Irreducible quantum group modules with finite dimensional weight spaces

PhD Dissertation

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Abstract

We classify all irreducible weight modules for a quantized enveloping algebra $U_q(\mathfrak{g})$ at most $q \in \mathbb{C}^*$ when the simple Lie algebra $\mathfrak{g}$ is not of type G_2. More precisely, our classificiation is carried out when q is either an odd root of unity or transcendental over $\mathbb{Q}$.

By a weight module we mean a finitely generated U_q-module which has finite dimensional weight spaces and is a sum of those. Our approach follows the procedures used by S. Fernando and O. Mathieu to solve the corresponding problem for semisimple complex Lie algebra modules. To achieve this we have to overcome a number of obstacles not present in the classical case.

In the process we also construct twisting functors rigerously for quantum group modules, study twisted Verma modules and show that these admit a Jantzen filtration with corresponding Jantzen sum formula.

Resumé

Vi klassificerer alle irreducible vægtmoduler for en kvantiseret indhyldningsalgebra $U_q(\mathfrak{g})$ for de fleste $q \in \mathbb{C}^*$ når Lie algebraen $\mathfrak{g}$ ikke er af type G_2. Vi klassificerer de irreducible moduler når q er en ulige enhedsrod og når q er transcendent over $\mathbb{Q}$.

Når vi skriver vægtmodul mener vi et endelig frembragt U_q-modul som har endelig dimensionelle vægtrum og som er en sum af disse. Vores fremgangsmåde følger fremgangsmåderne som S. Fernando og O. Mathieu har brugt til at løse det tilsvarende problem for semisimple Lie algebra moduler. For at opnå dette må vi løse adskillige problemer som ikke opstår i det klassiske tilfælde.

I processen konstruerer vi også twisting funktorer stringent for kvantegruppemoduler, undersøger tvistede Verma moduler og viser at disse har en Jantzen filtration med tilhørende Jantzen sum formel.

Introduction and overview of the dissertation

This dissertation is the collection of three papers in the following called P0, P1 and P2. P0 refers to the paper "Twisting functors for quantum group modules". P1 and P2 refer to the papers "Irreducible quantum group modules with finite dimensional weight spaces" I and II, respectively. Each paper has its own page numbering and numbering of lemmas, propositions, theorems etc. To help the reader there is a header on each page with the title of the paper. The papers are all posted at arXiv.org and have been submitted to journals. The arXiv numbers of the papers can be seen in the references at the end of this introduction.

Let $\mathfrak{g}$ be a semi-simple Lie algebra and let $U_q = U_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra as defined in [Jan96, Chapter 4]. The main goal of the dissertation is to classify all simple weight U_q-modules. By a weight module we mean a module that is a sum of its weight spaces with all weight spaces being finite dimensional. We achieve the desired classification in the case when q is an odd root of unity in P1 and in the case when q is transcendental in P2.

In the classical case i.e. for semi-simple Lie algebras the corresponding problem was solved by S. Fernando and O. Mathieu in the papers [Fer90] and [Mat00]. Some of the work in these two papers can be translated directly to the quantized enveloping algebra world but several results need to be proved in different ways. In the first paper P0 we do some detailed calculations necessary for the rest of the results. Furthermore in this part we define Arkhipov's twisting functors for modules over the quantized enveloping algebra in both the root of unity and non-root of unity cases. We then use the twisting functors to construct so-called twisted Verma modules. In the classical case H. H. Andersen and N. Lauritzen describe these in [AL03]. Then in the non-root of unity case we show that we can construct a Jantzen-type filtration of the twisted Verma modules in analogy with [AL03, Theorem 7.1]. This is shown in [And03] for integral weights. Here we show it for any (not necessarily integral) weight. In the following we won't need the twisted Verma modules but we do need the Jantzen filtrations (whose construction in our approach relies on those twists) for ordinary Verma modules with arbitrary weights in the classification of the so called admissible simple modules. For ordinary (not twisted) Verma modules this is not a new result. An entirely different proof can be found in [Jos95, Section 4.1.2-4.1.3]. The results of P0 were presented in the author's progress report in connection with his qualifying exam in June 2013. The progress report was approved and the author recieved his masters degree in mathematics on the basis of the report and the following oral exam. The paper has been slightly rewritten since then.

We distinguish in most sections between whether q is a root of unity or not. In the paper P1 we show (for roots of unity and non roots of unity respectively) how to reduce the classification of simple weight modules to the classification of two classes of modules: Simple finite dimensional modules over a subalgebra of U_q corresponding to the quantized enveloping algebra of a reductive Lie algebra and so called simple 'torsion free modules' over the quantized enveloping algebra of a simple Lie algebra. This involves very crucially defining, for a root β of the root system Φ, the concept of a module being β-finite or β-free. A module M is β-finite if all root vectors corresponding to β (defined in P1) act nilpotently on M. On the other hand a module is β-free if all root vectors corresponding to β act injectively on M. This is in analogy with the procedures in [Fer90] but we have to approach some of the proofs differently. For example in [Fer90] many results are shown that depends on roots without specifying a specific set of simple roots and positive roots. Then in later results a clever choice of base for the root system is chosen. When defining the quantized enveloping algebra U_q of a Lie algebra $\mathfrak{g}$ we first fix a set of simple roots and define U_q by generators and relations by using the corresponding simple root vectors as generators and requiring some relations between them. So we can not later make a different choice of basis like in [Fer90]. In the dissertation we solve this problem by considering certain twists of modules by braid operators corresponding to appropiate elements of the Weyl group W. Another related problem is that for a positive root β that is not simple we don't a priori have root vectors E_β and F_β. We can construct root vectors for all $\beta \in \Phi$ but the construction involves a choice of a reduced expression of w_0 — the longest element in W. For another reduced expression we get possibly different root vectors. Fortunately, the results where we need general root vectors turn out not to depend on the choice but of course we need to verify that this is the case so there is something extra to show here compared to the classical case.

For some other results Fernando uses algebraic geometric arguments and these can not be directly quantized. Instead we use concrete calculations for root vectors. To do these calculations we rely on formulas proved by G. Lusztig in [Lus90] in rank 2. The rank 2 case where $\mathfrak{g}$ is of type G_2 is different from all the other finite types in that there are three lengths of roots. In the dissertation we have ignored type G_2 entirely. To make the classification complete one would of course need to consider G_2 as well and this particular case should in principle be doable by similar methods, although the calculations get possibly very tedious. Especially since in [Lus90] the commutation formulas needed in some of the results of the dissertation are only calculated for types other than G_2. So the results in the dissertation are about all other finite types i.e. type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, $n \in \mathbb{Z}_{>0}$. The author has not made any calculations for G_2 and has instead focused on all the other cases.

Since the finite dimensional simple modules are well known the above reduction leaves us with the problem to classify the simple torsion free $U_q(\mathfrak{g})$-modules when $\mathfrak{g}$ is a simple Lie algebra. In the root of unity case the classification of torsion free modules can further be reduced to the classification of the classical (Lie algebra) simple torsion free modules and some finite dimensional simple modules so in this case we can reduce completely to the classical case treated in [Mat00]. We do this in P1. The reduction involves defining an analog of the coherent families defined in [Mat00]. A coherent family in [Mat00] is a huge $\mathfrak{g}$-module having weight spaces of all possible weights and with some requirement on a trace being polynomial in some parameter, see [Mat00, Section 4]. We define the analog of this for the quantized enveloping algebra. We then show that every infinite dimensional admissible simple module is a submodule of an appropiate semisimple coherent family. Torsion free simple modules are a specific case of admissible infinite dimensional simple modules. We then show that every one of these 'quantized coherent families' is a tensor product of some finite dimensional module and a Frobenius twist of an appropiate classical coherent family. In this way we have reduced to the classical case.

In the non-root of unity case we can't use the same trick to reduce to the classical case so we have to do some more work. This is done in P2. We follow the procedure in [Mat00]. Namely, for a given admissible simple infinite dimensional module L we construct a so called semisimple irreducible coherent family $\mathcal{EXT}(L)$ containing L as a submodule and then we show that each such semisimple irreducible coherent family contains an infinite dimensional admissible simple highest weight module $L(\lambda)$ with highest weight some λ such that $\mathcal{EXT}(L) \cong \mathcal{EXT}(L(\lambda))$. We thus reduce the classification of infinite dimensional admissible simple modules to the classification of infinite dimensional admissible simple highest weight modules. We show that these exist only in type A and C and we classify the weights λ such that $L(\lambda)$ is admissible. The final problem is then, given an infinite dimensional admissible simple highest weight module $L(\lambda)$, to find out precisely which submodules of $\mathcal{EXT}(L(\lambda))$ are torsion free. We complete the classification by doing this in type A and C seperately.

In the case when q is not a root of unity we don't define the concept of a general coherent family. Instead we define directly the module $\mathcal{EXT}(L)$ given a specific admissible infinite dimensional simple module L. We then proceed like in [Mat00] to show the analogies of the results in [Mat00]. Especially in the final classification in type A and C there are major differences between our approach and the one in [Mat00]. Here we do very concrete calculations involving specific chosen root vectors to show for a specific weight λ and a specific 'set of commuting roots' Σ precisely which 'twists' of $L(\lambda)_{F_\Sigma}$ are torsion free. In [Mat00] the final classification in type A and C can be done by looking at weight spaces. That is Mathieu classifies for a given irreducible coherent family $\mathcal{M}$ for which $t \in \mathfrak{h}^*/Q$, $\mathcal{M}[t]$ is simple and torsion free. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $Q = \mathbb{Z}\Phi \subset \mathfrak{h}^*$ is the root lattice and $\mathcal{M}[t] = \oplus_{\lambda \in t} \mathcal{M}_\lambda$. The modules in the dissertation which are analogs of the irreducible coherent families are $\mathcal{EXT}(L)$ where L is an infinite dimensional admissible simple module. These modules are 'larger' than their classical analogs. Let X be the set of weights (defined in all 3 papers) and let $Q = \mathbb{Z}\Phi$ denote the root lattice. We show in an example in P2 that we can have a torsion free module and a simple highest weight module both included in $\mathcal{EXT}(L(\lambda))[t]$ where $t = q^Q \in X/q^Q$. So just looking at weight spaces of $\mathcal{EXT}(L(\lambda))$ will not be enough in our case. Instead we concretely calculate the actions of most simple root vectors on the 'twists' of $L(\lambda)_{F_\Sigma}$ defined to construct $\mathcal{EXT}(L(\lambda))$ and show which twists give rise to a simple torsion free module.

A more detailed overview of the contents of each paper is given in their individual introductions.

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List of notation

Here we make a list of some of the notation used in the three papers of the dissertation. We refer to the papers by P0, P1 and P2. P0 refers to the paper "Twisting functors for quantum group modules". P1 and P2 refer to the papers "Irreducible quantum group modules with finite dimensional weight spaces" I and II, respectively.

$\mathbb{N}$	Here $\mathbb{N}$ contains 0. $\mathbb{N} = \{0, 1, 2, \dots\}$.
$\mathfrak{g}$	A semi-simple Lie algebra. In some sections we require $\mathfrak{g}$ to be simple.
Φ	The root system for $\mathfrak{g}$.
$\Pi = \{\alpha_1, \dots, \alpha_n\}$	A fixed set of simple roots for Φ.
Φ^+, Φ^-	The positive/negative roots of Φ respectively corresponding to the fixed set of simple roots.
Q	The root lattice $Q = \mathbb{Z}\Phi$.
W, s_i	The Weyl group corresponding to $\mathfrak{g}$. s_i is the simple reflection $s_i := s_{\alpha_i}$.
w_0	The longest element in W.
$(\cdot\|\cdot)$	A standard W-invariant bilinear form on $\mathfrak{h}^*$.
$\langle \cdot, \cdot^\vee \rangle$	$\langle \alpha, \beta^\vee \rangle = \frac{(\alpha\|\beta)}{(\beta\|\beta)}$.
Λ	The integral lattice consisting of elements $\mu \in \mathfrak{h}^*$ such that $(\mu\|\alpha) \in \mathbb{Z}$ for all $\alpha \in \Pi$.
A	$A = \mathbb{Z}[v, v^{-1}]$.
$U_v = U_v(\mathfrak{g})$	The quantized enveloping algebra over $\mathbb{Q}(v)$.
v_α	$v_\alpha = v^{\frac{(\alpha\|\alpha)}{2}}$.
U_A	Lusztig's A-form. The A-subalgebra of U_v generated by the divided powers $E_\alpha^{(n)}$, $F_\alpha^{(n)}$, $n \in \mathbb{N}$ and $K_\alpha^{\pm 1}$, $\alpha \in \Pi$.
$\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix}$, $c \in \mathbb{Z}$, $r \in \mathbb{N}$	$\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix} = \prod_{j=1}^{r} \frac{K_\alpha v_\alpha^{c+j-1} - K_\alpha^{-1} v_\alpha^{-c-j+1}}{v_\alpha^j - v_\alpha^{-j}}$.
$\mathbb{C}^*$	$\mathbb{C}^* = \mathbb{C} \backslash \{0\}$.
U_q, $q \in \mathbb{C}^*$	The quantized enveloping algebra over $\mathbb{C}$, $U_q = U_A \otimes_A \mathbb{C}_q$ where $\mathbb{C}_q$ is an A-algebra by sending v to $q \in \mathbb{C}^*$.
q_α	$q_\alpha = q^{\frac{(\alpha\|\alpha)}{2}}$.
U_q^-, U_q^0, U_q^+	U_q^- is generated by $F_\alpha^{(n)}$, $n \in \mathbb{N}$, U_q^+ is generated by $E_\alpha^{(n)}$, $n \in \mathbb{N}$. U_q^0 is generated by $K_\alpha^{\pm 1}$ and $\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix}$, $c \in \mathbb{Z}$, $r \in \mathbb{N}$.
T_w	In P0: The twisting functor corresponding to a $w \in W$, see Definition 3.5 in P0. In P1 and P2: The braid operator corresponding to $w \in W$.
R_w	In P0: The braid operator corresponding to $w \in W$.
S_w	The semiregular bimodule corresponding to a $w \in W$, see Section 1 in P0.
$S_v(F)$	$S_v(F) = U_{v(F)}/U_v$ where $U_{v(F)}$ is the Ore localization in the Ore set $\{F^n \| n \in \mathbb{N}\}$. See Section 3 in P0.

$l(w)$	The length of $w \in W$.
Let F_β be a root vector corresponding to β	Choose a reduced expression of w_0, $w_0 = s_{i_1} \cdots s_{i_N}$. Set $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ and $F_{\beta_j} = T_{s_{i_1} \cdots s_{i_{j-1}}}(F_{\alpha_{i_j}})$, $j \in \{1, \dots, N\}$. Then $\beta = \beta_j$ for some $j \in \{1, \dots, N\}$ and $F_\beta = F_{\beta_j}$ is a root vector corresponding to β. If $\beta \notin \Pi$ then this construction depends on the reduced expression chosen for w_0.
$[x, y]_v$	If $x \in (U_v)_\mu$ and $y \in (U_v)_\nu$ then $[x, y]_v = xy - v^{-(\mu,\nu)} yx$.
$\operatorname{ad}(F_\beta^i)(u)$, $\widetilde{\operatorname{ad}}(F_\beta^i)(u)$	$\operatorname{ad}(F_\beta^i)(u) := [[\dots[u, F_\beta]_v \dots]_v, F_\beta]_v$ and $\widetilde{\operatorname{ad}}(F_\beta^i)(u) := [F_\beta, [\dots, [F_\beta, u]_v \dots]]_v$ where the 'v-commutator' is taken i times from the left and right respectively, see Definition 2.11 in P0.
$\operatorname{ad}(F_\beta^{(i)})(u)$, $\widetilde{\operatorname{ad}}(F_\beta^{(i)})(u)$	$\operatorname{ad}(F_\beta^{(i)})(u) = ([i]!)^{-1} \operatorname{ad}(F_\beta^i)(u)$ and $\widetilde{\operatorname{ad}}(F_\beta^{(i)})(u) = ([i]!)^{-1} \widetilde{\operatorname{ad}}(F_\beta^i)(u)$, see Proposition 2.12 and the comments after in P0.
$U_{q(F_\beta)}$	The Ore localization of U_q in the set $\{F_\beta^r \mid r \in \mathbb{N}\}$.
Λ_l	$\Lambda_l = \{\mu \in \Lambda \mid 0 \leq \langle \mu, \alpha^\vee \rangle < l, \text{ for all } \alpha \in \Pi\}$.
X	The set of weights: The set of algebra homomorphisms $U_q^0 \to \mathbb{C}$. After Section 3 in P1 we restrict to type **1** modules so in this case $X = \Lambda_l \times \mathfrak{h}^*$ (see Lemma 3.3 and the comments after in P1).
$\operatorname{wt} M$	The weights of a given U_q-module M.
$M(\lambda)$	The Verma module with highest weight $\lambda \in X$.
$L(\lambda)$	The unique simple quotient of $M(\lambda)$.
$\operatorname{ch} M$	The character of a module M. See the comments after Definition 3.6 in P0.
$w.\lambda$	The dot-action of $w \in W$ on $\lambda \in X$, see the comments after Definition 3.6 in P0.
${}^w M$, ${}^{\overline{w}} M$	Twist of a U_q-module M by $w \in W$. See Definition 3.4 in P0, Definition 1.2 in P1 or Definition 1.2 in P2.
$M^w(\lambda)$	A twisted Verma module. $M^w(\lambda) = T_w M(w^{-1}.\lambda)$, see Definition 3.8 in P0.
DM, M a U_v-module	D is the duality functor on $U_v - \operatorname{Mod}$, see the comments after Definition 3.8 in P0.
$F_\beta^{(-n)}$	In the non root of unity case $F_\beta^{(-n)} = [n]_\beta! F_\beta^{-n}$, see Section 4 in P0. In the root of unity case see the comment after Lemma 5.2 in P1.
$\mathcal{F}$	See Definition 1.3 in P1.

$M^{[\beta]}$, $\beta \in \Phi$	$M^{[\beta]} = \{m \in M \mid \dim \left\langle E_\beta^{(n)} \mid n \in \mathbb{N} \right\rangle m < \infty\}$ if $\beta \in \Phi^+$ and $M^{[\beta]} = \{m \in M \mid \dim \left\langle F_{-\beta}^{(n)} \mid n \in \mathbb{N} \right\rangle m < \infty\}$ if $\beta \in \Phi^-$. The definition is independent of the choice of root vector E_β or $F_{-\beta}$, see Definition 2.5 in P1 and the comments after.
β-finite	A U_q-module M is β-finite if $M^{[\beta]} = M$ (see Proposition 2.2 and Proposition 3.6 in P1).
β-free	A U_q-module M is β-free if $M^{[\beta]} = 0$.
T_M, F_M	$F_M = \{\beta \in \Phi \mid M^{[\beta]} = M\}$, $T_M = \{\beta \in \Phi \mid M^{[\beta]} = 0\}$, see Definition 2.7 in P1.
$\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{u}$, $\mathfrak{u}^-$, $\mathfrak{p}^-$	$\mathfrak{p}$ is a parabolic Lie subalgebra of $\mathfrak{g}$. $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ is the Levi part and $\mathfrak{u}$ is the nilpotent part. $\mathfrak{u}^-$ is the nilpotent part of the opposite parabolic subalgebra $\mathfrak{p}^-$. See the comments before Definition 2.12 in P1.
$U_q(\mathfrak{p})$, $U_q(\mathfrak{l})$, $U_q(\mathfrak{u})$, $U_q(\mathfrak{u}^-)$	See the comments before Definition 2.12 in P1.
$U_q(\tau)$, $U_q(\mathfrak{t})$	See the comments after Proposition 2.17 in P1.
$\mathcal{M}(N)$	If N is a $U_q(\mathfrak{l})$-module then $\mathcal{M}(X) = U_q \otimes_{U_q(\mathfrak{p})} N$, see Definition 2.12 in P1.
$L(N)$	If N is a $U_q(\mathfrak{l})$-module then $L(N)$ is the unique simple submodule of $\mathcal{M}(N)$, see Proposition 2.16 in P1.
$M^{\mathfrak{u}}$	If M is a U_q-module then $M^{\mathfrak{u}} = \{m \in M \mid xm = \varepsilon(x)m,\ x \in U_q(\mathfrak{u})\}$, see Definition 2.13 in P1.
Σ	A set of commuting roots, see Definition 5.5 in P1 or Definition 4.13 in P2.
F_Σ	An Ore subset corresponding to Σ, see Corollary 5.8 in P1 (for q an odd root of unity) or Definition 4.19 in P2 (for q a non root of unity).
$U_{q(F_\Sigma)}$	The Ore localization of U_q in the Ore set F_Σ.
$\operatorname{Supp}(L,\mu)$, $\operatorname{Supp}_{\text{ess}}(L,\mu)$	See Definition 6.5 in P1.
$\operatorname{Supp}(L)$	See Definition 4.21 in P2.
$C(L)$	The cone corresponding to a simple module L, see Definition 5.12 in P1 and Definition 4.1 in P2.
$\psi_{F_\Sigma,\nu}.M$	See Definition 5.9 in P1.
$\varphi_{F_\Sigma,\mathbf{b}}.M$	See Definition 4.19 in P2.
$M^{[l]}$	See Definition 5.16 in P1.
T^*	$T^* = \mathfrak{h}^*/Q$, see Definition 6.2 in P1.
M^{ss}	The semisimple module with the same composition factors as M.
$\mathcal{EXT}(L)$	See Proposition 6.7 in P1 (root of unity case) or the comments before Lemma 5.4 in P2 (non root of unity case).

Twisting functors for quantum group modules

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Abstract

We construct twisting functors for quantum group modules. First over the field $\mathbb{Q}(v)$ but later over any $\mathbb{Z}[v, v^{-1}]$-algebra. The main results in this paper are a rigerous definition of these functors, a proof that they satisfy braid relations and applications to Verma modules.

Keywords: Quantum Groups; Quantized Enveloping Algebra; Twisting Functors; Representation Theory; Jantzen Filtration; Twisted Verma Modules

1 Introduction

Twisting functors were first introduced by S. Arkhipov (as a preprint in 2001 and published in [Ark04]). H. Andersen quantized the construction of twisting functors in [And03]. Each twisting functor T_w is defined via a so called semi-regular bimodule S_v^w. By the definition in [And03] its right module structure is not clear. Our first goal is to demonstrate that S_v^w is in fact a bimodule. We verify this by constructing an explicit isomorphism to an inductively defined right module. The calculations are in fact rather complicated and involve several manipulations with root vectors, see Section 2 below. At the same time these calculations will be essential in [Ped15a] and [Ped15b].

Once we have established the definition of the twisting functors we prove that they satisfy braid relations, see Proposition 3.11. In the ordinary (i.e. non-quantum) case the corresponding result was obtained by O. Khomenko and V. Mazorchuck in [KM05]. Our approach is similar but again the quantum case involves new difficulties, see Section 3. This section also contains an explicit proof of the fact that, for the longest word $w_0 \in W$, the twisting functor T_{w_0} takes a Verma module to its dual, see Theorem 3.9,

The above results have several applications in the representation theory of quantum group: They enable us to construct so called twisted Verma modules and Jantzen filtrations of (twisted) Verma modules with arbitrary (non-integral) weights and to derive the sum formula for these. In turn this simplifies the linkage principle in quantum category $\mathcal{O}_q$, q being a non-root of unity in an arbitrary field.

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1.2 Notation

In this paper we work with a quantum group over a semisimple Lie algebra $\mathfrak{g}$ defined as in [Jan96]. Let Φ (resp. Φ^+ and Φ^-) denote the roots (resp. positive and negative roots) and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots. The quantum group has generators $\{E_\alpha, F_\alpha, K_\alpha | \alpha \in \Pi\}$ with relations as found in [Jan96]. Let $Q = \mathbb{Z}\Phi$ denote the root lattice. Let (a_{ij}) be the cartan matrix for $\mathfrak{g}$ and let $(\cdot|\cdot)$ be the standard invariant bilinear form. Let $\Lambda = \operatorname{span}_{\mathbb{Z}} \{\omega_1, \dots, \omega_n\} \subset \mathfrak{h}^*$ be the integral lattice where $\omega_i \in \mathfrak{h}^*$ is the fundamental weights defined by $(\omega_i|\alpha_j) = \delta_{ij}$. At first we work with the quantum group $U_v(\mathfrak{g})$ defined over $\mathbb{Q}(v)$ but later we will specialize to an abitrary field and any nonzero q in the field. This is done by considering Lusztig's A-form U_A where $A = \mathbb{Z}[v, v^{-1}]$, see Section 4. For any A-algebra R; $U_R = U_A \otimes_A R$. We will later need the automorphism ω of U_v and the antipode S defined as in [Jan96] along with the definition of quantum numbers $[n]_\beta$ and quantum binomial coefficients. We use the notation $E^{(r)} = \frac{E^r}{[r]!}$ and similarly for F. The Weyl group W is generated by the simple reflections $s_i = s_{\alpha_i}$. As usual we define for a weight $\mu \in \Lambda$ the weight space $(U_v)_\mu := \{u \in U_v | K_\alpha u = v^{(\alpha|\mu)} u \text{ for all } \alpha \in \Pi\}$. For a $\mu \in Q$, K_μ is defined as follows: $K_\mu = \prod_{i=1}^n K_{\alpha_i}^{a_i}$ if $\mu = \sum_{i=1}^n a_i \alpha_i$. There is a braid group action on the quantum group U_v usually denoted by T_{s_i} where s_i is the reflection with respect to the simple root α_i. In this paper we will reserve the T for twisting functors so we will call this braid group action R instead. That is we have automorphisms R_{s_i} such that

$$\begin{aligned}
R_{s_i} E_{\alpha_i} &= - F_{\alpha_i} K_{\alpha_i} \\
R_{s_i} E_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v_{\alpha_i}^{-s} E_{\alpha_i}^{(r)} E_{\alpha_j} E_{\alpha_i}^{(s)}, \text{ if } i \neq j \\
R_{s_i} F_{\alpha_i} &= - K_{\alpha_i}^{-1} E_{\alpha_i} \\
R_{s_i} F_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v_{\alpha_i}^{s} F_{\alpha_i}^{(s)} F_{\alpha_j} F_{\alpha_i}^{(r)}, \text{ if } i \neq j \\
R_{s_i} K_\mu &= K_{s_i(\mu)}.
\end{aligned}$$

Our definition of braid operators follows the definition in [Jan96]. Note that this definition differs slightly from the original definition in [Lus90] (cf. [Jan96, Warning 8.14]).

The inverse to R_{s_i} is given by

$$\begin{aligned}
R_{s_i}^{-1} E_{\alpha_i} &= - K_{\alpha_i}^{-1} F_{\alpha_i} \\
R_{s_i}^{-1} E_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v_{\alpha_i}^{-s} E_{\alpha_i}^{(s)} E_{\alpha_j} E_{\alpha_i}^{(r)}, \text{ if } i \neq j \\
R_{s_i}^{-1} F_{\alpha_i} &= - E_{\alpha_i} K_{\alpha_i} \\
R_{s_i}^{-1} F_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v_{\alpha_i}^{s} F_{\alpha_i}^{(r)} F_{\alpha_j} F_{\alpha_i}^{(s)}, \text{ if } i \neq j \\
R_{s_i}^{-1} K_\mu &= K_{s_i(\mu)}.
\end{aligned}$$

For $w \in W$ with a reduced expression $s_{i_1} \cdots s_{i_r}$, R_w is defined as $R_{s_{i_1}} \cdots R_{s_{i_r}}$. This is independent of the reduced expression of w. An important property of the braid operators is that if $\alpha_{i_1}, \alpha_{i_2} \in \Pi$ and $w(\alpha_{i_1}) = \alpha_{i_2}$ then $R_w(F_{\alpha_{i_1}}) = F_{\alpha_{i_2}}$. These properties are proved in Chapter 8 in [Jan96].

For a reduced expression $s_{i_1}\cdots s_{i_N}$ of w_0 we can make an ordering of all the positive roots by defining

$$\beta_j := s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j}), \quad j=1,\dots,N$$

In this way we get $\{\beta_1,\dots,\beta_N\}=\Phi^+$. We could just as well have used the opposite reduced expression $w_0 = s_{i_N}\cdots s_{i_1}$. In the following we will sometimes use the numbering $s_{i_1}\cdots s_{i_N}$ and sometimes the numbering $s_{i_N}\cdots s_{i_1}$. Note that if $w=s_{i_1}\cdots s_{i_r}$ and we expand this to a reduced expression $s_{i_1}\cdots s_{i_r}s_{i_{r+1}}\cdots s_{i_N}$ we get $\{\beta_1,\dots,\beta_r\}=\Phi^+\cap w(\Phi^-)$. We can define 'root vectors' $F_{\beta_j}, j=1,\dots,N$ by

$$F_{\beta_j} := R_{s_{i_1}}\cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}}).$$

Note that this definition depends on the chosen reduced expression. For a different reduced expression we might get different root vectors. As mentioned above if $\beta\in\Pi$ then the root vector F_β defined above is the same as the generator with the same notation (cf. e.g [Jan96, Proposition 8.20]) so the notation is not ambigious in this case. Let $w\in W$ and let $s_{i_r}\cdots s_{i_1}$ be a reduced expression of w. Define F_{β_j} by choosing a reduced expression $s_{i_1}\cdots s_{i_r}s_{i_{r+1}}\cdots s_N$ of w_0 starting with the reduced expression $s_{i_1}\cdots s_{i_r}$ of w^{-1}. We define a subspace $U_v^-(w)$ of U_v^- as follows:

$$U_v^-(w) := \operatorname{span}_{\mathbb{Q}(v)}\left\{F_{\beta_1}^{a_1}\cdots F_{\beta_r}^{a_r} | a_j\in\mathbb{N}\right\}$$

where $F_{\beta_j}=R_{s_{i_1}}\cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$ as before. The definition of $U_v^-(w)$ seems to depend on the reduced expression of w. But the subspace is independent of the chosen reduced expression. This is shown in [Jan96, Proposition 8.22]. We will show below that $U_v^-(w)$ is a subalgebra of U_v^- and that

$$U_v^-(w) = \operatorname{span}_{\mathbb{Q}(v)}\left\{F_{\beta_r}^{a_r}\cdots F_{\beta_1}^{a_1} | a_j\in\mathbb{N}\right\}.$$

For a subalgebra $N\subset U_v$ we define $N^* = \bigoplus_\mu N_\mu^*$ (i.e. the graded dual) with the action given by $(uf)(x)=f(xu)$ for $u\in U_v$, $f\in N^*$, $x\in N$. We define 'the semiregular bimodule' $S_v^w := U_v\otimes_{U_v^-(w)} U_v^-(w)^*$. Proving that this is a U_v-bimodule will be the first main result of this paper. We will show that there exists a right module structure on S_v^w such that as a right module S_v^w is isomorphic to $U_v^-(w)^*\otimes_{U_v^-(w)} U_v$.

2 Calculations with root vectors

Let $A=\mathbb{Z}[v,v^{-1}]$. Lusztigs A-form is defined to be the A subalgebra of U_v generated by the divided powers $E_{\alpha_i}^{(n)}$ and $F_{\alpha_i}^{(n)}$ for $n\in\mathbb{N}$ and $K_i^{\pm1}$.

We want to define $U_A^-(w) = \operatorname{span}_A\left\{F_{\beta_1}^{(a_1)}\cdots F_{\beta_r}^{(a_r)} | a_i\in\mathbb{N}\right\}$ where the F_{β_i} are defined from a reduced expression of w like earlier. We have $U_v^-(w_0)=U_v^-$ so we want a similar property over A: $U_A^-(w_0)=U_A^-$ where U_A^- is the A-subalgebra generated by $\{F_{\alpha_i}^{(n)} | n\in\mathbb{N}, i=1,\dots,n\}$. This is shown very similar to the way it is shown for U_v in [Jan96].

Lemma 2.1 *Assume $\mathfrak{g}$ does not contain any G_2 components:*

1. *The subspace* $U_A(w) := \operatorname{span}_A \left\{ F_{\beta_1}^{(a_1)} \cdots F_{\beta_r}^{(a_r)} | a_i \in \mathbb{N} \right\}$ *depends only on* w*, not on the reduced expression chosen for* w*.*

2. *Let* α *and* β *be two distinct simple roots. If* w *is the longest element in the subgroup of* W *generated by* s_α *and* s_b *then the span defined as before is the subalgebra of* U_A *generated by* $F_\alpha^{(a)}$ *and* $F_\beta^{(b)}$*,* $a, b \in \mathbb{N}$*.*

Proof. Claim 2. is shown on a case by case basis. We will show first that the second claim implies the first.

We show this by induction on $l(w)$. If $l(w) \leq 1$ then there is only one reduced expression of w and there is nothing to show. Assume $l(w) > 1$ and that w has two reduced expressions $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}$ and $w = s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_r}$. We can assume that we can get from one of the reduced expression to the other by an elementary braid move ($s_\alpha s_\beta \cdots = s_\beta s_\alpha \cdots$). Set $\alpha = \alpha_1$ and $\gamma = \gamma_1$.

If $\alpha = \gamma$, set $w' = s_\alpha w$. Then the subspace spanned by the elements as in the lemma is for both expressions equal to:

$$\left(\sum_{a \geq 0} F_\alpha^{(a)} \right) \cdot R_{s_\alpha}(U_A^-(w')) \tag{1}$$

If $\alpha \neq \gamma$ then the elementary move must take place at the beginning of the reduced expression for both reduced expressions. Let w'' be the longest element generated by s_α and s_γ then we must have $w = w''w'$ for some w' with $l(w'') + l(w') = l(w)$ and the reduced expression for w' in both reduced expressions are equal whereas the reduced expressions for w'' are the two possible combinations for the two different reduced expressions. So the span of the products is given by $U_A^-(w') R_{w''}(U_A^-(w''))$ which is independent of the reduced expression by the second claim.

We turn to the proof of the second claim: First assume we are in the simply laced case. Then $w = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$. Lets work with the reduced expression $s_\alpha s_\beta s_\alpha$. The other situation is symmetric by changing the role of α and β. We want to show that

$$B := \left\langle F_\alpha^{(n_1)}, F_\beta^{(n_2)} | n_1, n_2 \in \mathbb{N} \right\rangle_A = \operatorname{span}_A \left\{ F_\alpha^{(a_1)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(a_3)} | a_i \in \mathbb{N} \right\} =: V \tag{2}$$

where $F_{\alpha+\beta}^{(a)} = R_\alpha(F_\beta^{(a)})$. By [Lus90] section 5 we have that $F_{\alpha+\beta}^{(a)} \in U_A^-$ for all $a \in \mathbb{N}$ and we see that

$$F_\beta^{(k)} F_\alpha^{(k')} = \sum_{t,s \geq 0} (-1)^s v^{-tr-s} F_\alpha^{(r)} F_{\alpha+\beta}^{(s)} F_\beta^{(t)}$$

where the restrictions on the sum is $s+t = k'$ and $s+t = k$. Lusztig calculates for the E_α's but just use the anti-automorphism Ω (defined in Section 1 of [Lus90]) on the results to get the corresponding formulas for the F's. Also we get the $(-1)^s$ from the fact that (using the notation of [Lus90]) $E_{12} = -R_{\alpha_2}(E_{\alpha_1})$ because of the difference in the definition of the braid operators. Since $F_{\alpha+\beta}^{(a)} \in U_A^-$ we have that $V \subset B$. If we show that V is invariant by multiplication from the left with $F_\alpha^{(a)}$ and $F_\beta^{(a)}$ for all $a \in \mathbb{N}$ then we must have $B \subset V$. For $F_\alpha^{(a)}$ this is clear.

For $F_\beta^{(k)}$, $k \in \mathbb{N}$ we use the formula above:

$$\begin{aligned} F_\beta^{(k)} F_\alpha^{(a_1)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(a_3)} &= \sum_{t,s\geq 0} (-1)^s v^{-d(tr+s)} F_\alpha^{(r)} F_{\alpha+\beta}^{(s)} F_\beta^{(t)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(a_3)} \\ &= \sum_{t,s\geq 0} (-1)^s v^{-d(tr+s)+dta_2} F_\alpha^{(r)} F_{\alpha+\beta}^{(s)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(t)} F_\beta^{(a_3)} \\ &= \sum_{t,s\geq 0} (-1)^s v^{-d(tr+s)+dta_2} \begin{bmatrix} s+a_2 \\ s \end{bmatrix} \begin{bmatrix} t+a_3 \\ t \end{bmatrix} F_\alpha^{(r)} F_{\alpha+\beta}^{(s+a_2)} F_\beta^{(t+a_3)}. \end{aligned}$$

We see that $F_\beta^{(k)} V \subset V$ so $V = B$.

In the non simply laced case we have to use the formulas in [Lus90] section 5.3 (d)-(i) but the idea of the proof is the same. If there were similar formulas for the G_2 case it would be possible to show the same here. I do not know if similar formulas can be found in this case. The important part is just that if you 'v-commute' two of the 'root vectors' $F_{\beta_i}^{(k)}$ and $F_{\beta_j}^{(k')}$ you get something that is still in U_A. □

Lemma 2.2

$$U_A^-(w_0) = U_A^-$$

Proof. It is clear that $U_A^-(w_0) \subset U_A^-$. We want to show that $F_\alpha^{(k)} U_A^-(w_0) \subset U_A(w_0)$ for all $\alpha \in \Pi$.

$U_A^-(w_0)$ is independent of the chosen reduced expression so we can choose a reduced expression for w_0 such that s_α is the last factor. Then the first root vector F_{β_1} is equal to F_α. Then it is clear that $F_\alpha^{(k)} U_A^-(w_0) \subset U_A^-(w_0)$. Since this was for an abitrary simple root α the proof is finished. (This argument is sketched in the appendix of [Lus90].) □

Corollary 2.3 *We get a basis of U_A^- by the products of the form $F_{\beta_1}^{(a_1)} \cdots F_{\beta_N}^{(a_N)}$ where $a_1, \dots, a_N \in \mathbb{N}$.*

Corollary 2.4 $U_A^-(w) = U_v^-(w) \cap U_A$.

Proof. Assume the length of w is r and define for $k = (k_1, \dots, k_r) \in \mathbb{N}^r$

$$F^{(k)} = F_{\beta_1}^{(k_1)} \cdots F_{\beta_r}^{(k_r)}.$$

It is clear that $U_A^-(w) \subseteq U_v^-(w) \cap U_A$. Assume $x \in U_v^-(w) \cap U_A$. Since $x \in U_v^-(w)$ we have constants $c_k \in \mathbb{Q}(v)$, $k \in \mathbb{N}^r$ such that

$$x = \sum_{k \in \mathbb{N}^r} c_k F^{(k)}.$$

Assume the length of w_0 is N and denote for $n \in \mathbb{N}^N$, $F^{(n)}$ like above for w. $U_v^-(w) \cap U_A \subseteq U_v^-(w_0) \cap U_A = U_A^-(w_0)$ ($U_A^-(w_0) \subset U_v^-(w_0) \cap U_A$ clearly and $U_A^-(w_0)$ is invariant under multiplication by U_A^-.) so there exists $b_n \in A$, $n \in \mathbb{N}^N$ such that

$$x = \sum_{k \in \mathbb{N}^N} b_k F^{(k)}.$$

But then we have two expressions of x in $U_v^-(w)$ expressed as a linear combination of basis elements. So we must have that the multindieces b_k are zero on coordinates $\geq r$ and that all the c_k are actually in A. This proves the corollary. □

Definition 2.5 *Let $x \in (U_v)_\mu$ and $y \in (U_v)_\gamma$ then*

$$[x, y]_v := xy - v^{-(\mu|\gamma)} yx.$$

Proposition 2.6 *For $x_1 \in (U_v)_{\mu_1}$, $x_2 \in (U_v)_{\mu_2}$ and $y \in (U_v)_\gamma$ we have*

$$[x_1 x_2, y]_v = x_1 [x_2, y]_v + v^{-(\gamma|\mu_2)} [x_1, y]_v x_2$$

and

$$[y, x_1 x_2]_v = v^{-(\gamma|\mu_1)} x_1 [y, x_2]_v + [y, x_1]_v x_2.$$

Proof. Direct calculation. □

We have the following which corresponds to the Jacobi identity. Note that setting $v = 1$ recovers the usual Jacobi identity for the commutator.

