlying physics and in order to obtain our desired representations, all we need is an understanding of the line bundles arising from prequantization. Details on the entire method can be found in e.g. [Woo92], [AE05].

2.1 Preliminaries

2.1.1 Connections in principal bundles

Definition 2.1. Let M be a manifold and G a Lie group. A principal G-bundle over M is a manifold P satisfying the following:

1. There is a free right action of G on P such that M is the quotient space of P under this action, and the quotient $\pi : P \to P/G = M$ is smooth.
2. Furthermore, P is locally trivializable; that is, every point of M has a neighbourhood U with an equivariant diffeomorphism $\pi^{-1}(U) \to U \times G$ covering the identity on M.

Let $\pi : P \to M$ be a principal G-bundle over a manifold M. For $p \in P$, let $i_p : G \to P$ denote the map $i_p(g) = p \cdot g$, and similarly let $r_g : P \to P$ denote the map $i_g(p) = p \cdot g$. For every $X = X_e \in \mathfrak{g} \cong T_e(G)$, let $X_p^* = (di_p)_e(X_e)$. This defines a vector field on P which is vertical in the sense that $d\pi(X^*) = 0$. In other words, for every $p \in P$, we have a short exact sequence

$$0 \to \mathfrak{g} \xrightarrow{(di_p)_e} T_pP \xrightarrow{d\pi_p} T_{\pi(p)}M \to 0.$$

Now, a connection on P defines a notion of horizontality in the principal bundle through a smooth choice of G-equivariant splittings of this exact sequence for all $p \in P$. More precisely, we consider the following:

Definition 2.2. A *connection* on a principal bundle $G \to M$ is a $\mathfrak{g}$-valued 1-form A on P such that

1. $A(X^*) = X$ for all $X \in \mathfrak{g}$.
2. A is G-equivariant in the sense that for all $g \in G$, we have $r_g^*(A) = \mathrm{Ad}_{g^{-1}} A$.

Thus, if we let $V_p = \ker(d\pi) \subseteq T_pP$ denote the space of all vertical vectors in T_pP, then $T_pP = V_p \oplus H_p$, where $H_p = \ker(A_p)$ is the *horizontal subspace* of T_pP. Conversely, for every n-dimensional distribution H on $P \to M$ satisfying that $d\pi_p|_{H_p} : H_p \to T_{\pi(p)}M$ is an isomorphism and that $H_{p\cdot g} = d(r_g)H_g$, there is a unique connection A on $P \to M$ with $\ker(A_p) = H_p$ for all p.

Throughout the rest of this manuscript, $\mathcal{A}_P$ will denote the set of connections on the principal bundle $P \to M$.

2.1.2 Curvature of connections

We now introduce the concept of curvature of a principal bundle connection. We do this by instead considering it as an affine connection in an associated vector bundle, and describing the curvature as the failure of a certain sequence to be a chain complex.

Definition 2.3. Let $P \to M$ be a principal G-bundle, and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on a finite dimensional vector space V. The *vector bundle associated to* P *by* ρ is the quotient space

$$P \times_\rho V = P \times V/\sim,$$

where $(p, v) \sim (p \cdot g, \rho(g^{-1})v)$ with projection $\pi : P \times_\rho V \to M$ given by $\pi([(p, v)]) = p$. In particular, for the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, the associated vector bundle is denoted Ad_P and called the *adjoint bundle*.

Remark 2.4. Let $E = P \times_\rho V$ be the vector bundle associated to a principal G-bundle P by ρ. Associate to a G-equivariant map $f : P \to V$, i.e. a map satisfying $f(p \cdot g) = \rho(g^{-1})f(p)$, a section φ_f of E by letting $\varphi_f(x) = [(p, f(p))]$ for some $p \in \pi^{-1}(x)$. This association is a bijection.

This observation allows us to associate to every connection A in $P \to M$ an affine connection in the adjoint bundle Ad_P as follows: let $\varphi : P \to V$ be the G-equivariant map corresponding to a section of Ad_P, and let X be a vector field on M. Let $p \in P$, and let $\tilde{X}_p$ be the unique horizontal lift of $X_{\pi(p)}$ to T_pP, given by the connection A. Now, define a G-equivariant map $\nabla_X^A \varphi : P \to V$ by letting

$$\nabla_X^A \varphi(p) = d\varphi_p(\tilde{X}_p) \in T_{\varphi(p)}V \cong V.$$

Recall that any affine connection $\nabla : C^\infty(M, E) \to \Omega^1(M, E)$ in a vector bundle $E \to M$ extends to a map $\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ (see e.g. [Wel08, p. 74]) and gives a sequence

$$\cdots \to \Omega^{k-1}(M, E) \xrightarrow{\nabla} \Omega^k(M, E) \xrightarrow{\nabla} \Omega^{k+1}(M, E) \to \cdots . \tag{2.1}$$

In a local frame f over $U \subseteq M$ for E, this extension is given by

$$\nabla\xi(f) = d\xi(f) + \theta(f) \wedge \xi(f),$$

where $\xi \in \Omega^p(U,E)$, and $\theta(f) = \theta(\nabla, f) \in \Omega^1(U, \mathrm{Hom}(E,E))$ is the connection matrix associated to ∇. In general, we will not have $\nabla \circ \nabla = 0$, and we let $F_\nabla \in \Omega^2(M, \mathrm{Hom}(E,E))$ denote the *curvature* of ∇ given by

$$F_\nabla \varphi = \nabla\nabla\varphi$$

for $\varphi \in C^\infty(M,E)$. In a local frame f, the curvature is given in terms of the connection matrix: ignoring the notational dependence on f, we find that

$$\begin{aligned} F_\nabla \xi &= (d+\theta)(d+\theta)\xi = (d+\theta)(d\xi + \theta \cdot \xi) \\ &= d^2\xi + \theta \cdot (d\xi) + d(\theta \cdot \xi) + \theta \wedge (\theta \cdot \xi) \\ &= \theta \cdot (d\xi) + d\theta \cdot \xi - \theta \cdot (d\xi) + (\theta \wedge \theta)\xi \\ &= d\theta \cdot \xi + (\theta \wedge \theta) \cdot \xi \end{aligned}$$

for all $\xi \in C^\infty(U,E)$. That is, we have the following:

Lemma 2.5. *In a local frame f,*

$$F_\nabla(f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

We are now in a position to describe the curvature of a connection in a principal G-bundle.

Definition 2.6. Let A be a connection in $P \to M$, and let ∇^A be the induced connection in Ad_P. The *curvature* F_A of A is the curvature $F_{\nabla^A} \in \Omega^2(M, \mathrm{Hom}(\mathrm{Ad}_P, \mathrm{Ad}_P))$.

We end this section by giving a few alternative descriptions of the curvature of A. Let $\Omega^k(P;\mathfrak{g}) = \Omega^k(P \times \mathfrak{g})$ be the space of $\mathfrak{g}$-valued k-forms on P, and let $\Omega^k(P;\mathfrak{g})^G$ be the subset of those k-forms that are G-equivariant. The pullback of the adjoint bundle Ad_P under $\pi : P \to M$ is the trivial bundle $P \times \mathfrak{g} \to P$. The pullback $\tilde{\nabla}^A$ of ∇^A to $P \times \mathfrak{g}$ in fact restricts to the G-equivariant forms on P and gives rise to a sequence

$$\cdots \to \Omega^{k-1}(P;\mathfrak{g})^G \xrightarrow{\tilde{\nabla}^A} \Omega^k(P;\mathfrak{g})^G \xrightarrow{\tilde{\nabla}^A} \Omega^{k+1}(P;\mathfrak{g})^G \to \cdots . \tag{2.2}$$

Proposition 2.7. *We have*

$$\tilde{\nabla}^A \tilde{\nabla}^A \varphi = [(dA + \tfrac{1}{2}[A \wedge A]) \wedge \varphi]$$

for $\varphi \in \Omega^k(P;\mathfrak{g})$, where $[\cdot,\cdot]$ denotes the bracket on $\mathfrak{g}$. Let $F_{\tilde{\nabla}^A}$ be the curvature of $\tilde{\nabla}^A$, i.e. $F_{\tilde{\nabla}^A} = \tilde{\nabla}^A \circ \tilde{\nabla}^A$. Then $\pi^*(F_A) = F_{\tilde{\nabla}^A}$.

Finally, using the G-equivariance of A, it is possible to view the curvature F_A as an element of $\Omega^2(M, \mathrm{Ad}_P)$. Namely, for $q \in M$, $p \in \pi^{-1}(q)$, and vector fields X, X' on M, define

$$\tilde{F}_A(X_q, X'_q)(p) = (dA + \tfrac{1}{2}[A \wedge A])((d\pi|_{H_p})^{-1}X_q, (d\pi|_{H_p})^{-1}X'_q).$$

This (well-)defines a G-equivariant map $P \to \mathfrak{g}$, and by Remark 2.4 we can consider $\tilde{F}_A \in \Omega^2(M, \mathrm{Ad}_P)$. In the following proposition, $[\cdot,\cdot] : \mathrm{Ad}_P \otimes \mathrm{Ad}_P \to \mathrm{Ad}_P$ is defined by $[(p,v),(p,v')] = (p,[v,v'])$ on representatives (p,v) and (p,v').

Proposition 2.8. *Let $\tilde{F}_A \in \Omega^2(M, \mathrm{Ad}_P)$ be the 2-form defined above. Then for every $\varphi \in \Omega^k(M, \mathrm{Ad}_P)$, we have*

$$\nabla^A \nabla^A \varphi = [F_A \wedge \varphi],$$

and $\pi^*(\tilde{F}_A) = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P;\mathfrak{g})^G$.

Conversely, $d\mathrm{Ad} : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ induces a map

$$\Omega^k(M, \mathrm{Ad}_P) \to \Omega^k(M, \mathrm{Hom}(\mathrm{Ad}_P, \mathrm{Ad}_P))$$

mapping the $\tilde{F}_A$ of Proposition 2.8 to the F_A of Definition 2.6. For a proof of the above results, see [Him10]. Because of this, we will abuse notation and also refer to the element $\pi^*(\tilde{F}_A) \in \Omega^2(P; \mathfrak{g})^G$ as the curvature of A and simply denote it F_A.

Definition 2.9. A connection A on a principal G-bundle $P \to M$ is called *flat*, if $F_A = 0$ pointwise. The space of all flat connections on P is denoted $\mathcal{F}_P$.

Notice that we could have simply defined $F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P; \mathfrak{g})^G$, but we stress the fact that the curvature of A is exactly the obstruction to the sequence (2.1) being a complex.

2.2 Prequantum line bundles

Throughout this section, let (M, ω) denote a symplectic manifold with a symplectic form $\omega \in \Omega^2(M, \mathbb{R})$.

Definition 2.10. A *prequantum line bundle* on (M, ω) is a triple $(\mathcal{L}, \nabla, (\cdot, \cdot))$ consisting of a complex line bundle $\mathcal{L} \to M$ with a Hermitian structure $(\cdot, \cdot)$, and a compatible connection ∇ satisfying the prequantum condition

$$F_\nabla = -i\omega.$$

A necessary and sufficient condition for the existence of a prequantum line bundle on M is that $[\frac{\omega}{2\pi}] \in \mathrm{Im}(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}))$. See e.g. [Woo92].

Let $(\mathcal{L}, \nabla, (\cdot, \cdot))$ be a prequantum line bundle on a compact symplectic manifold M, and let $\mathcal{H}^k = C^\infty(M, \mathcal{L}^{\otimes k})$ be the space of all smooth sections of $\mathcal{L}^{\otimes k}$. This is an inner product space with the L^2-inner product

$$\langle s_1, s_2 \rangle = \frac{1}{n!} \int_M (s_1, s_2)_k \, \omega^n, \tag{2.3}$$

where $2n$ is the dimension of M, and $(\cdot, \cdot)_k$ is the Hermitian structure on $\mathcal{L}^{\otimes k}$ induced by that on $\mathcal{L}$. The relevant Hilbert space in this context is the L^2-completion of $\mathcal{H}^k$. For our purposes though, the main point of interest lies in a certain finite-dimensional subspace of $\mathcal{H}^k$, constructed as follows.

Assume that $\mathcal{T}$ is a smooth manifold smoothly parametrizing Kähler structures on M. That is, assume that there is a map $I : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ mapping $\sigma \mapsto I_\sigma$ such that for every $\sigma \in \mathcal{T}$, (M, ω, I_σ) is Kähler, and such that I is smooth in the sense that it defines a smooth section of the pullback bundle $\pi_M^*(\mathrm{End}(TM))$ over $\mathcal{T} \times M$, where $\pi_M : \mathcal{T} \times M \to M$ denotes the projection onto M. We denote by M_σ the Kähler manifold (M, ω, I_σ). With the complex structure on M_σ, ω has type $(1, 1)$, and by the prequantum condition, the line bundles $\mathcal{L}^{\otimes k}$ canonically obtain the structures of holomorphic line bundles (see [Kob97, Prop. 3.7]) denoted $\mathcal{L}_\sigma^{\otimes k}$. Now, let

$$H_\sigma^k = H^0(M_\sigma, \mathcal{L}_\sigma^{\otimes k}) = \{s \in \mathcal{H}_k \mid \nabla_\sigma^{0,1} s = 0\}$$

be the space of holomorphic sections of $\mathcal{H}^k$. Using the splitting

$$T^*M \otimes \mathbb{C} = T_\sigma^* \oplus \bar{T}_\sigma^*,$$

of the complexified cotangent bundle into $\pm i$-eigenspaces of I_σ, the operator $\nabla_\sigma^{0,1}$ is the composition

$$C^\infty(M, \mathcal{L}^{\otimes k}) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes \mathcal{L}^{\otimes k}) \xrightarrow{\pi_{I_\sigma}^{0,1}} C^\infty(M, \bar{T}_\sigma^* \otimes \mathcal{L}^{\otimes k}),$$

which can be identified with the operator $\bar{\partial}_\sigma$ giving rise to the holomorphic structure on $\mathcal{L}^{\otimes k}$.

The process of restricting the space of sections using Kähler structures on M is known as *Kähler quantization*.

2.3 Hitchin's connection

Our next objective is to understand the dependence of H^k_σ on the complex structures σ. More precisely, consider the trivial bundle $\mathcal{T} \times \mathcal{H}^k \to \mathcal{T}$. It is a non-trivial fact that the vector spaces H^k_σ are all finite-dimensional, and we now assume that they form a finite rank subbundle $\mathcal{V}_k$ of $\mathcal{T} \times \mathcal{H}^k$. Our goal is to find a connection in $\mathcal{T} \times \mathcal{H}^k$ preserving the subbundle $\mathcal{V}_k$. Let $\boldsymbol{\nabla}^t$ denote the trivial connection in the vector bundle $\mathcal{T} \times C^\infty(M, \mathcal{L}^{\otimes k})$. Then the composition of $\boldsymbol{\nabla}^t$ with the fibre-wise projection $\pi^k_\sigma : \mathcal{H}^k \to H^k_\sigma$ defines a connection which preserves $\mathcal{V}_k$ by construction. However, this connection is non-flat and thus not suited for our purposes.

Hitchin's idea was to consider instead the connection $\boldsymbol{\nabla}^H$ in $\mathcal{T} \times \mathcal{H}^k$ defined by

$$\boldsymbol{\nabla}^H_V = \boldsymbol{\nabla}^t_V - u(V) \tag{2.4}$$

for vector fields V on $\mathcal{T}$, where $u(V)_\sigma \in \mathrm{Diff}^{(2)}(M, \mathcal{L}^{\otimes k})$ is a second order differential operator, and to analyze under which conditions on u the connection $\boldsymbol{\nabla}^H$ will preserve $\mathcal{V}_k$. When the bundle is preserved, we refer to $\boldsymbol{\nabla}^H$ as a *Hitchin connection*. Whereas this is not always the case, under certain conditions one can find explicit formulas for u. Here, we recall briefly the construction of [And12b] (see also [Gam10]), giving such conditions on the symplectic manifold and the family of complex structures.

Denote by g_σ the Kähler metric on (M_σ, ω) given by

$$g_\sigma(X, Y) = \omega(X, I_\sigma Y)$$

for $X, Y \in C^\infty(M, TM_\mathbb{C})$. For a vector field V on $\mathcal{T}$, let $V[I] : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM_\mathbb{C}))$ denote the derivative of I along V. Since I^2 is constant, the Leibniz rule implies that $V[I]$ anti-commutes with I, and so it is type-interchanging,

$$V[I]_\sigma \in C^\infty(M, (T^*_\sigma \otimes \bar{T}_\sigma) \oplus (\bar{T}^*_\sigma \otimes T_\sigma)),$$

and we may write $V[I]_\sigma = V[I]''_\sigma \oplus V[I]'_\sigma$ according to this splitting. Assume now that $\mathcal{T}$ is a complex manifold and that $V'[I]_\sigma = V[I]'_\sigma$, $V''[I]_\sigma = V[I]''_\sigma$, where V' and V'' are the $(1,0)$ and $(0,1)$ parts of V respectively. Define $\tilde{G}(V) \in C^\infty(M, TM_\mathbb{C} \otimes TM_\mathbb{C})$ by

$$V[I] = \tilde{G}(V)\omega,$$

where juxtaposition denotes contraction of tangent and cotangent vectors. Andersen shows that $\tilde{G}(V)$ captures the variation of the Kähler metrics as

$$V[g] = \omega\tilde{G}(V)\omega,$$

so that $\tilde{G}(V) \in C^\infty(M, S^2(TM_\mathbb{C}))$ and through type considerations that we may define $G(V) \in C^\infty(M, S^2(T_\sigma))$ by

$$\tilde{G}(V) = G(V) + \bar{G}(V).$$

Define now the *Ricci potential* $F_\sigma \in C^\infty(M, \mathbb{R})$ by

$$\mathrm{Ric}_\sigma = \mathrm{Ric}^H_\sigma + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma,$$

where $\mathrm{Ric}_\sigma \in \Omega^{1,1}(M_\sigma)$ is the Ricci form of the Levi-Civita connection $\hat{\nabla}_\sigma$ on T_σ, Ric_σ^H its harmonic part, and where F_σ is normalized so that $\int_M F_\sigma \, \omega^m = 0$. Viewing $G(V)$ as a bundle morphism $G(V) : T_\sigma^* \to T_\sigma$, define $\Delta_{G(V)}$ to be the composition

$$\begin{aligned} C^\infty(M, \mathcal{L}^k) &\xrightarrow{\nabla_\sigma^{1,0}} C^\infty(M, T_\sigma^*, \mathcal{L}^k) \xrightarrow{G\otimes\mathrm{Id}} C^\infty(M, T_\sigma \otimes \mathcal{L}^k) \\ &\xrightarrow{\hat{\nabla}_\sigma^{1,0}\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla_\sigma^{1,0}} C^\infty(M, T_\sigma^* \otimes T_\sigma \otimes \mathcal{L}^k) \xrightarrow{\mathrm{tr}} C^\infty(M, \mathcal{L}^k). \end{aligned} \tag{2.5}$$

Finally, let $n \in \mathbb{Z}$ with $2k + n \neq 0$ and define

$$u(V) = \frac{1}{2k+n} o(V) + V'[F], \tag{2.6}$$

where

$$o(V) = \frac{1}{2}\Delta_{G(V)} - \nabla_{G(V)dF} - nV'[F]. \tag{2.7}$$

Theorem 2.11 ([And12b]). *Assume that I is rigid in the sense that*

$$\bar{\partial}_\sigma(G(V)) = 0$$

for all vector fields V on $\mathcal{T}$ and all $\sigma \in \mathcal{T}$. If moreover the first Chern class of (M, ω) is $c_1(M) = n[\frac{\omega}{2\pi}]$ and $H^1(M, \mathbb{R}) = 0$, then $\boldsymbol{\nabla}^H$, defined by (2.4) *with u given by* (2.6)*, is a Hitchin connection.*

A natural question concerns the flatness of the Hitchin connection of Theorem 2.11. We will see that in certain special cases, and indeed in the cases relevant for us, it is possible to find a projectively flat connection in the bundle $\mathcal{V}_k$. Here, a connection is called *projectively flat*, if its induced parallel transport defines isomorphisms of fibres up to scalar multiplication. For an evaluation of the curvature of the Hitchin connection in the general setting above, we refer to [Gam10].

Remark 2.12. In [AGL12], the authors show that by passing to the metaplectic correction – which involves twisting the prequantum line bundles $\mathcal{L}^k$ by the square root of the canonical bundle of M – the condition of Theorem 2.11 that M is compact and that $c_1(M) = n[\frac{\omega}{2\pi}]$ may be replaced by the vanishing of the second Stiefel–Whitney class, and the authors are able to provide in the compact case a more explicit formula for the corresponding u.

2.4 Toeplitz operators

Whereas we will not need it directly in this chapter, the theory of Toeplitz operators turns out to provide a convenient tool to describe asymptotics of quantum representations, as we will see later.

In the following $\|\cdot\|$ denotes the operator norms induced by the inner product on H_σ^k.

Definition 2.13. Let $f \in C^\infty(M)$. The *Toeplitz operator* $T_{f,\sigma}^{(k)} : \mathcal{H}_k \to H_\sigma^k$ is defined by

$$T_{f,\sigma}^{(k)}(s) = \pi_\sigma^k(f \cdot s).$$

We will also abuse notation slightly and write $T_{f,\sigma}^{(k)} = T_{f,\sigma}^{(k)}|_{H_\sigma^k}$. Likewise, for two complex structures σ_0, σ_1, define $T_{f,(\sigma_0,\sigma_1)}^{(k)} : H_{\sigma_0}^k \to H_{\sigma_1}^k$ by

$$T_{f,(\sigma_0,\sigma_1)}^{(k)} = T_{f,\sigma_1}^{(k)}|_{H_{\sigma_0}^k}.$$

As we will primarily be interested in the Toeplitz operators because of their abilities to describe asymptotics, we list the following properties, due to Bordemann, Meinrenken and Schlichenmaier [BMS94, Thm. 4.1], and Andersen [And07, Thm. 19].

Theorem 2.14. *Let $f \in C^\infty(M)$. Then for complex structures σ, σ_0, and σ_1, we have*

$$\lim_{k\to\infty} \|T^{(k)}_{f,\sigma}\| = \lim_{k\to\infty} \|T^{(k)}_{f,(\sigma_0,\sigma_1)}\| = \sup_{x\in M} |f(x)|,$$

both sequences moreover being bounded from above by $\sup_{x\in M}|f(x)|$.

The following result, which we will refer to in the proof of Theorem 6.7, is due to Schlichenmaier.

Theorem 2.15. *Let $f_1, f_2 \in C^\infty(M)$. Then, for any $\sigma \in \mathcal{T}$,*

$$T^{(k)}_{f_1,\sigma} T^{(k)}_{f_2,\sigma} \sim \sum_{l=0}^{\infty} T^{(k)}_{c_l(f_1,f_2),\sigma} k^{-l},$$

where $c_l(f_1, f_2) \in C^\infty(M)$ are uniquely determined functions, and $c_0(f_1, f_2) = f_1 f_2$. Here, $\sim$ means that

$$\|T^{(k)}_{f_1,\sigma} T^{(k)}_{f_2,\sigma} - \sum_{l=0}^{L} T^{(k)}_{c_l(f_1,f_2),\sigma} k^{-l}\| = O(k^{-(L+1)})$$

for all positive integers L.

Remark 2.16. The coefficients c_l closely relates geometric quantization to deformation quantization as they may utilized to define a star product, the so-called *Berezin–Toeplitz star product*, on M by

$$f \star g = \sum_{l=0}^{\infty} c_l(f, g) \left(\frac{1}{k}\right)^l.$$

We will not need the relation to deformation quantization and refer to [KS01] for the details.

2.5 Quantization of moduli spaces

The space we will be interested in quantizing is the space of flat connections in a trivializable principal G-bundle on a given surface. Before going into the details, we review the general picture and some of the central results.

2.5.1 The moduli space of flat connections

Let $\pi : P \to M$ be a principal G-bundle. We first describe how, given a connection A in $P \to M$, any curve in M can be lifted to a unique horizontal curve in P.

Lemma 2.17. *Let A be a connection in $P \to M$. Let $\alpha : [0,1] \to M$ be a smooth curve with starting point α_0, and let $p_0 \in \pi^{-1}(\alpha_0)$. Then there exists a unique smooth lift $\beta : [0,1] \to P$ of α with starting point p_0, such that $\dot\beta_t$ is a horizontal lift of $\dot\alpha_t$.*

Proof. Let $\gamma : [0,1] \to P$ be any lift of α. We want to find a smooth curve $\beta_t = \gamma_t \cdot g_t : I \to P$, where $g_t : [0,1] \to G$, so that $A(\dot\beta_t) = 0$ for all $t \in [0,1]$. We see that

$$A(\dot\beta_t) = A((dr_{g_t})\dot\gamma_t + (di_{\gamma_t})\dot g_t) = \mathrm{Ad}_{g_t^{-1}}(\dot\gamma_t) + (dl_{g_t^{-1}})(\dot g_t),$$

where $l_g : G \to G$ denotes left multiplication in G. Thus, $A(\dot\beta_t) = 0$ if and only if

$$A(\dot\gamma_t) = -(dr_{g_t^{-1}})\dot g_t.$$

Solving this equation with initial condition $\gamma_0 \cdot g_0 = p_0$ gives the desired horizontal lift β_t. □

If $\alpha : [0,1] \to M$ is a loop, $\alpha(0) = \alpha(1) = x_0$, the starting and ending points of the lift β defined above are both in the fibre P_{x_0} over x_0. Thus there is a g so that $\beta(1) = \beta(0) \cdot g$. This g is called the *holonomy of A along α with respect to* p_0 and is denoted $g = \mathrm{hol}_{A,p_0}(\alpha)$. The holonomy defines a map

$$\mathrm{hol}_{A,p_0} : \mathrm{Loops}(M, x_0) \to G$$

for any given $p_0 \in \pi^{-1}(x_0)$.

The space $\mathcal{A}_P$ of all connections in the principal G-bundle $P \to M$ is too big for our purposes, and in this section we restrict it using the natural symmetry arising from the G-action.

Definition 2.18. A *principal bundle homomorphism* between two principal G-bundles P and P' is a G-equivariant bundle homomorphism. If $P = P'$ it is called a *gauge transformation* of the bundle. Denote by $\mathcal{G}_P$ the group of all gauge transformations $P \to P$.

Remark 2.19. To every G-equivariant map $u : P \to G$, $p \mapsto u_p$, we associate a gauge transformation $\Phi : P \to P$ by letting $\Phi(p) = p \cdot u_p$. Here, $g \in G$ acts on itself on the right by $h \mapsto g^{-1}hg$. This association is a bijection.

The group $\mathcal{G}_P$ acts on $\mathcal{A}_P$ via pullback, and the action preserves $\mathcal{F}_P$. For a G-equivariant map $u : P \to G$, we write this action $A \mapsto A \cdot u$.

Definition 2.20. The *moduli space of flat connections on a principal G-bundle $P \to M$* is the space $\mathcal{M}_P = \mathcal{F}_P / \mathcal{G}_P$.

Before indulging in the question on how to quantize $\mathcal{M}_P$, we give a group theoretical description of it using the holonomy map. For a proof of the following results, see e.g. [Him10, Prop. 3.10.1 and Thm. 3.10.4].

Proposition 2.21. *Let A be a flat connection in P, and assume that M is connected. Let $x_0 \in M$, let $p_0 \in \pi^{-1}(x_0)$, and let α be a loop in M. Up to conjugation in G, the association $A \mapsto \mathrm{hol}_{A,p_0}(\alpha)$ is independent of the base point x_0, the choice of lift p_0, the gauge tranformation class of the connection A, and the homotopy class of α. In other words, we have a well-defined map*

$$\mathrm{hol} : \mathcal{M}_P \to \mathrm{Hom}(\pi_1(M), G)/G,$$

where G acts on $\mathrm{Hom}(\pi_1(M), G)$ on the right by $(\rho \cdot g)(\alpha) = g^{-1}\rho(\alpha)g$.

Definition 2.22. A *flat principal G-bundle* on a manifold M is a pair (P, A) consisting of a principal G-bundle $P \to M$ and a flat connection A in P. Two flat principal G-bundles (P, A) and (P', A') are called isomorphic if there is a principal bundle homomorphism $\Phi : P \to P'$ such that $A = \Phi^*(A')$. The set $\mathcal{M}^G(M)$ of isomorphism classes is called the *moduli space of flat principal G-bundles on M*.

Theorem 2.23. *The map $\mathcal{M}^G(M) \to \mathrm{Hom}(\pi_1(M), G)/G$ mapping $[(P, A)]$ to $[\mathrm{hol}_A]$ is a bijection.*

2.5.2 The Chern–Simons line bundle

The goal of this section is to give a sketch of the construction that a certain subset of the moduli space is prequantizable. We follow [Fre95] and [RSW89].

From now on, assume always that G is a simple, connected, simply connected, and compact Lie group. In this case, it is well-known that any principal G-bundle over M, where $\dim M \leq 3$, is trivializable. Let M be an oriented compact 3-manifold with boundary $\partial M = \Sigma$, and let $P \to M$ be a principal G-bundle. Trivializing the bundle $P \cong M \times G$ by a trivialization $p \mapsto (\pi(p), g_p)$ is equivalent to choosing a section $s : M \to P$ through the identification $p \cdot g_p = s(\pi(p))$. Using such a section, the pullback of connections determines an identification $\mathcal{A}_P \cong \Omega^1(M;\mathfrak{g})$, and likewise we can identify $\mathcal{G}_P \cong C^\infty(M,G)$.

Let $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{R}$ be an Ad-invariant inner product on $\mathfrak{g}$, and define for a connection $A \in \mathcal{A}_P$ with curvature $F_A \in \Omega^2(P;\mathfrak{g})$ the *Chern–Simons form* $\alpha(A) \in \Omega^3(P)$ by

$$\alpha(A) = \langle A \wedge F_A\rangle - \tfrac{1}{6}\langle A \wedge [A\wedge A]\rangle.$$

Definition 2.24. Let $s : M \to P$ be a trivialization of $P \to M$. The *Chern–Simons functional* or *Chern–Simons action* of A is then given by

$$\mathrm{CS}_s(A) = \int_M s^*(\alpha(A)) \in \mathbb{R}$$

Let $\theta \in \Omega^1(G;\mathfrak{g})$ be the *Maurer–Cartan form* defined by $\theta(v) = (dl_{g^{-1}})v \in \mathfrak{g}$ for $v \in T_gG$. The next proposition describes the behaviour of the Chern–Simons functional under gauge transformation (see [Fre95, Prop. 2.10]).

Proposition 2.25. *Let $\Phi : P \to P$ be a gauge transformation associated to a map $u : P \to G$ and let $\theta_u = (u\circ s)^*\theta$. Then for $A \in \Omega^1(M;\mathfrak{g})$,*

$$\begin{aligned}\mathrm{CS}_{\Phi\circ s}(A) &= \mathrm{CS}_s(\Phi^*A)\\ &= \mathrm{CS}_s(A) + \int_{\partial M}\langle \mathrm{Ad}_{(u\circ s)^{-1}}A\wedge\theta_u\rangle - \int_M \tfrac{1}{6}\langle\theta_u\wedge[\theta_u\wedge\theta_u]\rangle.\end{aligned}$$

Assume from now on that $\langle\cdot,\cdot\rangle$ is normalized so that

$$-\int_G \tfrac{1}{6}\langle\theta\wedge[\theta\wedge\theta]\rangle = 1.$$

Then the last integral of Proposition 2.25 is an integer.

Definition 2.26. In the case where M is closed we obtain the Chern–Simons action

$$\mathrm{CS}_s : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}/\mathbb{Z}.$$

Since any two sections are related by a gauge transformation, by Proposition 2.25 this function is independent of s and will be denoted CS.

Our next goal is to associate in the non-closed case, $\partial M = \Sigma$, $Q = P|_\Sigma$, a certain complex line bundle $\mathcal{L}_Q$ over $\mathcal{A}_Q$ and use the Chern–Simons action to define a lift of the action of $\mathcal{G}_Q$ to $\mathcal{L}_Q$, ultimately giving rise to a line bundle over a subset of the quotient $\mathcal{A}_Q/\mathcal{G}_Q$. We first need the following (see [Fre95, Lem. 2.12]).

Lemma 2.27. *For any gauge transformation $g : M \to G$, the functional*

$$W_{\partial M}(g) = \int_M -\tfrac{1}{6}\langle g^*\theta\wedge[g^*\theta\wedge g^*\theta]\rangle \pmod 1$$

depends only on the restriction of g to ∂M.

Let $\mathcal{G}_Q$ denote the space of gauge transformations in Q. Fix again a trivialization s of P, let $A \in \mathcal{A}_Q \cong \Omega^1(\Sigma;\mathfrak{g})$ and let $g : \Sigma \to G$ be a gauge transformation in Q. Choose extensions $\tilde{A}$ and $\tilde{g}$ of A and g to a connection respectively a gauge transformation in P. By Proposition 2.25 and Lemma 2.27, the function $\Theta : \mathcal{A}_Q \times \mathcal{G}_Q \to \mathrm{U}(1)$ given by

$$\Theta(A,g) = \exp(2\pi i(\mathrm{CS}_s(\tilde{g}^*\tilde{A}) - \mathrm{CS}_s(\tilde{A}))) \tag{2.8}$$

depends neither on the choice of extensions of A and g, nor of the preliminary choice of trivialization. One can show that Θ satisfies the cocycle condition

$$\Theta(g^*A,h)\Theta(A,g) = \Theta(A,gh).$$

We turn now to the question of defining a symplectic structure on (an appropriate subspace of) the moduli space. We do this through a quotient construction, considering a symplectic structure on $\mathcal{A}_Q$. Notice that $\mathcal{A}_Q$ is really an affine infinite-dimensional manifold, modelled on $\Omega^1(\Sigma;\mathfrak{g})$. We ignore all technical details necessary to deal with such objects and simply notice that for a given connection $A \in \mathcal{A}_Q$, there is an identification $T_A\mathcal{A}_Q \cong \Omega^1(\Sigma;\mathfrak{g})$. There then is a natural symplectic form ω on $\mathcal{A}_Q$, invariant under $\mathcal{G}_Q$, defined by

$$\omega(\eta_1,\eta_2) = -\int_\Sigma \langle \eta_1 \wedge \eta_2 \rangle \tag{2.9}$$

for $\eta_1,\eta_2 \in \Omega^1(\Sigma;\mathfrak{g})$. Using a similar identification $T_{\mathrm{Id}}\mathcal{G}_Q \cong C^\infty(\Sigma,\mathfrak{g})$, a moment map $\mu : \mathcal{A} \to C^\infty(\Sigma;\mathfrak{g})^*$ for the action of $\mathcal{G}_Q$ on $\mathcal{A}_Q$ is given by

$$\mu_\xi(A) = 2\int_\Sigma \langle F_A \wedge \xi \rangle,$$

for $\xi \in C^\infty(\Sigma,\mathfrak{g})$, and $A \in \mathcal{A}_Q$ with curvature $F_A \in \Omega^2(\Sigma;\mathfrak{g})$. Notice now that the (infinite-dimensional analogue of the) Marsden–Weinstein quotient

$$\mathcal{M}_Q = \mu^{-1}(\{0\}) /\!\!/ \mathcal{G}_Q$$

is exactly the moduli space $\mathcal{M}_Q$ of flat connections on Q up to gauge equivalence.

Let $\mathcal{A}_Q^*$ be the subset of $\mathcal{A}_Q$ consisting of flat *irreducible* connections in Q, i.e. flat connections A such that ∇^A is injective, and let $\mathcal{M}_Q^* = \mathcal{A}_Q^*/\mathcal{G}_Q$. This space turns out to be an open subset of $\mathcal{M}$ obtaining naturally the structure of a symplectic manifold through the quotient construction.

Now, let $\tilde{\mathcal{L}}_Q = \mathcal{A}_Q \times \mathbb{C}$ be the trivial line bundle over $\mathcal{A}_Q$ and lift the action of $\mathcal{G}_Q$ to $\tilde{\mathcal{L}}_Q$ using Θ. There is then a connection B on $\tilde{\mathcal{L}}_Q$ given in a trivialization $s : \Sigma \to Q$ by

$$(B_s)_A(\eta) = \int_\Sigma \langle A \wedge \eta \rangle,$$

for $A \in \mathcal{A}_Q \cong \Omega^1(\Sigma;\mathfrak{g})$, $\eta \in T_A\mathcal{A}_Q \cong \Omega^1(\Sigma;\mathfrak{g})$. One checks that this indeed defines a connection on $\tilde{\mathcal{L}}_Q$, constructed to satisfy $F_B = \frac{i}{2\pi}\omega$. What is less obvious is that this connection is preserved by the lifted action of $\mathcal{G}_Q$ and induces a connection $\bar{B}$ on the line bundle $\mathcal{L} \to \mathcal{M}_Q^*$ defined to be all equivalence classes of elements of $\mathcal{A}_Q^* \times \mathbb{C}$ under the relation

$$(A,z) \sim (g^*A, \Theta(A,g)z)$$

for all $g \in \mathcal{G}_Q$. The line bundle $\mathcal{L}$ carries a Hermitian structure since Θ is U(1)-valued, and $\bar{B}$ is compatible with this structure. Thus we finally obtain the following:

Theorem 2.28. *Let Σ be a closed oriented surface and $Q \to \Sigma$ a principal G-bundle. Then the moduli space $\mathcal{M}_Q^*$ of flat irreducible connections is prequantizable.*

Freed furthermore discusses the case of a surface with boundary, which is slightly more involved. An important special case is obtained from the space of connections on a surface with one boundary component, having prescribed central holonomy about the boundary component: assume that Σ is a compact surface with genus $g \geq 2$ and a single boundary component. The fundamental group of Σ is then freely generated by α_i, β_i, $i = 1,\dots,g$. Let $p \in \partial\Sigma$, and let $\gamma = \prod_{i=1}^g [\alpha_i,\beta_i] \in \pi_1(\Sigma,p)$ be the class of a loop going once around the boundary. Let $G = \mathrm{SU}(N)$, and let $d \in \mathbb{Z}_N$ be relatively prime to N, or let $(N,d) = (2,0)$ if $g = 2$. Let $D = e^{2\pi i d/N} I \in \mathrm{SU}(N)$, and define

$$\mathrm{Hom}_d(\pi_1(\Sigma,p),\mathrm{SU}(N)) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma,p),\mathrm{SU}(N)) \mid \rho(\gamma) = D\}.$$

The conjugation action acts on this subspace of $\mathrm{Hom}(\pi_1(\Sigma,p),\mathrm{SU}(N))$, and one finds that it consists of irreducible representations. The resulting moduli space

$$\mathcal{M}^d_{\mathrm{SU}(N)} = \mathrm{Hom}_d(\pi_1(\Sigma,p),\mathrm{SU}(N))/\mathrm{SU}(N)$$

is a smooth compact manifold that does not depend on p, and by the construction of Freed [Fre95], it admits a prequantum line bundle.

2.5.3 Teichmüller space and Hitchin's connection

Let Σ be a compact surface, and let $\mathcal{C}(\Sigma)$ be the space of conformal equivalence classes of Riemannian metrics on Σ. Recall that two metrics are called conformally equivalent if they are related by multiplication by a positive function. The group $\mathrm{Diff}(\Sigma)$ of orientation-preserving diffeomorphisms of Σ acts on $\mathcal{C}(\Sigma)$ by pullback.

Definition 2.29. The *Teichmüller space* of Σ is the quotient

$$\mathcal{T}(\Sigma) = \mathcal{C}(\Sigma)/\mathrm{Diff}_0(\Sigma).$$

It is well-known that there is a bijective correspondence between elements of $\mathcal{C}(\Sigma)$ and complex structures on Σ – see e.g. [Jos02]. For that reason, Teichmüller space is often refered to as the space of complex structures on Σ. We note here the well-known fact that $\mathcal{T}(\Sigma)$ is a contractible space and carries a natural complex structure.

From now on, let $P \to \Sigma$ be a principal G-bundle, and let $G = \mathrm{SU}(N)$. As the notation suggests, we want $\mathcal{T}$ to parametrize complex structures on the space $\mathcal{M}^*$ of irreducible flat $\mathrm{SU}(N)$-connections on Σ. A Riemannian metric (or a complex structure) on Σ gives rise to a Hodge star operator $* : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$. Extending this to an operator on Ad_P-valued 1-forms and using the fact that $H^1(\Sigma, \mathrm{Ad}_P)$ identifies with the space $\ker(d_A) \cap \ker(*d_A*)$ of harmonic Ad_P-valued 1-forms, the operator $*$ acts on $T_{[A]}\mathcal{M}^* \cong H^1(\Sigma, \mathrm{Ad}_P)$. Furthermore, $*$ satifies $*^2 = -1$, and defines an almost complex structure $I_\sigma = -*$ on $\mathcal{M}^*$. By work of Narasimhan and Seshadri [NS64], this almost complex structure is integrable. Finally, it can be seen to be compatible with the symplectic structure defined previously and so $\mathcal{M}^*$ obtains the structure of a Kähler manifold $(\mathcal{M}^*, \omega, I_\sigma)$.

This defines a map $I : \mathcal{C}(\Sigma) \to C^\infty(\mathcal{M}^*, \mathrm{End}(T\mathcal{M}^*))$. The group $\mathrm{Diff}(\Sigma)$ also acts on $\mathcal{M}^*$ via pullback or via its action on $\pi_1(\Sigma)$ and it induces an action of $\mathrm{Diff}(\Sigma)$ on $C^\infty(\mathcal{M}^*, \mathrm{End}(T\mathcal{M}^*))$. The map I is equivariant with respect to this action, and $\mathrm{Diff}_0(\Sigma)$ acts trivially on $\mathcal{M}^*$, so we obtain a map

$$I : \mathcal{T}(\Sigma) \to C^\infty(\mathcal{M}^*, \mathrm{End}(TM^*))$$

parametrizing Kähler structures on $\mathcal{M}^*$. In this case, the bundle $\mathcal{V}$ over $\mathcal{T}$ with fibres $\mathcal{V}_\sigma = H^0(\mathcal{M}^*, \mathcal{L}_\sigma^{\otimes k})$ does form a finite rank vector bundle, and the existence of a projectively flat connection in this bundle – as outlined in Section 2.3 – was proved by Hitchin [Hit90] (for $g > 2$ and by van Geemen and de Jong [vGdJ98] for $g = 2$), and independently by Axelrod, Della Pietra, and Witten [ADPW91]. Likewise, Andersen's general contruction summarized in Theorem 2.11 provides a projectively flat connection in this case. We will refer to the connection obtained from any of these constructions as the *Hitchin connection*.

In the case where Σ is a compact surface with genus $g \geq 2$ and one boundary component, the space $\mathcal{M} = \mathcal{M}^d_{\mathrm{SU}(N)}$ carries the structure of a compact Kähler manifold $\mathcal{M}_\sigma = (\mathcal{M}, \omega, I_\sigma)$ for every $\sigma \in \mathcal{T}(\Sigma)$. This moduli space is known to satisfy the conditions of Theorem 2.11, and we may appeal to Andersen's construction to obtain an explicit expression for the Hitchin connection in this case.

Chapter 3
Topological quantum field theory

3.1 Historical background

We begin this chapter with a brief discussion of how Chern–Simons theory heuristically gives rise to the notion of topological quantum field theory (TQFT). Let G be as in Section 2.5.2, and recall that for a closed 3-manifold M with a principal G-bundle $P \to M$, the Chern–Simons action defines a map

$$\mathrm{CS} : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}/\mathbb{Z}.$$

Witten, in the late 80's, interpreted this action as the Lagrangian of a quantum field theory. One important object of these is the *partition function*, in this case given by the path integral

$$Z^{\mathrm{phys}}_{k,G}(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} \exp(2\pi i k \mathrm{CS}(A))\, \mathcal{D}A,$$

for $k \in \mathbb{N}$. From a mathematical point of view, this is ill-defined, as there is currently no way to canonically make sense of the integral over the infinite-dimensional space $\mathcal{A}_P/\mathcal{G}_P$, but Witten argues on the physical level of rigour that it defines a topological invariant of the 3-manifold M. It is worth noting that using the path integral, this invariant can be extended formally to an invariant of pairs (M, L), where L is a link in M, as follows: to every component L_i of L we associate a finite-dimensional representation R_i of G – referred to in the following as a *colouring* or a *labelling* of L_i – and define the *Wilson loop variable*

$$W_{R,A}(L) = \prod_i \mathrm{tr}(R_i(\mathrm{hol}_A(L_i))) \tag{3.1}$$

and the *Wilson loop path integral*

$$Z^{\mathrm{phys}}_{k,G}(M, L, R) = \int_{\mathcal{A}_P/\mathcal{G}_P} W_{R,A}(L) \exp(2\pi i k \mathrm{CS}(A))\, \mathcal{D}A.$$

In this notation, the quantity

$$\langle W_R(L)\rangle = \frac{Z^{\mathrm{phys}}_{k,G}(M, L, R)}{Z^{\mathrm{phys}}_{k,G}(M)}$$

is referred to as the *Wilson loop expectation value*.

In the non-closed case, $\partial M = \Sigma \neq \emptyset$, the aim is to associate to the boundary a vector space $V(\Sigma)$, which from a physical point of view represents physical states on Σ, and to the 3-manifold M a vector $Z_{k,G}^{\mathrm{phys}}(M) \in V(\Sigma)$ which physically represents the time evolution of states. Let $P \to M$ be a principal G-bundle with a trivialization $s : M \to P$. As (a large subset of) the moduli space $\mathcal{M}$ of irreducible connections in $P|_\Sigma \to \Sigma$ is prequantizable, we may apply the general construction outlined in Section 2.2 to obtain the vector space $V(\Sigma) = H^0(\mathcal{M}, \mathcal{L}^{\otimes k})$ of holomorphic sections of the line bundle $\mathcal{L}^{\otimes k} \to \mathcal{M}$. Heuristically, we obtain a vector in this space in the following way: for each $[A] \in \mathcal{M}$, let $\mathcal{A}_A$ be the connections on M restricting to A on on ∂M, and let $\mathcal{G}' \subseteq \mathcal{G}_P$ denote the gauge transformations restricting to gauge transformations on $P|_\Sigma$. Now, let

$$Z_{k,G}^{\mathrm{phys}}(M)([A]) = \int_{\mathcal{A}_A/\mathcal{G}'} \exp(2\pi i k \mathrm{CS}_s(A'))\, \mathcal{D}A'.$$

By the construction of $\mathcal{L}$, it turns out that this formally gives rise to a holomorphic section of $\mathcal{L}^{\otimes k} \to \mathcal{M}$. Again however, this is of course ill-defined as the integral is. Trying to axiomatize the physical formalism, one arrives at the notion of a TQFT. In this chapter, we first describe TQFT from a general point of view and go on to consider several rigorous constructions of such theories.

For more details on the general picture, see e.g. [Oht02, App. F], [Ati90] or Witten's original paper [Wit89].

3.2 The axiomatic point of view

A full mathematical axiomatization of Witten's notion of a TQFT was first put forward by Atiyah, [Ati88]. As we will see later, in the various realizations of the axioms some adjustments are necessary, but the main philosophy of Atiyah's TQFTs will survive; the reader should thus be aware that the properties listed in this section will be replaced later in this chapter. In Atiyah's picture, a *$(d+1)$-dimensional TQFT (Z, V) over a field* Λ, which we will also refer to as an *anomaly-free TQFT* for a reason to be explained in Section 3.3.5, consists of the following data: to every (possibly empty) closed oriented smooth d-manifold Σ, we associate a vector space $V(\Sigma)$ over Λ, and to every (possibly empty) compact oriented smooth $(d+1)$-manifold M, we associate an element $Z(M) \in V(\partial M)$. The associations satisfy the following axioms:

1. (Z, V) is functorial: to every orientation-preserving diffeomorphism
$$f : \Sigma \to \Sigma'$$
of n-dimensional manifolds, we associate a linear isomorphism
$$V(f) : V(\Sigma) \to V(\Sigma')$$
satisfying, for a composition of $f : \Sigma \to \Sigma'$ and $g : \Sigma' \to \Sigma''$, that
$$V(g \circ f) = V(g) \circ V(f).$$
If f extends to an orientation-preserving diffeomorphism $M \to M'$ with $\partial M = \Sigma$, $\partial M' = \Sigma'$, then
$$V(f)(Z(M)) = Z(M').$$

2. (Z, V) is involutive: let $-\Sigma$ denote the n-manifold Σ with the opposite orientation. Then $V(-\Sigma) = V(\Sigma)^*$.

3. (Z, V) is multiplicative: for disjoint unions of surfaces, $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$, and if $M = M_1 \cup_{\Sigma_3} M_2$ is obtained by gluing two $(d+1)$-manifolds M_1, M_2 with boundaries $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup -\Sigma_3$ along Σ_3 (see Figure 3.1), then
$$Z(M) = \langle Z(M_1), Z(M_2) \rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing

$$V(\Sigma_1)\otimes V(\Sigma_3)\otimes V(\Sigma_3)^*\otimes V(\Sigma_2)\to V(\Sigma_1)\otimes V(\Sigma_2).$$

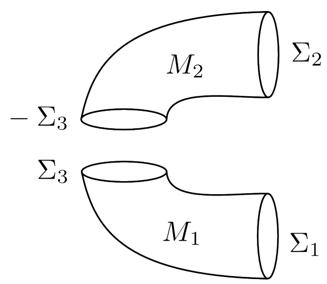


Figure 3.1: The gluing axiom.

The last axiom allows us to think of (Z,V) as a functor from a cobordism category to the category of vector spaces. Namely, if $\partial M=\Sigma_0\cup -\Sigma_1$ for (possibly empty) n-manifolds Σ_0 and Σ_1, then we will view $Z(M)$ as an element

$$Z(M)\in V(\Sigma_0)^*\otimes V(\Sigma_1)=\mathrm{Hom}(V(\Sigma_0),V(\Sigma_1)).$$

The axiom also shows that $V(\emptyset)=V(\emptyset)\otimes V(\emptyset)$, so $V(\emptyset)$ is either trivial or isomorphic to Λ. We will explicitly require the latter. Similarly, $Z(\emptyset)\in\Lambda$ is either 0 or 1, and we explicitly require that $Z(\emptyset)=1$. Finally, the axioms also imply that $Z(\Sigma\times I)\in\mathrm{End}(V(\Sigma))$ is idempotent, and we shall require that $Z(\Sigma\times I)=\mathrm{id}_{V(\Sigma)}$.

3.2.1 Mapping class group representations from TQFTs

The above axioms hint at how to obtain representations of mapping class groups of closed surfaces from a $(2+1)$-dimensional TQFT. Namely, let Σ be a closed oriented surface, and let $f\in\Gamma(\Sigma)$ be the mapping class of an orientation-preserving diffeomorphism $\varphi:\Sigma\to\Sigma$. Put $\rho(f)=V(\varphi):V(\Sigma)\to V(\Sigma)$, and let

$$M_\varphi=\Sigma\times[0,\tfrac{1}{2}]\cup_\varphi\Sigma\times[\tfrac{1}{2},1]$$

be the mapping cylinder of φ obtained by gluing together two copies of $\Sigma\times\{\frac{1}{2}\}$ using φ.

Proposition 3.1. *The map $\rho:\Gamma(\Sigma)\to\mathrm{End}(V(\Sigma))$ is a well-defined representation of $\Gamma(\Sigma)$. Furthermore, if f is the mapping class of φ as above, then $\rho(f)=Z(M_\varphi)$.*

Proof. Let $\varphi_t:\Sigma\to\Sigma$ be an isotopy between orientation-preserving diffeomorphisms φ_0 and φ_1. The map $\Sigma\times I\to\Sigma\times I$ given by

$$(x,t)\mapsto(\varphi_1\varphi_t^{-1}(x),t)$$

extends the map $\varphi_1\varphi_0^{-1}\sqcup\mathrm{id}:\Sigma\sqcup-\Sigma\to\Sigma\sqcup-\Sigma$, and it follows from the axioms that $V(\varphi_1\varphi_0^{-1})=\mathrm{id}$ and $V(\varphi_1)=V(\varphi_0)$.

To prove the last statement, notice that $\varphi\sqcup\mathrm{id}:\Sigma\sqcup-\Sigma\to\Sigma\sqcup-\Sigma$ extends to an orientation-preserving diffeomorphism $\Sigma\times I\to M_\varphi$. It follows that

$$Z(M_\varphi)=V(\varphi)=\rho(f).$$

□

In particular, the axioms imply that for a 3-manifold M with boundary $\partial M = \Sigma$, the action of a mapping class $[\varphi]$ on a vector $Z(M) \in V(\Sigma)$ is given by

$$\rho(f)(Z(M)) = Z(M \cup_\Sigma M_\varphi). \tag{3.2}$$

Let

$$T_\varphi = (\Sigma \times [0,1])/(x,0) \sim (\varphi(x),1)$$

denote the *mapping torus* of a diffeomorphism $\varphi : \Sigma \to \Sigma$, referred to as the *monodromy* of the mapping torus. It then follows from the axioms that $Z(T_\varphi) = \operatorname{tr} V(\varphi)$. In particular one finds the dimensions of the TQFT vector spaces as

$$\dim V(\Sigma) = \operatorname{tr} V(\mathrm{id}|_{V(\Sigma)}) = Z(\Sigma \times S^1).$$

In general, the TQFT defines an invariant of closed oriented $(n+1)$-dimensional manifolds, often called a *quantum invariant*. The so-called *universal construction*, which we will discuss in Section 3.4.4, gives a criterion for extending an invariant of closed manifolds to a TQFT functor. The main example in this report is the skein theoretical construction of [BHMV95]. Before going into details with this construction, we turn to the powerful abstract category theoretical construction of [Tur10]. This construction summarizes the work by Reshetikhin and Turaev [RT90], [RT91], giving the historically first concrete realization of Witten's TQFT.

3.3 The Reshetikhin–Turaev TQFT

3.3.1 Ribbon categories and graphical calculus

We first set up the relevant categorical framework. The notion of a ribbon category will encompass the structures relevant for our constructions.

Definition 3.2. A *ribbon category* is a monoidal category $\mathcal{V}$ with a braiding c, a twist θ, and a compatible duality $(*, b, d)$.

Let us discuss these concepts one at a time.

Definition 3.3. A *monoidal category* is a category $\mathcal{V}$ with a covariant functor $\otimes : \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ associating to two objects V and W an object $V \otimes W$ and to morphisms $f : V \to V'$ and $g : W \to W'$ a morphism $f \otimes g : V \otimes W \to V' \otimes W'$, such that the following properties are satisfied:

1. There exists a unit object $\mathbf{1}$ satisfying $V \otimes \mathbf{1} = V$, and $\mathbf{1} \otimes V = V$ for all objects V in $\mathcal{V}$.
2. For triples U, V, W of objects, we have $(U \otimes V) \otimes W = U \otimes (V \otimes W)$.
3. For morphisms f in $\mathcal{V}$, $f \otimes \mathrm{id}_\mathbf{1} = \mathrm{id}_\mathbf{1} \otimes f = f$.
4. For triples f, g, h of morphisms, we have $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

Remark 3.4. The above definition is really that of a *strict* monoidal category. More generally one could consider categories, where the above equalities are replaced by fixed isomorphisms. By general abstract nonsense, nothing is lost by requiring equality, and we will avoid the more general case completely.

We now introduce the notion of graphical calculus, and discuss it in parallel with the axioms of a ribbon category. A morphism

$$f : W_1 \otimes \cdots \otimes W_m \to V_1 \otimes \cdots \otimes V_n$$

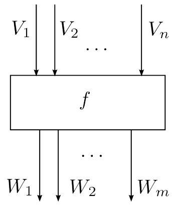


Figure 3.2: A morphism represented by a diagram.

in $\mathcal{V}$ is represented by a diagram with a box with several downwards pointing arrows attached as in Figure 3.2. The cases where either m or n is 0 (or both are) will be allowed as well; here, the corresponding tensor product will be $\mathbf{1}$ by convention. Composition of morphisms is represented by stacking diagrams on top of each other, and tensoring morphims is represented by placing diagrams next to each other; an example is illustrated in Figure 3.3 where $g : U \to W'$ and $f : W \otimes W' \to V$ are two morphisms, and the diagram represents the morphism

$$f \circ (\mathrm{id}_W \otimes g) : W \otimes U \to V.$$

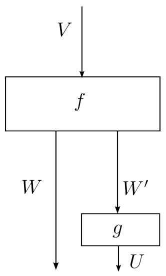


Figure 3.3: Composition and tensor products of morphisms.

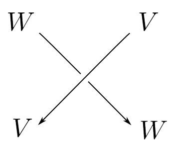


Figure 3.4: The braiding isomorphisms $c_{V,W}$.

Definition 3.5. A *braiding* in $\mathcal{V}$ is a family $c = \{c_{V,W} : V \otimes W \to W \otimes V\}$ of isomorphisms, represented in graphical calculus by the diagram in Figure 3.4, satisfying the following properties, where U, V, W, V', W' are objects of $\mathcal{V}$, and $f : V \to V'$, $g : W \to W'$ are morphisms:

1. $c_{U,V \otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$.
2. $c_{U \otimes V,W} = (c_{U,V} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W})$.
3. $(g \otimes f)c_{V,W} = c_{V',W'}(f \otimes g)$.