Proposition 2.7 *for $x \in (U_v)_\mu$, $y \in (U_v)_\nu$ and $z \in (U_v)_\gamma$ we have*

$$[[x, y]_v, z]_v = [x, [y, z]_v]_v - v^{-(\mu|\nu)} [y, [x, z]_v]_v + v^{-(\nu|\mu+\gamma)} \left(v^{(\nu|\mu)} - v^{-(\nu|\mu)} \right) [x, z]_v y$$

Proof. Direct calculation. □

For use in the theorem below define:

Definition 2.8 *Let $A = \mathbb{Z}[v, v^{-1}]$ and let A' be the localization of A in* [2] *(and/or* [3]*) if the Lie algebra contains any B_n, C_n or F_4 part (resp. any G_2 part). Let $w \in W$ have a reduced expression $s_{i_r} \cdots s_{i_1}$. Define β_j and F_{β_j}, $i = 1, \cdots, r$ as above: $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ and $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$. We define*

$$U_{A'}^-(w) = \operatorname{span}_{A'} \left\{ F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} | a_1, \ldots, a_r \in \mathbb{N} \right\}$$

This subspace is independent of the reduced expression for w. This can be proved in the same way as Lemma 2.1 using the rank 2 calculations done in [Lus90].

The main tool that will be used in this project is the following theorem from [DP93, thm 9.3] originally from [LS91, Proposition 5.5.2]:

Theorem 2.9 *Let F_{β_j} and F_{β_i} be defined as above. Let $i < j$. Let $A = \mathbb{Z}[v, v^{-1}]$ and let A' be the localization of A in* [2] *(and/or* [3]*) if the Lie algebra contains any B_n, C_n or F_4 part (resp. any G_2 part). Then*

$$[F_{\beta_j}, F_{\beta_i}]_v = F_{\beta_j} F_{\beta_i} - v^{-(\beta_i|\beta_j)} F_{\beta_i} F_{\beta_j} \in \operatorname{span}_{A'} \left\{ F_{\beta_{i+1}}^{a_{i+1}} \cdots F_{\beta_{j-1}}^{a_{j-1}} \right\}$$

Proof. We shall provide the details of the proof sketched in [DP93]. The rank 2 case is handled in [Lus90]. Note that in [Lus90] we see that when $\mu = 2$ (in his notation) we get second divided powers and when $\mu = 3$ we get third divided powers. This is one reason why we need to be able to divide by [2] and [3].

So we assume the rank 2 case is proven. In particular we can assume there is no G_2 component. Let $k \in \mathbb{N}$, $k < j$. Then $[F_{\beta_j}, F_{\beta_k}] = R_{s_{i_1}} \cdots R_{s_{i_{k-1}}} [R_{s_{i_k}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}}), F_{\alpha_{i_k}}]_v$ so we can assume in the above that $i = 1$. We can then assume that $j > 2$ because otherwise we would be in the rank 2 case. We will show by induction over $l \in \mathbb{N}$ that

$$[F_{\beta_t}, F_{\beta_1}]_v = F_{\beta_t} F_{\beta_1} - v^{-(\beta_1|\beta_t)} F_{\beta_1} F_{\beta_t} \in \operatorname{span}_{A'} \left\{ F_{\beta_2}^{a_2} \cdots F_{\beta_{t-1}}^{a_{t-1}} \right\}$$

for all $1 < t \leq l$. The induction start $l = 2$ is the rank 2 case. Assume the induction hypothesis that

$$[F_{\beta_t}, F_{\beta_1}]_v = F_{\beta_t} F_{\beta_1} - v^{-(\beta_1|\beta_t)} F_{\beta_1} F_{\beta_t} \in \operatorname{span}_{A'} \left\{ F_{\beta_2}^{a_2} \cdots F_{\beta_{t-1}}^{a_{t-1}} \right\}$$

for $t \leq l$. We need to prove the result for $l+1$. We have $\beta_{l+1} = s_{i_1} \cdots s_{i_l}(\alpha_{i_{l+1}})$. Now define $i = i_l$ and $j = i_{l+1}$. Set $w = s_{i_1} \cdots s_{i_{l-1}}$. So $\beta_{l+1} = w s_i(\alpha_j)$ and $F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j})$. Define $\alpha = \alpha_{i_1}$. We need to show that

$$[R_w R_{s_i}(F_{\alpha_j}), F_\alpha]_v \in \operatorname{span}_{A'} \left\{ F_{\beta_2}^{a_2} \cdots F_{\beta_l}^{a_l} \right\}.$$

We divide into cases:

Case 1) $(\alpha_i|\alpha_j) = 0$: In this case $R_w R_{s_i}(F_{\alpha_j}) = R_w(F_{\alpha_j})$. Since $s_i s_j = s_j s_i$ there is a reduced expression for w_0 starting with $s_{i_1} \cdots s_{l-1} s_j s_i$. So the induction hypothesis gives us that $[R_w(F_{\alpha_j}), F_{\alpha_1}]_v$ can be expressed by linear combinations of ordered monomials involving only $F_{\beta_2} \cdots F_{\beta_{l-1}}$.

Case 2) $(\alpha_i|\alpha_j) = -1$ and $l(ws_j) > l(w)$: In this case $w s_i s_j(\alpha_i) = w(\alpha_j) > 0$ so there is a reduced expression for w_0 starting with $s_{i_1} \cdots s_{i_{l-1}} s_i s_j s_i = s_{i_1} \cdots s_{i_{l-1}} s_j s_i s_j$. So we have by induction that $[R_w(F_{\alpha_j}), F_\alpha]_v$ is a linear combination of ordered monimials only involving $F_{\beta_2} \cdots F_{\beta_{l-1}}$.

Observe that we have

$$\begin{aligned} F_{\beta_{l+1}} =& R_w R_{s_i}(F_{\alpha_j}) \\ =& R_w(F_{\alpha_j} F_{\alpha_i} - v F_{\alpha_i} F_{\alpha_j}) \\ =& R_w(F_{\alpha_j}) F_{\beta_l} - v F_{\beta_l} R_w(F_{\alpha_j}) \\ =& [R_w(F_{\alpha_j}), F_{\beta_l}]_v \end{aligned}$$

so by Proposition 2.7 we get

$$\begin{aligned} [F_{\beta_{l+1}}, F_\alpha]_v =& [[R_w(F_{\alpha_j}), F_{\beta_l}]_v, F_\alpha]_v \\ =& [R_w(F_{\alpha_j}), [F_{\beta_l}, F_\alpha]_v]_v - v^{-(w(\alpha_j)|\beta_l)} [F_{\beta_l}, [R_w(F_{\alpha_j}), F_\alpha]_v]_v \\ & + v^{-(\beta_l|\alpha + w(\alpha_j))} \left(v^{-1} - v\right) [R_w(F_{\alpha_j}), F_\alpha]_v F_{\beta_l}. \end{aligned}$$

By induction (and Proposition 2.6) $[R_w(F_{\alpha_j}), [F_{\beta_l}, F_\alpha]_v]_v$ and $[F_{\beta_l}, [R_w(F_{\alpha_j}), F_\alpha]_v]_v$ are linear combinations of ordered monomials containing only $F_{\beta_2}, \dots, F_{\beta_{l-1}}$ so we have proved this case.

Case 3) $(\alpha_i|\alpha_j) = -1$ and $l(ws_j) < l(w)$: In this case write $u = ws_j$. We claim $l(us_i) > l(u)$. Assume $l(us_i) < l(u)$ then

$$l(w) + 2 = l(w s_i s_j) = l(u s_j s_i s_j) = l(u s_i s_j s_i) < l(u) + 2 = l(w) + 1$$

A contradiction. So there is a reduced expression of w_0 starting with us_i. We have $F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_u(F_{\alpha_i})$ so we get

$$[F_{\beta_{l+1}}, F_\alpha]_v = [R_u(F_{\alpha_i}), F_\alpha]_v$$

Now we claim that either $u^{-1}(\alpha) = \alpha_j$ or $u^{-1}(\alpha) < 0$: Indeed $w^{-1}(\alpha) < 0$ so $u^{-1}(\alpha)$ is < 0 unless $w^{-1}(\alpha) = -\alpha_j$ in which case we get $u^{-1}(\alpha) = s_j w^{-1}(\alpha) = s_j(-\alpha_j) = \alpha_j$. If $\alpha = u(\alpha_j)$ we get

$$[R_u(F_{\alpha_i}), F_\alpha]_v = R_u([F_{\alpha_i}, F_{\alpha_j}]_v) = R_u(R_{s_j}(F_{\alpha_i})) = R_w(F_{\alpha_i}) = F_{\beta_l}$$

In the other case we know from induction that

$$[R_u(F_{\alpha_i}), F_\alpha]_v \in U^-_{A'}(u^{-1})$$

Now $U^-_{A'}(u^{-1}) \subset U^-_{A'}(s_j u^{-1}) = U^-_{A'}(w^{-1})$ so we get that $[R_u(F_{\alpha_i}), F_\alpha]_v$ can be expressed as a linear combination of monomials involving $F_\alpha = F_{\beta_1}$ and the terms $F_{\beta_2} \cdots F_{\beta_{l-1}}$. Assume that a monomial of the form $F_\alpha^m F_{\beta_2}^{a_2} \cdots F_{\beta_{l-1}}^{a_{l-1}}$ appears with nonzero coefficient. The weights of the left and right hand side must agree so we have $ws_i(\alpha_j) + \alpha = \sum_{k=2}^{l-1} a_k \beta_k + m\alpha$ or

$$ws_i(\alpha_j) = \sum_{k=2}^{l-1} a_k \beta_k + (m-1)\alpha$$

Since $w^{-1}(\beta_k) < 0$ for $k = 1, 2, \dots l-1$ (and $\alpha = \beta_1$) we get

$$\alpha_i + \alpha_j = w^{-1} w s_i(\alpha_j) = \sum_{k=2}^{l-1} a_k w^{-1}(\beta_k) + (m-1) w^{-1}(\alpha) < 0.$$

Which is a contradiction.

Case 4) $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, $(\alpha_i|\alpha_j) = -2$ and $l(ws_j) > l(w)$: Here we get

$$F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_w(F_{\alpha_j} F_{\alpha_i} - v^2 F_{\alpha_i} F_{\alpha_j}) = R_w(F_{\alpha_j}) F_{\beta_l} - v^2 F_{\beta_l} R_w(\alpha_j) = [R_w(F_{\alpha_j}), F_{\beta_l}]_v$$

From here the proof goes exactly as in case 2.

Case 5) $\langle \alpha_j, \alpha_i^\vee \rangle = -2$, and $l(ws_j) > l(w)$: First of all since $l(ws_j) > l(w)$ we can deduce that $l(ws_i s_j s_i s_j) = l(w) + 4$: We have $-\beta_{l+1} + 2ws_i s_j(\alpha_i) = ws_i s_j s_i(\alpha_j) = w(\alpha_j) > 0$ showing that we must have $ws_i s_j(\alpha_i) > 0$.

We have

$$F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_w(F_{\alpha_i} F_{\alpha_j}^{(2)} - vF_{\alpha_j} F_{\alpha_i} F_{\alpha_j} + v^2 F_{\alpha_j}^{(2)} F_{\alpha_i})$$

We claim that we have

$$R_{s_i}(F_{\alpha_j}) = \frac{1}{[2]} \left(R_{s_i} R_{s_j}(F_{\alpha_i}) F_{\alpha_i} - F_{\alpha_i} R_{s_i} R_{s_j}(F_{\alpha_i}) \right)$$

This is shown by a direct calculation. First note that

$$R_{s_i} R_{s_j}(F_{\alpha_i}) = R_{s_j}^{-1} R_{s_j} R_{s_i} R_{s_j}(F_{\alpha_i}) = R_{s_j}^{-1}(F_{\alpha_i}) = F_{\alpha_j} F_{\alpha_i} - v^2 F_{\alpha_i} F_{\alpha_j}$$

So

$$\begin{aligned} R_{s_i} R_{s_j}(F_{\alpha_i}) F_{\alpha_i} - F_{\alpha_i} R_{s_i} R_{s_j}(F_{\alpha_i}) =& F_{\alpha_j} F_{\alpha_i}^2 - v^2 F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} - F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} + v^2 F_{\alpha_i}^2 F_{\alpha_j} \\ =& F_{\alpha_j} F_{\alpha_i}^2 - v[2] F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} + v^2 F_{\alpha_i}^2 F_{\alpha_j} \\ =& [2] R_{s_i}(F_{\alpha_i}). \end{aligned}$$

Therefore

$$\begin{aligned} F_{\beta_{l+1}} =& \frac{1}{[2]} \left(R_w R_{s_i} R_{s_j}(F_{\alpha_i}) F_{\beta_l} - F_{\beta_l} R_w R_{s_i} R_{s_j}(F_{\alpha_i}) \right) \\ =& \frac{1}{[2]} [R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_{\beta_l}]_v \\ =& \frac{1}{[2]} [[R_w(F_{\alpha_j}), F_{\beta_l}]_v, F_{\beta_l}]_v \end{aligned}$$

By Proposition 2.7 and the above we get

$$\begin{aligned}[R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_\alpha]_v =&[[R_w(F_{\alpha_j}), F_{\beta_l}]_v, F_\alpha]_v \\ =&[R_w(F_{\alpha_j}), [F_{\beta_l}, F_\alpha]_v]_v - v^2[F_{\beta_l}, [R_w(F_{\alpha_j}), F_\alpha]_v]_v \\ &+v^{2-(\alpha|\beta_l)}\left(v^{-2}-v^2\right)[R_w(F_{\alpha_j}), F_\alpha]_v F_{\beta_l}\end{aligned}$$

which by induction is a linear combination of ordered monomials involving only $F_{\beta_2}, \dots, F_{\beta_l}$. Using Proposition 2.7 again we get

$$\begin{aligned}[2][F_{\beta_{l+1}}, F_\alpha]_v =&[[R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_{\beta_l}]_v, F_\alpha]_v \\ =&[R_w R_{s_i} R_{s_j}(F_{\alpha_i}), [F_{\beta_l}, F_\alpha]_v]_v - [F_{\beta_l}, [R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_\alpha]_v]_v\end{aligned}$$

which by induction and the above is a linear combination of ordered monomials involving only $F_{\beta_2}, \dots, F_{\beta_l}$.

Case 6) $(\alpha_i|\alpha_j) = -2$, $l(ws_j) < l(w)$ and $l(ws_j s_i) < l(ws_j)$: Set $u = ws_j s_i$. We claim $l(us_i) = l(us_j) > l(u)$. Indeed suppose the contrary then $l(w)+2 = l(ws_i s_j) = l(us_i s_j s_i s_j) < l(u)+4 = l(w)+2$. We reason like in case 3): We have $F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_u R_{s_i} R_{s_j} R_{s_i}(F_{\alpha_j}) = R_u(F_{\alpha_j})$. Now either $u^{-1}(\alpha) = \alpha_i$, $u^{-1}(\alpha) = s_i(\alpha_j)$ or $u^{-1}(\alpha) < 0$. If $u^{-1}(\alpha) < 0$ we get by induction that $[F_\alpha, R_u(F_{\alpha_j})]_v$ is in $U_{A'}^-(u^{-1}) \subset U_{A'}^-(w^{-1})$ and by essentially the same weight argument as in case 3) we are done.

If $\alpha = u(\alpha_i)$ then

$$\begin{aligned}[R_u(F_{\alpha_j}), F_\alpha]_v =&[R_u(F_{\alpha_j}), R_u(F_{\alpha_i})]_v \\ =&R_u(F_{\alpha_j}F_{\alpha_i} - v^2 F_{\alpha_i}F_{\alpha_j}) \\ =&\begin{cases} R_u R_{s_i}(F_{\alpha_j}) & \text{if } \langle \alpha_j, \alpha_i^\vee \rangle = -1 \\ R_u R_{s_i} R_{s_j}(F_{\alpha_i}) & \text{if } \langle \alpha_j, \alpha_i^\vee \rangle = -2 \end{cases}\end{aligned}$$

So $[F_\alpha, R_u(F_{\alpha_j})]_v \in U_{A'}^-(s_i s_j s_i u^{-1}) = U_{A'}^-(s_i w^{-1})$. Assume we have a monomial of the form $F_\alpha^m F_{\beta_2}^{a_2} \cdots F_{\beta_l}^{a_l}$ with m nonzero in the expression of $[R_u(F_{\alpha_j}), F_\alpha]_v$. Then

$$ws_i(\alpha_j) = \sum_{k=2}^{l} a_k \beta_k + (m-1)\alpha$$

and we get

$$\alpha_j = \sum_{k=2}^{l} a_k s_i w^{-1}(\beta_k) + (m-1)s_i w^{-1}(\alpha) < 0.$$

A contradiction.

If $\alpha = us_i(\alpha_j)$ then

$$\begin{aligned}[R_u(F_{\alpha_j}), F_\alpha]_v =&R_u[F_{\alpha_j}, R_{s_i}(F_{\alpha_j})]_v \\ =&R_u(F_{\alpha_j}R_{s_i}(F_{\alpha_j}) - v^{-2}R_{s_i}(F_{\alpha_i})F_{\alpha_j}) \\ =&R_u(R_{s_i}R_{s_j}R_{s_i}(F_{\alpha_j})R_{s_i}(F_{\alpha_j}) - v^{-2}R_{s_i}(F_{\alpha_j})R_{s_i}R_{s_j}R_{s_i}(F_{\alpha_j}))\end{aligned}$$

Which is in $U_{A'}^-(s_i s_j s_i u^{-1}) = U_{A'}^-(s_i w^{-1})$ by the rank 2 case. By the same weight argument as above we are done.

Case 7) $(\alpha_i|\alpha_j) = -2$, $l(ws_j) < l(w)$ and $l(ws_js_i) > l(ws_j)$: Set $u = ws_j$. Like in case 3) we get that either $u^{-1}(\alpha) = \alpha_j$ or $u^{-1}(\alpha) < 0$. If $\alpha = u(\alpha_j)$:

$$[F_{\beta_{l+1}}, F_\alpha]_v = R_u[R_{s_j}R_{s_i}(F_{\alpha_j}), F_{\alpha_j}]_v \in U^-_{A'}(s_is_ju^{-1}) = U^-_{A'}(s_iw^{-1})$$

And by a weight argument as above we are done.

If $u^{-1}(\alpha) < 0$ then $\alpha = \beta'_i$ for some $i \in \{1, \ldots, l-2\}$ where the β'_is are defined as above but using a reduced expression of u. Set $\beta'_{l-1} = u(\alpha_j)$, $\beta'_l = us_j(\alpha_i)$ and $\beta'_{l+1} = us_js_i(\alpha_j) = ws_i(\alpha_j) = \beta_{l+1}$. Then

$$[F_{\beta_{l+1}}, F_\alpha]_v = [F_{\beta'_{l+1}}, F_{\beta'_i}]_v \in U^-_{A'}(s_is_ju^{-1}) = U^-_{A'}(s_iw^{-1})$$

by induction and by a weight argument as above we are done. □

Lemma 2.10 *Let $w_0 = s_{i_1} \cdots s_{i_N}$ and let $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$ let $l, r \in \{1, \ldots, N\}$ with $l \leq r$. Then*

$$\operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_r}_{\beta_r} \cdots F^{a_l}_{\beta_l} | a_j \in \mathbb{N} \right\} = \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_l}_{\beta_l} \cdots F^{a_r}_{\beta_r} | a_j \in \mathbb{N} \right\}$$

and the subspace is invariant under multiplication from the left by F_{β_i}, $i = l, \ldots, r$.

Proof. If $r - l = 0$ the lemma obviously holds. Assume $r - l > 0$. For $k \in \mathbb{N}^{r-l}$, $k = (k_l, \ldots, k_r)$ let $F^k = F^{k_l}_{\beta_l} \cdots F^{k_r}_{\beta_r}$. We will prove the statement that $F^k \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_r}_{\beta_r} \cdots F^{a_l}_{\beta_l} | a_j \in \mathbb{N} \right\}$ by induction over $k_l + \cdots + k_r$. If $k = 0$ the statement holds. We have

$$F^k = F_{\beta_j} F^{k_j - 1}_{\beta_j} F^{k_{j+1}}_{\beta_{j+1}} \cdots F^{k_r}_{\beta_r}.$$

By induction $F^{k_j-1}_{\beta_j} F^{k_{j+1}}_{\beta_{j+1}} \cdots F^{k_r}_{\beta_r} \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_r}_{\beta_r} \cdots F^{a_l}_{\beta_l} | a_j \in \mathbb{N} \right\}$ so if we show that $F_{\beta_j} F^{b_r}_{\beta_r} \cdots F^{b_l}_{\beta_l} \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_r}_{\beta_r} \cdots F^{a_l}_{\beta_l} | a_i \in \mathbb{N} \right\}$ for all b_i, $i = l, \ldots r$ then we have shown the first inclusion.

We use downwards induction on j and induction on $b_1 + \cdots + b_r$. If $j = r$ then this is obviously true. If $j < r$ we use theorem 2.9 to conclude that

$$F_{\beta_r} F_{\beta_j} - v^{-(\beta_r|\beta_j)} F_{\beta_j} F_{\beta_r} \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_{r-1}}_{\beta_{r-1}} \cdots F^{a_{j+1}}_{\beta_{j+1}} | a_i \in \mathbb{N} \right\}$$

If $b_r = 0$ the induction over j finishes the claim. We get now if $b_r \neq 0$

$$F_{\beta_j} F^{b_r}_{\beta_r} \cdots F^{b_l}_{\beta_l} = v^{(\beta_r|\beta_j)} \left(F_{\beta_r} F_{\beta_j} F^{b_r - 1}_{\beta_r} \cdots F^{b_l}_{\beta_l} + \Sigma F^{b_r-1}_{\beta_r} \cdots F^{b_l}_{\beta_l} \right)$$

where $\Sigma \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_{r-1}}_{\beta_{r-1}} \cdots F^{a_{j+1}}_{\beta_{j+1}} | a_i \in \mathbb{N} \right\}$. By the induction on $b_r + \cdots + b_l$ $F_{\beta_j} F^{b_r-1}_{\beta_r} \cdots F^{b_l}_{\beta_l} \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_r}_{\beta_r} \cdots F^{a_l}_{\beta_l} | a_i \in \mathbb{N} \right\}$ and the induction on j ensures that $\Sigma F^{b_r-1}_{\beta_r} \cdots F^{b_l}_{\beta_l} \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_r}_{\beta_r} \cdots F^{a_l}_{\beta_l} | a_i \in \mathbb{N} \right\}$ since Σ contains only elements generated by $F_{\beta_{r-1}} \cdots F_{\beta_l}$.

We have now shown that

$$\operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_l}_{\beta_l} \cdots F^{a_r}_{\beta_r} | a_j \in \mathbb{N} \right\} \subset \operatorname{span}_{\mathbb{Q}(v)} \left\{ F^{a_r}_{\beta_r} \cdots F^{a_l}_{\beta_l} | a_j \in \mathbb{N} \right\}$$

The other inclusion is shown symmetrically. In the process we also proved that the subspace is invariant under left multiplication by F_{β_j}. □

Remark *The above lemma shows that $U_v^-(w)$ is an algebra.*

Definition 2.11 *Let $\beta \in \Phi^+$ and let F_β be a root vector corresponding to β. Let $u \in U_q$. Define $\operatorname{ad}(F_\beta^i)(u) := [[\dots[u, F_\beta]_v \dots]_v, F_\beta]_v$ and $\widetilde{\operatorname{ad}}(F_\beta^i)(u) := [F_\beta, [\dots, [F_\beta, u]_v \dots]]_v$ where the 'v-commutator' is taken i times from the left and right respectively.*

Proposition 2.12 *Let $u \in (U_A)_\mu$, $\beta \in \Phi^+$ and F_β a corresponding root vector. Set $r = \langle \mu, \beta^\vee \rangle$. Then in U_A we have the identity*

$$\operatorname{ad}(F_\beta^i)(u) = [i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(n)} u F_\beta^{(i-n)}$$

and

$$\widetilde{\operatorname{ad}}(F_\beta^i)(u) = [i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(i-n)} u F_\beta^{(n)}$$

Proof. This is proved by induction. For $i = 0$ this is clear. The induction step for the first claim:

$$\begin{aligned}
&[i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(n)} u F_\beta^{(i-n)} F_\beta \\
&- v_\beta^{-r-2i} F_\beta [i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(n)} u F_\beta^{(i-n)} \\
=&[i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)} [i+1-n] F_\beta^{(n)} u F_\beta^{(i+1-n)} \\
&- [i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)-r-2i} [n+1] F_\beta^{(n+1)} u F_\beta^{(i-n)} \\
=&[i]_\beta! \sum_{n=0}^{i+1} (-1)^n v_\beta^{n(-i-r)} \left(v_\beta^n [i+1-n] + v_\beta^{n-i-1} [n] \right) F_\beta^{(n)} u F_\beta^{(i+1-n)} \\
=&[i+1]_\beta! \sum_{n=0}^{i+1} (-1)^n v_\beta^{n(-i-r)} F_\beta^{(n)} u F_\beta^{(i+1-n)}.
\end{aligned}$$

The other claim is shown similarly by induction. □

So we can define $\operatorname{ad}(F_\beta^{(i)})(u) := ([i]!)^{-1} \operatorname{ad}(F_\beta^i)(u) \in U_A$ and $\widetilde{\operatorname{ad}}(F_\beta^{(i)})(u) := ([i]!)^{-1} \widetilde{\operatorname{ad}}(F_\beta^i)(u) \in U_A$.

Proposition 2.13 *Let $a \in \mathbb{N}$, $u \in (U_A)_\mu$ and $r = \langle \mu, \beta^\vee \rangle$. In U_A we have the identities*

$$\begin{aligned}
u F_\beta^{(a)} &= \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} F_\beta^{(a-i)} \operatorname{ad}(F_\beta^{(i)})(u) \\
&= \sum_{i=0}^{a} (-1)^i v_\beta^{a(r+i)-i} F_\beta^{(a-i)} \widetilde{\operatorname{ad}}(F_\beta^{(i)})(u)
\end{aligned}$$

and

$$
\begin{aligned}
F_\beta^{(a)}u =& \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} \widetilde{\mathrm{ad}}(F_\beta^{(i)})(u) F_\beta^{(a-i)} \\
=& \sum_{i=0}^{a} (-1)^i v_\beta^{a(r+i)-i} \,\mathrm{ad}(F_\beta^{(i)})(u) F_\beta^{(a-i)}
\end{aligned}
$$

Proof. This is proved by induction. For $a=0$ this is obvious. The induction step for the first claim:

$$
\begin{aligned}
[a+1]_\beta u F_\beta^{(a+1)} =& u F_\beta^{(a)} F_\beta \\
=& \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} F_\beta^{(a-i)} \,\mathrm{ad}(F_\beta^{(i)})(u) F_\beta \\
=& \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)-r-2i} [a+1-i]_\beta F_\beta^{(a+1-i)} \,\mathrm{ad}(F_\beta^{(i)})(u) \\
&+ \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} [i+1]_\beta F_\beta^{(a-i)} \,\mathrm{ad}(F_\beta^{(i+1)})(u) \\
=& \sum_{i=0}^{a} v_\beta^{(i-a-1)(r+i)-i} [a+1-i]_\beta F_\beta^{(a+1-i)} \,\mathrm{ad}(F_\beta^{(i)})(u) \\
&+ \sum_{i=1}^{a+1} v_\beta^{(i-a-1)(r+i-1)} [i]_\beta F_\beta^{(a+1-i)} \,\mathrm{ad}(F_\beta^{(i)})(u) \\
=& \sum_{i=0}^{a+1} v_\beta^{(i-a-1)(r+i)} \left(v_\beta^{-i} [a+1-i]_\beta + v_\beta^{a+1-i} [i] \right) F_\beta^{(a+1-i)} \,\mathrm{ad}(F_\beta^{(i)})(u) \\
=& [a+1]_\beta \sum_{i=0}^{a+1} v_\beta^{(i-a-1)(r+i)} F_\beta^{(a+1-i)} \,\mathrm{ad}(F_\beta^{(i)})(u).
\end{aligned}
$$

So the induction step for the first identity is done. The three other identities are shown similarly by induction. $\square$

Let $s_{i_1} \dots s_{i_N}$ be a reduced expression of w_0 and construct root vectors F_{β_i}, $i=1,\dots,N$. In the rest of the section F_{β_i} refers to the root vectors constructed as such. In particular we have an ordering of the root vectors.

Proposition 2.14 *Let $1 \leq i < j \leq N$ and $a,b \in \mathbb{Z}_{>0}$.*

$$
[F_{\beta_j}^b, F_{\beta_i}^a]_v \in \mathrm{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_i}^{a_i} \cdots F_{\beta_j}^{a_j} | a_l \in \mathbb{N}, a_i < a, a_j < b \right\}.
$$

Proof. From Theorem 2.9 we get the $a=1$, $b=1$ case. We will prove the general case by 2 inductions.

If $j-i=1$ then $[F_{\beta_j}, F_{\beta_i}^a]_v = 0$ for all a. We will use induction over $j-i$.

We have by Proposition 2.6 that

$$
[F_{\beta_j}, F_{\beta_i}^a]_v = v^{-((a-1)\beta_i|\beta_j)} F_{\beta_i}^{a-1} [F_{\beta_j}, F_{\beta_i}]_v + [F_{\beta_j}, F_{\beta_i}]_v F_{\beta_i}^{a-1}.
$$

The first term is in the correct subspace by Theorem 2.9. On the second we use the fact that $[F_{\beta_i}, F_{\beta_j}]_v$ only contains factors $F_{\beta_{i+1}}^{a_i} \cdots F_{\beta_{j-1}}^{a_{j-1}}$ and the induction over $j-i$ as well as induction over a to conclude that we can commute the $F_{\beta_i}^{a-1}$ to the correct place and be in the correct subspace.

Now just make a similar kind of induction on $i-j$ and b to get the result that

$$[F_{\beta_j}^b, F_{\beta_i}^a]_v \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_i}^{a_i} \cdots F_{\beta_j}^{a_j} | a_l \in \mathbb{N}, a_i < a, a_j < b \right\}. \qquad \square$$

Corollary 2.15 *Let $1 \leq i < j \leq N$ and $a, b \in \mathbb{Z}_{>0}$.*

$$[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v \in \operatorname{span}_A \left\{ F_{\beta_i}^{(a_i)} \cdots F_{\beta_j}^{(a_j)} | a_l \in \mathbb{N}, a_i < a, a_j < b \right\}.$$

Proof. Proposition 2.14 tells us that there exists $c_k \in \mathbb{Q}(v)$ such that

$$[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v = \sum_k c_k F_{\beta_i}^{(a_i^k)} \cdots F_{\beta_j}^{(a_j^k)}$$

with $a_i^k < a$ and $a_j^k < b$ for all k. But since $[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v \in U_A^-$ there exists $b_k \in A$ such that

$$[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v = \sum_k b_k F_{\beta_1}^{(a_1^k)} \cdots F_{\beta_N}^{(a_N^k)}.$$

Now we have two expressions of $[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v$ in terms of a basis of $U_{\mathbb{Q}(v)}^-$. So we must have that the c_k's are equal to the b_k's. Hence $c_k \in A$ for all k $\square$

Lemma 2.16 *Let $n \in \mathbb{N}$. Let $1 \leq j < k \leq N$.*
$\operatorname{ad}(F_{\beta_j}^{(i)})(F_{\beta_k}^{(n)}) = 0$ *and* $\widetilde{\operatorname{ad}}(F_{\beta_k}^{(i)})(F_{\beta_j}^{(n)}) = 0$ *for $i \gg 0$.*

Proof. We will prove the first assertion. The second is proved completely similar. We can assume $\beta_j = 1$ because

$$\operatorname{ad}(F_{\beta_j}^{(i)})(F_{\beta_k}^{(n)}) = T_{s_{i_1}} \cdots T_{s_{i_{j-1}}} \left(\operatorname{ad}(F_{\alpha_{i_j}}^{(i)})(T_{s_{i_j}} \cdots T_{s_{i_{k-1}}}(F_{\alpha_{i_k}}^{(n)})) \right).$$

So we assume $\beta_j = \beta_1 =: \beta \in \Pi$ and $\alpha := \beta_k = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_j}) \in \Phi^+$. We have

$$\operatorname{ad}(F_\beta)(F_\alpha^{(n)}) \in \operatorname{span}_A \left\{ F_{\beta_2}^{(a_2)} \cdots F_{\beta_k}^{(a_k)} | a_l \in \mathbb{N}, a_k < n \right\},$$

hence the same must be true for $\operatorname{ad}(F_\beta^{(i)})(F_\alpha^{(n)})$. By homogenity if the monomial $F_{\beta_2}^{(a_2)} \cdots F_{\beta_k}^{(a_k)}$ appears with nonzero coefficient then we must have

$$i\beta + n\alpha = \sum_{s=2}^{k} a_s \beta_s$$

or equivalently

$$(n - a_k)\alpha = \sum_{s=2}^{k-1} a_s \beta_s - i\beta.$$

Use s_β on this to get

$$(n - a_k) s_\beta(\alpha) = \sum_{s=2}^{s-1} a_s s_\beta(\beta_s) + i\beta.$$

By the way the β_s's are chosen $s_\beta(\beta_s) > 0$ for $1 < s < k$. So this implies that a positive multiple $(n - a_j)$ of a positive root must have $i\beta$ as coefficient. If we choose i greater than nd where d is the maximal possible coefficient of a simple root in any positive root then this is not possible. Hence we must have for $i > nd$ that $\mathrm{ad}(F_\beta^{(i)})(F_\alpha^{(n)}) = 0$. □

In the next lemma we will need to work with inverse powers of some of the F_β's. We know from e.g. [And03] that $\{F_\alpha^a | a \in \mathbb{N}\}$, $\alpha \in \Pi$ is a multiplicative set so we can take the Ore localization in this set. Since R_w is an algebra isomorphism of U_v we can also take the Ore localization in one of the 'root vectors' F_{β_j}. We will denote the Ore localization in F_β by $U_{v(F_\beta)}$.

Lemma 2.17 *Let $\beta \in \Phi^+$ and F_β a root vector. Let $u \in (U_v)_\mu$ be such that $\widetilde{\mathrm{ad}}(F_\beta^i)(u) = 0$ for $i \gg 0$. Let $a \in \mathbb{N}$ and set $r = \langle \mu, \beta^\vee \rangle$. Then in the algebra $U_{v(F_\beta)}$ we get*

$$uF_\beta^{-a} = \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-i-a} \widetilde{\mathrm{ad}}(F_\beta^i)(u)$$

and if $u' \in (U_v)_\mu$ is such that $\mathrm{ad}(F_\beta^i)(u') = 0$ for $i \gg 0$

$$F_\beta^{-a} u' = \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta \mathrm{ad}(F_\beta^i)(u') F_\beta^{-i-a}.$$

Proof. First we want to show that

$$\widetilde{\mathrm{ad}}(F_\beta^i)(u) F_\beta^{-1} = \sum_{k=i}^{\infty} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\mathrm{ad}}(F_\beta^k)(u). \tag{3}$$

Remember that $\widetilde{\mathrm{ad}}(F_\beta^k)(u) = 0$ for k big enough so this is a finite sum. This is shown by downwards induction on i. If i is big enough this is $0 = 0$. We have

$$F_\beta \widetilde{\mathrm{ad}}(F_\beta^i)(u) = \widetilde{\mathrm{ad}}(F_\beta^{i+1})(u) + v_\beta^{-r-2i} \widetilde{\mathrm{ad}}(F_\beta^i)(u) F_\beta$$

so

$$\begin{aligned}
\widetilde{\mathrm{ad}}(F_\beta^i)(u) F_\beta^{-1} &= F_\beta^{-1} \widetilde{\mathrm{ad}}(F_\beta^{i+1})(u) F_\beta^{-1} + v_\beta^{-r-2i} F_\beta^{-1} \widetilde{\mathrm{ad}}(F_\beta^i)(u) \\
&= \sum_{k=i+1}^{\infty} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\mathrm{ad}}(F_\beta^k)(u) + v_\beta^{-r-2i} F_\beta^{-1} \widetilde{\mathrm{ad}}(F_\beta^i)(u) \\
&= \sum_{k=i}^{\infty} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\mathrm{ad}}(F_\beta^k)(u).
\end{aligned}$$

Setting $i = 0$ in the above we get the induction start:

$$uF_\beta^{-1} = \sum_{k \geq 0} v_\beta^{-r-2k} F_\beta^{-k-1} \widetilde{\mathrm{ad}}(F_\beta^k)(u).$$

For the induction step assume

$$uF_\beta^{-a} = \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-i} \widetilde{\mathrm{ad}}(F_\beta^i)(u).$$

Then

$$
\begin{aligned}
uF_\beta^{-a-1} &= \sum_{i\geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-i} \widetilde{\operatorname{ad}}(F_\beta^i)(u) F_\beta^{-1} \\
&= \sum_{i\geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-i} \sum_{k\geq i} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\operatorname{ad}}(F_\beta^k)(u) \\
&= \sum_{k\geq 0}\sum_{i=0}^{k} v_\beta^{-(a+1)r-(a+1)i-2k} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-1-k} \widetilde{\operatorname{ad}}(F_\beta^k)(u) \\
&= \sum_{k\geq 0} v_\beta^{-(a+1)r-(a+2)k} \left(\sum_{i=0}^{k} v_\beta^{-(a+1)i+ak} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta \right) F_\beta^{-a-1-k} \widetilde{\operatorname{ad}}(F_\beta^k)(u).
\end{aligned}
$$

The induction is finished by observing that

$$
\begin{aligned}
\sum_{i=0}^{k} v_\beta^{-(a+1)i+ak} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta &= v_\beta^{ak} + \sum_{i=1}^{k} v_\beta^{-(a+1)i+ak} \left(v_\beta^{i} \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta - v_\beta^{a+i} \begin{bmatrix} a+i-1 \\ i-1 \end{bmatrix}_\beta \right) \\
&= v_\beta^{ak} + \sum_{i=1}^{k} v_\beta^{-ai+ak} \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta - \sum_{i=1}^{k} v_\beta^{-a(i-1)+ak} \begin{bmatrix} a+i-1 \\ i-1 \end{bmatrix}_\beta \\
&= v_\beta^{ak} + \sum_{i=1}^{k} v_\beta^{-ai+ak} \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta - \sum_{i=0}^{k-1} v_\beta^{-ai+ak} \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta \\
&= \begin{bmatrix} a+k \\ k \end{bmatrix}_\beta .
\end{aligned}
$$

The other identity is shown similarly by induction. □

Definition 2.18 *Let $\beta \in \Phi^+$ and let β be F_β a root vector. We define for $n \in \mathbb{N}$ in $U_{v(F_\beta)}$*

$$F_\beta^{(-n)} = [n]! F_\beta^{-n}$$

i.e. $F_\beta^{(-n)} = \left(F_\beta^{(n)}\right)^{-1}$.

Corollary 2.19 *Let $\beta \in \Phi^+$ and F_β a root vector. Let $u \in (U_v)_\mu$ be such that $\widetilde{\operatorname{ad}}(F_\beta^{(i)})(u) = 0$ for $i \gg 0$. Let $a \in \mathbb{N}$ and set $r = \langle \mu, \beta^\vee \rangle$. Then in the algebra $U_{v(F_\beta)}$ we get*

$$uF_\beta^{(-a)}F_\beta^{-1} = \sum_{i\geq 0} v_\beta^{-(a+1)r-(a+2)i} F_\beta^{(-i-a)} F_\beta^{-1} \widetilde{\operatorname{ad}}(F_\beta^{(i)})(u)$$

and if $u' \in (U_v)_\mu$ is such that $\operatorname{ad}(F_\beta^{(i)})(u') = 0$ for $i \gg 0$

$$F_\beta^{(-a)}F_\beta^{-1}u' = \sum_{i\geq 0} v_\beta^{-(a+1)r-(a+2)i} \operatorname{ad}(F_\beta^{(i)})(u') F_\beta^{(-i-a)} F_\beta^{-1}.$$

3 Twisting functors

In this paper we are following the paper [And03] closely. The definition of twisting functors for quantum group modules given later and the ideas in this section are mostly coming from this paper.

We will start by showing that the semiregular bimodule S_v^w is a bimodule isomorphic to $U_v^-(w)^* \otimes_{U_v^-(w)} U_v$ as a right module.

Recall how $U_v(w)$, S_v^w and $S_v(F)$ are defined: Let $s_{i_r} \cdots s_{i_1}$ be a reduced expression for w and $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$ as usual then

$$U_v^-(w) = \mathrm{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} | a_i \in \mathbb{N} \right\},$$

$$S_v^w = U_v \otimes_{U_v^-(w)} U_v^-(w)^*$$

and for $F \in U_v^-$ such that $\{F^a | a \in \mathbb{N}\}$ is a multiplicative set

$$S_v(F) = U_{v(F)}/U_v$$

where $U_{v(F)}$ denotes the Ore localization in the multiplicative set $\{F^a | a \in \mathbb{N}\}$.

In the following proposition we will define a left U_v isomorphism between S_v^w and $S_v(F_{\beta_r}) \otimes_{U_v} S_v^{w'}$ where $w' = s_{i_r} w$. We will need some notation. Let $m \in \mathbb{N}$. We denote by $f_m^{(r)} \in (\mathbb{Q}(v)[F_{\beta_r}])^*$ the linear function defined by $f_m^{(r)}(F_{\beta_r}^a) = \delta_{m,a}$. We will drop the (r) from the notation in most of the following. For $g \in U_v^-(w')^*$ we define $f_m \cdot g$ to be the linear function defined by: For $x \in U_v^-(w')$, $(f_m \cdot g)(x F_{\beta_r}^a) = f_m(F_{\beta_r}^a) g(x)$. From the definition of $U_v^-(w)$ and because we are taking *graded* dual every $f \in U_v^-(w)^*$ is a linear combination of functions on the form $f_m \cdot g$ for some $m \in \mathbb{N}$ and $g \in U_v^-(w')$ (by induction this implies that every function in $U_v^-(w)$ is a linear combination of functions of the form $f_{m_r}^{(r)} \cdots f_{m_2}^{(2)} \cdot f_{m_1}^{(1)}$ for some $m_1, \ldots, m_r \in \mathbb{N}$). Note that the definition of f_m makes sense for $m < 0$ but then $f_m = 0$.

Proposition 3.1 *Assume $w = s_{i_k} \cdots s_{i_1} = s_{i_k} w'$, where k is the length of w, then as a left U_v module*

$$S_v^w \cong S_v(F_{\beta_k}) \otimes_{U_v} S_v^{w'}$$

by the following left U_v isomorphism

$$\varphi_k : S_v^w \to S_v(F_{\beta_k}) \otimes_{U_v} S_v^{w'}$$

defined by:

$$\varphi_k(u \otimes f_m \cdot g) = u F_{\beta_k}^{-m-1} K_{\beta_k} \otimes (1 \otimes g), \quad u \in U_v, m \in \mathbb{N}, g \in U_v^-(w')^*.$$

The inverse to φ_k is the left U_v-homomorphism $\psi_k : S_v(F_{\beta_k}) \otimes_{U_v} S_v^{w'} \to S_v^w$ given by:

$$\psi_k(u F_{\beta_k}^{-m} \otimes (1 \otimes g)) = v^{(m\beta_k|\beta_k)} u K_{\beta_k}^{-1} \otimes f_{m-1} \cdot g, \quad u \in U_v . m \in \mathbb{N}, g \in U_v^-(w')^*.$$

Proof. The question is if φ_k is welldefined. Let $f = f_m \cdot g$. We need to show that the recipe for $u F_{\beta_j} \otimes f$ is the same as the recipe for $u \otimes F_{\beta_j} f$ for $j = 1, \ldots, k$.

For $j=k$ this is easy to see. Assume from now on that $j<k$. We need to figure out what $F_{\beta_j} f$ is. We have by Proposition 2.13 (setting $r=\langle \beta_j,\beta_k^\vee\rangle$)

$$\begin{aligned}(F_{\beta_j} f)(xF_{\beta_k}^a) &= f(xF_{\beta_k}^a F_{\beta_j})\\ &= f\left(x\sum_{i=0}^{a} v_\beta^{(i-a)(r+i)}\begin{bmatrix} a\\ i\end{bmatrix}_\beta \widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})F_{\beta_k}^{a-i}\right)\\ &= \left(\sum_{i=0}^{a} v_\beta^{-m(r+i)}\begin{bmatrix} m+i\\ i\end{bmatrix}_\beta f_{m+i}\cdot\left(\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})g\right)\right)(xF_{\beta_k}^a)\\ &= \left(\sum_{i\geq 0} v_\beta^{-m(r+i)}\begin{bmatrix} m+i\\ i\end{bmatrix}_\beta f_{m+i}\cdot\left(\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})g\right)\right)(xF_{\beta_k}^a)\end{aligned}$$

so

$$F_{\beta_j} f = \sum_{i\geq 0} v_\beta^{-m(r+i)}\begin{bmatrix} m+i\\ i\end{bmatrix}_\beta f_{m+i}\cdot\left(\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})g\right).$$

Note that the sum is finite because of Lemma 2.16.

On the other hand we have that $uF_{\beta_j}\otimes f$ is sent to (using Lemma 2.17)

$$\begin{aligned}&uF_{\beta_j}F_{\beta_k}^{-m-1}K_{\beta_k}\otimes(1\otimes g)\\ &= u\sum_{i\geq 0} v_{\beta_k}^{-(m+1)r-(m+2)i}\begin{bmatrix} m+i\\ i\end{bmatrix}_\beta F_{\beta_k}^{-i-m-1}\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})K_{\beta_k}\otimes(1\otimes g)\\ &= u\sum_{i\geq 0} v_{\beta_k}^{-mr-mi}\begin{bmatrix} m+i\\ i\end{bmatrix}_\beta F_{\beta_k}^{-i-m-1}K_{\beta_k}\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})\otimes(1\otimes g).\end{aligned}$$

Using the fact that $\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})$ can be moved over the first and the second tensor we see that the two expressions $uF_{\beta_j}\otimes f$ and $u\otimes F_{\beta_j} f$ are sent to the same.

So φ_k is a welldefined homomorphism. It is clear from the construction that φ_k is a U_v homomorphism.

We also need to prove that ψ_k is welldefined. We prove that $uF_{\beta_k}^{-m}F_{\beta_j}\otimes(1\otimes g)$ is sent to the same as $uF_{\beta_k}^{-m}\otimes(1\otimes F_{\beta_j}g)$ by induction over $k-j$. If $j=k-1$ we see from Lemma 2.17 and Theorem 2.9 that $F_{\beta_{k-1}}F_{\beta_k}^{-a}=v^{-(a\beta_k|\beta_{k-1})}F_{\beta_k}^{-a}F_{\beta_{k-1}}$ and therefore $uF_{\beta_k}^{-m}F_{\beta_{k-1}}\otimes(1\otimes g)$ is sent to

$$\begin{aligned}&v^{(m\beta_k-\beta_j|\beta_k)+(m\beta_k|\beta_{k-1})}uK_{\beta_k}^{-1}F_{\beta_{k-1}}\otimes f_{m-1}\cdot g\\ &= v^{(m\beta_k+(m-1)\beta_{k-1}\beta_k)}uK_{\beta_k}^{-1}\otimes F_{\beta_{k-1}}(f_{m-1}\cdot g).\end{aligned}$$

Note that because we have $\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j})=0$ for all $i\geq 1$ we get $F_{\beta_{k-1}}(f_{m-1}\cdot g)=v^{-(\beta_{k-1}|(m-1)\beta_k)}f_{m-1}\cdot(F_{\beta_{k-1}}g)$. Using this we see that $uF_{\beta_k}^{-m}F_{\beta_{k-1}}\otimes(1\otimes g)$ is sent to the same as $uF_{\beta_k}^{-m}\otimes(1\otimes F_{\beta_{k-1}}g)$.

Now assume $j-k>1$. To calculate what $uF_{\beta_k}^{-m}F_{\beta_j}\otimes(1\otimes g)$ is sent to we need to calculate $F_{\beta_k}^{-m}F_{\beta_j}$. By Lemma 2.17

$$F_{\beta_k}^{-m}F_{\beta_j} = v^{mr}F_{\beta_j}F_{\beta_k}^{-m} - \sum_{i\geq 1} v_\beta^{-(m+1)i}\begin{bmatrix} m+i-1\\ i\end{bmatrix}_\beta F_{\beta_k}^{-m-i}\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(u).$$

So

$$uF_{\beta_k}^{-m}F_{\beta_j}\otimes(1\otimes g)=u\Bigg(v_\beta^{mr}F_{\beta_j}F_{\beta_k}^{-m} - \sum_{i\geq 1} v_\beta^{-(m+1)i}\begin{bmatrix} m+i-1 \\ i \end{bmatrix}_\beta F_{\beta_k}^{-m-i}\widetilde{\mathrm{ad}}(F_{\beta_k}^i)(u)\Bigg)\otimes(1\otimes g).$$

By the induction over $k-j$ (remember that $\widetilde{\mathrm{ad}}(F_{\beta_k}^i)(u)$ is a linear combination of ordered monomials involving only the elements $F_{\beta_{j+1}}\cdots F_{\beta_{k-1}}$) this is sent to the same as

$$u\left(v_\beta^{mr}F_{\beta_j}F_{\beta_k}^{-m}\otimes(1\otimes g)-\sum_{i\geq 1}v_\beta^{-(m+1)i}\begin{bmatrix} m+i-1 \\ i \end{bmatrix}_\beta F_{\beta_k}^{-m-i}\otimes(1\otimes\widetilde{\mathrm{ad}}(F_{\beta_k}^i)(u)g)\right)$$

which is sent to

$$\begin{aligned}
&u\Bigg(v_\beta^{mr+2m}F_{\beta_j}K_{\beta_k}^{-1}\otimes f_{m-1}\cdot g \\
&\qquad - K_{\beta_k}^{-1}\otimes\sum_{i\geq 1}v_\beta^{2(m+i)-(m+1)i}\begin{bmatrix} m+i-1 \\ i \end{bmatrix}_\beta \otimes f_{m+i-1}\cdot(\widetilde{\mathrm{ad}}(F_{\beta_k}^i)(u)g)\Bigg)\\
&= v_\beta^{2m}uK_{\beta_k}^{-1}\Bigg(v_\beta^{(m-1)r}F_{\beta_j}\otimes f_{m-1}\cdot g \\
&\qquad -1\otimes\sum_{i\geq 1}v_\beta^{-(m-1)i}\begin{bmatrix} m+i-1 \\ i \end{bmatrix}_\beta\otimes f_{m+i-1}\cdot(\widetilde{\mathrm{ad}}(F_{\beta_k}^i)(u)g)\Bigg)\\
&= v^{(m\beta_k|\beta_k)}uK_{\beta_k}^{-1}\otimes f_{m-1}\cdot(F_{\beta_j}g).
\end{aligned}$$

But this is what $uF_{\beta_k}^{-m}\otimes(1\otimes F_{\beta_j}g)$ is sent to. We have shown by induction that ψ_k is well defined. It is easy to check that ψ_k is the inverse to φ_k. □

Proposition 3.2 *Let $s_{i_r}\cdots s_{i_1}$ be a reduced expression of $w\in W$. There exists an isomorphism of left U_v-modules*

$$S_v^w\cong S_v(F_{\beta_r})\otimes_{U_v}\cdots\otimes_{U_v}S_v(F_{\beta_1})$$

Proof. The proof is by induction of the length of w. Note that $S_v^e=U_v\otimes_k k^*\cong U_v$ so Proposition 3.1 with $w'=e$ gives the induction start.

Assume the length of w is $r>1$. By Proposition 3.1 we have $S_v^w\cong S_v(F_{\beta_r})\otimes_{U_v}S_v^{w'}$. By induction $S_v^{w'}\cong S_v(F_{\beta_{r-1}})\otimes_{U_v}\cdots\otimes_{U_v}S_v(F_{\beta_1})$. This finishes the proof. □

We can now define a right action on S_v^w by the isomorphism in Proposition 3.2. By first glance this might depend on the chosen reduced expression for w. But the next proposition proves that this right action does not depend on the reduced expression chosen.

Proposition 3.3 *As a right U_v module $S_v^w\cong U_v^-(w)^*\otimes_{U_v}U_v$.*

Proof. All isomorphisms written in this proof are considered to be right U_v isomorphisms. This is proved in a very similar way to Proposition 3.1. We will sketch the proof here.

For $l \in \{1, \dots, N\}$ define $S_v^l = (U_v^l)^* \otimes_{U_v^l} U_v$ where $U_v^l = \operatorname{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_l}^{a_l} \cdots F_{\beta_r}^{a_r} | a_i \in \mathbb{N} \right\}$. Note that $S_v^1 = U_v^-(w)^* \otimes_{U_v} U_v$. We want to show that $(U_v^l)^* \otimes_{U_v^l} U_v \cong S_v^{l+1} \otimes_{U_v} S_v(F_{\beta_l})$. If we prove this we will have $S_v^1 \cong S_v^2 \otimes_{U_v} S_v(F_{\beta_1}) \cong \cdots\cdots \cong S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1}) \cong S_v^w$ as a right module and we are done.

Let $r = \langle \beta_j, \beta_l^\vee \rangle$. From Proposition 2.13 we have

$$F_{\beta_j} F_{\beta_l}^a = \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} \begin{bmatrix} a \\ i \end{bmatrix}_\beta F_{\beta_l}^{a-i} \operatorname{ad}(F_{\beta_l}^i)(F_{\beta_j})$$

and by Lemma 2.17 we have

$$F_{\beta_l}^{-a} F_{\beta_j} = \sum_{i \geq 0} v_{\beta_l}^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_{\beta_l} \operatorname{ad}(F_{\beta_l}^i)(F_{\beta_r}) F_{\beta_l}^{-i-a}.$$

We define the right homomorphism φ_l from $(U_v^l)^* \otimes_{U_v^l} U_v$ to $S_v^{l+1} \otimes_{U_v} S_v(F_{\beta_l})$ by

$$\varphi_l(g \cdot f_{m_l} \otimes u) = (g \otimes 1) \otimes K_{\beta_l} F_{\beta_l}^{-m_l-1} u.$$

Like in the previous propisition we can use the above formulas to show that this is well defined and we can define an inverse like in the previous proposition only reversed. The inverse is:

$$\psi_l((g \otimes 1) \otimes F_{\beta_l}^{-m-1} u) = v^{-((m+1)\beta_l|\beta_l)} g \cdot f_m \otimes K_{\beta_l}^{-1} u. \qquad \square$$

So we have now that S_v^w is a bimodule isomorphic to $U_v \otimes_{U_v^-(w)} U_v^-(w)^*$ as a left module and isomorphic to $U_v^-(w)^* \otimes_{U_v^-(w)} U_v$ as a right module. We want to examine the isomorphism between these two modules. For example what is the left action of K_α on $f \otimes 1 \in (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$.

Assume $f = f_{m_r}^{(r)} \cdots f_{m_1}^{(1)}$ i.e. that $f(F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r}) = \delta_{m_1,a_1} \cdots \delta_{m_r,a_r}$. Then we get via the isomorphism $(U_v^-(w))^* \otimes_{U_v^-(w)} U_v \cong S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1})$ that $f \otimes u$ is sent to

$$K_{\beta_r} F_{\beta_r}^{-m_r-1} \otimes \cdots \otimes K_{\beta_1} F_{\beta_1}^{-m_1-1} u.$$

We want to investigate what this is sent to under the isomorphism $S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1}) \cong U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$. To do this we need to commute u with $F_{\beta_1}^{-m_1-1}$, then $F_{\beta_2}^{-m_2-1}$ and so on. So we need to find $\widetilde{u}$ and $m_1', \dots, m_r'$ such that

$$K_{\beta_r} F_{\beta_r}^{-m_r-1} \cdots K_{\beta_1} F_{\beta_1}^{-m_1-1} u = \widetilde{u} K_{\beta_r} F_{\beta_r}^{-m_r'-1} \cdots K_{\beta_1} F_{\beta_1}^{-m_1'-1}$$

or equivalently

$$u F_{\beta_1}^{m_1'+1} K_{\beta_1}^{-1} \cdots F_{\beta_r}^{m_r'+1} K_{\beta_r}^{-1} = F_{\beta_1}^{m_1+1} K_{\beta_1}^{-1} \cdots F_{\beta_r}^{m_r+1} K_{\beta_r}^{-1} \widetilde{u}.$$

Assume we have found such $\widetilde{u}$ and $m_1', \dots, m_r'$ then the above tensor is sent to

$$v^{\sum_{i=1}^r ((m'+1)\beta_i|\beta_i)} \widetilde{u} \otimes \widetilde{f}$$

where $\widetilde{f} = f^{(r)}_{m'_r} \cdots f^{(1)}_{m'_1}$. So in conclusion we have that $f \otimes u \in (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$ maps to $v^{\sum_{i=1}^r ((m'+1)\beta_i|\beta_i)} \widetilde{u} \otimes \widetilde{f} \in U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$ where $\widetilde{f}$ and $\widetilde{u}$ are defined as above.

We have a similar result the other way: $u \otimes f \in U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$ maps to $v^{-\sum_{i=1}^r ((m+1)\beta_i|\beta_i)} \overline{u} \otimes \overline{f} \in (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$. So if we want to figure out the left action of u on a tensor $f \otimes 1$ we need to first use the isomorpism $(U_v^-(w))^* \otimes_{U_v^-(w)} U_v \to U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$ then use u on this and then use the isomorphism $U_v \otimes_{U_v^-(w)} (U_v^-(w))^* \to (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$ back again.

In particular if $u = K_\alpha$ we have $\overline{f} = f$ and $\overline{u} = v^{\sum_{i=1}^r ((m_i+1)\beta_i|\beta_i)} K_\alpha$. Note that if $f = f^{(m_r)}_{m_r} \cdots f^{(1)}_{m_1}$ then the grading of f is $\sum_{i=1}^r m_i\beta_i$ so $K_\alpha(f \otimes 1) = v^{\left(\gamma + \sum_{i=1}^r \beta_i|\alpha\right)} f \otimes K_\alpha$ for $f \in (U_v^-(w))^*_\gamma$.

Definition 3.4 *Let $w \in W$. For a U_v-module M define a 'twisted' version of M called ${}^wM$. The underlying space is M but the action on ${}^wM$ is given by: For $m \in M$ and $u \in U_v$*

$$u \cdot m = R_{w^{-1}}(u)m.$$

Note that if $w, s \in W$ and $l(sw) > l(w)$ then ${}^s({}^wM) = {}^{sw}M$ since for $u \in U_v$ and $m \in {}^s({}^wM)$: $u \cdot m = R_s(u) \cdot m = R_{w^{-1}}(R_s(u))m = R_{(sw)^{-1}}(u)m$.

Definition 3.5 *The twisting functor T_w associated to an element $w \in W$ is the following:*

$T_w : U_v-\mathrm{Mod} \to U_v-\mathrm{Mod}$ *is an endofunctor on* $U_v-\mathrm{Mod}$*. For a* U_v*-module* M*:*

$$T_wM = {}^w(S_v^w \otimes_{U_v} M).$$

Definition 3.6 *Let M be a U_v-module and $\lambda : U_v^0 \to \mathbb{Q}(v)$ a character (i.e. an algebra homomorphism into $\mathbb{Q}(v)$). Then*

$$M_\lambda = \{m \in M | \forall u \in U_v^0, um = \lambda(u)m\}.$$

Let X denote the set of characters. Let $\operatorname{wt} M$ denote all the weights of M, i.e. $\operatorname{wt} M = \{\lambda \in X | M_\lambda \neq 0\}$. We define for $\mu \in \Lambda$ the character v^μ by $v^\mu(K_\alpha) = v^{(\mu|\alpha)}$. We also define $v^\mu_\beta = v^{\frac{(\beta|\beta)}{2}\mu}$. We say that M only has integral weights if all its weights are of the form v^μ for some $\mu \in \Lambda$.

W acts on X by the following: For $\lambda \in X$ define $w\lambda$ by

$$(w\lambda)(u) = \lambda(R_{w^{-1}}(u)).$$

Note that $wv^\mu = v^{w(\mu)}$.

We will also need the dot action. It is defined as such: For a weight $\mu \in X$ and $w \in W$, $w.\mu = v^{-\rho} w(v^\rho \mu)$ where $\rho = \frac{1}{2}\sum_{\beta \in \Phi} \beta$ as usual. The Verma module $M(\lambda)$ for $\lambda \in X$ is defined as $M(\lambda) = U_v \otimes_{U_v^{\geq 0}} \mathbb{Q}(v)_\lambda$ where $\mathbb{Q}(v)_\lambda$ is the onedimensional module with trivial U_v^+ action and U_v^0 action by λ (i.e. $K_\mu \cdot 1 = \lambda(K_\mu)$). $M(\lambda)$ is a highest weight module generated by $v_\lambda = 1 \otimes 1$.

Note that $R_{w^{-1}}$ sends a weight space of weight μ to the weight space of weight $w(\mu)$ since if we have a vector m with weight μ in a module M we get in ${}^wM$ that

$$K_\alpha \cdot m = R_{w^{-1}}(K_\alpha)m = K_{w^{-1}(\alpha)}m = v^{(w^{-1}(\alpha)|\mu)}m = v^{(\alpha|w(\mu))}m.$$

We define the character of a U_v-module M as usual: The character is a map $\operatorname{ch} M : X \to \mathbb{N}$ given by $\operatorname{ch} M(\mu) = \dim M_\mu$. Let e^μ be the delta function $e^\mu(\gamma) = \delta_{\mu,\gamma}$. We will write $\operatorname{ch} M$ as the formal infinite sum

$$\operatorname{ch} M = \sum_{\mu \in X} \dim M_\mu e^\mu.$$

For more details see e.g. [Hum08]. Note that if we define $w(\sum_\mu a_\mu e^\mu) = \sum_\mu a_\mu e^{w(\mu)}$ then $\operatorname{ch} {}^w M = w(\operatorname{ch} M)$ by the above considerations.

Proposition 3.7

$$\operatorname{ch} T_w M(\lambda) = \operatorname{ch} M(w.\lambda)$$

Proof. To determine the character of $T_w M(\lambda)$ we would like to find a basis. We will do this by looking at some vectorspace isomporphisms to a space where we can easily find a basis. Then use the isomorphisms back again to determine what the basis looks like in $T_w M(\lambda)$. So assume $w = s_{i_r} \cdots s_{i_1}$ is a reduced expression for w. Expand to a reduced expression $s_{i_N} \cdots s_{i_{r+1}} s_{i_r} \cdots s_{i_1}$ for w_0. Let $U_v^w = \operatorname{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_{r+1}}^{a_{r+1}} \cdots F_{\beta_N}^{a_N} | a_i \in \mathbb{N} \right\}$. Set $k = \mathbb{Q}(v)$. We have the canonical vector space isomorphisms

$$\begin{aligned} U_v^-(w)^* \otimes_{U_v^-(w)} U_v \otimes_{U_v} U_v \otimes_{U_v^{\geq 0}} k_\lambda \cong & U_v^-(w)^* \otimes_{U_v^-(w)} U_v \otimes_{U_v^{\geq 0}} k_\lambda \\ \cong & U_v^-(w)^* \otimes_k U_v^w \otimes_k k_\lambda. \end{aligned}$$

The map from the last vectorspace to the first is easily seen to be $f \otimes u \otimes 1 \mapsto f \otimes u \otimes 1 \otimes 1 = f \otimes u \otimes v_\lambda$, $f \in U_v^-(w)^*$, $u \in U_v^w$ and $v_\lambda = 1 \otimes 1 \in U_v \otimes_{U_v^{\geq 0}} k_\lambda = M(\lambda)$ is a highest weight vector in $M(\lambda)$.

So we see that a basis of $T_w M(\lambda) = {}^w(U_v^-(w)^* \otimes_{U_v^-(w)} U_v \otimes_{U_v} M)$ is given by the following: Choose a basis $\{f_i\}_{i \in I}$ for $U_v^-(w)^*$ and a basis $\{u_j\}_{j \in J}$ for U_v^w. Then a basis for $T_w M(\lambda)$ is given by

$$\{f_i \otimes u_j \otimes v_\lambda\}_{i \in I, j \in J}.$$

So we can find the weights of $T_w M(\lambda)$ by examining the weights of $f \otimes u \otimes v_\lambda$ for $f \in U_v^-(w)^*$ and $u \in U_v^w$. By the remarks before this proposition we have that $K_\alpha(f \otimes 1) = v^{(\gamma + \sum_{i=1}^r \beta_i | \alpha)} f \otimes K_\alpha$ for $f \in U_v^-(w)^*_{v^\gamma}$ so for such f and for $u \in (U_v^w)_{v^\mu}$ the weight of $f \otimes u \otimes v_\lambda$ is $v^{\gamma + \mu + \sum_{i=1}^r \beta_i} \lambda$. After the twist with w the weight is $v^{w(\gamma+\mu)} w.\lambda$. The weights γ and μ are exactly such that $w(\gamma) < 0$ and $w(\mu) < 0$ so we see that the weights of $T_w M(\lambda)$ are $\{v^\mu w.\lambda | \mu < 0\}$ each with multiplicity $\mathcal{P}(\mu)$ where $\mathcal{P}$ is Kostant's partition function. This proves that the character is the same as the character for the Verma module $M(w.\lambda)$. □

Definition 3.8 *Let $\lambda \in X$ and $M(\lambda)$ the Verma module with highest weight λ. Let $w \in W$. We define*

$$M^w(\lambda) = T_w M(w^{-1}.\lambda).$$

Recall the duality functor $D : U_v - \text{Mod} \to U_v - \text{Mod}$. For a U_v module M, $DM = \operatorname{Hom}(M, \mathbb{Q}(v))$ is the graded dual module with action given by $(xf)(m) = f(S(\omega(m)))$ for $x \in U_v$, $f \in DM$ and $m \in M$. By this definition we have $\operatorname{ch} DM = \operatorname{ch} M$ and $D(DM) = M$.

Theorem 3.9 *Let w_0 be the longest element in the Weyl group. Let $\lambda \in X$. Then*

$$T_{w_0} M(\lambda) \cong DM(w_0.\lambda)$$

Proof. We will show that $DT_{w_0}M(w_0.\lambda) \cong M(\lambda)$ by showing that $DT_{w_0}M(w_0.\lambda)$ is a highest weight module with highest weight λ. We already know that the characters are equal by Proposition 3.7 so all we need to show is that $DT_{w_0}M(w_0.\lambda)$ has a highest weight vector of weight λ that generates the whole module over U_v. Consider the function $g_\lambda \in DM^{w_0}(\lambda)$ given by:

$$g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \ldots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = \begin{cases} 1 & \text{if } a_N = \cdots = a_1 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $F_{\beta_N}^{-a_N-1} \otimes \ldots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}$ with $a_i \in \mathbb{N}$ defines a basis for $M^{w_0}(\lambda)$ so this defines a function on $M^{w_0}(\lambda)$. In the proof of Proposition 3.7 we see that a basis is given by $f \otimes 1 \otimes v_\lambda \in U_v^-(w_0) \otimes U_v \otimes M(\lambda) = T_{w_0}M(\lambda)$. We know that elements of the form $f_{m_N}^{(N)} \cdots f_{m_1}^{(1)}$ defines a basis of $(U_v^-)^* = U_v^-(w_0)^*$. Under the isomorphisms of Proposition 3.3 $f_{m_N}^{(N)} \cdots f_{m_1}^{(1)} \otimes 1 \otimes v_{w_0.\lambda}$ is sent to

$$K_{\beta_N} F_{\beta_N}^{-m_N-1} \otimes \cdots \otimes K_{\beta_1} F_{\beta_1}^{-m_1-1} \otimes v_{w_0.\lambda} \in S_v(F_{\beta_N}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1}) \otimes_{U_v} M(w_0.\lambda).$$

If we commute all the K's to the right to the v_λ we get some non-zero multiple of

$$F_{\beta_N}^{-m_N-1} \otimes \cdots \otimes F_{\beta_1}^{-m_1-1} \otimes v_{w_0.\lambda}.$$

So we have shown that $\{F_{\beta_N}^{-m_N-1} \otimes \cdots \otimes F_{\beta_1}^{-m_1-1} \otimes v_{w_0.\lambda} | m_i \in \mathbb{N}\}$ is a basis of $M^{w_0}(\lambda)$.

The action on a dual module DM is given by $uf(u') = f(S(\omega(u)u'))$. Remember that the action on $M^{w_0}(\lambda)$ is twisted by R_{w_0} so we get that

$$ug_\lambda(F_{\beta_N}^{-a_N-1} \otimes \ldots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = g_\lambda(R_{w_0}(S(\omega(u)))F_{\beta_N}^{-a_N-1} \otimes \ldots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}).$$

In particular for $u = K_\mu$ we get

$$\begin{aligned} K_\mu g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \ldots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) &= g_\lambda(K_{w_0(\mu)} F_{\beta_N}^{-a_N-1} \otimes \ldots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \\ &= v^c(w_0.\lambda)(K_{w_0(\mu)}) g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \ldots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \end{aligned}$$

where

$$c = (w_0(\mu)| \sum_{i=1}^N a_i \beta_i + \sum_{i=1}^N \beta_i).$$

we have

$$\begin{aligned} v^c(w_0.\lambda)(K_{w_0(\mu)}) =& v^{(w_0(\mu)| \sum_{i=1}^N a_i\beta_i + \sum_{i=1}^N \beta_i)} \left(v^{-\rho} w_0(v^\rho \lambda)\right) (K_{w_0(\mu)}) \\ =& v^{(w_0(\mu)| \sum_{i=1}^N a_i\beta_i + 2\rho)} v^{-(\rho | w_0(\mu))} (v^\rho \lambda)(K_\mu) \\ =& v^{(w_0(\mu)| \sum_{i=1}^N a_i\beta_i + \rho)} v^{(\rho|\mu)} \lambda(K_\mu) \\ =& v^{(w_0(\mu)| \sum_{i=1}^N a_i\beta_i + \rho)} v^{-(\rho | w_0(\mu))} \lambda(K_\mu) \\ =& v^{(w_0(\mu)| \sum_{i=1}^N a_i\beta_i)} \lambda(K_\mu). \end{aligned}$$

Setting the a_i's equal to zero we get $\lambda(K_\mu)$. So g_λ has weight λ. We want to show that g_λ generates $DM^{w_0}(\lambda)$ over U_v.

Let $M \in \mathbb{N}^N$, $M = (m_1, \dots, m_N)$. An element in $DM^{w_0}(\lambda)$ is a linear combination of elements of the form g_M defined by:

$$g_M(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = \delta_{a_1,m_1} \cdots \delta_{a_N,m_N}.$$

This is because of the way the dual module is defined (as the graded dual). We want to show that $g_M \in U_v g_\lambda$ by using induction over $m_1 + \cdots m_N$. Note that $g_{(0,\dots,0)} = g_\lambda$ so this gives the induction start. Assume $M = (m_1, \dots, m_N) \in \mathbb{N}^N$. Let j be such that $m_N = \cdots = m_{j+1} = 0$ and $m_j > 0$. By induction we get for $M' = (0, \dots, 0, m_j - 1, m_{j-1}, \dots, m_1)$ that $g_{M'} \in U_v g_\lambda$. Now let $u_j = \omega(S^{-1}(R_{w_0}^{-1}(F_{\beta_j})))$. Then

$$u_j g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = g_\lambda(F_{\beta_j} F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}).$$

From Lemma 2.17 we get for $r > j$ (setting $k = \langle \beta_j, \beta_r^\vee \rangle$)

$$F_{\beta_j} F_{\beta_r}^{-a} = v_{\beta_r}^{-ak} F_{\beta_r}^{-a} + \sum_{i \geq 1} v_{\beta_r}^{-ak-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_{\beta_r} F_{\beta_r}^{-i-a} \widetilde{\mathrm{ad}}(F_{\beta_r}^i)(u).$$

But $g_{M'}$ is zero on every $F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}$ where one of the a_i's with $i > j$ is strictly greater than zero. This coupled with the observation above gives us that

$$\begin{aligned} & u_j g_{M'}(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \\ =& g_{M'}(v^c F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_j}^{-(a_j-1)-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \\ =& v^c g_M(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_j}^{-a_j-1} \otimes \cdots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \end{aligned}$$

where c is some constant coming from the commutations. We see that $g_M = v^{-c} u_j g_{M'}$ which finishes the induction step.

So in conclusion we have that $DM^{w_0}(\lambda)$ is a highest weight module with highest weight λ. So we have a surjection from $M(\lambda)$ to $DM^{w_0}(\lambda)$. But since the two modules have the same character and the weight spaces are finite dimensional the surjection must be an isomorphism. □

Proposition 3.10 *Let M be a U_v-module, $\beta \in \Phi^+$ and let $w \in W$. Assume $s_{i_r} \cdots s_{i_1}$ is a reduced expression of w and $F_\beta = R_{s_{i_1}} \cdots R_{s_{i_r}}(F_\alpha)$ for some $\alpha \in \Pi$ such that $l(s_\alpha w) > l(w)$ (so we have $w(\beta) = \alpha$). Then*

$${}^w(S_v(F_\beta) \otimes_{U_v} M) \cong S_v(F_\alpha) \otimes_{U_v} {}^w M.$$

Proof. Define the map $\varphi : S_v(F_\alpha) \otimes {}^w M \to {}^w(S_v(F_\beta) \otimes M)$ by

$$\varphi(u F_\alpha^{-m} \otimes m) = R_{w^{-1}}(u) F_\beta^{-m} \otimes m.$$

This is obivously a U_v-homomorphism if it is welldefined and it is a bijection because $R_{w^{-1}}$ is a U_v-isomorphism. We have to check that if $u F_\alpha^{-m} = u' F_\alpha^{-m'}$ then $R_{w^{-1}}(u) F_\beta^{-m} = R_{w^{-1}}(u') F_\alpha^{-m'}$ and that $\varphi(u F_\alpha^{-m} u' \otimes m) = \varphi(u F_\alpha^{-m} \otimes R_{w^{-1}}(u') m)$ but $u F_\alpha^{-m} = u' F_\alpha^{-m'}$ if and only if $F_\alpha^{m'} u = F_\alpha^m u'$. Using the isomorhpism $R_{w^{-1}}$ on this we get $F_\beta^{m'} R_{w^{-1}}(u) = F_\beta^m R_{w^{-1}}(u')$ which implies

$R_{w^{-1}}(u)F_\beta^{-m} = R_{w^{-1}}(u')F_\beta^{-m'}$. For the other equation: Since we only have the definition of φ on elements on the form $uF_\alpha^{-m}\otimes m$ assume $F_\alpha^{-m}u' = \widetilde{u}F_\beta^{-\widetilde{m}}$. This is equivalent to $u'F_\alpha^{\widetilde{m}} = F_\alpha^m\widetilde{u}$. Use $R_{w^{-1}}$ on this to get $R_{w^{-1}}(u')F_\beta^{\widetilde{m}} = F_\beta^m\widetilde{u}$ or equivalently $F_\beta^{-m}R_{w^{-1}}(u) = R_{w^{-1}}(\widetilde{u})F_\alpha^{-\widetilde{m}}$. Now we can calculate:

$$\begin{aligned}\varphi(uF_\alpha^{-m}u'\otimes m) =& \varphi(u\widetilde{u}F_\alpha^{-\widetilde{m}}\otimes m)\\ =& R_{w^{-1}}(u\widetilde{u})F_\beta^{-\widetilde{m}}\otimes m\\ =& R_{w^{-1}}(u)R_{w^{-1}}(\widetilde{u})F_\beta^{-\widetilde{m}}\otimes m\\ =& R_{w^{-1}}(u)F_\beta^{-m}\otimes R_{w^{-1}}(u')m = \varphi(uF_\alpha^{-m}\otimes R_{w^{-1}}(u)m).\end{aligned}$$ □

Proposition 3.11 *$w\in W$. If s is a simple reflection such that $sw > w$ then*

$$T_{sw} = T_s\circ T_w.$$

Proof. Let α be the simple root corresponding to the simple reflection s. By Proposition 3.2 we get for M a U_v-module:

$$\begin{aligned}T_{sw}M = {}^{sw}(S_v^{sw}\otimes_{U_v} M) \cong& {}^{sw}(S_v(R_{w^{-1}}(F_\alpha))\otimes_{U_v} S_v^w\otimes_{U_v} M)\\ \cong& {}^{s}({}^{w}(S_v(R_{w^{-1}}(F_\alpha))\otimes_{U_v} S_v^w\otimes_{U_v} M))\\ \cong& {}^{s}(S_v(F_\alpha)\otimes_{U_v} {}^{w}(S_v^w\otimes_{U_v} M))\end{aligned}$$

where the last isomorphism is the one from Proposition 3.10. □

4 Twisting functors over Lusztigs A-form

We want to define twisting functors so they make sense to apply to U_A modules. Note first that the maps R_s send U_A to U_A.

Recall that for $n\in\mathbb{N}$ with $n>0$ and F_β a root vector we have defined in $U_{v(F_\beta)}$

$$F_\beta^{(-n)} = [n]_\beta! F_\beta^{-n} \tag{4}$$

i.e. $F_\beta^{(-n)} = \left(F_\beta^{(n)}\right)^{-1}$.

Definition 4.1 *Let s be a simple reflection corresponding to a simple root α. Let S_A^s be the U_A-sub-bimodule of $S_v^s = S_v(F_\alpha)$ generated by the elements $\{F_\alpha^{(-n)}F_\alpha^{-1}|n\in\mathbb{N}\}$.*

Note that $S_A^s\otimes_A\mathbb{Q}(v) = S_v^s$.