As an example, it follows from the definitions that

$$(\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) = (c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W).$$

This is not obvious from the algebraic statements above but is a natural consequence of the existence of graphical calculus and Theorem 3.9 below.

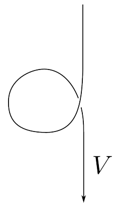


Figure 3.5: The twist isomorphisms θ_V.

Definition 3.6. A *twist* in $(\mathcal{V}, c)$ is a family $\theta = \{\theta_V : V \to V\}$ of isomorphisms, represented diagrammatically as in Figure 3.5, satisfying the relations

$$\begin{aligned} \theta_{V\otimes W} &= c_{W,V}c_{V,W}(\theta_V \otimes \theta_W), \\ \theta_V f &= f\theta_U \end{aligned}$$

for objects V, U, W and morphisms $f : U \to V$ in $\mathcal{V}$.

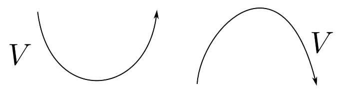


Figure 3.6: The duality morphisms b_V and d_V respectively.

Definition 3.7. A *duality* in $\mathcal{V}$ associates to every object V of $\mathcal{V}$ an object V^* of $\mathcal{V}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbf{1}$, satisfying the following:

1. $(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$.
2. $(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$.

A duality $(*, b, d)$ in a category $(\mathcal{V}, c, \theta)$ with braiding and twist is called *compatible* with the braiding and twist if furthermore

$$(\theta_V \otimes \mathrm{id}_{V^*})b_V = (\mathrm{id}_V \otimes \theta_{V^*})b_V.$$

In graphical calculus, a downward arrow coloured by an object V^* will be used interchangeably with upward arrows coloured by V, and we will represent b_V and d_V by the diagrams in Figure 3.6.

3.3.2 The Reshetikhin–Turaev functor

The graphical calculus hints that we might be able to associate to an oriented graph in $\mathbb{R}^2 \times [0, 1]$ – where strings are coloured by objects of $\mathcal{V}$, and possibly containing a number of boxes, called coupons – a pair of objects in $\mathcal{V}$, and a morphism between them. The existence of the Reshetikhin–Turaev functor tells us that this is indeed the case. In this subsection, this is made somewhat more precise.

A *band* is a homeomorphic image in $\mathbb{R}^2 \times [0, 1]$ of $[0, 1] \times [0, 1]$, an *annulus* is a homeomorphic image of $S^1 \times [0, 1]$, and a *coupon* is a band with distinguished bases $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$.

Definition 3.8. A *ribbon* (k, l)*-graph* (or a *ribbon graph* for short) is an oriented surface embedded in $\mathbb{R}^2 \times [0, 1]$, decomposed into a union of directed bands, directed annuli, and coupons, such that bases of bands meet either the planes $\mathbb{R}^2 \times \{0, 1\}$ or the bases of coupons; see Figure 3.7 (and see [Tur10] for a more precise statement). Here, k bands meet $\mathbb{R}^2 \times \{0\}$, and l bands meet $\mathbb{R}^2 \times \{1\}$.

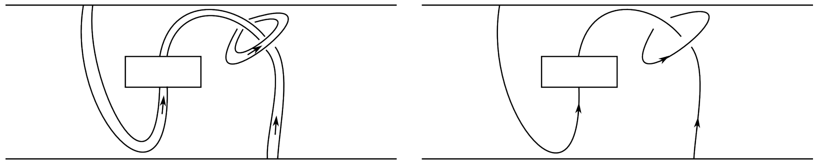


Figure 3.7: An example of a $(1,1)$-ribbon graph and how to represent it by a 1-dimensional graph using the blackboard framing.

To simplify drawings in what follows, we will often picture bands and annuli in ribbon graphs using the blackboard framing as in Figure 3.7; here, the 1-dimensional graph represents the ribbon graph obtained by letting bands and annuli be parallel to the plane of the picture. This is possible since ribbon graphs are always assumed to be orientable. Note that the theory of ribbon graphs therefore in particular contains the theory of banded oriented links in S^3.

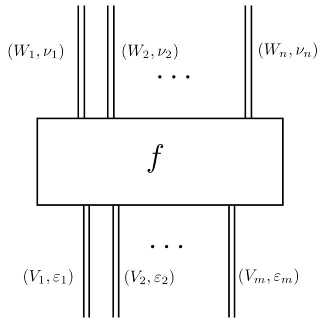


Figure 3.8: The colouring of a coupon.

Assume that $\mathcal{V}$ is a monoidal category with duality. A colouring[1] of a ribbon graph is an association of an object of $\mathcal{V}$ to every non-coupon band and every annulus, and an association of a morphism

$$f : V_1^{\varepsilon_1} \otimes \cdots \otimes V_m^{\varepsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n}$$

to every coupon meeting bands with colours V_i, W_j as in Figure 3.8. Here, $\varepsilon_i, \nu_j \in \{-1, 1\}$, and $V^1 = V$, $V^{-1} = V^*$ for an object V. For every V_i, W_j, the appropriate sign is determined by the orientation of the bands in question, the sign being positive if the band is directed downwards, and negative otherwise. This turns the set of coloured ribbon graphs into a monoidal category denoted $\mathrm{Rib}_\mathcal{V}$. Objects are sequences $((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m))$, where $\varepsilon_i \in \{-1, 1\}$, and morphisms $\eta \to \eta'$ are *isotopy types* of coloured ribbon graphs meeting in $\mathbb{R}^2 \times \{0\}$ the sequence η, and similarly in $\mathbb{R}^2 \times \{1\}$ the sequence η', where the numbers ε_i determine the orientations of the bands. Here composition is given by stacking graphs and tensor product is given by placing graphs next to each other.

Theorem 3.9. *Let $\mathcal{V}$ be a ribbon category. There is a covariant functor*

$$F = F_\mathcal{V} : \mathrm{Rib}_\mathcal{V} \to \mathcal{V}$$

preserving tensor product, transforming (V, ε) to V^ε, and transforming morphisms to the natural corresponding morphisms in $\mathcal{V}$: namely, consider the diagrams of graphical calculus as ribbon graphs, taking parallels as above. Then the functor transforms these graphs to the morphisms of $\mathcal{V}$ represented by the diagrams. Furthermore, imposing a few conditions outlined in [Tur10], the functor is unique.

[1] In the language of [Tur10], this is a v-colouring.

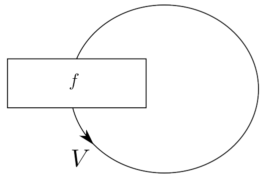


Figure 3.9: The coloured ribbon graph Ω_f corresponding under F to $\mathrm{tr}(f)$.

For a morphism $f : V \to V$ in a ribbon category, define the *quantum trace* $\mathrm{tr}(f) : \mathbf{1} \to \mathbf{1}$ by

$$\mathrm{tr}(f) = d_V c_{V,V^*}((\theta_V f) \otimes \mathrm{id}_{V^*}).$$

Then $F(\Omega_f) = \mathrm{tr}(f)$, where Ω_f is the coloured ribbon graph shown in Figure 3.9. It follows immediately that $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ and $\mathrm{tr}(f \otimes g) = \mathrm{tr}(g \otimes f)$. Define the *quantum dimension*

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V).$$

Then $\dim(V \otimes W) = \dim(V)\dim(W)$.

3.3.3 Modular categories

As already mentioned, our quest is to find invariants of 3-manifolds fitting into the framework of TQFT. The operator invariants of coloured ribbon graphs constructed above are too general to be useful for this purpose; rather, the ribbon categories used in the construction should satisfy several other conditions, summed up in the notion of a modular category.

Definition 3.10. An *Ab-category* is a monoidal category $\mathcal{V}$ such that the set $\mathrm{Hom}(V, W)$ of morphisms $V \to W$ has the structure of an additive abelian group and such that composition is bilinear. In particular, this holds for $\mathrm{End}(\mathbf{1})$, and this group obtains the structure of a (commutative) ring, with multiplication being given by composition; it is denoted K and called the *ground ring* of $\mathcal{V}$.

Definition 3.11. An object V of a ribbon Ab-category $\mathcal{V}$ is called *simple*, if the map $k \mapsto k \otimes \mathrm{id}_V$ defines a bijection $K \to \mathrm{Hom}(V, V)$.

Definition 3.12. Let $\{V_i\}_{i\in I}$ be a family of objects in a ribbon Ab-category $\mathcal{V}$. An object V of $\mathcal{V}$ is *dominated* by $\{V_i\}$, if there exists a finite subfamily $\{V_{i(r)}\}_r$ and morphisms $f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)}$ such that $\mathrm{id}_V = \sum_r f_r g_r$.

Definition 3.13. A *modular category* is a ribbon Ab-category $\mathcal{V}$ with a finite set of simple objects $\{V_i\}_{i\in I}$ satisfying the following axioms:

1. $\mathbf{1} \in \{V_i\}_{i\in I}$.
2. For any $i \in I$, there exists $i^* \in I$ such that V_{i^*} is isomorphic to $(V_i)^*$.
3. All objects of $\mathcal{V}$ are dominated by $\{V_i\}_{i\in I}$.
4. The matrix S with entries $S_{i,j} = \mathrm{tr}(c_{V_j,V_i} c_{V_i,V_j}) \in K$ is invertible over K.

Let $(\mathcal{V}, \{V_i\}_{i\in I})$ be a modular category. We are now in a position to define the link invariant that will give rise to our 3-manifold invariant. Let L be an oriented banded m-component link in S^3, viewed as a link in $\mathbb{R}^2 \times [0, 1]$. Let $\mathrm{col}(L)$ be the finite set of all possible colourings of L by elements of $\{V_i\}_{i\in I}$. For $\lambda \in \mathrm{col}(L)$, let $\Gamma(L, \lambda)$ be the coloured ribbon $(0,0)$-graph obtained from L by colouring it according to λ. Now $F(\Gamma(L,\lambda))$ is an element of K, and we define

$$\{L\} = \sum_{\lambda \in \mathrm{col}(L)} \prod_{n=1}^{m} \dim(\lambda(L_n)) F(\Gamma(L,\lambda)) \in K.$$

Note that this expression does not depend on the orientation of L.

3.3.4 Surgery and 3-manifold invariants

Let L be an m-component banded link in S^3. Choose a regular closed neighbourhood U of L, consisting of m disjoint solid tori $U_1, \dots, U_m$. Each of these are homeomorphic to $S^1 \times D^2$ with boundary homeomorphic to $S^1 \times S^1$. Choose homeomorphisms $h_i : S^1 \times S^1 \to S^1 \times S^1$ and form the space

$$M_L = (S^3 \setminus U) \cup_{h_i} (\sqcup_{i=1}^m D^2 \times S^1)$$

which is the disjoint union of $S^3 \setminus U$ and m copies of solid tori $D^2 \times S^1$, these glued together along their common boundary $\sqcup_{i=1}^m S^1 \times S^1$ using the homeomorphisms h_i. The resulting topological space M_L is a closed orientable manifold. The space M_L constructed as such depends of course on the homeomorphisms involved in the gluing. Using the banded structure of L, one canonically obtains particular homeomorphisms h_i depending only on L, and the resulting surgery is referred to as *integral surgery*. Using this, we say that M_L is *obtained by surgery on S^3 along L.*

Theorem 3.14 (Lickorish, Wallace)**.** *Any closed connected oriented 3-manifold can be obtained by (integral) surgery on S^3 along a banded m-component link.*

Let L be an oriented m-component banded link. Let $\sigma(L)$ be the signature of the linking matrix $A_{ij} = \mathrm{lk}(L_i, L_j)$ consisting of linking numbers of the components. Here, the linking number of a banded knot $L_i = K \times [0,1]$ with itself is defined to be the linking number of its boundary knots $K \times \{0\}$ and $K \times \{1\}$. Note that $\sigma(L)$ is independent of the orientations of the components.

Let as before $(\mathcal{V}, \{V_i\}_{i\in I})$ be a modular category. Assume that there is an element $\mathcal{D} \in K$ called a *rank* satisfying $\mathcal{D}^2 = \sum_{i\in I}(\dim(V_i))^2$. Since the V_i are simple, each twist θ_{V_i} can be identified with an element $v_i \in K$, which is invertible, since θ_{V_i} is an isomorphism. Now, set

$$\Delta = \sum_{i\in I} v_i^{-1} (\dim(V_i))^2 \in K.$$

The dimensions $\dim(V_i)$ are invertible because of axiom (4) of a modular category, and it follows that both $\mathcal{D}$ and Δ are invertible in K.

We can now define the quantum invariant of 3-manifolds.

Theorem 3.15 (Reshetikhin–Turaev)**.** *Let M be a closed connected oriented 3-manifold obtained by surgery on S^3 along a banded link L. Then*

$$\tau(M) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \{L\} \in K$$

is a topological invariant of M.

Finally, we extend the invariant to an invariant of 3-manifolds containing coloured ribbon $(0,0)$-graphs. Let M be a closed connected oriented 3-manifold containing a coloured ribbon graph Ω. Assume that M is the result of surgery along an m-component oriented link L in

S^3, and assume by applying isotopy that Ω does not meet the regular neighbourhood U of L used in the surgery, so we can view Ω as a ribbon graph in $S^3 \setminus U$. Now, let

$$\{L, \Omega\} = \sum_{\lambda \in \mathrm{col}(L)} \prod_{n=1}^{m} \dim(\lambda(L_n)) F(\Gamma(L, \lambda) \cup \Omega) \in K.$$

As before, this will not depend on the orientation of L or the numbering of the components.

Theorem 3.16. *Let* (M, Ω) *be as above. Then*

$$\tau(M, \Omega) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \{L, \Omega\} \in K$$

is a topological invariant of the pair (M, Ω).

The proof of this theorem uses the theorem of Kirby stating that closed oriented manifolds obtained by surgery along banded links L and L' are homeomorphic (by orientation-preserving homeomorphism) if and only if L and L' are related by a certain sequence of moves on banded links called Kirby moves.

3.3.5 Modular functors and TQFT

Following [Tur10, Chap. III, IV], we now discuss how the invariants defined above fit with the wishlist of Section 3.2. One major disparity will be the fact that the gluing axiom will only hold up to a projective factor. From the physical viewpoint, this may be seen as an incarnation of the ill-definedness of the path integral measure (see [Fre09]), and it will imply that – without further modification – the mapping class group representations of Section 3.2.1 will only be projective; we return to a fuller discussion of these in Chapter 4.

Let for the rest of this section $\mathcal{V}$ be a modular category with simple objects $\{V_i\}_{i \in I}$ and ground ring K.

Definition 3.17. A *labelled marked surface* (Σ, l, λ) (which we will often write Σ or (Σ, λ) when there is no risk of confusion) is a closed oriented surface Σ together with a finite set $l = (p_1, \dots, p_n)$ of *labelled points* in Σ, choices of projective tangent vectors $v_p \in \mathbb{P}T_pS$ at each point in l, as well as choices of *labels* or *colours* $\lambda_j \in I$ and signs $\varepsilon_j \in \{\pm 1\}$ for each p_j.

A *morphism* of labelled marked surfaces Σ_1 and Σ_2 is a pair (f, s), where f is the isotopy class of a diffeomorphism $\Sigma_1 \to \Sigma_2$ preserving the labelled points, their projective tangent vectors, their signs, and their labels, and $s \in \mathbb{Z}$ is an integer.

In order to most easily be able to make sense of morphisms induced by cobordisms, it will be convenient to relate marked labelled surfaces to a particular standard surface: given the data of a connected marked labelled surface, Turaev [Tur10, Sect. IV.1.2] defines a reference surface Σ_0 with the same genus, marked points and marked point data as Σ (see Figure 3.10). A *parametrized surface* is a connected labelled marked surface Σ together with a morphism $\Sigma_0 \to \Sigma$. A *morphism* of parametrized surfaces is a morphism of the underlying labelled marked surfaces commuting with the parametrizations. These notions extend to non-connected surfaces in the obvious way.

Definition 3.18. A (2-dimensional) *modular functor* associates to each parametrized surface Σ a K-module $V(\Sigma)$ and to every morphism $f : \Sigma_1 \to \Sigma_2$ of parametrized surfaces a morphism $f_* : V(\Sigma_1) \to V(\Sigma_2)$ satisfying the following axioms:

- Functoriality: $(fg)_* = f_* g_*$.
- Disjoint union: $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes_K V(\Sigma_2)$, commutative and associative in the obvious senses and natural in the sense that $(f \sqcup g)_* = f_* \otimes g_*$ for morphisms $f : \Sigma_1 \to \Sigma_1'$, $g : \Sigma_2 \to \Sigma_2'$.

- Empty surface: $V(\emptyset) = K$ and that $V(Y) = V(\emptyset \sqcup Y) = V(\emptyset) \otimes_K V(Y)$ is induced by $V(\emptyset) = K$.
- Duality: for any labelled marked surface (Σ, λ), there is a bilinear pairing
$$\langle \cdot, \cdot \rangle : V(\Sigma, \lambda) \otimes_K V(-\Sigma, \lambda^*) \to K,$$
where $(-\Sigma, \lambda^*)$ denotes the labelled marked surface of opposite orientation and dual labels. The pairing is assumed to be compatible in the natural sense with morphisms, disjoint union and reversal of orientation.

For a connected parametrized surface (Σ, l, λ) of genus g, let

$$V(\Sigma, l, \lambda) = \bigoplus_{(\mu_1,\dots,\mu_g)\in I^g} \operatorname{Hom}\left(\mathbf{1}, V_{\lambda_1}^{\varepsilon_1} \otimes \cdots \otimes V_{\lambda_n}^{\varepsilon_n} \otimes \bigotimes_{r=1}^{g} (V_{\mu_r} \otimes V_{\mu_r}^*)\right). \tag{3.3}$$

For a non-connected parametrized surface Σ, let $V(\Sigma)$ be the (non-ordered) tensor product of the modules associated by (3.3) to each component. To a morphism $f : \Sigma \to \Sigma'$ of parametrized surfaces, let $f_* : V(\Sigma) \to V(\Sigma')$ be the trivial map.

Proposition 3.19 ([Tur10])**.** *The functor* $(\Sigma \mapsto V(\Sigma), f \mapsto f_*)$ *is modular.*

Definition 3.20. A *decorated* 3*-manifold* (M, Ω) is a compact oriented 3-manifold M with parametrized boundary containing a coloured ribbon graph Ω meeting the boundary surface in the distinguished points (see Figure 3.10) so that points of sign $+1$ (resp. -1) meet bands whose orientation are direct inwards (resp. outwards), the banded structure coinciding with the choice of tangent vectors at each point. Moreover, colours of the bands agree with the colours of the points.

We are now finally in a position to define the TQFT associated to a given modular category. Assume that $(M, \partial_- M, \partial_+ M)$ is a triple of a decorated 3-manifold (M, Ω) with parametrized boundary $\partial M = (-\partial_- M) \sqcup \partial_+ M$. We define a map

$$\tau(M, \Omega) = \tau((M, \Omega), \partial_- M, \partial_+ M) : V(\partial_- M) \to V(\partial_+ M) \tag{3.4}$$

as follows: for each component of the boundary surfaces $\partial_- M, \partial_+ M$ and each choice of $f \in V(\partial_- M)$, $g \in V(\partial_+ M)^*$, we fill in a standard handlebody containing a ribbon graph coloured by f and g in the natural way using the boundary parametrizations, so as to obtain a *closed* 3-manifold $\hat{M}$ containing a ribbon graph $\hat{\Omega}(f, g)$. We then let

$$\tau(M, \Omega)(f, g) = \tau(\hat{M}, \hat{\Omega}(f, g)).$$

See Figure 3.10 for an illustration of this construction, and see [Tur10, Sect. IV.1] for the (many) details left out in this account. We can now state the following, highly non-trivial, result which makes precise the relation with Atyiah's TQFT axioms; for a proof, see [Tur10, Thm. IV.1.9, Thm.IV.4.3].

Theorem 3.21. *The function* $(M, \partial_- M, \partial_+ M) \mapsto \tau(M)$ *extends the modular functor* V *to a non-degenerate topological quantum field theory in the following sense:*

- *Naturality: if* $(M_1, \partial_- M_1, \partial_+ M_1)$ *and* $(M_2, \partial_- M_2, \partial_+ M_2)$ *are decorated* 3*-manifolds and* $f : M_1 \to M_2$ *is a diffeomorphism preserving the boundary structure, then* $(f|_{\partial_+(M_1)})_* \circ \tau(M_1) = \tau(M_2) \circ (f|_{\partial_-(M_1)})_*$.
- *Multiplicativity: if* M *is the disjoint union of* M_1 *and* M_2*, then* $\tau(M) = \tau(M_1) \otimes \tau(M_2)$.
- *Gluing: if* M *is obtained by gluing* M_1 *and* M_2 *along* $f : \partial_+(M_1) \to \partial_-(M_2)$*, then*
$$\tau(M) = (\mathcal{D}\Delta^{-1})^m \tau(M_2) f_* \tau(M_1)$$
for some $m \in \mathbb{Z}$.

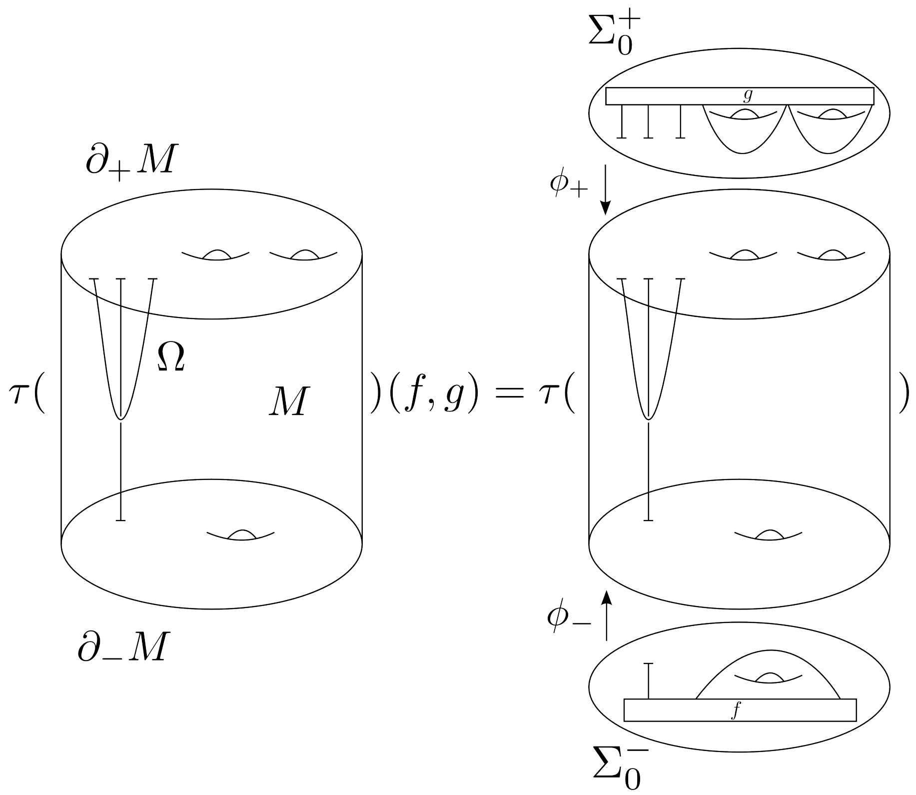


Figure 3.10: The construction of $\tau(M,\Omega)(f,g)$: one obtains from (M,Ω) a closed 3-manifold $\hat{M}$ by gluing in standard handlebodies with boundaries Σ_0^+ and Σ_0^- according to the parametrizations $\varphi^\pm : \Sigma_0^\pm \to \partial_\pm M$. The handlebodies contain coupons coloured by f and g. Orientations depend on the choices of signs at the marked points and are omitted in the figure.

- *Normalization: let (Σ, l, λ) be a labelled marked surface, and let $\Sigma \times [0,1]$ be the decorated 3-manifold with parametrized boundary $\Sigma \times \{0\} = -\Sigma$, $\Sigma \times \{1\} = \Sigma$ and the ribbon graph formed by the cylinders $\Sigma \times l$. Then*

$$\tau(\Sigma \times [0,1]) = \mathrm{id}|_{V(\Sigma)}.$$

- *Non-degeneracy: the space $V(\Sigma)$ is generated by $\tau(M,\Sigma')$, where (M,Σ') runs over all decorated 3-manifolds with parametrized boundary Σ' and all morphisms $f : \Sigma \to \Sigma'$.*

Remark 3.22. The integer m appearing in the gluing formula above may be determined explicitly by endowing the decorated surfaces with even further structure. How to do so is shown in [Tur10, Sects. IV.3–4]. For our purposes, it will suffice to simply consider the gluing axiom to be true up to a power of $\mathcal{D}\Delta^{-1}$, these constants being calculated explicitly in Section 4.3.3.

3.3.6 Modular categories from quantum groups

As we have seen, we may define a TQFT for any modular category. For completeness, we briefly include the main points of the construction by Reshetikhin and Turaev of the modular category using quantum groups. A streamlined introduction giving all details is available in [Tur10, Ch. XI].

For any given Hopf algebra A over a commutative unital ring K, one might consider the category of representations of A, denoted $\mathrm{Rep}(A)$, with objects being finite rank A-modules and morphisms being A-homomorphisms. This category is a monoidal Ab-category.

Furthermore, the category has a natural duality pairing. To provide this category with a braiding, the algebra should further have the structure of a *quasitriangular* Hopf algebra; this means that there is a distinct element $R \in A^{\otimes 2}$ (often refered to as an R-matrix) satisfying certain conditions. Finally, to get a twist in $\mathrm{Rep}(A)$, one fixes an element v in the center of A satisfying again particular conditions. With all of these in place, $\mathrm{Rep}(A)$ acquires the structure of a ribbon Ab-category. For the representation category to be a modular category, we furthermore need a finite collection of finite rank A-modules $\{V_i\}_{i\in I}$ that are *simple*, in the sense that their only endomorphisms are multiplications by scalars and which furthermore satisfy the following conditions:

1. For some element $0 \in I$, we have $V_0 = K$ (where A acts by the Hopf algebra counit).

2. For every $i \in I$, there exists $i^* \in I$ so that V_{i^*} and $(V_i)^*$ are isomorphic.

3. For every $k, l \in I$ the tensor product $V_k \otimes V_l$ splits as a finite direct sum of V_i, $i \in I$, and a module V satisfying $\mathrm{tr}_q(f) = 0$ for any $f \in \mathrm{End}(V)$. Here, tr_q denotes the trace defined in Section 3.3.2.

4. Denoting by $S_{i,j}$ the quantum trace of $x \mapsto \mathrm{flip}_{A,A}(R)Rx$ on $V_i \otimes V_j$, we obtain an invertible matrix $[S_{i,j}]_{i,j\in I}$. Here, $\mathrm{flip}_{V,W} : V \otimes W \to W \otimes V$ is the homomorphism defined by $v \otimes w \mapsto w \otimes v$.

We thus obtain exactly what we are looking for: if a collection of A-modules as above exists, then $\mathrm{Rep}(A)$ has a (non-full) modular subcategory.

It thus remains to construct Hopf algebras satisfying all of the above conditions. These arise in the language of quantum groups. While quantum groups can be defined for general simple Lie algebras, let us consider only $\mathfrak{sl}_2$ for which the construction is known to work out. The quantum group $U_q(\mathfrak{sl}_2)$ is defined to be the algebra over $\mathbb{C}$ generated by elements K, K^{-1}, E, F with relations

$$\begin{gathered} K^{-1}K = KK^{-1} = 1, \\ KE = q^{-1}EK, \quad KF = qFK, \\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \end{gathered}$$

Assume for simplicity that q is a primitive l'th root of unity with l even; this turns out to be the most simple setting, and in general one cannot hope for the construction to work. The quantum group $U_q(\mathfrak{sl}_2)$ is not quite good enough for our purposes, and we consider instead the quotient $\tilde{U}_q(\mathfrak{sl}_2)$ of $U_q(\mathfrak{sl}_2)$ by the two-sided ideal generated by $E^{l/2}, F^{l/2}, K^l - 1$. Then $\tilde{U}_q(\mathfrak{sl}_2)$ can be endowed with an R-matrix and a twist satisfying the necessary conditions for its representation category to give rise to a modular category. Here, the role of the simple modules described above will be played by certain irreducible $\tilde{U}_q(\mathfrak{sl}_2)$-modules. The construction for $\mathfrak{sl}_2$ was first carried out in [RT90], [RT91], and has since be generalized to other Lie algebras. All in all, we obtain for every even l a TQFT by the general construction outlined in the previous section.

3.4 Skein theory of the Kauffman bracket

We now switch gears again and turn to the skein theoretical construction of the quantum invariant which is essentially equivalent to the one by Reshetikhin–Turaev but more suited for our purposes. The version we describe was constructed in [BHMV92], [BHMV91] and proven to lead to a TQFT in [BHMV95].

3.4.1 Skein modules

Definition 3.23. Let M be a compact oriented 3-manifold. The *Kauffman module* $K(M)$ of M is the $\mathbb{Z}[A, A^{-1}]$-module generated by all isotopy classes of banded links in M quotiented by the skein relations shown in Figure 3.11. Here, equivalences are assumed to take place in some small 3-ball in M. Elements of $K(M)$ are called *skeins*.

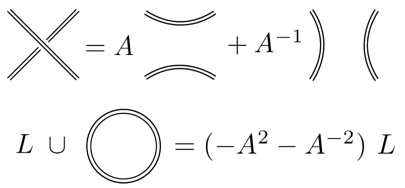


Figure 3.11: The skein relations.

It is well-known that

$$K(S^3) \cong \mathbb{Z}[A, A^{-1}].$$

The skein class of a banded link L in S^3 is called its *Kauffman bracket* and is denoted $\langle L \rangle$. For a closed surface Σ, $K(\Sigma \times I)$ is a $\mathbb{Z}[A, A^{-1}]$-algebra with multiplication given by stacking copies of $\Sigma \times I$. For the solid torus, the Kauffman module will be denoted $\mathcal{B}$ and is the algebra

$$\mathcal{B} = K(S^1 \times I \times I) \cong \mathbb{Z}[A, A^{-1}][z].$$

Under the latter isomorphism, z^n corresponds to n parallel unknotted longitudes in the solid torus[2]. The Kauffman module of a disjoint union of n solid tori is $\mathcal{B}^{\otimes n}$.

Definition 3.24. Let L be an n-component banded link in S^3 with ordered components $L_1, \ldots, L_n$, and let $z^{a_i}, \ldots, z^{a_n}$ be monomials in $\mathcal{B}$. Let $\langle z^{a_1}, \ldots, z^{a_n} \rangle_L$ be the result of replacing each L_i by a_i parallel copies and taking the Kauffman bracket of the resulting link in S^3. Extending linearly, we obtain the *meta-bracket*

$$\langle \cdot, \ldots, \cdot \rangle_L : \mathcal{B}^{\otimes n} \to \mathbb{Z}[A, A^{-1}].$$

Diagrammatically, this value will be pictured as in Figure 3.12. We say that L_i is *coloured* by a_i.

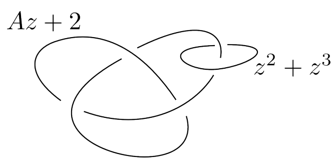


Figure 3.12: The diagrammatic notation for the metabracket of a 2-component link whose components are coloured by $Az + 2$ and $z^2 + z^3$.

3.4.2 The quantum invariant

Let $t : \mathcal{B} \to \mathcal{B}$ be the map induced on $\mathcal{B}$ by a *right* twist about a meridian of the solid torus. We then obtain the following (see [BHMV92]).

[2] Here and in the following, we will not explicitly distinguish between banded links and isotopy classes of banded links.

Lemma 3.25. *There is a basis e_i, $i = 0, 1, \dots,$ of $\mathcal{B}$ consisting of monic polynomials satisfying $e_0 = 1$, $e_1 = z$, $ze_j = e_{j+1} + e_{j-1}$. The e_i are eigenvectors for t with eigenvalues $\mu_i = (-1)^i A^{i^2+2i}$.*

This basis arises naturally from idempotents in the Temperley–Lieb algebra, as we will see in the next section.

From now on, we will assume that A is a primitive root of unity of order $2p$ for some $p \geq 1$. In other words, we will consider the above construction with coefficients in the ring $\Lambda_p = \mathbb{Z}[A, A^{-1}]/\varphi_{2p}(A)$, where φ_{2p} is the $2p$'th cyclotomic polynomial, and we let $\mathcal{B}_p = \mathcal{B} \otimes \Lambda_p$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $\mathcal{B}_p$ defined by using the meta-bracket on the Hopf link, let N_p denote the left-kernel of this form, and let $V_p = \mathcal{B}_p / N_p$.

Theorem 3.26 ([BHMV92])**.** *The meta-bracket factors through $V_p^{\otimes n}$, and t descends to a map $t : V_p \to V_p$. Furthermore, V_p is a finite-dimensional algebra of rank p for $p = 1, 2$ and of rank $\lfloor (p-1)/2 \rfloor$ for $p \geq 3$.*

We are now in a position to define the 3-manifold invariant. For a banded link in S^3, we denote by $b_+(L)$ and $b_-(L)$ the number of positive, respectively negative, eigenvalues of the linking matrix of L as defined in Section 3.3.4. Define an element of V_p (except for $p = 2$, where it is an element of $V_2 \otimes \mathbb{Z}[\frac{1}{2}]$) by $\Omega_1 = 1$, $\Omega_2 = 1 + \frac{z}{2}$, and

$$\Omega_p = \sum_{i=0}^{n-1} \langle e_i \rangle e_i$$

for $p \geq 3$, where $n = \lfloor (p-1)/2 \rfloor$ is the rank of V_p. It turns out that $\langle t^{\pm 1}(\Omega_p) \rangle$ are invertible elements of $\Lambda_p[\frac{1}{p}]$. Now, let M_L be a closed oriented 3-manifold obtained by integral surgery along L, and define an element of $\Lambda_p[\frac{1}{p}]$ by

$$\theta_p(M_L) = \frac{\langle \Omega_p, \dots, \Omega_p \rangle_L}{\langle t(\Omega_p) \rangle^{b_+(L)} \langle t^{-1}(\Omega_p) \rangle^{b_-(L)}}. \tag{3.5}$$

Theorem 3.27 ([BHMV92])**.** *The expression θ_p defines a topological invariant of closed oriented 3-manifolds.*

The proof once again relies on Kirby's theorem on how links giving rise to homeomorphic manifolds via surgery are related by certain moves. The right hand side of (3.5) is almost directly seen to be an invariant under Kirby moves, since in fact Ω_p satisfies $\langle t^{\pm 1}(\Omega_p), t^{\pm 1}(b) \rangle = \langle t^{\pm 1}(\Omega_p) \rangle \langle b \rangle$ for all $b \in \mathcal{B}$ (see also [BHMV92, Prop. 2.1]).

As before, we can extend the above invariant to an invariant of 3-manifolds with banded links. Let $K \subseteq M_L$ be a banded link viewed as a banded link in $S^3 \setminus L$.

Theorem 3.28 ([BHMV91])**.** *The element of $\Lambda_p[\frac{1}{p}]$ defined by*

$$\theta_p(M_L, K) = \frac{\langle \Omega_p, \dots, \Omega_p, z, \dots, z \rangle_{L \cup K}}{\langle t(\Omega_p) \rangle^{b_+(L)} \langle t^{-1}(\Omega_p) \rangle^{b_-(L)}}$$

is a topological invariant of closed oriented 3-manifolds containing banded links.

3.4.3 The Temperley–Lieb algebra

Definition 3.29. The *n'th Temperley–Lieb algebra*, denoted TL_n, is the Kauffman module $K(I \times I \times I, 2n)$ of the unit ball, where we fix $2n$ ordered small intervals on the boundary. That is, TL_n consists of isotopy classes of banded tangles meeting the boundary in the $2n$ intervals. The algebra structure is given by using the ordering to the divide the $2n$ intervals into two sets of n intervals, and gluing together copies of $I \times I \times I$, such that n intervals are glued to n intervals.

The Temperley–Lieb algebra TL_n is generated by n elements $1, a_1, \ldots, a_{n-1}$ shown in Figure 3.13. Here, as before, arcs in the diagram will represent bands parallel to the plane of the diagram. By definition, an integer i next to an arc corresponds to taking i parallels of the arc in the plane.

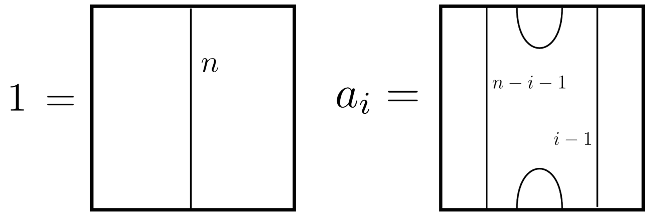


Figure 3.13: Generators of the Temperley–Lieb algebra TL_n.

Assume as before that A is a primitive $2p$'th root of unity, $p > 2$, so that $A^2 - A^{-2}$ is invertible. Define Δ_i by

$$\Delta_i = (-1)^i \frac{A^{2(i+1)} - A^{-2(i+1)}}{A^2 - A^{-2}}.$$

If A is chosen so that all $\Delta_0, \ldots, \Delta_{n-1}$ are non-zero, then there exists a non-zero element $f^{(n)} \in TL_n$, called the n'th *Jones–Wenzl idempotent*, satisfying

$$(f^{(n)})^2 = f^{(n)}, \qquad f^{(n)} e_i = e_i f^{(n)} = 0, \;\; 1 \le i \le n-1.$$

In particular, these will exist if A^4 is not a k'th root of unity for any $k \le n$. Diagrammatically, we denote $f^{(n)}$ as in Figure 3.14. The proof of the existence of the $f^{(n)}$ goes by inductively defining $f^{(n+1)}$ as in Figure 3.15. This recursive formula is due to Wenzl.

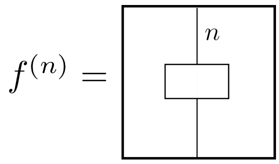


Figure 3.14: The diagram for a Jones–Wenzl idempotent.

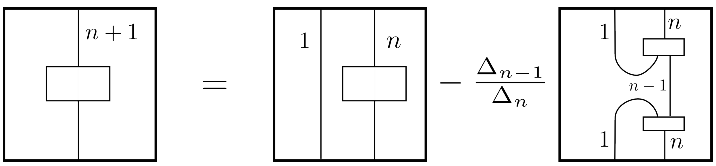


Figure 3.15: Wenzl's recursive formula for $f^{(n+1)}$.

We define a map $\mathrm{TL}_n \to \mathcal{B}$ by placing $I \times I \times I$ in the solid torus and taking the closure, i.e. joining the n intervals on the top to the n intervals on the bottom by parallel arcs in the torus, encircling the torus. The image of $f^{(n)}$ is exactly the e_n used to define Ω_p, and $\langle e_n \rangle = \Delta_n$. The last equality follows by induction on the equality from Lemma 3.25 since $\langle z^k \rangle = (-A^2 - A^{-2})^k$.

3.4.4 The universal construction and TQFT

We turn now to the question of how to turn a 3-manifold invariant into a TQFT. The main focus here will be on a slightly modified version of the invariant above. We begin by generalizing the axioms presented in Section 3.2, following [BHMV95].

Consider a functor (Z,V) from a cobordism category associating to a d-manifold Σ this time a k-module for some commutative unital ring k with a conjugation mapping $\lambda \mapsto \overline{\lambda}$. To an equivalence class of a cobordism M between two d-manifolds, $\partial M = -\Sigma_1 \sqcup \Sigma_2$, we associate a linear map $Z(M) : \Sigma_1 \to \Sigma_2$. Here, cobordisms M_1 and M_2 between Σ_1 and Σ_2 are considered equivalent if there is an isomorphism $M_1 \to M_2$ acting identically on the boundary. The isomorphism will of course be required to preserve whatever structure the cobordisms are endowed with; typically they will simply be diffeomorphisms, but as we will see, cobordisms might contain certain extra structures.

Assume that $V(\emptyset) = k$. In this case, when $\partial M = \Sigma$, we write $Z(M)$ for $Z(M)(1) \in V(\Sigma)$, and for closed M, let $\langle M\rangle$ denote the corresponding element of k. We will call (Z,V) a *quantization functor* if furthermore there is a non-degenerate sesquilinear form $\langle\cdot,\cdot\rangle_\Sigma$ on $V(\Sigma)$ such that for any $(d+1)$-manifolds M_1, M_2 with $\partial M_1 = \partial M_2 = \Sigma$, we have

$$\langle Z(M_1), Z(M_2)\rangle_\Sigma = \langle M_1 \cup_\Sigma (-M_2)\rangle.$$

If the set $\{Z(M) \mid \partial M = \Sigma\}$ generates $V(\Sigma)$, we say that the functor is *cobordism generated*.

On the other hand, a k-valued invariant $\langle\cdot\rangle$ of closed $(d+1)$-manifolds is called *multiplicative* if $\langle M_1 \sqcup M_2\rangle = \langle M_1\rangle\langle M_2\rangle$ and it is called *involutive* if $\langle -M\rangle = \overline{\langle M\rangle}$.

Note that the invariant coming from a quantization functor is always both multiplicative and involutive. In fact, the converse is also true:

Proposition 3.30 (The universal construction). *Any multiplicative and involutive invariant $\langle\cdot\rangle$ of closed cobordisms extends to a unique cobordism generated quantization functor.*

Proof. Let Σ be a d-manifold, and let $\mathcal{V}(\Sigma)$ be the k-module freely generated by all cobordisms from $\emptyset$ to Σ. For $M_1, M_2 \in \mathcal{V}(\Sigma)$, put

$$\langle M_1, M_2\rangle_\Sigma = \langle M_1 \cup_\Sigma (-M_2)\rangle.$$

This extends to a form on $\mathcal{V}(\Sigma)$ which is sesquilinear by the involutivity of the invariant. Let $V(\Sigma)$ be the quotient of $\mathcal{V}(\Sigma)$ by the left kernel of this form. The form then descends to a non-degenerate sesquilinear form on $V(\Sigma)$. Finally, if M is a cobordism from Σ_1 to Σ_2, define $Z(M) : V(\Sigma_1) \to V(\Sigma_2)$ by

$$Z(M)([M']) = M \cup_{\Sigma_1} M'.$$

Then $Z(M)$ is well-defined. Since furthermore multiplicativity of the invariant ensures that $V(\emptyset) = k$, the associations (Z,V) give a unique cobordism generated quantization functor. □

Note that for any cobordism generated quantization functor we obtain a map

$$V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \sqcup \Sigma_2). \tag{3.6}$$

We are now able to give the refined definition of a TQFT.

Definition 3.31. A cobordism generated quantization functor is called a *topological quantum field theory* if the map (3.6) is an isomorphism, if $V(\Sigma)$ is free of finite rank for all Σ, and if the pairing $\langle\cdot,\cdot\rangle_\Sigma$ determines an isomorphism $V(\Sigma) \to V(\Sigma)^*$.

As was the case in Theorem 3.21, the invariants θ_p constructed in Section 3.4.2 need to be modified slightly to fit into the framework of TQFT. More precisely, the invariants have so-called *framing anomalies*, and the axioms hold only up to invertible scalar factors. The solution to this is to extend the cobordism category, requiring the manifolds in question to have certain extra structures. For our purposes, the relevant cobordism category is the category $C_2^{p_1}$ of smooth closed oriented 2-manifolds (Σ, l) with p_1-structure (see [BHMV95, App. B] for details on these) and containing a set of banded intervals l, with bordisms being compact smooth oriented 3-manifolds (M, L) extending the p_1-structures of the boundary components and containing banded links L meeting the intervals specified in the boundary. In this category, cobordisms are equivalent if they are diffeomorphic through an orientation-preserving diffeomorphism, having isotopic links and p_1-structures homotopic relative to the boundary. We will not need these p_1-structures explicitly and rather than defining them precisely, we only observe the following:

Proposition 3.32. *For any closed 3-manifold M there is a one-to-one correspondence, denoted σ, from homotopy classes of p_1-structures on M to the integers. For 2-manifolds, p_1-structures are unique up to homotopy.*

Let $k_p = \mathbb{Z}[A, A^{-1}, \kappa, \frac{1}{p}]/(\varphi_{2p}(A), \kappa^6 - u)$, where $u = A^{-6-p(p+1)/2}$. Then k_p is a ring with involution $A \mapsto A^{-1}$, $\kappa \mapsto \kappa^{-1}$. Define $\eta \in k_p$ by $\eta = \kappa^3$ for $p = 1$, $\eta = (1 - A)\kappa^3/2$ for $p = 2$, and

$$\eta = \tfrac{1}{2}(A\kappa)^3(A^2 - A^{-2})p^{-1}\sum_{m=1}^{2p}(-1)^m A^{-m^2},$$

for $p \geq 3$. Let $b_i(M)$ denote the i'th Betti number of a manifold M, and σ the map of Proposition 3.32. We then obtain the following main theorem of [BHMV95].

Theorem 3.33. *Let $M = (M, \alpha, L)$ be a closed 3-manifold with p_1-structure α and L a banded link, and write $(M, L) = \bigsqcup_{i=1}^n (M_i, L_i)$, where the M_i are connected components of M. Then the expression*

$$\langle M\rangle_p = \eta^{b_0(M)+b_1(M)}\kappa^{\sigma(\alpha)}\prod_{i=1}^n \theta_p(M_i, L_i),$$

defines a multiplicative and involutive invariant on $C_2^{p_1}$ and thus gives rise to a quantization functor (Z_p, V_p). Furthermore, if $p > 2$ is even, the functor is a TQFT.

Remark 3.34. The resulting TQFT is commonly referred to as the SU(2)-TQFT as it is constructed to realize Witten's Chern–Simons theory with gauge group $G = \mathrm{SU}(2)$. In the case where p is odd, the above theorem fails to hold, as the map

$$V_p(\Sigma_1, l_1) \otimes V_p(\Sigma_2, l_2) \to V_p((\Sigma_1, l_1) \sqcup (\Sigma_2, l_2))$$

fails to be an isomorphism, unless one of the l_i has an even number of components. Thus, in this case, we obtain a TQFT, called the SO(3)-TQFT by restricting to the cobordism category $C_2^{p_1}$(even) whose objects are surfaces containing an even number of embedded intervals and morphisms are as before.

Note also that the TQFT constructed in this section fits into the general framework of modular functors, even though we took a different approach. This will be clear in Section 3.5, but see also [Tur10, Ch. XII] for details.

Throughout the rest of this report, we will make a somewhat gross abuse of notation in the case of even p and write (Z_k, V_k) for the TQFT (Z_p, V_p), where $p = 2k + 4$. That is, for every $k = 1, 2, \ldots$, we let (Z_k, V_k) denote the SU(2)-*TQFT at level* k obtained by evaluating the Kauffman bracket at a primitive $4k + 8$'th root of unity. This discrepancy is common throughout the litterature and is made explicit here in order to line up our various constructions. As we will primarily be dealing with the TQFT (Z_k, V_k), this should not cause any confusion.

3.4.5 Properties of the TQFT

We first note the following general fact about TQFTs which should be true by the general philosophy of Section 3.2.1. See [BHMV95, Thm. 1.2].

Theorem 3.35. *Let (Z, V) be a TQFT in the sense of Definition 3.31, and let M be a cobordism from Σ to Σ. Let M_Σ be the manifold obtained by identifying the two copies of Σ. Then $\langle M_\Sigma \rangle = \operatorname{tr} Z(M)$.*

In order to be able to do explicit calculations, we describe a basis of the $V_p(\Sigma, l)$ in terms of handlebodies containing particular banded links.

Throughout the rest of this section, assume that p is an even number $p = 2n + 2$, and let $C_p = \{0, \dots, n-1\}$. A triple (a, b, c) of elements from C_p is called *admissible*, if $a+b+c$ is even, $|a-b| \leq c \leq a+b$, and $a+b+c < 2n$. A *colouring* of a surface (Σ, l) in $C_2^{p_1}$ is an assignment of a colour $c_j \in C_p$ to every component l_j of l. A *banded trivalent graph* G in a cobordism M of $C_2^{p_1}$ is a graph contained in an oriented surface $SG \subseteq M$ such that

1. The graph G meets ∂M transversally in the set of 1-valent vertices of G.

2. Every vertex of G in the interior of M has valency 2 or 3.

3. The surface SG is a regular neighbourhood of G in SG, and $SG \cap \partial M$ is a regular neighbourhood of $G \cap \partial M$ in $SG \cap \partial M$.

A *colouring* of a banded trivalent graph G is a colouring of edges of G by elements of C_p such that colours of edges meeting in 2-valent vertices coincide, and such that colours of edges meeting a 3-valent vertex form an admissible triple.

We view a coloured graph G of M as a skein in the following way: the graph determines a collection of coloured embedded intervals l in ∂M. Let l_c be the *expansion* of l in Σ obtained by taking c_j parallel copies of the component l_j, where c_j is the colouring of l_j. The *expansion* of (M, G) is the element of the module $K(M, l_c)$ (consisting of skeins in M meeting l_c) obtained by splitting the graph G into a union of I-shaped, O-shaped, and Y-shaped pieces and performing certain replacements: an I-shaped piece is a single edge coloured by $i \in C_p$ with two boundary vertices and gets replaced by the skein coloured by the idempotent $f^{(i)}$, viewed as a skein element in a ball in M. An O-shaped piece consists of a single i-coloured edge and a 2-valent vertex and similarly gets replaced by e_i, the closure of $f^{(i)}$. Finally, the Y-shaped pieces consist of 3 edges meeting in a trivalent vertex and here we make the expansion in Figure 3.16. The admissibility constraints ensure that this last assignment is possible, and piecing the various components back together we obtain an element of $K(M, l_c)$. Furthermore, this element does not depend on the decomposition of G since the $f^{(i)}$ are idempotents. As always, the banded structures of the banded graphs are implicit in all diagrams.

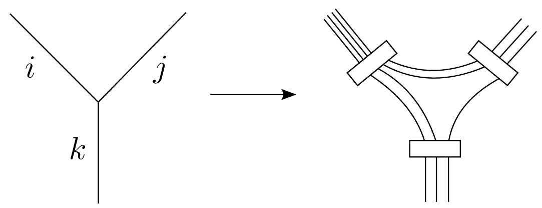


Figure 3.16: A Y-shaped piece with edges coloured by i, j, and k. In this example, $i = 4$, $j = 3$, $k = 3$.

We can define an invariant of closed oriented 3-manifolds with p_1-structures containing a banded trivalent graph by expanding the graph and using the invariant $\langle \cdot \rangle_p$. The invariant obtained this way will be multiplicative and involutive and thus defines a quantization functor (Z_p^c, V_p^c) on the extended category $C_2^{p_1,c}$, where surfaces and cobordisms have coloured structure. Since the invariants coincide, for a surface with structure (Σ, l) viewed as a surface with coloured structure by colouring the components of l by $1 \in C_p$, the vector spaces $V_p^c(\Sigma)$ and $V_p(\Sigma)$ will coincide as well. Therefore, we simply write $Z_p = Z_p^c$, $V_p = V_p^c$, and $\langle \cdot, \cdot \rangle_\Sigma$ for the sesquilinear form. Furthermore, for a given cobordism with coloured structure, we will not distinguish between it and its expansion, when the meaning is clear from the context.

Let (Σ, l, c) be a surface with coloured structure, and let γ be a simple closed curve in Σ. Denote by ${}_i\Sigma(\gamma)_j$ the result of cutting Σ along γ, capping off the two boundary components by disks containing 1-component banded intervals coloured by i and j. Representing an element of $V({}_i\Sigma(\gamma)_i)$ by a manifold M, we obtain an element of $V(\Sigma)$ represented by the manifold M' obtained by identifying the two disks, so $V({}_i\Sigma(\gamma)_i)$ embeds in $V(\Sigma)$.

Theorem 3.36 (Coloured splitting theorem). *Let $\gamma \subseteq \Sigma$ as above. Cutting along γ gives an orthogonal decomposition*

$$V_p(\Sigma) = \bigoplus_{i=0}^{n-1} V_p({}_i\Sigma(\gamma)_i).$$

We end this section by describing explicitly bases of the vector spaces $V_p(\Sigma, l, c)$.

Theorem 3.37. *Let (Σ, l, c) be a connected closed surface with coloured structure. Let H be a handlebody with boundary Σ, and let G be a coloured banded graph with only 1-valent and 3-valent vertices, such that 1-valent vertices correspond to the intervals l_i, and such that H is as a tubular neighborhood of G (see Figure 3.17). Then $V_p(\Sigma, l, c)$ has an orthogonal basis consisting of colourings of G compatible with the colouring of l in the sense that colourings of edges incident to an interval in Σ are coloured by the colour of the interval.*

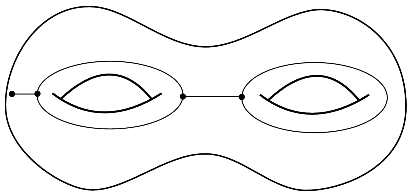


Figure 3.17: The standard basis graph of a surface (Σ, l, c). Here Σ has genus 2 and l consists of a single component.

3.5 A modular category from the Homflypt polynomial

In this section, we will review, following [Bla00] and [AU11], Blanchet's construction of a modular functor from the Homflypt polynomial, which shall give rise to what we will call the quantum SU(N)-invariant. The result will be equivalent to the one obtained from the quantum group construction, and for $N = 2$ it will essentially be equivalent to the one described in the previous section, using the skein theory of the Kauffman bracket. Indeed, the construction will very closely emulate the ideas of Section 3.4.

3.5.1 The Hecke category

Definition 3.38. Let K be an integral domain containing invertible elements a, v, s such that more $s - s^{-1}$ is invertible. The *(relative) Homflypt skein module* $\mathcal{H}(M, l)$ of an oriented 3-manifold M containing a set of *framed points* $l \subseteq \partial M$ together with projective tangent directions and choices of signs ε_p at points $p \in l$ is the K-module generated by isotopy classes of framed links L in M with $L \cap \partial M = l$, the framings of L and l agreeing, modulo the *Homflypt relations* in Figure 3.18. As before, all links pictured will be given the blackboard framing.

Here, we take a *framed link* to mean an oriented link together with a non-singular normal vector field up to homotopy. Unlike in Section 3.4, the orientation of the link will play an essential role.

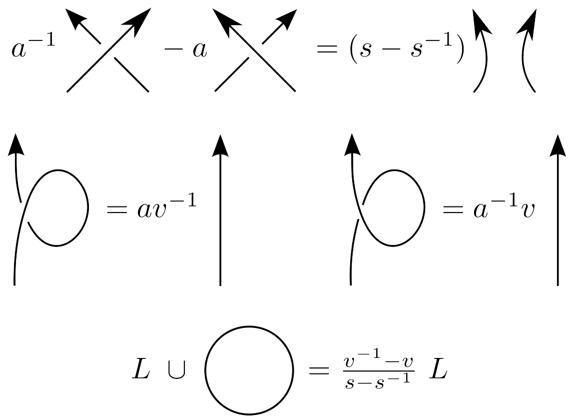


Figure 3.18: The Homflypt relations.

Definition 3.39. The *Hecke category* H is the category whose objects are 2-discs equipped with framed points. If $\alpha = (D^2, l_0)$ and $\beta = (D^2, l_1)$ are objects of H, then

$$\mathrm{Hom}_{\mathrm{H}}(\alpha, \beta) = \mathrm{H}(\alpha, \beta) := \mathcal{H}(D^2 \times [0,1], l_0 \times \{0\} \sqcup -l_1 \times \{1\}),$$

where $-l$ denotes the set of framed points of l with signs reversed.

For $f \in \mathrm{H}(\alpha, \beta)$, $g \in \mathrm{H}(\beta, \gamma)$, we use the notation $fg \in \mathrm{H}(\alpha, \gamma)$ for their composition. The embedding of two disks D^2_{-1} and D^2_1 into D^2 given by

$$D^2_{-1} \sqcup D^2_1 \to D^2 : z \mapsto \pm 1/2 + z/4$$

turns H into a monoidal category with a natural braiding, twist and duality, cf. Section 3.3.3 (see [Bla00, p. 196] for details).

We will need the following notation: for $n \in \mathbb{N}$ denote by n the object consisting of points $-1 + (2j-1)/n \in D^2$, $j = 1, \ldots, n$, framed along the real axis, and let $\mathrm{H}_\alpha = \mathrm{End}_{\mathrm{H}}(\alpha)$ for an object α of H. Furthermore, define the *quantum integer* and *quantum factorial* of $n \in \mathbb{N}$ by

$$[n] = \frac{s^n - s^{-n}}{s - s^{-1}}, \quad [n]! = \prod_{j=1}^{n} [j],$$

respectively.

3.5.2 Young idempotents

The main goal of this section is to form from the Hecke category a modular category $\mathrm{H}^{\mathrm{SU}(N),k}$ depending on integers $N \geq 2$ and $k \geq 1$. The set of simple objects of the category will be given by Young diagrams.

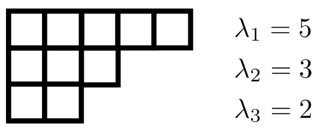


Figure 3.19: An example of a Young diagram.

A *Young diagram* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p \geq 1)$ of size $n = |\lambda|$ is the diagram (see Figure 3.19) associated to a partition of $n = \lambda_1 + \cdots + \lambda_p$. We will think of λ as being made up of *cells* in $\{(i,j) \mid 1 \leq i \leq p,\ 1 \leq j \leq \lambda_i\}$. To each cell $c \in \lambda$ of a Young diagram λ, associate the *hook-length* $\mathrm{hl}(c)$ and *content* $\mathrm{cn}(c)$ given by

$$\mathrm{hl}(c) = \lambda_i + \lambda_j^\vee - i - j + 1, \quad \mathrm{cn}(c) = j - i,$$

where $\lambda_j^\vee$ denotes the length of the j'th column of λ. Moreover, we define the *hook-length* of λ by

$$[\mathrm{hl}(\lambda)] = \prod_{c\in\lambda}[\mathrm{hl}(c)].$$

As in Section 3.4, special attention will be paid to idempotents associated naturally to the simple objects of the modular functor. Assume in the following that $[n]!$ is invertible in K. For a permutation $\sigma \in S_n$, let $w_\sigma \in B_n$ denote the positive crossing braid corresponding to σ as well as the element of H_n determined by this braid. Define $f_n, g_n \in \mathrm{H}_n$ by

$$f_n = \frac{1}{[n]!} s^{-n(n-1)/2} \sum_{\sigma\in S_n} (as^{-1})^{-l(\sigma)} w_\sigma,$$
$$g_n = \frac{1}{[n]!} s^{n(n-1)/2} \sum_{\sigma\in S_n} (-as)^{-l(\sigma)} w_\sigma,$$

where $l(\sigma)$ is the length of σ with respect to the standard presentation of S_n. The f_n and g_n form idempotents [Bla00, Prop. 1.1] of H_n whose meaning is most naturally understood in terms of the representation theory of S_n; not needing this interpretation we take the sanity of the definition as a given.

For a Young diagram λ, let $\square_\lambda$ be the object in H given by the union of the points $(j+il)/(n+1) \in D^2$, where (j,l) runs over the cells of λ, each point framed horizontally. That is, we simply view the diagram as a subset of D^2 in a natural way. Let $F_\lambda, G_\lambda \in \mathrm{H}_{\square_\lambda}$ denote the endomorphisms formed by placing $[\lambda_i]!f_{\lambda_i}$ along each row λ and $[\lambda_j^\vee]!g_{\lambda_j^\vee}$ along each column j (see Figure 3.20).

Definition 3.40. For a Young diagram λ of hook-length $[\mathrm{hl}(\lambda)] \neq 0$, define the idempotent associated to λ by

$$y_\lambda = [\mathrm{hl}(\lambda)]^{-1} F_\lambda G_\lambda.$$

That y_λ is an idempotent is the content of [Bla00, Prop. 1.6].

3.5.3 Modular structure

Let $\mathcal{C}$ denote a set of Young diagrams λ with $[\mathrm{hl}(\lambda)] \neq 0$, and let $\mathrm{H}^{\mathcal{C}}$ denote the category whose objects are triples $\alpha = (D^2, l, \lambda)$, where $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in \mathcal{C}^m$, and $l = (l_1, \dots, l_m)$ are m framed points in the interior of D^2. For an object α of $\mathrm{H}^{\mathcal{C}}$, define its expansion $E(\alpha)$ to be the object of H obtained by embedding $\square_{\lambda^{(j)}}$ in a neighbourhood of l_j according to the framing. Let $\pi_\alpha = y_{\lambda^{(1)}} \otimes \cdots \otimes y_{\lambda^{(m)}}$ and define the module $\mathrm{H}^{\mathcal{C}}(\alpha,\beta)$ by

$$\mathrm{H}^{\mathcal{C}}(\alpha,\beta) = \pi_\alpha \mathrm{H}(E(\alpha), E(\beta))\pi_\beta.$$

Extending the structure of H to $\mathrm{H}^{\mathcal{C}}$, $\mathrm{H}^{\mathcal{C}}$ obtains the structure of a ribbon category. In order to construct from this a modular category that is as universal as possible – in the sense of containing as many simple modules as possible – Blanchet follows the purification procedure outlined in [Tur10, Ch. XI]. As we will not need these technical details, we only state the result and once again note for the sceptical reader that all definitions in the following are indeed sensible (see [Bla00, Sect. 2]).

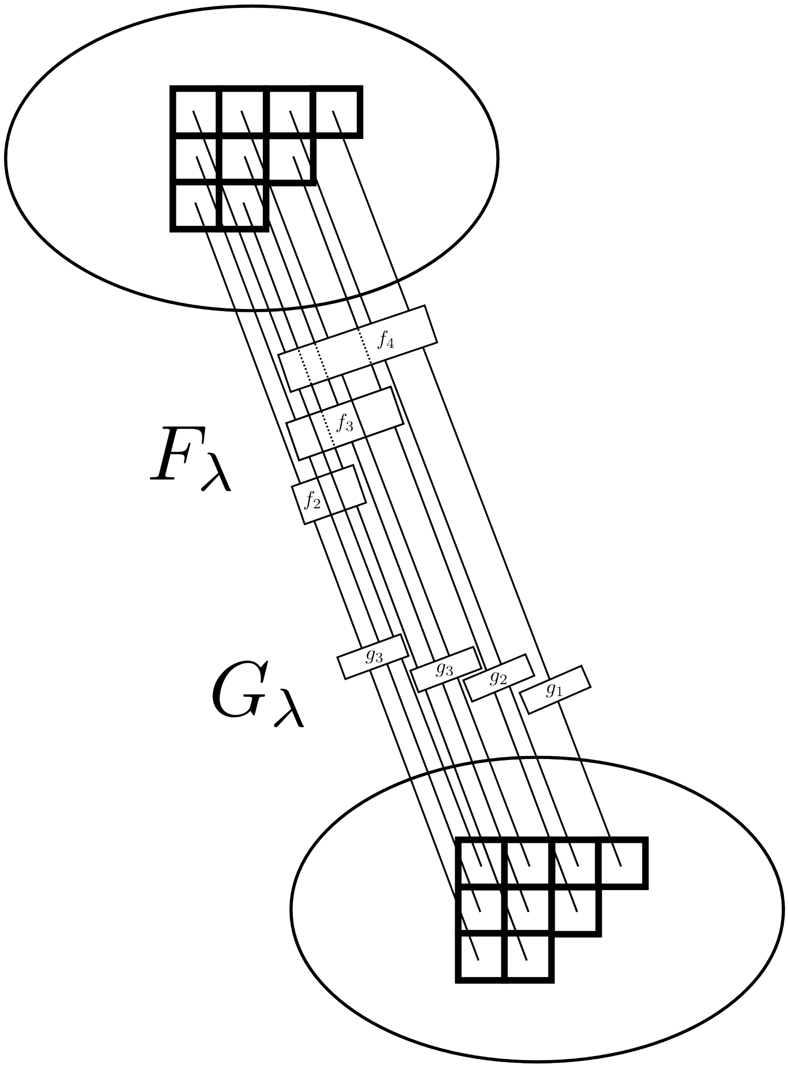


Figure 3.20: An example of the morphisms F_λ and G_λ forming the idempotent y_λ. In the figure, the factors $[\lambda_i]$ and $[\lambda_j^\vee]$ as well as all framings are omitted.

Let $N \geq 2$ and $k \geq 1$. In the following, assume that a is a primitive root of unity of order $2N(k+N)$, that $s = a^{-N}$, and that $v = s^{-N}$. Let

$$\begin{aligned} Y_{N,k} &= \{(\lambda_1 \geq \cdots \geq \lambda_p \geq 1) \mid \lambda_1 \leq k,\ p < N\}, \\ \bar{Y}_{N,k} &= \{(\lambda_1 \geq \cdots \geq \lambda_p \geq 1) \mid \lambda_1 \leq k,\ p \leq N\}, \\ \mathcal{C}_{N,k} &= \{(\lambda_1 \geq \cdots \geq \lambda_p \geq 1) \mid \lambda_1 + \lambda_1^\vee \leq k + N\} \end{aligned}$$

be three sets of Young diagrams, and let $\mathrm{H}^{\mathrm{SU}(N),k}$ be the category whose objects are those of $\mathrm{H}^{\mathcal{C}_{N,k}}$ (noting implicitly that this is well-defined), and whose morphisms are

$$\mathrm{H}^{\mathrm{SU}(N),k}(\alpha,\beta) = \mathrm{H}^{\mathcal{C}_{N,k}}(\alpha,\beta)/\{f \in \mathrm{H}^{\mathcal{C}_{N,k}}(\alpha,\beta) \mid \mathrm{tr}(fg) = 0 \ \ \forall g \in \mathrm{H}^{\mathcal{C}_{N,k}}(\beta,\alpha)\}$$

for objects α and β. Here tr denotes the quantum trace defined in Section 3.3.2. For $\lambda \in \bar{Y}_{N,k}$, define $\lambda^\star \in Y_{N,k}$ to be the Young diagram obtained by taking the complement of λ in λ_1^N and rotating the result 180 degrees. The following is [Bla00, Thm. 2.11]:

Theorem 3.41. *The category $H^{\mathrm{SU}(N),k}$ with simple objects $\lambda \in Y_{N,k}$ is modular, and the dual λ^* of λ is isomorphic to $\lambda^\star$.*

With Theorem 3.41 we may now appeal to Theorem 3.16 to obtain a 3-manifold invariant. As we still have not specified the value of a, we do have some freedom of choice here, and the value chosen below is chosen in order to obtain invariants that are in agreement with those obtained heuristically by Witten [Wit89].

Definition 3.42. Let $N \geq 2$, $k \geq 1$, and let

$$a = \exp(-\pi i/(N(N+k))). \tag{3.7}$$

Then the *quantum* $\mathrm{SU}(N)$*-invariant* $Z_k^{\mathrm{SU}(N)}(M, L)$ of a closed oriented 3-manifold M containing a coloured oriented link L is the invariant obtained by applying Theorem 3.16 to the modular category of Theorem 3.41 with the specified value of a. The underlying TQFT will be denoted $(Z_k^{\mathrm{SU}(N)}, V_k^{\mathrm{SU}(N)})$.

3.6 Relations between the invariants

At this point, having given various accounts of 3-manifold invariants that should all emulate the same physical object, and having in the progress grossly overloaded the term *TQFT*, we take a look back and mention how the invariants are connected. The following result is essentially [Oht02, Thm. H.3].

Theorem 3.43. *Let $k \geq 1$, let $N = 2$, and let a be a primitive root of unity of order $4k+8$. The invariants of closed 3-manifolds containing no links arising from the modular category of Section 3.3.6 with parameter $q = a^{-1/4}$, from Theorem 3.33 with skein variable $A = a^{-1}$, and from Blanchet's modular category of Theorem 3.41 with parameter a all agree. When the 3-manifold contains an n-component link L, the invariant of Theorem 3.33 differs from the others by a factor $(-1)^n$.*

Proof. Since all invariants in question are constructed from link invariants, the claim boils down to showing that the skein relations and normalizations of the invariants agree up to the claimed power of -1. For the quantum group invariant and Kauffman bracket invariant, this is carried out in [Oht02, App. H], and the agreement with the invariants of Theorem 3.41 follows by the same reasoning. □

With this in mind, when there is no risk of confusion, we will often write $Z_k = Z_k^{\mathrm{SU}(N)}$ for the quantum $\mathrm{SU}(N)$-invariant.

Chapter 4

Quantum representations

4.1 Definitions of quantum representations

In the previous chapter, we saw several different ways to construct 3-manifold invariants fitting into the framework of TQFT, all of which should give rise to projective mapping class group representations by the method outlined in Section 3.2.1. We begin this chapter by briefly gathering the main points of the various constructions in order to establish the conventions that will be used throughout the remainder of this manuscript.

4.1.1 Mapping class group representations from modular categories

Recall first that for any modular category, we constructed in Theorem 3.21 a non-degenerate topological quantum field theory (τ, V). Let (Σ, l, λ) be a parametrized labelled marked surface of genus g, let n be the cardinality of l, and let $f \in \Gamma_{g,n}$ be a diffeomorphism of the complement of a small open neighbourhood of l, which we will extend trivially to obtain a diffeomorphism of Σ preserving the labelled points and their associated tangent vectors. Let M_f denote the cylinder $\Sigma \times [0,1]$ with the bottom boundary $\Sigma \times \{0\}$ parametrized by f, its top boundary $\Sigma \times \{1\}$ parametrized by id, and containing a ribbon graph as in Theorem 3.21. Define $\rho(f) : V(\Sigma) \to V(\Sigma)$ by

$$\rho(f) = \tau(M_f).$$

As in Section 3.2.1, this map depends only on the isotopy class of the diffeomorphism f. Suppose now that f, h are two such diffeomorphisms. Then by Theorem 3.21 (see also [Tur10, Sect. IV.5]),

$$\rho(fh) = (\mathcal{D}\Delta^{-1})^m \rho(f)\rho(h)$$

for some integer m, depending on f and h, where $\mathcal{D}$ and Δ are defined in Section 3.3.4. Thus we obtain this way a projective representation of $\Gamma_{g,n}$. As before, this projective ambiguity might be solved by passing to surfaces containing more structure (see [Tur10, Sects. IV.3–4]) whereby one obtains a *linear* representation, not of the mapping class group but a central extension hereof.

Definition 4.1. The *quantum* $\mathrm{SU}(N)$*-representation* $\rho_k^{\mathrm{SU}(N)} : \Gamma_{g,n} \to \mathbb{P}\mathrm{Aut}(V_k^{\mathrm{SU}(N)})$ is the representation obtained by the above procedure applied to Blanchet's modular category, cf. Definition 3.42.

4.1.2 Skein theory revisited

As these will be our main point of reference, we discuss now two ways the skein theoretical construction of TQFTs gives rise to projective representations of the mapping class group of a surface. Let $p > 2$ be even, and let (Z_p, V_p) be the corresponding TQFT given by Theorem 3.33.

Let (Σ, l, c) be a surface with coloured structure, and let (H, G) be one of the basis elements of Theorem 3.37. That is, H is a handlebody with boundary Σ, and G is a banded trivalent graph in H endowed with a colouring compatible with that of Σ. By the general construction in Section 3.2.1, an action of the mapping class group is given by gluing to H the mapping cylinder M_φ for a homeomorphism $\varphi : \Sigma \to \Sigma$. In the case where $l = \emptyset$, this immediately defines a new element of $V_p(\Sigma, l, c)$, and in the case $l \neq \emptyset$, we extend the coloured structure in H to $H \cup M_\varphi$ by extending the graph G to a graph

$$G \cup (l \times [0, \tfrac{1}{2}] \cup_\varphi l \times [\tfrac{1}{2}, 1]) \subseteq H \cup M_\varphi,$$

and colouring the new edge compatibly. Here, mapping classes are assumed to preserve the coloured structure; equivalently, we consider as in Section 4.1.1 mapping classes of (Σ, l, c) viewed as a surface with boundary, each boundary component encircling a component of l.

Now, it is known that the mapping cylinder of a Dehn twist about a curve γ in Σ can be presented by surgery on $\Sigma \times [0, 1]$ along the curve $\tilde{\gamma} \times \{\frac{1}{2}\}$. Here, $\tilde{\gamma}$ is the banded link obtained by a full negative twist to the curve γ viewed as a link with the blackboard framing with respect to the surface $\Sigma \times \{\frac{1}{2}\}$. In the language of surgery, $\tilde{\gamma}$ is the curve γ with framing -1 with respect to $\Sigma \times \{\frac{1}{2}\}$. Thus, from the skein theoretical construction of TQFT, the action $\rho_p(t_\gamma)$ of a Dehn twist about γ on (H, G) is given by adding the banded link $\tilde{\gamma}$ coloured by Ω_p to the handlebody as in Figure 4.1. We could now simply define the projective action of a general mapping class f by writing it as a product of Dehn twists, $f = t_{\alpha_1} \cdots t_{\alpha_n}$, and letting

$$\rho_p(f) = \rho_p(t_{\alpha_1}) \cdots \rho_p(t_{\alpha_n}).$$

Working in the skein module of the handlebody, it is now a feasible task to compute the action of a word of Dehn twists on V_p by hand, at least for small enough values of p.

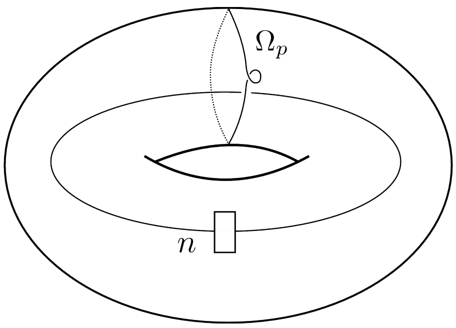


Figure 4.1: The action on V_p of a meridian twist in the torus on the handlebody element containing the skein corresponding to e_n. We analyze this example further in Lemma 4.9.

4.1.3 Roberts' construction

Roberts [Rob94] has given an alternative description of the mapping class group action which in some cases is more suited for direct calculations. Furthermore, its construction is significantly more elementary, lending itself not to the abstract setup of TQFT and surgeries but only to concrete manipulations with handlebodies.

For a compact oriented 3-manifold M, denote by $K_\xi(M)$ the complex vector space obtained from $K(M)$ by the homomorphism $A \mapsto \xi$ for a non-zero complex number ξ. Assume in the following that ξ is a primitive root of unity of order $4k + 8$. In this case, $K_\xi(S^3)$ is isomorphic to $\mathbb{C}$. Let Σ be a closed oriented surface embedded into S^3, such that its complement is a union of two handlebodies H and H'. We define a bilinear form

$$\langle \cdot, \cdot \rangle : K_\xi(H) \times K_\xi(H') \to K_\xi(S^3) = \mathbb{C}$$

on generators as follows: if $x \in K_\xi(H)$, $x' \in K_\xi(H')$ represent links L, L' in H, H' then $\langle x, x' \rangle$ is given by the value of $L \cup L'$ in $K(S^3)$, considering H and H' as subsets of S^3. Taking the quotient by the left kernel in $K_\xi(H)$ and right kernel in $K_\xi(H')$ we obtain a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : V_k(\Sigma) \times V_k'(\Sigma) \to \mathbb{C}.$$

It turns out that the $V_k(\Sigma)$ are finite-dimensional vector spaces, and by Proposition 1.9 of [BHMV95], they are isomorphic to the ones arising from the Kauffman bracket TQFT.

We now proceed to describe the action of the Dehn twists on $V_k(\Sigma)$. Let K denote the set of Dehn twists about curves in Σ bounding discs in H, and let K' be the set of those bounding discs in H'. Elements of K extend in a unique way to homeomorphisms of H, giving rise to an action of such Dehn twists on $K_\xi(H)$ preserving the left kernel of the above form. Therefore, the group generated by the Dehn twists in K act on $V_k(\Sigma)$. Denote this action by ρ_k. To describe the action of any element of the mapping class group, it now suffices to describe the action by elements of K', since elements of $K \cup K'$ generate the mapping class group by Theorem 1.11. For an element $f' \in K'$, define $\rho_k(f')$ by

$$\langle \rho_k(f')(x), y \rangle = \langle x, (f')^{-1}(y) \rangle,$$

for $x \in V_k(\Sigma)$, $y \in V_k'(\Sigma)$. Since the form is non-degenerate, this determines ρ_k on the group generated by K'.

An element of the mapping class group could be written as a word in Dehn twists in more than one way, and for a mapping class $f = t_{\alpha_1} \cdots t_{\alpha_n}$, $t_{\alpha_i} \in K \cup K'$, one should verify that

$$\rho_k(f) := \rho_k(t_{\alpha_1}) \cdots \rho_k(t_{\alpha_n}),$$

is well-defined – at least up to a scalar factor, so that it gives rise once again to a projective representation of $\Gamma(\Sigma)$. This is Theorem 3.12 of [Rob94]. As projective representations $\rho_k : \Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}V_k)$, the ρ_k agree with the ones arising from the Kauffman bracket construction of TQFT.

This construction extends immediately to the case where the surface and handlebodies have coloured structure, and we can use the basis for $V_k(\Sigma)$ of Theorem 3.37 to obtain an explicit expression for the projective representation in this case.

Remark 4.2. There is a general procedure to turn projective representations into honest ones. If $\rho : G \to PGL(V)$ is a projective representation of a group G on a vector space V with ground field F, there exists a central extension $0 \to A \to \tilde{G} \to G \to 0$ and a representation $\sigma : \tilde{G} \to GL(V)$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & \tilde{G} \arrow[r, "\pi"] \arrow[d, "\sigma"] & G \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & F^* \arrow[r, "\mathrm{diag}"] & GL(V) \arrow[r] & PGL(V) \arrow[r] & 0
\end{tikzcd}$$

Rather than considering the projective representations as homomorphisms to projective linear groups, it is some times more natural to consider them as lifts of homomorphisms determined on the generators. For example, in the case of the torus with mapping class group generators t_a and t_b, the projective ambiguity turns out to lie completely in the relation $(t_a t_b)^6 = 1$, and one can lift the projective representation to the central extension B_3, once again generated by t_a and t_b but now with a single relation $t_a t_b t_a = t_b t_a t_b$. This generalizes to the higher genus case to some extent – see [MR95].

4.2 Quantum representations from geometric quantization

We now return to the question of how to use geometric quantization to construct quantum representations of the mapping class group and compare the resulting representations with those arising from topological quantum field theory. Throughout this section, let again $G = \mathrm{SU}(N)$, and let Σ be a compact oriented surface of genus $g \geq 2$. One reference for the following description is [And92].

Assume first that Σ is closed, and let

$$\mathcal{M}_\sigma = (\mathcal{M}^*, \omega, I_\sigma)$$

be the Kähler manifolds with holomorphic line bundles $\mathcal{L}_\sigma^k \to \mathcal{M}_\sigma$ arising as in Section 2.5.3, with Kähler structures parametrized by Teichmüller space $\mathcal{T}$. Here and in the following, we write $\mathcal{L}^k = \mathcal{L}^{\otimes k}$. Let $\mathcal{V}_k \to \mathcal{T}$ be the vector bundle over $\mathcal{T}$ with fibre $\mathcal{V}_{k,\sigma} = H^0(\mathcal{M}_\sigma, \mathcal{L}_\sigma^k)$, and let $\sigma \in \mathcal{T}$ be fixed.

Recall that $\mathrm{Diff}(\Sigma)$ acts on $\mathcal{M}^*$, and that the action of $\mathrm{Diff}_0(\Sigma) \subseteq \mathrm{Diff}(\Sigma)$ is trivial. It follows that the mapping class group $\Gamma(\Sigma)$ acts on $\mathcal{M}^*$, and this action lifts to an action on $\mathcal{L}^k$ in the following way: let $\tilde{\mathcal{L}} = \mathcal{A} \times \mathbb{C}$ be the trivial bundle over $\mathcal{A}$, and define a map $\Psi : \mathcal{A} \times \mathrm{Diff}(\Sigma) \to \mathrm{U}(1)$ by

$$\Psi(A, f) = \exp(2\pi i(\mathrm{CS}(\widetilde{f^*A}) - \mathrm{CS}(\tilde{A}))),$$

extending A and f^*A to connections $\tilde{A}$ and $\widetilde{f^*A}$ in a principal bundle over a 3-manifold with boundary Σ. This map satisfies

$$\Theta(f^*A, g \circ f)\Psi(A, f) = \Psi(g^*A, f)\Theta(A, g),$$

where Θ is the map (2.8) used to construct the prequantum line bundle. Furthermore, it can be proved that $\Psi(A, f) = 1$ for $f \in \mathrm{Diff}_0(\Sigma)$. Thus the map $\mathcal{L}_A \to \mathcal{L}_{f^*A}$ mapping $(A, z) \mapsto (f^*A, \Psi(A, f)z)$ projects to an action of $\Gamma(\Sigma)$ on the line bundle $\mathcal{L} \to \mathcal{M}^*$.

In fact, this action determines for a mapping class $f \in \Gamma(\Sigma)$ a map

$$f^* : H^0(\mathcal{M}_\sigma, \mathcal{L}_\sigma^k) \to H^0(\mathcal{M}_{f^*\sigma}, \mathcal{L}_{f^*\sigma}^k).$$

Now, choose a path γ in $\mathcal{T}$ from $f^*\sigma$ to σ, and let

$$P_{f^*\sigma,\sigma} : H^0(\mathcal{M}_{f^*\sigma}, \mathcal{L}_{f^*\sigma}^k) \to H^0(\mathcal{M}_\sigma, \mathcal{L}_\sigma^k)$$

denote parallel transport in $\mathcal{V}_k$, determined by the Hitchin connection. Since the connection is projectively flat, this depends on the path chosen only up to scalar multiplication. Thus, the composition

$$\rho_k^N(f) = P_{f^*\sigma,\sigma} f^* \in \mathbb{P}\mathrm{Aut}(H^0(\mathcal{M}_\sigma, \mathcal{L}_\sigma^k))$$

determines a projective representation of $\Gamma(\Sigma)$. Using again the projective flatness of the connection, this is seen to depend on σ only up to conjugation. We could also use the connection to canonically identify all fibres with the space $\mathbb{P}\tilde{V}_k$ of covariant constant sections in $\mathbb{P}\mathcal{V}_k$ and in this way obtain projective representations

$$\rho_k^N : \Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}\tilde{V}_k).$$

In the case where Σ has a single boundary component and $d \in \mathbb{Z}_N$ satisfies $\gcd(N, d) = 1$ or $(N, d) = (2, 0)$, the mapping class group of the surface Σ still acts on the corresponding moduli space $\mathcal{M}_{\mathrm{SU}(N)}^d$ (see [And06]), and one may lift this action to the corresponding line bundle over $\mathcal{M}_{\mathrm{SU}(N)}^d$, as defined in Section 2.5.2. Therefore, we obtain projective representations

$$\rho_k^{N,d} : \Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}\tilde{V}_k), \tag{4.1}$$

where $\tilde{V}_k$ is constructed as before. These have the advantage over the representations described in the closed case, that the moduli space $\mathcal{M}^d_{\mathrm{SU}(N)}$ is a compact smooth manifold, which allows for a study of quantum representations using tools from symplectic geometry and Toeplitz quantization. As an example, this allowed Andersen to prove that these quantum representations are asymptotically faithful, cf. Theorem 4.11 below, and we will apply similar methods in Section 6.2 below.

4.2.1 Relation between the quantum representations

Just like we saw that the quantum invariants of the previous chapter were all essentially equivalent, the projective representations arising from geometric quantization and topological quantum field theory have several similarities, some of which are recorded in this and the next section. One immediate similarity is the fact that the dimensions of the representation spaces agree and are given by the *Verlinde formula* (several references are relevant here, but see e.g. [Wit91] for a discussion of the Verlinde formula and its relevance for the study of moduli spaces):

Theorem 4.3. *Let $\tilde{V}_k(\Sigma_g)$ be the representation spaces obtained by the above construction for $N = 2$ for a closed genus $g \geq 2$ surface, let $V_k(\Sigma_g)$ denote the vector spaces constructed from the Kauffman bracket, and let $V_k^{\mathrm{SU}(2)}(\Sigma_g)$ denote the vector spaces obtained from Blanchet's modular category. Then*

$$\dim \tilde{V}_k(\Sigma_g) = \dim V_k(\Sigma_g) = \dim V_k^{\mathrm{SU}(2)}(\Sigma_g) = \sum_{j=0}^{k} S_{0j}^{2g-2}, \tag{4.2}$$

where

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(i+1)(j+1)}{k+2}\right)$$

are the entries of the S-matrix (see Section 4.3.3 below for a further discussion). In the case where Σ_g contains coloured structure (l,c), let $V_k(\Sigma_g,l,c)$ be the space associated to (Σ_g,l,c) by the Kauffman bracket construction, and $V_k^{\mathrm{SU}(2)}(\Sigma_g,l,c)$ the space associated to Σ_g, viewed in the obvious way as a marked labelled surface. Let s be the number of coloured points and c_j the colour corresponding to l_j. Then

$$\dim V_k(\Sigma_g,l,c) = \dim V_k^{\mathrm{SU}(2)}(\Sigma_g,l,c) = \sum_{j=0}^{k} \frac{1}{S_{0j}^{2g-2+s}} \prod_{i=1}^{s} S_{c_i j}. \tag{4.3}$$

Similar formulas hold for $V_k^{\mathrm{SU}(N)}$, $N > 2$, but we will not need those (see [Bla00, Sect. 2.3.1]). In fact, the Verlinde formula (4.2) is a polynomial in k of degree $3g-3$ when $g \geq 2$ and degree 1 if $g = 1$; this polynomial was given explicitly through a residue formula argument by Szenes [Sze95].

What is less clear is whether the corresponding quantum representations are equivalent. In Section 4.3 we consider the case $g = 1$ in further detail, and we end this section by a discussion on known correspondences.

In fact, probably the quantum representations that are the most well-understood by physicists arise from neither of the previuos constructions, but from what is known as *conformal field theory*. Here, one constructs a modular functor which associates to labelled marked surfaces so-called spaces of *conformal blocks* and constructs a projective representation of mapping class groups on those via a monodromy construction, not entirely unlike the one of Section 4.2. Now, Andersen and Ueno ([AU07a], [AU07b], [AU12], [AU11]) have shown that the modular functor of conformal field theory and the one described in Section 3.5 are in fact equivalent.

On the other hand, Laszlo [Las98] has shown that the projective representations arising from conformal field theory agree with those obtained from the monodromy of the Hitchin connection for closed surfaces Σ_g of genus $g > 2$. For marked labelled surfaces, the connection with geometric quantization is less clear, and to the best of our knowledge, no complete account on the existence of any such relation exists. We touch upon this question again in Section 5.6.2.

4.3 The torus case

As noted in the introduction, we will be particularly interested in understanding the quantum invariants of mapping tori of torus diffeomorphisms. This on the other hand boils down to understanding the quantum representations of Γ_1. In this section, we collect most of the notation used in the rest of the manuscript and show that the projective representations of Γ_1 from conformal field theory agree with those arising from Blanchet's modular category (for the T-matrices in the case of $G = \mathrm{SU}(N)$, but for the S-matrices only for $G = \mathrm{SU}(2)$). This equivalence follows abstractly from the isomorphism [AU11] mentioned in the previous section, but we show here how to obtain it through direct calculation. Recall (or see [KT08, App. A] for an argument) that $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We refer to the quantum representations of these elements as the *S-matrix* and the *T-matrix* respectively.

4.3.1 Invariants of torus mapping tori

Let $\Sigma = S^1 \times S^1$ be the torus and let $\mu = \{1\} \times S^1, \nu = S^1 \times \{1\}$ be the meridional resp. longitudinal curves in Σ, oriented so that their algebraic intersection number is $i(\mu,\nu) = 1$, and let Σ have the orientation determined by the ordered pair (μ, λ). For a simple closed curve γ in Σ, let t_γ denote the left Dehn twist about γ, so that in the standard identification of Γ_1 with $\mathrm{SL}(2,\mathbb{Z})$ via its action on homology, we have

$$t_\mu = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad t_\nu = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Furthermore, let $\iota = -\mathrm{id} \in \mathrm{SL}(2,\mathbb{Z})$ be the mapping class of elliptic involution. As in Section 3.4.5, the TQFT vector space $V_k^{\mathrm{SU}(N)}(\Sigma)$ associated to Σ has a natural basis

$$(D^2 \times S^1, \hat{y}_\lambda) \in V_k(\Sigma) = \mathcal{H}(D^2 \times S^1)/\sim$$

consisting of handlebodies containing longitudinal framed links coloured by $\lambda \in Y_{N,k}$. Recall here that the labelling set $Y_{N,k}$ consists of all Young diagrams $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p \geq 1)$ with $\lambda_i \leq k$ and $p < N$, and $\hat{y}_\lambda$ is defined by tracing the idempotent y_λ as in [Bla00, p. 200]. For a Young diagram λ as above, we index its cells (i,j) and write

$$\lambda = \{(i,j) \mid 1 \leq i \leq p, \, 1 \leq j \leq \lambda_i\}.$$

A simple linear algebra argument shows that any element of $\mathrm{SL}(2,\mathbb{Z})$ with trace ± 2 is conjugate to

$$\begin{pmatrix} \pm 1 & -b \\ 0 & \pm 1 \end{pmatrix}$$

for some $b \in \mathbb{Z}$. In particular, since ι is central, the Dehn twist relation $f t_\gamma f^{-1} = t_{f(\gamma)}$ (see Lemma 1.7) and the change of coordinate principle tell us that any mapping torus over a torus with monodromy of trace ± 2 is homeomorphic to either $M^b = T_{t_\mu^b}$ or $\tilde{M}^b = T_{\iota t_\mu^b}$ for some $b \in \mathbb{Z}$. To understand the quantum invariants of these 3-manifolds, it therefore suffices to describe $\rho_k(t_\mu)$ and $\rho_k(\iota)$.

The action of t_μ on $V_k(\Sigma)$ can be understood as in Sections 4.1.2-4.1.3 via the surgery description of Z_k. More precisely, M_{t_μ} can be described as surgery on a meridional curve in $\Sigma \times \{\frac{1}{2}\} \subseteq \Sigma \times [0,1]$ with framing 1 relative to $\Sigma \times \{\frac{1}{2}\}$. Denoting this curve by L, the definition of the Z_k implies that

$$Z_k(M_{t_\mu}) = \Delta^{-1} Z_k(\Sigma \times [0,1], L_\omega) \in \mathrm{End}(V_k(\Sigma)),$$

where Δ is as in Section 3.3.4, i.e. it is the Homflypt polynomial of the unlink with a single positive twist, coloured by Blanchet's surgery element ω (see [Bla00]), and L_ω likewise denotes the link L coloured by ω. Thus, gluing the mapping cylinder M_{t_μ} to $(H, \hat{y}_\lambda)$ has the net effect of adding to H a meridional curve encircling $\hat{y}_\lambda$ coloured by ω with a single positive twist. Since ω is defined to behave well under Kirby moves, we can remove it at the cost of giving $\hat{y}_\lambda$ a *negative* twist and multiplying the result by Δ.

now, the result of giving $\hat{y}_\lambda$ a positive twist is given by [Bla00, Prop. 1.11 (b)]. It follows from this result that $\rho_k(t_\mu)$ is diagonal in the basis given by the $\hat{y}_\lambda$ and has entries

$$(\rho_k(t_\mu))_{\lambda,\lambda'} = a^{-|\lambda|^2 + N^2|\lambda| + 2N\sum_{(i,j)\in\lambda} \mathrm{cn}(i,j)} \delta_{\lambda,\lambda'}, \tag{4.4}$$

where the sum is over all cells (i,j) of the Young diagram λ, $\mathrm{cn}(i,j) = j - i$ is the content of (i,j), and $|\lambda|$ denotes the number of cells in λ. In conclusion,

$$Z_k(T_{t_\mu^b}) = \sum_{\lambda \in Y_{N,k}} (\rho_k(t_\mu^b))_{\lambda,\lambda} = \sum_{\lambda \in Y_{N,k}} a^{b(-|\lambda|^2 + N^2|\lambda| + 2N\sum_{(i,j)\in\lambda} \mathrm{cn}(i,j))}. \tag{4.5}$$

The action of ι on $V_k(\Sigma)$ can be described similarly. We refer to [Tur10, IV.5.4] for the topological details (notice that Turaev uses the opposite torus orientation but that this makes no difference in representing ι – for the twist, one has to be a bit more careful) and note that the result is

$$(\rho_k(\iota))_{\lambda,\lambda'} = \delta_{\lambda,(\lambda')^\star},$$

where * denotes the involution on $Y_{N,k}$ defined in Section 3.5. In particular this implies that for $N = 2$, ι is in the (projective) kernel of every ρ_k. In general, it follows that

$$Z_k(T_{\iota t_\mu^b}) = \mathrm{tr}\, Z_k(C_{\iota t_\mu^b}) = \mathrm{tr}(\rho_k(\iota)\rho_k(t_\mu^b)) = \mathrm{tr}\, \rho_k(t_\mu)^b = \sum_{\lambda \in Y_{N,k}} \rho_k(t_\mu)^b_{\lambda,\lambda} \delta_{\lambda,\lambda^\star},$$

and in particular that

$$Z_k^{\mathrm{SU}(2)}(T_{\iota t_\mu^b}) = Z_k^{\mathrm{SU}(2)}(T_{t_\mu^b}). \tag{4.6}$$

4.3.2 The T-matrix

Let as always $G = \mathrm{SU}(N)$. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus in G. Denote by $\langle \cdot, \cdot \rangle$ the basic inner product on $\mathfrak{t}^*$ such that the highest root (and therefore all roots), denoted α_m, has length $\sqrt{2}$. Denote by Λ^R the root lattice in $\mathfrak{t}^*$ and by Λ^w the lattice dual to Λ^R under the basic inner product. We refer to Λ^w as the weight lattice, silently identifying elements of $\mathfrak{t}^*$ and $\mathfrak{t}$ using the basic inner product. The weight lattice has a basis consisting of fundamental weights Λ_i, $i = 1, \dots, N-1$. Now, let $k \in \mathbb{Z}_{\geq 0}$ be a level, and let

$$c = \frac{(\dim G)k}{k+N}$$

be the *level k central charge*.

The relevant labelling set P_k in conformal field theory is the set of highest weight integrable representations of the loop group LG, which in the above notation is given by

$$P_k = \{\lambda \in \Lambda^w \cap P_+ \mid \langle \lambda, \alpha_m \rangle \leq k\},$$

where P_+ denotes the positive Weyl chamber. Now Γ_1 acts on the complex vector space spanned by these labels. The T-matrix at level k is given by (see e.g. [GW86] or [Kac90]) the diagonal matrix

$$T^{\mathrm{CFT}}_{\lambda,\lambda'} = \delta_{\lambda,\lambda'} \exp\left(\frac{\pi i}{r}\langle \lambda+\rho, \lambda+\rho\rangle - \frac{\pi i}{N}\langle \rho, \rho\rangle\right), \tag{4.7}$$

for $\lambda, \lambda' \in P_k$, where $r = k + N$. We will argue that this matrix differs from the T-matrix of Blanchet's modular functor by the scalar factor $\exp(2\pi i c/24)$.

Now let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{N-1} \geq 0) \in Y_{N,k}$ be a Young diagram with at most k columns. Corresponding to this Young diagram is the weight $\lambda = \sum_i (\lambda_i - \lambda_{i+1})\Lambda_i \in P_k$ giving a bijection $Y_{N,k} \to P_k$ (see Figure 4.2 for the case $N = 3$, $k = 3$). Here, we let $\lambda_N = 0$. We prove the following seemingly well-known lemma, expressing the quadratic Casimir of a weight in terms of the length and contents of a Young diagram.

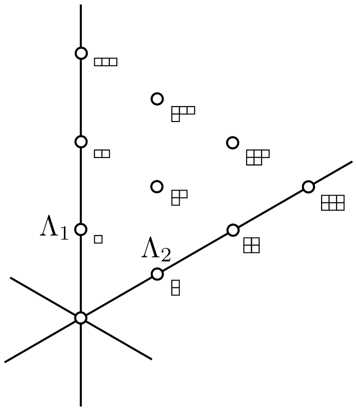


Figure 4.2: The correspondence between P_3 for $\mathfrak{sl}_3$ and $Y_{3,3}$.

Lemma 4.4 ([AJ12], Lem. A.1.1)**.** *Under the above correspondence between weights and Young diagrams,*

$$\langle \lambda+\rho, \lambda+\rho\rangle - \frac{\dim \mathrm{SU}(N)\cdot N}{12} = -\frac{1}{N}\left(|\lambda|^2 - N^2|\lambda| - 2N \sum_{(i,j)\in\lambda} \mathrm{cn}(i,j)\right).$$

Here $|\lambda| = \sum_{i=1}^{N-1} \lambda_i$ denotes the number of cells in λ and should not be confused with $\sqrt{\langle \lambda, \lambda\rangle}$.

Proof. First of all, note that by Freudenthal's strange formula

$$\langle \lambda+\rho, \lambda+\rho\rangle - \frac{\dim \mathrm{SU}(N)\cdot N}{12} = \langle \lambda, \lambda\rangle + 2\langle \lambda, \rho\rangle. \tag{4.8}$$

It is well-known (see e.g. [Hum79, Lemma 13.3.A]) that $\rho = \sum_{j=1}^{N-1} \Lambda_j$, and so the right hand side of (4.8) becomes

$$\langle \lambda, \lambda\rangle + 2\langle \lambda, \rho\rangle = \sum_{i,j=1}^{N-1} (\lambda_i - \lambda_{i+1})(\lambda_j - \lambda_{j+1})\langle \Lambda_i, \Lambda_j\rangle + 2\sum_{i,j=1}^{N-1} (\lambda_i - \lambda_{i+1})\langle \Lambda_i, \Lambda_j\rangle.$$

Now $\langle \Lambda_i, \Lambda_j \rangle$ is simply the (i,j)'th entry of the inverse of the Cartan matrix of SU(N), and so can be written

$$\langle \Lambda_i, \Lambda_j \rangle = \min(i,j) - \frac{ij}{N}.$$

Note also that

$$\sum_{(i,j)\in\lambda} \mathrm{cn}(i,j) = \sum_{i=1}^{N-1} \frac{\lambda_i(\lambda_i+1)}{2} - i\lambda_i.$$

Thus, expressed in terms of λ_i, the formula of the lemma is

$$\sum_{i,j=1}^{N-1} (\lambda_i - \lambda_{i+1})(\lambda_j - \lambda_{j+1} + 2)\left(N\min(i,j) - ij\right)$$
$$= -|\lambda|^2 + N^2|\lambda| + 2N\left(\sum_{i=1}^{N-1} \frac{\lambda_i(\lambda_i+1)}{2} - i\lambda_i\right).$$

Let LHS(λ), RHS(λ) denote the left hand side and right hand side of this equation respectively. It suffices to show that LHS(0) = RHS(0), which is obvious, and that the difference of the two expressions is invariant under $\lambda \to \lambda + \Lambda_l =: \bar{\lambda}$ for all $1 \leq l \leq N-1$, viewing λ as an element in the weight lattice. Under this transformation, the Young diagram becomes $\lambda_i \to \lambda_i + 1$ for $i \leq l$ and $\lambda_i \to \lambda_i$ for $i > l$. One easily finds that

$$\mathrm{RHS}(\bar{\lambda}) - \mathrm{RHS}(\lambda) = -(l^2 + 2|\lambda| l) + N^2 l + 2N \sum_{i=1}^{l} (\lambda_i + 1 - i),$$

and that

$$\mathrm{LHS}(\bar{\lambda}) - \mathrm{LHS}(\lambda) = 2\sum_{j=1}^{l-1} (\varepsilon_j + 1)(N-l)j + 2\sum_{j=l+1}^{N-1} (\varepsilon_j+1)(N-j)l + (2\varepsilon_l + 3)(Nl - l^2)$$
$$= 2\sum_{j=1}^{l} (\varepsilon_j + 1)(N-l)j + 2\sum_{j=l+1}^{N-1} (\varepsilon_j+1)(N-j)l + Nl - l^2,$$

where $\varepsilon_j = \lambda_j - \lambda_{j+1}$. To see the latter, simply notice that the transformation is chosen such that $\varepsilon_l \to \varepsilon_l + 1$ and that the sum in LHS(λ) changes only when $i = l$ or $j = l$. Rewriting the expressions slightly, it now suffices to prove that

$$-2|\lambda| l + N^2 l + 2N\sum_{j=1}^{l} (\lambda_j - j) + Nl$$
$$= 2\sum_{j=1}^{l} (\lambda_j - \lambda_{j+1} + 1)(N-l)j + 2\sum_{j=l+1}^{N-1} (\lambda_j - \lambda_{j+1} + 1)(N-j)l.$$

The right hand side may be further rewritten as

$$2\sum_{j=1}^{l} \varepsilon_j (N-l)j + 2\sum_{j=l+1}^{N-1} \varepsilon_j (N-j)l + N(-l^2 + Nl),$$

and we therefore need to prove that

$$-2|\lambda| l + 2N\sum_{j=1}^{l} (\lambda_j - j) + Nl = 2\sum_{j=1}^{l} (\lambda_j - \lambda_{j+1})(N-l)j + 2\sum_{j=l+1}^{N-1} (\lambda_j - \lambda_{j+1})l - Nl^2.$$

For $l = 1$ this is easily checked and we proceed by induction on l, assuming that the equality holds true for some $l < N - 1$. Under $l \to l + 1$, the excess term on the left hand side is

$$-2|\lambda| + 2N(\lambda_{l+1} - (l+1)) + N,$$

and on the right hand side, it is

$$-2\sum_{j=1}^{l} \varepsilon_j j + 2\varepsilon_{l+1}(N - (l+1)) + 2\sum_{j=l+2}^{N-1} \varepsilon_j (N - j) - 2Nl - N.$$

Finally, that these two excess terms agree follows from a calculation in terms of ε_j, using that

$$\lambda_j = \sum_{m=j}^{N-1} \varepsilon_m, \quad |\lambda| = \sum_{j=1}^{N-1} j\varepsilon_j.$$

□

From Lemma 4.4 it follows that the T-matrix T^{CFT} defined in (4.7) agrees with the inverse of (4.4) up to a factor of $\exp(2\pi ic/24)$. More precisely, let

$$f_r(\lambda) = -\frac{\pi i}{rN}\left(|\lambda|^2 - N^2|\lambda| - 2N\sum_{(i,j)\in\lambda} \mathrm{cn}(i,j)\right),$$

and note that

$$\begin{aligned}
\exp\left(\frac{\pi i}{r}\langle\lambda+\rho,\lambda+\rho\rangle\right) &= \exp\left(f_r(\lambda) + \frac{\pi i N \dim G}{12r}\right) \\
&= \exp\left(f_r(\lambda) - \frac{\pi i \dim G(r-N)}{12r} + \pi i \frac{\dim G}{12}\right) \\
&= \exp\left(f_r(\lambda) - \frac{\pi i c}{12} + \frac{\pi i}{N}\langle\rho,\rho\rangle\right).
\end{aligned}$$

Thus, combining our choice of a in (3.7) with (4.4), we obtain

$$\rho_k(t_\mu)_{\lambda,\lambda'} = \exp(-f_r(\lambda)) = \exp(-2\pi ic/24)(T^{\mathrm{CFT}}_{\lambda,\lambda'})^{-1}. \tag{4.9}$$

4.3.3 The S-matrix

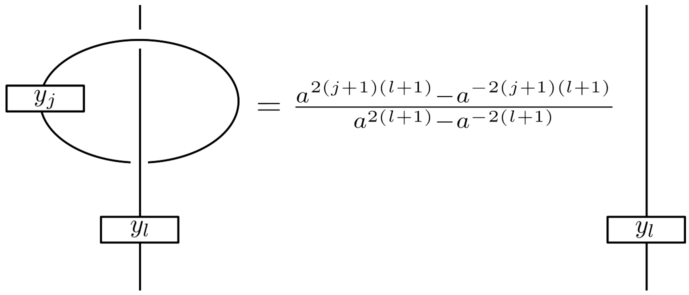
Extending further our analogies between the theories, we now consider the S-matrices. Let in the following $N = 2$. The following results could be extracted from the litterature, using that our skein relations agree with those of $U_q(\mathfrak{sl}_2)$ (cf. e.g. [Oht02]), but are included here for the sake of completeness.

In conformal field theory, the S-matrix is given by

$$S^{\mathrm{CFT}}_{jl} = \sqrt{\frac{2}{r}}\sin\left(\frac{\pi(j+1)(l+1)}{r}\right), \tag{4.10}$$

where $j, l = 0, \dots, k$. Let us find the corresponding matrix arising from Blanchet's modular category. Recall also that we denote elements of $Y_{2,k}$ simply by their number of cells, and note that in this case, all the simple objects of the modular category are self-dual. Thus in particular, by [Bla00, Prop. 2.6], the framed Homflypt polynomial of coloured framed links is invariant under choice of orientation of any of the components of the link, and as such we leave out the orientations in the pictures drawn below.

Lemma 4.5 ([AJ12], Lem. A.2.1)**.** *Let $0 \leq j \leq k$, $0 \leq l \leq k$. Then we have the following relation of links viewed as elements of the relative version (cf. [Bla00, p. 195]) of $\mathcal{H}(I \times I \times I)$:*

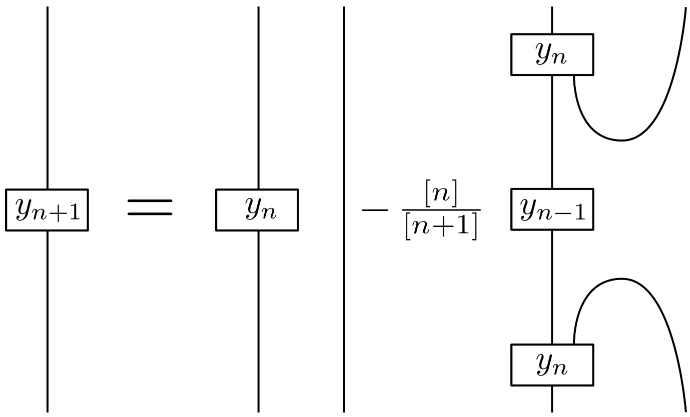


Proof. The proof is a direct translation of the similar proof in the Kauffman skein module (see e.g. [Lic97, Lemma 14.2]), where one obtains the same result up to a factor of $(-1)^{j+l}$.

Recall from Section 3.5 that y_j is the composition of a number of quasi-idempotents, most of which are the identity in the case $N = 2$. One finds that $y_j = f_j$, in the notation of Section 3.5. Recall that we write $[j] = (a^{2j} - a^{-2j})/(a^2 - a^{-2})$. Using that

$$-[j-1] + [j](a^2 + a^{-2}) = [j+1],$$

the recursive relation given for $y_j = f_j$ can be rewritten to look as follows:



Of course, this resembles closely Wenzl's recursive definition of the Jones–Wenzl idempotents. Now, a simple inductive argument using the quantum dimensions given in [Bla00, Prop. 1.14], proves the lemma in the case of $l = 0$. This can be used to give a Chebyshev polynomial style recursive relation for the idempotents traced in $\mathcal{H}(S^1 \times I \times I)$ (which turns out to be exactly the same as the one for the Jones–Wenzl idempotents, but with all signs positive), which by the exact same argument as the one for [Lic97, Lemma 14.2] can be used to prove the lemma. □

Following [Tur10], let s be the matrix whose (j, l)'th entry is the Homflypt polynomial evaluated on the Hopf link with components coloured j and l. It follows immediately from Lemma 4.5 that

$$s_{jl} = \frac{a^{2(j+1)(l+1)} - a^{-2(j+1)(l+1)}}{a^{2(l+1)} - a^{-2(l+1)}}[l+1] = \frac{a^{2(j+1)(l+1)} - a^{-2(j+1)(l+1)}}{a^2 - a^{-2}}.$$

Now, according to [Tur10], the S-matrix is given by $\mathcal{D}^{-1}s$, where $\mathcal{D}$ is the rank of the modular category, satisfying $\mathcal{D}^2 = \sum_{j=0}^{k} s_{0j}^2$. We find that for $k \geq 1$,

$$\begin{aligned}\mathcal{D}^2 &= \sum_{j=0}^{k} [j+1]^2 = \frac{1}{(a^{-2}-a^2)^2} \sum_{j=1}^{k+1} (a^{-4j} + a^{4j} - 2) \\ &= \frac{1}{(2i\sin(\pi/r))^2}(-1-1-2(r-1)) = \frac{-2r}{-4\sin^2(\pi/r)} = \frac{r}{2}\frac{1}{\sin^2(\pi/r)}.\end{aligned}$$

A natural[1] choice of rank is thus

$$\mathcal{D} = \sqrt{\frac{r}{2}} \frac{1}{\sin(\pi/r)},$$

and the S-matrix becomes

$$S_{jl} = \mathcal{D}^{-1} s_{jl} = \sqrt{\frac{2}{r}} \sin\left(\frac{\pi(j+1)(l+1)}{r}\right) = S_{jl}^{\mathrm{CFT}}.$$

Remark 4.6. Recall that the framing anomaly of the level k quantum $\mathrm{SU}(N)$-invariant is a power of $\mathcal{D}^{-1}\Delta$. For $N = 2$,

$$\Delta = \sum_{j=0}^{k} T_{jj}^{-1} [j+1]^2.$$

By the well-known quadratic reciprocity law for general Gauss sums (see Theorem 5.14), we find

$$\begin{aligned}\Delta &= \frac{a}{(a^2-a^{-2})^2} \sum_{j=1}^{k-1} a^{-j^2-4j} + a^{-j^2+4j} - 2a^{-j^2} \\ &= \frac{a}{(a^2-a^{-2})^2} (\sqrt{2r}\exp(\pi i(2r-16)/(8r)) - 2\sqrt{r/2}\exp(\pi i/4)) \\ &= -\frac{e^{\frac{\pi i}{2r}}}{\sin^2(\pi/r)} \sqrt{2r}\exp(\pi i/4)\left(\exp(-2\pi i/r) - 1\right).\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{D}^{-1}\Delta &= -\frac{1}{2}\frac{\exp(-\pi i/(2r))}{\sin(\pi/r)} \exp(\pi i/4)(\exp(-2\pi i/r) - 1) \\ &= \exp\left(-3\frac{\pi i}{2r}\right)\exp(3\pi i/4) = \left(\exp\left(\frac{2\pi i c}{24}\right)\right)^3.\end{aligned}$$

Remark 4.7. We have made explicit the difference between the specialization (3.7) and the corresponding ones in the Kauffman bracket theory to make it clear how one needs to be a little cautious in comparing the theories and using known results from skein theory. Indeed one also obtains a modular category from the Kauffman bracket [Tur10, Ch. XII] which corresponds closely to Blanchet's modular category. Turaev obtains the S-matrix (4.10) through the choice $A = \pm i e^{\pi i/(2r)}$, thus getting rid of all signs, but in this specialization, the twist coefficients of the theories will not agree. On the other hand, by the proof of Lemma 4.5, if we let $A = a^{-1}$, we obtain the same T-matrix as with Blanchet's modular category, but the (i,j)'th entry of the resulting S-matrix will differ from Blanchet's by a factor of $(-1)^{i+j}$. This in particular means that the two representations are equivalent, through conjugation by the diagonal matrix $(-1)^j \delta_{ij}$.

[1] Compare again this computation with the corresponding one in the theory coming from the Kauffman bracket. Here one obtains the same result but with the opposite sign, making the choice of square root seem somewhat less canonical.

Remark 4.8. It would be nice to generalize Lemma 4.5 and see that the S-matrices of the various theories agree in the general case $G = \mathrm{SU}(N)$, but as we do not use it in the main body of the manuscript, we leave the problem at this remark and refer to [ML03] for a description of the Homflypt polynomial of coloured Hopf links.

4.3.4 Geometric quantization for the torus

Whereas we have previously only dealt with geometric quantization for moduli spaces of surfaces with genus $g \geq 2$, we may in fact describe it explicitly in the torus case. We will not need this construction explicitly but it ties together our previous discussion, and part of the notation will be used in Section 5.4. Here, we follow [Jef92, App. A.3] (see also [ADPW91]).

For $G = \mathrm{SU}(N)$, the moduli space of the torus can be identified with $(T \times T)/W$, where T is a maximal torus in $\mathrm{SU}(N)$ with an action of the Weyl group W, acting diagonally on $T \times T$. While we will not have reason to do so, in explicit terms we could think of T as the maximal torus consisting of diagonal matrices, and the action of $W \cong S_N$ on $T \times T$ as simultaneous permutation of the diagonal entries in both factors.

Now, the moduli space has singularities and rather than describing a line bundle on this singular space, one constructs a line bundle on the covering $T \times T$ with an action of S_N. More precisely, let as before $\mathfrak{t} = \operatorname{Lie} T$, and let $\mathbb{A} = \mathfrak{t} \oplus \mathfrak{t}$ so that $T \times T = \mathbb{A}/\Lambda$ for the lattice $\Lambda = \Lambda^R \oplus \Lambda^R$ (see [Jef92, App. A]), using the basic inner product $\langle \cdot, \cdot \rangle$ to identify $\mathfrak{t}$ and $\mathfrak{t}^*$, and we view $\mathbb{A}$ as the subspace of the connections on the torus with $A_0 = 0$ being the trivial connection. We define a symplectic form on $\mathbb{A}$ by

$$\omega((\xi_1, \eta_1), (\xi_2, \eta_2)) = 2\pi(\langle \xi_1, \eta_2 \rangle - \langle \xi_2, \eta_1 \rangle).$$

Now, consider the trivial bundle $\mathbb{A} \times \mathbb{C} \to \mathbb{A}$ quotiented out by the relation

$$(A, v) \sim (A + \lambda, e_\lambda(A) v)$$

for $v \in \mathbb{C}$, $\lambda \in \Lambda$. Here,

$$e_\lambda(A) = \varepsilon(\lambda) \exp(-\tfrac{i}{2}\omega(A - A_0, \lambda)),$$

and ε is a *theta-characteristic* $\varepsilon : \Lambda \to \{\pm 1\}$ satisfying

$$\varepsilon(\lambda_1 + \lambda_2) = \varepsilon(\lambda_1)\varepsilon(\lambda_2)(-1)^{\omega(\lambda_1, \lambda_2)/(2\pi)}.$$

Denote by $\mathcal{L}$ the resulting line bundle on $T \times T$. This carries a natural lifted action of the Weyl group, given by the Γ-equivariant trivial action on the trivial bundle over $\mathbb{A}$,

$$w(A, z) = (A_0 + w(A - A_0), z).$$

Note that in the above construction, different choices of theta-characteristics turn out to give rise to equivalent line bundles (cf. [Jef92, Rem. A.3]).