Proposition 4.2 *In $U_v(\mathfrak{sl}_2)$ let E,K,F be the usual generators and define as in [Lus90] the elements*

$$\begin{bmatrix}K;c\\ t\end{bmatrix} = \prod_{n=1}^{t}\frac{Kv^{c-n+1}-K^{-1}v^{-c+n-1}}{v^s-v^{-s}}.$$

Then

$$F^{(-s)}F^{-1}E^{(r)} = \sum_{t=0}^{r}E^{(r-t)}\begin{bmatrix}K;r-s-t-2\\ t\end{bmatrix}F^{(-s-t)}F^{-1}.$$

Proof. This is proved by induction over r. We define as in [Jan96]

$$[K;c] = \begin{bmatrix} K;c \\ 1 \end{bmatrix} = \frac{Kv^c - K^{-1}v^{-c}}{v - v^{-1}}.$$

From [Jan96] we get $EF^{s+1} = F^{s+1}E + [s+1]F^s[K,-s]$ so

$$F^{-s-1}E = EF^{-s-1} + [s+1]F^{-1}[K;-s]F^{-s-1} = EF^{-s-1} + [s+1][K;-2-s]F^{-s-2}$$

and multiplying with $[s]!$ we get

$$F^{(-s)}F^{-1}E = EF^{(-s)}F^{-1} + [K;-2-s]F^{(-s-1)}F^{-1}.$$

This is the induction start. The rest is the induction step. In the process you have to use that

$$\frac{1}{[r]}\left([r-t]\begin{bmatrix} K;r-s-t \\ t \end{bmatrix} + \begin{bmatrix} K;r-1-s-t \\ t-1 \end{bmatrix}[K;-s-t]\right) = \begin{bmatrix} K;r-s-t-1 \\ t \end{bmatrix}$$

or equivalently that

$$[r-t][K;r-s-t] + [t][K;-s-t] = [r][K;r-s-2t].$$

This can be shown by a direct calculation. □

We could have proved this in the other way around instead too to get

Proposition 4.3

$$E^{(r)}F^{(-s)}F^{-1} = \sum_{t=0}^{r} F^{(-s-t)}F^{-1}\begin{bmatrix} K;s+t-r+2 \\ t \end{bmatrix}E^{(r-t)}.$$

The above and Corollary 2.19 shows that $S_A(F)$ is a bimodule. We can now define the twisting functor T_s^A corresponding to s:

Definition 4.4 *Let s be a simple reflection corresponding to a simple root α. The twisting functor $T_s^A : U_A\text{-Mod} \to U_A\text{-Mod}$ is defined by: Let M be a U_A module, then*

$$T_s^A(M) = {}^s(S_A(F_\alpha) \otimes_{U_A} M).$$

Note that $T_s^A(M) \otimes_A \mathbb{Q}(v) = T_s(M \otimes_A \mathbb{Q}(v))$ so that if M is a $\mathbb{Q}(v)$ module then $T_s^A = T_s$ on M.

We want to define the twisting functor for every $w \in W$ such that if w has a reduced expression $w = s_{i_r} \cdots s_{i_1}$ then $T_w^A = T_{s_{i_r}}^a \circ \cdots \circ T_{s_{i_1}}^A$. As before we define a 'semiregular bimodule' $S_A^w = U_A \otimes_{U_A^-(w)} U_A^-(w)^*$ and show this is a bimodule isomorphic to $S_A(F_{\beta_r}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$.

Theorem 4.5 *$S_A^w := U_A \otimes_{U_A^-(w)} U_A^-(w)^*$ is a bimodule isomorphic to $S_A(F_{\beta_r}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$ and the functors T_s^A, $s \in \Pi$ satisfy braid relations.*

Proof. Note that $U_A^-(w)$ can be seen as an A-submodule of $U_v^-(w)$ and similarly $U_A^-(w)^*$ can be seen as a submodule of $U_v^-(w)^*$. So we have an injective A homomorphism

$$S_A^w \to S_v^w.$$

Assume the length of w is r and $w = s_{i_r}w'$, $l(w') = r-1$. We want to show that the isomorphism φ_r from Proposition 3.2 restricts to an isomorphism $S_A^w \to S_A(F_{\beta_r}) \otimes_{U_A} S_A^{w'}$.

Assume $f \in U_A^-(w)$ is such that $f = g \cdot f'_m$ meaning that $f(xF_{\beta_r}^{(n)}) = g(x)\delta_{m,n}$, ($x \in U_A^-(w')$, $n \in \mathbb{N}$) where $g \in U_A^-(w')^*$. Then $f'_m = [m]_{\beta_r}! f_m$ where f_m is defined like in Proposition 3.2 and for $u \in U_A$ we have therefore

$$\varphi_r(u \otimes f) = uF_{\beta_r}^{(-m)}F_{\beta_r}^{-1} \otimes (1 \otimes g)$$

which can be seen to lie in $S_A(F_{\beta_r}) \otimes_{U_A} S_A^{w'}$. The inverse also restricts to a map to the right space:

$$\begin{aligned}\psi_r(uF_{\beta_r}^{(-m)}F_{\beta_r}^{-1} \otimes (1 \otimes g)) =&\psi_r(u[m]_{\beta_r}!F_{\beta_r}^{-m-1} \otimes (1 \otimes g))\\ =&[m]_{\beta_r}!u \otimes f_m \cdot g\\ =&u \otimes f'_m \cdot g.\end{aligned}$$

The maps are well defined because they are restrictions of well defined maps and it is easy to see that they are inverse to each other.

As in the generic case we get a right module action on S_A^w in this way. This is the right action coming from S_v^w restricted to S_A^w. So now we have $S_A^w = S_A(F_{\beta_r}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$. Showing that the twisting functors then satisfy braid relations is done in the same way as in Proposition 3.11. □

Now we can define $T_w^A = T_{s_{i_r}}^A \circ \cdots \circ T_{s_{i_1}}^A$ if $w = s_{i_r} \cdots s_{i_1}$ is a reduced expression of w. By the previous theorem there is no ambiguity in this definition since the T_s^A's satisfy braid relations.

It is now possible for any A algebra R to define twisting functors U_R-Mod$\to$ U_R-Mod. Just tensor over A with R.

F.x. let $R = \mathbb{C}$ with $v \mapsto 1$. $S_A(F_\beta) \otimes_A \mathbb{C}$ is just the normal $S^s = U_{(y_\beta)}/U$ via the isomorphism $uF_\beta^{(-n)}F_\beta^{-1} \otimes 1 \mapsto \overline{u}y_\beta^{-n-1}$ where $\overline{u}$ is given by the isomorphism between $U_A^- \otimes_A \mathbb{C}$ and U^-.

Theorem 4.6 *Let R be an A-algebra with $v \in A$ being sent to $q \in R\backslash\{0\}$. Let $\lambda : U_R^0 \to R$ be an R-algebra homomorphism and let $M_R(\lambda) = U_R \otimes_{U_R^{\geq 0}} R_\lambda$ be the U_R Verma module with highest weight λ where R_λ is the rank 1 free $U_R^{\geq 0}$-module with $U_R^{>0}$ acting trivially and U_R^0 acting as λ. Let $D : U_R \to U_R$ be the duality functor on $U_R - \mathrm{Mod}$ induced from the duality functor on $U_A \to U_A$. Then*

$$T_{w_0}^R M_R(\lambda) \cong DM_R(w_0.\lambda).$$

Proof. The proof is the almost the same as the proof of Theorem 3.9. We have by Corollary 2.19 (setting $k = \langle \beta_j, \beta_r^\vee \rangle$)

$$F_{\beta_j}F_{\beta_r}^{(-a)}F_{\beta_r}^{-1} = q^{-(a+1)(\beta_r|\beta_j)}F_{\beta_r}^{(-a)}F_{\beta_r}^{-1}F_{\beta_j} + \sum_{j \geq 1} q_{\beta_r}^{-(a+1)k-(a+2)i}F_{\beta_r}^{(-a-i)}F_{\beta_r}^{-1}\widetilde{\mathrm{ad}}(F_{\beta_r}^{(i)})(u).$$

Define for $M = (m_1, \ldots, m_N) \in \mathbb{N}$ the function

$$g_M(F_{\beta_N}^{(-a_N)}F_{\beta_N}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)}F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) = \begin{cases} 1 & \text{if } a_1 = m_1 \ldots a_N = m_N \\ 0 \text{ otherwise} . \end{cases}$$

Note that $g_{(0,\dots,0)} = g_\lambda$ from Theorem 3.9. In particular it has weight λ. We want to show that $DM_R^{w_0}(\lambda) = U_R g_{(0,\dots,0)}$. We use induction on the number of nonzero entries in M. Assume j is such that $m_N = \cdots = m_{j+1} = 0$ and $m_j = n > 0$. Let $M' = (0,\dots,0,0,m_{j-1},\dots,m_1)$. By induction $g_{M'} \in U_R g_{(0,\dots,0)}$.

Set $u = \omega(S^{-1}(R_{w_0}^{-1}(F_{\beta_j}^{(n)})))$. Then

$$
\begin{aligned}
& u g_{M'}(F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) \\
=& g_{M'}(F_{\beta_j}^{(n)} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) \\
=& g_{M'}(\frac{1}{[n]_{\beta_j}!} F_{\beta_j}^{n} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) \\
=& g_{M'}(q^{c_1} \frac{1}{[n]_{\beta_j}!} F_{\beta_j}^{n-1} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \cdots \otimes F_{\beta_j} F_{\beta_j}^{(-a_j)} F_{\beta_j}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) \\
& \vdots \\
=& g_{M'}(q^{c_n} \frac{1}{[n]_{\beta_j}!} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \cdots \otimes F_{\beta_j}^{n} F_{\beta_j}^{(-a_j)} F_{\beta_j}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) \\
=& \begin{cases} g_{M'}(q^{c_n} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \cdots \otimes \begin{bmatrix} a_j \\ n \end{bmatrix}_{\beta_j} F_{\beta_j}^{(-(a_j-n))} F_{\beta_j}^{-1} \otimes \cdots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) & \text{if } n \leq a_j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

for some appropiate integers $c_1,\dots,c_n \in \mathbb{Z}$. $g_{M'}$ is nonzero on this only when $n = a_j$. So we get in conclusion that $u g_{M'} = v^{-c_n} g_M$. This finishes the induction step. □

5 $\mathfrak{sl}_2$ calculations

Assume $\mathfrak{g} = \mathfrak{sl}_2$. Let $r \in \mathbb{N}$. Let $M_A(v^r)$ be the $U_A(\mathfrak{sl}_2)$ Verma module with highest weight $v^r \in \mathbb{Z}$ i.e. $M_A(v^r) = U_A \otimes_{U_A^{\geq 0}} A_{v^r}$ where A_{v^r} is the free $U_A^{\geq 0}$-module of rank 1 with U_A^+ acting trivially and $K \cdot 1 = q^r$. Inspired by [And03] we see that in $\mathfrak{sl}_2$ we have for $r \in \mathbb{Z}$ the homomorphism $\varphi : M_A(v^r) \to DM_A(v^r)$ given by:

Let $\{w_i = F^{(i)} w_0\}$ be a basis for $M_A(\lambda)$ where w_0 is a highest weight vector in $M_A(v^r)$ and let $\{w_i^*\}$ be the dual basis in $DM_A(\lambda)$. Then

$$\varphi(w_i) = (-1)^i v^{i(i-1)-ir} \begin{bmatrix} r \\ i \end{bmatrix} w_i^*.$$

Checking that this is indeed a homomorphism of U_A algebras is a straightforward calculation.

By Theorem 4.6 we see that $DM_A(v^r) = M_A^s(v^r)$. In the following section we will try to say something about the composition factors of a Verma module so it is natural to consider first $\mathfrak{sl}_2$ Verma modules.

Definition 5.1 *Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $r \in \mathbb{N}$. Then $H_A(v^r)$ is defined to be the free*

$U_A(\mathfrak{sl}_2)$-module of rank $r+1$ with basis $e_0, \ldots, e_r$ defined as follows:

$$\begin{aligned} Ke_i =& v^{r-2i}e_i, \begin{bmatrix} K;c \\ t \end{bmatrix} e_i = \begin{bmatrix} r-2i+c \\ t \end{bmatrix} e_i \\ E^{(n)}e_i =& \begin{bmatrix} i \\ n \end{bmatrix} e_{i-n}, \quad n \in \mathbb{N} \\ F^{(n)}e_i =& \begin{bmatrix} r-i \\ n \end{bmatrix} e_{i+n}, \quad n \in \mathbb{N} \end{aligned}$$

for $i = 0, \ldots, r$. Where $e_{<0} = 0 = e_{>r}$.

Lemma 5.2 *Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $r \in \mathbb{N}$. Then we have a short exact sequence:*

$$0 \to DM_A(v^{-r-2}) \to M_A(v^r) \to H_A(v^r) \to 0.$$

Proof. We use the fact that $DM_A(v^{-r-2}) = T_s^A M_A(v^r)$ by Theorem 4.6. Let $e_i = F^{(i)}w_0$ where w_0 is a heighest weight vector in $M_A(v^r)$. We will construct a U_A-homomorphism $\text{span}_A \{e_i | i > r\} \to DM_A(-r-2)$. Let τ be as defined in [Jan96] Chapter 4. Note that in $U_{A(F)}$ $S(\tau(F))$ is invertible so we can consider S and τ as automorphisms of $U_{A(F)}$. We define a map by

$$e_{r+i} \mapsto (-1)^{r+i} S(\tau(F^{(-i-1)}))w_0$$

Note that for $\mathfrak{sl}_2$ $R_s = S \circ \tau \circ \omega$. Using this and the formula in Proposition 4.2 it is straightforward to check that this is a U_A-homomorphism. □

If we specialize to an A-algebra R with R being a field where v is sent to a non-root of unity $q \in R$ we get that $M_R(q^k) = U_R \otimes_{U_A} M_A(v^k)$ is simple for $k < 0$. So in the above with $r \in \mathbb{N}$, $DM_R(q^{-r-2}) = M_R(q^{-r-2}) = L_R(q^{-r-2})$ and actually we see also that $H_R(q^r) = L_R(q^r)$. So there is an exact sequence

$$0 \to L_R(q^{-r-2}) \to M_R(q^r) \to L_R(q^r) \to 0.$$

So the composition factors in $M_R(q^r)$ are $L_R(q^r)$ and $L_R(q^{-r-2}) = L_R(s.q^r)$ where s is the simple reflection in the Weyl group of $\mathfrak{sl}_2$.

6 Jantzen filtration

In this section we will work with the field $\mathbb{C}$ and send v to a non root of unity $q \in \mathbb{C}^*$. We define $U_q = U_A \otimes_A \mathbb{C}_q$ where $\mathbb{C}_q$ is the A-algebra $\mathbb{C}$ with v being sent to q. These results compare to the results in [And03] and [AL03].

Let λ be a weight i.e. an algebra homomorphism $U_q^0 \to \mathbb{C}$ and let $M(\lambda) = U_q \otimes_{U_q^{\geq 0}} \mathbb{C}_\lambda$ be the Verma module of highest weight λ. Consider the local ring $B = \mathbb{C}[X]_{(X-1)}$ and the quantum group $U_B = U_A \otimes_A B$. We define $\lambda X : U_q^0 \to B$ to be the weight defined by $(\lambda X)(K_\mu) = \lambda(K_\mu)X$ and we define $M_B(\lambda X) = U_B \otimes_{U_B^{\geq 0}} B_{\lambda X}$ to be the Verma module with highest weight λX. Note that $M_B(\lambda X) \otimes_B \mathbb{C} \cong M(\lambda)$ when we consider $\mathbb{C}$ as a B-algebra via the specialization $X \mapsto 1$

For a simple root $\alpha_i \in \Pi$ we define $M_{B,i}(\lambda X) := U_B(i) \otimes_{U_B^{\geq 0}} B_{\lambda X}$, where $U_B(i)$ is the subalgebra generated by $U_B^{\geq 0}$ and F_{α_i}. We define $M_{B,i}^{s_i}(\lambda) :=$

${}^{s_i}((U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*) \otimes_{U_B(i)} M_{B,i}(s_i.\lambda))$ where the module $(U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*)$ is a $U_B(i)$-bimodule isomorphic to $S_{B,i}(F_{\alpha_i}) = (U_B(i))_{(F_{\alpha_i})}/U_B(i)$ by similar arguments as earlier.

Proposition 6.1 *There exists a nonzero homomorphism $\varphi : M_B(\lambda X) \to M_B^{s_\alpha}(\lambda X)$ which is an isomorphism if $q^\rho\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have a short exact sequence*

$$0 \to M_B(\lambda X) \xrightarrow{\varphi} M_B^{s_\alpha}(\lambda X) \to M(s_\alpha.\lambda) \to 0$$

where we have identified the cokernel $M_B^{s_\alpha}(s_\alpha.\lambda X)/(X-1)M_B(s_\alpha.\lambda X)$ with $M(s_\alpha.\lambda)$.

Furthermore there exists a nonzero homomorphism $\psi : M_B^{s_\alpha}(\lambda X) \to M_B(\lambda X)$ which is an isomorphism if $q^\rho\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have a short exact sequence

$$0 \to M_B^{s_\alpha}(\lambda X) \xrightarrow{\psi} M_B(\lambda X) \to M(\lambda)/M(s_\alpha.\lambda) \to 0.$$

Proof. We will first define a map from $M_{B,i}(\lambda X)$ to

$$M_{B,i}^{s_i}(\lambda X) = {}^{s_i}\left((U_B(i))_{(F_\alpha)}/U_B(i) \otimes_{U_B} M_{B,i}(s_\alpha.\lambda X)\right).$$

Setting $\lambda' = \lambda X$ define

$$\varphi(F_\alpha^{(n)} v_{\lambda'}) = a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'}$$

where

$$a_n = (-1)^n q_\alpha^{-n(n+1)} \lambda'(K_\alpha)^n \prod_{t=1}^n \frac{q_\alpha^{1-t}\lambda'(K_\alpha) - q_\alpha^{t-1}\lambda'(K_\alpha)^{-1}}{q_\alpha^t - q_\alpha^{-t}}.$$

So we need to check that this is a homomorphism: First of all for $\mu \in Q$.

$$\begin{aligned}
K_\mu \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} =& a_n K_{s_\alpha(\mu)} F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& q^{(n+1)(s_\alpha(\mu)|\alpha)} (s_\alpha.\lambda')(K_{s_\alpha(\mu)}) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& q^{-(n+1)(\mu|\alpha)} q^{-(\rho|s_\alpha(\mu))} q^{(\rho|\mu)} \lambda'(K_\mu) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& q^{-(n+1)(\mu|\alpha)} q^{-(\rho-\alpha|\mu)} q^{(\rho|\mu)} \lambda'(K_\mu) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& q^{-n(\mu|\alpha)} \lambda'(K_\mu) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& \varphi(K_\mu F_\alpha^{(n)} v_{\lambda'}).
\end{aligned}$$

We have

$$\begin{aligned}
E_\alpha \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} =& a_n R_{s_i}(E_{\alpha_i}) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& - a_n F_\alpha K_\alpha F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& - q_\alpha^{2(n+1)} s_\alpha.\lambda'(K_\alpha) [n]_\alpha a_n F_\alpha^{(-n+1)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=& - q_\alpha^{2n} \lambda'(K_\alpha^{-1}) [n]_\alpha a_n F_\alpha^{(-n+1)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(E_\alpha F_\alpha^{(n)} v_{\lambda'}) =& \varphi\left(F_\alpha^{(n-1)} \frac{q_\alpha^{1-n} K_\alpha - q_\alpha^{n-1} K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} v_{\lambda'}\right) \\
=& \left(a_{n-1} F_\alpha^{(-n+1)} F_\alpha^{-1} \frac{q_\alpha^{1-n}\lambda'(K_\alpha) - q_\alpha^{n-1}\lambda'(K_\alpha)^{-1}}{q_\alpha - q_\alpha^{-1}}\right) \otimes v_{\lambda'}
\end{aligned}$$

so we see that $\varphi(E_\alpha F_\alpha^{(n)} v_{\lambda'}) = E_\alpha \cdot \varphi(F_\alpha^{(n)} v_{\lambda'})$. Clearly $\varphi(E_{\alpha'} F_\alpha^{(n)} v_{\lambda'}) = 0 = E_{\alpha'} \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{\lambda'}$ for any simple $\alpha' \neq \alpha$ so what we have left is F_α: By Proposition 4.3

$$\begin{aligned}
&F_\alpha \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=&a_n R_{s_i}(F_\alpha) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=&-a_n K_\alpha^{-1} E_\alpha F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=&-a_n K_\alpha^{-1} F_\alpha^{(-n-1)} F_\alpha^{-1} [K_\alpha; n+2] \otimes v_{s_\alpha.\lambda'} \\
=&-a_n q_\alpha^{-2(n+2)} s_\alpha.\lambda'(K_\alpha^{-1}) \frac{q_\alpha^{n+2} s_\alpha.\lambda'(K_\alpha) - q_\alpha^{-n-2} s_\alpha.\lambda'(K_\alpha)^{-1}}{q_\alpha - q_\alpha^{-1}} F_\alpha^{(-n-1)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} \\
=&-a_n q_\alpha^{-2(n+1)} \lambda'(K_\alpha) \frac{q_\alpha^{n} \lambda'(K_\alpha^{-1}) - q_\alpha^{-n} \lambda'(K_\alpha)}{q_\alpha - q_\alpha^{-1}} F_\alpha^{(-n-1)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(F_\alpha F_\alpha^{(n)} v_\lambda) =&[n+1]_\alpha \varphi(F_\alpha^{(n+1)} v_\lambda) \\
=&[n+1]_\alpha a_{n+1} F_\alpha^{(-n-1)} F_\alpha^{-1} \otimes v_\lambda
\end{aligned}$$

so we see that $\varphi(F_\alpha F_\alpha^{(n)} v_\lambda) = F_\alpha \cdot \varphi(F_\alpha^{(n)} v_\lambda)$.

Now note that if $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$ then $X-1$ does not divide a_n for any $n \in \mathbb{N}$ implying that a_n is a unit. So when $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$, φ is an isomorphism. If $\lambda(K_\alpha) = \varepsilon q_\alpha^r$ for some $\varepsilon \in \{\pm 1\}$ and $r \in \mathbb{N}$ we see that $X-1$ divides a_n for any $n > r$ so the image of φ is

$$\operatorname{span}_B \left\{ F_\alpha^{(-n)} F_\alpha \otimes v_{s_\alpha.\lambda'} | n \leq r \right\} + (X-1) \operatorname{span}_B \left\{ F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} | n > r \right\}.$$

Thus the cokernel $M_{B,i}^{s_i}(\lambda)/\operatorname{Im}\varphi$ is equal to

$$\operatorname{span}_B \left\{ F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} | n > r \right\} / (X-1) \operatorname{span}_B \left\{ F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'} | n > r \right\}$$

which is seen to be isomorphic to $M_{B,i}^{s_i}(s_i.\lambda')/(X-1)M_{B,i}^{s_i}(s_i.\lambda')$.

If $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$ then obviously we can define an inverse to φ, $\psi : M_{B,i}^{s_i}(\lambda') \to M_{B,i}(\lambda')$. If $\lambda(K_\alpha) = \varepsilon q^r$ for some $\varepsilon \in \{\pm 1\}$ and some $r \in \mathbb{N}$ we define $\psi : M_{B,i}^{s_i}(\lambda') \to M_{B,i}(\lambda')$ by

$$\psi(F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha.\lambda'}) = \frac{(X-1)}{a_n} F_\alpha^{(n)} v_{\lambda'}$$

(note that for all λ and all $n \in \mathbb{N}$, $(X-1)^2 \nmid a_n$ so $\frac{(X-1)}{a_n} \in B$). This implies $\varphi \circ \psi = (X-1)\,\mathrm{id}$ and $\psi \circ \varphi = (X-1)\,\mathrm{id}$. Using that φ is a U_q-homomorphism we show that ψ is: For $u \in U_q$ and $v \in M_{B,i}^{s_i}(\lambda')$:

$$(X-1)\psi(uv) = \psi(u\varphi(\psi(v))) = \psi(\varphi(u\psi(v))) = (X-1)u\psi(v).$$

Since B is a domain this implies $\psi(uv) = u\psi(v)$.

We see that $X-1$ divides $\frac{X-1}{a_n}$ for any $n \leq r$ so the image of ψ is

$$(X-1) \operatorname{span}_B \left\{ F_\alpha^{(n)} v_{\lambda'} | n \leq r \right\} + \operatorname{span}_B \left\{ F_\alpha^{(n)} v_{\lambda'} | n > r \right\}.$$

Thus the cokernel $M_{B,i}(\lambda)/\operatorname{Im}\psi$ is equal to

$$\operatorname{span}_B\left\{F_\alpha^{(n)}v_{\lambda'}|n\leq r\right\}/(X-1)\operatorname{span}_B\left\{F_\alpha^{(n)}v_{\lambda'}|n\leq r\right\}$$

which is seen to be isomorphic to

$$M_{B,i}(\lambda)/M_{B,i}(s_\alpha.\lambda).$$

Now we induce to the whole quantum group: We have that

$$M_B(\lambda')=U_B\otimes_{U_B(i)}M_{B,i}(\lambda')$$

and

$$\begin{aligned}M_B^{s_i}(\lambda')=&{}^{s_i}\left((U_B\otimes_{U_B(s_i)}U_B(s_i)^*)\otimes_{U_B}U_B\otimes_{U_B^{\geq 0}}B_{\lambda'}\right)\\ \cong&{}^{s_i}\left((U_B\otimes_{U_B(i)}U_B(i)\otimes_{U_B(s_i)}U_B(s_i)^*)\otimes_{U_B^{\geq 0}}B_{\lambda'}\right)\\ \cong&U_B\otimes_{U_B(i)}{}^{s_i}\left((U_B(i)\otimes_{U_B(s_i)}U_B(s_i)^*)\otimes_{U_B^{\geq 0}}B_{\lambda'}\right)\\ \cong&U_B\otimes_{U_B(i)}M_{B,i}^{s_i}(\lambda')\end{aligned}$$

so by inducing to U_B-modules using the functor $U_B\otimes_{U_B(i)}-$ we get a map $\varphi:M_B(\lambda')\to M_B^{s_i}(\lambda')$ and a map $\psi:M_B^{s_i}(\lambda')\to M_B(\lambda')$. This functor is exact on $M_{B,i}(\lambda')$ and $M_{B,i}^{s_i}(\lambda')$ so the proposition follows from the above calculations. □

Proposition 6.2 *Let $\lambda:U_q^0\to\mathbb{C}$ be a weight. Set $\lambda'=\lambda X$. Let $w\in W$ and $\alpha\in\Pi$ such that $w(\alpha)>0$. There exists a nonzero homomorphism $\varphi:M_B^w(\lambda')\to M_B^{ws_\alpha}(\lambda')$ that is an isomorphism if $q^\rho\lambda(K_{w(\alpha)})\notin\pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have the short exact sequence*

$$0\to M_B^w(\lambda')\overset{\varphi}{\to}M_B^{ws_\alpha}(\lambda')\to M^w(s_{w(\alpha)}.\lambda)\to 0$$

where the cokernel $M_B^{ws_\alpha}(s_{w(\alpha)}.\lambda')/(X-1)M_B^{ws_\alpha}(s_{w(\alpha)}.\lambda')$ is identified with $M^w(s_{w(\alpha)}.\lambda)$.

Furthermore there exists a nonzero homomorphism $\psi:M_B^{ws_\alpha}(\lambda X)\to M_B^w(\lambda X)$ which is an isomorphism if $q^\rho\lambda(K_{w(\alpha)})\notin\pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have a short exact sequence

$$0\to M_B^{ws_\alpha}(\lambda')\overset{\psi}{\to}M_B^w(\lambda')\to M^w(\lambda)/M^w(s_{w(\alpha)}.\lambda)\to 0.$$

Proof. Let $\mu=w^{-1}.\lambda$ and $\mu'=\mu X$ then from Proposition 6.1 we get a homomorphism $M_B(\mu')\to M_B^{s_\alpha}(\mu')$ and a homomorphism $M_B^{s_\alpha}(\mu')\to M_B(\mu')$. Observe that

$$\begin{aligned}q^\rho\mu(K_\alpha)=&w^{-1}.\lambda(K_\alpha)\\ =&w^{-1}(q^\rho\lambda)(K_\alpha)\\ =&q^{(\rho|w(\alpha))}\lambda(K_{w(\alpha)})\\ =&\left(q^\rho\lambda\right)\left(K_{w(\alpha)}\right)\end{aligned}$$

so $M_B(\mu') \to M_B^{s_\alpha}(\mu')$ and $M_B^{s_\alpha}(\mu') \to M_B(\mu')$ are isomorphisms if $(q^\rho\lambda)(K_{w(\alpha)}) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have the short exact sequences

$$0 \to M_B(\mu') \to M_B^{s_\alpha}(\mu') \to M(\mu') \to 0$$

and

$$0 \to M_B^{s_\alpha}(\mu') \to M_B(\mu') \to M(\mu')/M(s_\alpha.\mu') \to 0.$$

Now we use the twisting functor T_w on the homomorphisms $M_B(\mu') \to M_B^{s_\alpha}(\mu')$ and $M_B^{s_\alpha}(\mu') \to M_B(\mu')$ to get homomorphisms $\varphi : M_B^w(\lambda) \to M_B^{ws_\alpha}(\lambda)$ and $\psi : M_B^{ws_\alpha}(\lambda) \to M_B^w(\lambda)$ (using the fact that $T_w \circ T_{s_\alpha} = T_{ws_\alpha}$). We are done if we show that T_w is exact on Verma modules. But

$$\begin{aligned}
T_w M_B(\mu') &= {}^w\left((U_B^-(w)^* \otimes_{U_B^-(w)} U_B) \otimes_{U_B} U_B \otimes_{U_B^{\geq 0}} B_{\mu'}\right) \\
&\cong {}^w\left((U_B^-(w)^* \otimes_{U_B^-(w)} U_B) \otimes_{U_B^{\geq 0}} B_{\mu'}\right) \\
&\cong {}^w\left(U_B^-(w)^* \otimes_{U_B^-(w)} U_B^- \otimes_{\mathbb{C}} B_{\mu'}\right)
\end{aligned}$$

as vectorspaces and U_B^0 modules. Observing that U_B^- is free over $U_B^-(w)$ we get the exactness. □

Fix a weight $\lambda : U_q^0 \to \mathbb{C}$ and a $w \in W$. Define $\Phi^+(w) := \Phi^+ \cap w(\Phi^-) = \{\beta \in \Phi^+ | w^{-1}(\beta) < 0\}$ and $\Phi^+(\lambda) := \{\beta \in \Phi^+ | q^\rho\lambda(K_\beta) \in \pm q^{\mathbb{Z}}\}$. Choose a reduced expression of $w_0 = s_{i_1} \cdots s_{i_N}$ such that $w = s_{i_n} \cdots s_{i_1}$. Set

$$\beta_j = \begin{cases} -ws_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), & \text{if } j \leq n \\ ws_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), & \text{if } j > n. \end{cases}$$

Then $\Phi^+ = \{\beta_1, \ldots, \beta_N\}$ and $\Phi^+(w) = \{\beta_1, \ldots, \beta_n\}$. We denote by $\Psi_B^w(\lambda)$ the composite

$$M_B^w(\lambda X) \stackrel{\varphi_1^w(\lambda)}{\to} M_B^{ws_{i_1}}(\lambda X) \stackrel{\varphi_2^w(\lambda)}{\to} \cdots \stackrel{\varphi_N^w(\lambda)}{\to} M_B^{ww_0}(\lambda X)$$

where the homomorphisms are the ones from Proposition 6.2 i.e. the first n homomorphisms are the ψ's and the last $N - n$ homomorphisms are the φ's from Proposition 6.2. We denote by $\Psi^w(\lambda)$ the U_q-homomorphism $M^w(\lambda X) \to M^{ww_0}(\lambda X)$ induced by tensoring the above U_B-homomorphism with $\mathbb{C}$ considered as a B module by $X \mapsto 1$.

In analogy with Theorem 7.1 in [AL03] and Proposition 4.1 in [And03] we have

Theorem 6.3 *Let $\lambda : U_q^0 \to \mathbb{C}$ be a weight. Let $w \in W$. Then there exists a filtration of $M^w(\lambda)$, $M^w(\lambda) \supset M^w(\lambda)^1 \supset \cdots \supset M^w(\lambda)^r$ such that $M^w(\lambda)/M^w(\lambda)^1 \cong \operatorname{Im}\Psi^w(\lambda) \subset M^{ww_0}(\lambda)$ and*

$$\begin{aligned}
\sum_{i=1}^r \operatorname{ch} M^w(\lambda)^i = &\sum_{\beta \in \Phi^+(\lambda) \cap \Phi^+(w)} (\operatorname{ch} M(\lambda) - \operatorname{ch} M(s_\beta.\lambda)) \\
&+ \sum_{\beta \in \Phi^+(\lambda) \setminus \Phi^+(w)} \operatorname{ch} M(s_\beta.\lambda).
\end{aligned}$$

Proof. Set $\lambda' = \lambda X$. Define for $i \in \mathbb{N}$

$$M_B^w(\lambda')^i = \{m \in M_B^w(\lambda') | \Psi_B^w(\lambda)(m) \in (X-1)^i M_B^{ww_0}(\lambda')\}.$$

Set $M^w(\lambda)^i = \pi(M_B^w(\lambda')^i)$ where $\pi : M_B^w(\lambda) \to M^w(\lambda)$ is the canonical homomorphism from $M_B^w(\lambda)$ to $M_B^w(\lambda)/(X-1)M_B^w(\lambda) \cong M^w(\lambda)$. This defines a filtration of $M^w(\lambda)$. We have $M^w(\lambda)^{N+1} = 0$ so the filtration is finite.

Let $\mu : U_q^0 \to \mathbb{C}$ be a weight. Set $\mu' = \mu X$. The maps $\varphi_j^w(\lambda)$ restrict to weight spaces. Denote the restriction $\varphi_j^w(\lambda)_{\mu'}$. Let $\Psi_B^w(\lambda)_{\mu'} : M_B^w(\lambda)_{\mu'} \to M_B^{ww_0}(\lambda)_{\mu'}$ be the restriction of $\Psi_B^w(\lambda)$ to the μ' weight space. We have a nondegenerate bilinear form $(-,-)$ on $M(\lambda')_{\mu'}$ given by $(x,y) = \left(\Psi_B^w(\lambda)_{\mu'}(x)\right)(y)$. It is nondegenerate since $\Psi_B^w(\lambda)$ is injective. Let $\nu : B \to \mathbb{C}$ be the $(X-1)$-adic valuation i.e. $\nu(b) = m$ if $b = (X-1)^m b'$, $(X-1) \nmid b'$. We have by [Hum08, Lemma 5.6] (originally Lemma 5.1 in [Jan79])

$$\sum_{j \geq 1} \dim(M_j)_\mu = \nu(\det \Psi_B^w(\lambda)_{\mu'}).$$

Clearly $\nu(\det \Psi_B^w(\lambda)_{\mu'}) = \sum_{j=1}^N \nu(\det \varphi_j^w(\lambda)_{\mu'})$ and the result follows when we show:

$$\nu(\det \varphi_j^w(\lambda)_{\mu'}) = \dim_{\mathbb{C}} \left(\operatorname{coker} \varphi_j^w(\lambda)_{\mu'}\right).$$

Fix $\varphi := \varphi_j^w(\lambda)_{\mu'}$ and let M and N be the domain and codomain respectively. M and N are free B modules of finite rank. Let d be the rank. We can choose bases $m_1, \ldots, m_d$ and $n_1, \ldots, n_d$ such that $\varphi(m_i) = a_i n_i$, $i = 1, \ldots, d$ for some $a_i \in B$. Set $C = \operatorname{coker} \varphi \cong \bigoplus_{i=1}^d B/(a_i)$ and set $C_{\mathbb{C}} = C \otimes_B (B/(X-1)B) = C \otimes_B \mathbb{C}$ where $\mathbb{C}$ is considered a B-module by $X \mapsto 1$. Note that

$$B/(a_i) \otimes_B \mathbb{C} = \begin{cases} \mathbb{C}, & \text{if } (X-1)|a_i \\ 0, & \text{otherwise} \end{cases}$$

so $\dim_{\mathbb{C}} C_{\mathbb{C}} = \#\{i | \nu(a_i) > 0\}$. Since there exists a $\psi : N \to M$ such that $\varphi \circ \psi = (X-1)\,\mathrm{id}$ we get $\nu(a_i) \leq 1$ for all i. So then $\dim_{\mathbb{C}} C_{\mathbb{C}} = \nu(\det \varphi)$ and the claim has been shown. □

7 Linkage principle

Let R be a field that is an A-algebra and $q \in R$ the nonzero element that v is sent to. As usual we can define the Verma modules: Assume $\lambda : U_R^0 \to R$ is a homomorphism. Then we define $M_R(\lambda) = U_R \otimes_{U_R^{\geq 0}} R_\lambda$ where R_λ is the onedimensional R-module with trivial action from U_R^+ and U_R^0 acting as λ. There is a unique simple quotient $L_R(\lambda)$ of $M_R(\lambda)$.

Let $\alpha = \alpha_i \in \Pi$. Consider the parabolic Verma module $M_{R,i}(\lambda) := U_R(i) \otimes_{U_R^{\geq 0}} R_\lambda$, where $U_R(i)$ is the submodule generated by $U_R^{\geq 0}$ and F_{α_i}. We get a map $M_{R,i}(\lambda) \to M_{R,i}^s(\lambda) := {}^s((U_R(i) \otimes_{U_R(s_i)} U_R(s_i)^*) \otimes_{U_R(i)} M_{R,i}(s.\lambda))$ where the module $(U_R(i) \otimes_{U_R(s_i)} U_R(s_i)^*)$ is a $U_R(i)$-bimodule by the similar arguments as earlier. Inducing to the whole quantum group and using T_w we get a homomorphism

$$M_R^w(\lambda) \to M_R^{ws_\alpha}(\lambda)$$

So we can construct a sequence of homomorphisms $\varphi_1, \dots, \varphi_N$

$$M_R(\lambda) \xrightarrow{\varphi_1} M_R^{s_{i_1}}(\lambda) \xrightarrow{\varphi_2} \cdots\cdots \xrightarrow{\varphi_N} M_R^{w_0}(\lambda) = DM_R(\lambda).$$

We denote the composition by Ψ. Note that the image of Ψ must be the unique simple quotient $L_R(\lambda)$ of $M_R(\lambda)$ since every map $M(\lambda) \to DM(\lambda)$ maps to the unique simple quotient of $M(\lambda)$ (by the usual arguments e.g. like in [Hum08, Theorem 3.3]).

First we want to consider some facts about the map $\varphi : M_R^w(\lambda) \to M_R^{ws_\alpha}(\lambda)$. Let $M_\alpha(\lambda)$ denote the $U_R(\mathfrak{sl}(2))$ Verma module with highest weight $\lambda(K_\alpha)$. We will use the notation $M_{p_\alpha}(\lambda)$ for the parabolic $U_R(i)$ Verma module $U_R(i) \otimes_{U_R^{\geq 0}} R_\lambda$. The map φ was constructed by first inducing the map of parabolic modules and then using the twisting functor T_w.

Assume the sequence of $U_R(\mathfrak{sl}_2)$ modules $M_\alpha(\lambda) \to M_\alpha^s(\lambda) \to Q_\alpha(\lambda) \to 0$ is exact (i.e. $Q_\alpha(\lambda)$ is the cokernel of the map $M_\alpha(\lambda) \to M_\alpha^s(\lambda)$). Inflating to the parabolic situation we get an exact sequence $M_{p_\alpha}(\lambda) \to M_{p_\alpha}^s(\lambda) \to Q_{p_\alpha}(\lambda) \to 0$ where $Q_{p_\alpha}(\lambda)$ is just the inflation of $Q_\alpha(\lambda)$ to the corresponding parabolic module.