Let as before $P_k \subseteq \mathfrak{t}$ denote the level k label set of TQFT. To construct the desired basis elements of $H^0(T \times T, \mathcal{L}^k)$, let $\tau \in \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ define a complex structure on $T \times T$ and define for $\gamma \in P_k$ the *theta-function*

$$\theta_{\gamma,k}(\tau, u) = \sum_{\alpha \in \Lambda^R} \exp\left(\pi i k \tau \left|\alpha + \frac{\gamma}{k}\right|^2 + 2\pi i k \left\langle u, \alpha + \frac{\gamma}{k} \right\rangle\right).$$

Now, these functions are not Weyl group invariant and so do not define the sections of $\mathcal{L}^k \to T \times T/W$ that we are interested in. Instead, we use them to define the *Weyl anti-invariant theta-functions* $\theta^-_{\gamma,k}$ by

$$\theta^-_{\gamma,k} = \sum_{w \in W} \det(w) \theta_{w(\gamma),k}.$$

Let $H^{(k)}$ be the bundle over the upper half plane whose fibre over τ is $H^0((T\times T)_\tau,\mathcal{L}^k)^W$. A basis for the space $\mathcal{H}^{(k)}$ of covariantly constant sections of $H^{(k)}$ is given by the *Weyl invariant theta functions*

$$\psi_{\gamma,k}=\frac{\theta^-_{\gamma+\rho,k+N}}{\theta^-_{\rho,N}}$$

for $\gamma\in P_k$. In particular, the label set P_k naturally labels basis vectors of $\mathcal{H}^{(k)}$ and gives us a natural identification with the space $V_k^{\mathrm{SU}(N)}$. In [ADPW91], the authors further define on $\mathcal{H}^{(k)}$ a particular inner product (see p. 871) with respect to which the $\psi_{\gamma,k}$ are orthogonal.

4.4 Kernels and images

One of our first goals will be to understand the algebraic properties of the quantum representations and to analyze two conjectures involving their kernels and images. Note that for a surface Σ, we are interested in the kernel and image of ρ_k as a map $\Gamma(\Sigma)\to\mathbb{P}\mathrm{Aut}(V_k(\Sigma))$. That is, a mapping class $f\in\Gamma(\Sigma)$ is said to be in the *(projective) kernel* of ρ_k, if $\rho_k(f)$ is a scalar multiple of the identity, and we can make sense of the *order* of an element in the usual way. Here and throughout this section, in order to draw from some of the properties outlined in Section 3.4, we will be dealing with the quantum representations of the Kauffman bracket TQFT.

4.4.1 Dehn twists

It follows from Corollary 1.5 and the following result that none of the ρ_k are faithful.

Lemma 4.9. *Dehn twists about non-separating curves in a closed oriented surface Σ with empty coloured structure have order $4k+8$ in the projective representation ρ_k, $k\geq 2$.*

Proof. In the case $g=0$ all curves are separating, and there is nothing to prove.

Consider the case $g=1$. It is enough to prove the statement for a single non-separating curve γ_0 as for general non-separating curve γ there exists a homeomorphism φ taking γ_0 to γ by the change of coordinates principle. Then, by Lemma 1.7,

$$\rho_k(\gamma)=\rho_k(\varphi)\rho_k(\gamma_0)\rho_k(\varphi)^{-1},$$

and the orders of $\rho_k(\gamma)$ and $\rho_k(\gamma_0)$ will coincide, since conjugation and taking powers commute. Let γ_0 denote the meridian. It follows as in Section 4.3.1 – or from Lemma 3.25 and skein theory considerations (or from Roberts' construction of ρ_k) – that in the basis $e_0,\dots,e_k$ of V_k, the action of t_{γ_0} is given by

$$\rho_k(t_{\gamma_0})=\mathrm{diag}(\mu_0,\dots,\mu_k)^{-1}$$

up to a scalar, where the μ_j are given in Lemma 3.25. Clearly $\rho_k(t_{\gamma_0})^{4k+8}$ is the identity, and we only need to prove that the order of $\rho_k(t_{\gamma_0})$ is not less than $4k+8$. If some power n of this matrix is a scalar times the identity, then this scalar is 1 since $\mu_0=1$. Now,

$$\mu_m=(-1)^mA^{m^2+2m}=A^{(2k+4)m+m^2+2m}=A^{(2k+6)m+m^2},$$

and we are done, if we can prove the following claim: if there exists an n such that $4k+8$ divides $((2k+6)m+m^2)n$ for all $m=0,\dots,k$, then $n\in(4k+8)\mathbb{Z}$. Since $k\geq 2$, it suffices to prove that

$$1=\gcd(4k+8,(2k+6)1+1^1,(2k+6)2+2^2)=\gcd(4k+8,2k+7,4k+16).$$

Now if a natural number a divides $4k+8$ and $4k+16$, it divides 8 and is either 1 or even, and no even numbers divide $2k+7$.

The case $g \geq 2$ follows from the $g = 1$ case by Theorem 3.36, as one can find a separating curve in the surface, splitting the surface into a disjoint union of a genus $g-1$ surface and a torus. The corresponding TQFT vector space splits into a number of vector spaces associated to coloured surfaces, and the mapping class group action respects this splitting. Now it suffices to consider the action of the meridional curve in the torus with one 0-coloured interval, which is exactly the one we considered before. □

Remark 4.10. In the case $k = 1$, the order in Lemma 4.9 is 4, since $\mu_1^4 = A^{12} = 1$.

For surfaces with coloured structure and for separating curves, pretty much anything goes, as the following examples illustrate.

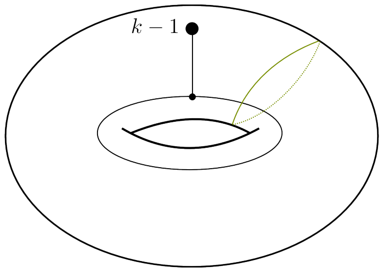


Figure 4.3: A torus with a single coloured interval.

Consider the torus Σ_1 with coloured structure (l, c), where l consists of a single component. Let G be the graph in the solid torus shown in Figure 4.3. If k is even, and the component is coloured by k, there is only a single admissible colouring of G, and the corresponding projective representation is trivial. In general, when l consists of a single component coloured by an even number i,

$$\dim V_k(\Sigma_1, l, c) = k - i + 1, \tag{4.11}$$

as is easily seen by considering the admissibility constraints on G. Now, if k is odd, and l is coloured by $i = k - 1$, we claim that the Dehn twist t_a about the meridian shown in the image is represented by a matrix of order 4. In this case, the only admissible colourings of the non-coloured edge of G are $i/2$ and $i/2+1$. Now t_a acts diagonally on the corresponding basis vectors by multiplication by $\mu_{i/2}$ and $\mu_{i/2+1}$ respectively. To prove that t_a^4 is in the kernel of ρ_k^{k-1}, we only have to realize that $\mu_{i/2}^4 = \mu_{i/2+1}^4$, which follows from the following simple calculation:

$$\begin{aligned}\mu_{i/2}^4 \mu_{i/2+1}^{-4} &= (A^{i^2/4+i})^4 (A^{i^2/4+i+1+i+2})^{-4} = A^{4i-8i-12} \\ &= A^{-(4i+12)} = A^{-(4k+8)} = 1.\end{aligned}$$

Now, the same holds true for any other non-separating curve by the argument of Lemma 4.9.

This argument appears to be hard to generalize to the higher genus case, as the number of admissible colourings grows rapidly. It is, however, not specific to the case where l has one component, and a similar argument works for the torus containing the coloured structure shown in Figure 4.4; once again, the Dehn twists about the curves pictured all act with order 4.

As the level increases, so does the dimensions of V_k (when $g \geq 1$), and in the light of the Lemma 4.9, one might expect the representations to be increasingly faithful. The following result was first proven in the setup of geometric quantization by Andersen [And06].

Theorem 4.11 (Asymptotic faithfulness of $\rho_k^{N,d}$)**.** *Let Σ be a compact surface of genus $g \geq 2$ with one boundary component, and let $\rho_k^{N,d}$ be the projective representation of* $\Gamma(\Sigma)$

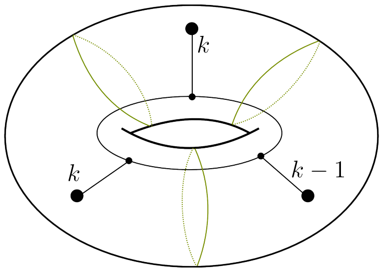


Figure 4.4: A torus with three intervals coloured $(k, k, k-1)$.

from (4.1)*. Then*

$$\bigcap_{k=1}^{\infty} \ker(\rho_k^{N,d}) = \begin{cases} \{1, \iota\}, & \text{if } g = 2, N = 2, d = 0, \\ \{1\}, & \text{otherwise,} \end{cases}$$

where ι denotes the hyperelliptic involution.

Sketch of proof. Let $\mathcal{M}_\sigma = (\mathcal{M}^d_{\mathrm{SU}(N)}, \omega, I_\sigma)$. A main point of the proof is that the only elements of $\Gamma(\Sigma)$ which act trivially on $\mathcal{M}$ are the elements specified on the right hand side in the theorem, which Andersen deduces by showing that elements acting trivially on $\mathcal{M}$ also act trivially on the corresponding $\mathrm{SL}(N, \mathbb{C})$-moduli space, which reduces the problem to well-known Teichmüller theory. Parallel transport in $\mathcal{V}_k$ induces a parallel transport in $\mathrm{End}(\mathcal{V}_k)$ (see [And06] or Theorem 6.7 below). Let $\varphi \in \Gamma(\Sigma)$, denote by φ^* the action of φ on $\mathcal{M}$, and let $f \in C^\infty(\mathcal{M})$ be any smooth function on $\mathcal{M}$. We have the following commutative diagram.

$$\begin{tikzcd} H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma) \arrow[r, "\varphi^*"] \arrow[d, "T^k_{f,\sigma}"'] & H^0(\mathcal{M}_{\varphi^*\sigma}, \mathcal{L}^k_{\varphi^*\sigma}) \arrow[r, "P_{\varphi^*\sigma,\sigma}"] \arrow[d, "T^k_{f\circ\varphi^*,\varphi^*\sigma}"'] & H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma) \arrow[d, "P_{\varphi^*\sigma,\sigma} T^k_{f\circ\varphi^*,\varphi^*\sigma}"] \\ H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma) \arrow[r, "\varphi^*"] & H^0(\mathcal{M}_{\varphi^*\sigma}, \mathcal{L}^k_{\varphi^*\sigma}) \arrow[r, "P_{\varphi^*\sigma,\sigma}"] & H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma) \end{tikzcd}$$

Let $\varphi \in \cap_{k=1}^\infty \ker(\rho_k^{N,d})$, and let us see that φ acts trivially on $\mathcal{M}$. By construction of $\rho_k^{N,d}$, we have that $P_{\varphi^*\sigma,\sigma} \circ \varphi^*$ is a scalar multiple of the identity, and by the above diagram, $T^{(k)}_{f,\sigma} = P_{\varphi^*\sigma,\sigma} T^{(k)}_{f\circ\varphi^*,\varphi^*\sigma}$. Andersen proves that for any two points $\sigma_0, \sigma_1 \in \mathcal{T}$, the Toeplitz operators satisfy

$$\|P_{\sigma_0,\sigma_1} T^{(k)}_{f,\sigma_0} - T^{(k)}_{f,\sigma_1}\| = O(k^{-1}).$$

From this it follows that

$$\begin{aligned} \lim_{k\to\infty} \|T^{(k)}_{f - f\circ\varphi^*,\sigma}\| &= \lim_{k\to\infty} \|T^{(k)}_{f,\sigma} - T^{(k)}_{f\circ\varphi^*,\sigma}\| \\ &= \lim_{k\to\infty} \|P_{\varphi^*\sigma,\sigma} T^{(k)}_{f\circ\varphi^*,\varphi^*\sigma} - T^{(k)}_{f\circ\varphi^*,\sigma}\| = 0. \end{aligned}$$

Now, by Theorem 2.14, $f - f \circ \varphi^* = 0$, so φ acts trivially on $\mathcal{M}$. □

Later, using only methods of skein theory, Freedman, Walker, and Wang [FWW02], and more recently Marché and Narimannejad [MN08], have shown that in the SU(2)-case, the same result is true for closed surfaces.

Theorem 4.12 (Asymptotic faithfulness of ρ_k). *Let Σ_g be a closed connected surface of genus g, and let ρ_k denote the projective representation obtained from the Kauffman bracket TQFT. Then*

$$\bigcap_{k=1}^{\infty} \ker(\rho_k) = \begin{cases} \{1, \iota\} & \text{if } g = 1, 2, \\ \{1\} & \text{otherwise.} \end{cases}$$

4.4.2 Conjectures and experiments

In light of the previous section, we arrive at the following conjecture.

Conjecture 4.13. *Let Σ be a surface of genus $g \geq 1$ with n coloured intervals and $2g+n > 2$. Then the kernel of the corresponding quantum representation ρ_k at level k will be generated by powers of Dehn twists of all possible curves together with central elements. For non-separating curves in non-coloured surfaces, the powers are $4k + 8$.*

This conjecture is hard to prove or disprove directly, since in general it is a non-trivial task to determine whether or not a general mapping class can be written as a word in the specific powers of Dehn twists. The reason for excluding the case $g = 1$, $n = 0$ from Conjecture 4.13 is the following.

Proposition 4.14. *For a torus with empty structure, the statement of Conjecture 4.13 fails to hold.*

Proof. For a group G and a natural number n, let G^n denote the normal subgroup of G generated by all n'th powers of elements in G. In [New62], it is shown that the group $\mathrm{PSL}(2,\mathbb{Z})/\mathrm{PSL}(2,\mathbb{Z})^n$ has infinite order for $n = 6{\cdot}72 = 432$. Since the surjective composition

$$\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})/\mathrm{PSL}(2,\mathbb{Z})^n$$

factors over $\mathrm{SL}(2,\mathbb{Z})^n$, it follows that $\mathrm{SL}(2,\mathbb{Z})/\mathrm{SL}(2,\mathbb{Z})^n$ has infinite order.

The normal subgroup $\langle t_\alpha^n \rangle$ generated by n'th powers of all possible Dehn twists about curves in the torus is obviously contained in the group Γ_1^n generated by *all* n'th powers. Since Γ_1 is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$, the group $\Gamma_1/\langle t_\alpha^{432} \rangle$ has infinite order. The same will be true for $\Gamma_1/\langle t_\alpha^{432}, H_1 \rangle$, where H_1 denotes the mapping class of elliptic involution.

In the torus case, the only curves giving rise to non-trivial Dehn twists are the non-separating ones. Thus, if the Conjecture 4.13 were true, we would therefore obtain an isomorphism

$$\Gamma_1/\langle t_\alpha^{4k+8}, H_1 \rangle \to \rho_k(\Gamma_1).$$

It is a theorem by Gilmer, [Gil99], that the images $\rho_k(\Gamma_1)$ are finite for all levels, giving a contradiction at level $k = 106$, since $4 \cdot 106 + 8 = 432$. $\square$

The finiteness of the image of the quantum representations of surfaces with no coloured structure, used in the above proof above for $g = 1$, turns out to be unique for the torus, as follows from the following result by Masbaum, [Mas99].

Theorem 4.15. *For a surface Σ with genus $g \geq 2$, the image of ρ_k is infinite whenever $k \neq 1, 2, 4, 8$.*

Sketch of proof. In [Mas99], Masbaum constructs explicitly a mapping class in the sphere containing four intervals coloured by the number 1 whose image under ρ_k is infinite in the specified range. By Theorem 3.36, we can view this mapping class as a mapping class on Σ, still acting with infinite order, proving the theorem. We recall the construction of the mapping class. The vector space $V_k(S^2, l, (1,1,1,1))$ is two-dimensional for all levels $k \geq 2$ generated by the two handlebodies with coloured structure shown in Figure 4.5. Let a and b be the curves in $(S^2, l, (1,1,1,1))$ shown in Figure 4.6, and let $w = t_a^{-1} t_b$. Making a change of basis, Masbaum finds an explicit representation matrix M_k for the action of w and proves that, for a particular embedding $A \mapsto \xi$ of the ground ring into $\mathbb{C}$, the matrix has trace $|\mathrm{tr}(M_k)| > 2$ for $k \neq 1, 2, 4, 8$. Thus, since the vector spaces V_k were two-dimensional, at least one eigenvalue λ_k of M_k satisfies $|\lambda_k| > 1$. Note that since the anomalies in an embedding will be complex roots of unity, and whereas the eigenvalues will change under scalar multiplication by these, their absolute values will not, and we conclude that M_k has infinite order. □

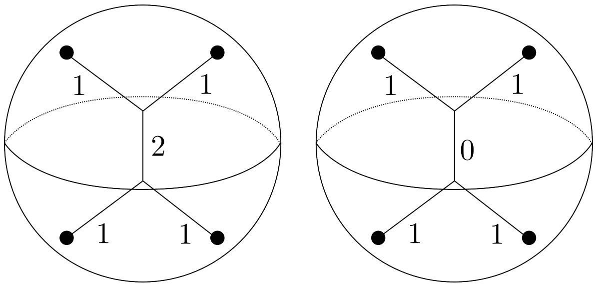


Figure 4.5: The generators of $V_k(S^2, l, (1,1,1,1))$.

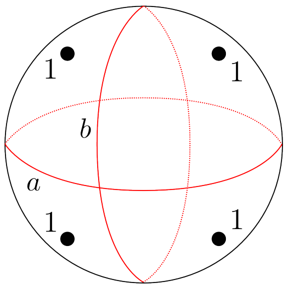


Figure 4.6: The curves a and b used in Theorem 4.15.

The same example was carried out with the geometric construction of the quantum representations in [LPS13]. Here, the images of the mapping class group in $V_k(S^2, l, (1,1,1,1))$ are described explicity and all proven finite when $k = 1, 2, 4, 8$.

Norbert A'Campo and Gregor Masbaum has written a PARI/GP tool[2] to calculate explicitly the matrices for the actions of certain Dehn twists in a particular handlebody basis of V_k for low levels k (and to be more precise, calculations often become infeasible around $k = 14$). The software uses the fact that to evaluate general coloured link diagrams, it suffices to be able to evaluate coloured theta graphs and tetrahedral graphs, whose evaluations are computed in [MV94]. One important feature of the software is that all calculations are done over $\mathbb{Z}[A, A^{-1}]$ and thus not subject to rounding errors. Using this, we can check some of the cases left out by the results above, improving Masbaum's result ever so slightly.

Proposition 4.16. *For a surface Σ with genus $g \geq 2$, the image of ρ_k is infinite for $k \neq 1, 2, 4$.*

Computer assisted proof. By Theorem 4.15, it suffices to consider the case $k = 8$. Consider the mapping class $w = t_\alpha^2 t_\beta^{-2}$ acting on $V_8(\Sigma_1, l, 2)$ (see Figure 4.7). By a calculation involving the above mentioned software,

$$\mathrm{tr}(\rho_8(t_a^2 t_b^{-2})) = A^{14} + 2A^{12} + A^{10} - 2A^8 - 3A^6 + 2A^2 + 1. \tag{4.12}$$

[2]The software is currently available at `http://www.geometrie.ch/TQFT/`

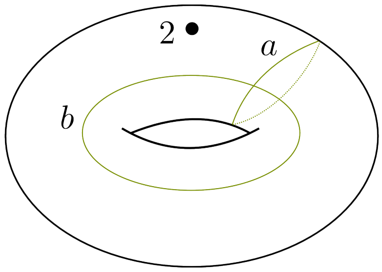


Figure 4.7: The curves a and b used in Proposition 4.16.

By (4.11), we have $\dim V_8(\Sigma, l, 2) = 7$, and letting the parameter A be

$$A = \exp(3 \cdot 2\pi i/(8 \cdot 4 + 8)),$$

the absolute value of (4.12) is strictly greater than 7, implying as before that $t_a^2 t_b^{-2}$ has infinite order in ρ_8. The result follows by Theorem 3.36. □

In the case of a surface containing a single coloured interval, we can also use the software to immediately describe the orders of elements as functions of the level and colouring, so as to obtain intuition about their behaviours. Let ρ_k^i denote the quantum representation associated to a genus $g = 1$ surface containing an interval coloured i. Tables 4.1 and 4.2 describe the orders of two particular mapping classes. A dash represents an order greater than 100.

$k \backslash i$	0	2	4	6	8	10	12
2	3	1					
3	15	3					
4	12	3	1				
5	12	–	3				
6	12	–	3	1			
7	6	18	–	3			
8	30	–	30	3	1		
9	15	–	–	–	3		
10	12	12	–	–	3	1	
11	21	–	–	–	–	3	
12	24	–	–	–	–	3	1
13	60	–	–	–	–	–	3

Table 4.1: The orders of $\rho_k^i(t_a t_b^{-1})$.

$k \backslash i$	0	2	4	6	8	10	12
2	8	1					
3	6	2					
4	12	12	1				
5	4	–	2				
6	16	–	16	1			
7	18	–	–	2			
8	12	–	60	20	1		
9	10	–	–	–	2		
10	24	–	–	–	24	1	
11	6	–	–	–	–	2	
12	8	–	–	–	–	28	1
13	6	–	–	–	–	–	2

Table 4.2: The orders of $\rho_k^i(t_a^2 t_b^{-1})$.

For example we note again from this that we really do need the assumption of no coloured structure in the last part of Conjecture 4.13. Namely, we see that $(t_a^2 t_b^{-1})^2 \in \ker \rho_7^2$, but since the mapping class group in this case is

$$\Gamma_{1,1} \cong \langle t_a, t_b \mid t_a t_b t_a = t_b t_a t_b \rangle,$$

we have a well-defined homomorphism $l : \Gamma_{1,1} \to \mathbb{Z}$ mapping t_a and t_b to 1. Now, if Conjecture 4.13 were true in this case, we should expect that $l(\ker \rho_k^i) \subseteq \gcd(6, 4k+8)\mathbb{Z}$, since $l((t_a t_b)^3) = 6$, but $l((t_a^2 t_b^{-1})^2) = 2 \notin 6\mathbb{Z}$.

In view of these considerations, one might also hope for the last part of Conjecture 4.13 to hold in the coloured case when the image of the relevant quantum representation is infinite; no contradictions have been found by computer search in this case.

Chapter 5

Asymptotics of quantum representations

In this chapter we will discuss the large k asymptotics of the quantum representations considered in the previous chapters. The main reason for being interested in these lies in the fact that – physically – for high levels, which we will think of as $k = 1/\hbar$, the values of the path integrals localize to the space of classical solutions, which for Chern–Simons theory is the moduli space of flat connections. Thus, while the path integrals themselves are ill-defined, this allows for a rigorous study of their asymptotics which in particular may be compared with any of the rigorous constructions of TQFT from Chapter 3.

5.1 Physical motivation

We begin by briefly motivating the work of the following sections by recalling Witten's conjecture for the semi-classical approximation of Chern–Simons theory [Wit89]. This conjecture has been described by a number of authors and rather than repeating the entire analysis, we provide only enough details to motivate our mathematical study of the asymptotics of quantum invariants. For a more detailed account, we refer to e.g. [Wit89, Sect. 2], [Ati90, Sect. 7.2], [AS92], [Saw06], [And02, Sect. 7.2] [AH12, App. A], [Oht02, App. F.3]. Recall that our main objective is to study the Chern–Simons partition function

$$Z_{k,G}^{\mathrm{phys}}(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} \exp(2\pi i k \mathrm{CS}(A))\, \mathcal{D}A. \tag{5.1}$$

To tie in with our previous discussion, assume in the following that $G = \mathrm{SU}(N)$, and that $\mathfrak{g} = \mathrm{Lie}(G)$ is endowed with the Ad-invariant inner product of Section 4.3.2. The method of stationary phase roughly states that an oscillatory integral like the above localizes, as $k \to \infty$, to the critical values of its phase. For finite dimensional integrals this may be made precise as follows: if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with finitely many non-degenerate critical points and $g : \mathbb{R}^n \to \mathbb{R}$ is smooth and compactly supported, then (see [AH12, App. A], [Hör90, Sect. 7.7]) we have

$$\int_{\mathbb{R}^n} \exp(ikf(x))g(x)\,dx \sim_{k\to\infty} \left(\frac{2\pi}{k}\right)^{n/2} \sum_{x\in\mathrm{Crit}(f)} \exp\left(\frac{\pi i}{4}\operatorname{sign}\mathrm{Hess}_x(f)\right)\frac{\exp(ikf(x))g(x)}{\sqrt{|\det \mathrm{Hess}_x(f)|}}. \tag{5.2}$$

In this context, the critical points of f are referred to as the *stationary points* of f. Now, the stationary points of the Chern–Simons functional are exactly the flat connections (see [Saw06, Lemma 7.2]), and so by analogy of (5.2), one should expect – asymptotically – the main contribution of the integral of (5.1) to arise from flat connections on M. Suppose that α is a flat connection, and that $\alpha+\beta$, $\beta \in \Omega^1(M)\otimes\mathfrak{g}$, is a perturbation of α. Then

$$\mathrm{CS}(\alpha+\beta) = \mathrm{CS}(\alpha) + \frac{1}{8\pi^2}\int_M \mathrm{tr}(\beta\wedge\nabla^\alpha\beta + \tfrac{2}{3}\beta\wedge\beta\wedge\beta).$$

That is, the contribution to the path integral coming from the flat connection α is

$$\exp(2\pi ik\mathrm{CS}(\alpha))\int_{\beta\in\Omega^1(M)\otimes\mathfrak{g}} \exp\left(\frac{ik}{4\pi}\int_M \mathrm{tr}(\beta\wedge\nabla^\alpha\beta + \tfrac{2}{3}\beta\wedge\beta\wedge\beta)\right) d\beta. \tag{5.3}$$

For the rest of this document, we restrict our attention to the quadratic term of the latter integral, which is often referred to as the *1-loop term*. The study of this as well as the higher order terms arising from the cubic term of the integral is often colloquially referred to as *perturbative Chern–Simons theory*; a field it is beyond the scope of this thesis to give a thorough discussion of.

Turning again to the finite-dimensional case we observe that it follows from the well-known exact formula

$$\int_{\mathbb{R}} \exp(i\lambda x^2)\,dx = \sqrt{\frac{\pi}{|\lambda|}}\exp(\pm\pi i/4)$$

that for a non-degenerate quadratic form Q on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \exp(\tfrac{1}{2}ikQ(x,x))\,dx = \left(\frac{2\pi}{k}\right)^{n/2}\frac{\exp(\pi i\,\mathrm{sign}\,Q/4)}{\sqrt{|\det Q|}}.$$

We wish to apply this result to (5.3) with the operator $Q = -*\nabla^\alpha$, the self-adjoint operator on $\Omega^1(M)\otimes\mathfrak{g}$ satisfying

$$\frac{1}{2}ik\langle\beta, Q(\beta)\rangle = \frac{ik}{4\pi}\int_M \mathrm{tr}(\beta\wedge\nabla^\alpha\beta)\,d\beta$$

with respect to the pairing

$$\langle\beta_1,\beta_2\rangle = -\int_M \mathrm{tr}(\beta_1\wedge *\beta_2).$$

Now, Q is degenerate on $\nabla^\alpha\Omega^0(M)\otimes\mathfrak{g}$, but assuming further that α is isolated and non-degenerate, Q defines a non-degenerate form on $\Omega^1(M;\mathfrak{g})/\nabla^\alpha\Omega^0(M;\mathfrak{g})$. This reduction has the following finite-dimensional interpretation: if K is a compact Lie group acting on $\mathbb{R}^n$ preserving a possibly degenerate quadratic form Q, and $\mathcal{O}_x$ denotes the K-orbit of $x\in\mathbb{R}^n$, then K defines a map $\rho_x : K\to\mathcal{O}_x$ by $\rho_x(g) = g\dot{x}$. Let $B:\mathrm{Lie}(K)\to T_x\mathcal{O}_x$ denote the differential of ρ_x at the identity. Then one can show that

$$\int_{\mathbb{R}^n} \exp(\tfrac{1}{2}ikQ(x,x))\,dx = \mathrm{Vol}\,K\int_{\mathbb{R}^n/K}\exp(\tfrac{1}{2}ikQ(x,x))|\det(B^*B)|^{1/2}\,d\mu,$$

where B^* denotes the adjoint of B, $\mathrm{Vol}\,K$ is the volume of K with respect to a Haar measure, and $d\mu$ is the induced measure on the orbit space $\mathbb{R}^n/K$.

Pursuing this finite-dimensional analogy, Witten [Wit89] was able to describe the constituents of the semi-classical limit of the path integral in terms of mathematically well-known invariants, under the assumption that the flat connections are isolated modulo gauge. We arrive at the following conjecture, cited from [AH12, Conj. 1.3], which is a refinement of Witten's original calculation, cf. also [FG91, (1.36)], [Jef92, (5.1)]: assume first that the moduli space $\mathcal{M}$ of flat connections on M is smooth, and that the tangent space at a point $[A]\in\mathcal{M}$ is $H^1(\mathcal{M},\nabla^A)$. Associated to each $[A]\in\mathcal{M}$ is the so-called *Reidemeister torsion* (see [Fre92] for a definition as well as example calculations) whose square root we denote $\tau_M(A)^{1/2}\in\wedge^{\max}H^0(M,\nabla^A)\otimes(\wedge^{\max}H^1(M,\nabla^A))^*$. As $H^0(M,\nabla^A)\subseteq\mathfrak{g}$, the choice of basic inner product in $\mathfrak{g}$ allows us to canonically identify $\tau_M(A)^{1/2}$ as a volume on $H^1(M,\nabla^A)\cong T_{[A]}\mathcal{M}$, giving us a means of integration on $\mathcal{M}$. In the conjecture, $Z_k^G(M)$ denotes the quantum G-invariant of M.

Conjecture 5.1 (Semi-classical approximation conjecture). *The leading order of the semi-classical approximation of the quantum G-invariant of a closed oriented* 3*-manifold M is*

$$\begin{aligned} Z_k^G(M) \dot{\sim}_{r\to\infty} & \frac{1}{|Z(G)|}\exp\left(-\pi i \dim G \frac{1+b^1(M)}{4}\right) \\ & \cdot \sum_{c\in C_M} \int_{[A]\in\mathcal{M}_c} \tau_M(A)^{1/2} \exp(2\pi i r \mathrm{CS}(A)) \exp\left(-2\pi i\left(\frac{I_A}{4}+\frac{h_A^0+h_A^1}{8}\right)\right) \\ & \cdot r^{\operatorname{gmax}_{A\in M_c}(h_A^1-h_A^0)/2}. \end{aligned}$$

In this conjecture, $\dot{\sim}$ means that the left hand side should have an asymptotic expansion with the right hand side being the leading order term; we make this precise in the following section. In the right hand side, the $c \in C_M$ index the (finitely many) connected components $\mathcal{M}_c$ of $\mathcal{M}$, $r = k + h^\vee$, where $h^\vee$ is the dual Coxeter number of G (which for $G = \mathrm{SU}(N)$ is $h^\vee = N$), $b^1(M)$ denotes the first Betti number of M, $I_A \in \mathbb{Z}/8\mathbb{Z}$ denotes the so-called *spectral flow* of

$$\begin{pmatrix} *\nabla^{A(t)} & -\nabla^{A(t)}* \\ \nabla^{A(t)}* & 0 \end{pmatrix},$$

where $A(t)$ is a path from the trivial connections to A in the space of connections. Moreover, $h_A^i = \dim H^i(M, \nabla^A)$, and $\operatorname{gmax}_{A\in\mathcal{M}_c}$ denotes the *generic maximum* over $\mathcal{M}_c$, i.e. maximum over all Zariski open subsets of $\mathcal{M}_c$ with the property that $h_A^1 - h_A^0$ is constant on that subset. In particular, as all ingredients are mathematically well-defined, at least under the regularity conditions on $\mathcal{M}$, we are left with a means of testing on one hand the soundness of the heuristic argument leading to the conjecture, and on the other the sensibleness of the rigorously defined invariant.

5.2 The asymptotic expansion conjecture

Let $G = \mathrm{SU}(N)$, and let M be a compact oriented 3-manifold. Since any principal bundle over M is trivializable, we refer to the moduli space $\mathcal{M}$ of flat bundles on M simply as the moduli space of flat connections on M.

It is known that $\mathcal{M}$ has only finitely many connected components. Here, we assume that $\pi_1(M)$ has n generators and view $\mathcal{M}$ as a quotient of a subset of $\mathrm{SU}(2)^{\times n}$ with the natural topology using Theorem 2.23. If furthermore M is closed, the Chern–Simons action is known to be constant on connected components. Once again, in the case $G = \mathrm{SU}(N)$, the Chern–Simons action is given by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2}\int_M \operatorname{tr}(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A).$$

We can now formulate the following conjecture, which may be viewed as a counterpart to Conjecture 5.1 with no assumptions on $\mathcal{M}$. The conjecture first appeared in [And02, Sect. 7.2]; see also [AH12].

Conjecture 5.2 (The asymptotic expansion conjecture (AEC)). *Let M be a closed oriented 3-manifold. Let $r = k+N$, and let $\{c_0 = 0, \dots, c_m\}$ be the finitely many values of the Chern–Simons action on the moduli space of M. Then there exist $d_j \in \mathbb{Q}$, $b_j \in \mathbb{C}$, and $a_j^l \in \mathbb{C}$ for $j = 0, \dots, m$, $l = 1, 2, \dots$ such that*

$$Z_k(M) \sim_{k\to\infty} \sum_{j=0}^m \exp(2\pi i r c_j) r^{d_j} b_j \left(1 + \sum_{l=1}^\infty a_j^l r^{-l/2}\right)$$

in the sense that

$$\left| Z_k(M) - \sum_{j=0}^{m} \exp(2\pi i r c_j) r^{d_j} b_j \left(1 + \sum_{l=1}^{L} a_j^l r^{-l/2}\right)\right| = O(r^{d-(L+1)/2}),$$

for $L = 0, 1, \dots,$ *where* $d = \max_j d_j$.

Remark 5.3. If an asymptotic expansion as the above exists, it is well-known that the constants c_j, d_j, b_j, and a_j^l are uniquely determined – see [And12c] and [Han99, Theorem 8.2]. In particular they are topological invariants of the 3-manifold.

In comparison with Conjecture 5.1, it is natural to conjecture that the constants appearing in this conjecture have various topological interpretations in terms of e.g. Reidemeister torsion and spectral flow. We will be particularly interested in the behaviour of the d_j. For the following conjecture, see [And02, Sect. 7.2], [AH12].

Conjecture 5.4 (The growth rate conjecture)**.** *Let* c_j, d_j *be as above, and let* $\mathcal{M}_j$ *denote the subspace of* $\mathcal{M}$ *consisting of connections with Chern–Simons action* c_j*. Then*

$$d_j = \frac{1}{2} \operatorname*{gmax}_{[A]\in\mathcal{M}_j} (h_A^1 - h_A^0).$$

Combining the conjectures, we can describe the growth rate of $Z_k(M)$ as follows (compare with [Gar98]).

Conjecture 5.5. *Let* $d = \max_i\{d_i\}$ *be the largest of the* d_i *of Conjecture 5.4. Then*

$$|Z_k(M)| = O(k^d).$$

Remark 5.6. Recall that we in the definition of the quantum $\mathrm{SU}(N)$-invariants made a choice of the Homflypt polynomial variable a in order to set up the invariants to agree with those of [Wit89]. It is an interesting question – and one to which we offer no answer – how the above asymptotic expansions depend on the choice of a.

5.2.1 Known results on the AEC

Papers on these various conjectures are plenty, and we try here to give an overview of what is known on the subject but apologize in advance to any authors that might have been left out. Jeffrey, [Jef92], proved the AEC for all lens spaces as well as for mapping tori of Anosov torus homeomorphisms on a torus in the case $G = \mathrm{SU}(2)$ and for a certain subfamily of these mapping tori for general G. Jeffrey's result for Anosov torus homeomorphisms has been extended to general G in [Cha12]. Andersen, [And12c], proved the AEC for mapping tori of finite order diffeomorphisms of surfaces of genus $g \geq 2$ in the case $G = \mathrm{SU}(N)$, and in [AH12] the authors expanded on this result, identifying the leading order terms of the asymptotic expansion in terms of classical topological-geometric invariants. The case of lens spaces for $G = \mathrm{SU}(N)$ is discussed in [HT02] (see the comment preceding [HT02, Conj. 4.3]). Hikami, [Hik05], also considered the case of Brieskorn homology spheres for $G = \mathrm{SU}(2)$ in greater detail. The paper [KSV97] suggests numerical evidence *against* the conjectures for the 3-manifolds $S^3(4_1(-n/1))$, $n = 7, 16, 22$, demonstrating a contribution from a non-Chern–Simons-value phase of order -2 in the level. The asymptotics of quantum invariants of surgeries on the figure-eight knot have also been analyzed by Andersen and Hansen in [AH06].

Rozansky, [Roz96], discusses the AEC for general Seifert manifolds for $G = \mathrm{SU}(2)$. The methods used in that paper are somewhat heuristic and some error estimates have been left out. These are supplied in Hansen's PhD thesis [Han99] and collected in [Han01] and the preprint [Han05], proving the AEC for all Seifert manifolds in the case $G = \mathrm{SU}(2)$. Our present analysis should be viewed in this context, as the manifolds M^b and $\tilde{M}^b$ we will be considering are Seifert manifolds of symbol $\{b; (o_1, 1)\}$ and $\{(b; (n_2, 2)\}$ for $b \in \mathbb{Z}$ (see e.g. [Orl72, pp. 124–125]). The Thurston geometries of the manifolds are Euclidean when $b = 0$, and they have nil geometry for $n \neq 0$.

As far as preprints go, the results are as follows: in [CM11], Charles and Marché prove the SU(2)-AEC for the manifolds $S^3(4_1(p/q))$ with p not divisible by 4 and $H^1(M, \mathrm{Ad}_\rho) = 0$ for all irreducible flat connections $\rho : \pi_1(M) \to \mathrm{SU}(2)$. Using similar techniques, Charles [Cha11] proves the AEC for $S^3(T_{a,b}(p/q))$, whenever $p/q \neq 2abl/m$ for all l, m with a and b not dividing m, also in the SU(2) case. Here $T_{a,b}$ is the (a, b)-torus knot. Charles extends his results to other mapping tori in [Cha10a] under certain regularity conditions.

In [And12a] Andersen has recently given a geometric formula for the leading order asymptotics of quantum invariants of any 3-manifold, suited for a further asymptotic analysis.

5.2.2 The quantum invariants as invariants

Before turning to our new results, we include an intermezzo on the value of the quantum invariant as a 3-manifold invariant and implications of the AEC for distinguishing 3-manifolds.

It is well-known that the quantum SU(2)-invariants do not classify 3-manifolds up to orientation preserving homeomorphisms; see for instance [KB93], [Lic93] in which the authors have produced pairs of 3-manifolds whose quantum SU(2)-invariants agree for all levels. Much more recently, in [Fun13, Thm. 1.3], Funar constructs infinitely many pairs of mapping tori of Anosov homeomorphisms for which all the invariants coming from the modular tensor category construction described in the previous chapter agree pairwise. Inspired by this, one might ask the following, which to our knowledge is an open question:

Question 5.7. *Are there infinite (or just arbitrarily large) families of* 3*-manifolds for which the quantum G-invariants (or more generally all invariants constructed from modular tensor categories) agree?*

The AEC suggests that in looking for such a family, one could consider 3-manifolds whose moduli spaces of connections (or simply fundamental groups) are in one way or the other similar. For instance, the lens spaces $L(p, q)$ for fixed p have isomorphic fundamental groups, yet a computer search implementing Jeffrey's lens space computation [Jef92, Thm. 3.4] did not shed much light upon the question. It did, however, suggest the following (which it might be possible to prove directly from Jeffrey's formula).

Question 5.8. *Let $p > 2$ be a prime, and let $0 < n < m < p/2$. Is it true that*

$$Z_k^{\mathrm{SU}(2)}(L(p^2, np - 1)) = Z_k^{\mathrm{SU}(2)}(L(p^2, mp - 1))$$

for all $k \notin p\mathbb{N}$? This implies in particular that, for any fixed k_0, there are arbitrarily large sets of non-homeomorphic 3*-manifolds not distinguished by $\{Z_1, \dots, Z_{k_0}\}$.*

Note here that the fact that the spaces in the statement above are not homeomorphic follows directly from the classification of lens spaces. More precisely, for $p, q_1, q_2 > 0$ with $\gcd(q_1, p) = \gcd(q_2, p) = 1$, the lens spaces $L(p, q_1)$ and $L(p, q_2)$ are homeomorphic, if and only if

$$q_1 \equiv \pm q_2^{\pm 1} \bmod p.$$

On the other hand, Conjecture 5.1 could seem to suggest that the collection of quantum invariants contains the invariants appearing on the right hand side of the conjecture. This is not true in general, as for instance the quantum SU(2)-invariants can not distinguish the non-homeomorphic lens spaces $L(65, 8)$ and $L(65, 18)$ (see [Jef92, Rem. 3.9]) whereas Reidemeister torsion can. Indeed, the volumes of the moduli spaces of the two manifolds, given by integration of the Reidemeister torsions, are different, suggesting that the interplay between the various invariants is not entirely obvious.

It is an interesting question what exactly the geometric content of the quantum invariants is. We know from Thurston's Geometrization, as proved by Perelman, that the geometry of a closed 3-manifold is closely related to its fundamental group, and Conjectures 5.1 and 5.2 tell us how one might hope to understand quantum invariants of 3-manifolds through a study of the representation theory of their fundamental groups. In the best case, one could hope for a quantum analogue of geometrization. As an exemplification of this rather vague wish, we have the following conjecture.

Conjecture 5.9 (The quantum Poincaré conjecture)**.** *The collection of the quantum G-invariants $\{Z_k^G\}$ is an S^3-detector for fixed G.*

For a recent development on this conjecture, as well as its relation to the problem of detecting the unknot, see the preprint [And12a]. Tying this together with Conjecture 5.4, we propose the following

Conjecture 5.10. *If M satisfies Conjecture 5.2, then*

$$d = \max_j \{d_j\} = -\frac{\dim G}{2}$$

only if $M = S^3$.

The relevance to Conjecture 5.9 is the following (see [And02, p. 479]): if for a 3-manifold M satisfying Conjecture 5.4, $d = -\dim G/2$, then $\pi_1(M)$ has no non-central G-representation. Suppose now moreover that M is obtained from S^3 by $+1$-surgery along a knot K. Then by the main result of [KM04], K is the unknot and $M = S^3$.

5.2.3 Mapping tori of torus homeomorphisms

Let Σ be a closed surface, and let T_φ be the mapping torus for a homeomorphism in the mapping class $\varphi \in \Gamma(\Sigma)$. Then by Theorem 3.35, $Z_k(T_\varphi) = \operatorname{tr} \rho_k(\varphi)$, and we can use the framework of quantum representations to study the conjectures above. We consider the case of the mapping tori of homeomorphisms of a torus and aim at proving the asymptotic expansion conjecture for this family of 3-manifolds. It is well-known that every orientable torus bundle over S^1 is homeomorphic to such a mapping torus, and thus these are covered by our analysis.

The calculations have been carried out for the homeomorphisms $U \in \Gamma_1$ satisfying $|\operatorname{tr}(U)| > 2$, for which the result is the following theorem; see [Jef92, Thm. 4.1].

Theorem 5.11. *Let*

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \cong \mathrm{SL}(2, \mathbb{Z})$$

and assume that $|\operatorname{tr}(U)| > 2$. Then there exists a canonical choice of framing for T_U, and the quantum $\mathrm{SU}(2)$-invariant is given by

$$\begin{aligned} Z_k(T_U) = &\exp\left(\frac{2\pi i \psi(U)}{4r}\right) \operatorname{sgn}(d + a \mp 2) \sum_{\pm} \pm \frac{1}{2|c|\sqrt{|d + a \mp 2|}} \\ &\cdot \sum_{\beta=0}^{|c|-1} \sum_{\gamma=1}^{|d+a\mp 2|} \exp\left(2\pi i r \frac{-c\gamma^2 + (a-d)\gamma\beta + b\beta^2}{d + a \mp 2}\right), \end{aligned}$$

where $r = k + 2$, and $\psi(U) \in \mathbb{Z}$ depends only on U and is given by [Jef92, (4.4)].

Remark 5.12. Note that it follows from this that up to the framing correction – i.e. the number $\exp(\frac{2\pi i \psi(U)}{4r})$ – the sequence $\{Z_k(T_U)\}_k$ is periodic (with period being a divisor of $(d + a - 2)(d + a + 2)$), so in particular it is bounded. Note also that the mapping classes

$U \in \Gamma_1$ with $|\mathrm{tr}(U)| > 2$ are exactly the Anosov ones. We will discuss these further in Section 6.1.1.

We should also note that Jeffrey uses, in her definition of $Z_k(T_U)$, the representations of $\mathrm{SL}(2,\mathbb{Z})$ arising from conformal field theory but by the results of Section 4.3, these are essentially equivalent to those of interest to us.

For the rest of this section, let M^b and $\tilde{M}^b$ denote the 3-manifolds introduced in Section 4.3.1.

5.3 The case $G = \mathrm{SU}(2)$

5.3.1 The quantum invariants

The main technical tool we need is a generalization of quadratic reciprocity by Jeffrey [Jef92] which we recall here. For proofs, see [Jef91], [HT02, App.].

Theorem 5.13 (Jeffrey). *Let $r \in \mathbb{Z}$, let V be a real vector space of dimension l with an inner product $\langle\cdot,\cdot\rangle$, and let Λ be a lattice in V with dual lattice Λ^*. Let $B : V \to V$ be a self-adjoint automorphism and let $\psi \in V$. Assume that*

$$\tfrac{1}{2}r\langle\lambda, B\lambda\rangle,\ \langle\lambda, B\eta\rangle,\ r\langle\lambda,\psi\rangle,\ \tfrac{1}{2}r\langle\mu, B\mu\rangle,\ r\langle\mu,\xi\rangle,\ r\langle\mu,\psi\rangle \in \mathbb{Z}$$

for all $\lambda, \eta \in \Lambda$, $\mu, \xi \in \Lambda^*$. Then

$$\begin{aligned}\mathrm{Vol}(\Lambda^*) &\sum_{\lambda\in\Lambda/r\Lambda} \exp(\pi i\langle\lambda, B\lambda/r\rangle)\exp(2\pi i\langle\lambda,\psi\rangle) \\ &= \left(\det\frac{B}{i}\right)^{-1/2} r^{l/2} \sum_{\mu\in\Lambda^*/B\Lambda^*} \exp(-\pi i\langle\mu+\psi, rB^{-1}(\mu+\psi)\rangle).\end{aligned}$$

An immediate corollary of Jeffrey's quadratic reciprocity theorem is the following well-known formula for generalized Gauss sums (where we correct a minor typo in [Jef92]).

Theorem 5.14. *Let a,b,c be integers, $a \neq 0$, $c \neq 0$, and assume that $ac+b$ is even. Then*

$$\sum_{n=0}^{|c|-1} \exp(\pi i(an^2+bn)/c) = |c/a|^{1/2}\exp\left(\pi i\frac{|ac|-b^2}{4ac}\right)\sum_{n=0}^{|a|-1}\exp(-\pi i(cn^2+bn)/a).$$

As it does not complicate matters to any greater extent and might shed light on an asymptotic expansion conjecture for links (which to the our knowledge has never been formulated explicitly), we will be interested in pairs (M^b, L_λ), where L_λ is a λ-coloured link parallel to the meridional curve used in the surgery description of C_{t_μ}. More concretely, (M^b, L_λ) has the surgery description shown in Figure 5.1.

Note that the pairs (M^b, L_λ), $b \neq 0$, are exactly of the type that Beasley considers in [Bea09]. It could be interesting to compare the work of Beasley with the present results, but we make no such attempt.

Elements of $Y_{2,k}$ correspond simply to integers j, $0 \le j \le k$, and as before, we will simply write j for the Young diagram containing j cells.

Theorem 5.15 ([AJ12], Thm. 3.1.3). *For $k \geq j$ and $b \neq 0$, we have*

$$\begin{aligned}Z_k^{\mathrm{SU}(2)}(M^b, L_j) = \exp\left(\frac{\pi i b}{2r}\right)&\Bigg(\sqrt{\frac{r}{2|b|}}\exp(-\pi i\,\mathrm{sgn}(b)/4) \\ &\cdot\left[\sum_{n=0}^{|b|-1}\exp\left(2\pi i r\frac{n^2}{b}\right)\sum_{l=0}^{j}\exp\left(2\pi i\left(\frac{(2l-j)^2}{4br}+\frac{(2l-j)n}{b}\right)\right)\right] \\ &-\frac{j+1}{2}-\frac{(-1)^j(j+1)}{2}\exp\left(-\frac{\pi i}{2}br\right)\Bigg).\end{aligned}$$

where $r = k+2$.

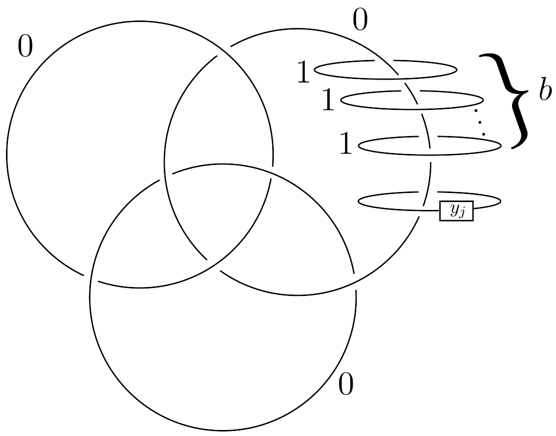


Figure 5.1: Surgery description of (M, L_j). In this picture a link component with a number next to it means surgery along that component with framing according to the number; this should not be confused with the colourings used for idempotents (cf. also Section 4.3), such as the one on the component coloured y_j in this picture.

Remark 5.16. Recall that χ_j, the composition of the character of the $(j+1)$-dimensional irreducible representation of SU(2) with $\exp(i\cdot)$ (see e.g. [Fol95, p. 143]), is given by

$$\chi_j(t) = \sum_{l=0}^{j} \exp(i(2l-j)t) = \begin{cases} j+1 & \text{if } t = 0, \\ (-1)^{j+1}(j+1) & \text{if } t = \pi, \\ \frac{\sin((j+1)t)}{\sin(t)} & \text{otherwise,} \end{cases}$$

for $t \in [0, 2\pi)$. Combining this with the description of the moduli space of flat connections of the manifolds M^b in terms of their holonomies, given in the proof of Proposition 5.19 below, one can directly relate the quantum invariant contributions coming from the links L_j to the holonomies of the corresponding connections.

Proof of Theorem 5.15. First of all, note that for $z \neq 0, \pm 1$,

$$\frac{z^{(j+1)} - z^{-(j+1)}}{z - z^{-1}} = z^{-j}\frac{z^{2j+2} - 1}{z^2 - 1} = \sum_{l=0}^{j} z^{2l-j}. \tag{5.4}$$

In the description of mapping cylinders as acting on $V_k(S^1 \times S^1)$ via gluing, adding the link L_j corresponds to applying the so-called curve operator, acting on the basis described in Section 4.3.1 by encircling the $\hat{y}_l$ by an unknot coloured y_j. This action is diagonal and described in Lemma 4.5. Applying the formula of Lemma 4.5 and (5.4) with $z = \exp(\pi i n/r)$, we therefore find that

$$\begin{aligned} Z_k(M^b, L_j) &= \sum_{n=0}^{k} a^{b(n^2+2n)} \frac{a^{2(n+1)(j+1)} - a^{-2(n+1)(j+1)}}{a^{2(n+1)} - a^{-2(n+1)}} \\ &= \exp\left(\frac{\pi i b}{2r}\right) \sum_{n=1}^{r-1} \sum_{l=0}^{j} \exp\left(\frac{\pi i}{r}(-\tfrac{1}{2}n^2 b + 2ln - jn)\right) \\ &= \exp\left(\frac{\pi i b}{2r}\right) \sum_{l=0}^{j} \sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r}(-n^2 b + 4ln - 2jn)\right). \end{aligned}$$

Quadratic reciprocity tells us that

$$\begin{aligned}\sum_{n=1}^{2r-1} \exp\left(\frac{\pi i}{2r}(bn^2 + 4ln - 2jn)\right) = \sqrt{\frac{2r}{|b|}} \exp\left(\frac{\pi i}{4}\operatorname{sgn}(b)\right) \exp\left(-\frac{\pi i(4l-2j)^2}{8br}\right) \\ \cdot \left(\sum_{n=0}^{|b|-1} \exp\left(-\frac{\pi i}{b}(2rn^2 + (4l-2j)n)\right)\right) - 1.\end{aligned} \tag{5.5}$$

It now suffices to relate the sum on the left hand side of (5.5) to the corresponding one having upper limit $r-1$. Note now that mod $4r$, we have $bn^2 \equiv b(2r-n)^2$ and

$$4ln - 2jn \equiv -(4l-2j)(2r-n).$$

Making in the process a change of variables $n \to r - n$, we conclude from this that

$$\begin{aligned}\sum_{n=r+1}^{2r-1} \exp\left(\frac{\pi i}{2r}(bn^2 + 4ln - 2jn)\right) &= \sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r}(b(n+r)^2 + (4l-2j)(n+r))\right) \\ &= \sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r}(b(2r-n)^2 + (4l-2j)(2r-n))\right) \\ &= \sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r}(bn^2 - (4l-2j)n)\right).\end{aligned}$$

From this it follows that

$$\begin{aligned}&\sum_{n=1}^{2r-1} \exp\left(\frac{\pi i}{2r}(bn^2 + 4ln - 2jn)\right) \\ &\quad= \sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r}(bn^2 + 4ln - 2jn)\right) + (-1)^j \exp\left(\frac{\pi i}{2}br\right) \\ &\qquad+ \sum_{n=r+1}^{2r-1} \exp\left(\frac{\pi i}{2r}(bn^2 + 4ln - 2jn)\right) \\ &\quad= \sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r}bn^2\right)\left(\exp\left(\frac{\pi i}{2r}(4ln - 2jn)\right) + \exp\left(-\frac{\pi i}{2r}(4ln-2jn)\right)\right) \\ &\qquad+ (-1)^j \exp\left(\frac{\pi i}{2}br\right).\end{aligned}$$

Note now that even though the individual terms are not, the sum

$$\sum_{l=0}^{j} \exp\left(\frac{\pi i}{2r}(4ln - 2jn)\right)$$

is real as in (5.4). We therefore find that

$$\begin{aligned}2\sum_{l=0}^{j}\sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r}(bn^2 + 4ln - 2jn)\right) = \sum_{l=0}^{j}\left(\sum_{n=1}^{2r-1} \exp\left(\frac{\pi i}{2r}(bn^2 + 4ln - 2jn)\right)\right) \\ - (-1)^j \exp\left(\frac{\pi i}{2}br\right)\end{aligned}$$

and combining all of this, performing an overall conjugation, we obtain the claim of the theorem. $\square$

Remark 5.17. Had we chosen instead to work with the quantum invariants arising from the Kauffman bracket, the curve operator used in the proof would carry an extra factor of $(-1)^j$ and so we would have obtained the same result for the quantum invariants only up to a factor of $(-1)^j$. In particular, the results agree in the link-less case $j = 0$.

Letting $j = 0$ in Theorem 5.15 and using (4.6), we immediately obtain the following formula for the quantum SU(2)-invariants of the link-less manifolds.

Corollary 5.18 ([AJ12], Cor. 3.1.6)**.** *For $b \neq 0$, we have*

$$Z_k^{\mathrm{SU}(2)}(M^b) = Z_k^{\mathrm{SU}(2)}(\tilde{M}^b) = \exp\left(\frac{\pi i b}{2r}\right)\left(\sqrt{\frac{r}{2|b|}}\exp(-\pi i \operatorname{sgn}(b)/4)\sum_{n=0}^{|b|-1}\exp(2\pi i n^2/b) - \frac{1}{2} - \frac{\exp(-\pi i r b/2)}{2}\right).$$

Plots of various values of this invariant are shown in Figures 5.2–5.6.

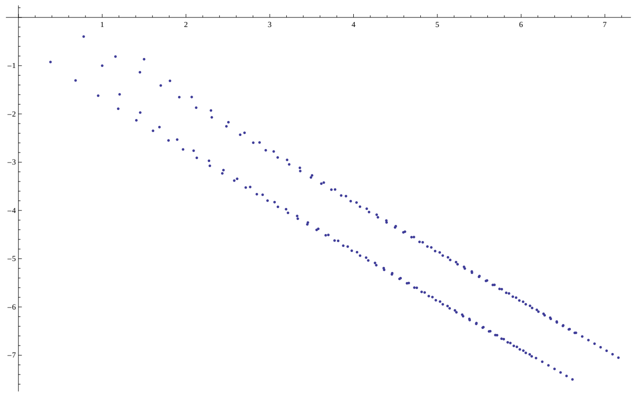


Figure 5.2: The values of $Z_k^{\mathrm{SU}(2)}(M^1)$ in the complex plane for $k = 1, \ldots, 200$.

5.3.2 The mapping torus moduli space

In the case of $G = \mathrm{SU}(2)$, we can describe the moduli space of flat connections completely explicitly and find the following.

Proposition 5.19 ([AJ12], Prop. 3.2.1)**.** *Let γ be the isotopy class of an essential simple closed curve in Σ_1, and let $b \in \mathbb{Z}$, $b \neq 0$. The moduli space $\mathcal{M}$ of flat* SU(2)*-connections on M^b can be described as follows:*

For b odd, it consists of a copy of the pillowcase, $\frac{|b|-1}{2}$ copies of the 2-torus T^2, as well as a component containing a single point.

For b even, it consists of 2 copies of the pillowcase and $\frac{|b|}{2} - 1$ copies of T^2.

The only irreducible connection is the one in the single point component for b odd. On the various components, the Chern–Simons action takes the following (not necessarily distinct) values:

$$\mathrm{CS}(\mathcal{M}) = \begin{cases} \{\frac{j^2}{b} \mid j = 0, \ldots, \frac{|b|-1}{2}\} \cup \{1 - \frac{b}{4}\} & \textit{if } b \textit{ is odd,} \\ \{\frac{j^2}{b} \mid j = 0, \ldots, \frac{|b|}{2}\} & \textit{if } b \textit{ is even.} \end{cases}$$

In this statement, *pillowcase* refers to the space $I \times I/(x \sim -x)$, which is also the SU(2)-moduli space of the torus and as a topological space is homeomorphic to S^2.

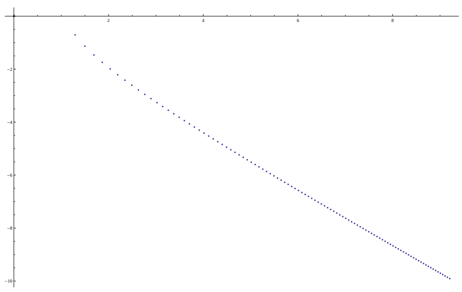


Figure 5.3: $Z_k^{\mathrm{SU}(2)}(M^2)$.

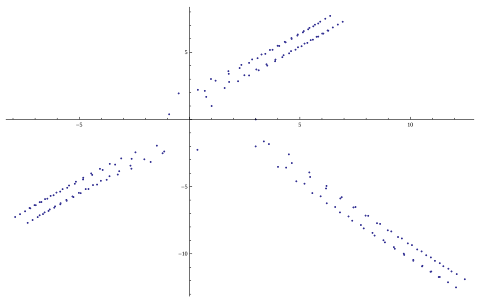


Figure 5.4: $Z_k^{\mathrm{SU}(2)}(M^3)$.

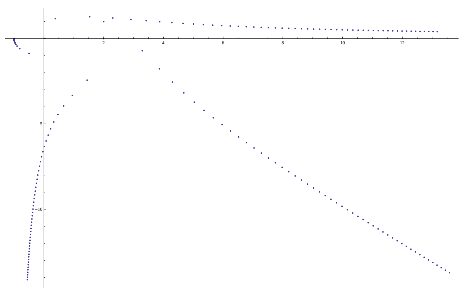


Figure 5.5: $Z_k^{\mathrm{SU}(2)}(M^4)$.

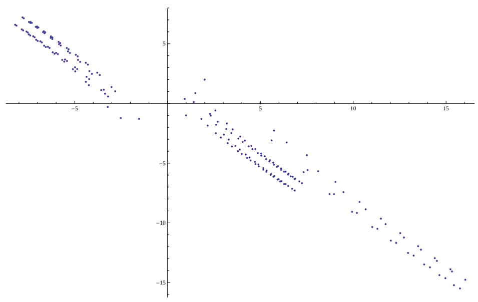


Figure 5.6: $Z_k^{\mathrm{SU}(2)}(M^5)$.

Proof. We can describe $\mathcal{M}$ by describing the representations of $\pi_1(T_{t_\gamma^b})$. In the following, we consider $m = -b$ as this lowers the total number of signs.

It is well-known (see e.g. [Jef92]) that for a mapping torus T_φ, $\varphi : \Sigma \to \Sigma$, the fundamental group is given by the twisted product

$$\pi_1(T_\varphi) = \mathbb{Z} \rtimes \pi_1(\Sigma),$$

where $\mathbb{Z}$ acts on $\pi_1(\Sigma)$ via φ. In our special case, the fundamental group therefore has the presentation

$$\pi_1(T_{t_\gamma^b}) = \langle \alpha, \beta, \delta \mid \alpha\beta = \beta\alpha, \delta\alpha\delta^{-1} = \alpha, \delta\beta\delta^{-1} = \alpha^m\beta \rangle.$$

Here, we simply note that two essential closed curves are homotopic if and only if they are isotopic (see e.g. [FM11, Prop. 1.10]), and we have simply let α be the homotopy class of any curve representing γ and choose β so that $i(\alpha, \beta) = 1$. The moduli space of flat connections is identified with a quotient of a subset of $\mathrm{SU}(2)^{\times 3}$ as

$$\mathcal{M} \cong \{(A, B, C) \in \mathrm{SU}(2)^{\times 3} \mid AB = BA, CAC^{-1} = A, CBC^{-1} = A^m B\}/\sim,$$

where $\sim$ denotes simultaneous conjugation. Since A and B commute for $[(A, B, C)] \in \mathcal{M}$, they both lie in the same maximal torus in $\mathrm{SU}(2)$, and by conjugating them simultaneously we may assume that they are both diagonal. In other words, they are both elements of $T := \mathrm{U}(1) \subseteq \mathrm{SU}(2)$. Here, for $a \in \mathrm{U}(1)$, we simply write a for the matrix $\mathrm{diag}(a, \bar{a})$ in $\mathrm{SU}(2)$. We now consider three cases.

Case 1. Assume that $A, B \in Z(\mathrm{SU}(2))$. In this case, $B = A^m B$, so $A^m = 1$, and so A must be the identity if m is odd.

Case 2. Assume that $A \notin Z(\mathrm{SU}(2))$. Then $C \in N(T)$, where $N(T)$ is the normalizer of T, which is given by $N(T) = T \cup L$, where

$$L = \left\{ \begin{pmatrix} 0 & \exp(2\pi i t) \\ -\exp(-2\pi i t) & 0 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

If $C \in T$ then $B = BA^m$, and $A^m = 1$ is the only restriction. If $C \in N(T) \setminus T = L$, conjugation by C corresponds to inversion of elements of T. Thus, for $C \in L$, we have $A^{-1} = A$ contradicting that $A \notin Z(\mathrm{SU}(2))$.

Case 3. Assume that $A \in Z(\mathrm{SU}(2)), B \notin Z(\mathrm{SU}(2))$. Again, $C \in N(T)$. If $C \in T$ we find again that $A^m = 1$, so $A = 1$ if m is odd. If $C \in L$, then $B^{-1} = BA^m$, and $B^2 = A^{-m} = A^m$, which is impossible for m even when $B \notin Z(\mathrm{SU}(2))$, but for m odd, and $A = -1$, we get a contribution for $B = \pm i$.

In conclusion, when m is odd,

$$\begin{aligned}\mathcal{M} \cong ((&\{1\} \times \{\pm 1\} \times \mathrm{SU}(2)) \cup (\{\exp(2\pi i j/m) \mid j = 0, \dots, |m| - 1\} \setminus \{1\} \times T \times T) \\ &\cup (\{1\} \times T \setminus \{\pm 1\} \times T) \cup (\{-1\} \times \{\pm i\} \times L))/\sim,\end{aligned}$$

and when m is even,

$$\begin{aligned}\mathcal{M} \cong ((&\{\pm 1\} \times \{\pm 1\} \times \mathrm{SU}(2)) \cup (\{ \exp(2\pi i j/m) \mid j = 0, \dots, |m| - 1\} \setminus \{\pm 1\} \times T \times T) \\ &\cup (\{\pm 1\} \times T \setminus \{\pm 1\} \times T))/\sim .\end{aligned}$$

In the case where m is odd, the last component is a union of two copies of T where all points are identified under conjugation since

$$\begin{aligned}\begin{pmatrix} 0 & \exp(2\pi i s) \\ -\exp(-2\pi i s) & 0 \end{pmatrix} \begin{pmatrix} 0 & \exp(2\pi i t) \\ -\exp(-2\pi i t) & 0 \end{pmatrix} \begin{pmatrix} 0 & \exp(2\pi i s) \\ -\exp(-2\pi i s) & 0 \end{pmatrix}^{-1} \\ = \begin{pmatrix} 0 & \exp(-2\pi i s + 4\pi i t) \\ -\exp(2\pi i s - 4\pi i t) & 0 \end{pmatrix}.\end{aligned}$$

This is the single point component of $\mathcal{M}$. If m is odd or even, for the quotients of the first and third component in the above description, it suffices to consider the quotient of $\{1\} \times T \times T$ or $\{\pm 1\} \times T \times T$ respectively, since we may first identify any element of $\mathrm{SU}(2)$ with its diagonalization. Now $T \times T$ is identified with the torus $\mathbb{R}^2/\mathbb{Z}^2$, and the only conjugation action left is the action by the Weyl group $W \cong \mathbb{Z}_2$ which acts on $\mathbb{R}^2/\mathbb{Z}^2$ by $(t, s) \mapsto (-t, -s)$. Therefore, $\mathcal{M}$ contains one or two of the pillowcase in the cases m odd or even respectively.

Finally, let $j \in \{0, \dots, |m| - 1\}$, and assume that $j/m \notin \{0, \frac{1}{2}\}$. Arguing as above, the only conjugation action left on $\{\exp(2\pi i j/m)\} \times T \times T$ is that of the Weyl group. Now, in this case, it acts non-trivially on the first factor, mapping $\exp(2\pi i j/m)$ to $\exp(-2\pi i j/m)$, and the resulting quotient becomes a number of copies of $T \times T$ as claimed.

Finding the values of the Chern–Simons action for $G = \mathrm{SU}(2)$ is a well-studied problem, and in our case, the values on the components can be found immediately using e.g. methods of [Jef92]. The result can also be seen as a special case of Proposition 5.28, where we elaborate on the available techniques.

The claim about reducibility follows from the fact that an $\mathrm{SU}(2)$-connection is reducible if and only if the corresponding representation has image contained in a maximal torus, which is the case for all representations above but the one mapping C into L. □

Corollary 5.20 ([AJ12], Cor. 3.2.2)**.** *The asymptotic expansion conjecture holds for M^b and $\tilde{M}^b$ for all $b \in \mathbb{Z}$, $b \neq 0$.*

Proof. As in the proof of Theorem 5.15, we note that

$$\exp(2\pi i r((b-1) - (j-1))^2/b) = \exp(2\pi i r j^2/b), \tag{5.6}$$

for $j = 0, \dots, |b| - 1$. To prove the corollary, it is now a matter of rearranging the terms in the formula for $Z_k(T_{t_\gamma^b})$, obtained in Corollary 5.18.

Assume first that b is even. In this case,

$$\exp(-\pi i r b/2) = \exp(\pi i r b/2) = \exp\left(2\pi i r \left(\frac{|b|}{2}\right)^2 \frac{1}{b}\right),$$

and it follows from (5.6) that

$$Z_k(M^b) = \exp\left(\frac{\pi i b}{2r}\right)\left(\sqrt{\frac{r}{2|b|}}\exp(-\operatorname{sgn}(b)\pi i/4)\left(2\sum_{n=1}^{|b|/2-1}\exp(2\pi i r n^2/b)\right.\right.$$
$$\left.\left. + 1 + \exp(\pi i b r/2)\right) - \frac{\exp(-\pi i r b/2)}{2} - \frac{1}{2}\right)$$
$$= \exp\left(\frac{\pi i b}{2r}\right)\left(\sum_{n=1}^{|b|/2-1}\exp(2\pi i r n^2/b)\left[\sqrt{\frac{2r}{|b|}}\exp(-\operatorname{sgn}(b)\pi i/4)\right]\right.$$
$$+ \exp(2\pi i r\cdot 0/b)\left[\sqrt{\frac{r}{2|b|}}\exp(-\operatorname{sgn}(b)\pi i/4) - \frac{1}{2}\right]$$
$$\left. + \exp\left(2\pi i r\left(\frac{|b|}{2}\right)^2\frac{1}{b}\right)\left[\sqrt{\frac{r}{2|b|}}\exp(-\operatorname{sgn}(b)\pi i/4) - \frac{1}{2}\right]\right).$$

Now, one obtains the full asymptotic expansion of $Z_k(M^b)$ by introducing the Taylor series for $\exp(\pi i b/(2r))$ and $1/\sqrt{r}$, and one obtains the conjecture by comparing the resulting expression with the result of Proposition 5.19. For b odd, the exact same argument shows that

$$Z_k(M^b) = \exp\left(\frac{\pi i b}{2r}\right)\left(\sum_{n=1}^{(|b|-1)/2}\exp(2\pi i r n^2/b)\left[\sqrt{\frac{2r}{|b|}}\exp(-\operatorname{sgn}(b)\pi i/4)\right]\right.$$
$$\left. + \exp(2\pi i r 0/b)\left[\sqrt{\frac{r}{2|b|}}\exp(-\operatorname{sgn}(b)\pi i/4) - \frac{1}{2}\right] - \exp(-\pi i r b/2)\frac{1}{2}\right),$$

and once again the claim follows from Proposition 5.19.

The argument for $\tilde{M}^b$ is identical as one finds the exact same Chern–Simons values for these manifolds. We omit the details. □

Note that the proof of Corollary 5.20 gives us explicitly the leading order term of $Z_k(M^b)$, and in particular we are now able to turn to Conjecture 5.4 for M^b.

5.3.3 The growth rate conjecture

The main technical tool needed in proving the growth rate conjecture for the manifolds M^b and $\tilde{M}^b$ is the correspondence between deRham cohomology with twisted coefficients and group cohomology which we briefly recall.

Let G be any group. A *G-module* is an abelian group N with a left action of G. The elements of N invariant under the action will be denoted N^G. A *cocycle on G with values in N* is a map $u : G \to N$ satisfying the cocycle condition

$$u(gh) = u(g) + gu(h).$$

A *coboundary* is a cocycle of the form $g \mapsto \delta m(g) := m - gm$ for some $m \in N$. The set of cocycles is denoted $Z^1(G,N)$, and the set of coboundaries is denoted $B^1(G,N)$. We define the first cohomology group of G with coefficients of N as the quotient

$$H^1(G,N) = Z^1(G,N)/B^1(G,N).$$

Notice that an element of N satisfies $\delta m \equiv 0$ exactly when $m \in N^G$. We are led to define

$$H^0(G,N) = N^G.$$

Now, let $P \to M$ be a principal G-bundle over a closed connected oriented 3-manifold M, $G = \mathrm{SU}(N)$, and let $[A]$ be the gauge equivalence class of a flat connection in P, represented by a representation $\rho \in \operatorname{Hom}(\pi_1(M), G)$. The representation ρ defines a $\pi_1(M)$-module structure on $\mathfrak{g} = \operatorname{Lie}(G)$ through the composition $\operatorname{Ad} \circ \rho : \pi_1(M) \to \operatorname{Aut}(\mathfrak{g})$. Let as always $H^i(M, \operatorname{Ad}_P)$ denote the cohomology of the complex (2.1). The following result is well-known.

Lemma 5.21. *If M has contractible universal covering space, there are isomorphisms*

$$H^0(M, \operatorname{Ad}_P) \cong H^0(\pi_1(M), \mathfrak{g}), \quad H^1(M, \operatorname{Ad}_P) \cong H^1(\pi_1(M), \mathfrak{g}).$$

Consider the case where M is the mapping torus of a homeomorphism of a surface of genus $g \geq 1$. Let A be a flat connection in $P \to M$, and let ρ be a representative of $[A]$ in the moduli space $\operatorname{Hom}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$. The elements of $\mathfrak{su}(2)$ fixed by the action of $\pi_1(M)$ given by $\operatorname{Ad} \circ \rho$ are exactly those in the centralizer of the image $\rho(\pi_1(M))$, and so by Lemma 5.21,

$$h^0_A = \dim \operatorname{Lie}(Z(\rho(\pi_1(M)))).$$

Similarly, Lemma 5.21 gives a description of h^1_A using only the corresponding representation of $\pi_1(M)$.

Theorem 5.22 ([AJ12], Thm. 3.3.2)**.** *Let $\mathcal{M}_{j/b}$, $j = 0, \dots, \lceil \frac{|b|+1}{2} \rceil$, and $\mathcal{M}_{-b/4}$ be the components of the moduli space of M^b arising from Proposition 5.19, and let*

$$d'_i = \frac{1}{2} \operatorname*{gmax}_{[A] \in \mathcal{M}_i} (h^1_A - h^0_A).$$

Then $d'_{j/b} = \frac{1}{2}$ and $d'_{-b/4} = 0$. In particular, Conjecture 5.4 holds true in this case.

Proof. Again, we introduce $m = -b$. Abusing notation slightly, we write $\rho \in \mathcal{M}$ for the (conjugacy class of a) representation corresponding to a (gauge class of a) flat connection in $\mathcal{M}$. Let A, B, C denote the images of generators α, β, δ of $\pi_1(M^m)$ under ρ. Using the remark following Lemma 5.21, we find that if $\rho \in \mathcal{M}_{j/b}$, then $h^0_\rho = 1$ except in four or eight points in the cases where m is odd or even respectively, those points corresponding to $A, B, C = \pm 1$. When $A, B, C = \pm 1$, we have $h^0_\rho = 3$. For the representation $\rho \in \mathcal{M}_{-b/4}$, we have $h^0_\rho = 0$.

We now describe h^1_ρ. The cocycles $Z^1(\pi_1(M^{-m}), \mathfrak{su}(2))$ embed in $\mathfrak{su}(2)^3$ under the map

$$u \mapsto (u(\alpha), u(\beta), u(\delta)).$$

The image can be determined since cocycles map the three relators

$$R_1 = \alpha\beta\alpha^{-1}\beta^{-1}, \ R_2 = \alpha\delta\alpha^{-1}\delta^{-1}, \ R_3 = \delta\beta\delta^{-1}\alpha^{-m}\beta^{-1}$$

of our presentation of $\pi_1(M^{-m})$ to $0 \in \mathfrak{su}(2)^3$. One finds that $Z^1(\pi_1(M^{-m}), \mathfrak{su}(2))$ can be identified with the kernel of the map $R = (\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) : \mathfrak{su}(2)^3 \to \mathfrak{su}(2)^3$ determined by R_1, R_2, R_3 by the requirement that

$$\tilde{R}_i(u(\alpha), u(\beta), u(\delta)) = u(R_i).$$

Assume for simplicity that $m > 0$. Noting that in general,

$$u(g^{-1}) = -\operatorname{Ad}(\rho(g^{-1}))u(g),$$

the cocycle condition gives

$$\begin{aligned} u(R_1) &= u(\alpha) - \operatorname{Ad}(B)u(\alpha) - u(\beta) + \operatorname{Ad}(A)u(\beta), \\ u(R_2) &= u(\alpha) - \operatorname{Ad}(C)u(\alpha) - u(\delta) + \operatorname{Ad}(A)u(\delta), \\ u(R_3) &= -\operatorname{Ad}(B)(\sum_{n=0}^{m} \operatorname{Ad}(A^n))u(\alpha) - u(\beta) \\ &\quad + \operatorname{Ad}(C)u(\beta) + u(\delta) - \operatorname{Ad}(A^m B)u(\delta). \end{aligned}$$

Here, the first two equalities are immediate, and the last one follows from

$$\begin{aligned}
u(R_3) &= u(\delta) + \mathrm{Ad}(C)u(\beta\delta^{-1}\alpha^{-m}\beta^{-1}) \\
&= u(\delta) + \mathrm{Ad}(C)(u(\beta) + \mathrm{Ad}(B)u(\delta^{-1}\alpha^{-m}\beta^{-1})) \\
&= u(\delta) + \mathrm{Ad}(C)u(\beta) + \mathrm{Ad}(CB)(u(\delta^{-1}) + \mathrm{Ad}(C^{-1})u(\alpha^{-m}\beta^{-1})) \\
&= u(\delta) + \mathrm{Ad}(C)u(\beta) - \mathrm{Ad}(CBC^{-1})u(\delta) \\
&\quad + \mathrm{Ad}(CBC^{-1})(u(\alpha^{-m}) + \mathrm{Ad}(A^{-m})u(\beta^{-1})) \\
&= u(\delta) + \mathrm{Ad}(C)u(\beta) - \mathrm{Ad}(A^m B)u(\delta) \\
&\quad - \mathrm{Ad}(CBC^{-1}A^{-m})u(\alpha^m) - \mathrm{Ad}(CBC^{-1}A^{-m}B^{-1})u(\beta) \\
&= u(\delta) + \mathrm{Ad}(C)u(\beta) - \mathrm{Ad}(A^m B)u(\delta) - \mathrm{Ad}(B)u(\alpha^m) - u(\beta)
\end{aligned}$$

since, in general

$$u(g^m) = \sum_{n=0}^{m-1} \mathrm{Ad}(\rho(g)^n)u(g).$$

In other words, R is given by

$$R(x_1, x_2, x_3) = \begin{pmatrix} x_1 - \mathrm{Ad}(B)x_1 - x_2 + \mathrm{Ad}(A)x_2 \\ x_1 - \mathrm{Ad}(C)x_1 - x_3 + \mathrm{Ad}(A)x_3 \\ -\mathrm{Ad}(B)(\sum_{n=0}^{m} \mathrm{Ad}(A^n))x_1 - x_2 + \mathrm{Ad}(C)x_2 + x_3 - \mathrm{Ad}(A^m B)x_3 \end{pmatrix}.$$

Under this identification, the coboundaries $B^1(\pi_1(M^{-m}), \mathfrak{su}(2))$ become

$$\{(x - \mathrm{Ad}(A)x, x - \mathrm{Ad}(B)x, x - \mathrm{Ad}(C)x) \mid x \in \mathfrak{su}(2)\} \subseteq \ker R \subseteq \mathfrak{su}(2)^3.$$

Recall that $\mathfrak{su}(2)$ has a basis given by

$$\left\{ e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}.$$

Consider first the case $\rho \in \mathcal{M}_{j/b}$. Write

$$A = \begin{pmatrix} \exp(2\pi ij/m) & 0 \\ 0 & \exp(-2\pi ij/m) \end{pmatrix}, \quad B = \begin{pmatrix} \exp(2\pi is) & 0 \\ 0 & \exp(-2\pi is) \end{pmatrix},$$
$$C = \begin{pmatrix} \exp(2\pi it) & 0 \\ 0 & \exp(-2\pi it) \end{pmatrix}$$

for $j \in 0, \ldots, \lceil \frac{m+1}{2} \rceil$, and $s, t \in [0, 1)$. A direct computation shows that the matrix representation of R in the basis given above is

$$R = \begin{pmatrix} P - S(s) & -P + S(j/m) & 0 \\ P - S(t) & 0 & -P + S(j/m) \\ T(m, s) & -P + S(t) & P - S(s), \end{pmatrix},$$

where P, S, and T are given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S(r) = \begin{pmatrix} \cos(4\pi r) & -\sin(4\pi r) & 0 \\ \sin(4\pi r) & \cos(4\pi r) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$T(m, s) = \begin{pmatrix} -\eta\cos(4\pi s) & \eta\sin(4\pi s) & 0 \\ -\eta\sin(4\pi s) & -\eta\cos(4\pi s) & 0 \\ 0 & 0 & -m \end{pmatrix},$$
$$\eta = \sum_{n=0}^{m-1} \exp(4\pi ijn/m) = \begin{cases} m, & \text{if } j/m \in \{0, \frac{1}{2}\}, \\ 0, & \text{otherwise.} \end{cases}$$

One finds that $\dim(\ker R) = 6$ when $\frac{j}{m}, s, t \in \{0, \frac{1}{2}\}$ and that $\dim(\ker R) = 4$ otherwise. A similar computation shows that

$$B^1(\pi_1(M^{-m}), \mathfrak{g}) \cong \operatorname{span}\left\{\begin{pmatrix} 1-\cos(4\pi j/m) \\ -\sin(4\pi j/m) \\ 0 \\ 1-\cos(4\pi s) \\ -\sin(4\pi s) \\ 0 \\ 1-\cos(4\pi t) \\ -\sin(4\pi t) \\ 0 \end{pmatrix}, \begin{pmatrix} \sin(4\pi j/m) \\ 1-\cos(4\pi j/m) \\ 0 \\ \sin(4\pi s) \\ 1-\cos(4\pi s) \\ 0 \\ \sin(4\pi t) \\ 1-\cos(4\pi t) \\ 0 \end{pmatrix}, 0\right\},$$

so the subspace of coboundaries has dimension 0 when $\frac{j}{m}, s, t \in \{0, \frac{1}{2}\}$ and dimension 2 otherwise. Notice that by definition of the generic max, these finitely many special cases have no influence on d_i'.

Now, consider the case of $\rho \in \mathcal{M}_{-b/4}$, and write

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this case, R is given by

$$R = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ m & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -m & 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix}.$$

Now $\dim(\ker R) = 3$, and here we find that

$$B^1(\pi_1(M^{-m}), \mathfrak{g}) \cong \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}\right\},$$

so all cocycles are coboundaries. Relating m to b, we obtain

$$\begin{aligned} d_{j/b}' &= \tfrac{1}{2}((4-2)-1) = \tfrac{1}{2}, \\ d_{-b/4}' &= \tfrac{1}{2}((3-3)-0) = 0. \end{aligned}$$

□

5.3.4 Summary of torus bundles

Having proved the SU(2)-AEC for those torus homeomorphisms of trace ± 2, we collect now what is known for torus bundles in general. Recall the trichotomy of $\Gamma_1 \cong \mathrm{SL}(2, \mathbb{Z})$ into elements of trace $|\mathrm{tr}| < 2$ called periodic (or finite order), $|\mathrm{tr}| = 2$ called reducible, and $|\mathrm{tr}| > 2$ called hyperbolic (or Anosov). This, of course, not including the elements $\pm\mathrm{id} \in \mathrm{SL}(2, \mathbb{Z})$ which are both finite order.

As mentioned, Jeffrey [Jef92] proves the SU(2)-AEC for all hyperbolic elements providing a concrete expression for the quantum invariants; see Theorem 5.11. In fact, she gives another formula that can be used to obtain the quantum invariants of finite order elements as well. Up to conjugation, the only finite order elements are the identity and

$$\iota = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; f_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \; f_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; f_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

and their inverses (see [FM11, p. 201]), indexed here by their order in $\mathrm{SL}(2, \mathbb{Z})$. Using [Jef92, (4.7)], the quantum SU(2)-invariants of the corresponding mapping tori are (up to framing anomalies) given by

$$\begin{aligned}
Z_k(T_\iota) &= Z_k(T_{\mathrm{id}}) = r - 1, \\
Z_k(T_{f_3}) &= \frac{i}{2\sqrt{3}}(2\exp(-2\pi i r/3) + 1) + \frac{1}{2}\exp(-\pi i/2), \\
Z_k(T_{f_4}) &= \frac{1}{2}(\exp(\pi i r) + 1), \\
Z_k(T_{f_6}) &= \frac{i}{2\sqrt{3}}(2\exp(2\pi i r/3) + 1) + \frac{1}{2},
\end{aligned}$$

and similar expressions for the inverses. Thus, the AEC can readily be checked for finite order elements as well. We summarize these results in the following theorem[1].

Theorem 5.23 ([AJ12], Thm. 3.4.2)**.** *The asymptotic expansions of the* SU(2)*-Witten–Reshetikhin–Turaev invariants of torus bundles T_U, $U \in \mathrm{SL}(2, \mathbb{Z})$, are exact and in accordance with Conjecture 5.2. The phases and growth rates of the invariants are summarized for the conjugacy classes of* $\mathrm{SL}(2, \mathbb{Z})$ *in Table 5.1.*

$U \in \mathrm{SL}(2, \mathbb{Z})$	$\{c_j\}$	$\{d_j\}$
$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$	$\{0\}$	$\{1\}$
$\begin{pmatrix} \pm 1 & -b \\ 0 & \pm 1 \end{pmatrix}$, $b \neq 0$ even	$\{\frac{j^2}{b} \mid j = 0, \ldots, \frac{\lvert b \rvert}{2}\}$	$\{\frac{1}{2}\}$
$\begin{pmatrix} \pm 1 & -b \\ 0 & \pm 1 \end{pmatrix}$, b odd	$\{\frac{j^2}{b} \mid j = 0, \ldots, \frac{\lvert b \rvert - 1}{2}\} \cup \{-\frac{b}{4}\}$	$\{\frac{1}{2}\} \cup \{0\}$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\lvert a + d \rvert \neq 2$	$\left\{ \frac{-c\gamma^2 + (a-d)\gamma\beta + b\beta^2}{d + a \pm 2} \;\middle\vert\; \begin{array}{l} 0 \le \beta < c, \\ 0 < \gamma \le \lvert a + d \pm 2 \rvert \end{array} \right\}$	$\{0\}$

Table 5.1: Summary of phases and growth rates of quantum invariants of torus bundles.

5.4 The case $G = \mathrm{SU}(N)$

5.4.1 The quantum invariants

Throughout the following, we use the notation of Section 4.3. We will need the observation that

$$\begin{aligned}
P_k + \rho &= \{\mu = \lambda + \rho \mid \lambda \in P_+, \langle \lambda, \theta \rangle \le k\} \\
&= \{\lambda \in \mathrm{int}(P_+) \mid \langle \lambda, \theta \rangle < r\}.
\end{aligned}$$

[1]Note though that Jeffrey seems to use an orientation convention different than ours. Recall that reversing the orientation of the 3-manifold conjugates the quantum invariants and reversing the sign of the Chern–Simons values.

For this reason, we let $\tilde{P}_r = P_k + \rho$.

Theorem 5.24 ([AJ12], Thm. 4.1.1)**.** *The level k quantum* SU(3)*-invariants of M^b, $b \neq 0$, are given by*

$$\begin{aligned}
Z_k(M^b) = \exp(2\pi i/r)\Bigg(&-\frac{\sqrt{3}i}{18b} r \sum_{n=0}^{3|b|-1} \sum_{m=0}^{3|b|-1} \exp\left(2\pi i r \frac{n^2+m^2-nm}{b}\right) \\
&- \frac{1}{2}\sqrt{\frac{3}{2b}} \exp(-\pi i/4)\sqrt{r} \sum_{n=0}^{2|b|-1} \exp\left(\pi i r \frac{3n^2}{2b}\right) \\
&+ \frac{1}{3} + \frac{2}{3}\exp\left(-2\pi i r \frac{b}{3}\right)\Bigg),
\end{aligned}$$

where $r = k+3$.

Proof. Much of the proof holds for every N and to illustrate how one might proceed in general, we specialize only to $N = 3$ when necessary.

By (4.5) and (4.9), it suffices to calculate

$$\sum_{\lambda \in P_k} \exp\left(b\frac{\pi i}{r}\langle \lambda+\rho, \lambda+\rho\rangle\right),$$

and by the remark preceding the statement of the theorem, this sum may also be written as

$$\sum_{\lambda \in \tilde{P}_r} \exp\left(b\frac{\pi i}{r}\langle \lambda, \lambda\rangle\right). \tag{5.7}$$

Let $g(\lambda) = \exp\left(b\frac{\pi i}{r}\langle\lambda,\lambda\rangle\right)$ for $\lambda \in \Lambda^w$, in the notation of Section 4.3.

The idea of the following follows closely that of the similar theorem in [Jef92]. In this, Jeffrey tiles cubes in $\mathbb{R}^{N-1}$ by copies of $\tilde{P}_r$ (or more precisely, subsets of the weight space of the form $\{\sum x_i \Lambda_i \in \mathbb{R}^{N-1} \mid x_i > 0, \sum_{i=1}^{N-1} x_i < r\}$), generated by the action of a particular r-dependent subgroup of the affine Weyl group, noting that for some large enough number of tiles, one is able to apply Theorem 5.13 to the weights contained in the resulting cube, and it turns out that when summing $g(\lambda)$ over all weights in the entire cube, all tiles contribute the same; now, the main difference between what we do here and what is considered in [Jef92] is that in her proof, the boundaries of the tiles do not contribute to the sum, and so the calculation boils down to a combinatorial count of the tiles used so as to figure out the contribution from a single tile. In our calculation, the boundaries do contribute and so we need to be slightly more careful; the aim is to apply Theorem 5.13 with $\Lambda = \Lambda^w$, $r = 2rN$, $B = 2bN\mathrm{Id}$, $\psi = 0$, tiling a cube of side lengths $2Nr$ in $\mathbb{R}^{N-1}$ by simplices as above. See Figure 5.7 for the case $N = 3$; here, the cube of side lengths $6r$ has been split up into 72 2-dimensional triangles as well as a number of lower dimensional simplices. The point now is that the values of $g(\lambda)$ on $\Lambda^w/2rN\Lambda^w$ are determined by the values on a single tile, i.e. a single r-alcove and its boundary. This follows from the fact that $g(\lambda)$ is invariant under the action of the Weyl group and the translations $\lambda \mapsto \lambda + 2h_\alpha$, for any simple root $2h_\alpha$. The group generated by these actions is exactly the group of reflections in faces of the r-alcove. The root lattice of SU(N) has volume $\mathrm{Vol}(\Lambda^R) = \sqrt{N}$. This can be seen using a particular identification of the root system with $\mathbb{R}^{N-1}$, writing the i'th basis vector of the lattice as

$$-\sqrt{\frac{j-1}{j}}e_{j-1} + \sqrt{\frac{j+1}{j}}e_j,$$

where e_j is the j'th standard basis vector of $\mathbb{R}^{N-1}$ (and $e_0 = 0$). Application of Theorem 5.13 gives[2]

[2]Note that in [Jef92], the branch cut is along the negative real axis, and that one thus needs to be slightly careful with the signs on the right hand side of Theorem 5.13 when $N \equiv 3 \bmod 4$.

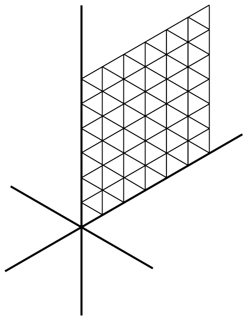


Figure 5.7: The tiling of $\Lambda^w/2Nr\Lambda^w$ by copies of $\tilde{P}_r$ for $N = 3$.

$$\sum_{\lambda\in\Lambda^w/2rN\Lambda^w} g(\lambda) = \frac{1}{\sqrt{N}}\sqrt{\left(\frac{i}{2bN}\right)^{N-1}}(2rN)^{(N-1)/2}\sum_{\mu\in\Lambda^R/2bN\Lambda^R}\exp\left(-\pi i r\frac{\langle\mu,\mu\rangle}{b}\right).$$

From now on, we specialize to the case $N = 3$ and elaborate on the general case in Remark 5.31. In this case $g(\lambda)$ is further invariant under $\lambda \mapsto \lambda + 3r\Lambda_i$, and the above simplifies to

$$\sum_{\lambda\in\Lambda^w/3r\Lambda^w} g(\lambda) = \frac{i}{b}\sqrt{3}r\sum_{\mu\in\Lambda^R/3b\Lambda^R}\exp\left(-\pi i r\frac{\langle\mu,\mu\rangle}{b}\right).$$

If $\mu = nh_1 + mh_2$, where h_1, h_2 are the simple (co)roots spanning Λ^R, we have

$$\langle\mu,\mu\rangle = 2n^2 + 2m^2 - 2nm.$$

The reflection invariance of $g(\lambda)$ implies that we can write $\sum_{\lambda\in\Lambda^w/3r\Lambda^w} g(\lambda)$ in terms of lower dimensional affine subspaces of the weight space. Namely,

$$\sum_{\lambda\in\Lambda^w/3r\Lambda^w} g(\lambda) = 18\sum_{\lambda\in\tilde{P}_r} g(\lambda) + 9\sum_{\lambda\in P_r^{(1)}} g(\lambda) - 2\sum_{\lambda\in P_r^{(0)}} g(\lambda),$$

where

$$\begin{aligned} P_r^{(1)} &= \{\lambda = a\Lambda_1 + 0\Lambda_2 \mid a = 0,\dots,3r-1\},\\ P_r^{(0)} &= \{\lambda = ar\Lambda_1 + br\Lambda_2 \mid a,b = 0,1,2\}. \end{aligned}$$

See Figure 5.8, where the colours indicate the various values of $g(\lambda)$. We find that

$$\sum_{\lambda\in P_r^{(0)}} g(\lambda) = 3\cdot 1 + 6\cdot\exp\left(\frac{2\pi i}{3}br\right),$$

and it follows from Theorem 5.14 that

$$\sum_{\lambda\in P_r^{(1)}} g(\lambda) = \sum_{n=0}^{3r-1}\exp\left(\frac{2\pi i}{3r}bn^2\right) = \sqrt{\frac{3r}{2b}}\exp(\pi i/4)\sum_{n=0}^{2|b|-1}\exp\left(-\pi i\frac{3rn^2}{2b}\right).$$

Putting this together, we obtain the claim of the theorem. □

In Figure 5.9 and Figure 5.10 we include plots of values of $Z_k^{\mathrm{SU}(N)}(M^1)$ for $N = 3, 4$ and $k = 1, \dots, 100$.

The case $b = 0$ is handled separately. It could be viewed as a special case of the Verlinde formula for $\mathrm{SU}(N)$ but we include an elementary argument for completeness.

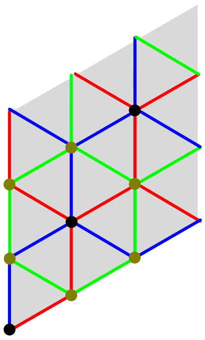


Figure 5.8: The division of $\Lambda^w/3r\Lambda^w$ into subsets in affine subspaces of lower dimensions.

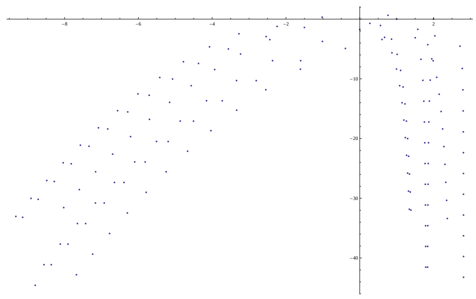


Figure 5.9: $Z_k^{\mathrm{SU}(3)}(M^1)$.

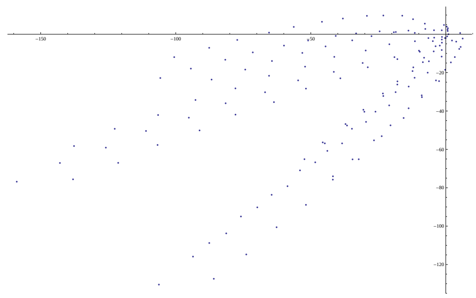


Figure 5.10: $Z_k^{\mathrm{SU}(4)}(M^1)$.

Proposition 5.25 ([AJ12], Prop. 4.1.2)**.** *The level k quantum* $\mathrm{SU}(N)$*-invariant of $M^0 = S^1 \times S^1 \times S^1$ is given by*

$$Z_k^{\mathrm{SU}(N)}(M^0) = \frac{1}{(N-1)!}(r-1)(r-2)\cdots(r-(N-1)).$$

where as always $r = k + N$.

Proof. In general, $Z_k(\Sigma \times S^1) = \dim V_k(\Sigma \times S^1)$, and it follows from the first part of the proof of Theorem 5.24 that $V_k(S^1 \times S^1)$ has basis given by elements of $\tilde{P}_r$. Elements of $\tilde{P}_r$ correspond to tuples $(a_1, \ldots, a_{N-1})$, where $a_i \in \mathbb{Z}_{>0}$ and $\sum_i a_i < r$. Let $f_N(r) = \#\tilde{P}_r$. It follows that

$$f_2(r) = r - 1, \quad f_{N+1}(r) = \sum_{l=1}^{r-1} f_N(l).$$

The claim then follows from induction on both N and r. □

For the trace -2 homeomorphisms, i.e. those giving rise to the manifolds $\tilde{M}^b$, matters become slightly more involved, as the trace sum is now to be performed only over the Young diagrams λ invariant under the involution $\lambda \mapsto \lambda^\star$. This boils down to summing over those λ invariant under $\lambda_i \mapsto \lambda_1 - \lambda_{N+1-i}$, $i = 1, \ldots, N-1$. Viewing $\lambda = \sum_i \varepsilon_i \Lambda_i$ as a weight, this corresponds to considering those λ with $\varepsilon_i = \varepsilon_{N-i}$, $i = 1, \ldots, N-1$. As an example of how this complicates the combinatorics, we note the following special case.

Proposition 5.26 ([AJ12], Prop. 4.1.3)**.** *We have*

$$\begin{aligned} Z_k^{\mathrm{SU}(3)}(\tilde{M}^1) = {} & \exp(2\pi i/r)\left(\left[\frac{1}{2}\left(\sqrt{r/2}\exp\left(-\frac{\pi i}{4}\right) - 1\right)\right]\right. \\ & \left. + \left[\frac{1}{2}\left(\sqrt{r/2}\exp\left(-\frac{\pi i}{4}\right) - \frac{1}{2}\right)\right]\exp(2\pi i r/4) - \frac{1}{4}\exp(-2\pi i r/2)\right). \end{aligned}$$

Proof. We proceed as in the proof of Theorem 5.24 and consider

$$\sum_{\lambda \in \tilde{P}_r} \exp\left(\frac{\pi i}{r}\langle \lambda, \lambda\rangle\right) \delta_{\lambda,\lambda^\star}.$$

By the comments before the statement of the proposition, in the case of SU(3), we find that $\delta_{\lambda,\lambda^\star} = 1$ if and only if $\lambda = n(\Lambda_1 + \Lambda_2) = n\rho$, and $n\rho \in \tilde{P}_r$ if and only if $0 < n \leq \lfloor \frac{r-1}{2} \rfloor$. Now $\langle n\rho, n\rho\rangle = 2n^2$ and so

$$\sum_{\lambda \in \tilde{P}_r} \exp\left(\frac{\pi i}{r}\langle \lambda, \lambda\rangle\right) \delta_{\lambda,\lambda^\star} = \sum_{n=1}^{\lfloor \frac{r-1}{2} \rfloor} \exp\left(\frac{\pi i}{r} 2n^2\right).$$

Case 1. Assume that $r \equiv 0 \bmod 4$ and let $r = 4s$. Then Theorem 5.14 immediately implies that

$$\sum_{n=1}^{\lfloor \frac{r-1}{2} \rfloor} \exp\left(\frac{\pi i}{r} 2n^2\right) = \sqrt{\frac{r}{2}} \exp\left(\frac{\pi i}{4}\right) - 1.$$

Case 2. Assume that $r \equiv 2 \bmod 4$ and let $r = 4s + 2$. We note that mod $4s + 2$, we have $n^2 \equiv (2s + 1 - n)^2 + (2s + 1)$ and thus

$$\sum_{n=1}^{\lfloor \frac{r-1}{2} \rfloor} \exp\left(\frac{\pi i}{r} 2n^2\right) = \sum_{n=1}^{2s} \exp\left(\frac{\pi i}{2s+1} n^2\right) = 0,$$

since the n'th and $(2s + 1 - n)$'th summands cancel.

Case 3. Assume that $r \equiv 1 \bmod 2$ and let $r = 2s + 1$. As above, mod $4s + 2$, we have $2n^2 \equiv 2(2s + 1 - n)^2$, and it follows that

$$\begin{aligned}\sum_{n=1}^{\lfloor \frac{r-1}{2} \rfloor} \exp\left(\frac{\pi i}{r} 2n^2\right) &= \sum_{n=1}^{s} \exp\left(\frac{\pi i}{2s+1} 2n^2\right) = \frac{1}{2}\sum_{n=1}^{2s} \exp\left(\frac{\pi i}{2s+1} 2n^2\right) \\ &= \frac{1}{2}\left(\sqrt{\frac{2s+1}{2}} \exp(\pi i/4)(1 + \exp(-\pi i(2s+1)/2)) - 1\right).\end{aligned}$$

The result follows. ☐

For $b = 0$, the situation is easier to handle.

Proposition 5.27 ([AJ12], Prop. 4.1.4)**.** *For $N = 2n$ even, we have*

$$\begin{aligned}Z_k^{\mathrm{SU}(N)}(\tilde{M}^0) = &\frac{1}{n!}\left(\left(\frac{r}{2} - \frac{n}{2}\right)\prod_{l=1}^{n-1}\left(\frac{r}{2} - l\right) + \prod_{l=1}^{n}\left(\frac{r+1}{2} - l\right)\right) \\ &+ \frac{1}{n!}\left(\left(\frac{r}{2} - \frac{n}{2}\right)\prod_{l=1}^{n-1}\left(\frac{r}{2} - l\right) - \prod_{l=1}^{n}\left(\frac{r+1}{2} - l\right)\right)\exp(\pi i r),\end{aligned}$$

and for $N = 2n - 1$ odd, we have

$$\begin{aligned}Z_k^{\mathrm{SU}(N)}(\tilde{M}^0) = &\frac{1}{2(n-1)!}\left(\prod_{l=1}^{n-1}\left(\frac{r}{2} - l\right) + \prod_{l=1}^{n-1}\left(\frac{r+1}{2} - l\right)\right) \\ &+ \frac{1}{2(n-1)!}\left(\prod_{l=1}^{n-1}\left(\frac{r}{2} - l\right) - \prod_{l=1}^{n-1}\left(\frac{r+1}{2} - l\right)\right)\exp(\pi i r).\end{aligned}$$

Proof. As in Proposition 5.26, we simply need to compute the number of elements of $\tilde{P}_r$ invariant under the involution. That is, tuples $(a_1, \dots, a_{N-1})$ with $0 < a_i$, $\sum_i a_i < r$ and $a_i = a_{N-i}$.

Case 1. If $N = 2n - 1$ is odd, and $r = 2s$ is even, this boils down to counting $b_i > 0$, $i < n$ with $\sum_i 2b_i < r = 2s$. This we already did in Proposition 5.25, and the result is

$$Z_k(\tilde{M}^0) = \frac{1}{(n-1)!} \prod_{l=1}^{n-1} \left(\frac{r}{2} - l\right).$$

Case 2. If $N = 2n - 1$ is odd, and $r = 2s + 1$ is odd, we can proceed as above, noting that now the requirement becomes $\sum_i 2b_i < 2s + 1$, or $\sum_i b_i < s + 1$, and the result in this case is

$$Z_k(\tilde{M}^0) = \frac{1}{(n-1)!} \prod_{l=1}^{n-1} \left(\frac{r+1}{2} - l\right).$$

This proves the claim for odd N.

Case 3. Assume now that $N = 2n$ is even. Let $f_n(r) = Z_k^{\mathrm{SU}(N)}(\tilde{M}^0)$ be the number of tuples $(b_1, \dots, b_n)$, $b_i > 0$ with $2\sum_{i=1}^{n-1} b_i + b_n < r$. Then obviously

$$f_1(r) = r - 1,$$

and f_n satisfies the recursive relation

$$f_{n+1}(r) = \begin{cases} \sum_{l=1}^{r/2-1} f_n(2l), & r \text{ even}, \\ \sum_{l=1}^{(r-1)/2} f_n(2l-1), & r \text{ odd}. \end{cases} \tag{5.8}$$

Introduce $g_n(s) = f_n(2s)$ and $h_n(s) = f_n(2s+1)$. Then (5.8) implies via induction on n and s that

$$g_n(s) = \frac{2}{n!} \left(s - \frac{n}{2}\right) \prod_{l=1}^{n-1} (s - l).$$

For the h_n, one can use a similar argument or simply refer to the proof of Proposition 5.25 and find that

$$h_n(s) = \frac{2}{n!} \prod_{l=1}^{n} (s - l).$$

Now the result follows from

$$f_n(r) = \frac{1}{2}((-1)^r + 1) g_n\left(\frac{r}{2}\right) + \frac{1}{2}((-1)^{r+1} + 1) h_n\left(\frac{r+1}{2}\right).$$

□

5.4.2 The Chern–Simons values

We turn now to the question of determining the possible $\mathrm{SU}(N)$-Chern–Simons values for the manifolds under consideration. Whereas it would be nice to have an analogue of Proposition 5.19 in the general case, we make do with a restriction on possible flat connections, following the proof of Proposition 5.19, rather than describing the moduli space explicitly.

Proposition 5.28 ([AJ12], Prop. 4.1.2)**.** *Let γ be the isotopy class of an essential closed curve in $S^1 \times S^1$, and let $b \in \mathbb{Z}$, $b \neq 0$. Let $\mathcal{M}$ be the moduli space of flat $\mathrm{SU}(N)$-connections on M^b. The Chern–Simons action takes the following (not necessarily distinct) values:*

Let $a_1, \dots, a_N \in \mathbb{Q}$ *with* $\sum_{l=1}^N a_l \in \mathbb{Z}$ *and* $ba_l \in \mathbb{Z}$ *for every* $l = 1, \dots, N$. *Then*

$$\frac{1}{2} b \left(\sum_{l=1}^N a_l^2 + \left(\sum_{l=1}^N a_l \right)^2 - 2a_N \sum_{l=1}^N a_l \right) \in \mathrm{CS}(\mathcal{M}),$$

and is the Chern–Simons value of a completely reducible flat connection; i.e. one whose holonomy is contained in a maximal torus.

The partially reducible and irreducible case: let $1 \leq i_1 \leq \cdots \leq i_r$, $1 \leq r < N$, *be integers with* $\sum_l i_l = N$, *and let* $a_1, \dots, a_r \in \mathbb{Q}$ *satisfy that* $\sum_{l=1}^r i_l a_l \in \mathbb{Z}$ *and that* $b i_l a_l \in \mathbb{Z}$ *for every* l. *Then*

$$-\frac{1}{2} b \sum_{l=1}^r i_l a_l (i_r a_r - a_l) + \frac{1}{4} \left((-1)^{b(1-a_r i_r) \sum_{l=1}^r i_l a_l} - 1 \right) \in \mathrm{CS}(\mathcal{M}),$$

and is the Chern–Simons value of a flat connection whose invariant subspaces have dimensions given by the i_l. *In either case,* $\exp(2\pi i a_l)$ *are the eigenvalues of the holonomy about* γ, *viewed – say – in* $S^1 \times S^1 \times \{\frac{1}{2}\}$.

Conversely, every value of the Chern–Simons action is of one of the above two forms.

Proof. We follow the approach of the proof of Proposition 5.19, and in particular we introduce once again $m = -b$ to reduce the total number of signs. The net result is a sign change of the Chern–Simons values. As in Proposition 5.19, we consider $(A, B, C) \in \mathrm{SU}(N)^{\times 3}$ with $AB = BA$, $AC = CA$, and $CBC^{-1} = A^m B$, ignoring in our notation the action of simultaneous conjugation when discussing flat connections. By the first relation, we may assume that both A and B lie in the maximal torus $T \subseteq \mathrm{SU}(N)$ consisting of diagonal $\mathrm{SU}(N)$-matrices, and we write $a_j = A_{jj}$, $b_j = B_{jj}$. The normalizer $N(T)$ of T consists of $N!$ components, naturally identified with the elements of the symmetric group S_N, as conjugating an element of T by an element of $N(T)$ permutes the diagonal elements accordingly. We write $N(T)_\sigma$ for the component of $N(T)$ corresponding to $\sigma \in S_N$.

Case 1. Assume that $A, B \in Z(\mathrm{SU}(N))$. Then $A^m = \mathrm{Id}$, and so $A = \exp(2\pi i \frac{j}{m})\mathrm{Id}$ for some j, $0 \leq j < b$ with $jN/m \in \mathbb{Z}$.

Case 2a. Assume that $A \notin Z(\mathrm{SU}(N))$, $C \in T$. Then $a_j = \exp(2\pi i \frac{\hat{a}_j}{m})$ for some $\hat{a}_j$ with $0 \leq \hat{a}_j < m$, not all equal, and satisfying $\frac{1}{m} \sum_j \hat{a}_j \in \mathbb{Z}$.

Case 2b. Assume that $A \notin Z(\mathrm{SU}(N))$, $C \notin T$. Then $C \in N(T) \setminus T$, say $C \in N(T)_\sigma$ for some $\sigma \neq \mathrm{id}$. Recall that conjugacy classes of S_N correspond to increasing sequences $1 \leq i_1 \leq \cdots \leq i_r$, $\sum_l i_l = N$, and assume for simplicity that

$$\sigma = (1\,2\,\dots\,n_1)(n_1 + 1\; n_1 + 2\,\dots\,n_2) \cdots (n_{r-1} + 1\,\dots\,n_r), \tag{5.9}$$

with $n_1 = i_1$, $n_l - n_{l-1} = i_l$. The conjugate cases are handled similarly. Then $AC = CA$ implies that, in block form, we have $A = \mathrm{diag}(\tilde{a}_1 \mathrm{Id}_{i_1}, \dots, \tilde{a}_r \mathrm{Id}_{i_r})$ with $\tilde{a}_j \in \mathrm{U}(1)$ satisfying $\prod_{j=1}^r \tilde{a}_j^{i_j} = 1$. For $l = 1, \dots, N$, let k_l be the smallest natural number such that $\sigma^{k_l}(l) = l$, that is, k_l is one of the i_m above. Then

$$b_l = b_{\sigma^{k_l}(l)} = (C^{k_l} B C^{-k_l})_{ll} = (A^{k_l b} C^{k_l} B C^{-k_l})_{ll} = a_l^{k_l m} b_l.$$

This implies that $\tilde{a}_j^{m i_j} = 1$ for all $j = 1, \dots, r$. Furthermore, the b_l must satisfy

$$b_{\sigma(l)} = A_{ll}^m b_l.$$

Case 3. Let $A \in Z(\mathrm{SU}(N))$, $B \notin Z(\mathrm{SU}(N))$ so that $A = a\mathrm{Id}$ for some $a \in \mathrm{U}(1)$, $a^N = 1$. As before, $C \in N(T)_\sigma$ for some $\sigma \in S_N$, and for every l, we have $b_{\sigma(l)} = a^m b_l$. Arguing exactly as above, we find that $a^{m k_l} = 1$ for every l, so $a^{m \gcd(k_l)} = 1$, once again limiting the possible values of a, and thus b_l, accordingly.

To our knowledge, explicit expressions for $\mathrm{SU}(N)$-Chern–Simons values of general flat connections exist only in a few very special cases. Perhaps most relevant to our study is [Nis98, Thm. 3.1], in which Nishi calculates the $\mathrm{SU}(N)$-Chern–Simons values of irreducible flat connections on general Seifert manifolds. However, many of the connections considered above are *not* irreducible. In another direction, [Jef92, Thm. 5.11] determines the $\mathrm{SU}(N)$-Chern–Simons values for a number of torus bundles; whereas this result can not be applied directly to our manifolds (as the map $wU - 1$, in the notation of Jeffrey, is not invertible), the proof of the theorem can.

Following Jeffrey and the notation of Section 4.3, we let $\mathbb{A} = \mathfrak{t} \oplus \mathfrak{t}$ and consider the lattice $\Lambda = \Lambda^R \oplus \Lambda^R \subseteq \mathbb{A}$. Thus it is natural to identify $T \times T = \mathbb{A}/\Lambda$, and consider elements of $\mathfrak{t}$ as traceless $N \times N$-matrices with entries being the coefficients of the simple (co)roots spanning Λ^R in such a way that $\exp(2\pi i \cdot) : \mathfrak{t} \to T$ maps Λ^R to $\mathrm{Id} \in T$. Recall that the basic symplectic form on $\mathbb{A}$ is

$$\omega((\xi_1, \eta_1), (\xi_2, \eta_2)) = 2\pi(\langle \xi_1, \eta_2 \rangle - \langle \xi_2, \eta_1 \rangle),$$

and note that the basic inner product is defined such that $\langle \lambda_1, \lambda_2 \rangle = -\operatorname{tr}(\lambda_1 \lambda_2)$. Finally, we will need the theta-characteristic $\varepsilon : \Lambda \to \{\pm 1\}$ satisfying

$$\varepsilon(\lambda_1 + \lambda_2) = \varepsilon(\lambda_1)\varepsilon(\lambda_2)(-1)^{\omega(\lambda_1, \lambda_2)/(2\pi)},$$

which we require to furthermore satisfy that $\varepsilon(h_\alpha, h_\beta) = 1$ for any pair of simple roots, all of them together spanning the lattice Λ. It follows that

$$\varepsilon(0,0) = \varepsilon(h_\alpha, 0) = \varepsilon(0, h_\alpha) = \varepsilon(\pm h_\alpha, \pm h_\beta) = 1$$

for simple coroots h_α, h_β. Moreover, $\varepsilon(\lambda, 0) = \varepsilon(0, \lambda) = 1$ for any $\lambda \in \Lambda^R$, as can easily be seen by induction. In general, for $(\lambda, \mu) \in \Lambda$, we have

$$\begin{aligned} \varepsilon(\lambda, \mu) &= \varepsilon((\lambda, 0) + (0, \mu)) = \varepsilon(\lambda, 0)\varepsilon(\mu, 0)(-1)^{\omega((\lambda,0),(0,\mu))/(2\pi)} \\ &= (-1)^{\langle \lambda, \mu \rangle} = (-1)^{\operatorname{tr}(\lambda\mu)}. \end{aligned} \tag{5.10}$$

Returning to our setup, note that any element $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ acts on $\mathfrak{t} \oplus \mathfrak{t}$ by

$$U(\lambda, \mu) = (a\lambda + b\mu, c\lambda + d\mu).$$

Let $(A, B, C) \in T \times T \times \mathrm{SU}(N)$ be a flat connection on M^m as described in the various cases above, and abusing notation let $(A, B) \in \mathfrak{t} \oplus \mathfrak{t}$ be such that $\exp(2\pi i A) = A$ and $\exp(2\pi i B) = B \in T$. Then in any case, there will exist $w \in W$ such that

$$wU(A, B) - (A, B) =: (\lambda, \mu) \in \Lambda^R. \tag{5.11}$$

Now, the discussion preceding [Jef92, Thm. 5.11] shows that

$$\exp(2\pi i \mathrm{CS}(A, B, C)) = \varepsilon(\lambda, \mu) \exp\left(\frac{i}{2}\omega((A, B), (\lambda, \mu))\right).$$

Combining this with (5.10), we finally obtain

$$\mathrm{CS}(A, B, C) = \frac{1}{2}(\langle A, \mu \rangle - \langle B, \lambda \rangle) - \frac{1}{4}((-1)^{\operatorname{tr}(\lambda\mu)} - 1). \tag{5.12}$$

Notice that this does not depend on C, nor – $\mathrm{mod}\, \mathbb{Z}$ – on the choice of $(A, B) \in \mathfrak{t} \oplus \mathfrak{t}$. We now return to our specific cases.