Inducing from a parabolic module to the whole module is done by applying the functor $M \mapsto U_R \otimes_{U(i)} M$. This is right exact so we get the exact sequence $M_R(\lambda) \to M_R^s(\lambda) \to Q_R(\lambda) \to 0$ where $Q_R(\lambda) = U_R \otimes_{U_R(i)} Q_{p_\alpha}(\lambda)$.

Assume we have a finite filtration of $Q_\alpha(\lambda)$:

$$0 = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q_\alpha(\lambda)$$

such that $Q_{i+1}/Q_i \cong L_\alpha(\mu_i)$. So we have after inflating:

$$0 = Q_{p_\alpha,0} \subset Q_{p_\alpha,1} \subset \cdots \subset Q_{p_\alpha,r} = Q_{p_\alpha}(\lambda)$$

such that $Q_{p_\alpha,i+1}/Q_{p_\alpha,i} \cong L_{p_\alpha}(\mu_i)$.

That is we have short exact sequences of the form

$$0 \to Q_{p_\alpha,i} \to Q_{p_\alpha,i+1} \to L_{p_\alpha}(\mu_i) \to 0.$$

Since induction is right exact we get the exact sequence

$$Q_{R,i} \to Q_{R,i+1} \to \overline{L_{p_\alpha}(\mu_i)} \to 0$$

where $Q_{R,i}$ is the induced module of $Q_{p_\alpha,i}$ and $\overline{L_{p_\alpha}(\mu_i)}$ is the induced module of $L_{p_\alpha}(\mu_i)$.

Starting from the top we have

$$Q_{R,r-1} \to Q_R(\lambda) \to \overline{L_{p_\alpha}(\mu_{r-1})} \to 0$$

so we see that the composition factors of $Q_R^{s_\alpha}(\lambda)$ are contained in the set of composition factors of $\overline{L_{p_\alpha(\mu_{r-1})}}$ and the composition factors of $Q_{R,r-1}$. By induction we get then that the composition factors of $Q_{R,r-1}$ are composition factors of $\overline{L_{p_\alpha}(\mu_i)}$, $i = 0, \dots, r-2$. The conclusion is that we can get a restriction on the composition factors of $Q_R(\lambda)$ by examining the composition factors of induced simple modules.

Let $L = L_{p_\alpha}(\mu)$ be a simple parabolic module and let $\overline{L}$ be the induction of L. Then because induction is right excact we have

$$M_R(\mu) \to \overline{L} \to 0.$$

So the composition factors of $\overline{L}$ are composition factors of $M_R(\mu)$. This gives us a restriction on the composition factors of $M_R(\lambda)$:

Use the above with $w^{-1}.\lambda$ in place of λ and use the twisting functor T_w^R on the exact sequence $M_R(w^{-1}.\lambda) \to M_R^s(w^{-1}.\lambda) \to Q_R(w^{-1}.\lambda) \to 0$ to get

$$M_R^w(\lambda) \to M_R^{ws}(\lambda) \to Q_R^w(\lambda) \to 0$$

where $Q_R^{ws}(\lambda) = T_w^R(Q_R(w^{-1}.\lambda))$. Add the kernel to get the 4-term exact sequence

$$0 \to K_R^{ws}(\lambda) \to M_R^w(\lambda) \to M_R^{ws}(\lambda) \to Q_R^{ws}(\lambda) \to 0$$

Since $\operatorname{ch} M_R^w(\lambda) = \operatorname{ch} M_R^{ws}(\lambda)$ we must have $\operatorname{ch} K_R^{ws}(\lambda) = \operatorname{ch} Q_R^{ws}(\lambda)$.

So we have a sequence of homomorphisms φ_i

$$M_R(\lambda) \overset{\varphi_1}{\to} M_R^s(\lambda) \overset{\varphi_2}{\to} \cdots \overset{\varphi_N}{\to} M_R^{w_0}(\lambda) = DM_R(\lambda)$$

and these maps each fit into a 4-term exact sequence

$$0 \to K_R^{ws}(\lambda) \to M_R^w(\lambda) \to M_R^{ws}(\lambda) \to Q_R^{ws}(\lambda) \to 0$$

where $\operatorname{ch} K_R^{ws}(\lambda) = \operatorname{ch} Q_R^{ws}(\lambda)$. In particular $M_R^w(\lambda) \to M_R^{ws}(\lambda)$ is an isomorphism if the corresponding $\mathfrak{sl}_2$ map $M_\alpha(w^{-1}.\lambda) \to DM_\alpha(w^{-1}.\lambda)(= M_\alpha^s(w^{-1}.\lambda))$ is an isomorphism. If the $\mathfrak{sl}_2$ map is not an isomorphism then we have a restriction on the composition factors that can get killed by the map $M_R(w^{-1}.\lambda) \to M_R^s(w^{-1}.\lambda)$ by the above. To get to the map $M_R^w(\lambda) \to M_R^{ws}(\lambda)$ we use T_w which is right exact so we get a restriction on the composition factors killed by $M_R^w(\lambda) \to M_R^{ws}(\lambda)$ too:

Fix α. From the above we know that a composition factor of $Q_R(\lambda)$ is a composition factor of $\overline{L_{p_\alpha}(\mu)}$ for some μ where $L_\alpha(\mu)$ is a composition factor of $M_\alpha(\lambda)$. Use this for $w^{-1}.\lambda$ and use T_w. So we get that a composition factor of $Q_R^{ws}(\lambda)$ is a composition factor of $T_w\overline{L_{p_\alpha}(\mu)}$ with μ as before. Since T_w is right exact we have that

$$T_w M_R(\mu) \to T_w\overline{L_{p_\alpha}(\mu)} \to 0$$

is exact. Since $\operatorname{ch} T_w M_R(\mu) = \operatorname{ch} M_R(w.\mu)$ we see that a composition factor of $Q_R^{ws}(\lambda)$ must be a composition factor of a Verma module $M_R(w.\mu)$ where μ is such that $L_\alpha(\mu)$ is a composition factor of $M_\alpha(w^{-1}.\lambda)$.

Definition 7.1 *We define a partial order on weights. We say $\mu \leq \lambda$ if $\mu^{-1}\lambda = q^{\sum_{i=1}^n a_i\alpha_i}$ for some $a_i \in \mathbb{N}$ where $\mu^{-1} : U_R^0 \to \mathbb{C}$ is the weight with $\mu^{-1}(K_\alpha) = \mu(K_\alpha^{-1})$ for all $\alpha \in \Pi$.*

For a weight ν of the form $\nu = q^{\sum_{i=1}^n a_i\alpha_i}$ with $a_i \in \mathbb{N}$ we call $\sum_{i=1}^n a_i$ the height of ν.

Note that for a Verma module $M(\lambda)$ we have $\mu \leq \lambda$ for all $\mu \in \operatorname{wt} M(\lambda)$ where $\operatorname{wt} M(\lambda)$ denotes the weights of $M(\lambda)$.

Definition 7.2 *Let $\mu, \lambda \in \Lambda$. Define $\mu \uparrow_R \lambda$ to be the partial order induced by the following: μ is less than λ if there exists a $w \in W$, $\alpha \in \Pi$ and $\nu \in \Lambda$ such that $\mu = w.\nu < \lambda$ and $L_\alpha(\nu)$ is a composition factor of $M_\alpha(w^{-1}.\lambda)$.*

i.e. $\mu \uparrow_R \lambda$ if there exists a sequence of weights $\mu = \mu_1, \ldots, \mu_r = \lambda$ such that μ_i is related to μ_{i+1} as above.

We have established the following:

Proposition 7.3 *If $L_R(\mu)$ is a composition factor of $M_R(\lambda)$ then $\mu \uparrow_R \lambda$.*

Proof. Choose a reduced expression of w_0 and construct the maps φ_i as above. If $L_R(\mu)$ is a composition factor of $M_R(\lambda)$ it must be killed by one of the maps φ_i since the image of Ψ is $L_R(\lambda)$. So $L_R(\mu)$ must be a composition factor of one of the modules $Q_R^w(\lambda)$. We make an induction on the height of $\mu^{-1}\lambda$. If $\mu^{-1}\lambda = 1$ then $\lambda = \mu$ and we are done. Otherwise we see that $L_R(\mu)$ is a composition factor of one of the $Q_R^w(\lambda)$'s. But every composition factor of $Q_R^w(\lambda)$ is a composition factor of $M(\nu)$ where $\nu \uparrow_R \lambda$ and $\nu < \lambda$. Since $\nu < \lambda$ the height of $\mu^{-1}\nu$ is less then the height of $\mu^{-1}\lambda$ so we are done by induction. $\square$

In the non-root of unity case $\uparrow_R$ is equivalent to the usual strong linkage: μ is strongly linked to λ if there exists a sequence μ_i with $\mu = \mu_1 < \mu_2 < \cdots < \mu_r = \lambda$ and $\mu_i = s_{\beta_i}.\mu_{i+1}$ for some positive roots β_i (remember that if $\beta = w(\alpha)$ then $s_\beta = w s_\alpha w^{-1}$).

In the nonroot of unity case we see that $M_\alpha(w^{-1}.\lambda)$ is simple if

$$q^\rho w^{-1}.\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}>0}.$$

Otherwise there is one composition factor in $M_\alpha(w^{-1}.\lambda)$ apart from $L_\alpha(w^{-1}.\lambda)$, namely $L_\alpha(s_\alpha w^{-1}.\lambda)$. So the composition factors of Q_R^w are composition factors of $M_R(w s_\alpha w^{-1}.\lambda) = M_R(s_{w(\alpha)}.\lambda)$. Actually $Q_R^w = M_R^{ws}(s_{w(\alpha)}.\lambda)$ in this case:

Lets consider the construction of the maps φ_i in the above. We start with the map $M_\alpha(\lambda) \to M_\alpha^s(\lambda)$ and then inflate to $M_{p_\alpha}(\lambda) \to M_{p_\alpha}^s(\lambda)$. In the case where q is not a root of unity it is easy to see that if $q^\rho\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}>0}$ then this is an isomorphism and otherwise the kernel (and the cokernel) is isomorphic to $M_{p_\alpha}(s.\lambda)$ which is a simple module. So after inducing we get the 4 term exact sequence

$$0 \to M_R(s.\lambda) \to M_R(\lambda) \to M_R^s(\lambda) \to M_R^s(s.\lambda) \to 0$$

since induction is exact on Verma modules. Use these observations on $w^{-1}.\lambda$ and the fact that T_w is exact on Verma modules and we get a map $M_R^w(\lambda) \to M_R^{ws}(\lambda)$ which is an isomorphism if $q^\rho\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}>0}$ and otherwise we have the 4-term exact sequence

$$0 \to M_R^w(s.\lambda) \to M_R^w(\lambda) \to M_R^{ws}(\lambda) \to M_R^{ws}(s.\lambda) \to 0$$

Theorem 7.4 *Let R be a field (any characteristic) and let $q \in R$ be a non-root of unity. R is an A-algebra by sending v to q. Let $\lambda : U_q^0 \to R$ be an algebra homomorphism.*

$M_R(\lambda)$ has finite Jordan-Holder length and if $L_R(\mu)$ is a composition factor of $M_R(\lambda)$ then $\mu \uparrow \lambda$ where $\uparrow$ is the usual strong linkage.

Proof. This will be proved by induction over $\uparrow$. If λ is anti-dominant (i.e. $q^\rho\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}>0}$ for all $\alpha \in \Pi$) then we get that all the maps φ_i are isomorphisms and so $M_R(\lambda)$ is simple. Now assume λ is not anti-dominant. A composition factor $L_R(\mu)$ must be killed by one of the φ_i's so must be a compostion factor of Q_R^w for some w. By the above calculations we see that if $q^\rho\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}>0}$ then $M_R^w(\lambda) \to M_R^{ws_\alpha}(\lambda)$ is an isomorphism and otherwise $Q_R^w = M_R^{ws_\alpha}(s_\alpha.\lambda)$. By induction all the Verma modules with highest weight μ strongly linked to λ has finite length and the composition factors are strongly linked to μ. This finishes the induction. $\square$

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Irreducible quantum group modules with finite dimensional weight spaces. I

Dennis Hasselstrøm Pedersen

Abstract

In this paper we classify all simple weight modules for a quantum group U_q at a complex odd root of unity q when the Lie algebra is not of type G_2. By a weight module we mean a finitely generated U_q-module which has finite dimensional weight spaces and is a sum of those. Our approach follows the procedures used by S. Fernando [Fer90] and O. Mathieu [Mat00] to solve the corresponding problem for semisimple complex Lie algebras.

1 Introduction and notation

Let $\mathfrak{g}$ be a simple complex Lie algebra not of type G_2. Let $q \in \mathbb{C}$ be a nonzero element and let $U_q := U_q(\mathfrak{g})$ be the quantum group over $\mathbb{C}$ with q as the quantum parameter (defined below). We want to classify all simple weight modules for U_q. In the papers [Fer90] and [Mat00] this is done for $\mathfrak{g}$-modules. Fernando proves in the paper [Fer90] that the classification of simple $\mathfrak{g}$ weight modules essentially boils down to classifying two classes of simple modules: The finite dimensional simple modules and the so called 'torsion free' simple modules. The classification of finite dimensional modules is well known in the classical case (as well as in the quantum group case) so the remaining problem is to classify the torsion free simple modules. Olivier Mathieu classifies these in the classical case in [Mat00]. The classification uses the concept of $\mathfrak{g}$ coherent families which are huge $\mathfrak{g}$ modules with weight vectors for every possible weight, see [Mat00, Section 4]. Mathieu shows that every torsion free simple module is a submodule of a unique irreducible semisimple coherent family and each of these irreducible semisimple coherent families contains a so-called admissible simple highest weight module as well. This reduces the classification to the classification of admissible simple highest weight modules.

1.1 Main results

In this paper we will first carry out the reduction done by Fernando to the quantum group case for q a non-root-of-unity and q an odd root of unity. Then we carry out the classification of torsion free simple module in the root of unity case. The corresponding classification of torsion free simple modules for generic q turns out to be much harder. We leave this to a subsequent paper [Ped15a].

We will follow closely the methods described in the two above mentioned papers. Many of the results can be directly translated from the classical case but in several cases we have to approach the problem a little differently. One of the first differences we encounter is the fact that in [Fer90] concepts are defined by using the root system without first choosing a base. Then later a base is chosen in an appropiate way. In the quantum group case we define the quantized enveloping algebra by first choosing a base of the root system and then defining the simple root vectors E_α, F_α, etc. This means that we can't later change the basis like in [Fer90]. The solution is to consider 'twists' of modules by Weyl group elements cf. definition 2.1. Another difference is the fact that we do not a priori have root vectors E_β for any positive root β unless β is simple. Root vectors can be constructed but the construction involves a choice of a reduced expression for the longest element of the Weyl group w_0. The root vectors constructed depend on this choice. So if we want to use root vectors to define our terms we should prove that our definitions are independent of the choice of the root vectors. Once the root vectors are defined we continue like in the classical case with some differences. Notably the proof of Proposition 2.11 is different. Here we reduce the problem to rank 2 calculations in the quantized enveloping algebra. This is also the main reason we exclude $\mathfrak{g}$ of type G_2 in this paper.

In the root of unity case the classification of simple weight modules reduces completely to the classical case as seen in Section 5. We use the same procedure as in [Mat00] to reduce the problem to classifying coherent families and then we show that all irreducible coherent families in the root of unity case can be constructed via classical $\mathfrak{g}$ coherent families.

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1.3 Notation

We will fix some notation: We denote by $\mathfrak{g}$ a fixed simple Lie algebra over the complex numbers $\mathbb{C}$. We assume $\mathfrak{g}$ is not of type G_2 to avoid unpleasant computations.

Fix a triangular decomposition of $\mathfrak{g}$: Let $\mathfrak{h}$ be a maximal toral subalgebra and let $\Phi \subset \mathfrak{h}^*$ be the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. Choose a simple system of roots $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$. Let Φ^+ (resp. Φ^-) be the positive (resp. negative) roots. Let $\mathfrak{g}^\pm$ be the positive and negative part of $\mathfrak{g}$ corresponding to the simple system Π. So $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$. Let W be the Weyl group generated by the simple reflections $s_i := s_{\alpha_i}$. For a $w \in W$ let $l(w)$ be the length of W i.e. the smallest amount of simple reflections such that $w = s_{i_1} \cdots s_{i_{l(w)}}$. Let $(\cdot|\cdot)$ be a standard W-invariant bilinear form on $\mathfrak{h}^*$ and $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha|\beta)}{(\beta|\beta)}$. Since $(\cdot|\cdot)$ is standard we have $(\alpha|\alpha) = 2$ for any short root $\alpha \in \Phi$. Let $Q = \operatorname{span}_{\mathbb{Z}} \{\alpha_1, \dots, \alpha_n\}$ denote the root lattice and $\Lambda = \operatorname{span}_{\mathbb{Z}} \{\omega_1, \dots, \omega_n\} \subset \mathfrak{h}^*$ the integral lattice where $\omega_i \in \mathfrak{h}^*$ are the fundamental weights defined by $(\omega_i|\alpha_j) = \delta_{ij}$.

Let $U_v = U_v(\mathfrak{g})$ be the corresponding quantized enveloping algebra defined over $\mathbb{Q}(v)$ as defined in [Jan96] with generators $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$, $\alpha \in \Pi$ and certain relations which can be found in Chapter 4 of [Jan96]. We define $v_\alpha = v^{(\alpha|\alpha)/2}$ (i.e. $v_\alpha = v$ if α is a short root and $v_\alpha = v^2$ if α is a long root) and for $n \in \mathbb{Z}$, $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$. Let $[n]_\alpha := [n]_{v_\alpha} = \frac{v_\alpha^n - v_\alpha^{-n}}{v_\alpha - v_\alpha^{-1}}$. We omit the subscripts when it is clear from the context. For later use we also define the quantum binomial coefficients: For $r \in \mathbb{N}$ and $a \in \mathbb{Z}$:

$$\begin{bmatrix} a \\ r \end{bmatrix}_v = \frac{[a][a-1]\cdots[a-r+1]}{[r]!}$$

where $[r]! := [r][r-1]\cdots[2][1]$. Let $A = \mathbb{Z}[v, v^{-1}]$ and let U_A be Lusztigs A-form defined in [Lus90], i.e. the A subalgebra generated by the divided powers $E_\alpha^{(n)} := \frac{1}{[n]_\alpha!} E_\alpha^n$, $F_\alpha^{(n)} := \frac{1}{[n]_\alpha!} F_\alpha^n$ and $K_\alpha^{\pm 1}$, $\alpha \in \Pi$.

Let $q \in \mathbb{C}$ be a nonzero complex number and set $U_q = U_A \otimes_A \mathbb{C}_q$ where $\mathbb{C}_q$ is the A-module equal to $\mathbb{C}$ as a vector space where v is sent to q. In the following sections we will distinguish between whether q is a root of unity or not.

We have a triangular decomposition of Lusztigs A-form $U_A = U_A^- \otimes U_A^0 \otimes U_A^+$ with U_A^- the A subalgebra generated by $\{F_\alpha^{(n)} | \alpha \in \Pi, n \in \mathbb{N}\}$ in U_A, U_A^+ the A subalgebra generated by $\{E_\alpha^{(n)} | \alpha \in \Pi, n \in \mathbb{N}\}$ in U_A and U_A^0 the A subalgebra generated by $\{K_\alpha^{\pm 1}, \begin{bmatrix} K_\alpha; c \\ r \end{bmatrix} | \alpha \in \Pi, c \in \mathbb{Z}, r \in \mathbb{N}\}$ in U_A where

$$\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix} := \prod_{j=1}^{r} \frac{K_\alpha v_\alpha^{c+1-j} - K_\alpha^{-1} v_\alpha^{-c-1+j}}{v_\alpha^j - v_\alpha^{-j}}.$$

For later use we also define $[K_\alpha; r] = \begin{bmatrix} K_\alpha; r \\ 1 \end{bmatrix}$. We have the corresponding triangular decomposition of U_q: $U_q = U_q^- \otimes U_q^0 \otimes U_q^+$ with $U_q^\pm = U_A^\pm \otimes_A \mathbb{C}_q$ and $U_q^0 = U_A^0 \otimes_A \mathbb{C}_q$.

For a $q \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ define $\begin{bmatrix} a \\ r \end{bmatrix}_q$ as the image of $\begin{bmatrix} a \\ r \end{bmatrix}_v$ in $\mathbb{C}$. We will omit the subscript from the notation when it is clear from the context. We define $q_\beta \in \mathbb{C}$ and $[n]_\beta \in \mathbb{C}$ as the image of $v_\beta \in A$ and $[n]_\beta \in A$, respectively abusing notation. Similarly, we will abuse notation and write $\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix}$ also for the image of $\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix} \in U_A$ in U_q. Define for $\mu \in Q$, $K_\mu = \prod_{i=1}^n K_{\alpha_i}^{a_i}$ if $\mu = \sum_{i=1}^n a_i \alpha_i$ with $a_i \in \mathbb{Z}$.

There is a braid group action on U_v which we will describe now. We use the definition from [Jan96, Chapter 8]. The definition is slightly different from the original in [Lus90, Theorem 3.1] (see [Jan96, Warning 8.14]). For each simple reflection s_i there is a braid operator that we will denote by T_{s_i} satisfying the following: $T_{s_i} : U_v \to U_v$ is a $\mathbb{Q}(v)$ automorphism and for $i \neq j \in \{1, \ldots, n\}$

$$\begin{aligned} T_{s_i}(K_\mu) &= K_{s_i(\mu)} \\ T_{s_i}(E_{\alpha_i}) &= -F_{\alpha_i} K_{\alpha_i} \\ T_{s_i}(F_{\alpha_i}) &= -K_{\alpha_i}^{-1} E_{\alpha_i} \\ T_{s_i}(E_{\alpha_j}) &= \sum_{i=0}^{-\langle \alpha_j, \alpha_i^\vee \rangle} (-1)^i v_{\alpha_i}^{-i} E_{\alpha_i}^{(r-i)} E_{\alpha_j} E_{\alpha_i}^{(i)} \\ T_{s_i}(F_{\alpha_j}) &= \sum_{i=0}^{-\langle \alpha_j, \alpha_i^\vee \rangle} (-1)^i v_{\alpha_i}^{i} F_{\alpha_i}^{(i)} F_{\alpha_j} F_{\alpha_i}^{(r-i)}. \end{aligned}$$

The inverse $T_{s_i}^{-1}$ is given by conjugating with the $\mathbb{Q}$-algebra anti-automorphism Ψ from [Lus90, section 1.1] defined as follows:

$$\Psi(E_{\alpha_i}) = E_{\alpha_i}, \quad \Psi(F_{\alpha_i}) = F_{\alpha_i}, \quad \Psi(K_{\alpha_i}) = K_{\alpha_i}^{-1}, \quad \Psi(v) = v.$$

The braid operators T_{s_i} satisfy braid relations so we can define T_w for any $w \in W$: Choose a reduced expression of w: $w = s_{i_1} \cdots s_{i_n}$. Then $T_w = T_{s_{i_1}} \cdots T_{s_{i_n}}$ is independent of the chosen reduced expression by [Lus90, Theorem 3.2]. We have $T_w(K_\mu) = K_{w(\mu)}$. Furthermore T_w restricts to an automorphism $T_w : U_A \to U_A$.

Let w_0 be the longest element in W and let $s_{i_1} \cdots s_{i_N}$ be a reduced expression of w_0. We define root vectors E_β and F_β for any $\beta \in \Phi^+$ by the following: First of all set

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), \text{ for } i = 1, \dots, N.$$

Then $\Phi^+ = \{\beta_1, \dots, \beta_N\}$. Set

$$E_{\beta_j} = T_{s_{i_1}} \cdots T_{s_{i_{j-1}}}(E_{\alpha_{i_j}})$$

and

$$F_{\beta_j} = T_{s_{i_1}} \cdots T_{s_{i_{j-1}}}(F_{\alpha_{i_j}}).$$

In this way we have defined root vectors for each $\beta \in \Phi^+$. These root vectors depend on the reduced expression chosen for w_0 above. For a different reduced expression we might get different root vectors. It is a fact that if $\beta \in \Pi$ then the root vectors E_β and F_β defined above are the same as the generators with the same notation (cf. e.g. [Jan96, Proposition 8.20]) so the notation is not ambigious in this case. By "Let E_β be a root vector" we just mean a root vector constructed as above for some reduced expression of w_0.

1.4 Basic definitions

Definition 1.1 *Let M be a U_q-module and $\lambda : U_q^0 \to \mathbb{C}$ a character (i.e. an algebra homomorphism into $\mathbb{C}$). Then the weight space M_λ is defined as*

$$M_\lambda = \{m \in M | \forall u \in U_q^0, um = \lambda(u)m\}.$$

Let X denote the set of characters of U_q^0. Let $\operatorname{wt} M$ denote all the weights of M, i.e. $\operatorname{wt} M = \{\lambda \in X | M_\lambda \neq 0\}$. If q is not a root of unity we define for $\mu \in \Lambda$ the character q^μ by $q^\mu(K_\alpha) = q^{(\mu|\alpha)}$ for any $\alpha \in \Pi$. We also define $q_\beta^\mu = q^{\frac{(\beta|\beta)}{2}\mu}$. We say that M only has integral weights if $\mu(K_\alpha) \in \pm q_\alpha^{\mathbb{Z}}$ for any $\alpha \in \Pi$, $\mu \in \operatorname{wt} M$.

If q is not a root of unity then U_q^0 is isomorphic to $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ and X can be identified with $(\mathbb{C}^*)^n$ by sending $\mu \in X$ to $(\mu(K_{\alpha_1}), \dots, \mu(K_{\alpha_n}))$. When q is a root of unity the situation is a bit more complex. We will show later that when q is a root of unity X can be identified with $S \times \Lambda_l \times \mathfrak{h}^*$ where S is the set of homomorphisms $Q \to \{\pm 1\}$ and Λ_l is a finite set depending on the order l of the root of unity. There is an action of W on X. For $\lambda \in X$ define $w\lambda$ by

$$(w\lambda)(u) = \lambda(T_{w^{-1}}(u)).$$

Note that $wq^\mu = q^{w(\mu)}$.

Definition 1.2 *Let M be a U_q-module and $w \in W$. Define the twisted module ${}^wM$ by the following:*

As a vector space ${}^wM = M$ but the action is given by twisting with w^{-1}: For $m \in {}^wM$ and $u \in U_q$:

$$u \cdot m = T_{w^{-1}}(u)m.$$

We also define ${}^{\overline{w}}M$ to be the inverse twist, i.e. for $m \in {}^{\overline{w}}M$, $u \in U_q$:

$$u \cdot m = T_{w^{-1}}^{-1}(u)m.$$

Hence for any U_q-module M, ${}^{\overline{w}}({}^wM) = M = {}^w({}^{\overline{w}}M)$.

Note that $\operatorname{wt} {}^wM = w(\operatorname{wt} M)$ and that ${}^w({}^{w'}M) \cong {}^{ww'}M$ for $w, w' \in W$ with $l(ww') = l(w) + l(w')$ because the braid operators T_w satisfy braid relations. Also ${}^{\overline{w}}({}^{\overline{w'}}M) \cong {}^{\overline{w'w}}M$.

Definition 1.3 *We define the category $\mathcal{F} = \mathcal{F}(\mathfrak{g})$ as the full subcategory of $U_q - \mathrm{Mod}$ such that for every $M \in \mathcal{F}$ we have*

1. *M is finitely generated as a U_q-module.*
2. *$M = \bigoplus_{\lambda \in X} M_\lambda$ and $\dim M_\lambda < \infty$.*

Note that the assignment $M \mapsto {}^wM$ is an endofunctor on $\mathcal{F}$ (in fact an auto-equivalence).

The goal of this paper is to classify all the simple modules in $\mathcal{F}$ in the case where $q \in \mathbb{C}$ is a root of unity. Our first step is a reduction to so called torsion free simple modules, see Definition 2.8. This reduction actually works for generic q as well and we treat that case first, see Section 2. Then in Section 3 we prove the corresponding reduction when q is a root of 1. To handle the torsion free simple modules we need some detailed calculations - found in [Ped15b] and recalled in Section 4 - on the commutation relations among quantum root vectors. Then we prove the classification of torsion free simple modules in Section 5 and Section 6. The classification for generic q turns out to be somewhat harder and will be the subject of a subsequent paper [Ped15a].

2 Nonroot of unity case: Reduction

In this section we fix a non-root-of-unity $q \in \mathbb{C}^*$.

Definition 2.1 *Let $M \in \mathcal{F}$ and let β be a root. M is called β-finite if for all $\lambda \in \operatorname{wt} M$ we have that $q^{\mathbb{N}\beta}\lambda \cap \operatorname{wt} M$ is a finite set. Here $q^{\mathbb{N}\beta}$ is the set $\{q^{i\beta} | i \in \mathbb{N}\}$ and $q^{i\beta}\lambda$ just means pointwise multiplication of characters.*

As an example consider a highest weight module M. For any positive root $\beta \in \Phi^+$, M is β-finite. If M is a Verma module then M is not β-finite for any negative root $\beta \in \Phi^-$.

Proposition 2.2 *Let $M \in \mathcal{F}$ and β a positive root. Let E_β be any choice of a root vector corresponding to β. Then the following are equivalent*

1. *M is β-finite.*

2. For all $m \in M$, $E_\beta^r m = 0$ for $r \gg 0$

Proof. Note that $E_\beta M_\lambda \subset M_{q^\beta \lambda}$. This shows that 1. implies 2.. Now assume 2. and assume M is not β-finite. Then we must have a $\lambda \in \operatorname{wt} M$, an increasing sequence $\{j_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$, weights $\mu_i = q^{j_i\beta}\lambda \in \operatorname{wt} M$ and weight vectors $0 \neq m_i \in M_{\mu_i}$ such that $E_\beta m_i = 0$. If $\lambda(K_\beta) = \pm q_\beta^j$ for some $j \in \mathbb{Z}$ then we can asssume without loss of generality that $j \in \mathbb{N}$ since otherwise we can replace λ by $q^{j_i\beta}\lambda$ for some sufficiently large j_i.

Now consider the subalgebra D of U_q generated by E_β, $K_\beta^{\pm 1}$ and F_β where F_β is the corresponding root vector to E_β (i.e. if $E_\beta = T_w(E_{\alpha_i})$ then $F_\beta = T_w(F_{\alpha_i})$). This is a subalgebra isomorphic to $U_{q_\beta}(\mathfrak{sl}_2)$. For each i we get a $U_{q_\beta}(\mathfrak{sl}_2)$-module Dm_i with highest weight μ_i. We claim that in each of those modules we have a weight vector $v_i \in Dm_i$ of weight λ:

To prove the claim it is enough to show that $F_\beta^{(j_i)} m_i \neq 0$ since F_β decreases the weight by β (i.e. $F_\beta M_\mu \subset M_{q^{-\beta}\mu}$). To show this we show that $E_\beta^{(j_i)} F_\beta^{(j_i)} m_i \neq 0$. In the following we will use Kac's formula:

$$E_\beta^{(r)} F_\beta^{(s)} = \sum_{j\geq 0} F_\beta^{(s-j)} \begin{bmatrix} K_\beta; 2j-r-s \\ j \end{bmatrix} E_\beta^{(r-j)}.$$

This is a well known formula that can be found in e.g. [Jan96, Lemma 1.7] (although in this reference it is written in a slightly different form).

$$\begin{aligned} E_\beta^{(j_i)} F_\beta^{(j_i)} m_i &= \sum_{s\geq 0} F_\beta^{(j_i-s)} \begin{bmatrix} K_\beta; 2s-2j_i \\ s \end{bmatrix} E_\beta^{(j_i-s)} m_i \\ &= \begin{bmatrix} K_\beta; 0 \\ j_i \end{bmatrix} m_i \\ &= \prod_{t=1}^{j_i} \frac{q_\beta^{1-t}\mu_i(K_\beta) - q_\beta^{t-1}\mu_i(K_\beta)^{-1}}{q_\beta^t - q_\beta^{-t}} m_i \\ &= \prod_{t=1}^{j_i} \frac{q_\beta^{2j_i+1-t}\lambda(K_\beta) - q_\beta^{-2j_i+t-1}\lambda(K_\beta)^{-1}}{q_\beta^t - q_\beta^{-t}} m_i. \end{aligned}$$

This is zero if and only if $\lambda(K_\beta) = \pm q_\beta^{-2j_i-1+t}$ for some $t = 1, \ldots, j_i$. Note that the power of q is negative in all cases here so this is not the case by the assumption above. So $F_\beta^{(j_i)} m_i \neq 0$ and we are done proving the claim. So we have $0 \neq v_i \in Dm_i$ of weight λ for $i \in \mathbb{N}$.

Consider the $U_{q_\beta}(\mathfrak{sl}_2)$ element $C_\beta = F_\beta E_\beta + \frac{q_\beta K_\beta + q_\beta^{-1} K_\beta^{-1}}{(q_\beta - q_\beta^{-1})^2}$. Then C_β acts on Dm_i by the scalar

$$\frac{q_\beta \mu_i(K_\beta) + q_\beta^{-1}\mu_i(K_\beta)^{-1}}{(q_\beta - q_\beta^{-1})}.$$

If C_β acts in the same way on Dm_i and Dm_k then we must have either $\mu_i(K_\beta) = \mu_k(K_\beta)$ (i.e. $i = j$) or $\mu_i(K_\beta) = q_\beta^{-2}\mu_j(K_\beta)^{-1}$. The second case implies that $\lambda(K_\beta) = \pm q_\beta^{-a}$ for some $a \in \mathbb{N}$ which we have ruled out above. So the vectors v_i are linearly independent. Hence M contains an infinite set of linearly independent vectors of weight λ. This contradicts the fact that $M \in \mathcal{F}$.□

Proposition 2.3 *Let β be a positive root and E_β a root vector corresponding to β. Let $M \in \mathcal{F}$. The set $M^{[E_\beta]} = \{m \in M | \dim \langle E_\beta \rangle m < \infty\}$ is a U_q-submodule of M.*

Proof. Assume first that β is a simple root. We want to show that for $v \in M^{[E_\beta]}$ we have for each $u \in U_q$, $uv \in M^{[E_\beta]}$. It is enough to show this for $u = F_\alpha$, $u = K_\alpha$ and $u = E_\alpha$ for all simple roots α. If $u = K_\alpha$ there is nothing to show since K_α acts diagonally on M. If $u = F_\alpha$ for $\alpha \neq \beta$ there is nothing to show since E_β and F_α commute. If $\alpha = \beta$ then we get the result from the identity

$$E_\alpha^{(r)} F_\alpha = F_\alpha E_\alpha^{(r)} + E_\alpha^{(r-1)}[K_\alpha; r-1]$$

found in e.g. [Jan96, section 4.4]. Finally if $u = E_\alpha$ and $\alpha \neq \beta$ then from the rank 2 calculations in [Lus90, section 5.3] we get:

- If $(\alpha|\beta) = 0$:
$$E_\beta^{(r)} E_\alpha = E_\alpha E_\beta^{(r)}.$$

- If $(\alpha|\beta) = -1$:
$$E_\beta^{(r)} E_\alpha = q^r E_\alpha E_\beta^{(r)} + q E_{\alpha+\beta} E_\beta^{(r-1)}$$
where $E_{\alpha+\beta} := T_{s_\alpha}(E_\beta)$.

- If $(\alpha|\beta) = -2$ and $\langle \alpha, \beta^\vee \rangle = -2$:
$$E_\beta^{(r)} E_\alpha = q^{2r} E_\alpha E_\beta^{(r)} + q^{r+1} E_{\alpha+\beta} E_\beta^{(r-1)} + q^2 E_{2\beta+\alpha} E_\beta^{(r-2)}$$
where $E_{\alpha+\beta} := T_{s_\alpha}(E_\beta)$ and $E_{2\beta+\alpha} := T_{s_\alpha} T_{s_\beta}(E_\alpha)$.

- If $(\alpha|\beta) = -2$ and $\langle \alpha, \beta^\vee \rangle = -1$: In this case we get from the calculations in [Lus90, section 5.3] that
$$E_\alpha E_\beta^{(r)} = q^{2r} E_\beta^{(r)} E_\alpha + q^2 E_\beta^{(r-1)} E_{\alpha+\beta}$$
where $E_{\alpha+\beta} := T_{s_\beta}(E_\alpha)$.
After using the $\mathbb{Q}$-algebra anti automorphism Ψ from [Lus90, section 1.1] we get
$$E_\beta^{(r)} E_\alpha = q^{2r} E_\alpha E_\beta^{(r)} + q^2 E'_{\alpha+\beta} E_\beta^{(r-1)}$$
where $E'_{\alpha+\beta} = \Psi(E_{\alpha+\beta}) = T_{s_\beta}^{-1}(E_\alpha)$.

In all cases we get that if $E_\beta^{(n)} m = 0$ for $n >> 0$ then $E_\beta^{(n)} E_\alpha m = 0$ for $n >> 0$. This proves that $uv \in \{m \in M | \dim \langle E_\beta \rangle m < \infty\}$ in this case also.

If β is not simple then $E_\beta = T_w(E_{\alpha'})$ for some simple root α' and some $w \in W$. Since T_w is an automorphism we have $T_w(U_q) = U_q$ so instead of proving the claim for $u = E_\alpha$, K_α and F_α we can show it for $u = T_w(E_\alpha)$, $T_w(K_\alpha)$ and $T_w(F_\alpha)$ so the claim follows from the calculations above. $\square$

Lemma 2.4 *Let E_β and E'_β be two choices of root vectors. Then $M^{[E_\beta]} = M^{[E'_\beta]}$*

Proof. Suppose we have two root vectors E_β and E'_β. By Proposition 2.3 and Proposition 2.2 we have $\dim \left\langle E'_\beta \right\rangle m < \infty$ for all $m \in M^{[E_\beta]}$ so $M^{[E_\beta]} \subset M^{[E'_\beta]}$. Symmetrically we have also $M^{[E'_\beta]} \subset M^{[E_\beta]}$. □

Definition 2.5 *Let β be a positive root and E_β a root vector corresponding to β. Define $M^{[\beta]} = \{m \in M | \dim \langle E_\beta \rangle m < \infty\}$.*

By Lemma 2.4 this definition is independent of the chosen root vector.

Everything here that is done for a positive root β can be done for a negative root just by replacing the E's with F's, i.e. for a negative root $\beta \in \Phi^-$, $M^{[\beta]} = \{m \in M | \dim \langle F_{-\beta} \rangle m < \infty\}$ and so on.

Definition 2.6 *Let $M \in \mathcal{F}$. Let $\beta \in \Phi$. M is called β-free if $M^{[\beta]} = 0$.*

Note that M is β-finite if and only if $M^{[\beta]} = M$ so β-free is, in a way, the opposite of being β-finite. Suppose $L \in \mathcal{F}$ is a simple module and β a root. Then by Proposition 2.3 L is either β-finite or β-free.

Definition 2.7 *Let $M \in \mathcal{F}$. Define $F_M = \{\beta \in \Phi | M$ is β-finite$\}$ and $T_M = \{\beta \in \Phi | M$ is β-free$\}$. For later use we also define $F_M^s := F_M \cap (-F_M)$ and $T_M^s := T_M \cap (-T_M)$ to be the symmetrical parts of F_M and T_M.*

Note that $\Phi = F_L \cup T_L$ for a simple module L and this is a disjoint union.

Definition 2.8 *A module M is called torsion free if $T_M = \Phi$.*

Proposition 2.9 *Let L be a simple module and β a root. L is β-free if and only if $q^{\mathbb{N}\beta} \operatorname{wt} L \subset \operatorname{wt} L$.*

Proof. Assume L is β-free and $\beta \in \Phi^+$. Let E_β be a corresponding root vector. The proof is similar for $\beta \in \Phi^-$ but with F instead of E. Then for all $0 \neq m \in L$, $E_\beta^{(r)} m \neq 0$. If $\lambda \in \operatorname{wt} L$ then there exists $0 \neq m_\lambda \in L_\lambda$ and since $E_\beta^{(r)} m_\lambda \in L_{q^{r\beta}\lambda}$ the implication follows. For the other way assume $q^{\mathbb{N}\beta} \operatorname{wt} L \subset \operatorname{wt} L$. Then L is clearly not β-finite. Since L is simple L must then be β-free. □

Proposition 2.10 *Let $L \in \mathcal{F}$ be a simple module. T_L is a closed subset of the roots Φ. That is if $\beta, \gamma \in T_L$ and $\beta + \gamma \in \Phi$. Then $\beta + \gamma \in T_L$.*

Proof. Since L is β-free we have $q^{\mathbb{N}\beta} \operatorname{wt} L \subset \operatorname{wt} L$ and since L is γ free we get further $q^{\mathbb{N}\gamma} q^{\mathbb{N}\beta} \operatorname{wt} L \subset \operatorname{wt} L$ so therefore $q^{\mathbb{N}(\beta+\gamma)} \operatorname{wt} L \subset \operatorname{wt} L$ hence L is $(\beta + \gamma)$ free. □

Proposition 2.11 *Let $M \in \mathcal{F}$ be a U_q-module. F_M is a closed subset of Φ. That is if $\beta, \gamma \in F_M$ and $\beta + \gamma \in \Phi$ then $\beta + \gamma \in F_M$.*

Proof. Let $\alpha, \beta \in F_M$ with $\alpha + \beta \in \Phi$. We have to show that $\alpha + \beta \in F_M$. First let us show the claim if the root system Φ is a rank 2 root system. In this case the claim will follow from the rank 2 calculations in [Lus90]. Assume $\Pi = \{\alpha_1, \alpha_2\}$. Assume first that we have $\alpha \in \Pi$ and $\beta \in \Phi^+$. We show below that we can always reduce to this situation. We can assume $\alpha = \alpha_1$ by renumbering if neccesary. We now have 5 possibilites:

Case 0) $(\alpha_1, \alpha_2) = 0$ is clear.

Case 1): $(\alpha_1|\alpha_2)=-1$. The only possibility for $\beta\in\Phi^+$ such that $\alpha+\beta$ is a root is $\beta=\alpha_2$. Set $E_{\alpha+\beta}=T_{s_\beta}(E_\alpha)$ then Lusztig shows in [Lus90, section 5.5] that

$$E_{\alpha+\beta}^{(k)}=\sum_{t=0}^{k}(-1)^t q^{-t}E_\beta^{(k-t)}E_\alpha^{(k)}E_\beta^{(t)}.$$

The difference in the definition of the braid operators between [Jan96] and [Lus90] means that we have to multiply the formula in [Lus90] by $(-1)^k$ since (using the notation of [Lus90]) $E_{12}=-E_{\alpha+\beta}$. Let $m\in M$. Then there exists a $T\in\mathbb{N}$ such that $E_\beta^{(t)}m=0$ for $t\geq T$ since M is β-finite. Let $m_t=E_\beta^{(t)}m$, $t=0,1,\dots,T$. For each m_t there is a $K_t\in\mathbb{N}$ such that $E_\alpha^{(k)}m_t=0$ for $k\geq K_t$ since M is α-finite. Set $K=\max\{T,K_0,\dots,K_T\}$ then the above identity shows that $E_{\alpha+\beta}^{(k)}m=0$ for $k\geq K$

Case 2): $\langle\alpha_1,\alpha_2^\vee\rangle=-2$. In this case $\beta=\alpha_2$ is the only possibility to choose $\beta\in\Phi^+$ such that $\alpha+\beta\in\Phi$. Set $E_{\alpha+\beta}=T_\alpha(E_\beta)$ then by [Lus90, section 5.5]:

$$E_{\alpha+\beta}^{(k)}=\sum_{t=0}^{k}(-1)^t q^{-2t}E_\alpha^{(k-t)}E_\beta^{(k)}E_\alpha^{(t)}$$

and the same argument as above works.

Case 3): $\langle\alpha_2,\alpha_1^\vee\rangle=-2$ and $\beta=\alpha_2$. Set $E_{\alpha+\beta}=T_\beta(E_\alpha)$ then

$$E_{\alpha+\beta}^{(k)}=\sum_{t=0}^{k}(-1)^t q^{-2t}E_\beta^{(k-t)}E_\alpha^{(k)}E_\beta^{(t)}$$

and the argument follows like in case 1) and 2).

Case 4): $\langle\alpha_2,\alpha_1^\vee\rangle=-2$ and $\beta=\alpha_1+\alpha_2$. In this case set $E_\beta=E_{\alpha_1+\alpha_2}=T_{\alpha_2}(E_{\alpha_1})$ and $E_{\alpha+\beta}=E_{2\alpha_1+\alpha_2}=T_{\alpha_2}T_{\alpha_1}(E_{\alpha_2})$. We want a property similar to the one in the other cases. We want to show that there exists $c_t\in\mathbb{Q}(q)$ such that

$$E_{2\alpha_1+\alpha_2}^{(k)}=\sum_{t=0}^{k}c_tE_{\alpha_1}^{(k-t)}E_{\alpha_1+\alpha_2}^{(k)}E_{\alpha_1}^{(t)}.$$

We will use notation like in [Lus90] so set $E_1=E_{\alpha_1}$, $E_{12}=E_{\alpha_1+\alpha_2}$ and $E_{112}=E_{2\alpha_1+\alpha_2}$. Let $k\in\mathbb{N}$. By 5.3 (h) in [Lus90]

$$E_1^{(k)}E_{12}^{(k)}=(-1)^kq^k\prod_{i=1}^{k}(q^{2i}+1)E_{112}^{(k)}+\sum_{s=0}^{k-1}(-1)^sq^{s-s(k-s)-s(t-s)}\left(\prod_{i=1}^{s}(q^{2i}+1)\right)E_{12}^{(k-s)}E_{112}^{(s)}E_1^{(k-s)}$$

so

$$E_{112}^{(k)}=(-1)^kc\left(E_1^{(k)}E_{12}^{(k)}-\sum_{s=0}^{k-1}(-1)^sq^{s-s(k-s)-s(t-s)}\left(\prod_{i=1}^{s}(q^{2i}+1)\right)E_{12}^{(k-s)}E_{112}^{(s)}E_1^{(k-s)}\right)$$

where $c=\left(q^k\prod_{i=1}^k(q^{2i}+1)\right)^{-1}$.

We will show by induction over $s<k$ that there exists $a_i\in\mathbb{Q}(q)$ such that

$$E_{12}^{(k-s)}E_{112}^{(s)}E_1^{(k-s)}=\sum_{i=0}^{s}a_iE_1^{(i)}E_{12}^{(k)}E_1^{(k-i)}.$$

The induction start $s = 0$ is obvious. Now observe that again from 5.3 (h) in [Lus90] we have for $s < k$:

$$\begin{aligned} E_1^{(s)} E_{12}^{(k)} =& (-1)^s q^{s-s(k-s)} \prod_{i=1}^{s} (q^{2i}+1) E_{12}^{(k-s)} E_{112}^{(s)} \\ &+ \sum_{n=0}^{s-1} (-1)^n q^{n-n(s-n)-n(k-n)} \left(\prod_{i=1}^{n} (q^{2i}+1) \right) E_{12}^{(k-n)} E_{112}^{(n)} E_1^{(s-n)}. \end{aligned}$$

So

$$E_{12}^{(k-s)} E_{112}^{(s)} = (-1)^s \left(q^{s-s(k-s)} \prod_{i=1}^{s} (q^{2i}+1) \right)^{-1} \left(E_1^{(s)} E_{12}^{(k)} - \sum_{n=0}^{s-1} (-1)^n b_n E_{12}^{(k-n)} E_{112}^{(n)} E_1^{(s-n)} \right)$$

where $b_n \in \mathbb{Q}(q)$ are the coefficients above. Hence

$$E_{12}^{(k-s)} E_{112}^{(s)} E_1^{(k-s)} = (-1)^s b E_1^{(s)} E_{12}^{(k)} E_1^{(k-s)} + \sum_{n=0}^{s-1} (-1)^{s+n} b'_n E_{12}^{(k-n)} E_{112}^{(n)} E_1^{(k-n)}$$

for some coefficients b and $b'_n \in \mathbb{Q}(q)$. This identity completes the induction over s.

So to sum up we have proven that there exists $c_t \in \mathbb{Q}(q)$ such that

$$E_{2\alpha_1+\alpha_2}^{(k)} = \sum_{t=0}^{k} c_t E_{\alpha_1}^{(k-t)} E_{\alpha_1+\alpha_2}^{(k)} E_{\alpha_1}^{(t)}.$$

(Note for later use in the root of unity case that the c_t are in the localization of $\mathbb{Z}[q, q^{-1}]$ in the elements $(q^{2i}+1)$ for $i \in \mathbb{N}$ which are nonzero unless q is an lth root of unity with l even). Now the proof goes as above.

The above 5 cases are the only possible cases with the above assumptions since we have excluded G_2.

We will now show how to reduce the problem to rank 2. Assume $\beta, \gamma \in F_M$ and $\beta + \gamma \in \Phi$. We will first show:

- There exists a $w \in W$ such that $w(\beta) \in \Pi$ and $w(\gamma) \in \Phi^+$.

Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced expression and let $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. Then $\Phi^+ = \{\beta_1, \ldots, \beta_N\}$. Assume first that both β and γ are positive. Then $\beta = \beta_j$ and $\gamma = \beta_r$ for some j and r. Without loss of generality we can assume $j < r$. Then we can set $w = s_{i_{j-1}} \cdots s_{i_1}$ in this case. If β and γ are both negative then $w_0(\beta)$ and $w_0(\gamma)$ are both positive and we can do as before. Assume $\beta < 0$ and $\gamma > 0$. Assume $\beta = -\beta_j$ and $\gamma = \beta_r$ for some j and r. Without loss of generality we can assume $j < r$. Then set $w = s_{i_j} \cdots s_{i_1}$. The claim has been shown.

Next we will show:

- There exists a $w \in W$ such that $w(\beta)$ and $w(\gamma)$ is contained in a rank 2 subsystem of the roots.

If $(\beta|\gamma) < 0$ then there exists a simple system Π' of Φ such that β and γ are in Π'. But since all simple system of a root system are W conjugate the claim follows. Assume $(\beta|\gamma) \geq 0$. Then $\langle \beta + \gamma, \gamma^\vee \rangle \geq \langle \gamma | \gamma^\vee \rangle = 2$ so

$s_\gamma(\beta+\gamma) = \beta+\gamma - \langle \beta+\gamma, \gamma^\vee \rangle \gamma \leq \beta - \gamma$. So $\beta-\gamma$ is a root in this case. Since we have excluded G_2 this means that the γ string through β is $\beta-\gamma, \beta, \beta+\gamma$ and therefore $\langle \beta+\gamma, \gamma^\vee \rangle = 2$ or equivalently $\langle \beta, \gamma^\vee \rangle = 0$. So $(\beta-\gamma|\gamma) = -(\gamma|\gamma) < 0$. Hence there is a simple system of roots Π' such that $\gamma, \beta-\gamma \in \Pi'$. So there exists w such that $w(\gamma)$ and $w(\beta-\gamma)$ are simple roots. Since $w(\beta) = w(\gamma) + w(\beta-\gamma)$ we see that $w(\beta)$ and $w(\gamma)$ are contained in a rank 2 subsystem of Φ. So the second claim is proven.

Note that ${}^wM$ is $w(\beta)$ and $w(\gamma)$ finite: Since $\operatorname{wt} {}^wM = w(\operatorname{wt} M)$ we have that a $\mu \in \operatorname{wt} {}^wM$ is of the form $\mu = w(\lambda)$ for some $\lambda \in \operatorname{wt} M$. Now $q^{\mathbb{N}w(\beta)}\mu \cap \operatorname{wt} {}^wM = w(q^{\mathbb{N}\beta}\lambda \cap \operatorname{wt} M)$ is finite because M was β-finite. All in all we get that for some w we have $w(\beta+\gamma) \in F_{{}^wM}$. But since $F_{{}^wM} = w(F_M)$ this shows that $\beta+\gamma \in F_M$.$\square$

Let L be a simple module. Since F_L and T_L are both closed subsets of Φ we get from [Fer90, Lemma 4.16] that $P_L := F_L \cup T_L^s$ is a parabolic subset of the roots - i.e. $P_L \cup (-P_L) = \Phi$ and P_L is a closed subset of Φ.

Since $P_L \cup (-P_L) = \Phi$ we must have for some $w \in W$, $\Phi^+ \subset w(P_L)$. From now on we will assume $\Phi^+ \subset P_L$ since otherwise we can just describe the module ${}^wL$ and then untwist once we have described this module. So we assume $P_L = \Phi^+ \cup \langle \Pi' \rangle$ where $\Pi' \subset \Pi$ and where $\langle \Pi' \rangle$ denotes the subset of Φ generated by Π', i.e. $\langle \Pi' \rangle = \mathbb{Z}\Pi' \cap \Phi$.

Let $\mathfrak{p}$ be the parabolic Lie algebra corresponding to P_L i.e. $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\beta \in P_L} \mathfrak{g}_\beta$ and let $\mathfrak{l}$ and $\mathfrak{u}$ be the Levi part and the nilpotent part of $\mathfrak{p}$ respectively i.e. $\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\beta \in P_L^s} \mathfrak{g}_\beta$ and $\mathfrak{u} = \bigoplus_{\beta \in P_L \setminus P_L^s} \mathfrak{g}_\beta$. We can define $U_q(\mathfrak{p})$, $U_q(\mathfrak{l})$ and $U_q(\mathfrak{u})$. Furthermore we can define $U_q(\mathfrak{u}^-)$ where $\mathfrak{u}^-$ is the nilpotent part of the opposite parabolic $\mathfrak{p}^-$ corresponding to $(-P_L)$. We have $U_q(\mathfrak{p}) = U_q(\mathfrak{l})U_q(\mathfrak{u})$ and $U_q(\mathfrak{g}) = U_q(\mathfrak{u}^-)U_q(\mathfrak{p})$.

Here is how we define the above subalgebras: (Defined like in [Pul06]) Assume $P_L = \Phi^+ \cup \langle \Pi' \rangle$. Let $w_0^{\mathfrak{l}}$ be the longest element in the Weyl group $W^{\mathfrak{l}}$ corresponding to Π'. Let w_0 be the longest element in W. Set $\overline{w} = w_0(w_0^{\mathfrak{l}})^{-1}$. Choose a reduced expression $w_0 = s_{j_1} \cdots s_{j_k} s_{i_1} \cdots s_{i_h}$ such that $w_0^{\mathfrak{l}} = s_{i_1} \cdots s_{i_h}$. Let $\{E_\beta, F_\beta | \beta \in \Phi^+\}$ be the root vectors defined by this reduced expression.

Set

$$\beta_t^1 = \beta_{t+k} = \overline{w} s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}), \quad t = 1, \ldots, h$$
$$\beta_t^2 = \beta_t = s_{j_1} \cdots s_{j_{t-1}}(\alpha_{j_t}), \quad t = 1, \ldots, k.$$

This means that

$$F_{\beta_t^1} = T_{\overline{w}} T_{s_{i_1}} \cdot T_{s_{i_{t-1}}}(F_{\alpha_{i_t}}), \quad t = 1, \ldots, h$$
$$F_{\beta_t^2} = T_{s_{j_1}} \cdot T_{s_{j_{t-1}}}(F_{\alpha_{j_t}}), \quad t = 1, \ldots, k$$

and similarly for the E's.

We define

$$U_q(\mathfrak{p}) = \left\langle E_{\beta_j}, K_\mu, F_{\beta_i^1} \right\rangle_{j=1,\ldots,N, \mu \in Q, i=1,\ldots h},$$

$$U_q(\mathfrak{l}) = \left\langle E_{\beta_i^1}, K_\mu, F_{\beta_i^1} \right\rangle_{\mu \in Q, i=1,\ldots h}$$

and

$$U_q(\mathfrak{u}) = \left\langle E_{\beta_i^2} \right\rangle_{i=1,\ldots,k}.$$

Similarly we define $U_q(\mathfrak{u}^-) = \left\langle F_{\beta_i^2} \right\rangle_{i=1,\dots,h}$. All of these are subalgebras of $U_q(\mathfrak{g})$ are independent of the chosen reduced expression of w_0 and $w_0^{\mathfrak{l}}$. Furthermore $U_q(\mathfrak{p})$ and $U_q(\mathfrak{l})$ are Hopf subalgebras of $U_q(\mathfrak{g})$ as stated in [Pul06, Proposition 5 and Lemma 2].

There is a Q grading on U_q with $\deg E_\alpha = \alpha$, $\deg F_\alpha = -\alpha$ and $\deg K_\beta^{\pm 1} = 0$ as described in e.g. [Jan96, section 4.7]. This induces a grading on $U_q^{\pm}$ and on $U_q(\mathfrak{u})$ and $U_q(\mathfrak{u}^-)$. We will define $U_q(\mathfrak{u})^{>0}$ and $U_q(\mathfrak{u}^-)^{<0}$ to be the subalgebras consisting of elements with nonzero degree (i.e. the augmentation ideals).

Definition 2.12 *Let $\mathfrak{p}$ be a standard parabolic sub Lie algebra of $\mathfrak{g}$ and let $\mathfrak{l}$, $\mathfrak{u}$ and $\mathfrak{u}^-$ be defined as above. Let N be a $U_q(\mathfrak{l})$-module. We define*

$$\mathcal{M}(N) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} N,$$

where N is considered as a $U_q(\mathfrak{p})$-module with $U_q(\mathfrak{u})$ acting trivially, i.e. through the coidentity $\varepsilon : U_q(\mathfrak{u}) \to \mathbb{C}$ sending everything of nonzero degree to zero.

Definition 2.13 *If M is a $U_q(\mathfrak{g})$-module we define*

$$M^{\mathfrak{u}} = \{m \in M | xm = \varepsilon(x)m,\ x \in U_q(\mathfrak{u})\}.$$

Proposition 2.14 *Let M be a $U_q(\mathfrak{g})$-module. $M^{\mathfrak{u}}$ is a $U_q(\mathfrak{l})$-module.*

Proof. We will show that for $u \in U_q(\mathfrak{l})$, $U_q(\mathfrak{u})^{>0}u \cap U_q(\mathfrak{g})U_q(\mathfrak{u})^{>0} \neq \emptyset$. This is true by simple grading considerations. We know that $U_q(\mathfrak{u})^{>0}u \subset U_q(\mathfrak{l})U_q(\mathfrak{u}) = U_q(\mathfrak{l})U_q(\mathfrak{u})^{>0} + U_q(\mathfrak{l})$. But the degree of a homogeneous element $u'u \in U_q^-$ with $u' \in U_q(\mathfrak{u})^{>0}$ cannot be in $\mathbb{Z}\Pi'$ since that would mean $u' \in U_q(\mathfrak{l})$. So $U_q(\mathfrak{u})^{>0}u \subset U_q(\mathfrak{l})U_q(\mathfrak{u})^{>0}$. □

Proposition 2.15 *Let N be a $U_q(\mathfrak{l})$-module and let M be a $U_q(\mathfrak{g})$-module. There are natural vector space isomorphisms*

$$\Phi = \Phi_{M,N} : \mathrm{Hom}_{U_q(\mathfrak{g})}(\mathcal{M}(N), M) \cong \mathrm{Hom}_{U_q(\mathfrak{l})}(N, M^{\mathfrak{u}}).$$

Proof. If $f : \mathcal{M}(N) \to M$ is a $U_q(\mathfrak{g})$-module map then $\Phi(f) : N \to M^{\mathfrak{u}}$ is defined by $\Phi(f) = f^{\mathfrak{u}} \circ (1 \otimes \mathrm{id}_N)$, where $1 \otimes \mathrm{id}_N : N \to \mathcal{M}(N)^{\mathfrak{u}}$ is given by $n \mapsto 1 \otimes n$ and $f^{\mathfrak{u}} : \mathcal{M}(N)^{\mathfrak{u}} \to M^{\mathfrak{u}}$ is the restriction of f to $\mathcal{M}(N)^{\mathfrak{u}}$.

The inverse map Ψ is given by: For $g : N \to M^{\mathfrak{u}}$, $\Psi(g)(u \otimes n) = ug(n)$. It is easy to check that Φ and Ψ are inverse to each other. □

Proposition 2.16 *If X is a simple $U_q(\mathfrak{l})$-module then $\mathcal{M}(X)$ has a unique simple quotient $L(X)$.*

Proof. The proof is exactly the same as the proof of Proposition 3.3 in [Fer90]: Suppose M is a submodule of $\mathcal{M}(X)$. If $0 \neq v \in M \cap (1 \otimes X)$ then $U_q v = U_q U_q(\mathfrak{l})v = U_q(1 \otimes X) = \mathcal{M}(X)$ so $M \cap (1 \otimes X) = 0$ for every proper submodule M. Let N be the sum of all proper submodules. N is proper since $N \cap (1 \otimes X) = 0$ and maximal since it is the sum of all proper submodules. □

Let $\mathcal{F}(\mathfrak{l})$ denote the full subcategory of $U_q(\mathfrak{l})$-modules that consists of modules that are finitely generated over U_q and are weight modules with finite dimensional weight spaces.

Proposition 2.17 *The maps $L : N \mapsto L(N)$ and $F : V \mapsto V^{\mathfrak{u}}$ determine a bijective correspondence between the simple modules in $\mathcal{F}(\mathfrak{l})$ and the simple modules M in $\mathcal{F}(\mathfrak{g})$ that have $M^{\mathfrak{u}} \neq 0$. L and F are inverse to each other.*

The second part of the proof is just a quantum version of the proof of Proposition 3.8 in [Fer90]. The first part is shown a little differently here.

Proof. First we will show that if V is a simple $U_q(\mathfrak{g})$-module with $V^{\mathfrak{u}} \neq 0$ then $V^{\mathfrak{u}}$ is a simple $U_q(\mathfrak{l})$-module: Assume $0 \neq V_1 \subset V^{\mathfrak{u}}$ is a $U_q(\mathfrak{l})$-submodule of $V^{\mathfrak{u}}$. We will show that $V_1 = V^{\mathfrak{u}}$. Since V is a simple $U_q(\mathfrak{g})$-module we have $V = U_q(\mathfrak{g})V_1$. Now as a vector space we have

$$\begin{aligned} V = U_q(\mathfrak{g})V_1 = U_q(\mathfrak{u}^-)U_q(\mathfrak{l})U_q(\mathfrak{u})V_1 &= U_q(\mathfrak{u}^-)U_q(\mathfrak{l})(U_q(\mathfrak{u})^{>0} + \mathbb{C})V_1 \\ &= U_q(\mathfrak{u}^-)U_q(\mathfrak{l})V_1 \\ &= U_q(\mathfrak{u}^-)V_1 \\ &= (U_q(\mathfrak{u}^-)^{<0} + \mathbb{C})V_1 \\ &= U_q(\mathfrak{u}^-)^{<0}V_1 + V_1. \end{aligned}$$

We are done if we show $U_q(\mathfrak{u}^-)^{<0}V^{\mathfrak{u}} \cap V^{\mathfrak{u}} = 0$. Observe that $U_q(\mathfrak{u}^-)^{<0}V^{\mathfrak{u}}$ is a $U_q(\mathfrak{l})$ module since $U_q(\mathfrak{l})U_q(\mathfrak{u}^-)^{<0} = U_q(\mathfrak{u}^-)^{<0}U_q(\mathfrak{l})$. Assume $v \in V^{\mathfrak{u}}$ and assume we have a $u' \in U_q(\mathfrak{u}^-)^{<0}$ such that $u'v \in V^{\mathfrak{u}}$. We can assume $u' \in (U_q(\mathfrak{u}^-)^{<0})_\gamma$ for some $\gamma \in Q$ and $v \in V_\mu$ for some $\mu \in X$. Assume $u'v \neq 0$. Then since V is simple there exists a $u \in U_q$ such that $uu'v = v$ but by weight considerations we must have $u \in (U_q)_{-\gamma} \subset U_q(\mathfrak{p}^-)U_q(\mathfrak{u})^{>0}$ so $uu'v = 0$ since $u'v \in V^{\mathfrak{u}}$. A contradiction.

Now assume N is a simple $U_q(\mathfrak{l})$ module. $L(N)^{\mathfrak{u}}$ is simple by the above. Let Φ be the isomorphism from Proposition 2.15 and consider $\Phi(p) : N \to L(N)^{\mathfrak{u}}$ where $p : \mathcal{M}(N) \to L(N)$ is the cannocial projection from $\mathcal{M}(N)$ to $L(N)$. Since Φ is an isomorphism the map $\Phi(p)$ is nonzero. Since N is simple by assumption and $L(N)^{\mathfrak{u}}$ is simple by the above we get that $\Phi(p)$ is an isomorphism.

Suppose V is a simple $U_q(\mathfrak{g})$-module such that $V^{\mathfrak{u}}$ is nonzero. Let $f = \Phi^{-1}(\mathrm{id}) : M(V^{\mathfrak{u}}) \to V$ where $\mathrm{id} : V^{\mathfrak{u}} \to V^{\mathfrak{u}}$ is the identity map. Then f is nonzero and therefore surjective because V is simple. But since $L(V^{\mathfrak{u}})$ is the unique simple quotient of $M(V^{\mathfrak{u}})$ we get $L(V^{\mathfrak{u}}) = V$. □

Let $\mathfrak{p}$ be a standard parabolic subalgebra of $\mathfrak{g}$ and define $U_q(\mathfrak{p})$, $\mathfrak{l}$, $U_q(\mathfrak{l})$ etc. as above. Let $\Phi^{\mathfrak{l}}$ be the roots corresponding to $\mathfrak{l}$ i.e. such that $\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi^{\mathfrak{l}}} \mathfrak{g}_\beta$. Then for $\beta \in \Phi^{\mathfrak{l}}$ and a $U_q(\mathfrak{l})$-module M we define β-finite, β-free, $M^{[\beta]}$ etc. as above. The definitions, lemmas and propositions above still hold in this case as long as we require $\beta \in \Phi^{\mathfrak{l}}$ so that we actually have root vectors $E_\beta, F_\beta \in U_q(\mathfrak{l})$. We define $T_M := \{\beta \in \Phi^{\mathfrak{l}} | M^{[\beta]} = 0\}$ and $F_M := \{\beta \in \Phi^{\mathfrak{l}} | M^{[\beta]} = M\}$ i.e. as before but only for roots in $\Phi^{\mathfrak{l}}$.

By now we have reduced the problem of classifying simple modules in $\mathcal{F}(\mathfrak{g})$ somewhat. If $L \in \mathcal{F}$ is a simple module we know that there exists some w such that $\Phi^+ \subset P_{{}^wL}$. Define $\mathfrak{l}$, $U_q(\mathfrak{l})$ from L etc. as above, then $\Phi^{\mathfrak{l}} = \langle \Pi' \rangle = F_L^s \cup T_L^s$ where Π' is the subset of simple roots such that $P_L = \Phi^+ \cup \langle \Pi' \rangle$. From the above we get then that ${}^wL$ is completely determined by the simple $U_q(\mathfrak{l})$-module $({}^wL)^{\mathfrak{u}}$. So we have reduced the problem to looking at simple $U_q(\mathfrak{l})$-modules N satisfying $\Phi^{\mathfrak{l}} = F_N^s \cup T_N^s$.

We claim that $\Pi' = \Pi'_{F_N^s} \cup \Pi'_{T_N^s}$ such that $F_N^s = \langle \Pi'_{F_N} \rangle$ and $T_N^s = \langle \Pi'_{T_N} \rangle$ and such that none of the simple roots in $\Pi'_{F_N^s}$ are connected to any simple root from $\Pi'_{T_N^s}$. Suppose $\alpha \in F_N^s$ is a simple root and suppose $\alpha' \in \Pi'$ is a simple root that is connected to α in the Dynkin diagram. So $\alpha + \alpha'$ is a root. There are two possibilities. Either $\alpha + \alpha' \in F_N$ or $\alpha + \alpha' \in T_N$. If $\alpha + \alpha' \in F_N$: Since F_N^s is symmetric we have $-\alpha \in F_N^s$ and since F_N is closed $\alpha' = \alpha + \alpha' + (-\alpha) \in F_N$. If $\alpha + \alpha' \in T_N$ and $\alpha' \in T_N$ then we get similarly $\alpha \in T_N$ which is a contradiction. So $\alpha' \in F_N$. We have shown that if $\alpha \in F_N$ then any simple root connected to α is in F_N also. So F_N and T_N contains different connected components of the Dynkin diagram for $\Phi^{\mathfrak{l}}$.

Let $\tau = c(\mathfrak{l}) \oplus \mathfrak{g}_{F_N^s} \oplus \mathfrak{h}_{F_N^s}$ and $\mathfrak{t} = \mathfrak{g}_{T_N^s} \oplus \mathfrak{h}_{T_N^s}$. Define

$$U_q(\tau) = \langle E_\alpha, K_\alpha, K_\beta, F_\alpha \rangle_{\alpha \in \Pi'_{F_N^s}, \beta \in \Phi \setminus \Phi^{\mathfrak{l}}}$$

and

$$U_q(\mathfrak{t}) = \langle E_\alpha, K_\alpha, F_\alpha \rangle_{\alpha \in \Pi'_{T_N^s}}.$$

Then by construction $U_q(\mathfrak{g}) \cong U_q(\tau) \otimes_{\mathbb{C}} U_q(\mathfrak{t})$ as a vector space via $u_1 \otimes u_2 \mapsto u_1 u_2$ for $u_1 \in U_q(\tau)$ and $u_2 \in U_q(\mathfrak{t})$.

To continue we want to use a result similar to [Lem69] Theorem 1 which says that there is a 1-1 correspondence between simple $U_q(\mathfrak{l})$-modules and simple $(U_q(\mathfrak{l}))_0$ modules. Since Lemire's result is for Lie algebras we will prove the same for quantum group modules but the proofs are essentially the same. In the following $\mathfrak{l}$ is the Levi part of some standard parabolic subalgebra $\mathfrak{p}$ and $U_q(\mathfrak{l})$ is defined as above. Note in particular that the results work for $\mathfrak{l} = \mathfrak{g}$ by choosing $\mathfrak{p} = \mathfrak{g}$. For easier notation we will set $C_q := (U_q(\mathfrak{l}))_0$.

Lemma 2.18 *Let V be a simple $U_q(\mathfrak{l})$-module and λ a weight of V. Then V_λ is a simple C_q-module.*

Proof. It is enough to show that for $v \in V_\lambda$ nonzero we have $V_\lambda = C_q v$ but this follows since $V_\lambda = (U_q(\mathfrak{l})v)_\lambda = (\bigoplus_\nu U_q(\mathfrak{l})_\nu v)_\lambda = U_q(\mathfrak{l})_0 v$ $\square$

Lemma 2.19 *Assume V_1 and V_2 are simple $U_q(\mathfrak{l})$-modules. Let $\lambda \in \operatorname{wt} V_1$ and assume $(V_1)_\lambda \cong (V_2)_\lambda$ as C_q-modules. Then $V_1 \cong V_2$.*

Proof. Let $0 \neq v_i \in (V_i)_\lambda$, $i = 1, 2$. Then $(V_i)_\lambda \cong C_q / \operatorname{Ann}_{C_q}(v_i)$ as C_q-modules since $(V_i)_\lambda$ is simple (Lemma 2.18). Let $M = \operatorname{Ann}_{C_q}(v_1)$, then M is a maximal left ideal in C_q since C_q/M is simple. We will show that there exists a unique maximal ideal M' of $U_q(\mathfrak{l})$ containing M. Let $M'' = U_q(\mathfrak{l})M$. Then $M'' \neq U_q(\mathfrak{l})$ because $M \neq C_q$ and so there is a maximal ideal M' containing M''. To show uniqueness we will show that $U_q(\mathfrak{l})/M''$ has a unique maximal submodule (and therefore a unique simple quotient). Clearly $U_q(\mathfrak{l})/M'' = \bigoplus_\gamma (U_q(\mathfrak{l})/M'')_\gamma$. Let N be a submodule of $U_q(\mathfrak{l})/M''$. Then $N = \bigoplus_\gamma N \cap (U_q(\mathfrak{l})/M'')_\gamma$. Since $(U_q(\mathfrak{l})/M'')_\lambda = (C_q/M) \cong (V_1)_\lambda$ is a simple C_q-module we have either $N \cap (U_q(\mathfrak{l})/M'')_\lambda = (U_q(\mathfrak{l})/M'')_\lambda$ or $N \cap (U_q(\mathfrak{l})/M'')_\lambda = 0$. In the first case we have $1 + M'' \in N$ and so $N = U_q(\mathfrak{l})/M''$. So all proper submodules of $U_q(\mathfrak{l})/M''$ have $N \cap (U_q(\mathfrak{l})/M'')_\lambda = 0$. Let N_0 be the sum of all proper submodules. Then this is the unique maximal submodule of $U_q(\mathfrak{l})/M''$. So there is a unique maximal submodule M' of $U_q(\mathfrak{l})$ containing M.

Set $M_i = \mathrm{Ann}_{C_q}(v_i)$. Then from the above we get unique maximal left ideals M_i' of $U_q(\mathfrak{l})$ containing M_i. By the uniqueness we have $M_i' = \mathrm{Ann}_{U_q(\mathfrak{l})}(v_i)$ and we have $V_i \cong U_q(\mathfrak{l})/M_i'$. Let $\varphi : C_q/M_1 \to C_q/M_2$ be the isomorphism between $(V_1)_\lambda$ and $(V_2)_\lambda$ and suppose $\varphi(1 + M_1) = x + M_2$. Then define $\Phi : U_q/M_1' \to U_q/M_2'$ by $\Phi(u + M_1') = ux + M_2'$. Then Φ is a $U_q(\mathfrak{l})$-isomorphism because Φ is a nonzero homomorphism between two simple modules. $\square$

Lemma 2.20 *Let $\lambda \in X$. Let N be a simple C_q-module such that $K_\alpha n = \lambda(K_\alpha)n$, for all $\alpha \in \Pi$ and $n \in N$. Then there exists a simple $U_q(\mathfrak{l})$-module V such that $N \cong V_\lambda$ as a C_q-module.*

Proof. Let $0 \neq n \in N$ and set $M = \mathrm{Ann}_{C_q}(n)$. Then there exists a maximal left ideal M' of $U_q(\mathfrak{l})$ like in the proof of Lemma 2.19. Set $V = U_q(\mathfrak{l})/M'$. This is a simple module since M' is maximal. We claim that $V_\lambda \cong N$ as C_q-modules. This follows from the fact that $C_q \cap M' = M$:

$M \subset C_q \cap M'$ by definition. Take any $x \in C_q \cap M'$ and assume $x \notin M$. Since M is maximal in C_q we must have $y \in C_q$ such that $yx - 1 \in M$ hence $1 \in M'$. This is a contradiction. So $M = C_q \cap M'$. $\square$

It now follows that we have just like Theorem 1 in [Lem69] the theorem:

Theorem 2.21 *Let $\lambda \in X$. There is a $1-1$ correspondence between simple $U_q(\mathfrak{l})$-modules V with weight $V_\lambda \neq 0$ and simple C_q modules with weight λ given by: For V a $U_q(\mathfrak{l})$-module, V_λ is the corresponding simple C_q-module.*

The next lemma we will prove is the equivalent of Lemma 4.5 in [Fer90]. The proof goes in almost exactly the same way.

Lemma 2.22 *Let L be a simple $U_q(\mathfrak{l})$-module. Let $U_q(\mathfrak{t})$ and $U_q(\tau)$ be defined as above. There exists a simple $U_q(\tau)$-module L_1 and a simple $U_q(\mathfrak{t})$-module L_2 such that $L \cong L_1 \otimes_{\mathbb{C}} L_2$ as a $U_q(\mathfrak{l}) = U_q(\tau) \otimes_{\mathbb{C}} U_q(\mathfrak{t})$ module. Furthermore if $\Pi'_{T_L^s} = \bigcup_{i=1}^s \Pi'_{(T_L^s)_i}$ where $\Pi'_{(T_L^s)_i}$ are the different connected components in $\Pi'_{T_L^s}$ set $\mathfrak{t}_i = \mathfrak{g}_{(T_L)_i} \oplus \mathfrak{h}_{(T_L)_i}$ and $U_q(\mathfrak{t}_i) = \langle F_\alpha, K_\alpha, E_\alpha \rangle_{\alpha \in \Pi'_{(T_L^s)_i}}$. Then $U_q(\mathfrak{t}) \cong U_q(\mathfrak{t}_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} U_q(\mathfrak{t}_s)$ and there exists simple $U_q(\mathfrak{t}_i)$-modules $(L_2)_i$ such that $L_2 \cong (L_2)_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (L_2)_s$ as $U_q(\mathfrak{t}_1) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} U_q(\mathfrak{t}_s)$-modules.*

Proof. Let λ be one of the weights of L. Then we know that $E := L_\lambda$ is a simple finite dimensional C_q-module. Let R (respectively R_1 and R_2) denote the image of C_q (respectively $U_q(\tau)_0$ and $U_q(\mathfrak{t})_0$) in $\mathrm{End}_{\mathbb{C}}(E)$. Since E is simple we have $R = \mathrm{End}_{\mathbb{C}}(E)$. Since $R_1 E \neq 0$ there exists a nontrivial R_1-submodule of $\mathrm{res}^R_{R_1} E$ and since E is finite dimensional there exists a *simple* R_1-submodule E_1 of $\mathrm{res}^R_{R_1} E$. The simplicity of E_1 implies that the representation $R_1 \to \mathrm{End}_{\mathbb{C}}(E_1)$ is surjective. The kernel of $R_1 \to \mathrm{End}_{\mathbb{C}}(E_1)$ must be $\mathrm{Ann}_{R_1}(E_1)$. But if this is nonzero then since $E = RE_1 = R_2 E_1$ and since R_1 and R_2 commutes we see that $\mathrm{Ann}_R(E)$ will be nonzero which is a contradiction since $R = \mathrm{End}_{\mathbb{C}}(E)$. So $R_1 \cong \mathrm{End}_{\mathbb{C}}(E_1)$ is simple. Similarly there exists a simple R_2-module E_2 and $R_2 \cong \mathrm{End}_{\mathbb{C}}(E_2)$ is simple. Now as in the proof of Lemma 4.5 in [Fer90] we get $R \cong R_1 \otimes R_2$ (using [ANT44, Theorem 7.1D]). Since $R = \mathrm{End}_{\mathbb{C}}(E)$ it has exactly one simple module up to isomorphism. This implies that $E \cong E_1 \otimes_{\mathbb{C}} E_2$ as R-modules.

Now set $L_1 = U_q(\tau)E_1$ and $L_2 = U_q(\mathfrak{t})E_2$. We have $L_\lambda = E \cong E_1 \otimes_{\mathbb{C}} E_2 = (L_1 \otimes_{\mathbb{C}} L_2)_\lambda$ and by Theorem 2.21 this implies that $L \cong L_1 \otimes_{\mathbb{C}} L_2$.