Case 1. This case really follows as a special case of considerations below but is included here to illustrate the general idea. It suffices to find the value of CS on one element of the component under consideration, so take $A = (j/m, \dots, j/m, j/m - Nj/m)$, where j satisfies the condition of the conclusion of our considerations, and let B be arbitrary, fitting with the assumptions of this case. Then in the notation of (5.12), we have

$$w = \mathrm{id}, \quad (\lambda, \mu) = ((0, \dots, 0), (j, \dots, j, j - Nj)),$$

and recalling that $\langle \cdot, \cdot \rangle = -\operatorname{tr}(\cdot\cdot)$, it follows from (5.12) that

$$\mathrm{CS}(A, B, C) = -\frac{1}{2}\left(\frac{Nj^2}{m} - \frac{Nj^2}{m} - \frac{Nj^2}{m} + \frac{N^2j^2}{m}\right) = -\frac{1}{2}N(N-1)\frac{j^2}{m}.$$

Case 2. Again, we may as well assume that σ is of the form (5.9): if (A, B, C) is a representation with $C \in N(T)_\sigma$, and $\eta \in S_N$, we find that for any $D \in N(T)_\eta$, $(DAD^{-1}, DBD^{-1}, DCD^{-1})$ is another representation with $DCD^{-1} \in N(T)_{\eta\sigma\eta^{-1}}$. Assume first that $\sigma \neq \mathrm{id}$. That is, we can assume that $\sigma(N) = N - k_N + 1 \neq N$, $\sigma^{-1}(N) = N - 1$. Let $a_1, \dots, a_r, b_1, \dots, b_N \geq 0$ be such that if

$$\begin{aligned} A &= \operatorname{diag}(a_1, \dots, a_1, \dots, a_r, \dots, a_r, a_r - m_A), \\ B &= \operatorname{diag}(b_1, b_2, \dots, b_N, b_N - m_B), \end{aligned}$$

where the a_l is repeated i_l times, and $m_A = \sum_l a_l$, $m_B = \sum_l b_l$, then $\exp(2\pi i A)$ and $\exp(2\pi i B)$ satisfy the conclusion of Case 2 or 3 above. This means that $m \gcd(i_l) a_l \in \mathbb{Z}$, and that we may write the last $i_r = k_N = N - \sigma(N) - 1$ entries of B as

$$c, \; c + a_r m, \; \dots, \; c + a_r m(i_r - 2), c + a_r m(i_r - 1) - m_B$$

for some $c \geq 0$. We find that $w = \sigma^{-1}$ satisfies the condition of (5.11) with (λ, μ) given as follows: for any diagonal matrix $(m_{ll})_l$ and $\sigma \in S_N$, let σM be the diagonal matrix whose l'th diagonal entry is the $m_{\sigma(l)}$, and let e_l be the matrix whose (l, l)'th entry is 1, all others being 0. Then

$$(\lambda, \mu) = (\sigma A - A, \sigma(mA + A) - B) \in \mathfrak{t} \oplus \mathfrak{t}.$$

Now, since $\operatorname{tr}((\sigma A)(\sigma B)) = \operatorname{tr}(AB)$ and $\operatorname{tr}(A(\sigma B)) = \operatorname{tr}((\sigma^{-1}A)B)$, it follows that

$$\langle A, \mu \rangle = -\operatorname{tr}(A(\sigma(mA + B) - B)) = -\operatorname{tr}(A(\sigma mA)) - \operatorname{tr}(A(\sigma B)) + \operatorname{tr}(AB),$$

and similarly that

$$\langle B, \lambda \rangle = -\operatorname{tr}(B(\sigma A)) + \operatorname{tr}(BA).$$

Now, note that $b_{\sigma^{-1}(N)} = ma_r(i_r - 2) + b_{\sigma(N)}$. Thus we find that

$$\begin{aligned} \langle A, \mu \rangle - \langle B, \lambda \rangle &= -m \operatorname{tr}(A(\sigma A)) + \operatorname{tr}((\sigma A - \sigma^{-1}A)B) \\ &= -m\left(\sum_l k_l a_l^2 - 2a_l m_A\right) - m_A b_\sigma(N) + m_A b_{\sigma^{-1}(N)} \\ &= -m\sum_l i_l a_l^2 + m_A m a_n k_N = m\left(i_r a_r \left(\sum_l i_l a_l\right) - \sum_l i_l a_l^2\right) \\ &= m\sum_l i_l a_l (i_r a_r - a_l). \end{aligned}$$

For the theta-characteristic, note first that

$$\begin{aligned} \operatorname{tr}(A^2 - A(\sigma A)) &= m_A^2, \\ \operatorname{tr}(AB - (\sigma A)B) &= m_A b_{\sigma(N)} - m_A b_N, \\ \operatorname{tr}(AB - A(\sigma B)) &= m_A b_{\sigma^{-1}(N)} - m_A b_N. \end{aligned}$$

From this we find that

$$\begin{aligned}(-1)^{\operatorname{tr}(\lambda\mu)} &= (-1)^{m\operatorname{tr}(A^2-A(\sigma A))+\operatorname{tr}(AB-A(\sigma B))+\operatorname{tr}(AB-(\sigma A)B)}\\ &= (-1)^{mm_A-2m_Ab_N+m_A(b_{\sigma(N)}+b_{\sigma^{-1}(N)})}\\ &= (-1)^{mm_A-2m_A(m_{\sigma^{-1}(N)}+ma_N-m_B)+m_A(b_{\sigma(N)}+b_{\sigma(N)})}\\ &= (-1)^{mm_A-m_A(ma_r(i_r-2)+b_{\sigma(N)})-2ma_rm_A+m_Ab_{\sigma(N)}}\\ &= (-1)^{mm_A-m_Ama_ri_r} = (-1)^{m(1-a_rk_r)\sum_l i_la_l}.\end{aligned}$$

Putting this together, we finally find that

$$\operatorname{CS}(A,B,C) = \frac{1}{2}m\sum_l i_la_l(i_ra_r-a_l) - \frac{1}{4}\left((-1)^{m(1-a_ri_r)\sum_l i_la_l}-1\right).$$

In the case $\sigma = \operatorname{Id}$, λ vanishes as in Case 1, and $\mu = mA$, and so

$$\begin{aligned}\operatorname{CS}(A,B,C) = \frac{1}{2}\langle A, mA\rangle &= -\frac{1}{2}\left(\sum_l a_l^2m + mm_a^2 - m_Ama_N - a_Nmm_A\right)\\ &= -\frac{1}{2}m\left(\sum_l a_l^2 + \left(\sum_l a_l\right)^2 - 2a_N\sum_l a_l\right).\end{aligned}$$

Case 3 follows as the special case of the above calculation where all a_l are equal. □

Example 5.29. For $N = 2$, we recover the considerations of Proposition 5.19. As a non-trivial but perhaps more illuminating example, for $N = 3$, $b = 1$, the moduli space becomes a union of sets of the form

$$\begin{gathered}\{\operatorname{Id}\}\times Z(\operatorname{SU}(3))\times\operatorname{SU}(3),\\ \{\operatorname{diag}(-1,-1,1)\}\times\{\operatorname{diag}(b,-b,-b^{-2})\mid b\in\operatorname{U}(1)\}\times N(T)_{(1\,2)},\\ \{\operatorname{diag}(-1,1,-1)\}\times\{\operatorname{diag}(b,-b^{-2},-b)\mid b\in\operatorname{U}(1)\}\times N(T)_{(1\,3)},\\ \{\operatorname{diag}(1,-1,-1)\}\times\{\operatorname{diag}(-b^{-2},b,-b)\mid b\in\operatorname{U}(1)\}\times N(T)_{(2\,3)},\\ \{e(\tfrac{1}{3})\operatorname{Id}\}\times\{\operatorname{diag}(e(1/3),e(2/3),1),\operatorname{diag}(e(2/3),1,e(1/3)),\operatorname{diag}(1,e(1/3),e(2/3))\}\times N(T)_{(1\,2\,3)},\\ \{e(\tfrac{2}{3})\operatorname{Id}\}\times\{\operatorname{diag}(e(2/3),e(1/3),1),\operatorname{diag}(e(1/3),1,e(2/3)),\operatorname{diag}(1,e(2/3),e(1/3))\}\times N(T)_{(1\,2\,3)},\\ \{e(\tfrac{1}{3})\operatorname{Id}\}\times\{\operatorname{diag}(e(1/3),1,e(2/3)),\operatorname{diag}(e(2/3),e(1/3),1),\operatorname{diag}(1,e(2/3),e(1/3))\}\times N(T)_{(1\,3\,2)},\\ \{e(\tfrac{2}{3})\operatorname{Id}\}\times\{\operatorname{diag}(e(2/3),1,e(1/3)),\operatorname{diag}(e(1/3),e(2/3),1),\operatorname{diag}(1,e(1/3),e(2/3))\}\times N(T)_{(1\,3\,2)},\\ \{\operatorname{Id}\}\times\{\operatorname{diag}(-1,-1,1)\}\times N(T)_{(1\,2)},\\ \{\operatorname{Id}\}\times\{\operatorname{diag}(-1,1,-1)\}\times N(T)_{(1\,3)},\\ \{\operatorname{Id}\}\times\{\operatorname{diag}(1,-1,-1)\}\times N(T)_{(2\,3)},\end{gathered}$$

up to conjugation. Here, we have used the notation $e(x) = \exp(2\pi ix)$. The resulting Chern–Simons values in this case are $0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0, 0$ respectively, as can be verified by the method explained in the proof.

Corollary 5.30 ([AJ12], Cor. 4.2.3)**.** *Conjecture 5.2 holds for the 3-manifolds M^b when $G = \operatorname{SU}(3)$.*

Proof. We will show that for $N = 3$, the values arising from Proposition 5.28 are exactly the phases of the expression in Theorem 5.24.

The irreducible case: in the notation of the second half of Proposition 5.28, we let $r = 1, i_1 = 3, a_1 = 1/3$. Then one finds the Chern–Simons value $-\frac{1}{3}b$, corresponding to the phase of the term of Theorem 5.24 whose growth rate of r is 0.

The partially reducible case: considering again the second half of Proposition 5.28, let $r = 2, i_1 = 1, i_2 = 2$, let $a_2 = n/(2b)$ for $n \in \{0, \ldots, 2b-1\}$, and let $a_1 = (b-n)/b$. Then through a short computation, the proposition gives the Chern–Simons value $\frac{3n^2}{4b}$, corresponding in Theorem 5.24 to the terms of growth rate $\frac{1}{2}$.

The completely reducible case: finally, consider the first half of Proposition 5.28 and let $a_1 = n/b$, $a_2 = m/b$, and $a_3 = -(n+m)/b$, where $n, m \in \{0, \ldots, 3b-1\}$. Then we find the Chern–Simons values

$$\frac{1}{b}(n^2 + m^2 + nm),$$

and the Corollary follows from the observation that

$$\begin{aligned}&\left\{\exp\left(2\pi i r\frac{n^2+m^2+mn}{b}\right) \,\middle|\, n, m = 0, \ldots, 3b-1\right\}\\ &= \left\{\exp\left(2\pi i r\frac{n^2+m^2-mn}{b}\right) \,\middle|\, n, m = 0, \ldots, 3b-1\right\}.\end{aligned}$$

□

Remark 5.31. From this argument, it is clear what should be expected to happen in the general case $G = \mathrm{SU}(N)$, $N > 3$. Namely, the sum $\sum_{\lambda \in \tilde{P}_r} g(\lambda)$ of the proof of Theorem 5.24 is expressed in terms of sums of elements of various affine subspaces of the weight space, each of these giving a contribution of a particular order in r, and whose corresponding phases are the Chern–Simons values of connections whose invariant subspaces have dimensions depending on the given subspace of the weight space under consideration: in the case $N = 3$, irreducible connections, partially reducible, and completely reducible connections correspond to 0-dimensional affine subspaces, 1-dimensional ones, and 2-dimensional subspaces (the alcoves themselves) as illustrated in Figure 5.8.

5.5 Higher genus Dehn twist mapping tori

In this section, we show how to possibly generalize the results on the quantum invariants from the previous setting to the case, where the 3-manifold is the mapping torus of a power of a Dehn twist of a non-separating curve on a general closed surface Σ_g. Whereas we obtain no final results pertaining to the full asymptotic expansion of the resulting quantum invariants, we illustrate possible ways of attack that we hope to apply in the future. Indeed the methods presented seem to work fairly generally; as far as obtaining formulas for the quantum invariants themselves, all we really need is full control of the eigenvalues of the quantum representations to reduce the study of asymptotics to that of particular Gauss sums – as was the case in the torus case. Thus in the best possible case, one might hope to apply the computational aspects of this problem to a fairly large class of mapping tori; e.g. all mapping tori of words in commuting Dehn twists. On the other hand, the method used generalizes our previously employed results on generalized Gauss sums.

5.5.1 Outline

Assume now that $G = \mathrm{SU}(2)$ and $g \geq 2$, and let t_γ denote the Dehn twist about a non-separating curve in Σ_g, and let us first rewrite the quantum SU(2)-invariant as follows.

Lemma 5.32. *We have*

$$Z_k(T_{t_\gamma}) = -\frac{1}{4}\exp\left(\frac{\pi i}{2r}\right)\sqrt{2r}\exp(-\pi i/4)\left(\frac{r}{2}\right)^{g-2}\sum_{n=1}^{r-1}\sin\left(\frac{\pi n}{r}\right)^{2-2g}(\exp(2\pi i n^2/r) - 1). \tag{5.13}$$

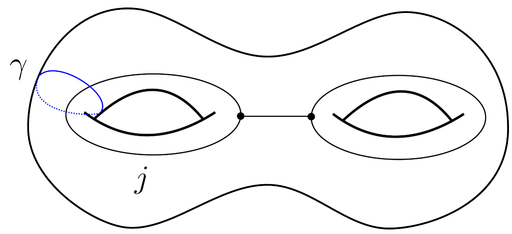


Figure 5.11: The left-most meridian γ in the case $g = 2$, as well as the standard handlebody graph basis, one edge coloured j.

Proof. Assume for simplicity that γ is the "left-most meridian" (see Figure 5.11) Let ${}_j\Sigma_j$ be the surface (with coloured structure) obtained from Σ_g by cutting along γ and assigning to each boundary component the colour j. By Theorem 3.36, the dimension of the vector space $V_k({}_j\Sigma_j)$ is exactly the number of basis elements in the standard graph basis of $V_k(\Sigma_g)$ having colour j on the left-most component (see Figure 5.11). This dimension is given by the (generalized) Verlinde formula (4.3),

$$\dim V_k({}_j\Sigma_j) = \left(\frac{r}{2}\right)^{g-2} \sum_{n=0}^{k} \sin\left(\frac{\pi(n+1)}{r}\right)^{2-2g} \sin\left(\frac{\pi(n+1)(j+1)}{r}\right)^2.$$

As before, let $\{\mu_j\}$ denote the eigenvalues of t_γ in the quantum SU(2)-representation ρ_k. Then

$$\begin{aligned} Z_k(T_{t_\gamma}) &= \operatorname{tr}(\rho_k(t_\gamma)) \\ &= \left(\frac{r}{2}\right)^{g-2} \sum_{n=0}^{k}\sum_{j=0}^{k} \sin\left(\frac{\pi(n+1)}{r}\right)^{2-2g} \sin\left(\frac{\pi(n+1)(j+1)}{r}\right)^2 \mu_j^{-1} \\ &= \left(\frac{r}{2}\right)^{g-2} \sum_{n=1}^{r-1} \sin\left(\frac{\pi n}{r}\right)^{2-2g} \sum_{j=1}^{r-1} \sin\left(\frac{\pi nj}{r}\right)^2 \mu_{j-1}^{-1}. \end{aligned}$$

To see (5.13), it thus suffices to show that

$$\sum_{j=1}^{r-1} \sin\left(\frac{\pi nj}{r}\right)^2 \mu_{j-1}^{-1} = -\frac{1}{4}\exp\left(\frac{\pi i}{2r}\right)(\sqrt{2r}\exp(-\pi i/4)(\exp(2\pi i n^2/r)-1)), \tag{5.14}$$

which really amounts to an elementary exercise of manipulating Gauss sums. First of all, notice that

$$\begin{aligned} \sum_{j=1}^{r-1} \sin\left(\frac{\pi nj}{r}\right)^2 \mu_{j-1}^{-1} &= \sum_{j=1}^{r-1} \exp\left(\frac{\pi i}{2r}\right) \sin^2\left(\frac{\pi nj}{r}\right) \exp\left(-\frac{\pi i}{2r}j^2\right) \\ &= -\frac{1}{4}\exp\left(\frac{\pi i}{2r}\right) \sum_{j=1}^{r-1} \left(\exp\left(\frac{2\pi i nj}{r}\right) + \exp\left(-\frac{2\pi i nj}{r}\right) - 2\right) \\ &\quad \cdot \exp\left(\frac{-\pi i}{2r}j^2\right). \end{aligned}$$

We consider the three terms in the latter sum individually. That is, we are interested in

$$\sum_{j=1}^{r-1} \exp\left(\pm\frac{2\pi i nj}{r}\right)\exp\left(-\frac{\pi i}{2r}j^2\right) = \sum_{j=1}^{r-1} \exp\left(\frac{\pi i}{2r}(\pm 4nj - j^2)\right). \tag{5.15}$$

Now, notice that by Theorem 5.16,

$$\begin{aligned}\sum_{j=0}^{2r-1} \exp\left(\frac{\pi i}{2r}(4nj - j^2)\right) &= \sqrt{2r}\exp\left(\pi i \frac{2r - (4n)^2}{-8r}\right) \\ &= \sqrt{2r}\exp(-\pi i/4)\exp(2\pi i n^2/r).\end{aligned} \tag{5.16}$$

One finds that

$$\sum_{j=1}^{r-1} \exp\left(\frac{\pi i}{2r}(\pm 4nj - j^2)\right) = \sum_{j=r+1}^{2r-1} \exp\left(\frac{\pi i}{2r}(\mp 4nj - j^2)\right).$$

This formula is good news, as most of the summands in the two sums in (5.15) appear in the left hand side of (5.16). Combining them with (5.16), we find that

$$\begin{aligned}&\sum_{j=1}^{r-1} \exp\left(\frac{\pi i}{2r}(+4nj - j^2)\right) + \sum_{j=1}^{r-1} \exp\left(\frac{\pi i}{2r}(-4nj - j^2)\right) \\ &\quad= \sqrt{2r}\exp(-\pi i/4)\exp(2\pi i n^2/r) - 1 - \exp\left(\frac{\pi i}{2r}(-4nr - r^2)\right) \\ &\quad= \sqrt{2r}\exp(-\pi i/4)\exp(2\pi i n^2/r) - 1 - \exp(-\pi i r/2).\end{aligned}$$

For the last summand, we find that

$$\begin{aligned}-2\sum_{j=1}^{r-1} \exp\left(-\frac{\pi i}{2r} j^2\right) &= -2\left(\sqrt{\frac{r}{2}}\exp(-\pi i/4) - \frac{\exp(-\pi i r/2)}{2} - \frac{1}{2}\right) \\ &= -\sqrt{2r}\exp(-\pi i/4) + \exp(-\pi i r/2) + 1.\end{aligned}$$

Putting together the above two calculations, we obtain (5.14). □

Note now that

$$\sum_{n=1}^{r-1} \sin\left(\frac{\pi n}{r}\right)^{2-2g}$$

is more or less the Verlinde formula itself, so in order to understand the asymptotic behaviour of the quantum invariant $Z_k(T_{t_\gamma})$, it would be most natural to describe that of the sum

$$\sum_{n=1}^{r} \sin\left(\frac{\pi n}{r}\right)^{2-2g} \exp(2\pi i n^2/r).$$

This expression bears striking resemblance to those being considered by Rozansky in [Roz96] but reapplying the methods of Rozansky ad verbum did not prove very useful. In the following, we discuss one possible way of attack due to Philippe Blanc.

5.5.2 Asymptotic analysis, take one

Lemma 5.33. *For integers $r > 1, g > 1$, $1 \leq n \leq \frac{r}{2} - 1$, we have*

$$\begin{aligned}&\frac{1}{2}\left(\sin(\pi n/r)^{2-2g}\exp(2\pi i n^2/r) + \sin(\pi(n+1)/r)^{2-2g}\exp(2\pi i(n+1)^2/r)\right) \\ &\quad= \exp(2\pi i(n+1)^2/r)\Psi_{r,g}(n+1) - \exp(2\pi i n^2/r)\Psi_{r,g}(n),\end{aligned}$$

where $\Psi_{r,g} : \{1, 2, \ldots, \lfloor \frac{r}{2} \rfloor\} \to \mathbb{C}$ is given by

$$\begin{aligned}\Psi_{r,g}(m) = \frac{i}{2}\int_0^\infty &\left(\sin\left(\frac{\pi(m - ix)}{r}\right)^{2-2g} \exp\left(2\pi\left(\frac{2m}{r} - \tfrac{1}{2}\right)x\right)\right. \\ &\left. - \sin\left(\frac{\pi(m + ix)}{r}\right)^{2-2g} \exp\left(-2\pi\left(\tfrac{2m}{r} - \frac{1}{2}\right)x\right)\right) \operatorname{csch}(\pi x)\exp(-2\pi i x^2/r)\, dx.\end{aligned}$$

Proof. Note first of all that the proof is based on the ideas of [Bla01] and as it will be clear, it can be generalized substantially. For our purposes, let $f(x) = \sin(x)^{2-2g}$, let

$$g_m(z) = f\left(\frac{\pi(m-ix)}{r}\right)\exp\left(2\pi\left(\frac{2m}{r}-\frac{1}{2}\right)\right)\operatorname{csch}(\pi x)\exp(-2\pi i x^2/r),$$

and let $\mathrm{PV}\int_{-\infty}^{\infty} = \lim_{\varepsilon\to 0}(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})$ denote the Cauchy principal value integral. First of all, we have that

$$\Psi_{r,g}(m) = \mathrm{PV}\int_{-\infty}^{\infty} g_m(x)\,dx.$$

Let C_R denote the positively oriented rectangle in $\mathbb{C}$ with vertices $R, -R, -R+i, R+i$. Note that $\sin\left(\frac{\pi(m-iz)}{r}\right)$ has poles in $ir\mathbb{Z} + im$, so since $1 \leq m \leq r/2$, g_m has no poles in the interior of C_R but has simple poles in 0 and i. Since $\operatorname{Res}_{z=0}\operatorname{csch}(\pi z) = \pi^{-1}$ and $\operatorname{Res}_{z=i}\operatorname{csch}(\pi z) = -\pi^{-1}$, we find that the residues of g_m in 0 and i are

$$\operatorname{Res}_{z=0} g_m(z) = \frac{f(\pi m/r)}{\pi}, \quad \operatorname{Res}_{z=i} g_m(z) = \exp\left(\frac{2\pi i(2m+1)}{r}\right)\frac{f(\pi(m+1)/r)}{\pi}.$$

Note also that $\int_R^{R+i} g(z)\,dz$ and $\int_{-R}^{-R+i} g(z)\,dz$ both tend to zero. Thus, letting $R \to \infty$, it follows from the (Cauchy principal value version of the) residue theorem that

$$\begin{aligned}\mathrm{PV}\int_{-\infty}^{\infty} g_m(x)\,dx &- \mathrm{PV}\int_{\infty}^{-\infty} g_m(x+i)\,dx \\ &= \pi i\left(\frac{f(\pi m/r)}{\pi} + \exp\left(\frac{2\pi i(2m+1)}{r}\right)\frac{f(\pi(m+1)/r)}{\pi}\right).\end{aligned} \tag{5.17}$$

Now, note also that

$$\mathrm{PV}\int_{-\infty}^{\infty} g_m(x+i)\,dx = \exp\left(2\pi i\frac{2n+1}{r}\right)\mathrm{PV}\int_{-\infty}^{\infty} g_{m+1}(x)\,dx.$$

Thus the claim follows immediately from (5.17) by multiplication by $-\frac{i}{2}\exp(2\pi i n^2/r)$. □

This lemma allows us to rewrite the sum of (5.13) in terms of integrals whose asymptotics might prove to be simpler to probe. For example, if r is odd, then

$$\begin{aligned}\sum_{n=1}^{r-1}\sin\left(\frac{\pi n}{r}\right)^{2-2g}\exp\left(\frac{2\pi i n^2}{r}\right) &= 2\sum_{n=1}^{\frac{r-1}{2}}\sin\left(\frac{\pi n}{r}\right)^{2-2g}\exp\left(\frac{2\pi i n^2}{r}\right) \\ = 2\Bigg(\frac{1}{2}\sin\left(\frac{\pi}{r}\right)^{2-2g}&\exp\left(\frac{2\pi i}{r}\right) + \frac{1}{2}\sin\left(\frac{\pi(r-1)}{2r}\right)\exp\left(\frac{2\pi i(r-1)^2}{4r}\right) \\ -\exp\left(\frac{2\pi i}{r}\right)&\Psi_{r,g}(1) + \exp\left(\frac{2\pi i(r-1)^2}{4r}\right)\Psi_{r,g}\left(\frac{r-1}{2}\right)\Bigg).\end{aligned}$$

Thus in this case it suffices to understand the asymptotics of $\Psi_{r,g}(1)$ and $\Psi_{r,g}(\frac{r-1}{2})$. For the sake of example, in the case $g = 2$, let us determine the first two terms of the expansion of $\Psi_{r,g}(1)$. Once again, the proof of the following lemma was outlined to us by Philippe Blanc.

Lemma 5.34. *We have*

$$\Psi_{r,2}(1) = \frac{1}{\pi^2}\left(\frac{1}{2}-\frac{\pi^2}{6}\right)r^2 + \frac{1}{\pi}(1-i)r^{3/2} + O(r).$$

Proof. For simplicity, introduce $\mu = 2/r$. In the following, we will write $O(\mu^p)$, $p \in \mathbb{R}$, for functions in $O(r^{-p})$ for $r \to \infty$. As we will only be concerned with the first few orders of the asymptotics, we introduce the real-valued functions $f_1, f_2 : (0,\infty) \to \mathbb{R}$ as well as r-dependent complex numbers Θ_1 and Θ_2 satisfying

$$\begin{aligned}
f_1(x) + if_2(x) &= \left(\sin\frac{\pi(1+ix)}{r}\right)^{-2} - \frac{r^2}{\pi^2}\frac{1}{(1+ix)^2},\\
\Theta_1 &= \frac{i}{2}\int_0^\infty \left(\frac{1}{(1-ix)^2}\exp\left(2\pi\left(\mu-\frac{1}{2}\right)x\right) - \frac{1}{(1+ix)^2}\exp\left(-2\pi\left(\mu-\frac{1}{2}\right)x\right)\right)\\
&\qquad\qquad \cdot \operatorname{csch}(\pi x)\exp(-\pi i\mu x^2)\,dx,\\
\Theta_2 &= i\int_0^\infty \left(f_1(x)\sinh\left(2\pi\left(\mu-\frac{1}{2}\right)x\right) - if_2(x)\cosh\left(2\pi\left(\mu-\frac{1}{2}\right)x\right)\right)dx.
\end{aligned}$$

Then by definition of $\Psi_{r,g}$, we have that

$$\Psi_{r,2}(1) = \frac{r^2}{\pi^2}\Theta_1 + \Theta_2.$$

Since for every finite t, the integral $\int_0^t \exp(-\pi i\mu x^2)\,dx$ is $O(\mu^{-1/2})$, the same is true for Θ_2, by the mean value theorem. These asymptotics are well-known but one reference which becomes relevant below is [GR65, Sect 8.25].

It now suffices to determine the asymptotics of Θ_1. Further splitting the non-oscillatory part of the integrand of Θ_1 into its real and imaginary parts, and noting that

$$i\frac{1-x^2}{(1+x^2)^2}\sinh(\lambda) - \frac{2x}{(1+x^2)^2}\cosh(\lambda) = \frac{i}{2}\left(\frac{1}{(1-ix)^2}\exp(\lambda) - \frac{1}{(1+ix)^2}\exp(-\lambda)\right)$$

for $\lambda \in \mathbb{C}$, we may write $\Theta_1 = A + B$, where

$$\begin{aligned}
A &= i\int_0^\infty \frac{1-x^2}{(1+x^2)^2}\sinh\left(2\pi\left(\mu-\frac{1}{2}\right)x\right)\operatorname{csch}(\pi x)\exp(-\pi i\mu x^2)\,dx,\\
B &= -\int_0^\infty \frac{2x}{(1+x^2)^2}\cosh\left(2\pi\left(\mu-\frac{1}{2}\right)x\right)\operatorname{csch}(\pi x)\exp(-\pi i\mu x^2)\,dx.
\end{aligned}$$

Finally, we split both of these expressions, allowing us to determine their leading order contributions. Write $A = A_1 + A_2$ according to the easily verified identity

$$\sinh\left(2\pi\left(\mu-\frac{1}{2}\right)x\right)\operatorname{csch}(\pi x) = -\exp(-2\pi\mu x) + \exp(-\pi x)\sinh(2\pi\mu x)\operatorname{csch}(\pi x).$$

Notice that up until this point, the assumption $g = 2$ has played no essential role but it drastically simplifies the following. Integrating by parts, we rewrite A_1 as

$$\begin{aligned}
A_1 &= -i\int_0^\infty \frac{1-x^2}{(1+x^2)^2}\exp(-2\pi\mu x - \pi i\mu x^2)\,dx\\
&= -i\left[\frac{x}{1+x^2}\exp(-2\pi\mu x - \pi i\mu x^2)\right]_{x=0}^\infty\\
&\quad + i(-2\pi\mu - 2\pi\mu ix)\int_0^\infty \frac{x}{1+x^2}\exp(-2\pi\mu - \pi i\mu x^2)\,dx\\
&= -2\pi i\mu\int_0^\infty \frac{x}{1+x^2}\exp(-2\pi\mu - \pi\mu ix^2)\,dx + 2\pi\mu ix\int_0^\infty \frac{x^2}{1+x^2}\exp(-2\pi\mu - \pi ix^2)\,dx\\
&= -2\pi i\mu\int_0^\infty \frac{x}{1+x^2}\exp(-2\pi\mu)\exp(\pi\mu ix^2)\,dx + 2\pi\mu\int_0^\infty \exp(-2\pi\mu x)\exp(-\pi\mu ix^2)\,dx\\
&\quad + 2\pi\mu\int_0^\infty \frac{1}{1+x^2}\exp(-2\pi\mu x)\exp(-\pi i\mu x^2)\,dx.
\end{aligned}$$

The third summand can be shown to be $O(\mu)$ by bounding it by its absolute value. The second summand we deal with later, and the first one we analyze in more detail. Following [GR65, Sect 8.25], let

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-x^2)\,dx, \quad S(x) = \frac{2}{\sqrt{2\pi}} \int_0^t \sin x^2\,dx, \quad C(x) = \frac{2}{\sqrt{2\pi}} \int_0^t \cos x^2\,dx$$

denote the probability integral and Fresnel integrals, respectively. By [GR65, (8.253.1), (8.256.1–4)], we find that

$$\begin{aligned}
&\int_0^\infty \exp(-2\pi\mu x)(\cos(\pi\mu x^2) - i\sin(\pi\mu x^2))\,dx \\
&= \frac{1}{\sqrt{\pi\mu}} \int_0^\infty \exp(-2\sqrt{\pi\mu}x)(\cos x^2 - i\sin x^2)\,dx \\
&= \frac{1}{\sqrt{2\mu}} \left(\frac{1}{2}(\cos(\pi\mu) - \sin(\pi\mu)) - \cos(\pi\mu)S(\sqrt{\pi\mu}) + \sin(\pi\mu)C(\sqrt{\pi\mu}) \right. \\
&\quad \left. -i\left(\frac{1}{2}(\cos(\pi\mu) + \sin(\pi\mu)) - \cos(\pi\mu)C(\sqrt{\pi\mu}) - \sin(\pi\mu)S(\sqrt{\pi\mu})\right)\right) \\
&= \frac{1}{\sqrt{2\mu}} \left(\frac{1}{2}(\exp(-\pi i\mu) - i\exp(-\pi i\mu)) + i\exp(\pi i\mu)\frac{\exp(\pi i/4)}{\sqrt{2}}\Phi(\sqrt{\pi\mu}\exp(-\pi i/4)) \right) \\
&= \frac{1}{\sqrt{2\mu}} \left(\frac{1}{2}(1-i) + i\left(\sqrt{\frac{2}{\pi}}\sqrt{\pi\mu}\right) + O(\mu) \right) \\
&= \frac{1}{2\sqrt{\mu}} \exp(-\pi i/4) + i + O(\mu^{1/2}).
\end{aligned}$$

On the other hand, A_2 might be bounded in absolute value by the integral

$$\int_0^\infty \sinh(2\pi\mu x)\operatorname{csch}(\pi x)\,dx,$$

which is in $O(\mu)$, so A_2 is $O(\mu)$, and summing up, we have

$$\begin{aligned}
A &= A_1 + A_2 \\
&= \pi\exp(-\pi i/4)\sqrt{\mu} - 2\pi i\mu \int_0^\infty \frac{x}{1+x^2} \exp(-2\pi\mu)\exp(\pi i\mu x^2)\,dx + O(\mu).
\end{aligned}$$

Now, following the same strategy as above, we write $B = B_1 + B_2$ using the identity

$$\cosh\left(2\pi\left(\mu - \frac{1}{2}\right)x\right)\operatorname{csch}(\pi x) = \coth(\pi x) + 2\sinh(\pi(\mu-1)x)\sinh(\pi\mu x)\operatorname{csch}(\pi x).$$

As before we find that B_2 is $O(\mu)$. Likewise,

$$\int_0^\infty \frac{2x}{(1+x^2)^2}(\exp(-\pi i\mu x^2) - 1)(\coth(\pi x) - 1)\,dx = O(\mu),$$

and it follows that

$$B_1 = -\int_0^\infty \frac{2x}{(1+x^2)^2}\coth(\pi x)\,dx - \int_0^\infty \frac{2x}{(1+x^2)^2}(\exp(-\pi i\mu x^2) - 1)\,dx.$$

By [GR65, (3.415.2), (8.366.8)],

$$\begin{aligned}
-\int_0^\infty \frac{2x}{(1+x^2)^2}\coth(\pi x)\,dx &= -\int_0^\infty \frac{2x}{(1+x^2)^2}\left(1 + \frac{2}{\exp(2\pi x) - 1}\right)dx \\
&= -1 - 4\left(-\frac{1}{8} - \frac{1}{4} + \frac{1}{4}\frac{\pi^2}{6}\right) = \frac{1}{2} - \frac{\pi^2}{6},
\end{aligned}$$

and integrating by parts,

$$-\int_0^\infty \frac{2x}{(1+x^2)^2}(\exp(-\pi i\mu x^2)-1)\,dx = \left[\frac{1}{1+x^2}(\exp(-\pi i\mu x^2)-1)\right]_{x=0}^\infty + 2\pi i\mu\int_0^\infty \frac{x}{1+x^2}\exp(-\pi i\mu x^2)\,dx.$$

Thus, all in all we have

$$B = B_1 + B_2 = \frac{1}{2} - \frac{\pi^2}{6} + 2\pi i\mu\int_0^\infty \frac{x}{1+x^2}\exp(-\pi i\mu x^2)\,dx + O(\mu),$$

and just as before, we have that

$$\mu\int_0^\infty \frac{x}{1+x^2}(1-\exp(2\pi\mu x))\exp(-\pi i\mu x^2)\,dx = O(\mu),$$

which altogether implies that

$$\Theta_1 = A + B = \frac{1}{2} - \frac{\pi^2}{6} + \pi\exp(-\pi i/4)\sqrt{\mu} + O(\mu).$$

Switching back to $r = 2/\mu$, we finally obtain

$$\Psi_{r,2}(1) = \frac{r^2}{\pi^2}\Theta + O\left(\frac{1}{r}\right) = \frac{r^2}{\pi^2}\left(\frac{1}{2} - \frac{\pi^2}{6} + \pi(1-i)r^{-1/2} + O\left(\frac{1}{r}\right)\right),$$

which is exactly the claim. □

5.5.3 Generalizing generalized Gauss sums

Much of the difficulty in the study of the previous sections lies in the fact that working out the asymptotics of functions $\Psi_{r,g}$ of Lemma 5.33 is non-trivial.

As such, it might very well be convenient to first better our understanding of $\dim V_k({}_j\Sigma_j)$. Recall that for the surfaces with no coloured structure, the dimensions $\dim V_k(\Sigma)$ are polynomials in k. Computer experimentation provides evidence that this continues to be the case for $\dim V_k({}_j\Sigma_j)$ (although the argument of [Sze95] does not appear to generalize ad verbum). More precisely, we note the following.

Conjecture 5.35. *Let $r = k+2$. For every $g \geq 1$ there exists a polynomial $p_g(r,j)$ of degree $3g-6$ in r (or 1 if $g=2$) and $2g-2$ in j such that*

$$\dim V_k({}_{j-1}(\Sigma_g)_{j-1}) = p_g(r,j)$$

for $k \in \mathbb{N}$ and $j = 0, 1, \dots, k$. For $g = 2, 3$,

$$\begin{aligned}\dim V_k({}_{j-1}(\Sigma_2)_{j-1}) &= j(r-j),\\ \dim V_k({}_{j-1}(\Sigma_3)_{j-1}) &= \frac{1}{6}rj(j-r)(j^2-rj-2).\end{aligned}$$

Assuming that this conjecture holds true, understanding the asymptotics of

$$Z_k(T_{t_\gamma^b}) = \exp\left(\frac{\pi i b}{2r}\right)\sum_{n=1}^{r-1} p^g(r,n)\exp\left(-\frac{\pi i}{2r}n^2 b\right)$$

boils down to understanding those of the generalized generalized Gauss sums

$$\sum_{n=1}^{r-1} n^m \exp\left(\frac{\pi i}{2r}n^2 b\right) \tag{5.18}$$

for $m = 0, \dots, 2g-2$. Assume for simplicity that $b = 1$ (but note that the proof of the following generalizes to all $b \neq 0$). We are now in a position to appeal to the methods of the previous section. As before, the following was pointed out to us by Philippe Blanc:

Lemma 5.36. *Let $p(x) = \frac{1}{4r}x^2$ and let $n \in \{1, \ldots, r-1\}$. Then for $m \geq 0$ if $n > 0$ and for $m > 0$ if $n = 0$ we have*

$$\begin{aligned}
&\frac{1}{2}(n^m \exp(2\pi i p(n)) + (n+1)^m \exp(2\pi i p(n+1))) \\
&\quad = \exp(2\pi i p(n+1))\varphi_m\left(n+1, \frac{n+1}{2r}\right) - \exp(2\pi i p(n))\varphi_m\left(n, \frac{n}{2r}\right),
\end{aligned}$$

where $\varphi_m : \mathbb{Z}_{\geq 0} \times [0,1] \to \mathbb{C}$ is the continuous function given by

$$\begin{aligned}
\varphi_m(\nu, \mu) = \frac{i}{2}\int_0^\infty &\left((\nu - ix)^m \exp(2\pi(\mu - \tfrac{1}{2})x) - (\nu + ix)^m \exp(-2\pi(\mu - \tfrac{1}{2})x)\right) \\
&\cdot \operatorname{csch}(\pi x) \exp\left(-\frac{\pi i}{2r}x^2\right) dx
\end{aligned}$$

for $(\nu, \mu) \in \mathbb{Z}_{\geq 0} \times (0,1)$.

Proof. The case $m = 0$ is [Bla01, Lemme 1]. The general case can be obtained by induction on m by differentiating the equation with respect to n, or by the same argument as in the proof of Lemma 5.33. □

This Lemma allows us to reduce (5.18) to

$$\sum_{n=1}^{r-1} n^m \exp\left(\frac{\pi i}{2r}n^2\right) = -\varphi_m(0,0) + \exp(\pi i r/2)\varphi_m(r, \tfrac{1}{2}) - \frac{1}{2}r\exp(\pi i r/2)$$

for $m > 1$. One will now hopefully be able to work out the asymptotics of the right hand side explicitly. We leave our present discussion with this remark and hope to investigate this problem further in the future.

5.6 Manifolds with links

So far, we have only been dealing with asymptotics of quantum invariants of closed 3-manifolds containing no links, but as we see from Theorem 5.15, it should be possible to make statements similar to that of the AEC in the case where the manifolds contain links and/or boundary components; moreover, from a physical point of view, these may be considered even more interesting than the asymptotics we have been considering so far, as they relate to expectation values of observables in Chern–Simons theory, as mentioned in Section 3.1. These asymptotics have been studied chiefly in the context of perturbative Chern–Simons theory of links in S^3, while litterature on the case for general 3-manifolds is significantly more rare. Relevant to our present studies is the work of Beasley [Bea09], who – building on previous work with Witten [BW05] – has considered in detail the case of Seifert manifolds containing particular links.

5.6.1 General manifolds

Notice that the Wilson loop variable (3.1) does not depend on the level k. In particular, we may follow the logic of Section 5.1 and try to compare the Wilson loop path integral with the mathematical invariant $Z_k(M, L, R)$, assuming that k is so large that all representations correspond to labels of the label set of the mathematical theory. Doing so and applying (5.2) directly leads us to the following conjecture, in which the notation is exactly as in Conjecture 5.1.

Conjecture 5.37 (Semi-classical approximation conjecture for manifolds with links)**.** *The leading order of the semi-classical approximation of the quantum G-invariant of a closed oriented 3-manifold M containing an n-component link L with colours $\lambda = (\lambda_1, \dots, \lambda_n)$ is*

$$\begin{aligned} Z_k^G(M, L, \lambda) \dot{\sim}_{r\to\infty} & \frac{1}{|Z(G)|} \exp\left(-\pi i \dim G \frac{1 + b^1(M)}{4}\right) \\ & \cdot \sum_{c \in C_M} \int_{[A] \in \mathcal{M}_c} \prod_{i=1}^n \mathrm{tr}_{\lambda_i}(\mathrm{hol}_A(L_i)) \tau_M(A)^{1/2} \exp(2\pi i r \mathrm{CS}(A)) \\ & \cdot \exp\left(-2\pi i \left(\frac{I_A}{4} + \frac{h_A^0 + h_A^1}{8}\right)\right) r^{\mathrm{gmax}_{A \in M_c}(h_A^1 - h_A^0)/2}. \end{aligned}$$

As in the case of the Asymptotic Expansion Conjecture, we might turn the above conjecture on its head and simply conjecture the existence of an asymptotic expansion for $Z_k^G(M, L, \lambda)$, but this would be little different from the AEC itself. Indeed, as Conjecture 5.37 suggests, the invariants c_j and d_j appearing in the AEC will be the same for all coloured links (L, λ) in M. In fact, we have already seen one occurence of this in Theorem 5.15; in Remark 5.16 we noted that the theorem is in accordance with Conjecture 5.37 as stated above. This is the most non-trivial piece of evidence we provide for now, but other cases exemplifying what one might expect include the following:

Example 5.38 (Links in S^3)**.** For S^3, and in the case of $G = \mathrm{SU}(2)$, the moduli space $\mathcal{M}$ consists of only the trivial connection A having $h_A^1 = 0$, $h_A^0 = 3$, and $\mathrm{CS}(A) = 0$. On the quantum side, we know that

$$Z_k(S^3) = \mathcal{D}^{-1} = \sqrt{\frac{2}{r}} \sin(\pi/r) \dot{\sim}_{r\to\infty} r^{-3/2} \pi \sqrt{2}.$$

Let L be an n-component link in S^3 coloured by $l = (l_1, \dots, l_n)$, where $l_j \in \{0, \dots, k\}$ corresponds to the $(j+1)$-dimensional irreducible representation of SU(2) through our usual identification. Then

$$\int_{\mathcal{M}} \prod_{i=1}^n \mathrm{tr}_{l_j}(\mathrm{hol}_A(L_j)) = \prod_{j=1}^n \dim(l_j) = \prod_{j=1}^n (l_j + 1),$$

and the discussion above tells us, that we should have

$$Z_k(S^3, L, l) \dot{\sim}_{r\to\infty} r^{-3/2} \pi \sqrt{2} \prod_{j=1}^n (l_j + 1).$$

This at first might be a bit surprising, as it means that the leading order asymptotics do not depend on anything but the colouring of the link. Recalling the skein relation defining the Jones polynomial, this is indeed the case, as, asymptotically, overcrossings tend to undercrossings (see [Lic97, Ex. 3.3]). Indeed, as far as obtaining knot invariants goes, only the lower order terms of the expansion of $Z_k(S^3, L, l)$ are interesting (see [Oht02, p. 421]).

Example 5.39 (S^1-fibres in trivial surface bundles)**.** Suppose now that $M = S^2 \times S^1$. In that case, the moduli space $\mathcal{M}^{\mathrm{SU}(2)}(M) = \mathrm{SU}(2)/\mathrm{SU}(2)$ consists of the conjugacy classes of SU(2). As a measure dA on this $\mathcal{M}^{\mathrm{SU}(2)}(M)$, we will take the measure induced by the Haar measure on SU(2), normalized to have volume 1 (we have not been able to find a reference for a calculation of the SU(2)-Reidemeister torsion on $S^2 \times S^1$, which should be straightforward).

Let $L_i = \{p_i\} \times S^1$, $i = 1, 2$, be two distinct (blackboard) framed links in M, and let $l_1, l_2 \in \{0, \dots, k\}$. Then it follows from The Schur Orthogonality Relations, or direct calculation, that

$$\int_{\mathcal{M}^{\mathrm{SU}(2)}(M)} \mathrm{tr}_{l_1}(\mathrm{hol}_{L_1}(A)) \, \mathrm{tr}_{l_2}(\mathrm{hol}_{L_2}(A)) \, dA = 0$$

if $l_1 \neq l_2$. This is nothing but the statement that the vector space associated by the TQFT to a sphere with two coloured points is 0-dimensional if the colours do not agree.

Similarly, one can argue that the vector space associated in SU(2)-TQFT to the torus with one coloured point is 0-dimensional whenever the colour is odd, and of course, more generally, the leading order asymptotics of the quantum invariants should give rise to the leading order term of the Verlinde formula (4.3); see also [Wit91, Sect. 3].

5.6.2 Geometric quantization for surfaces with coloured structure

So far, we have only discussed the quantum vector space associated to a surface by geometric quantization in the case of no coloured structure but in the following section, as well as in the last chapter of the dissertation, it will be useful to have a geometric description of $V_k(\Sigma, l, \lambda)$ for the case $l \neq \emptyset$.

Recall that in the case $l = \emptyset$, $V_k(\Sigma)$ was identified with the space $H^0(\mathcal{M}_\sigma, \mathcal{L}_\sigma^k)$ of holomorphic sections of a prequantum line bundle of the moduli space of flat G-connections on Σ.

In the present case, we will replace $\mathcal{M}_\sigma$ by a suitable moduli space taking into account the boundary conditions as specified by the choice of colours. Witten argued [Wit89, p. 372] (see also [Ati90, Sect. 5.3]) that the appropriate moduli space is the space $\mathcal{M}(\Sigma, l, \lambda)$ of s-equivalence classes of semi-stable parabolic bundles on Σ with flags at the points of l and flag types determined by λ. While definitely natural, we will not need this description in the following and could appeal to a theorem of Mehta and Seshadri which identifies the moduli space with the space of flat connections on $\Sigma \setminus l$ with particular boundary holonomies around the removed points. For the case of $G = \mathrm{SU}(N)$, this space may be described explicitly as follows. Following [DW97], let ∇_0 denote a base flat connection on a trivial G-bundle over a *once punctured* surface Σ such that in the natural coordinates of a disk neighbourhood of the puncture, ∇_0 is of the form $i\alpha\, d\theta$, having holonomy $\exp(2\pi i\alpha)$ about the puncture, $\alpha \in \mathfrak{g}$. Let $\mathcal{F}_{\delta,\alpha}$ denote the space of flat connections $\nabla_0 + A$ such that A and $\nabla_0 A$ are decaying exponentially at the puncture, and let $\mathcal{G}_\delta$ denote the space of gauge transformations g such that $\nabla_0 g$ and $\nabla_0^2 g$ have exponential decay at the puncture. Let $\mathcal{M}_{\delta,\alpha}$ denote the quotient $\mathcal{F}_{\delta,\alpha}/\mathcal{G}$. The authors show [DW97, Thm. 3.8] that $\mathcal{M}_{\delta,\alpha}$ is homeomorphic to the space of G-representations of $\pi_1(\Sigma)$ whose value at the boundary loop is exactly $\exp(2\pi i\alpha)$.

Let us briefly describe the connection with the TQFT constructions. Recall that $Y_{N,k}$ denotes the level k label set of Blanchet's modular functor, let $\mathrm{SU}(N)/\mathrm{SU}(N)$ be the set of conjugacy classes of $\mathrm{SU}(N)$, and let $\Psi_k : Y_{N,k} \to \mathrm{SU}(N)/\mathrm{SU}(N)$ be the map

$$\Psi_k(\lambda_1 \geq \cdots \geq \lambda_{N-1}) = \left[\exp(\tfrac{\pi i}{k}\mathrm{diag}(\lambda_1 - \lambda_2, \ldots, \lambda_{N-1} - \lambda_N, \lambda_N - \lambda_1))\right],$$

where $\lambda_N = 0$.

Recall that in general, the fundamental group of a punctured surface has presentation

$$\pi_1(\Sigma \setminus l) \cong \left\langle \alpha_j, \beta_j, \gamma_j \;\middle|\; \prod_{j=1}^{g} [\alpha_j, \beta_j] \prod_{j=1}^{n} \gamma_j = 1 \right\rangle,$$

where the γ_j corresponds to the loop going around the j'th puncture.

Definition 5.40. Let $\mathcal{M}^k(\Sigma, l, \lambda)$, $\lambda_i \in Y_{N,k}$, be the space of equivalence classes of representations

$$\mathcal{M}^k(\Sigma, l, \lambda) \cong \{\rho \in \mathrm{Hom}(\pi_1(\Sigma \setminus l), \mathrm{SU}(N)) \mid \rho(\gamma_j) \in \Psi_k(\gamma_j)\}/\mathrm{SU}(N).$$

A few remarks are in order. First of all, we notice that this space depends on k, in contrast with the closed surface case. Moreover, the space of G-representations makes sense for any prescribed boundary values, and so it can, and will, be desirable to understand these spaces with arbitratry fixed boundary values, rather than the k-dependent ones above. Note that one way to obtain this while preserving the relevance for TQFT is to let our colourings depend on the level, scaling them together with the levels. The physical importance of the resulting quantum invariants become somewhat less clear; for instance, if U denotes the unknot in S^3, and $\theta \in [0,1] \cap \mathbb{Q}$, then one easily finds that for the levels where the left hand side makes sense, we have

$$Z_k^{\mathrm{SU}(2)}(S^3, U, k\theta) \stackrel{\cdot}{\sim}_{k\to\infty} \begin{cases} r^{-3/2}\sqrt{2}\pi, & \text{if } \theta = 0, 1, \\ r^{-1/2}\sqrt{2}\sin(\pi\theta), & \text{if } \theta \in (0,1). \end{cases}$$

On the other hand, it should of course be possible to describe $\mathcal{M}^k(\Sigma, l, \lambda)$ in terms of flat connections well-behaved at the boundary as in the single boundary case. Then the form ω of (2.9) still makes sense, and we may ask when $\mathcal{M}^k(\Sigma, l, \lambda)$ is prequantizable. This will in general not be possible, as the periods of $\frac{\omega}{2\pi}$ are non-integral; however, Daskalopoulos and Wentworth find [DW97, Thm. 4.12] find that in the once punctured case the spaces $(\mathcal{F}_{\delta,\alpha}/\mathcal{G}_{0,\delta}, \frac{k}{2\pi}\omega)$ are prequantizable for the choices of labels relevant to the level k-theory; here, $\mathcal{G}_{0,\delta}$ denotes the subgroup of $\mathcal{G}_\delta$ consisting of gauge transformations that are the identity close to the puncture. Moreover, they construct in this case over the moduli space a prequantum line bundle, which we will refer to as a Chern–Simons line bundle, $\mathcal{L} \to \mathcal{F}_{\delta,\alpha}/\mathcal{G}_{0,\delta}$ through a cocycle construction (see [DW97, Sect. 5.1]) identical to that of Section 2.5.2. This construction it would be natural to generalize to the case of several boundary components. Some work in this direction is contained in [DW96].

In order to obtain mapping class group representations in this setup, we need to understand the lifting of the mapping class group action on the moduli space to the bundle $\mathcal{L}$. Having constructed $\mathcal{L}$ as a quotient of the trivial line bundle, one should here be able to repeat the construction of Section 4.2.

As in the closed case, one may use Hodge theory (see here also [And98]) to set up the Kähler quantization of the appropriate moduli spaces, and we need to understand the dependence of the construction on the various choices of complex structures thus obtained. Previously, this was exactly the role of the Hitchin connection, but to the best of our knowledge, no similar construction exists in the case of coloured structure. Constructing such a connection and showing that the resulting monodromy representation is equivalent to that arising from the various TQFT constructions is thus an interesting open problem and one we hope to pursue in the future; again, of course, the relevant framework is Andersen's construction as described in Section hitchinconnection. For now, we restrict ourselves to the case where the study of the Hitchin connection may be avoided entirely; the case of finite order mapping classes.

5.6.3 Finite order mapping tori

We end this chapter by describing in very rough terms current joint work in progress with Jørgen Ellegaard Andersen, Benjamin Himpel, Brendan McLellan and Johan Martens [AHJ+], where we wish to generalize the result of [And12c], which proved the AEC for finite order mapping tori, to finite order mapping tori with links.

Whereas we could examine the content of Conjecture 5.37 for these manifolds (and doing so would certainly be interesting), we take here a different route and instead generalize the argument of Andersen [And12c], from which we first recall the main points.

First of all, the reason for considering only finite order mapping classes is that, by a theorem of Nielsen (which may also be viewed as part of the Nielsen–Thurston classification), a finite order mapping class $f \in \Gamma_g$ will preserve a point $\sigma \in \mathcal{T}$. Thus one is in the position to use the setup of geometric quantization, as explained in Section 4.2, without having to deal with the Hitchin connection. That is, the quantum G-invariant $Z_k^G(T_f) = \operatorname{tr} \rho_k(f)$ is determined (sans framing) by the action of f on $H^0(\mathcal{M}_\sigma, \mathcal{L}_\sigma^k)$. Andersen is able to examine this action through the use of the Lefschetz–Riemann–Roch fixed point theorem due to Baum, Fulton, MacPherson, and Quart.

The setup we consider is the following: let (Σ, l, λ_k) be a closed surface with coloured structure and genus $g \geq 2$ such that $\Psi_k(\lambda_k)$ is constant (cf. the above discussion), and let

$f \in \Gamma_g$ be a finite order mapping class with a representative preserving setwise the coloured points, possibly permuting points of the same colour. Let as always T_f denote the mapping torus of f containing the coloured framed link $L = \{p_1, \ldots, p_n\} \times [0,1]/(x,0) \sim (f(x),1)$.

The goal is to examine the quantum G-invariants $Z_k^G(T_f, L)$ by cutting T_f along the surface $\Sigma \times \{0\} \subseteq T_f$ and study the action of f on $V_k(\Sigma, l, \lambda) = H^0((\mathcal{M}^k, l, \lambda_k)_\sigma, \mathcal{L}_\sigma^k)$ from the point of view of geometric quantization.