The second part of the lemma is proved in the same way. The only thing we used about $U_q(\tau)$ and $U_q(\mathfrak{t})$ was that $U_q(\mathfrak{l}) = U_q(\tau)U_q(\mathfrak{t})$ and that $U_q(\tau)_0$ and $U_q(\mathfrak{t})_0$ commutes. The same is true for $U_q(\mathfrak{t})$ and the $U_q(\mathfrak{t}_i)$'s. □

To summarize we have the following equivalent of Theorem 4.18 in [Fer90]:

Theorem 2.23 *Suppose $L \in \mathcal{F}$ is a simple $U_q(\mathfrak{g})$ module. Let $w \in W$ be such that $P_{^wL}$ is standard parabolic. With notation as above: $(^wL)^{\mathfrak{u}}$ is a simple $U_q(\mathfrak{l})$-module and this module decomposes into a tensor product $X_{\text{fin}} \otimes_{\mathbb{C}} X_{\text{fr}}$ where X_{fin} is a finite dimensional simple $U_q(\tau)$-module and X_{fr} is a torsion free $U_q(\mathfrak{t})$-module. Furthermore if $\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_s$ as a sum of ideals then $X_{\text{fr}} = X_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} X_s$ for some simple $U_q(\mathfrak{t}_i)$-modules.*

Given the pair $(X_{\text{fin}}, X_{\text{fr}})$ and the $w \in W$ defined above then L can be recovered as $^{\overline{w}}L(X_{\text{fin}} \otimes_{\mathbb{C}} X_{\text{fr}}))$.

So the problem of classifying simple modules in $\mathcal{F}$ is reduced to the problem of classifying finite dimensional simple modules of $U_q(\tau)$ and classifying torsion free simple modules of $U_q(\mathfrak{t})$ where $\mathfrak{t}$ is a simple Lie algebra. In the next section we will show that we can make the same reduction if q is an odd root of unity. The procedure is similar but there are some differences, e.g. because the $\mathfrak{sl}_2$ theory is a little different.

3 Root of unity case: Reduction

We will now consider the root of unity case. In this section $q \in \mathbb{C}$ will be assumed to be a primitive l'th root of unity where l is odd.

Lemma 3.1 *Let $\lambda \in X$ and $\alpha \in \Pi$. Then $\lambda(K_\alpha) = \pm q_\alpha^k$ for some $k \in \{0, \dots, l-1\}$.*

Proof. By Section 6.4 in [Lus90] we have the following relation in U_A:

$$\begin{bmatrix} K_\alpha; 0 \\ l-1 \end{bmatrix} \begin{bmatrix} K_\alpha; -l+1 \\ 1 \end{bmatrix} = \begin{bmatrix} l \\ l-1 \end{bmatrix}_{v_\alpha} \begin{bmatrix} K_\alpha; 0 \\ l \end{bmatrix}.$$

Since $\left[\begin{smallmatrix} l \\ l-1 \end{smallmatrix}\right]_{q_\alpha} = 0$ when q is an l'th root of unity we must have that either $q_\alpha^{-l+1}\lambda(K_\alpha) - q_\alpha^{l-1}\lambda(K_\alpha)^{-1} = 0$ or $q_\alpha^{1-k}\lambda(K_\alpha) - q_\alpha^{k-1}\lambda(K_\alpha)^{-1} = 0$ for some $k \in \{1, \dots, l-1\}$. Writing out what these equations imply we get that $\lambda(K_\alpha) = \pm q_\alpha^k$ for some $k \in \{0, \dots, l-1\}$. □

Definition 3.2

$$\Lambda_l = \{\lambda \in \Lambda | 0 \leq \langle \lambda, \alpha^\vee \rangle < l, \forall \alpha \in \Pi\}$$

Lemma 3.3 *Let $\lambda : U_q^0 \to \mathbb{C}$ be an algebra homomorphism. Then λ is completely determined by its values on K_α and $\left[\begin{smallmatrix} K_\alpha; 0 \\ l \end{smallmatrix}\right]$ with $\alpha \in \Pi$. Choosing a* *homomorphism $\sigma : Q \to \{\pm 1\}$, an element $\lambda^0 \in \Lambda_l$ and an element $\lambda^1 \in \mathfrak{h}^*$ determines a homomorphism $\lambda \in X$ as follows: For $\alpha \in \Pi$:*

$$\begin{aligned}\lambda(K_\alpha) =&\sigma(\alpha) q^{(\lambda^0|\alpha)}\\ \lambda(\begin{bmatrix} K_\alpha;0 \\ l\end{bmatrix}) =& \left\langle \lambda^1, \alpha^\vee \right\rangle.\end{aligned}$$

All algebra homomorphisms $\lambda : U_q^0 \to \mathbb{C}$ are of this form, i.e. $X = S \times \Lambda_l \times \mathfrak{h}^*$ in this case, where S is the set of homomorphisms $\sigma : Q \to \{\pm 1\}$.

Proof. We will use the relations for U_A from Section 6.4 of [Lus90]. Let $\beta \in \Pi$. If $\lambda(K_\beta) = d$ then $\lambda(K_\beta^{-1}) = d^{-1}$ and the value on $\left[\begin{smallmatrix} K_\beta;c \\ t\end{smallmatrix}\right] = \prod_{i=1}^t \frac{q_\beta^{c-i+1}K_\beta - q_\beta^{i-1-c}K_\beta^{-1}}{q_\beta^i - q_\beta^{-i}}$ for $0 \leq t < l$ is also determined. The relations

$$\begin{bmatrix} K_\beta;c \\ l\end{bmatrix} - \begin{bmatrix} K_\beta;c+1 \\ l\end{bmatrix} = -q_\beta^{c+1}K_\beta \begin{bmatrix} K_\beta;c \\ l-1\end{bmatrix}$$

determine the values on $\left[\begin{smallmatrix} K_\beta;c \\ t\end{smallmatrix}\right]$ for all $c \in \mathbb{Z}$ if the value on $\left[\begin{smallmatrix} K_\beta;0 \\ t\end{smallmatrix}\right]$ and the value on K_β is known. Finally if $c = rl + t$ with $0 \leq t < l$ we have

$$\begin{aligned}\begin{bmatrix} K_\beta;0 \\ rl+t\end{bmatrix} =& \begin{bmatrix} K_\beta;0 \\ rl\end{bmatrix}\begin{bmatrix} K_\beta;-rl \\ t\end{bmatrix}\\ =& r^{-1}\begin{bmatrix} K_\beta;0 \\ (r-1)l\end{bmatrix}\begin{bmatrix} K_\beta;-(r-1)l \\ l\end{bmatrix}\begin{bmatrix} K_\beta;-rl \\ t\end{bmatrix}\\ &\vdots\\ =&(r!)^{-1}\prod_{s=0}^{r-1}\begin{bmatrix} K_\beta;-sl \\ l\end{bmatrix}\begin{bmatrix} K_\beta;-rl \\ t\end{bmatrix}.\end{aligned}$$

So determining the value on K_β and $\left[\begin{smallmatrix} K_\beta;0 \\ l\end{smallmatrix}\right]$ determines the value on all of U_q^0.

If $\sigma, \lambda^0, \lambda^1$ is chosen as above it is easy to check that the relations from Section 6.4 in [Lus90] are satisfied. That all characters are of this form follows from Lemma 3.1. □

It can be noted in the above that $\lambda^1 = \lambda \circ \mathrm{Fr}' |_{\mathfrak{h}}$ where $\mathrm{Fr}' : U(\mathfrak{g}) \to U_q(\mathfrak{g}) / \left\langle K_\alpha^l - 1 | \alpha \in \Pi \right\rangle$ is the Frobenius map from [KL02]. We will restrict to modules of type **1** meaning $\sigma(\alpha) = 1$ for all $\alpha \in \Pi$ in the above. It is standard how to get from modules of type **1** to modules of any other type σ (cf. e.g. [Jan96, Section 5.1-5.4]).

Since we restrict to modules of type 1 we will assume from now on that $X = \Lambda_l \times \mathfrak{h}^*$. A weight $\lambda \in X$ will also be written as $(\lambda^0, \lambda^1) \in \Lambda_l \times \mathfrak{h}^*$.

Lemma 3.4 *Let $\lambda \in X$ with λ^0 and λ^1 defined as in Lemma 3.3. Let $\beta \in \Phi^+$, $c \in \mathbb{Z}$,*

$$\lambda(\begin{bmatrix} K_\beta;c+1 \\ l\end{bmatrix}) = \begin{cases} \lambda(\left[\begin{smallmatrix} K_\beta;c \\ l\end{smallmatrix}\right]) + 1 & \text{if } \left\langle \lambda^0, \beta^\vee\right\rangle + c \equiv -1 \mod l \\ \lambda(\left[\begin{smallmatrix} K_\beta;c \\ l\end{smallmatrix}\right]) & \text{otherwise}. \end{cases}$$

Proof. Set $a = \langle \lambda^0, \beta^\vee \rangle$. By $(b4)$ in Section 6.4 of [Lus90]

$$\begin{aligned}\lambda\left(\begin{bmatrix} K_\beta; c \\ l \end{bmatrix}\right) =& \lambda\left(\begin{bmatrix} K_\beta; c-1 \\ l \end{bmatrix} + q_\beta^c K_\beta \begin{bmatrix} K_\beta; c-1 \\ l-1 \end{bmatrix}\right) \\ =& \lambda\left(\begin{bmatrix} K_\beta; c-1 \\ l \end{bmatrix}\right) + q_\beta^{c+a} \begin{bmatrix} a+c-1 \\ l-1 \end{bmatrix}_{q_\beta}.\end{aligned}$$

$\begin{bmatrix} a+c-1 \\ l-1 \end{bmatrix}_{q_\beta}$ is zero unless $a+c-1 \equiv -1 \mod l$. If $a+c-1 \equiv -1 \mod l$ then $a+c \equiv 0 \mod l$ and so $q_\beta^{a+c} = 1$ and $\begin{bmatrix} a+c-1 \\ l-1 \end{bmatrix}_{q_\beta} = \begin{bmatrix} l-1 \\ l-1 \end{bmatrix}_{q_\beta} = 1$. $\square$

For a character $\lambda \in X$ and a $\mu \in Q$ we define $q^\mu \lambda$ as follows:

$$\begin{aligned}(q^\mu \lambda)(K_\alpha) =& q^{(\mu|\alpha)} \lambda(K_\alpha) = q_\alpha^{\langle \mu, \alpha^\vee \rangle} \lambda(K_\alpha) \\ (q^\mu \lambda)\left(\begin{bmatrix} K_\alpha; c \\ l \end{bmatrix}\right) =& \lambda\left(\begin{bmatrix} K_\alpha; c + \langle \mu, \alpha^\vee \rangle \\ l \end{bmatrix}\right).\end{aligned}$$

With this notation we get for a module M that $E_\alpha^{(r)} M_\lambda \subset M_{q^{r\alpha}\lambda}$ and $F_\alpha^{(r)} M_\lambda \subset M_{q^{-r\alpha}\lambda}$. Note also that $(q^{l\beta}\lambda)^1 = \lambda^1 + \beta$.

We use the same definitions as in Section 2:

Definition 3.5 *Let $M \in \mathcal{F}$ and let $\beta \in \Phi$. We call M β-finite if $q^{\mathbb{N}\beta}\lambda \cap \operatorname{wt} M$ is a finite set for all $\lambda \in \operatorname{wt} M$ where $q^{\mathbb{N}\beta}\lambda = \{q^{r\beta}\lambda | r \in \mathbb{N}\}$.*

The weight vectors E_β and F_β for positive β that are not simple are defined just as before by choosing a reduced expression of w_0. By [Lus90, Section 5.6] the divided powers $E_\beta^{(r)} := \frac{1}{[r]_\beta!} E_\beta^r$, $r \in \mathbb{N}$ are all contained in U_A and by abuse of notation we use the same symbol for the corresponding elements in U_q.

Proposition 3.6 *Let $M \in \mathcal{F}$ and let β be a positive root. Let E_β be any choice of root vector corresponding to β. Then the following are equivalent:*

1. *M is β-finite.*
2. *For all $m \in M$, $E_\beta^{(r)} m = 0$ for $r >> 0$*

Proof. Clearly 1. implies 2. since $E_\beta^{(r)} M_\lambda \subset M_{q^{r\beta}\lambda}$. Assume 2. and suppose M is not β finite.

We must have a $\lambda \in \operatorname{wt} M$, an increasing sequence $\{j_i\}_{i \in \mathbb{N}}$, weights $\mu_i = q^{j_i\beta}\lambda \in \operatorname{wt} M$ and weight vectors $m_i \in M_{\mu_i}$ such that $E_\beta^{(r)} m_i = 0$ for all $r \in \mathbb{N}\backslash\{0\}$. We can assume without loss of generality that if $\lambda\left(\begin{bmatrix} K_\beta; 0 \\ l \end{bmatrix}\right) \in \mathbb{Z}$ then $\lambda\left(\begin{bmatrix} K_\beta; 0 \\ l \end{bmatrix}\right) \in \mathbb{Z}_{>0}$ by Lemma 3.4.

Now consider the subalgebra D_β of U_q generated by $E_\beta^{(r)}$, $K_\beta^{\pm 1}$ and $F_\beta^{(r)}$ for $r \in \mathbb{N}$ where F_β is the root vector corresponding to E_β (i.e. if $E_\beta = T_w(E_{\alpha_i})$ then $F_\beta = T_w(F_{\alpha_i})$). For each i we get a D_β-module $D_\beta m_i$ with highest weight μ_i. We claim that in each of these modules we have at least one weight vector with one of the weights $\lambda, q^{-\beta}\lambda, \ldots, q^{-(l-1)\beta}\lambda$. So we want to show for each m_i that at least one of the vectors $F_\beta^{(j_i)} m_i, F_\beta^{(j_i+1)} m_i, \ldots, F_\beta^{(j_i+l-1)} m_i$ is nonzero.

We must have that one of the numbers $j_i, \ldots, j_i + l - 1$ is congruent to 0 modulo l. Lets call this number k. Say $k = rl$. Now we have

$$\begin{aligned} E_\beta^{(k)} F_\beta^{(k)} m_i =& \sum_{s \geq 0} F_\beta^{(k-s)} \begin{bmatrix} K_\beta; 2s-2k \\ s \end{bmatrix} E_\beta^{(k-s)} m_i \\ =& \begin{bmatrix} K_\beta; 0 \\ rl \end{bmatrix} m_i \\ =& \frac{1}{r!} \prod_{s=0}^{r-1} \begin{bmatrix} K_\beta; -sl \\ l \end{bmatrix} m_i \\ =& \frac{1}{r!} \prod_{s=0}^{r-1} (c_i - s)\, m_i \\ =& \binom{c_i}{r} m_i \end{aligned}$$

where $c_i = \mu_i \left(\left[\begin{smallmatrix} K_\beta; 0 \\ l \end{smallmatrix}\right] \right)$. To show that this is nonzero we must show that $c_i \notin \{0, \ldots, r-1\}$. If $\lambda \left(\left[\begin{smallmatrix} K_\beta; 0 \\ l \end{smallmatrix}\right] \right)$ is not an integer then this is automatically fullfilled. Otherwise we know $j_i = rl - t$ for some $t = 0, \ldots, l-1$. So $\mu_i = q^{(rl-t)\beta}\lambda$ and by Lemma 3.4

$$c_i = \mu_i \left(\begin{bmatrix} K_\beta; 0 \\ l \end{bmatrix} \right) = q^{(rl-t)\beta}\lambda \left(\begin{bmatrix} K_\beta; 0 \\ l \end{bmatrix} \right) = \lambda \left(\begin{bmatrix} K_\beta; 0 \\ l \end{bmatrix} \right) + r - 1 \geq r.$$

Since there are infinitely many m_i's we must have infinitely many weight vectors $\{v_j\}$ of weight one of the weights $\lambda, \lambda - \beta, \ldots, \lambda - (l-1)\beta$.

To show that they are linearly independent let $v_1, \ldots, v_n$ be a finite set of the above weight vectors. They are all of the form $F_\beta^{(k_i)} m_i$ for some i and some k_i. Assume v_n is the vector where the power k_n is maximal. Then $E_\beta^{(k_n)} v_i = 0$ for $i \neq n$ and $E_\beta^{(k_n)} v_n \neq 0$. It follows by induction on n that the set $\{v_1, \ldots, v_n\}$ is linearly independent. ☐

We define $M^{[\beta]} = \{m \in M | \dim \left\langle E_\beta^{(r)} | r \in \mathbb{N} \right\rangle m < \infty\}$. Proposition 2.3 and Lemma 2.4 carry over with the same proof. In particular $M^{[\beta]}$ is independent of the choice of root vector E_β. Again we call M β-free if $M^{[\beta]} = 0$. Again we can show everything with F's instead of E's if β is negative.

Propositions 2.9 and 2.10 carry over with almost identical proofs. Setting $l = 1$ in the propositions and their proofs below would make the proofs identical.

Proposition 3.7 *Let $M \in \mathcal{F}$ be a simple module and β a root. Then M is β-free if and only if $q^{\mathbb{N}l\beta} \operatorname{wt} M \subset \operatorname{wt} M$.*

Proof. Assume β is positive. If $q^{\mathbb{N}l\beta} \operatorname{wt} M \subset \operatorname{wt} M$ then M is clearly not β-finite and since M is simple we have by Proposition 2.3 that M is β-free in this case. For the other way assume M is β-free and assume we have a weight vector $0 \neq m \in M_\lambda$ such that $E_\beta^{(rl)} m = 0$ for some $r \in \mathbb{N}$. For any $i \in \mathbb{N}$, $\left[\begin{smallmatrix} i+rl \\ i \end{smallmatrix}\right]_\beta \neq 0$ so

$$E_\beta^{(rl+i)} m = \begin{bmatrix} i + rl \\ i \end{bmatrix}_\beta^{-1} E_\beta^{(i)} E_\beta^{(rl)} m = 0$$

But this implies that $m \in M^{[\beta]}$ which contradicts the assumption that M is β-free. If β is negative we do the same with F's instead of E's. $\square$

Proposition 3.8 *Let $L \in \mathcal{F}$ be a simple module. T_L is a closed subset of Φ.*

Proof. Assume $\beta, \gamma \in T_L$ with $\beta + \gamma \in \Phi$. Then since $\beta \in T_L$, $q^{\mathbb{N}l\beta} \operatorname{wt} L \subset \operatorname{wt} L$. Since $\gamma \in T_L$ we get then $q^{\mathbb{N}l\gamma} q^{\mathbb{N}l\beta} \operatorname{wt} L \subset \operatorname{wt} L$ so $q^{\mathbb{N}l(\beta+\gamma)} \operatorname{wt} L \subset \operatorname{wt} L$. $\square$

Proposition 3.9 *Let $L \in \mathcal{F}$ be a simple module. F_L and T_L are closed subsets of Φ and $\Phi = F_L \cup T_L$ (disjoint union).*

Proof. T_L is closed by Proposition 3.8. F_L is closed by the same proof as the proof of Proposition 2.11. Note that the constants in the proof of Proposition 2.11 that are inverted are all nonzero even when q is a l'th root of unity as long as l is odd. $\square$

We define P_L like in Section 2 and we assume like above that P_L is standard parabolic by considering ${}^wL$ for an appopriate $w \in W$. The subalgebras $U_q(\mathfrak{p})$, $U_q(\mathfrak{l})$, $U_q(\mathfrak{u})$, $U_q(\mathfrak{u}^-)$ etc. are defined as above but this time with divided powers. For example we have

$$U_q(\mathfrak{p}) = \left\langle E_{\beta_j}^{(r)}, K_\mu, F_{\beta_i^1}^{(r)} \right\rangle_{j=1,\ldots,N,\mu\in Q,i=1,\ldots,h,r\in\mathbb{N}}$$

and so on. Now the rest of the lemmas and proposition carry over with the same proofs as before and we have the following equivalent of Theorem 2.23:

Theorem 3.10 *Suppose $L \in \mathcal{F}$ is a simple $U_q(\mathfrak{g})$ module. Let $w \in W$ be such that $P_{{}^wL}$ is standard parabolic. With notation as above: $({}^wL)^{\mathfrak{u}}$ is a simple $U_q(\mathfrak{l})$-module and this module decomposes into a tensor product $X_{\text{fin}} \otimes_{\mathbb{C}} X_{\text{fr}}$ where X_{fin} is a finite dimensional simple $U_q(\tau)$-module and X_{fr} is a torsion free simple $U_q(\mathfrak{t})$-module. Furthermore if $\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_s$ as a sum of ideals then $X_{\text{fr}} = X_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} X_s$ as $U_q(\mathfrak{t}_1) \otimes \cdots \otimes U_q(\mathfrak{t}_s)$-module for some simple $U_q(\mathfrak{t}_i)$-modules X_i, $i = 1, \ldots, s$.*

Given the pair $(X_{\text{fin}}, X_{\text{fr}})$ and the $w \in W$ defined above then L can be recovered as ${}^{\overline{w}}(L(X_{\text{fin}} \otimes_{\mathbb{C}} X_{\text{fr}}))$.

So in the root of unity case we have also that to classify simple modules in $\mathcal{F}$ we just have to classify finite dimensional modules of $U_q(\tau)$ and 'torsion free' modules over $U_q(\mathfrak{t})$, where $\mathfrak{t}$ can be assumed to be a simple Lie algebra.

4 U_A formulas

In this section we recall from [Ped15b] some formulas for commuting root vectors with each other that will be used later. Note that in [Ped15b] the braid operators that we here call T_w are denoted by R_w. In [Ped15b] T_w denotes twisting functors.

Recall that $A = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate and U_A is the A-subspace of U_v generated by the divided powers $E_\alpha^{(n)}$, $F_\alpha^{(n)}$, $n \in \mathbb{N}$ and K_α, K_α^{-1}.

Definition 4.1 *Let $x \in (U_v)_\mu$ and $y \in (U_v)_\gamma$ then we define*

$$[x, y]_v := xy - v^{-(\mu|\gamma)} yx$$

Theorem 4.2 *Suppose we have a reduced expression of $w_0 = s_{i_1} \cdots s_{i_N}$ and define root vectors $F_{\beta_1}, \dots, F_{\beta_N}$. Let $i < j$. Let $A = \mathbb{Z}[v, v^{-1}]$ and let A' be the localization of A in $[2]$ if the Lie algebra contains any B_n, C_n or F_4 part. Then*

$$[F_{\beta_j}, F_{\beta_i}]_v = F_{\beta_j} F_{\beta_i} - v^{-(\beta_i|\beta_j)} F_{\beta_i} F_{\beta_j} \in \operatorname{span}_{A'} \left\{ F_{\beta_{j-1}}^{a_{j-1}} \cdots F_{\beta_{i+1}}^{a_{i+1}} \right\}$$

Proof. [LS91, Proposition 5.5.2]. Detailed proof also in [Ped15b, Theorem 2.9].□

Definition 4.3 *Define $\operatorname{ad}(F_\beta^i)(F_\alpha) := [[\dots[F_\alpha, F_\beta]_v \dots]_v, F_\beta]_v$ and $\widetilde{\operatorname{ad}}(F_\beta^i)(F_\alpha) := [F_\beta, [\dots, [F_\beta, F_\alpha]_v \dots]]_v$ where the commutator is taken i times from the left and right respectively.*

Proposition 4.4 *Let $u \in (U_A)_\mu$, $\beta \in \Phi^+$ and F_β a corresponding root vector. Set $r = \langle \mu, \beta^\vee \rangle$. Then in U_A we have the identity*

$$\operatorname{ad}(F_\beta^i)(u) = [i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(n)} u F_\beta^{(i-n)}$$

and

$$\widetilde{\operatorname{ad}}(F_\beta^i)(u) = [i]_\beta! \sum_{n=0}^{i} (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(i-n)} u F_\beta^{(n)}$$

Proof. Proposition 1.8 in [Ped15b]. □

So we can define $\operatorname{ad}(F_\beta^{(i)})(u) := ([i]!)^{-1} \operatorname{ad}(F_\beta^i)(u) \in U_A$ and $\widetilde{\operatorname{ad}}(F_\beta^{(i)})(u) := ([i]!)^{-1} \widetilde{\operatorname{ad}}(F_\beta^i)(u) \in U_A$.

Proposition 4.5 *Let $a \in \mathbb{N}$, $u \in (U_A)_\mu$ and $r = \langle \mu, \beta^\vee \rangle$. In U_A we have the identities*

$$\begin{aligned} u F_\beta^{(a)} &= \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} F_\beta^{(a-i)} \operatorname{ad}(F_\beta^{(i)})(u) \\ &= \sum_{i=0}^{a} (-1)^i v_\beta^{a(r+i)-i} F_\beta^{(a-i)} \widetilde{\operatorname{ad}}(F_\beta^{(i)})(u) \end{aligned}$$

and

$$\begin{aligned} F_\beta^{(a)} u &= \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} \widetilde{\operatorname{ad}}(F_\beta^{(i)})(u) F_\beta^{(a-i)} \\ &= \sum_{i=0}^{a} (-1)^i v_\beta^{a(r+i)-i} \operatorname{ad}(F_\beta^{(i)})(u) F_\beta^{(a-i)} \end{aligned}$$

Proof. Proposition 1.9 in [Ped15b]. □

Proposition 4.6 *For $x_1 \in (U_A)_{\mu_1}$, $x_2 \in (U_A)_{\mu_2}$ and $y \in (U_A)_\gamma$ we have*

$$[y, x_1 x_2]_v = x_1 [y, x_2]_v + v^{-(\gamma|\mu_2)} [y, x_1]_v x_2$$

and

$$[x_1 x_2, y]_v = v^{-(\gamma|\mu_1)} x_1 [x_2, y]_v + [x_1, y]_v x_2$$

Proof. Direct calculation. □

Let $s_{i_1} \dots s_{i_N}$ be a reduced expression of w_0 and construct root vectors F_{β_i}, $i = 1, \dots, N$. In the next lemma F_{β_i} refers to the root vectors constructed as such. In particular we have an ordering of the root vectors.

Lemma 4.7 *Let $n \in \mathbb{N}$. Let $1 \leq j < k \leq N$.*
$\mathrm{ad}(F_{\beta_j}^{(i)})(F_{\beta_k}^{(n)}) = 0$ *and* $\widetilde{\mathrm{ad}}(F_{\beta_k}^{(i)})(F_{\beta_j}^{(n)}) = 0$ *for $i \gg 0$.*

Proof. Lemma 1.11 in [Ped15b]. □

5 Ore localization and twists of localized modules

In this section q will be a complex primitive l'th root of unity with l odd. Recall that we will assume $X = \Lambda_l \times \mathfrak{h}^*$ in this case restricting to modules of type 1. For an element $\lambda \in X$ we define $\lambda^0 \in \Lambda_l$ and $\lambda^1 \in \mathfrak{h}^*$ as in Lemma 3.3 such that $\lambda(K_\alpha) = q^{(\lambda^0|\alpha)}$ and $\lambda(\left[\begin{smallmatrix} K_\alpha;0 \\ l \end{smallmatrix}\right]) = \langle \lambda^1, \alpha^\vee \rangle$ for $\alpha \in \Pi$ and we will also write $\lambda = (\lambda^0, \lambda^1) \in X$.

Lemma 5.1 *Let β be a positive root and F_β a corresponding root vector. The set*

$$\{r! F_\beta^{(rl)} | r \in \mathbb{N}\} = \{\left(F_\beta^{(l)}\right)^r | r \in \mathbb{N}\}$$

is an Ore subset of U_q.

Proof. We can assume β is simple since otherwise $F_\beta = T_w(F_\alpha)$ for some $\alpha \in \Pi$ and some $w \in W$ and $T_w(U_q) = U_q$. Since $r! F_\beta^{(rl)} k! F_\beta^{(kl)} = (r+k)! F_\beta^{(rl+kl)}$ the set is multiplicative and does not contain 0. We will show the Ore property for each generator of U_q. First consider $\alpha \in \Pi$ a simple root not equal to β. Let $n \in \mathbb{N}$. We have the following identities for $r \geq 1$ (cf. Proposition 4.5)

$$\begin{aligned}
r! F_\beta^{(rl)} E_\alpha^{(n)} =& E_\alpha^{(n)} r! F_\beta^{(rl)} \\
r! F_\beta^{(rl)} K_\alpha^{\pm 1} =& K_\alpha^{\pm 1} r! F_\beta^{(rl)} \\
F_\alpha^{(n)} r! F_\beta^{(rl)} =& r! F_\beta^{(rl)} F_\alpha^{(n)} \\
& + \sum_{k=0}^{r-1} \sum_{i=kl+1}^{kl+l} c_i (r-k-1)! F_\beta^{(rl-kl-l)} F_\beta^{(kl+l-i)} \mathrm{ad}(F_\beta^{(i)})(F_\alpha^{(n)})
\end{aligned}$$

where $c_i = q_\beta^{i(i+\langle \alpha, \beta^\vee \rangle)} r(r-1) \cdots (r-k)$. Finally we have the $\mathfrak{sl}_2$ identities for $0 \leq i \leq l$:

$$\begin{aligned}
r! F_\beta^{(rl)} F_\beta^{(n)} =& F_\beta^{(n)} r! F_\beta^{(rl)} \\
E_\beta^{(i)} r! F_\beta^{(rl)} =& r! F_\beta^{(rl)} E_\beta^{(i)} + \sum_{t=1}^{i} r(r-1)! F_\beta^{(rl-l)} F_\beta^{(l-t)} E_\beta^{(i-t)} \begin{bmatrix} K_\beta; i - rl \\ t \end{bmatrix}_\beta .
\end{aligned}$$

So we have shown that it is an Ore set. □

We will denote the Ore localization of U_q in the above set by $U_{q(F_\beta)}$. For a U_q-module M we define $M_{(F_\beta)} := U_{q(F_\beta)} \otimes_{U_q} M$. We write the inverse of $F_\beta^{(rl)}$, $r \in \mathbb{N}$ as $F_\beta^{(-rl)}$ i.e. $F_\beta^{(-rl)} = r!\left(r!F_\beta^{(rl)}\right)^{-1} \in U_{q(F_\beta)}$.

Lemma 5.2 *Let $\lambda = (\lambda^0, \lambda^1) \in \Lambda_l \times \mathfrak{h}^*$, $\beta \in \Phi^+$ and let F_β be a corresponding root vector. Let I_λ be the left $U_{q(F_\beta)}$-ideal $U_{q(F_\beta)}\{(u - \lambda(u))|u \in U_q^0\}$. Then there exists, for each $b \in \mathbb{C}$, an automorphism of $U_{q(F_\beta)}$-modules $\psi^\lambda_{F_\beta,b} : U_{q(F_\beta)}/I_\lambda \to U_{q(F_\beta)}/I_{(\lambda^0,\lambda^1+b\beta)}$ such that for $u \in U_{q(F_\beta)}$ and $i \in \mathbb{N}$, $\psi^\lambda_{F_\beta,i}(u + I_\lambda) = F_\beta^{(-il)} u F_\beta^{(il)} + I_{(\lambda^0,\lambda^1+i\beta)}$ and the map $b \mapsto \psi^\lambda_{F_\beta,b}(u + I_\lambda)$ is polynomial in b. Furthermore $\psi^{(\lambda^0,\lambda^1+b\beta)}_{F_\beta,b'} \circ \psi^\lambda_{F_\beta,b} = \psi^\lambda_{F_\beta,b+b'}$ for $b, b' \in \mathbb{C}$.*

Proof. If β is not simple then $F_\beta = T_w(F_\alpha)$ for some simple root $\alpha \in \Pi$. Then we define $\psi^\lambda_{F_\beta,b}(u) = T_w(\psi^{w\lambda}_{F_\alpha,b}(T_w^{-1}(u)))$ where $T_w^{-1}(F_\beta^{(-l)}) = F_\alpha^{(-l)}$ and $T_w(F_\alpha^{(-l)}) = F_\beta^{(-l)}$. So we assume from now on that $\beta \in \Pi$.

We define $\psi^\lambda_{F_\beta,b}$ on generators: For $\alpha \in \Pi\backslash\{\beta\}$ and $n \in \mathbb{N}$

$$\begin{aligned}
\psi^\lambda_{F_\beta,b}(E_\alpha^{(n)}) =& E_\alpha^{(n)} \\
\psi^\lambda_{F_\beta,b}(F_\alpha^{(n)}) =& F_\alpha^{(n)} \\
& - \sum_{k\geq 0} \binom{b}{k+1} \sum_{i=kl+1}^{kl+l} q_\beta^{i(\langle \alpha|\beta^\vee\rangle - i)} F_\beta^{(-kl-l)} F_\beta^{(kl+l-i)} \operatorname{ad}(F_\beta^{(i)})(F_\alpha^{(n)}) \\
\psi^\lambda_{F_\beta,b}(K_\alpha^{\pm 1}) =& \lambda(K_\alpha^{\pm 1}) \\
\psi^\lambda_{F_\beta,b}(F_\beta^{(n)}) =& F_\beta^{(n)} \\
\psi^\lambda_{F_\beta,b}(E_\beta) =& E_\beta + bF_\beta^{(-l)} F_\beta^{(l-1)} [\langle \lambda_0, \beta^\vee\rangle + 1]_\beta \\
\psi^\lambda_{F_\beta,b}(E_\beta^{(l)}) =& E_\beta^{(l)} + bF_\beta^{(-l)} \sum_{t=1}^{l-1} F_\beta^{(l-t)} E_\beta^{(l-t)} \begin{bmatrix} \langle \lambda_0, \beta^\vee \rangle \\ t \end{bmatrix}_\beta + bF_\beta^{(-l)}(\langle \lambda_1, \beta^\vee\rangle + 1 - b) \\
\psi^\lambda_{F_\beta,b}(K_\beta^{\pm 1}) =& \lambda(K_\beta^{\pm 1}).
\end{aligned}$$

The sum given in the formula for $F_\alpha^{(n)}$ is finite by Lemma 4.7. It is easy to check that $\psi^\lambda_{F_\beta,i}(u + I_\lambda) = F_\beta^{(-il)} u F_\beta^{(il)} + I_{(\lambda^0,\lambda^1+b\beta)}$ for $i \in \mathbb{N}$ and it is seen from the formulas that $b \mapsto \psi^\lambda_{F_\beta,b}(u + I_\lambda)$ is polynomial. So $\psi^\lambda_{F_\beta,b}$ satisfies the generating relations of U_q for $b \in \mathbb{N}$ hence it satisfies the generating relations for all $b \in \mathbb{C}$ because $\psi^\lambda_{F_\beta,b}(u)$ is polynomial in b. Similarly we can show the rest of the claims by using the fact that $b \mapsto \psi^\lambda_{F_\beta,b}(u)$ is polynomial. $\square$

We will define a twist of the action of a $U_{q(F_\beta)}$-module:

Definition 5.3 *Let M be a $U_{q(F_\beta)}$-module. We define $\psi_{F_\beta,b}.M$ to be the module equal to M as a vector space with action twisted via $\psi_{F_\beta,b}$: For $m \in M$ we denote the corresponding element in $\psi_{F_\beta,b}.M$ by $\psi_{F_\beta,b}.m$. Let $\lambda = (\lambda^0, \lambda^1) \in \operatorname{wt} M$ and assume $m \in M_\lambda$. We have a homomorphism of $U_{q(F_\beta)}$-modules $\pi : U_{q(F_\beta)}/I_\lambda \to M$ defined by sending $u + I_\lambda$ in $U_{q(F_\beta)}/I_\lambda$ to um. We define for $u \in U_{q(F_\beta)}$:*

$$u \cdot \psi_{F_\beta,b}.m = \psi_{F_\beta,b}.\left(\pi(\psi^{(\lambda^0,\lambda^1-b\beta)}_{F_\beta,b}(\overline{u}))\right)$$

where $\overline{u} = u + I_{(\lambda^0,\lambda^1-b\beta)} \in U_{q(F_\beta)}/I_{(\lambda^0,\lambda^1-b\beta)}$.

Lemma 5.4 *Let M be a $U_{q(F_\beta)}$-module. Let $r \in \mathbb{Z}$.*

$$\psi_{F_\beta,r}.M \cong M.$$

Furthermore for $\lambda = (\lambda^0, \lambda^1) \in \operatorname{wt} M$ we have as $(U_{q(F_\beta)})_0$-modules

$$\psi_{F_\beta,r}.M_\lambda \cong M_{(\lambda^0,\lambda^1 - r\beta)}.$$

Proof. The isomorphism in both cases is given by $\psi_{F_\beta,r}.m \mapsto F_\beta^{(rl)}m$. Using the fact that $\psi_{F_\beta,r}^{(\lambda^0,\lambda^1-r\beta)}(u + I_{(\lambda^0,\lambda^1-r\beta)}) = F_\beta^{(-rl)}uF_\beta^{(rl)} + I_\lambda$ it is easy to show that this is a homomorphism and the inverse is given by multiplying by $F_\beta^{(-rl)}$.□

Definition 5.5 *Let $\Sigma \subset \Phi^+$. Then Σ is called a set of commuting roots if there exists an ordering of the roots in Σ; $\Sigma = \{\beta_1, \dots, \beta_s\}$ such that for some reduced expression of w_0 and corresponding construction of the root vectors F_β we have: $[F_{\beta_j}, F_{\beta_i}]_q = 0$ for $1 \leq i < j \leq s$.*

For any subset $I \subset \Pi$, let Q_I be the subgroup of Q generated by I, Φ_I the root system generated by I , $\Phi_I^+ = \Phi^+ \cap \Phi_I$ and $\Phi_I^- = -\Phi_I^+$.

We have the following equivalent of Lemma 4.1 in [Mat00]:

Lemma 5.6 *1. Let $I \subset \Pi$ and let $\alpha \in I$. There exists a set of commuting roots $\Sigma' \subset \Phi_I^+$ with $\alpha \in \Sigma'$ such that Σ' is a basis of Q_I.*

2. Let J, F be subsets of Π with $F \neq \Pi$. Let $\Sigma' \subset \Phi_J^+ \backslash \Phi_{J\cap F}^+$ be a set of commuting roots which is a basis of Q_J. There exists a set of commuting roots Σ which is a basis of Q such that $\Sigma' \subset \Sigma \subset \Phi^+ \backslash \Phi_F^+$

Proof. The first part of the proof is just combinatorics of the root system so it is identical to the first part of the proof of Lemma 4.1 in [Mat00]: Let us first prove assertion 2.: If J is empty we can choose $\alpha \in \Pi \backslash F$ and replace J and Σ' by $\{\alpha\}$. So assume from now on that $J \neq \emptyset$. Set $J' = J \backslash F$, $p = |J'|$, $q = |J|$. Let $J_1, \dots, J_k$ be the connected components of J and set $J_i' = J' \cap J_i$, $F_i = F \cap J_i$, and $\Sigma_i' = \Sigma \cap \Phi_{J_i}$, for any $1 \leq i \leq k$. Since $\Sigma' \subset \Phi_J$ is a basis of Q_J, each Σ_i' is a basis of Q_{J_i}. Since Σ_i' lies in $\Phi_{J_i}^+ \backslash \Phi_{F_i}^+$, the set $J_i' = J_i \backslash F_i$ is not empty. Hence J' meets every connected component of J. Therefore we can write $J = \{\alpha_1, \dots, \alpha_q\}$ in such a way that $J' = \{\alpha_1, \dots, \alpha_p\}$ and, for any s with $p + 1 \leq s \leq q$, α_s is connected to α_i for some $i < s$. Since Π is connected we can write $\Pi \backslash J = \{\alpha_{q+1}, \dots, \alpha_n\}$ in such a way that, for any $s \geq q + 1$, α_s is connected to α_i for some i with $1 \leq i < s$. So $\Pi = \{\alpha_1, \dots, \alpha_n\}$ such that for $s > p$ we have that α_s is connected to some α_i with $1 \leq i < s$.