Our work thus consists of the following main points:

1. On one hand, in order to apply the Lefschetz–Riemann–Roch fixed point theorem, we need that $H^i((\mathcal{M}^k, l, \lambda_k)_\sigma, \mathcal{L}_\sigma^k) = 0$ for all $i > 0$ for large enough values of k. That this is the case is known, not for the hypothetical Chern–Simons line bundle, but in the context of algebraic geometry for the so-called Quillen line bundle over the (equivalent) moduli space of parabolic bundles. In fact, the algebrogeometric construction we relate to that of conformal blocks, making explicit the connection with TQFT through the work of Andersen and Ueno.

2. On the other hand, and with this in mind, it suffices to relate the Quillen construction to the Chern–Simons construction, and in particular to figure out the exact role of Chern–Simons theory by working out in detail the construction outlined in Section 5.6.2.

Chapter 6
The AMU conjecture

6.1 Pseudo-Anosov mapping classes

It is natural to examine exactly how the Nielsen–Thurston classification is reflected by the quantum representations. By Theorem 4.11 the collection of quantum representations on a closed surface logically detects the classification entirely, but one could wish for further criteria describing more precisely how the trichotomy becomes apparent in the representations. In [And08], Andersen describes an exclusion principle method for picking out pseudo-Anosov mapping classes using quantum representations. In [AMU06], the authors consider the mapping class group of a sphere with four coloured intervals and show – by relating the quantum representations to a well-known homological representation of the mapping class group – that pseudo-Anosov elements act with infinite order in the quantum representations at high enough levels, and that the quantum representations furthermore determine the stretch factors of the pseudo-Anosovs. The authors conjecture that this happens in general.

Conjecture 6.1 ([AMU06])**.** *Let Σ be a surface of genus $g \geq 0$ with n coloured intervals, $2g+n > 2$, and let φ be a pseudo-Anosov mapping class. Then there exists for every N a k_0 such that $\rho_k^{\mathrm{SU}(N)}(\varphi)$ has infinite order for $k > k_0$. Furthermore, the $\rho_k^{\mathrm{SU}(N)}$ determine the stretch factor of φ.*

Remark 6.2. Note that the conjecture as stated in [AMU06] makes no assumptions on the colours chosen, and the authors prove the conjecture for a four times punctured sphere with the simplest non-trivial colouring $1 \in \{0, \dots, k\}$ at all points in the case of $\mathrm{SU}(2)$ (and a slight variation on this theme for $\mathrm{SU}(N)$).

For fixed genus, the k_0 of the conjecture can become arbitrarily large, as for every k the pseudo-Anosov $t_{a_1}^{4k+8} t_{a_2}^{-4k-8} t_{a_3}^{4k+8} t_{a_4}^{-4k-8} t_{a_5}^{4k+8}$ in Γ_2 is in $\ker \rho_k$. Tables 4.1–4.2 also suggest that for low levels, we should expect the elements to alternate between having finite and infinite order.

The work contained in Section 6.1 has appeared partially in [AJ12], and the details not already included in this paper will appear in an updated version. The results in Section 6.2 will appear in a joint paper with Andersen, currently in preparation.

6.1.1 Asymptotic expansion and stretch factors

Since the collection of all quantum representations determine the mapping class group (up to central elements) by Theorem 4.11, if Conjecture 5.2 holds for the mapping torus T_φ of a pseudo-Anosov homeomorphism φ on a surface Σ, we might expect to be able to read off the stretch factors directly from the expansion, thus testing the last part of Conjecture 6.1.

As motivation for the following sections, let us first see what information may be extracted from the results of [Jef92] in the case of the mapping torus of a torus; this turns out to be particularly easy (but recall that the first part of Conjecture 6.1 can not be true for the torus, since ρ_k has finite image for all k in this case by the previously mentioned result of Gilmer).

Proposition 6.3. *Let $\varphi : \Sigma_1 \to \Sigma_1$ be an Anosov mapping class of the closed torus, given by the* $\mathrm{SL}(2,\mathbb{Z})$ *matrix*

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the stretch factor λ of φ is given by

$$\lambda = |Z_{n(a^2+2ad+d^2-6)}(T_\varphi)|^{-2}$$

for all $n \in \mathbb{N}$.

Proof. In the torus case, the stretch factor λ is nothing but the spectral radius of φ. This is a fundamental fact but can also be seen as a consequence of the construction of incidence matrices in Section 1.3.1. In other words,

$$\lambda = \max_{\pm} \left| \frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2} \right|.$$

By Theorem 5.11,

$$\begin{aligned} |Z_{n(a^2+2ad+d^2-6)}(T_\varphi)| &= |Z_{n((a+d+2)(a+d-2)-2)}(T_\varphi)| \\ &= \left| e^{2\pi i \psi(U)/(4n(a+d+2)(a+d-2))} \sum_{\pm} \pm \frac{1}{2|c|\sqrt{|d+a\mp 2|}} \sum_{\beta=0}^{|c|-1} \sum_{\gamma=1}^{|d+a\mp 2|} 1 \right| \\ &= \left| \sum_{\pm} \pm \frac{1}{2\sqrt{|d+a\mp 2|}} |d+a\mp 2| \right| \\ &= \left| \frac{\sqrt{|d+a-2|} - \sqrt{|d+a+2|}}{2} \right|. \end{aligned}$$

It follows that

$$|Z_{n(a^2+2ad-6)}(T_\varphi)|^2 = \min_{\pm} \left| \frac{1}{2}\left(a+d \pm \sqrt{(a+d)^2-4} \right) \right| = \lambda^{-1}.$$

□

In hopes of generalizing this result to other surfaces, let us briefly discuss a different reason why Proposition 6.3 is true. In her discussion of the semi-classical approximation conjecture for torus bundles, Jeffrey notes the following: for mapping tori of Anosov torus homeomorphisms, the mapping torus moduli space may be understood in terms of fixed points of the action of the Anosov on the moduli space (cf. Section 6.2 below), the path integral localizes to the fixed point set, and the Reidemeister torsion of a connection is given in terms of the differential of the moduli space action (see [Jef91, Prop. 3.10], [Jef92, Prop. 5.6]). Now, in the torus case for $G = \mathrm{SU}(2)$, as we saw in the previous chapter, the action on the moduli space $(T \times T)/W$ is essentially the action on the torus itself, and it is therefore not surprising that we can recover the stretch factor: the sequence chosen in Proposition 6.3 is set up to reduce the quantum invariant to a sum of the Reidemeister torsion contributions.

On the other hand, the fixed point set itself carries information about the dynamics of the Anosov in the sense of the following proposition.

Proposition 6.4. *For $G = \mathrm{SU}(2)$ and $\varphi \in \mathrm{SL}(2,\mathbb{Z})$ an Anosov mapping class on the torus with stretch factor $\lambda > 1$, let c_m denote the number of fixed points of the action of φ^m on the moduli space of the torus. Then $c_{m+1}/c_m \to \lambda$ as $m \to \infty$.*

Proof. By the discussion preceding [Jef92, Prop. 5.12], the number of fixed points of the action of $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, $\operatorname{tr} U \neq \pm 2$, is

$$|2 + a + d| + |2 - a - d| - n,$$

where n is the number of points in $\mathbb{R}^2/\mathbb{Z}^2$ fixed by both U and $-U$. Now of course, we have $a + d = \lambda + \lambda^{-1}$, where λ is an eigenvalue of U, and the result follows since if $\lambda(\varphi^m)$ denotes the largest eigenvalue of φ^m, then

$$\lim_{m\to\infty} \frac{c_{m+1}}{c_m} = \lim_{m\to\infty} \frac{\lambda(\varphi^{m+1})}{\lambda(\varphi^m)} = \lambda.$$

□

The quantum counterpart of this proposition is the following.

Proposition 6.5. *Let $G = \mathrm{SU}(2)$, and let $\varphi \in \mathrm{SL}(2,\mathbb{Z})$ be an Anosov mapping class on the torus with stretch factor $\lambda > 1$. Let $k_n = a_n^2 + 2a_n d_n + d_n^2 - 6$, where a_n and d_n are defined by $\varphi^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Then*

$$\lim_{n\to\infty} \sqrt[n]{|Z_{k_n}(T_{\varphi^n})|} = \lambda^{-1/2}.$$

Proof. As in the proof of Propostion 6.3, we find that

$$|Z_{k_n}(T_{\varphi^n})| = \left| \frac{\sqrt{|d_n + a_n - 2|} - \sqrt{|d_n + a_n + 2|}}{2} \right|.$$

The result follows immediately, as $\lim_{n\to\infty} a_n/a_{n-1} = \lim_{n\to\infty} d_n/d_{n-1} = \lambda$. □

This proposition illustrates how one might hope to illuminate the general AMU conjecture by considering the quantum representations of iterates of pseudo-Anosovs. In general, we propose the following.

Conjecture 6.6 (Stretch Factor Conjecture)**.** *Let $G = \mathrm{SU}(N)$, let Σ_g be a closed genus g surface, and let $\varphi \in \Gamma_g$ with stretch factor λ. Then there exists a rational number $c \in \mathbb{Q}$, and a sequence $\{k_n\}_n \subseteq \mathbb{N}$ such that*

$$\lim_{n\to\infty} \sqrt[n]{|Z^G_{k_n}(T_{\varphi^n})|} = \lambda^c.$$

In Proposition 6.5, the k_n were chosen such that $(k_n+2)\mathrm{CS}(A) \in \mathbb{Z}$ for all flat connections A on T_{φ^n}, and if one knew that all values of the Chern–Simons functional on any given mapping torus were rational, this condition on the k_n would be a natural addition to the above conjecture. More generally, one could assume e.g. that the k_n were chosen to satisfy

$$|\exp(2\pi i(k_n + N)\mathrm{CS}(A)) - 1| < 1/n$$

for all A.

In the following sections, we discuss various ideas on how to generalize the above to general surfaces.

6.2 Fixed points in moduli spaces

6.2.1 Isolated fixed points

In this section we analyze the quantum representations of those $\varphi \in \Gamma_g$ whose induced action on moduli space has isolated non-degenerate fixed points.

Let in the following $\mathcal{M} = \mathcal{M}^d_{\mathrm{SU}(N)}$ be one of the moduli spaces considered in Section 2.5.2 which we know are both compact and smooth, and for which a Hitchin connection exists by the prescription of Theorem 2.11. Let $\mathcal{L}$ be its prequantum line-bundle, and let us consider the action of Γ_g defined through the monodromy of the (projectively flat) Hitchin connection in the bundle $\mathcal{V}_k$ over Teichmüller space $\mathcal{T}$ whose fibre at a point $\sigma \in \mathcal{T}$ is $H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma)$ as in Section 4.2. Then $\varphi \in \Gamma_g$ acts via the composition

$$\rho_k(\varphi) : H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma) \xrightarrow{\varphi^*} H^0(\mathcal{M}_{\varphi^*\sigma}, \mathcal{L}^k_{\varphi^*\sigma}) \xrightarrow{P_{\varphi^*\sigma,\sigma}} H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma), \tag{6.1}$$

where σ is arbitrary, and $P_{\varphi^*\sigma,\sigma}$ is the parallel transport isomorphism of the Hitchin connection, depending on the choice of path from $\varphi^*\sigma$ to σ only up to a projective factor. The map φ^* is defined as in Section 4.2.

Recall that the Toeplitz operator

$$T^{(k)}_{f,(\sigma_0,\sigma_1)} : H^0(\mathcal{M}_{\sigma_0}, \mathcal{L}^k_{\sigma_0}) \to H^0(\mathcal{M}_{\sigma_1}, \mathcal{L}^k_{\sigma_1})$$

of a smooth function $f \in C^\infty(\mathcal{M})$ is given by

$$T^{(k)}_{f,(\sigma_0,\sigma_1)} = T^{(k)}_{f,\sigma_1}|_{H^0(\mathcal{M}_{\sigma_0}, \mathcal{L}^k_{\sigma_0})} = \pi^k_{\sigma_1} M_f|_{H^0(\mathcal{M}_{\sigma_0}, \mathcal{L}^k_{\sigma_0})},$$

where M_f denotes the multiplication operator on $C^\infty(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma)$, and π^k_σ denotes orthogonal projection onto $H^0(\mathcal{M}, \mathcal{L}^k)$.

Recall from Section 2.3 the definition of the Ricci potential $F : \mathcal{T} \to C^\infty(\mathcal{M}, \mathbb{R})$, and for any smooth curve $\gamma : [0,1] \to \mathcal{T}$, let

$$\dot\gamma(t) = \dot\gamma(t)' + \dot\gamma(t)''$$

denote the splitting of its tangent field into holomorphic and antiholomorphic parts, and let $g : [0,1] \to C^\infty(\mathcal{M})$ be the curve of functions satisfying

$$\dot g(t) = -(\dot\gamma(t)')[F]g(t) \tag{6.2}$$

and $g(0) \equiv 1$. The following theorem is essentially [And07, Thm. 20]. Note that in [And07], it is assumed – for other reasons – that $(N,d) = (2,1)$.

Theorem 6.7. *Let $\gamma : [0,1] \to \mathcal{T}$ be a smooth curve from σ_0 to σ_1, let P_γ denote the parallel transport of the Hitchin connection $\boldsymbol{\nabla}^H$ along γ, and let $g_\gamma = g(1) \in C^\infty(M)$ be the function determined by* (6.2)*. Then*

$$\|P_\gamma - T^{(k)}_{g_\gamma,(\sigma_0,\sigma_1)}\| = O\left(\frac{1}{k}\right),$$

where $\|\cdot\|$ denotes the operator norm as in Section 2.4.

Proof. The proof amounts to an analysis of the variation of Toeplitz operators, as was the case in the proof of Theorem 4.11. As will therefore be clear, the proof is an adaptation of [And06, Prop. 6], building upon calculations carried out in [And06], [And12b], [AG11], and [Gam10, Ch. 7]. Rather than repeating these, we present the main upshot and work backwards to obtain the required estimate.

For $f, g \in C^\infty(\mathcal{M})$, define

$$f \mathbin{\tilde\star} g = \sum_{j=0}^\infty \tilde c_j(f,g)\frac{1}{k^j}$$

where the $\tilde{c}_j$ are determined by

$$\sum_{j=0}^{\infty} \tilde{c}_j(f,g)\left(\frac{1}{k+n/2}\right)^j = \sum_{j=0}^{\infty} c_j(f,g)\left(\frac{1}{k}\right)^j,$$

where on the other hand n is as in Theorem 2.11 and $c_j(f,g)$ are the coefficients of the Berezin–Toeplitz star product of Remark 2.16.

For a vector field V on $\mathcal{T}$ and a family of smooth functions f, parametrized by $\mathcal{T}$, define a formal operator $A(V)$ by

$$A(V)(f) = V[F] \tilde{\star} f - V[F]f + (c_0(V) \tilde{\star} f - c(V)(f))\left(\frac{1}{k}\right),$$

where, in the notation of Section 2.3,

$$c(V) = \frac{1}{4}\left(\Delta_{\tilde{G}(V)} - 2\nabla_{\tilde{G}(V)dF} - \Delta_{\tilde{G}(V)}(F) + 2nV[F]\right),$$

and c_0 is the zero-order part of c, given by

$$c_0(V) = \frac{1}{2}(-\Delta_{\tilde{G}(V)}(F) + nV[F]).$$

Moreover, the action of $\Delta_{\tilde{G}(V)}$ on functions is given as in (2.5), replacing the prequantum connection in $\mathcal{L}^k$ by the trivial connection in the trivial line bundle over $\mathcal{M}$. Denote by $\boldsymbol{\nabla}^{H,\mathrm{End}}$ the connection in $\mathrm{End}(\mathcal{V}_k)$ induced by $\boldsymbol{\nabla}^H$ which is given by

$$(\boldsymbol{\nabla}_V^{H,\mathrm{End}}\Phi)(s) = \boldsymbol{\nabla}_V^H\Phi(s) - \Phi(\boldsymbol{\nabla}_V^H s)$$

for $V \in T(\mathcal{T})$, and sections s and Φ of $\mathcal{V}_k$ and $\mathrm{End}(\mathcal{V}_k)$ respectively. Let $T_f^{(k)}$ denote the section of $\mathrm{End}(\mathcal{V}_k)$ given by $\sigma \mapsto T_{f,\sigma}^{(k)}$. Then we have the crucial estimate (see e.g. [Gam10, (7.13)]) that pointwise,

$$\left\| \boldsymbol{\nabla}_V^{H,\mathrm{End}}(T_f^{(k)}) - T_{V[f]}^{(k)} - \sum_{j=0}^{m} T_{A_j(V)(f)}^{(k)}\left(\frac{1}{k+n/2}\right)^j \right\| = O\left(\frac{1}{k^{m+1}}\right) \tag{6.3}$$

for all $V \in T(\mathcal{T})$ and all $m \geq 0$, where here A_j denotes the part of A of degree $-j$ in k. We will only need the estimate to first order but note that it may be applied to analyze the lower order asymptotics of the parallel transport operator[1].

One particular ingredient – which we will need below – in the derivation of (6.3) is that

$$\pi_\sigma^k f o(V) = 2\pi_\sigma^k c(V')f, \tag{6.4}$$

for any $\sigma \in \mathcal{T}$ and $V \in T(\mathcal{T})$, where here o is given by (2.7).

Note now that by definition A_0 vanishes. Thus, applying the above estimate along $\gamma(t)$ to the tangent field $\dot{\gamma}(t)$ and the functions $g(t)$, we obtain $C_0(t) > 0$, independent of k such that for any section s of $H^0(\mathcal{M}_{\sigma_0}, \mathcal{L}_{\sigma_0}^k)$,

$$\left| \boldsymbol{\nabla}_{\dot{\gamma}(t)}^{H,\mathrm{End}}(T_{g(t)}^{(k)})s - T_{\dot{g}(t)}^{(k)}s \right| \leq \frac{C(t)}{k}|s|, \tag{6.5}$$

where $|\cdot|$ denotes the norm induced by the L^2-inner product (2.3). By using respectively the definition of the Hitchin connection, equation (6.4), the definition of $g(t)$, the left hand side

[1] Indeed, in the context of studying quantum representations, this might provide a possible way of attack at the AEC, which we hope to include in a future study.

of (6.5) may be written as

$$\begin{aligned}\left|\boldsymbol{\nabla}^{H,\mathrm{End}}_{\dot\gamma(t)}(T^{(k)}_{g(t)})s - T^{(k)}_{\dot g(t)}s\right| &= \left|\boldsymbol{\nabla}^{H}_{\dot\gamma(t)}T^{(k)}_{g(t),\gamma(t)}s - \pi^k_{\gamma(t)}(\dot g(t) + g(t)u(\dot\gamma(t)))s\right| \\ &= \left|\boldsymbol{\nabla}^{H}_{\dot\gamma(t)}T^{(k)}_{g(t),\gamma(t)}s - \pi^k_{\gamma(t)}\left(\dot g(t) + (\dot\gamma(t)')[F]g(t) + \frac{1}{k+n/2}c(\dot\gamma(t)')g(t)\right)s\right| \\ &= \left|\boldsymbol{\nabla}^{H}_{\dot\gamma(t)}T^{(k)}_{g(t),\gamma(t)}s - \frac{1}{k+n/2}\pi^k_{\gamma(t)}c(\dot\gamma(t)')g(t)s\right|\end{aligned}$$

By Theorem 2.14, we obtain $C_1(t) > 0$ such that

$$\frac{1}{k+n/2}\left|\pi^k_{\gamma(t)}c(\dot\gamma(t)')g(t)s\right| \leq \frac{1}{k+n/2}\|T^{(k)}_{c(\dot\gamma(t)')g(t),\gamma(t)}\||s| \leq \frac{C_1(t)}{k}|s|.$$

Now one may use Theorem 2.14 to show that the bounds $C_0(t)$ and $C_1(t)$ may be taken uniformly over $[0,1]$ (see also [And06, Prop. 1]), and so the triangle inequality implies that

$$\left|\boldsymbol{\nabla}^{H}_{\dot\gamma(t)}T^{(k)}_{g(t),\gamma(t)}s\right| \leq \frac{C}{k}|s| \tag{6.6}$$

for some $C > 0$ and all $t \in [0,1]$. This is exactly [And07, Prop. 2], and we may now simply repeat the arguments of the proof of [And07, Thm. 20]. We include this proof for completeness.

We first introduce a new inner product $\langle\cdot,\cdot\rangle_F$ on sections of $\mathcal{V}_k$ by

$$\langle s_1, s_2\rangle_F = \frac{1}{m!}\int_{\mathcal{M}}(s_1,s_2)_k\exp(-F)\,\omega^m,$$

where $2m = \dim_{\mathbb{R}}\mathcal{M}$. Let $|\cdot|_F$ denote the induced norm. By [And06, Lem. 1], $\langle\cdot,\cdot\rangle_F$ is equivalent to $\langle\cdot,\cdot\rangle$ uniformly in k. We need it however to apply the following estimate, given in [And06, Prop. 2]. Let

$$E(V)(s) = V|s|_F^2 - \langle\boldsymbol{\nabla}^H_V s, s\rangle_F - \langle s, \boldsymbol{\nabla}^H_V s\rangle_F$$

for $V \in T(\mathcal{T})$, and sections s of $\mathcal{V}_k$. Then for any compact subset K of $\mathcal{T}$ and any vector field V defined over K, there exists $C_2 > 0$ such that

$$|E(V)(s)| \leq \frac{C_2}{k+n/2}|s|_F^2, \tag{6.7}$$

for all sections s of $\mathcal{V}_k$ over K, and all tangent fields V.

Define $\Theta_k(t) : H^0(\mathcal{M}_{\sigma_0}, \mathcal{L}^k_{\sigma_0}) \to H^0(\mathcal{M}_{\gamma(t)}, \mathcal{L}^k_{\gamma(t)})$ by

$$\Theta_k(t) = P_{\sigma_0,\gamma(t)} - T_{g(t),\gamma(t)},$$

where $P_{\sigma_0,\gamma(t)}$ denotes the parallel transport of $\boldsymbol{\nabla}^H$ from σ_0 to $\gamma(t)$ along γ. Let now $s \in H^0(\mathcal{M}_{\sigma_0}, \mathcal{L}^k)$ with $|s|_F = 1$ and define a differentiable function $n_k : [0,1] \to \mathbb{R}_{\geq 0}$ by

$$n_k(t) = |\Theta_k(t)s|_F^2.$$

By applying the definition of $E(\dot\gamma(t))$ followed by the definition of $\Theta_k(t)$, we find that

$$\begin{aligned}\frac{dn_k}{dt} &= \langle\boldsymbol{\nabla}^H_{\dot\gamma(t)}(\Theta_k(t)s), \Theta_k(t)s\rangle_F + \langle\Theta_k(t)s, \boldsymbol{\nabla}^H_{\dot\gamma(t)}(\Theta_k(t)s)\rangle_F + E(\dot\gamma(t))(\Theta_k(t)s) \\ &= -\langle\boldsymbol{\nabla}^H_{\dot\gamma(t)}T^{(k)}_{g(t),\gamma(t)}s, \Theta_k(t)s\rangle_F - \langle\Theta_k(t)s, \boldsymbol{\nabla}^H_{\dot\gamma(t)}T^{(k)}_{g(t),\gamma(t)}s\rangle_F + E(\dot\gamma(t))(\Theta_k(t)s).\end{aligned}$$

By applying (6.6) and (6.7), this implies that there exists $C_3 > 0$ such that

$$\begin{aligned}\left|\frac{dn_k}{dt}\right| &\leq 2|\boldsymbol{\nabla}^H_{\dot\gamma(t)}T^{(k)}_{g(t),\gamma(t)}s|_F|\Theta_k(t)s|_F + |E(\dot\gamma(t))(\Theta_k(t)s)| \\ &\leq \frac{C_3}{k}(\sqrt{n_k} + n_k).\end{aligned}$$

This on the other hand implies that

$$n_k(t) \leq \left(\exp\left(\frac{C_3 t}{2k}\right) - 1\right)^2,$$

which implies the claim of the theorem as

$$|P_\gamma s - T^{(k)}_{g(1),\sigma_1} s|_F = |\Theta_k(1)s|_F = \sqrt{n_k(1)} \leq \exp\left(\frac{C_3}{2k}\right) - 1 = O\left(\frac{1}{k}\right).$$

□

Assume now that the graph of φ intersects the diagonal in $\mathcal{M} \times \mathcal{M}$ transversally. This in particular means that fixed point set $|\mathcal{M}|^\varphi$ of the action of φ on $\mathcal{M}$ consists of isolated points.

We will apply Theorem 6.7 to analyze $\mathrm{tr}(\rho_k(\varphi))$ by studying the integral kernel, known as the Bergman kernel, of the projection π^k_σ; as such, the following analysis closely follows that of [And07, Sect. 4].

More precisely, the *Bergman kernel* $K^{(k)}_\sigma \in C^\infty(\mathcal{M} \times \mathcal{M}, \mathcal{L}^k \boxtimes (\mathcal{L}^k)^*)$ satisfies

$$\pi^k_\sigma s(x) = \int_{\mathcal{M}} K^{(k)}_\sigma(x,y) s(y) \frac{\omega^n(y)}{n!(2\pi)^n}$$

for any smooth section $s \in C^\infty(\mathcal{M}, \mathcal{L}^k)$. Here, as always, ω is the symplectic form on $\mathcal{M}$, and $2n = \dim_{\mathbb{R}} \mathcal{M}$. The idea in the following will be to employ well-known results on asymptotics of the Bergman kernel; concisely, the kernel vanishes off the diagonal in $\mathcal{M} \times \mathcal{M}$ and its near-diagonal expansion is well-understood.

Moreover, notice that for any operator T on $C^\infty(\mathcal{M}, \mathcal{L}^k)$ of trace class with respect to the L^2 inner product and with kernel $K \in C^\infty(\mathcal{M} \times \mathcal{M}, \mathcal{L}^k \boxtimes (\mathcal{L}^k)^*)$, we have

$$\mathrm{tr}(T) = \int_{\mathcal{M}} K(x,x) \frac{\omega^n(x)}{n!(2\pi)^n},$$

where we implicitly use the identification $K(x,x) \in \mathcal{L}_x \otimes \mathcal{L}^*_x \cong \mathbb{C}$ through the hermitian pairing on $\mathcal{L}_x$. This equality follows immediately from the observation that if $\{s_i\}$ is an orthonormal basis of $C^\infty(\mathcal{M}, \mathcal{L}^k)$ and $T = \sum_{j,l} c_{jl} s_j \otimes s^*_l$, then $K(x,x') = \sum_{j,l} c_{jl} s_j(x) \otimes s_l(x')^*$, the duals being with respect to the L^2 and prequantum hermitian pairings respectively, and both sides equal $\sum_l c_{ll}$.

Thus, to understand $\mathrm{tr}\, \rho_k(\varphi)$, $\varphi \in \Gamma_g$, we will want to express the integral kernel of $\rho_k(\varphi)$ in terms of the Bergman kernel. Let $g_\gamma \in C^\infty(\mathcal{M})$ be the function obtained by applying Theorem 6.7 to the curve γ from $\varphi^*\sigma$ to σ used in (6.1). For $s \in H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma)$ and $x \in \mathcal{M}$, we then have

$$\begin{aligned}
(\rho_k(\varphi)s)(x) &\sim \int_{\mathcal{M}} K^{(k)}_\sigma(x,y) g_\gamma(y) \tilde{\varphi}(s(\varphi^{-1}(y))) \frac{\omega^n(y)}{n!(2\pi)^n} \\
&= \int_{\mathcal{M}} K^{(k)}_\sigma(x,\varphi(y)) g_\gamma(\varphi(y)) \tilde{\varphi}(s(y)) \frac{\omega^n(y)}{n!(2\pi)^n},
\end{aligned}$$

where $\sim$ here and in the following will denote equality to leading order in k. Now in particular, as the Bergman kernel decays faster than any power of k away from the diagonal (see e.g. [KS01]), this shows that the above integral will localize to the fixed point set of the action of φ on $\mathcal{M}$. More precisely, let $\tilde{\varphi}$ denote the lift of φ defined in Section 4.2, define $K^{(k),\varphi}_\sigma \in C^\infty(\mathcal{M} \times \mathcal{M}, \mathcal{L}^k \boxtimes (\mathcal{L}^k)^*)$ by

$$K^{(k),\varphi}_\sigma(x,\varphi(y)) s(\varphi(y)) = K^{(k)}_\sigma(x,\varphi(y)) \tilde{\varphi}(s(y)),$$

and choose for every fixed point $x \in |\mathcal{M}|^\varphi$ disjoint chart neighbourhoods U_x of x trivializing $\mathcal{L}$. Let $U = \bigsqcup_{x\in|\mathcal{M}|^\varphi} U_x$ and let $s : U \to \mathcal{L}^*$ be a holomorphic frame. Note first that

$$\operatorname{tr}\rho_k(\varphi) \sim \int_U K_\sigma^{(k),\varphi}(x,\varphi(x))g_\gamma(\varphi(x))\frac{\omega^n}{n!(2\pi)^n}.$$

Adopting the notation of [KS01], and if necessary restricting to smaller open subsets U_x, we obtain functions $\chi, \tilde{b}_r \in C^\infty(U \times U)$, $r \geq 0$, such that $K_\sigma^{(k)}$ has an asymptotic expansion (see [KS01, Thm. 5.6]) on $U \times U$ given by

$$K_\sigma^{(k)}(x,y) \sim k^n \exp(k\chi(x,y))\sum_{r=0}^\infty \frac{\tilde{b}_r(x,y)}{k^r},$$

in the sense that

$$\sup_{(x,y)\in E}\left|K_\sigma^{(k)}(x,y) - k^n\exp(k\chi(x,y))\sum_{r=0}^{m-1}\frac{\tilde{b}_r(x,y)}{k^r}\right| = O(k^{n-m})$$

for all compact subsets $E \subset U \times U$ and all $m > 0$. Here $\chi(x,x) = 0$ and $\operatorname{Re}\chi(x,y) < 0$ for $x \neq y$. Moreover, for every $x_0 \in U$, $y \mapsto \chi(x_0,y)$ has non-degenerate critical point at $y = x_0$, and $\tilde{b}_0(x,x) = 1$. In the definition of χ and in the relation to [KS01], we have implicitly used the trivialization s; for a condensed explanation of the (non-)dependence on s, see [And07, p. 12]. Since for $x \in |\mathcal{M}|^\varphi$, we have

$$K_\sigma^{(k),\varphi}(x,x) = K_\sigma^{(k)}(x,x)\operatorname{tr}(\tilde{\varphi} : \mathcal{L}_x^k \to \mathcal{L}_x^k),$$

an application for each U_x of the method of stationary phase with complex valued phase function – cf. e.g. [Hör90, Sect. 7.7] and the discussion preceding [KS01, Prop. 5.3] – made possible by the abovementioned properties of χ, immediately implies the following, where L_x denotes the Hessian of $y \mapsto \chi(y,\varphi(y))$ at x. Assume for now that $\det L_x \neq 0$ for all $x \in |\mathcal{M}|^\varphi$ but note that this assumption is likely made redundant by the assumptions on φ, cf. Question 6.12 below.

Theorem 6.8. *Assume that the graph of φ intersects the diagonal in $\mathcal{M} \times \mathcal{M}$ diagonally. Then the leading order asymptotics of $\operatorname{tr}(\rho_k(\varphi))$ are given by*

$$\operatorname{tr}(\rho_k(\varphi)) = \sum_{x\in|\mathcal{M}|^\varphi}\frac{\operatorname{tr}(\tilde{\varphi} : \mathcal{L}_x^k \to \mathcal{L}_x^k)g_\gamma(x)\exp(i\operatorname{sign}L_x/4)}{\sqrt{|\det L_x|}} + O\left(\frac{1}{k}\right).$$

To tie this together with our study of the AEC, let us now see how the quantities of the right hand side may be understood in the closed surface case. That is, we are interested in the $\mathrm{SU}(N)$-moduli space $\mathcal{M}(T_\varphi)$ of the mapping torus T_φ. The map $r : \mathcal{M}(T_\varphi) \to \mathcal{M}(\Sigma)$ given by restriction of a connection to $\Sigma\times\{0\}$ expresses $\mathcal{M}(T_\varphi)$ as a (singular) fibration over the set $|\mathcal{M}|^\varphi$ of fixed points of the action of φ on $\mathcal{M}(\Sigma)$. Moreover, if for $[\rho] \in |\mathcal{M}(\Sigma)|^\varphi$, we choose $g \in \mathrm{SU}(N)$, such that ρ extends to a flat connection on T_φ whose holonomy along the fibre γ of T_φ is g, we find that $r^{-1}([\rho]) \cong Z_\rho g/Z_\rho$, where Z_ρ denotes the centraliser of the image of ρ, and the action is by conjugation. This is a simple group theoretical fact proven in e.g. [Jef91, Prop. 3.9] [And12c, Sect. 6], and the bijection is given by mapping $\tilde{\rho} \in r^{-1}([\rho])$ to $\tilde{\rho}(\gamma)^{-1} \in Z_\rho g$.

The following result, which we in fact employed in the torus case in the proof of Proposition 5.28, follows almost immediately from the construction of Section 4.2 (see also e.g. [Jef92, Prop. 5.5], [And12c, Lem. 7.1]).

Proposition 6.9. *For $A \in r^{-1}(\rho)$,*

$$\exp(2\pi ik\mathrm{CS}(A)) = \operatorname{tr}(\tilde{\varphi} : \mathcal{L}_\rho^k \to \mathcal{L}_\rho^k).$$

Likewise, the square root $\tau_A^{1/2}$ of the Reidemeister torsion may be given in this case in terms of the action of φ on the moduli space. The following is [Jef92, Prop. 5.6 (b)].

Proposition 6.10. *Assume as above that the action of φ has isolated fixed points in $\mathcal{M}(\Sigma)$. Then for $\rho \in |\mathcal{M}|^\varphi$, we have*

$$\int_{A \in r^{-1}(\rho)} \tau_A^{1/2} = \int_{A \in r^{-1}(\rho)} \frac{1}{\sqrt{|\det(d\varphi|_{\mathcal{N}_\rho^*} - \mathrm{id})|}} \, d\,\mathrm{Vol},$$

where Vol *is induced by the metric on G, $\mathcal{N}_\rho^*$ denotes the orthocomplement of the kernel of the action of $d\varphi_\rho - \mathrm{id}$ on $H^1(\Sigma, \nabla^\rho)$, and $d\varphi$ denotes the linear map induced by φ.*

Remark 6.11. A similar formula is shown for irreducible – but not necessarily isolated – connections on finite order mapping tori in [AH12, (5.1)]. It is an interesting problem to come up with a general form of the formula of the proposition.

Question 6.12. *At this point, several natural questions have appeared, all of which we hope to approach in future work:*

1. *Firstly, it would be interesting to carry out the above analysis in the framework of metaplectic correction, as mentioned in Remark 2.12. For example, comparing Theorem 6.8 and Proposition 6.9, we note that we lack the shift $k \to k + N$ appearing in Conjecture 5.2. This shift appears naturally in metaplectic correction (see e.g. [AGL12, Thm. 1.3]). Moreover, the equivalent of Theorem 6.7 will be slightly simpler, as the u in this case contains no term of degree zero in k. Indeed, our analysis should also be compared to that of [Cha10a] carried out in a setup similar to the above.*

   In our case, we expect to be able to see this shift occuring naturally in the function g_γ, which may be given explicitly by solving (6.2)*. It will therefore be natural to study the* 3*-dimensional interpretation of this function.*

2. *Secondly, comparing Theorem 6.8 and Proposition 6.10 (or [Jef92, Prop. 5.6 (b)]), it is clear that it should be possible to relate $|\det(L_x)|$ to $|\det(d\varphi|_{\mathcal{N}_\rho^*} - \mathrm{id})|$ (see also [Cha10b, Lem. 4.3.3]). Likewise, for pseudo-Anosov homeomorphisms, the result is hopefully explicitly related to their stretch factors.*

3. *Thirdly, it is an interesting question when the regularity assumption of Theorem 6.8 holds. In the torus case, we have seen that for powers of Dehn twists, the assumption does not hold (which immediately generalizes to the higher genus case), while for Anosov homeomorphisms it does. One could hope that for (some) general pseudo-Anosov homeomorphisms, the assumption would be true. A source of possible counterexamples is mentioned in Section 6.2.4 below. On the other hand, it would be interesting to try to remove the assumption altogether, performing the necessary modifications to the analysis.*

4. *Finally, in Theorem 6.8, we have given only the leading order asymptotics of* $\mathrm{tr}(\rho_k(\varphi))$*. As we saw in Theorem 6.7, the parallel transport of $\boldsymbol{\nabla}^H$ may be understood to all orders, and likewise, the lower order terms of the expansion of the Bergman kernel are well-understood (see [KS01], [Xu12] and the references contained herein), so it would be natural to extend Theorem 6.8 to provide a full asymptotic expansion of* $\mathrm{tr}(\rho_k(\varphi))$*.*

6.2.2 The four times punctured sphere

Motivated by the above discussion, we will now analyze several explicit examples of actions of mapping class groups on moduli spaces. Consider first the four times punctured sphere Σ_0^4. It is well-known, see [Gol97], [Mag80], that the moduli space of flat SU(2)-connections on this surface, whose holonomy around the four boundary components are assumed to be

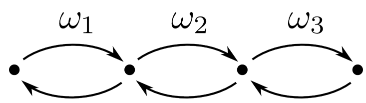


Figure 6.1: The generators ω_1, ω_2,and ω_3 of Γ_0^4.

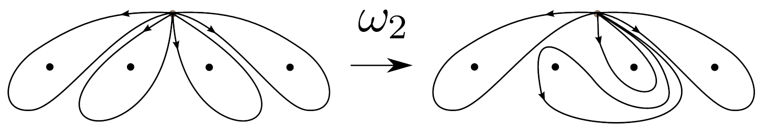


Figure 6.2: The action of ω_2 on $\pi_1(\Sigma_{0,4})$.

in fixed generical conjugacy classes of SU(2), is 2-dimensional and homeomorphic to S^2. More precisely, its fundamental group has the standard presentation

$$\pi_1(\Sigma_0^4) = \langle A, B, C, D \mid ABCD = 1\rangle,$$

where A, B, C, D denote isotopy classes of loops isotopic to curves around each of the four punctures, orientations set up accordingly, and the functions

$$\begin{gathered} a = \operatorname{tr}(A),\ b = \operatorname{tr}(B),\ c = \operatorname{tr}(C),\ d = \operatorname{tr}(D), \\ x = \operatorname{tr}(AB),\ y = \operatorname{tr}(BC),\ z = \operatorname{tr}(CA) \end{gathered}$$

define, for fixed values of $a, b, c, d \in [-2, 2]$, the above mentioned homeomorphism with S^2 for generic values of a, b, c, d, by identifying the moduli space of flat connections $\mathcal{M}_{(a,b,c,d)}$ with fixed boundary data (a, b, c, d) with those $(x, y, z) \in \mathbb{R}^3$ satisfying

$$\begin{aligned} x^2 + y^2 + z^2 + xyz = {} & (ab + cd)x + (ad + bc)y + (ac + bd)z \\ & - (a^2 + b^2 + c^2 + d^2 + abcd - 4). \end{aligned}$$

We are interested in the action of the mapping class group

$$\begin{gathered} \Gamma_0^4 = \langle \omega_1, \omega_2, \omega_3 \mid \omega_1\omega_3 = \omega_3\omega_1,\ \omega_1\omega_2\omega_1 = \omega_2\omega_1\omega_2,\ \omega_2\omega_3\omega_2 = \omega_3\omega_2\omega_3, \\ \omega_1\omega_2\omega_3^2\omega_1\omega_2\omega_3 = 1,\ (\omega_1\omega_2\omega_3)^4 = 1\rangle \end{gathered}$$

on $\mathcal{M}_{a,b,c,d}$ given by pullback of flat connections or, in the terms of representations of π_1, precomposition. Here, the generators are those described by [Bir75, Thm. 4.5] (see Figure 6.1). We include here the actions of the inverses of the generators for ease of computer implementation.

Proposition 6.13. *Assume that $a = b = c = d$. Then the action on $\mathcal{M}_a = \mathcal{M}_{(a,a,a,a)}$ of the generators of Γ_0^4 and their inverses are given in Goldman's coordinates by*

$$\begin{gathered} \omega_1^* = \omega_3^* : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ 2a^2 - xy - z \\ y \end{pmatrix},\ \omega_2^* : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z \\ y \\ 2a^2 - x - yz \end{pmatrix}, \\ (\omega_1^{-1})^* = (\omega_3^{-1})^* : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ z \\ 2a^2 - xz - y \end{pmatrix},\ (\omega_2^{-1})^* : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2a^2 - z - xy \\ y \\ x \end{pmatrix}. \end{gathered}$$

In general, only the subgroup $\langle t_1 = \omega_1^2 = \omega_3^2, t_2 = \omega_2^2\rangle \subseteq \Gamma_0^4$ acts on $\mathcal{M}_{(a,b,c,d)}$, and the action is given by

$$t_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ ad + bc - x(ac + bd - xy - z) - y \\ ac + bd - xy - z \end{pmatrix},$$
$$t_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} ab + cd - x - yz \\ y \\ ac + bd - y(ab + cd - x - yz) - z \end{pmatrix},$$
$$t_1^{-1} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ ad + bc - xz - y \\ ac + bd - x(ad + bc - xz - y) - z \end{pmatrix},$$
$$t_2^{-1} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} ab + cd - (ac + bd - xy - z)y - x \\ y \\ bd + ac - xy - z \end{pmatrix}.$$

Proof. Consider $\omega_1 \in \Gamma_0^4$. On the level of the fundamental group, ω_1 maps

$$A \mapsto B,\ B \mapsto B^{-1}AB,\ C \mapsto C,\ D \mapsto D.$$

and its inverse ω^{-1} maps

$$A \mapsto ABA^{-1},\ B \mapsto A,\ C \mapsto C,\ D \mapsto D.$$

The induced action on the moduli space in the coordinates above can be found via application of the general $\mathrm{SL}(2,\mathbb{C})$ trace identities (see [Mag80, p. 93])

$$\begin{aligned} \operatorname{tr}(UV) + \operatorname{tr}(UV^{-1}) &= \operatorname{tr}(U)\operatorname{tr}(V), \\ \operatorname{tr}(UVU^{-1}V^{-1}) &= \operatorname{tr}(U)^2 + \operatorname{tr}(V)^2 + \operatorname{tr}(UV)^2 - \operatorname{tr}(U)\operatorname{tr}(V)\operatorname{tr}(UV) - 2. \end{aligned}$$

With this, the calculation is elementary, but we have chosen to include all details, first of all because the calculation is instructive, but also because our result for the action of ω_2^2 disagrees slightly with that of [Gol97]. In terms of the coordinates (x, y, z, a, b, c, d), $\omega_1^{\pm 1}$ interchange a and b, keep c and d fixed and act on the three holonomy functions $x, y, z : \mathcal{M}_{(a,b,c,d)} \to [-2, 2]$ by

$$\begin{aligned}
(\omega_1)^* x &= \operatorname{tr}((\omega_1)_*(A)(\omega_1)_* B) = \operatorname{tr}(BB^{-1}AB) = x, \\
(\omega_1^{-1})^* x &= \operatorname{tr}(ABA^{-1}A) = x, \\
(\omega_1)^* y &= \operatorname{tr}(B^{-1}ABC) = \operatorname{tr}(B^{-1})\operatorname{tr}(ABC) - \operatorname{tr}(BABC) \\
&= bd - (\operatorname{tr}(BA)\operatorname{tr}(BC) - \operatorname{tr}(A^{-1}B^{-1}BC)) \\
&= bd - (xy - (\operatorname{tr}(A^{-1})\operatorname{tr}(C) - \operatorname{tr}(AC))) = bd + ac - xy - z, \\
(\omega_1^{-1})^* y &= \operatorname{tr}(AC) = z, \\
(\omega_1)^* z &= (\omega_1^{-1})^* z = \operatorname{tr}(CB) = y, \\
(\omega_1^{-1})^* z &= \operatorname{tr}(ABA^{-1}C) = x\operatorname{tr}(A^{-1}C) - \operatorname{tr}(ABC^{-1}A) \\
&= x(ac - z) - (\operatorname{tr}(ABC^{-1})a - \operatorname{tr}(BC^{-1})) \\
&= x(ac - z) - ((xc - d)a - (bc - y)) = ad + bc - xz - y.
\end{aligned}$$

Similarly, the other generators of Γ_0^4 determine the automorphisms of the fundamental group given by

$$\begin{aligned}
&(\omega_2)_*(B) = C,\ (\omega_2^{-1})_*(B) = BCB^{-1}, \quad (\omega_2)_*(C) = C^{-1}BC,\ (\omega_2^{-1})_*(C) = B, \\
&(\omega_3)_*(C) = D,\ (\omega_3^{-1})_*(C) = CDC^{-1}, \quad (\omega_3)_*(D) = D^{-1}CD,\ (\omega_3^{-1})_*(D) = C.
\end{aligned}$$

and the actions on the moduli space are given by

$$
\begin{aligned}
(\omega_2)^* x &= \operatorname{tr}(AC) = z, \\
(\omega_2^{-1})^* x &= \operatorname{tr}(ABCB^{-1}) = bd + ac - xy - z, \\
(\omega_2)^* y &= \operatorname{tr}(CC^{-1}BC) = y, \\
(\omega_2^{-1})^* y &= \operatorname{tr}(BCB^{-1}B) = y, \\
(\omega_2)^* z &= \operatorname{tr}(AC^{-1}BC) = \operatorname{tr}(AC^{-1})y - \operatorname{tr}(CA^{-1}BC) \\
&= (ac - z)y - (c(ay - d) - \operatorname{tr}(C^{-1}A^{-1}BC)) \\
&= (ac - z)y - (c(ay - d) - (ab - x)) = ab + cd - x - yz, \\
(\omega_2^{-1})^* z &= \operatorname{tr}(BA) = x, \\
(\omega_3^{\pm 1})^* x &= \operatorname{tr}(AB) = x, \\
(\omega_3)^* y &= \operatorname{tr}(BD) = \operatorname{tr}(BC^{-1}B^{-1}A^{-1}) \\
&= (bc - y)x - \operatorname{tr}(BC^{-1}AB) = (bc - y)x - (\operatorname{tr}(BC^{-1}A)b - (ac - z)) \\
&= (bc - y)x - ((xc - d)b - (ac - z)) = ac + bd - xy - z, \\
(\omega_3)^* z &= \operatorname{tr}(DA) = \operatorname{tr}(C^{-1}B^{-1}A^{-1}A) = y, \\
(\omega_3^{-1})^* z &= \operatorname{tr}(CDC^{-1}A) = \operatorname{tr}(B^{-1}A^{-1}C^{-1}A) \\
&= x(ca - z) - \operatorname{tr}(ABC^{-1}A) \\
&= ad + bc - xz - y.
\end{aligned}
$$

□

Example 6.14. With the above, the problem of studying the fixed point sets of concrete pseudo-Anosovs becomes a problem of solving polynomial equations. Perhaps the simplest pseudo-Anosov is $\varphi = t_2^{-1} t_1 \in \Gamma_{0,4}$, which in the case $a = b = c = d = 0$ maps (x, y, z) to

$$
\begin{pmatrix}
-x - (-y + x(xy + z))(xy + z - x(-y + x(xy + z))) \\
-y + x(xy + z) \\
xy + z - x(-y + x(xy + z))
\end{pmatrix}.
$$

The action of φ^m is visualized in Figures 6.3–6.5 for $m = 1, 2, 3$. Here, $a = b = c = d = 0$, and we consider the action of φ on the curve $\gamma(t) = (t, -t, 2) \in \mathcal{M}_{(0,0,0,0)}$, $t \in [-2, 2]$, shown in red. The images $(\varphi^m)^*\gamma$ are shown in green for 10^5 values of t. For $m = 1, 2$, the blue dots represent fixed points of the action of φ^m. In Figure 6.6 we plot $(\varphi^{\pm 2})^*\gamma$ and note that, as in the torus case, it would appear that action induced by a pseudo-Anosov is pseudo-Anosov itself.

We may in fact describe explicit formulas for the quantum representations of the four times punctured sphere using the approach of skein theory. Let $k \in \mathbb{N}$, and let $n \in \{0, \dots, k\}$. Then the action of Γ_0^4 on $V_n^k := V^k(S^2, P, (n, n, n, n))$ is given as follows: there is a basis v_l of V_n^k given by graphs in B^3, as in Figure 6.7. These graphs are admissible for even l, $0 \le l \le 2\min(k - n, n)$, and in particular $\dim V_n^k = \min(k - n, n) + 1$.

Let $\omega_1, \omega_2, \omega_3$ be the generators of Γ_0^4 described in the previous sections acting on V_n^k through the projective quantum representations $\rho_n^k : \Gamma_0^4 \to \mathbb{P}\mathrm{Aut}(V_n^k)$. This action is most easily described in the setup of skein theory; the action of ω_i on v_l is given by adding to B^3 a shell $\Sigma_0^4 \times I$ containing the appropriate framed braid (see Figure 6.7). In the following,

$$
\begin{Bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Bmatrix}
$$

denotes the *quantum* 6-*j*-*symbol*, defined by the last identity in Figure 6.8, and for which an explicit formula may be found in [MV94].

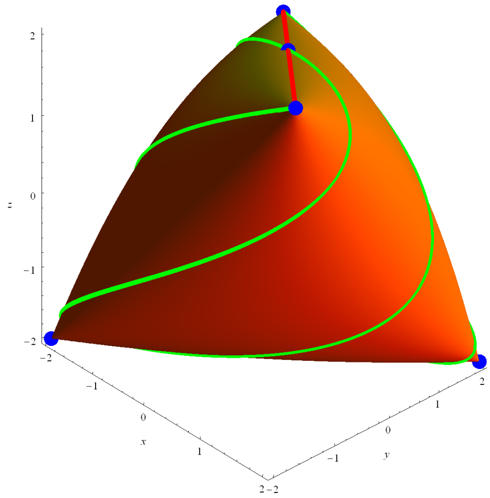


Figure 6.3: $\varphi^*\gamma$.

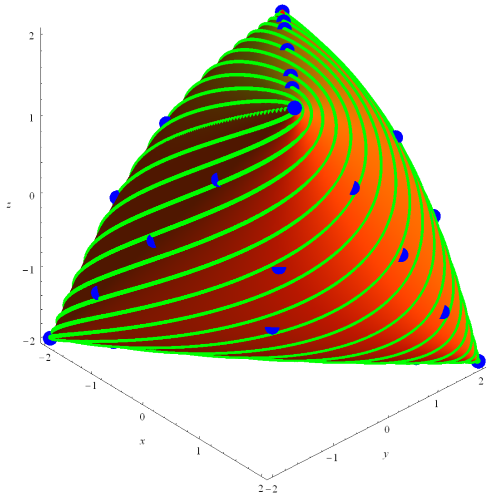


Figure 6.4: $(\varphi^2)^*\gamma$.

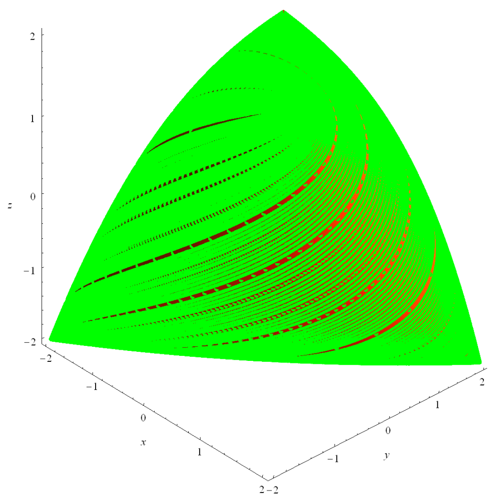


Figure 6.5: $(\varphi^3)^*\gamma$.

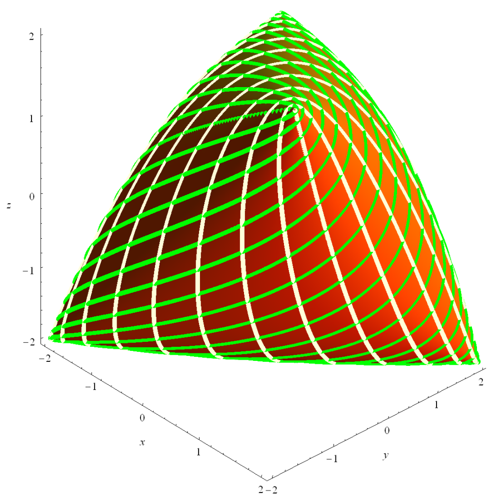


Figure 6.6: $(\varphi^2)^*\gamma$ and $(\varphi^{-2})^*\gamma$.

Lemma 6.15. *The quantum representations ρ_n^k are given explicitly by*

$$\rho_n^k(\omega_1)v_l = \rho_n^k(\omega_3)v_l = \lambda_{nl}v_l,$$
$$\rho_n^k(\omega_2)v_l = \sum_{j,m \ even}^{k} \lambda_{nj} \begin{Bmatrix} n & n & m \\ n & n & j \end{Bmatrix} \begin{Bmatrix} n & n & j \\ n & n & l \end{Bmatrix} v_m,$$

where

$$\lambda_{nj} = (-1)^{(2n-j)/2} A^{-n(n+2)} A^{j(j+2)/2}.$$

Proof. This is an elementary calculation using the well-known skein theoretical identities in Figure 6.8, collected here for ease of reference. For proofs of all of these, we refer to [KL94], [MV94]. Notation is set up to agree with [MV94], and in particular we have used

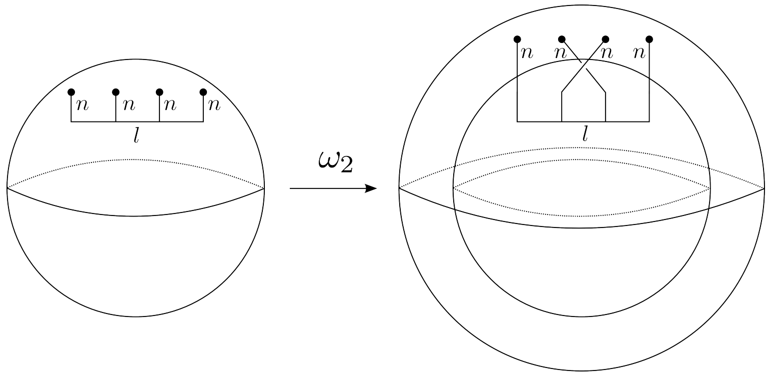


Figure 6.7: The basis elements v_l of V_n^k, and the action of ω_2 on v_l.

that $\delta(j;n,n) = \lambda_{nj}$, and that

$$\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix} = \frac{\langle i \rangle \left\langle \begin{matrix} i & b & c \\ j & d & a \end{matrix} \right\rangle}{\langle i, a, d \rangle \langle i, b, c \rangle}.$$

□

The special case $n = 1$ will be considered in more detail in Section 6.4.

Remark 6.16. That $\omega_1 \omega_3^{-1}$ acts by the identity in the quantum representation should be seen as a consequence of the fact that its action on the corresponding moduli space (see Section 6.4) is trivial, as we noted in Proposition 6.13.

6.2.3 The once punctured torus

Consider now the case of the once punctured torus Σ_1^1 with one boundary component. Its fundamental group is $\pi_1(\Sigma_1^1) = \langle a, b, c \mid ABA^{-1}B^{-1} = C \rangle$, and letting $l \in [-2, 2]$, the moduli space $\mathcal{M}_l$ of flat SU(2)-connections whose holonomy about C has trace l can be identified with

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - xyz - 2 = l\}$$

with coordinates $x = \operatorname{tr}(A)$, $y = \operatorname{tr}(B)$, $z = \operatorname{tr}(AB)$; see e.g. [Gol97], [Bro98]. Note that for $l = 2$, we recover the (singular) moduli space of the closed torus, for $l = -2$, the moduli space consists of a single point, and for $l \in (-2, 2)$, we obtain concentric 2-spheres. Moreover, for $l \in (-2, 2)$, all connections are irreducible: if $\rho \in \mathcal{M}_l$, $l \in (-2, 2)$ were reducible, we obtain

$$l = \operatorname{tr}(\rho(c)) = \operatorname{tr}([\rho(a), \rho(b)]) = \operatorname{tr}(\mathrm{Id}) = 2.$$

In this case, the mapping class group is $\Gamma_1^1 \cong \mathrm{SL}(2, \mathbb{Z})$ and is generated by the Dehn twists t_a, t_b, t_c (see p. 5). The induced action on the moduli space is

$$(t_a)^*(x, y, z) = (x, z, xz - y), \quad (t_b)^*(x, y, z) = (xy - z, y, x)$$

as one might find using the trace formula from before. For instance, t_a maps $a \mapsto a$, $b \mapsto ba$, $c \mapsto c$, so letting tr denote the trace in the appropriate representation,

$$\begin{aligned} \operatorname{tr}((t_a)_* a) &= x, \\ \operatorname{tr}((t_a)_* b) &= \operatorname{tr}(ba) = z, \\ \operatorname{tr}((t_a)_*(ab)) &= \operatorname{tr}(aba) = \operatorname{tr}(ba^2) = \operatorname{tr}(ba)\operatorname{tr}(a) - \operatorname{tr}(b) = xz - y. \end{aligned}$$

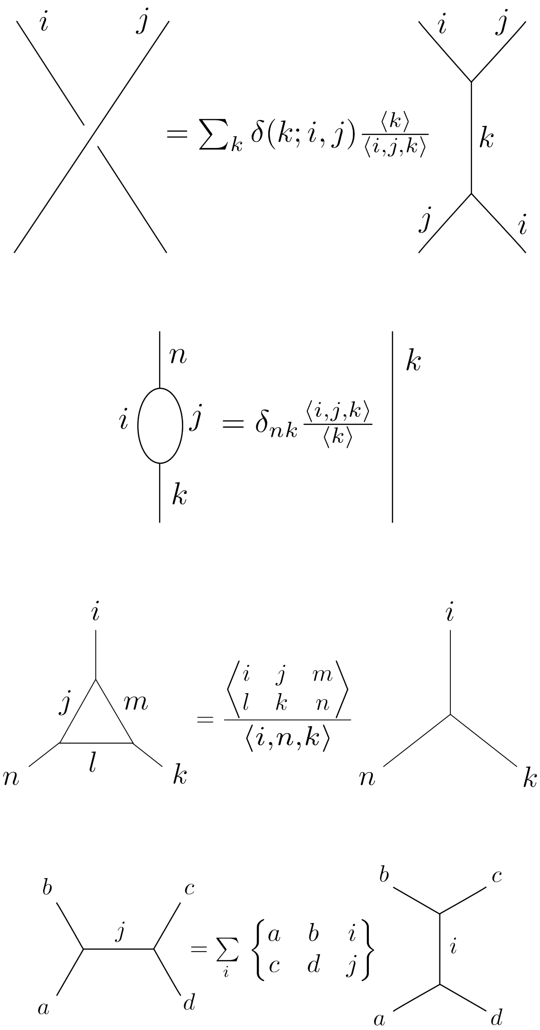


Figure 6.8: The skein theoretical identities used in the proof of Lemma 6.15.

Now, Brown, [Bro98], has shown that for an Anosov map φ, the set of fixed points of φ on the full moduli space $\mathcal{M} = \bigcup_l \mathcal{M}_l$ consists of algebraic sets of dimension 1, intersecting $\mathcal{M}_l$ transversally for almost all values of l. Moreover, by the symplectic eigenvalue theorem, for a fixed point p in $\mathcal{M}$, the eigenvalues of $d\varphi_p$ are in $U(1) \cup \mathbb{R} \setminus \{0\}$. In the limiting case $l = 2$, the action on the moduli space $(\mathbb{R}^2/\mathbb{Z}^2)/\pm$ is essentially given by the action on the torus $\mathbb{R}^2/\mathbb{Z}^2$ itself, and for a smooth fixed point $p \in \mathcal{M}_2$, the eigenvalues of $d\varphi_p$ are real, and the largest eigenvalue is the stretch factor of φ.

Example 6.17. Let $\varphi = (t_b)^{-1} t_a$ be the pseudo-Anosov map corresponding to $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ in $\mathrm{SL}(2, \mathbb{Z})$ with $\varphi_*(c) = c$. The induced map on the full moduli space $\mathcal{M}$ is

$$\varphi^*(x, y, z) = (z, zy - x, z(yz - x) - y),$$

and the number of fixed points on $\mathcal{M}_l$ is 3, 2 and 1 in the cases $l = 2$, $l \in (-2, 2)$, and $l = -2$ respectively. For the non-trivial fixed points of the $l = 2$ shell, we already know that the eigenvalues of $d\varphi$ are the stretch factors of φ. On the other hand, Brown [Bro98, Thm. 4.4] argues that the eigenvalues at $l = -2$ are all roots of unity of order $n < 5$. In our particular case of $\varphi = (t_b)^{-1} t_a$, the eigenvalues are $\exp(\pm 2\pi i/3)$. This example shows that in order to obtain information about stretch factors in the case of punctured surfaces, one must likely invoke some sort of limit where boundary holonomies tend to e.g. the identity.

6.2.4 Liftings of toral pseudo-Anosovs

Another interesting result by Brown, [Bro03a], relating to the above discussion is the construction of pseudo-Anosovs on a closed genus g Riemann surface covering the once punctured torus and lifting a power of a given pseudo-Anosov on this torus. Brown shows that for any given genus $g \geq 2$ and any given pseudo-Anosov $\varphi : \Sigma_1^1 \to \Sigma_1^1$ on the once punctured torus, there exists $n > 0$ such that φ^n lifts to a map $\tilde{\varphi^n} : \Sigma_g \to \Sigma_g$ of a particular branched covering $\Sigma_g \to \Sigma_1^1$, the branch point having degree g, such that $\tilde{\varphi^n}$ is pseudo-Anosov with stretch factor exactly the stretch factor of φ^n. Moreover, the action of φ^n on the moduli space of flat connections on Σ_1^1 having order g boundary holonomy embeds in the moduli space of Σ_g, and the action of $\tilde{\varphi^n}$ restricts to that of φ^n. In particular, one might hope to understand the fixed point sets of this particular family of pseudo-Anosovs in terms of the genus 1 data alone.

Example 6.18. One particular example considered and evaluated by Brown is the pseudo-Anosov $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ mapping $A \mapsto ABA^2BA^2$, $B \mapsto BA^2$, $C \mapsto C$. Through Brown's construction, this map lifts to the automorphism of $\pi_1(\Sigma_2) = \langle a, b, c, d \mid [a, b][c, d] \rangle$ given by

$$\begin{aligned} a &\mapsto dcba^2baba, \quad & b &\mapsto ba, \\ c &\mapsto badc^2dcdc, \quad & d &\mapsto dc. \end{aligned}$$

This lift is described explicitly but for our purposes we simply note that the covering map $\Sigma_2 \to \Sigma_{1,1}$ induces the homomorphism $a \mapsto A^2$, $b \mapsto B$, $c \mapsto CA^2C^{-1}$, $d \mapsto CBC^{-1}$ of fundamental groups.

The moduli space of flat connections on $\Sigma_{1,1}$ whose boundary holonomy has order $g = 2$ is nothing but the union of the $l = \pm 2$ shells described above. As the connection ρ_{-2} corresponding to $l = -2$ is always fixed, this one will be of particular interest. Note that ρ_{-2} may be represented as

$$\rho_{-2}(A) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho_{-2}(B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_{-2}(C) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This then induces the representation $\rho : \pi_1(\Sigma_2) \to \mathrm{SU}(2)$ given by

$$\begin{aligned} \rho(a) = \rho(c) &= \rho_{-2}(A^2) = -\mathrm{Id}, \\ \rho(b) = \rho(d) &= \rho_{-2}(B) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Now, [Bro03b, Cor. 5.3] notes that this particular connection is smooth in $\mathcal{M}(\Sigma_2)$, even though it is not reducible, and that the induced action on the pseudo-Anosov possibly has fixed directions, such that the fixed point is not isolated in the set of all fixed points. This example it would be interesting to verify further as it does not seem completely obvious: although the moduli space may be identified with $\mathbb{C}P^3$ as a topological space, the representation theoretical construction of it seems to suggest that it has singularities; an elementary calculation along the lines of Theorem 5.22 shows that $\dim H^1(\pi_1(\Sigma_2), \mathrm{Ad} \circ \rho) = 8 > 6$. We leave a further analysis hereof to a future study.

6.3 Stretch factors via Heegaard decompositions

So far, we have only been considering how quantum invariants of mapping tori and the study of fixed point sets in moduli spaces may shed light on the AMU conjecture. A different approach would be to consider instead the 3-manifolds obtained through Heegaard decompositions specified by a choice of mapping class. The quantum G-invariant $Z_k(M)$ of the Heegaard split 3-manifold $M = H_g \cup_\varphi \bar{H}_g$, where H_g denotes the standard genus g handlebody, $\partial H_g = \Sigma_g$, and $\varphi \in \Gamma_g$ may be obtained from $\rho_k(\varphi)$ through our usual identification of H_g with a vector in $V_k(\Sigma_g)$.

In this section, we briefly study the classical case, setting out to analyze the action of iterates of pseudo-Anosov mapping classes on the moduli spaces of flat G-connections. To obtain some ideas of what to expect in general, let us again consider the action of Anosov maps on the torus T^2 before moving on to the case of higher genus. More precisely, let $\varphi \in \Gamma_1$, and let $M_n = H_1 \cup_{\varphi^n} \bar{H}_1$ be the 3-manifold obtained by gluing together two solid genus 1 handlebodies, matching orientations appropriately. Then by definition, M_n has Heegaard genus at most 1 and is so either a lens space, $S^2 \times S^1$, or S^3. Consider the matrix representation $\varphi \in \mathrm{SL}(2,\mathbb{Z})$ of φ corresponding to the usual action of φ on the homology of the torus. Now, since λ is the only eigenvalue greater than 1, we have now that $\varphi^n - \lambda\varphi^{n-1} \to 0$, entry-wise. Assume for simplicity that

$$\varphi = \begin{pmatrix} p & d \\ -q & -b \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

with $0 < |q| < p$. Then, possibly excluding finitely many low values of n, M^n will be a lens space $L(p_n, q_n)$, where (p_n, q_n) are given by

$$\varphi^n = \begin{pmatrix} p_n & d_n \\ -q_n & -b_n \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z});$$

see e.g. [Jef92]. Now, the discussion above implies that asymptotically, both p_n/p_{n-1} and q_n/q_{n-1} will tend to λ.

Let us now consider what happens to the moduli spaces of the manifolds under consideration. Recall that the fundamental group of a lens space $L(p,q)$ is $\pi_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$. We will consider the cases $G = \mathrm{U}(1)$, and $G = \mathrm{SU}(2)$. In the case $G = \mathrm{U}(1)$,

$$\mathcal{M}^{\mathrm{U}(1)}(M) = \mathrm{Hom}(\pi_1(M), \mathrm{U}(1)) = \mathrm{Hom}(H_1(M), \mathrm{U}(1)),$$

and so we should be able to extract the information contained in the symplectic mapping class group representation, but nothing more; thus, whereas the non-abelian case is the interesting one, the abelian case will illustrate the general principles.

For a discussion of the geometric quantization of these abelian moduli spaces and the role of Toeplitz operators in this theory – which we will not need in this manuscript – we refer to [And05] and [AB11].

6.3.1 $G = \mathrm{U}(1)$, $g = 1$

In this case,

$$\mathcal{M}^{\mathrm{U}(1)}(L(p,q)) = \mathrm{Hom}(\pi_1(L(p,q)), \mathrm{U}(1)) = \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathrm{U}(1))$$

consists of p elements $A_0, \dots, A_{p-1}$, determined by their value on the generator 1 of $\mathbb{Z}/p\mathbb{Z}$,

$$A_m(1) = \exp(2\pi i m/p). \tag{6.8}$$

In particular, this implies that

$$0 < \lim_{n\to\infty} \frac{|\mathcal{M}^{\mathrm{U}(1)}(M^n)|}{\lambda^n} < \infty. \tag{6.9}$$

That the above is true for the case $g = 1$ is also not particularly surprising if one considers $\mathcal{M}^G(M^n)$ as the intersection

$$\mathcal{M}^G(H_1) \cap (\varphi^n)^* \mathcal{M}^G(H_1) \subseteq \mathcal{M}^G(\Sigma_1).$$

On $\mathcal{M}^G(\Sigma_1) = (T \times T)/W$, which for the case of $G = \mathrm{U}(1)$ is just the torus itself, the action of the mapping class is given by the action on the torus as before. For $G = \mathrm{SU}(2)$, we have $T = \mathrm{U}(1)$, and we will see more or less the same thing happen.

Now, in the case of higher genus surfaces, the resulting 3-manifold moduli spaces are not necessarily finite, and so (6.9) will not make sense in general. One might however hope for a similar thing happening to the number of connected components, and so we pose the following

Question 6.19. *For which groups G and which pseudo-Anosovs $\varphi : \Sigma_g \to \Sigma_g$ with stretch factors λ, will one have that*

$$0 < \lim_{n\to\infty} \frac{|\pi_0(\mathcal{M}^G(M^n))|}{\lambda^{dn}} < \infty,$$

where $M^n = H_g \cup_\varphi \bar{H}_g$, and $d \in \mathbb{Q}$ depends on G and g?

We will see that this is not the case for $G = \mathrm{U}(1)$ before moving on to the case of $G = \mathrm{SU}(2)$, but before doing so, note that we could ask the similar question for the *volume* of the moduli space, $\mathrm{Vol}(\mathcal{M}^G(M^n))$, rather than just a topological count on it. Recall that the square root of the Reidemeister torsion defines a density on the moduli space of flat connections, which has been computed explicitly for lens spaces (see e.g. [FG91] and the preprint [McL12] for references on the following). For the case $G = \mathrm{U}(1)$, the volume of the moduli space is simply

$$\mathrm{Vol}(\mathcal{M}^{\mathrm{U}(1)}(L(p,q))) = \sum_{m=0}^{p-1} \tau_{A_m}^{1/2} = \sum_{m=0}^{p-1} \frac{1}{\sqrt{p}} = \sqrt{p},$$

and we find that

$$0 < \lim_{n\to\infty} \frac{\mathrm{Vol}(\mathcal{M}^{U(1)}(M^n))}{\lambda^{n/2}} < \infty.$$

Again, we may pose as a general question when something like this holds.

Question 6.20. *For which groups G and which pseudo-Anosovs $\varphi : \Sigma_g \to \Sigma_g$ with stretch factors λ, will one have that*

$$0 < \lim_{n\to\infty} \frac{\mathrm{Vol}(\mathcal{M}^G(M^n))}{\lambda^{dn}} < \infty,$$

where $M^n = H_g \cup_\varphi \bar{H}_g$, and $d \in \mathbb{Q}$ is some rational number, depending on g and G?

6.3.2 $G = \mathrm{U}(1)$, $g > 1$

Now, consider any orientation-preserving homeomorphism $\varphi : \Sigma_g \to \Sigma_g$, and recall that the action of φ on holonomy determines an element $h(\varphi) = \mathrm{Sp}(2g,\mathbb{Z})$, where we in the following use the standard symplectic basis $H_1(\Sigma_g,\mathbb{Z}) = \mathbb{Z}\langle \alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\rangle$, where α_i,β_j are the standard generators satisfying that α_i is in the kernel of the map $H_1(H_g,\mathbb{Z}) \to H_1(\Sigma_g,\mathbb{Z})$ induced by the inclusion $\Sigma_g \to H_g$. We will use α_i,β_j to denote both the corresponding curves, homology classes and homotopy classes which should cause no confusion. Now, for any 3-manifold with Heegaard decomposition $M = H_g \cup_\varphi \bar{H}_g$, the fundamental group of M has rank at most g and has a presentation

$$\pi_1(M) \cong \langle \alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g \mid \alpha_i = 0, \varphi_*(\alpha_i) = 0,\ i = 1,\dots,g\rangle. \tag{6.10}$$

As before, $\mathrm{U}(1)$-representations of $\pi_1(M)$ factor through $H_1(M,\mathbb{Z})$ and so $\mathcal{M}^{\mathrm{U}(1)}(M)$ is determined by $h(\varphi)$. More precisely, if we write

$$h(\varphi) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in the basis above, we find that

$$\mathcal{M}^{\mathrm{U}(1)} = \left\{ (x_1, \dots, x_g) \in [0,1)^g \;\middle|\; \sum_{j=1}^{g} c_{ij} x_i = 0 \bmod 1,\ i = 1, \dots g \right\},$$

where $c_{ij} = (C)_{ij}$. The map is given through the identification (6.10) by $A(\beta_j) = \exp(2\pi i x_j)$ for a flat connection $A : \pi_1(M) \to \mathrm{U}(1)$. Now, if $\det C = 0$, this of course means that $\mathcal{M}^{\mathrm{U}(1)}(M)$ is infinite. In particular, this is true whenever φ is in the kernel of h; in other words that φ is in the Torelli group $\mathcal{I}(\Sigma_g) = \ker h$. Moreover, as noted in Section 1.3.1, Thurston constructed pseudo-Anosovs in $\mathcal{I}(\Sigma_g)$ (see also [MS07]) and of course we can not detect any information about such maps from their action on homology. If, however C is invertible, we find that $\mathcal{M}^{\mathrm{U}(1)}$ is finite and in bijection with the set

$$C([0,1)^g) \cap \mathbb{Z}^g,$$

whose cardinality is $|\det C|$. This happens for the interesting family of pseudo-Anosovs whose action on homology furthermore determine their stretch factors, such as those discussed in Section 1.3.1: consider for simplicity the case $g = 2$. Let φ be such a pseudo-Anosov; e.g., assume that φ is orientable. Let λ_1, λ_2, $|\lambda_1| \geq |\lambda_2| > 1$ be eigenvalues of $h(\varphi)$, so that $|\lambda_1|$ is the stretch factor of φ. We claim that

$$0 < \lim_{n \to \infty} \frac{|\mathcal{M}^{\mathrm{U}(1)}(M^n)|}{|\lambda_1 \lambda_2|^n} < \infty.$$

This follows from an elementary but slightly tedious calculation of $|\det(C_n)|$, defined by $h(\varphi^n) = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$, where one first diagonalizes $h(\varphi^m) = h(\varphi)^m$, and notes that in the concrete expression for $|\det(C_n)|$, all occurrences of λ_1^{2n} cancel.

6.3.3 $G = \mathrm{SU}(2), g = 1$

We return to our study of lens spaces and consider Anosov maps $\varphi : \Sigma_1 \to \Sigma_1$. Recall again that $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$, and that this time around, $\mathcal{M}^{\mathrm{SU}(2)}(L(p,q))$ consists of $\lfloor (p+1)/2 \rfloor$ flat connections $A_m : \pi_1(L(p,q)) \to \mathrm{SU}(2)$, given up to conjugacy by

$$A_m = \begin{pmatrix} \exp(2\pi i m/p) & 0 \\ 0 & \exp(-2\pi i m/p) \end{pmatrix},$$

for $m = 0, \dots, \lfloor p/2 \rfloor$. Letting once again M^n denote the 3-manifold $H_1 \cup_{\varphi^n} \bar{H}_1$, we find, exactly as in Section 6.3.2 that

$$0 < \lim_{n \to \infty} \frac{|\mathcal{M}^{\mathrm{SU}(2)}(M^n)|}{\lambda^n} < \infty.$$

Remark 6.21. More generally, for $G = \mathrm{SU}(N)$, $N \geq 2$, we have that

$$\mathcal{M}^{\mathrm{SU}(N)}(L(p,q)) = \#\{(n_1, \dots, n_N) \mid n_i \in \mathbb{Z}, 0 \leq n_i < p,\ \sum_{i=1}^{N} n_i \in p\mathbb{Z}\}/S_N,$$

where S_N acts by permutation. This number is in $\Theta(p^{N-1})$, so in Question 6.19, the relevant choice of d is $\operatorname{rank} G$ for $G = \mathrm{U}(1)$, or $G = \mathrm{SU}(N)$. The identification is given by mapping $(n_1, \dots, n_N)$ to the conjugacy class of connections containing the one whose value on 1 is the diagonal matrix

$$\operatorname{diag}(\exp(2\pi i n_1/p), \dots, \exp(2\pi i n_N/p)).$$

The question about volumes becomes slightly more interesting here. Assume that $N = 2$. The central connections A_0 and (when p is even) $A_{p/2}$ have Reidemeister torsions (correcting a small typo in [Jef92]) whose square roots are given by

$$\tau_{A_0}^{1/2} = 1/p,\ \tau_{A_{p/2}}^{1/2} = 4/p.$$

See also e.g. [FG91]. Thus, for large powers of φ, these will not contribute to the volume asymptotics. For the reducible but non-central SU(2)-connections

$$A_1, \dots, A_{\lfloor (p-1)/2 \rfloor} : \mathbb{Z}/p\mathbb{Z} \to \mathrm{SU}(2),$$

Freed and Gompf show that

$$\tau_{A_m}^{1/2} = \frac{4}{\sqrt{p}} \left| \sin\left(\frac{2\pi i}{p}\right) \sin\left(\frac{2\pi q^* m}{p}\right) \right|,$$

where q^* is the inverse of q mod p. Now, $(q_n)^*$ is one of the entries of φ^n, so as before, $(q_n)^* - \lambda(q_{n-1})^* \to 0$, so all in all, to answer Question 6.20 positively in this case, we need to show that the quantity

$$y_n = \sum_{m=1}^{\lfloor \lambda^n p/2 \rfloor - 1} \left| \sin\left(\frac{2\pi i}{\lambda^n p}\right) \sin\left(\frac{2\pi q^* m}{p}\right) \right|$$

satisfies $y_n = \Theta(\lambda^n)$. We explain heuristically why this is the case. The above sum may be rewritten as

$$y_n = \sum_{x \in I_n} |\sin(2\pi i x) \sin(2\pi q^* \lambda^n x)|,$$

where $I_n = \{1, \dots, \lfloor \lambda^n p/2 \rfloor - 1\}/(\lambda^n p) \subseteq [0, \frac{1}{2}]$. As $n \to \infty$, I_n becomes dense in $[0, \frac{1}{2}]$, and so one might hope to approximate

$$y_n \sim_{n\to\infty} \lambda^n p \int_0^{\frac{1}{2}} |\sin(2\pi i x) \sin(2\pi q^* \lambda^n x)| \,.$$

Note however that this is not simply a Riemann sum approximation as the integrand depends on the subdivision of $[0, \frac{1}{2}]$ under consideration, and so a rigorous argument would supply the relevant error estimates. The claim then follows if one furthermore notices that the integral converges to $2/\pi^2$.

6.3.4 $G = \mathrm{SU}(2)$, $g > 1$

Let us finally consider an example of how manners are complicated in the general case. Consider the family of pseudo-Anosovs $\varphi : \Sigma_2 \to \Sigma_2$ defined by

$$t_{a_1}^{n_1} t_{a_2}^{-n_2} t_{a_3}^{n_3} t_{a_4}^{n_4} t_{a_5}^{-n_5},$$

and let $\alpha_1, \alpha_2, \beta_1, \beta_2$ denote the generators of the fundamental group $\pi_1(\Sigma_2)$, oriented as in Figure 6.9 so that the non-trivial actions of the left Dehn twists making up φ are given by

$$t_{a_1}\beta_1 = \beta_1\alpha_1,\ t_{a_2}^{-1}\alpha_1 = \beta_1\alpha_1,\ t_{a_3}\beta_1 = \alpha_1\alpha_2\beta_1,$$
$$t_{a_3}\beta_2 = \alpha_1\alpha_2\beta_2,\ t_{a_4}^{-1}\alpha_2 = \alpha_2\beta_2,\ t_{a_5}\beta_2 = \beta_2\alpha_2.$$

Notice that, for ease of computer implementation, the choice of generators is such that positive words are mapped to positive words under the action of this particular class of pseudo-Anosov homeomorphisms. With these choices,

$$\pi_1(\Sigma_2) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1^{-1}, \beta_1][\alpha_2, \beta_2]^{-1} \rangle.$$

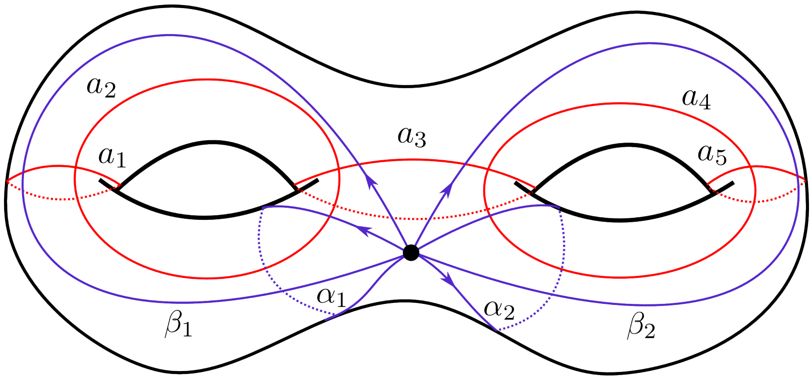


Figure 6.9: The generators $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $\pi_1(\Sigma_2)$.

Example 6.22. The complexity of the fundamental groups corresponding 3-manifolds – as given by (6.10) – grows quickly in iterates of φ. Take e.g. all n_i's to be 1. Then

$$\pi_1(M^1) = \{1\},$$
$$\pi_1(M^2) = \langle \beta_1, \beta_2 \mid 1 = \beta_1^4 = (\beta_1\beta_2)^4 \rangle,$$
$$\pi_1(M^3) = \langle \beta_1, \beta_2 \mid 1 = \beta_1^5\beta_2\beta_1^{11} = \beta_1\beta_2\beta_1^5(\beta_2\beta_1)^3\beta_2\beta_1^5(\beta_2\beta_1)^2\beta_2\beta_1^5(\beta_2\beta_1)^3\beta_2\beta_1^5(\beta_2\beta_1)^2\beta_2 \rangle.$$

Of course, the lengths of the words arising as relations for the fundamental group in this way are closely related to the stretch factor of the pseudo-Anosov being iterated; see [FLP79, Exposé 10]. Other 3-manifolds with curious fundamental groups may be constructed in this way. Take e.g. the pseudo-Anosov φ with $n_1 = n_3 = n_5 = 1$ and $n_2 = a, n_4 = b$. Then one easily finds that

$$\pi_1(H_2 \cup_\varphi \bar{H}_2) = \langle \beta_1, \beta_2 \mid 1 = \beta_1^a = \beta_2^b \rangle = (\mathbb{Z}/a\mathbb{Z}) * (\mathbb{Z}/b\mathbb{Z}).$$

6.4 Jones representations of mapping class groups

We end off this thesis by going back to the roots of quantum topology, discussing possible spin-offs of the study of genus 0 quantum representations by viewing them as specializations of the so-called Jones representations of braid groups. These representations have formed a huge field of research in their own right, and we make no attempt to give a full overview; rather, we discuss how the work of [AMU06] may be put in this framework and how one might hope to generalize the results of [AMU06] to other punctured spheres. The observations made in the following have all arised from discussions with Jens Kristian Egsgaard.

As will be clear from the constructions below, the Jones representation of the braid group B_n may be viewed as the restriction of a sum of quantum SU(2)-representations of an n times punctured disc, capped off to form an $n+1$ times punctured sphere, the punctures viewed as the marked points with a natural choice of framing and having colours $1 \in \{0, \dots, k\}$ at the first n punctures and varying colour at the puncture obtained from the capping. One main difference is that we make no assumption on the parameters a, A or q appearing in the constructions of the quantum representations, and we define the Jones representations explicitly to avoid technicalities dealing with the identifications so as to obtain representations well-defined for all but finitely many choices of complex values of the parameters in question; when no choice is made, the representation is referred to as *generic*.

More precisely, we will consider the *generic Jones representation* of the braid group B_n in terms of its standard presentation

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| > 1 \rangle. \tag{6.11}$$

We will describe the representation in terms of both the skein theory of the Kauffman bracket as well as the representation theory of quantum groups. In either case, it is a representation $\mathcal{V}$, splitting as

$$\mathcal{V} = \bigoplus_d \mathcal{V}_d \otimes \mathcal{V}_d^*,$$

where the $\mathcal{V}_d$ are the above mentioned quantum representations; that is, we will describe natural actions ρ^d of B_4 on each $\mathcal{V}_d$, such that the representation on $\mathcal{V}$ splits as

$$\bigoplus_d \rho^{d,n} \otimes \mathrm{id} = \bigoplus_d \bigoplus_{i=1,\dots,v_d} \rho^{d,n},$$

where $v_d = \dim \mathcal{V}_d$.

6.4.1 The Kauffman bracket approach

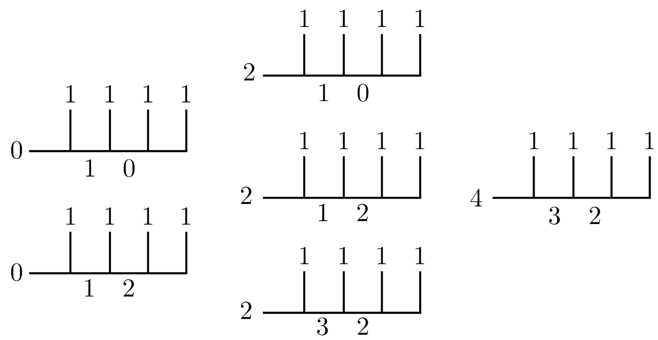


Figure 6.10: The basis for $\mathcal{V}_0 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4$.

In this case, $\mathcal{V}$ is the reduced Temperley–Lieb algebra TL_n, the reduced skein module of a 3-ball $D^2 \times I$ with $2n$ marked points $(p_1, \dots, p_n) \times \{\pm 1\}$, an algebra structure given by stacking, and with the usual action of the braid group. The space $\mathcal{V}_d$ is the space of skeins in a 3-ball, meeting its boundary sphere in $n+1$ ordered points labelled $(d, 1, \dots, 1)$. On this space, the action $\rho^{d,n}$ of B_n is given by adding a shell $S^2 \times I$, extending the d-coloured point trivially, and acting on the remaining points in the obvious way (cf. also Figure 6.7). The spaces V_d have bases given by trivalent graphs, expandable into Jones–Wenzl idempotents to give skeins in the ball in exactly the same way as in Section 3.4.

Example 6.23. Consider the case $n = 4$. Choosing the particular basis in Figure 6.10, the action of B_4 on $\mathcal{V}_0 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4$ is given by

$$\sigma_1 \mapsto \begin{pmatrix} -A^{-3} & 0 \\ 0 & A \end{pmatrix} \oplus \begin{pmatrix} A & 0 & 0 \\ 0 & \frac{A^9}{1+A^4+A^8} & \frac{(1+A^4)^2(1+A^8)}{A(1+A^4+A^8)^2} \\ 0 & A^{-1} & -\frac{1}{A^3+A^7+A^{11}} \end{pmatrix} \oplus (A),$$

$$\sigma_2 \mapsto \begin{pmatrix} \frac{A^5}{1+A^4} & A^{-1} - \frac{A^3}{(1+A^4)^2} \\ A^{-1} & -\frac{1}{A^3+A^7} \end{pmatrix} \oplus \begin{pmatrix} \frac{A^5}{1+A^4} & A^{-1} - \frac{A^3}{(1+A^4)^2} & 0 \\ \frac{1}{A} & -\frac{1}{A^3+A^7} & 0 \\ 0 & 0 & A \end{pmatrix} \oplus (A),$$

$$\sigma_3 \mapsto \begin{pmatrix} -A^{-3} & 0 \\ 0 & A \end{pmatrix} \oplus \begin{pmatrix} -A^{-3} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} \oplus (A).$$

All calculations involved are elementary and can be derived by using the identities shown in Figure 6.11 (see e.g. [Lic97, Figure 13.5], [FWW02, Figure 2], [KL94]). In fact, the calculation for the special case $d = 0$ is a special case of Lemma 6.15.

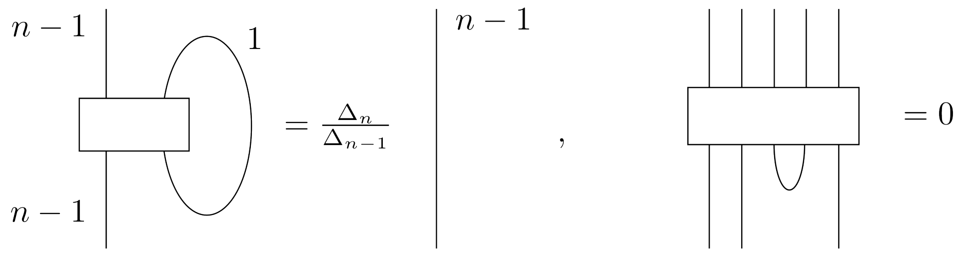


Figure 6.11: The skein identities needed to evaluate the Jones representations explicitly.

6.4.2 Quantum group approach

Let $U_q(\mathfrak{sl}_2)$ be the quantum group of $\mathfrak{sl}_2$ with generic parameter q, with its usual generators E, F, and K. Let $V_{d-1} = \mathbb{C}^d$ denote the irreducible d-dimensional $U_q(\mathfrak{sl}_2)$-module given (see e.g. [Oht02, p. 90]) in the standard basis $(e_i)_{i=0}^{d-1}$ by

$$\begin{aligned}
E.e_i &= \begin{cases} [d-i+2]e_{i-1}, & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\
F.e_i &= \begin{cases} [i+1]e_{i+1}, & \text{if } i < d-1, \\ 0, & \text{if } i = d, \end{cases} \\
K.e_i &= q^{(d-2i-1)/2}.
\end{aligned}$$

Here, $[m] = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$. The tensor product of two such modules is defined via the comultiplication (see [Oht02, p. 87]) $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ given by

$$\begin{aligned}
\Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, \\
\Delta(E) &= E \otimes K + 1 \otimes E, \\
\Delta(F) &= F \otimes 1 + K^{-1} \otimes F.
\end{aligned}$$

Now, in the notation of the introduction to this section, let $\mathcal{V} = \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V_1^{\otimes n})$, and let $\mathcal{V}_d = \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V_d, V_1^{\otimes n})$. The braid group generator $\sigma_i \in B_n$ acts on $\mathcal{V}_d$ by post-composition by the map $R_i : V_1^{\otimes n} \to V_1^{\otimes n}$ given explicitly by applying (see [Oht02, p. 92]) the *R-matrix*

$$R = \begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & 0 & q^{-1/4} & 0 \\ 0 & q^{-1/4} & q^{1/4} - q^{-3/4} & 0 \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix} : V_1^{\otimes 2} \to V_1^{\otimes 2}$$

to the i'th and $i+1$'st tensor product factors of $V_1^{\otimes n}$. Now, it is immediate that each R_i preserves the eigenspaces of the action of K on $V_1^{\otimes n}$, and the action on $\mathcal{V}_d$ is equivalent to the action given by R on

$$W_d = \ker(K - q^{d/2}) \cap \ker(E) \subseteq V_1^{\otimes n}.$$

Denote the representations arising in this way by $\rho_q^{d,n} : B_n \to \mathrm{GL}(\dim \mathcal{V}_d, \mathbb{Z}[q^{\pm 1/4}])$.

Example 6.24. Consider again the case $n = 4$. Denoting the basis vectors of $V_1^{\otimes 4}$ by

$e_{ijkl} = e_i \otimes e_j \otimes e_k \otimes e_l$ for $i,j,k,l \in \{0,1\}$, one readily finds that

$$\begin{aligned}
W_0 = \operatorname{span}(&q^{1/2}e_{0101} - e_{0110} - e_{1001} + q^{-1/2}e_{1010},\\
&q^{1/2}e_{0011} + qe_{0101} - (q^{1/2}+q^{-1/2})e_{0110}\\
&- (q^{1/2}+q^{-1/2})e_{1001} + q^{-1}e_{1010} + q^{-1/2}e_{1100}),\\
W_2 = \operatorname{span}(&q^{1/4}e_{0001} - q^{-1/4}e_{0010}, q^{1/4}e_{0010} - q^{-1/4}e_{0100}, q^{1/4}e_{1000} - q^{-1/4}e_{0100}),\\
W_4 = \operatorname{span}(&e_{0000}),
\end{aligned}$$

giving explicit bases for the W_d. Determining the action of B_4 on $W_0 \oplus W_2 \oplus W_4$ is thus reduced to an exercise in linear algebra. As an example, denote by v the first basis vector of the above basis for W_0. Then the action of σ_1 on v is

$$\begin{aligned}
R_1(v) &= q^{1/2}q^{-1/4}e_{1001} - q^{1/4}e_{1010} - (q^{1/4} - q^{-3/4})e_{1001} - q^{-1/4}e_{0101}\\
&\quad + q^{-1/2}(q^{-1/4}e_{0110} + (q^{1/4} - q^{-3/4})e_{1010})\\
&= -q^{-3/4}v.
\end{aligned}$$

One thus easily finds that in these bases, the action of B_4 on $W_0 \oplus W_2 \oplus W_4$ is given by

$$\begin{aligned}
\sigma_1 &\mapsto \begin{pmatrix} -q^{-3/4} & -(q^{-5/4}+q^{-1/4}+q^{3/4}) \\ 0 & q^{1/4} \end{pmatrix} \oplus \begin{pmatrix} q^{1/4} & 0 & 0 \\ 0 & q^{1/4} & 0 \\ 0 & q^{-1/4} & -q^{-3/4} \end{pmatrix} \oplus (q^{1/4}),\\
\sigma_2 &\mapsto \begin{pmatrix} -q^{-3/4} & 0 \\ q^{-1/4} & q^{1/4} \end{pmatrix} \oplus \begin{pmatrix} q^{1/4} & 0 & 0 \\ q^{-1/4} & -q^{-3/4} & q^{-1/4} \\ 0 & 0 & q^{1/4} \end{pmatrix} \oplus (q^{1/4}),\\
\sigma_3 &\mapsto \begin{pmatrix} -q^{3/4} & -(q^{-5/4}+q^{-1/4}+q^{3/4}) \\ 0 & q^{1/4} \end{pmatrix} \oplus \begin{pmatrix} -q^{-3/4} & q^{-1/4} & 0 \\ 0 & q^{1/4} & 0 \\ 0 & 0 & q^{1/4} \end{pmatrix} \oplus (q^{1/4}).
\end{aligned}$$

In the following, we will discuss the kernels of the representations $\rho_q^{n,d}$. Note here that it is already known that braid groups are linear, i.e. that they admit faithful representations (see [Big01], [Kra02]):

Theorem 6.25. *The Lawrence–Krammer representation* $\rho^{\mathrm{LK}} : B_n \to \mathrm{GL}\left(\binom{n}{2}, \mathbb{Z}\langle t^{\pm 1}, q^{\pm 1}\rangle\right)$ *given in a basis* $v_{j,k}$, $1 \leq j < k \leq n$ *by*

$$\rho^{\mathrm{LK}}(v_{j,k}) = \begin{cases} v_{j,k} & i \notin \{j-1, j, k-1, k\},\\ qv_{i,k} + (q^2-q)v_{i,j} + (1-q)v_{j,k} & i = j-1,\\ v_{j+1,k} & i = j \neq k-1,\\ qv_{j,i} + (1-q)v_{j,k} - (q^2-q)tv_{i,k} & i = k-1 \neq j,\\ v_{j,k+1} & i = k,\\ -tq^2 v_{j,k} & i = j = k-1. \end{cases}$$

is faithful.

6.4.3 Observations

In order to tie together our present analysis with known results, we first note that [Bir75, Thm. 4.5] identifies Γ_0^n, $n \geq 2$, with a quotient of B_n as

$$\begin{aligned}
\Gamma_0^n = \langle \omega_1, \ldots, \omega_{n-1} \mid\ &\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1},\\
&\omega_i\omega_j = \omega_j\omega_i \quad |i-j| > 1,\\
&\omega_1 \cdots \omega_{n-2}\omega_{n-1}^2\omega_{n-2}\cdots\omega_1 = 1,\\
&(\omega_1 \cdots \omega_{n-1})^n = 1\rangle.
\end{aligned}$$

Denote by $l : B_n \to \mathbb{Z}$ the exponent sum with respect to the presentation (6.11). Then we notice that the representation $q^{l/4}\rho_q^{0,4}$ descends to a representation

$$\rho : \Gamma_0^4 = B_4/\langle \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1,\ (\sigma_1\sigma_2\sigma_3)^4\rangle \to \mathrm{GL}(2, \mathbb{Z}[q^{\pm 1/4}]),$$

equivalent by construction to the representation, also denoted ρ, given on [AMU06, p. 484], and in particular we know by [AMU06, p. 485] that

$$\ker(\rho) = \langle \omega_1\omega_3^{-1}, (\omega_1\omega_2\omega_3)^2\rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \tag{6.12}$$

This on the other hand also tells us, that the projective kernel $\ker_P(\rho_q^{0,4})$ of $\rho_q^{0,4}$ is

$$\ker_P(\rho_q^{0,4}) = \langle \sigma_1\sigma_3^{-1}, (\sigma_1\sigma_2\sigma_3)^2, \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1\rangle \subseteq B_4. \tag{6.13}$$

Lemma 6.26. $\rho_q^{2,4}|_{\langle \sigma_1\sigma_3^{-1}, (\sigma_1\sigma_2\sigma_3)^2\rangle}$ *is faithful.*

Proof. Let $a = \sigma_1\sigma_3^{-1}$ and $b = (\sigma_1\sigma_2\sigma_3)^2$, and let $\varphi \in \ker \rho_q^{2,4} \cap \langle a, b\rangle$. In the braid group, we have the relations $ab = ba^{-1}$, $a^{-1}b = ba$, (since, for instance, these relations hold in the Lawrence–Krammer representation), and b^2 is the generator of the center of B_4 (see [FM11, p. 247]). Thus we may assume that $\varphi = b^n a^m$ for $n, m \in \mathbb{Z}$. Notice that $\det \rho_q^{2,4}(b) = q^{-3/2}$ and $\det \rho_q^{2,4}(a) = 1$, so

$$1 = \det \rho_q^{2,4}(\varphi) = q^{-3n/2}$$

implies that $n = 0$. On the other hand, by the explicit formulas for $\rho_q^{2,4}(\sigma_1)$ and $\rho_q^{2,4}(\sigma_3)$, we see that the $(1,1)$'th entry of $\rho_q^{2,4}(a^m)$ is $(-q)^m$, and we conclude that $m = 0$ and thus that $\varphi = \mathrm{id}$. □

Notice also that

$$\langle \sigma_1\sigma_3^{-1}, (\sigma_1\sigma_2\sigma_3)^2, \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1\rangle = \langle \sigma_1\sigma_3^{-1}, (\sigma_1\sigma_2\sigma_3)^2, \sigma_2\sigma_1\sigma_3^{-1}\sigma_2\rangle.$$

Let a, b be as above, and let $c = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2$. To show that the Jones representation of B_4 is faithful, it thus suffices to show that any word in the generators of (6.13) containing c and in the kernel of $\rho_q^{2,4}$ must necessarily be trivial in B_4.

The group $\langle a, c\rangle$ is free of rank 2 and is exactly the kernel of the homomorphism $B_4 \to B_3$ mapping $\sigma_1 \mapsto \sigma_1, \sigma_2 \mapsto \sigma_2, \sigma_3 \mapsto \sigma_1$ [DFG82, p. 406].

In fact, just like $q^{l/4}\rho_q^{0,4}$, for a particular choice of q, was proven in [AMU06] to be equivalent to a particular holonomy representation of M, one finds that $q^{-l/4}\rho_q^{2,4}$ is equivalent to the reduced Burau representation. This fact appears to be well-known (see e.g. the comment before [FWW02, Thm. 5.1]), but is easily seen as a consequence of our particular choices of bases. Recall (see e.g. [FK13, Def. 2.3]) that for $n \geq 3$, the reduced Burau representation $\beta_q : B_n \to \mathrm{GL}(n-1, \mathbb{Z}[q^{\pm 1}])$ is given by

$$\begin{aligned}
\beta_q(\sigma_1) &= \begin{pmatrix} -q & 1 \\ 0 & 1 \end{pmatrix} \oplus \mathbf{1}_{n-3}, \\
\beta_q(\sigma_j) &= \mathbf{1}_{j-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ q & -q & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathbf{1}_{n-j-2}, \quad 1 < j < n-1, \\
\beta_q(\sigma_{n-1}) &= \mathbf{1}_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ q & -q \end{pmatrix}.
\end{aligned}$$

Lemma 6.27. *The Jones representation* $q^{-l/4}\rho_q^{n-2,n}$ *is equivalent to the Burau representation* $\beta_{q^{-1}}$.

Proof. In the notation from above, let $w_i = e_{0\dots010\dots0} \in V_1^{\otimes n}$ (with the 1 at the i'th place). Then $E.w_i = q^{(n-i)/2}e_{0\dots0}$, and so the vectors $v_i = q^{1/4}w_{i+1} - q^{-1/4}w_i$, $i = 1, \dots n-1$, form an ordered basis of $\ker(K - q^{(n-2)/2}) \cap \ker(E) \subseteq V_1^{\otimes n}$. A direct computation shows that

$$q^{-l/4}\rho_q^{n-2,n}(\sigma_1) = \begin{pmatrix} -q^{-1} & q^{-1/2} \\ 0 & 1 \end{pmatrix} \oplus \mathbf{1}^{n-3},$$

$$q^{-l/4}\rho_q^{n-2,n}(\sigma_j) = \mathbf{1}^{j-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ q^{-1/2} & -q^{-1} & q^{-1/2} \\ 0 & 0 & 1 \end{pmatrix} \oplus \mathbf{1}_{n-j-2}, \quad 1 < j < n-1,$$

$$q^{-l/4}\rho_q^{n-2,n}(\sigma_{n-1}) = \mathbf{1}^{n-3} \oplus \begin{pmatrix} 1 & 0 \\ q^{-1/2} & -q^{-1} \end{pmatrix}.$$

Conjugating by the diagonal matrix $q^{j/2}\delta_{jk}$ gives the desired result. □

Now, it is an interesting question whether or not the Jones representation is faithful for general B_n, but a natural place to start searching for elements of the kernel would be the kernel of the Burau representation, which is known not to be faithful for $n \geq 5$, [Big99]. Indeed, Bigelow produces an explicit non-trivial element of the kernel; by an explicit computer calculation using formulas obtained as in Section 6.4.1 we found that this particular elements in $\ker(\rho^{2,6})$ thus obtained is not in the kernel of $\rho^{0,6}$.

On the other hand, it is unknown whether or not the Burau representation is faithful for $n = 4$. It is however known [SKL02] (see also [LS05]) that for $n = 4$, any non-trivial element of the kernel of β_q must be pseudo-Anosov in B_4, that is, pseudo-Anosov as a mapping class of the four times punctured disk. Thus, in order to show that $\ker \rho_q^{2,4} \cap \ker\langle a, b, c\rangle$ is trivial, it will suffice to show that no element of $\langle a, b, c\rangle$ is pseudo-Anosov. Therefore it becomes very natural to study what information about pseudo-Anosovs is encaptured by $\rho_q^{2,4}$, optimally obtaining a result like [AMU06, Corollary 5.8] where it is shown that stretch factors of those braids descending to pseudo-Anosovs on the four times punctured sphere can be obtained from particular specializations of $\rho_q^{0,4}$.

Obtaining this for $\rho_q^{2,4}$ might be a bit much to hope for: obtaining any information for the entire class of pseudo-Anosovs, necessarily no pseudo-Anosov would be in the kernel of $\rho_q^{2,4}$ at all, which would then imply the faithfulness of the Burau representation in this case.

We end off this remark by giving two examples of what one might expect to happen.

Example 6.28. The braid $\varphi = \sigma_1\sigma_2^{-1} \in B_4$ descends to a pseudo-Anosov on the four times punctured sphere by [AMU06, Lemma 3.6] (as $\mathrm{tr}(h(\varphi)) = 3$, in the notation of [AMU06]). The eigenvalues of $\rho_{-1}^{2,4}(\varphi)$ are $(\lambda, \lambda^{-1}, 1)$, where λ is the stretch factor of φ.

The braid $\varphi = \sigma_1\sigma_2\sigma_3^{-1} \in B_4$ is the stretch factor minimizing pseudo-Anosov of B_4 (see [LT11]), but it does not descend to a pseudo-Anosov (as $\mathrm{tr}(h(\varphi)) = 2$), and the only eigenvalue of $\rho_{-1}^{2,4}(\varphi)$ is $\exp(\pi i/4)$. In [SKL02], the authors discuss further the specialization $q = -1$ and its relevance for the faithfulness of the Burau representation. In particular, they classify pseudo-Anosovs of B_4 into three classes, referred to as *trigoned*, *2-cusped*, or *(3+1)*, based on the type of their invariant foliations (see [SKL02, Sect. 3.2] for details), and argue that the stretch factor of a 2-cusped pseudo-Anosov is given by its spectral radius in β_{-1}, and that the kernel of β_{-1} consists of trigoned elements.

Indeed, inspired by this and with evidence based on computer search, it would seem that for any $\varphi \in B_4$, the eigenvalues of $\rho_{-1}^{0,4}$ are also eigenvalues of $\rho_{-1}^{2,4}$.

At this point, it is worth discussing the equivalent problem in the study of the mapping class group action on moduli spaces, as this provides us with a good hint of what the kernels might be. Recall the geometric proof of Asymptotic Faithfulness (cf. Theorem 4.11), where we saw that asymptotically, the projective kernels of the quantum representations were related to the kernels of the mapping class group action on the corresponding moduli space, which on the other hand we may understand through Teichmüller theory. Now, in the present case, the relevant moduli spaces are the level-dependent spaces described in Section 5.6.2, and so the analogy breaks down somewhat. On the other hand, recall from (6.12) that the representation $q^{l/4}\rho^{0,4}$ on Γ_0^4 has kernel $\langle \omega_1\omega_3^{-1}, (\omega_1\omega_2\omega_3)^2\rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This is interesting since this subgroup is exactly the kernel of the action of Γ_0^4 on its corresponding Teichmüller space (see [FM11, p. 344]). Now, as we will discuss in more detail below, this story generalizes to spheres with even numbers of punctures, and it is known that the kernel of the action of Γ_0^n on Teichmüller space is trivial for $n > 4$, so if one is able to prove that the projective kernel of the corresponding Jones representation $\rho_q^{0,2n}$ of Γ_0^{2n} is exactly the kernel of the Teichmüller space action, it follows immediately that

$$\ker_P(\rho_q^{0,2n}) = \langle \sigma_1 \cdots \sigma_{n-1}\sigma_{n-1}\cdots\sigma_1,\ (\sigma_1\cdots\sigma_{n-1})^n\rangle.$$

The corresponding statement for $\rho_q^{d,n}$ would be that the elements of the kernel would be those acting trivially on the moduli space for the $2n+1$ punctured sphere with holonomy conditions specified appropriately: $2n$ of the punctures have holonomies contained in the same SU(2) conjugacy class, and the holonomy a about the remaining puncture is allowed to vary, so that one obtains a one-parameter family of moduli spaces $\mathcal{M}_a$, all of which are homeomorphic whenever a is small but non-trivial (see [Jef94, Prop. 6.7]). Here, the relation between the projective kernels of the corresponding $\rho_q^{d(a),n}$ and the kernels of the action on $\mathcal{M}_a$ and the corresponding Teichmüller space is an interesting non-trivial question which we leave for a future study.

6.4.4 Jones representations for higher genus surfaces

Now, as was already noticed by Jones [Jon87, Sect. 10], one can also use the Jones representations of braid groups to obtain representations of particular subgroups of the mapping class group of a higher genus surface by using the fact that one can choose a system of Dehn twists satisfying the braid relations of Proposition 1.9. The construction goes as follows:

Jones showed that for n even, it is always possible to rescale the summands $\rho_q^{0,n}$ of the Jones representations of B_n (corresponding in the language of [Jon87] to the representations obtained by using the rectangular Young diagram consisting of 2 rows and $n/2$ columns) by a fractional power of q to obtain a representation $\tilde{\rho}_q$ of the quotient Γ_0^n.

Now on the other hand, we have seen how one might find a system of simple closed curves $\gamma_1, \dots, \gamma_{2g+1}$ on a genus g surface such that the Dehn twists about these curves satisfy $t_{\gamma_i}t_{\gamma_j}t_{\gamma_i} = t_{\gamma_j}t_{\gamma_i}t_{\gamma_j}$ for $|i-j| = 1$ and $t_{\gamma_i}t_{\gamma_j} = t_{\gamma_j}t_{\gamma_i}$ for $|i-j| > 1$. As we have already noticed, in the particular case of $g = 2$, these $2g+1$ Dehn twists actually generate the group. Indeed, the Birman–Hilden presentation (1.2) of Γ_2 shows that we obtain a well-defined homomorphism $\Gamma_2 \to \Gamma_0^6$ by mapping $t_{\gamma_i} \to \sigma_i$. More generally, the Birman–Hilden Theorem (see [FM11, Sect 9.4]) states that there is a short exact sequence

$$1 \to \mathbb{Z}_2 \to C_{\Gamma_g}(\iota) \xrightarrow{p} \Gamma_0^{2g+2} \to 1, \tag{6.14}$$

where $C_{\Gamma_g}(\iota)$ is the centralizer of the hyperelliptic involution ι in Γ_g, commonly referred to as the *symmetric mapping class group* or the *hyperelliptic mapping class group* and which for $g = 1, 2$ coincides with the entire group. The hyperelliptic mapping class groups are known to be linear by a result of Korkmaz, [Kor00].

For our purposes, we note that we obtain this way representations of $C_{\Gamma_g}(\iota)$, which we will also refer to as the *Jones representations*, through composition with the representations $\tilde{\rho}_q$ of Γ_0^{2g+2}. The case of $g = 1$ is exactly that covered by [AMU06]. We turn now to a brief discussion of the representations arising in the case $g = 2$.

Lemma 6.29. *The Jones representation $\tilde{\rho}_q$ of Γ_2 detect the stretch factors of orientable pseudo-Anosovs. Moreover, these have infinite order in $\tilde{\rho}_{-1}$.*

Proof. Recall first that by Theorem 1.22 that for orientable pseudo-Anosovs, the stretch factors are determined by their action on homology. Kasahara [Kas01, Lemma 2.1] noted that the representation $\tilde{\rho}_{-1}$ is equivalent to the action of Γ_2 on $\wedge^2 H_1(\Sigma_2,\mathbb{Z})/\omega\mathbb{Z}\otimes\mathrm{sgn}$, where ω is the algebraic intersection pairing, and $\mathrm{sgn} : t_{\gamma_i} \mapsto (-1)\mathrm{Id}$.

Now we simply note that if $\lambda_1 \geq \lambda_2 \geq \lambda_2^{-1} \geq \lambda_1^{-1}$ are the eigenvalues of the action on $H_1(\Sigma_2,\mathbb{Z})$ of an element φ of the family of mapping classes being considered, then the corresponding eigenvalues of the action of φ on $\wedge^2 H_1(\Sigma_2,\mathbb{Z})/\omega\mathbb{Z}$ are

$$\lambda_1\lambda_2 \geq \lambda_1\lambda_2^{-1} \geq 1 \geq \lambda_1^{-1}\lambda_2 \geq \lambda_1^{-1}\lambda_2^{-1},$$

and the stretch factor λ_1 of φ is given as the square root of the product of the two greatest eigenvalues of $\tilde{\rho}_{-1}(\varphi)$. □

Question 6.30. *Is there a similar homological interpretation of $\rho_{-1}^{0,2n}$ in general?*

Lemma 6.31. *The Birman–Hilden isomorphism $\Gamma_2/\langle\iota\rangle \to \Gamma_0^6$ of* (6.14) *preserves the Nielsen–Thurston types of mapping classes. That is, periodic, reducible and pseudo-Anosov mapping classes are mapped to periodic, reducible and pseudo-Anosov mapping classes respectively. Moreover, in the pseudo-Anosov case, the map is stretch factor preserving.*

Proof. First note that the statement is well-posed as composition with the involution does not change the Nielsen–Thurston type of a given mapping class. That finite order elements correspond to finite order elements is obvious (noting that a lift of a homeomorphism on Σ_0^6 might have double the order of the original homeomorphism). Let $\pi : \Sigma_2 \to \Sigma_0^6$ denote the projection given by the involution. If $f \in \Gamma_2$ is a homeomorphism lifting a periodic homeomorphism $p(f) \in \Gamma_0^6$ preserving a multicurve $\{\gamma_i\}$ in Γ_0^6 avoiding the branch points, then there is a family of disjoint curves $\{\tilde{\gamma}_i^{\pm}\}$ in Σ_2 with $\pi(\tilde{\gamma}_i^{\pm}) = \gamma_i$ such that either $f(\tilde{\gamma}_i^{\pm}) = \tilde{\gamma}_i^{\pm}$ or $f(\tilde{\gamma}_i^{\pm}) = \tilde{\gamma}_i^{\mp}$. This implies that f^2 is reducible.

That the image of a reducible map under p is reducible itself is the content of [Whi00, Thm. 5.1].

Now if $f \in \Gamma_2$ is pseudo-Anosov, then $p(f)$ must also be pseudo-Anosov by the Nielsen–Thurston classification and the above considerations (as also f^2 is pseudo-Anosov). That the lift of a pseudo-Anosov is pseudo-Anosov with the same stretch factor is a general fact (see [FM11, Sect. 11.3.3, Sect. 14.1.1, Sect. 14.2.1]). □

Theorem 6.32. *The AMU conjecture is true for the orientable pseudo-Anosov elements of Γ_0^6.*

Proof. The pseudo-Anosovs in question lift to orientable pseudo-Anosov homeomorphisms of Γ_2 (see [FM11, Sect. 11.3.3]). The result follows from Lemma 6.29 by the same argument as in [AMU06, Sect. 5] where it was shown how to obtain ρ_{-1} as a limit of the level k quantum SU(2)-representations as $k \to \infty$. □

More generally, the AMU conjecture holds true for any pseudo-Anosov of $\Gamma_{0,6}$ whose lift to Γ_2 has the property that its stretch factor is determined by its action on homology. A concrete class of examples for which this holds is the image $p(\mathcal{W})$ of the special pseudo-Anosovs $\mathcal{W} \subseteq \Gamma_2$ discussed in Section 1.3.1.

We end this section by remarking that it is an interesting problem to relate the faithful Lawrence–Krammer–Bigelow representations to the braid group representations obtained from the representation theory of quantum groups or equivalently from the study of the KZ equation (see [JK11] and [Koh12] for recent work in this direction).

Index

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