Let $\Sigma' = \{\beta_1, \dots, \beta_q\}$. We will define $\beta_{q+1}, \dots, \beta_l$ inductively such that for each $s \geq q$, $\{\beta_1, \dots, \beta_s\}$ is a commuting set of roots which is a basis of $\Phi_{\{\alpha_1,\dots,\alpha_s\}}$. So assume we have defined $\beta_1, \dots, \beta_s$. Let w_s be the longest word in $s_{\alpha_1}, \dots, s_{\alpha_s}$ and let w_{s+1} be the longest word in $s_{\alpha_1}, \dots, s_{\alpha_{s+1}}$. Choose a reduced expression of w_s such that the corresponding root vectors $\{F_\beta\}$ satisfies $[F_{\beta_j}, F_{\beta_i}]_q = 0$ for $i < j$. Choose a reduced expression of $w_{s+1} = w_s w'$ starting with the above reduced expression of w_s. Let N_s be the length of w_s and N_{s+1} be the length of w_{s+1}. So we get an ordering of the roots generated by $\{\alpha_1, \dots, \alpha_{s+1}\}$: $\Phi_{\{\alpha_1,\dots,\alpha_{s+1}\}}^+ = \{\gamma_1, \dots, \gamma_{N_s}, \gamma_{N_s+1}, \dots, \gamma_{N_{s+1}}\}$ with $\Phi_{\{\alpha_1,\dots,\alpha_s\}}^+ = \{\gamma_1, \dots, \gamma_{N_s}\}$. Consider $\gamma_{N_s+1} = w_s(\alpha_{s+1})$. Since w_s only consists of the simple reflections corresponding to $\alpha_1,\dots,\alpha_s$ we must have that $\gamma_{N_s+1} = \alpha_{s+1} + \sum_{i=1}^s m_i\alpha_i$ for some coefficients $m_i \in \mathbb{N}$. So $\{\beta_1,\dots,\beta_s,\gamma_{N_s+1}\}$ is a basis of $\Phi_{\{\alpha_1,\dots,\alpha_{s+1}\}}$. From Theorem 4.2 we get for $1 \leq i \leq s$

$$[F_{\gamma_{N_s+1}}, F_{\beta_i}]_q \in \operatorname{span}_{\mathbb{C}} \left\{ F_{\gamma_2}^{a_2}\cdots F_{\gamma_{N_s}}^{a_{N_s}} | a_i \in \mathbb{N} \right\}.$$

But since $\{\gamma_1,\dots,\gamma_{N_s}\} = \Phi^+_{\{\alpha_1,\dots,\alpha_s\}}$ and since $\gamma_{N_s+1} = \alpha_{s+1} + \sum_{i=1}^s m_i\alpha_i$ we get $[F_{\gamma_{N_s+1}}, F_{\beta_i}]_q = 0$.

All that is left is to show that $\gamma_{N_s+1} \notin \Phi_F$. By the above we must have that α_{s+1} is connected to some $\alpha_i \in J'$. We will show that the coefficient of α_i in γ_{N_s+1} is nonzero. Otherwise $(\gamma_{N_s+1}|\alpha_i) < 0$ and so $\gamma_{N_s+1} + \alpha_i \in \Phi_{\{\alpha_1,\dots,\alpha_{s+1}\}}$ and by Theorem 1 in [Pap94], $\gamma_{N_s+1} + \alpha_i = \gamma_j$ for some $1 < j \leq s$. This is impossible since $\gamma_{N_s+1} + \alpha_i \notin \Phi_{\{\alpha_1,\dots,\alpha_s\}}$. So we can set $\beta_{s+1} = \gamma_{N_s+1}$ and the induction step is finished.

To prove assertion 1. it can be assumed that $I = \Pi$. Thus assertion 1. follows from assertion 2. with $J = \{\alpha\}$ and $F = \emptyset$. $\square$

Lemma 5.7 *Let $\Sigma = \{\beta_1,\dots,\beta_n\}$ be a set of commuting roots with corresponding root vectors $F_{\beta_1},\dots,F_{\beta_n}$, then $F_{\beta_1}^{(l)},\dots,F_{\beta_n}^{(l)}$ commute.*

Proof. Calculating in U_v for $i < j$ we get using Proposition 4.6

$$[F_{\beta_j}^{(l)}, F_{\beta_i}^{(l)}]_v = \frac{1}{([l]_v!)^2}[F_{\beta_j}^l, F_{\beta_i}^l]_v = 0$$

hence $v^{(l\beta_i|l\beta_j)}F_{\beta_i}^{(l)}F_{\beta_j}^{(l)} - F_{\beta_j}^{(l)}F_{\beta_i}^{(l)} = 0$ in U_A. Since $[F_{\beta_i}^{(l)}, F_{\beta_j}^{(l)}]_q = q^{l^2(\beta_i|\beta_j)}F_{\beta_i}^{(l)}F_{\beta_j}^{(l)} - F_{\beta_j}^{(l)}F_{\beta_i}^{(l)} = F_{\beta_i}^{(l)}F_{\beta_j}^{(l)} - F_{\beta_j}^{(l)}F_{\beta_i}^{(l)}$ we have proved the lemma. $\square$

Corollary 5.8 *Let $\Sigma = \{\beta_1,\dots,\beta_n\}$ be a set of commuting roots with corresponding root vectors $F_{\beta_1},\dots,F_{\beta_n}$. The set*

$$\begin{aligned} F_\Sigma :=& \{r_1! F_{\beta_1}^{(r_1 l)}\cdots r_n! F_{\beta_n}^{(r_n l)} | r_1,\dots,r_n \in \mathbb{N}\} \\ =& \{\left(F_{\beta_1}^{(l)}\right)^{r_1}\cdots\left(F_{\beta_n}^{(l)}\right)^{r_n} | r_1,\dots,r_n \in \mathbb{N}\} \end{aligned}$$

is an Ore subset of U_q.

Proof. This follows from Lemma 5.7 and Lemma 5.1. $\square$

We let $U_{q(F_\Sigma)}$ denote the Ore localization of U_q in the Ore subset F_Σ. For a U_q-module M we define $M_{F_\Sigma} = U_{q(F_\Sigma)} \otimes_{U_q} M$.

Definition 5.9 *Let $\Sigma = \{\beta_1,\dots,\beta_n\}$ be a set of commuting roots that is a basis of Q with a corresponding Ore subset F_Σ. Let $\nu \in \mathfrak{h}^*$, $\nu = \sum_{i=1}^n a_i\beta_i$ for some $a_i \in \mathbb{C}$. For a $U_{q(F_\Sigma)}$-module M we define $\psi_{F_\Sigma,\nu}.M = \psi_{F_{\beta_1},a_1}\circ\cdots\circ\psi_{F_{\beta_n},a_n}.M$.*

Corollary 5.10 *Let Σ be a set of commuting roots that is a basis of Q. Let $\mu \in Q$ and let M be a $U_{q(F_\Sigma)}$-module. Then*

$$\psi_{F_\Sigma,\mu}.M \cong M$$

as $U_{q(F_\Sigma)}$-modules. Also for $\lambda = (\lambda^0,\lambda^1) \in \operatorname{wt} M$:

$$\psi_{F_\Sigma,\mu}.M_\lambda \cong M_{(\lambda^0,\lambda^1+\mu)}$$

as $(U_{q(F_\Sigma)})_0$-modules.

Proof. Since Σ is a basis of Q we can write $\mu = \sum_{\beta \in \Sigma} a_\beta \beta$ for some $a_\beta \in \mathbb{Z}$. So the corollary follows from Lemma 5.4. $\square$

Definition 5.11 *A module $M \in \mathcal{F}$ is called admissible if its weights are contained in a single coset of $(\Lambda_l \times \mathfrak{h}^*)/(\Lambda_l \times Q)$ and if the dimensions of the weight spaces are uniformly bounded. M is called admissible of degree d if d is the maximal dimension of the weight spaces in M.*

Of course all finite dimensional simple modules are admissible but the interesting admissible modules are the infinite dimensional admissible simple modules. In particular simple torsion free modules in $\mathcal{F}$ are admissible. We show later that each infinite dimensional simple module L gives rise to a 'coherent family' $\mathcal{EXT}(L)$ containing at least one simple highest weight module that is admissible of the same degree.

We need the equivalent of Lemma 3.3 in [Mat00]. Some of the proofs leading up to this are more or less the same as in [Mat00] but for completenes we include it here as well.

Definition 5.12 *A cone C is a finitely generated submonoid of the root lattice Q containing 0. If L is a simple module define the cone of L, $C(L)$, to be the submonoid of Q generated by T_L.*

Lemma 5.13 *Let $L \in \mathcal{F}$ be an infinite dimensional simple module. Then the group generated by the submonoid $C(L)$ is Q.*

Compare [Mat00] Lemma 3.1

Proof. First consider the case where $T_L \cap (-F_L) = \emptyset$. Then in this case we have $\Phi = T_L^s \cup F_L^s$. Since F_L^s and T_L^s contain different connected components of the Dynkin diagram and since L is simple and infinite we must have $\Phi = T_L^s$ and therefore $C(L) = Q$.

Next assume $T_L \cap (-F_L) \neq \emptyset$. By Lemma 4.16 in [Fer90] $P_L = T_L^s \cup F_L$ and $P_L^- = T_L \cup F_L^s$ are two opposite parabolic subsystems of Φ. So we have that $T_L \cap (-F_L)$ and $(-T_L) \cap F_L$ must be the roots corresponding to the nilradicals $\mathfrak{v}^\pm$ of two opposite parabolic subalgebras $\mathfrak{p}^\pm$ of $\mathfrak{g}$. Since we have $\mathfrak{g} = \mathfrak{v}^+ + \mathfrak{v}^- + [\mathfrak{v}^+, \mathfrak{v}^-]$ we get that $T_L \cap (-F_L)$ generates Q. Since $C(L)$ contains $T_L \cap (-F_L)$ it generates Q. $\square$

Definition 5.14 *Let $x \geq 0$ be a real number. Define $\rho_l(x) = \operatorname{Card} B_l(x)$ where $B_l(x) = \{\mu \in lQ | \sqrt{(\mu|\mu)} \leq x\}$ and $lQ = \{l\mu \in Q | \mu \in Q\}$.*

Let $M \in \mathcal{F}$ be a weight module with all weights lying in a single coset of $(\Lambda_l \times \mathfrak{h}^*)/(\Lambda_l \times Q)$ say $(0, \lambda^1) + (\Lambda_l \times Q)$. The density of M is

$$\delta_l(M) = \liminf_{x \to \infty} \rho_l(x)^{-1} \sum_{\mu^0 \in \Lambda_l, \mu^1 \in B_l(x)} \dim M_{q^{\mu^1}(\mu^0, \lambda^1)}.$$

For a cone C we define $\delta(C) = \liminf_{x \to \infty} \rho_1(x)^{-1} \operatorname{Card}(C \cap B_1(x)) = \liminf_{x \to \infty} \rho_l(x)^{-1} \operatorname{Card}(lC \cap B_l(x))$ where $lC = \{lc \in Q | c \in C\}$.

Lemma 5.15 *There exists a real number $\varepsilon > 0$ such that $\delta_l(L) > \varepsilon$ for all infinite dimensional simple modules L.*

Proof. Note that since $q^{lC(L)}\lambda \subset \operatorname{wt} L$ for all $\lambda \in \operatorname{wt} L$ we have $\delta_l(L) \geq \delta(C(L))$.

Since $C(L)$ is the cone generated by T_L and $T_L \subset \Phi$ (a finite set) there can only be finitely many different cones.

Since there are only finitely many different cones attached to infinite simple dimensional modules and since any cone C that generates Q has $\delta(C) > 0$ we conclude via Lemma 5.13 that there exists an $\varepsilon > 0$ such that $\delta_l(L) > \varepsilon$ for all infinite dimensional simple modules. $\square$

Definition 5.16 *Let M be a $\mathfrak{g}$-module. We can make M into a U_q-module by the Frobenius homomorphism: We define $M^{[l]}$ to be the U_q-module equal to M as a vector space and with the action defined as follows: For $m \in M$, $\alpha \in \Pi$,*

$$\begin{aligned} K_\alpha^{\pm 1} m =& m \\ E_\alpha m =& 0 \\ E_\alpha^{(l)} m =& e_\alpha m \\ F_\alpha m =& 0 \\ F_\alpha^{(l)} m =& f_\alpha m \end{aligned}$$

where e_α is a root vector of $\mathfrak{g}$ of weight α and f_α is such that $[e_\alpha, f_\alpha] = h_\alpha$. The above defines an action of U_q on M by Theorem 1.1 in [KL02].

Proposition 5.17 *Let $\lambda = (\lambda^0, \lambda^1) \in X$ and let $L(\lambda)$ be the unique simple highest weight module with weight λ. Then $L(\lambda) \cong L_{\mathbb{C}}(\lambda^1)^{[l]} \otimes L((\lambda^0, 0))$ where $L_{\mathbb{C}}(\lambda^1)$ denotes the unique simple $\mathfrak{g}$-module of highest weight λ^1.*

Proof. The proof of Theorem 3.1 in [AM15] works here in exactly the same way. $\square$

Lemma 5.18 *Let $M \in \mathcal{F}$ be an admissible module. Then M has finite Jordan-Hölder length.*

Proof. As M is admissible, we have $\delta_l(M) < \infty$. For any exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, we have $\delta_l(M_2) \geq \delta_l(M_1) + \delta_l(M_3)$. Let Y be the set of all $\mu \in \Lambda$ such that $|\langle \mu, \alpha^\vee \rangle| \leq 1$ for all $\alpha \in \Pi$. By Proposition 5.17 and the classification of classical simple finite dimensional $\mathfrak{g}$-modules any finite dimensional U_q-module L has $L_{(\mu^0,\mu^1)} \neq 0$ for some $\mu^0 \in \Lambda_l$ and some $\mu^1 \in Y$. It follows like in [Mat00, Lemma 3.3] that the length of M is finite and bounded by $A + \delta_l(M)/\varepsilon$ where $A = \sum_{\mu^0 \in \Lambda_l, \mu^1 \in Y} \dim M_{(\mu^0,\mu^1)}$ and ε is the constant from Lemma 5.15. $\square$

Lemma 5.19 *Let M be an admissible module. Let $\Sigma \subset \Phi^+$ be a set of commuting roots and F_Σ a corresponding Ore subset. Assume $-\Sigma \subset T_M$. Then for $\lambda = (\lambda^0, \lambda^1) \in X$:*

$$\dim (M_{F_\Sigma})_\lambda = \max_{\mu \in \mathbb{Z}\Sigma} \{\dim M_{(\lambda^0, \lambda^1 + \mu)}\}$$

and if $\dim M_\lambda = \max_{\mu \in \mathbb{Z}\Sigma}\{\dim M_{(\lambda^0,\lambda^1+\mu)}\}$ then $(M_{F_\Sigma})_\lambda \cong M_\lambda$ as $(U_q)_0$-modules.

In particular if $\Sigma \subset T_M$ as well then $M_{F_\Sigma} \cong M$ as U_q-modules.

Compare to Lemma 4.4(ii) in [Mat00].

Proof. We have $\Sigma = \{\beta_1, \dots, \beta_r\}$ for some $\beta_1, \dots, \beta_r \in \Phi^+$. Let $F_{\beta_1}, \dots, F_{\beta_r}$ be corresponding q-commuting root vectors. Let $\lambda \in X$ and set

$$d = \max_{\mu \in \mathbb{Z}\Sigma} \{\dim M_{(\lambda^0, \lambda^1 + \mu)}\}.$$

Let V be a finite dimensional subspace of $(M_{F_\Sigma})_\lambda$. Then there exists a homogenous element $s \in F_\Sigma$ such that $sV \subset M$. Let $\nu \in \mathbb{Z}\Sigma$ be the degree of s. So $sV \subset M_{q^\nu \lambda}$ hence $\dim sV \leq d$. Since s acts injectively on M_{F_Σ} we have $\dim V \leq d$. Now the first claim follows because $F_\beta^{(\pm l)}$ acts injectively on M_{F_Σ} for all $\beta \in \Sigma$.

We have an injective U_q-homomorphism from M to M_{F_Σ} sending $m \in M$ to $1 \otimes m \in M_{F_\Sigma}$ that restricts to a $(U_q)_0$-homomorphism from M_λ to $(M_{F_\Sigma})_\lambda$. If $\dim M_\lambda = d$ then this is surjective as well. So it is an isomorphism. The last claim follows because $\pm\Sigma \subset T_M$ implies $\dim M_\lambda = \dim M_{q^\mu \lambda}$ for any $\mu \in \mathbb{Z}l\Sigma$; so $M_\lambda \cong (M_{F_\Sigma})_\lambda$ for any $\lambda \in X$. Since M is a weight module this implies that $M \cong M_{F_\Sigma}$ as U_q-modules. □

Lemma 5.20 *Let $L \in \mathcal{F}$ be a simple $U_q(\mathfrak{sl}_2)$ module. Then the weight spaces of L are all 1-dimensional.*

Proof. For $\mathfrak{sl}_2$ there is only one simple root α and we will denote the root vectors E_α and F_α by E and F respectively. Similarly $K^{\pm 1} = K_\alpha^{\pm 1}$. Consider the Casimir element $C = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}$. Let $\lambda \in \operatorname{wt} L$ and let $c \in \mathbb{C}$ be an eigenvalue of C on L_λ. Consider the eigenspace $L(c) = \{v \in L_\lambda | Cv = cv\}$. Then $F^{(l)}E^{(l)}$ acts on this space since C commutes with all elements from $U_q(\mathfrak{sl}_2)$. Choose an eigenvector $v_0 \in L(c)$ for $F^{(l)}E^{(l)}$. We will show by induction that $E^{(n)}F^{(n)}v_0 \in \mathbb{C}v_0$ for all $n \in \mathbb{N}$. The induction start $n = 0$ is obivous. Let $n \in \mathbb{N}$ and assume $n = i + rl$ with $0 \leq i < l$. If $i \neq 0$ then $[n] \neq 0$ and we have:

$$\begin{aligned}
E^{(n)}F^{(n)}v_0 =& \frac{1}{[n]^2} E^{(n-1)} EF F^{(n-1)} v_0 \\
=& \frac{1}{[n]^2} E^{(n-1)} \left(C - \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} \right) F^{(n-1)} v_0 \\
=& \frac{1}{[n]^2} E^{(n-1)} F^{(n-1)} \left(c - \frac{q^{\left(\lambda^0|\alpha\right)+1-2n} + q^{2n-1-\left(\lambda^0|\alpha\right)}}{(q-q^{-1})^2} \right) v_0
\end{aligned}$$

where α is the simple root. So the claim follows by induction. In the case that $i = 0$ we have

$$\begin{aligned}
E^{(n)}F^{(n)}v_0 =& \frac{1}{r} E^{(rl-l)} E^{(l)} F^{(rl)} v_0 \\
=& \frac{1}{r} E^{(rl-l)} \sum_{t=0}^{l} F^{(rl-t)} \begin{bmatrix} K; 2t - rl - l \\ t \end{bmatrix} E^{(l-t)} v_0 \\
=& \frac{1}{r} E^{(rl-l)} \sum_{t=0}^{l} F^{(rl-l)} F^{(l-t)} E^{(l-t)} \begin{bmatrix} K; l - rl \\ t \end{bmatrix} v_0 \\
=& \frac{1}{r} E^{(rl-l)} F^{(rl-l)} \left(F^{(l)}E^{(l)} + \sum_{t=1}^{l-1} F^{(l-t)} E^{(l-t)} \begin{bmatrix} \left(\lambda^0|\alpha\right) + l - rl \\ t \end{bmatrix} + \left\langle \lambda^1, \alpha^\vee \right\rangle + 1 - r \right) v_0
\end{aligned}$$

Since v_0 is an eigenvector for $F^{(l)}E^{(l)}$ we have only left to consider the action of $F^{(i)}E^{(i)}$ for $1 \leq i < l$. But we can show like above that $F^{(i)}E^{(i)}v_0 \in \mathbb{C}v_0$ by using that $C = FE + \frac{qK+K^{-1}q^{-1}}{(q-q^{-1})^2}$.

Now since L is simple we must have that L_λ is a simple $(U_q)_0$-module (Lemma 2.18). So L_λ is generated by v_0. Since $E^{(n)}F^{(n)}v_0 \in \mathbb{C}v_0$ for all $n \in \mathbb{C}$ we get $\dim L_\lambda = 1$. □

Lemma 5.21 *Let L be a simple infinite dimensional admissible module. Let $\beta \in (T_L^s)^+$. Then there exists a $b \in \mathbb{C}$ such that $\psi_{F_\beta,b}.L_{F_\beta}$ contains a simple admissible U_q-submodule L' with $T_{L'} \subset T_L$ and $\beta \notin T_{L'}$.*

Proof. By Lemma 5.19 $L \cong L_{F_\beta}$ as U_q-modules so we will write L instead of L_{F_β} when taking twist. The $U_{q(F_\beta)}$-module structure on L coming from the isomorphism. Let D_β be the subalgebra of U_q generated by $E_\beta^{(n)}$, $K_\beta^{\pm 1}$, $F_\beta^{(n)}$, $n \in \mathbb{N}$. Then D_β is isomorphic to the algebra $U_{q_\beta}(\mathfrak{sl}_2)$. Let $v \in L$ and consider the D_β-module $D_\beta v$. Since L is admissible so is $D_\beta v$. So $D_\beta v$ has a simple D_β-submodule V by Lemma 5.18.

Let $v \in V$ be a weight vector such that $E_\beta v = 0$ (such a v always exists since $E_\beta^l = 0$). Assume λ is the weight of v. By Lemma 5.20 $F_\beta^{(l)}E_\beta^{(l)}v = cv$ for some $c \in \mathbb{C}$.

Then by (the proof of) Lemma 5.2 we get

$$
\begin{aligned}
&F_\beta^{(l)}E_\beta^{(l)}\psi_{F_\beta,b}.v \\
&\quad = \psi_{F_\beta,b}.\left(c + b\sum_{t=1}^{l-1} F_\beta^{(l-t)}E_\beta^{(l-t)} \begin{bmatrix} \langle \lambda_0, \beta^\vee \rangle \\ t \end{bmatrix}_\beta + b(\langle \lambda_1, \beta^\vee \rangle + 1 - b)\right) v \\
&\quad = \psi_{F_\beta,b}.\left(c + b(\langle \lambda_1, \beta^\vee \rangle + 1 - b)\right) v.
\end{aligned}
$$

Since $\mathbb{C}$ is algebraically closed the polynomial in b, $c+b(\langle \lambda_1, \beta^\vee \rangle + 1 - b)$ has a zero. Assume from now on that $b \in \mathbb{C}$ is chosen such that $c+b(\langle \lambda_1, \beta^\vee \rangle + 1 - b) = 0$.

Thus $\psi_{F_\beta,b}.L$ contains an element $v' = \psi_{F_\beta,b}.v$ such that $F_\beta^{(l)}E_\beta^{(l)}v' = 0$ and since $F_\beta^{(l)}$ acts injectively on $\psi_{F_\beta,b}.L$, we have $E_\beta^{(l)}v' = 0$. Set $V = \{m \in \psi_{F_\beta,b}.L | E_\beta^{(N)}m = 0, N >> 0\} = (\psi_{F_\beta,b}.L)^{[\beta]}$. By Proposition 2.3 this is a U_q-submodule of the U_q-module $\psi_{F_\beta,b}.L$. It is nonzero since $v' \in V$. By Lemma 5.18 V has a simple U_q-submodule L'.

We have left to show that $T_{L'} \subset T_L$. Assume $\gamma \in T_{L'}$. Then $q^{l\mathbb{N}\gamma} \operatorname{wt} L' \subset \operatorname{wt} L'$ by Proposition 3.7. But since $\operatorname{wt} L' \subset \{(\lambda^0, \lambda^1 - b\beta) | (\lambda^0, \lambda^1) \in \operatorname{wt} L\}$ we get for some $\nu \in \operatorname{wt} L$, $\{(\nu^0, \nu^1 - b\beta + r\gamma) | r \in \mathbb{N}\} \subset \{(\lambda^0, \lambda^1 - b\beta) | (\lambda^0, \lambda^1) \in \operatorname{wt} L\}$ or equivalently $q^{l\mathbb{N}\gamma}\nu \subset \operatorname{wt} L$. But this shows that $\gamma \notin F_L$ and since L is a simple U_q-module this implies that $\gamma \in T_L$. By construction we have $\beta \notin T_{L'}$. □

Lemma 5.22 *Let $L \in \mathcal{F}$ be a simple module. Then there exists a $w \in W$ such that $w(F_L \backslash F_L^s) \subset \Phi^+$ and $w(T_L \backslash T_L^s) \subset \Phi^-$.*

Proof. Lemma 4.16 in [Fer90] tells us that there exists a basis B of the root system Φ such that the antisymmetrical part, $F_L \backslash F_L^s$, of F_L is contained in the positive roots Φ_B^+ corresponding to the basis B and the antisymmetrical part, $T_L \backslash T_L^s$, of T_L is contained in the negative roots Φ_B^- corresponding to the basis. Since all bases of a root system are W-conjugate the claim follows. □

Lemma 5.23 *Let L be an infinite dimensional admissible simple module. Let $w \in W$ be such that $w(F_L \backslash F_L^s) \subset \Phi^+$. Let $\alpha \in \Pi$ be such that $-\alpha \in w(T_L)$ (such an α always exists). Then there exists a commuting set of roots Σ with $\alpha \in \Sigma$ which is a basis of Q such that $-\Sigma \subset w(T_L)$.*

Proof. Set $L' = {}^wL$. Since $w(T_L) = T_{^wL} = T_{L'}$ we will just work with L'. Then $F_{L'} \backslash F_{L'}^s \subset \Phi^+$.

Note that it is always possible to choose a simple root $\alpha \in -T_{L'}$ since L' is infinite dimensional: If this was not possible we would have $\Phi^- \subset F_{L'}$. But since $F_{L'} \backslash F_{L'}^s \subset \Phi^+$ this implies $F_{L'} = \Phi$.

Set $F = F_{L'}^s \cap \Pi$. Since L' is infinite dimensional $F \neq \Pi$. By Lemma 5.6 2. applied with $J = \{\alpha\} = \Sigma'$ there exists a commuting set of roots Σ that is a basis of Q such that $\Sigma \subset \Phi^+ \backslash \Phi_F^+$. Since $F_{L'} \backslash F_{L'}^s \subset \Phi^+$ we have $\Phi^- = T_{L'}^- \cup (F_{L'}^s)^-$. To show $-\Sigma \subset T_{L'}$ we show $\left(\Phi^- \backslash \Phi_F^-\right) \cap F_{L'}^s = \emptyset$ or equivalently $(F_{L'}^s)^- \subset \Phi_F^-$.

Assume $\beta \in F_{L'}^s \cap \Phi^+$, $\beta = \sum_{\alpha \in \Pi} a_\alpha \alpha$, $a_\alpha \in \mathbb{N}$. The height of β is the sum $\sum_{\alpha \in \Pi} a_\alpha$. We will show by induction on the height of β that $-\beta \in \Phi_F^-$. If the height of β is 1 then β is a simple root and so $\beta \in F$. Clearly $-\beta \in \Phi_F^-$ in this case. Assume the height of β is greater than 1. Let $\alpha' \in \Pi$ be a simple root such that $\beta - \alpha'$ is a root. There are two possibilities: $-\alpha' \in T_{L'}$ or $\pm\alpha' \in F_{L'}^s$.

In the first case where $-\alpha' \in T_{L'}$ we must have $-\beta + \alpha' \in F_{L'}^s$ since if $-\beta + \alpha' \in T_L$ then $-\beta = (-\beta + \alpha') - \alpha' \in T_{L'}$ because $T_{L'}$ is closed (Proposition 3.8). So $\beta - \alpha' \in F_{L'}^s$ and $\beta \in F_{L'}^s$. Since $F_{L'}$ is closed (Proposition 2.11) we get $-\alpha' = (\beta - \alpha') - \beta \in F_L$ which is a contradiction. So the first case ($-\alpha' \in T_{L'}$) is impossible.

In the second case since $F_{L'}$ is closed we get $\pm(\beta - \alpha') \in F_{L'}$ i.e. $\beta - \alpha' \in F_{L'}^s$. By the induction $-(\beta - \alpha') \in \Phi_F^-$ and since $-\beta = -(\beta - \alpha') - \alpha'$ we are done.□

6 Coherent families

As in the above section q is a complex primitive l'th root of unity with l odd in this section. For $\lambda \in X$ we write $\lambda = (\lambda^0, \lambda^1)$ like above.

Lemma 6.1 *Let $M, N \in \mathcal{F}$ be semisimple U_q-modules. If $\mathrm{Tr}^M = \mathrm{Tr}^N$ then $M \cong N$.*

Proof. Theorem 7.19 in [Lam01] states that this is true for modules over a *finite dimensional* algebra. So we will reduce to the case of modules over a finite dimensional algebra. Let L be a composition factor of M and λ a weight of L. Then the multiplicity of the U_q-composition factor L in M is the multiplicity of the $(U_q)_0$-composition factor L_λ in M_λ by Theorem 2.21. M_λ is a finite dimensional $(U_q)_0$-module. Let I be the kernel of the homomorphism $(U_q)_0 \to \mathrm{End}_{\mathbb{C}}(M_\lambda)$ given by the action of $(U_q)_0$. Then $(U_q)_0/I$ is a finite dimensional $\mathbb{C}$ algebra and M_λ is a module over $(U_q)_0/I$. Furthermore since $\mathrm{Tr}^M(\lambda, u) = 0$ for all $u \in I$ the trace of an element $u \in (U_q)_0$ is the same as the trace of $u + I \in (U_q)_0/I$ on M_λ as a $(U_q)_0/I$-module. So if $\mathrm{Tr}^M = \mathrm{Tr}^N$ the multiplicity of L_λ in M_λ and N_λ are the same and hence the multiplicity of L in M is the same as in N. □

Definition 6.2

$$T^* = \mathfrak{h}^*/Q.$$

By Corollary 5.10 it makes sense to write $\psi_{F_\beta,t}.M$ for $t \in T^*$ up to isomorphism for a $U_{q(F_\Sigma)}$-module M.

Definition 6.3 *A (quantized) coherent family is a U_q-module $\mathcal{M}$ such that for all $\mu \in \Lambda_l$:*

- $\dim \mathcal{M}_{(\mu,\nu)} = \dim \mathcal{M}_{(\mu,\nu')}$ *for all $\nu, \nu' \in \mathfrak{h}^*$.*
- *For all $u \in (U_q)_0$, the map $\mathfrak{h}^* \ni \nu \mapsto \operatorname{Tr} u|_{\mathcal{M}_{(\mu,\nu)}}$ is polynomial.*

For a coherent family $\mathcal{M}$ and $t \in T^*$ define

$$\mathcal{M}[t] = \bigoplus_{\mu^0 \in \Lambda_l, \mu^1 \in t} \mathcal{M}_{(\mu^0,\mu^1)}.$$

$\mathcal{M}$ is called irreducible if there exists a $t \in T^*$ such that $\mathcal{M}[t]$ is a simple U_q-module.

Lemma 6.4 *Let $\mathcal{M}$ be a coherent family. Let $\mu \in \Lambda_l$. Then the set Ω of all weights $\nu \in \mathfrak{h}^*$ such that the $(U_q)_0$-module $\mathcal{M}_{(\mu,\nu)}$ is simple is a Zariski open subset of $\mathfrak{h}^*$.*

If $\mathcal{M}$ is irreducible then $\Omega \neq \emptyset$ if $\mathcal{M}_{(\mu,\nu)} \neq 0$ for any $\nu \in \mathfrak{h}^*$ (equivalently for all $\nu \in \mathfrak{h}^*$).

Proof. If $\mathcal{M}_{(\mu,\nu)} = 0$ for all $\nu \in \mathfrak{h}^*$ then $\Omega = \emptyset$. Assume $\dim \mathcal{M}_{(\mu,\nu)} = d > 0$ for all $\nu \in \mathfrak{h}^*$. If $\mathcal{M}$ is irreducible there exists $t \in T^*$ such that $\mathcal{M}[t]$ is a simple U_q-module. Then for $\nu \in t$, $\mathcal{M}_{(\mu,\nu)} = \mathcal{M}[t]_{(\mu,\nu)}$ is a simple U_q-module by Theorem 2.21. So in this case $\Omega \neq \emptyset$.

Now the proof goes exactly like in [Mat00, Lemma 4.7]: The $(U_q)_0$-module $\mathcal{M}_{(\mu,\nu)}$ is simple if and only if the bilinear map $B_\nu : (U_q)_0 \times (U_q)_0 \ni (u,v) \mapsto \operatorname{Tr}(uv|_{\mathcal{M}_{(\mu,\nu)}})$ has maximal rank d^2. For any finite dimensional subspace $E \subset (U_q)_0$ the set Ω_E of all ν such that $B_\nu|_E$ has rank d^2 is open. Therefore $\Omega = \cup_E \Omega_E$ is open. □

Definition 6.5 *Let L be an admissible U_q-module and let $\mu \in \Lambda_l$.*

$$\operatorname{Supp}(L,\mu) = \{\nu \in \mathfrak{h}^* | \dim L_{(\mu,\nu)} > 0\}$$

and

$$\operatorname{Supp}_{\text{ess}}(L,\mu) = \{\nu \in \operatorname{Supp}(L,\mu) | \dim L_{(\mu,\nu)} \text{ is maximal in } \{\dim L_{(\mu,\nu')} | \nu' \in \mathfrak{h}^*\}\}.$$

Definition 6.6 *Let M be an admissible module. Define M^{ss} to be the unique (up to isomorphism) semisimple module with the same composition factors as M.*

Let V be a U_q-module such that $V = \bigoplus_{i \in I} V_i$ for some index set I and some admissible U_q-modules V_i. Then $V^{ss} = \bigoplus_{i \in I} V_i^{ss}$.

Proposition 6.7 *Let L be an infinite dimensional admissible simple U_q-module. Then there exists a unique (up to isomorphism) semisimple irreducible coherent family $\mathcal{EXT}(L)$ containing L.*

Proof. Let $w \in W$ be such that $w(F_L \backslash F_L^s) \subset \Phi^+$ and Σ a set of commuting roots that is a basis of Q such that $-\Sigma \subset w(T_L)$ (Exists by Lemma 5.22 and Lemma 5.23) with corresponding Ore subset F_Σ. Set

$$\mathcal{EXT}(L) := \left(\bigoplus_{t \in T^*} {}^{\overline{w}} \left(\psi_{F_\Sigma,t} . ({}^w L)_{F_\Sigma} \right) \right)^{ss}.$$

For each $t \in T^*$ choose a representative $\nu_t \in t$. As a $(U_{q(F_\Sigma)})_0$-module

$$\mathcal{EXT}(L) = \bigoplus_{t \in T^*} {}^{\overline{w}} \left(\psi_{F_\Sigma,\nu_t} . ({}^w L)_{F_\Sigma} \right)^{ss}.$$

Define $Y := \{\mu \in \Lambda_l | \operatorname{Supp}({}^w L, \mu) \neq \emptyset\}$. For each $\mu \in Y$ let $\lambda_\mu \in \operatorname{Supp}_{\mathrm{ess}}({}^w L, \mu)$. By Corollary 5.10

$$({}^w L)_{F_\Sigma} \cong \bigoplus_{\mu \in Y} \bigoplus_{\nu \in Q} \psi_{F_\Sigma,\nu} . (({}^w L)_{F_\Sigma})_{(\mu,\lambda_\mu)}$$

as $(U_{q(F_\Sigma)})_0$-modules.

So we have the following $(U_{q(F_\Sigma)})_0$-module isomorphisms:

$$\begin{aligned} \mathcal{EXT}(L) &\cong \bigoplus_{\mu \in Y} \bigoplus_{t \in T^*} \bigoplus_{\nu \in Q} {}^{\overline{w}} \left(\psi_{F_\Sigma,\nu_t+\nu} . (({}^w L)_{F_\Sigma})_{(\mu,\lambda_\mu)} \right)^{ss} \\ &\cong \bigoplus_{\mu \in Y} \bigoplus_{\nu \in \mathfrak{h}^*} {}^{\overline{w}} \left(\psi_{F_\Sigma,\nu} . (({}^w L)_{F_\Sigma})_{(\mu,\lambda_\mu)} \right)^{ss}. \end{aligned}$$

Let $u \in (U_q)_0$ and $\mu \in Y$. Then we see from the above and Lemma 5.19 that

$$\operatorname{Tr} u|_{\mathcal{EXT}(L)_{(\mu,\nu)}} = \operatorname{Tr} \psi^{(\mu,\lambda_\mu)}_{F_\Sigma,\nu-\lambda_\mu}(T_w^{-1}(u))|_{({}^w L)_{(\mu,\lambda_\mu)}}.$$

By Lemma 5.2 this is polynomial in $\nu - \lambda_\mu$ hence also polynomial in ν. We know that this polynomial is determined in all ν such that $\nu - \lambda_\mu \in \operatorname{Supp}_{\mathrm{ess}}(L, \mu)$. $\operatorname{Supp}_{\mathrm{ess}}(L, \mu)$ is Zariski dense in $\mathfrak{h}^*$ because $\lambda_\mu - \mathbb{N}\Sigma \subset \operatorname{Supp}_{\mathrm{ess}}(L, \mu)$ and Σ is a basis of Q. So Tr is determined on all of $\mathcal{EXT}(L)$ by L. For any $(\mu, \nu) \in X$ we have

$$\begin{aligned} \dim \mathcal{EXT}(L)_{(\mu,\lambda_\mu+\nu)} &= \dim \left(\psi_{F_\Sigma,\nu} . (({}^w L)_{F_\Sigma})_{(\mu,\lambda_\mu)} \right)^{ss} \\ &= \dim(({}^w L)_{F_\Sigma})_{(\mu,\lambda_\mu)} \end{aligned}$$

so $\mathcal{EXT}(L)$ is a coherent family.

Assume $\mathcal{M}$ is a semisimple irreducible coherent family containing L. Let $\mu \in Y$. By Lemma 6.4 the set Ω_1 of $\nu \in \mathfrak{h}^*$ such that $\mathcal{EXT}(L)_{(\mu,\nu)}$ is simple and the set Ω_2 of $\nu \in \mathfrak{h}^*$ such that $\mathcal{M}_{(\mu,\nu)}$ is simple are non-empty open subsets of $\mathfrak{h}^*$ ($\Omega_1 \neq \emptyset$ because $\mathcal{EXT}(L)_{(\mu,\nu)} = L_{(\mu,\nu)}$ for $\nu \in \operatorname{Supp}_{\mathrm{ess}}(L, \mu)$). So their intersection $\Omega_1 \cap \Omega_2$ is open and non-empty (since any Zariski open set of $\mathfrak{h}^*$ is Zariski dense in $\mathfrak{h}^*$). Since $\operatorname{Supp}_{\mathrm{ess}}(L, \mu)$ is Zariski dense we get that there exists a $\nu \in \Omega_1 \cap \Omega_2 \cap \operatorname{Supp}_{\mathrm{ess}}(L, \mu)$ such that $\mathcal{M}_{(\mu,\nu)}$ and $\mathcal{EXT}(L)_{(\mu,\nu)}$ are simple. Since $L_{(\mu,\nu)} \subset \mathcal{M}_{(\mu,\nu)}$ and $L_{(\mu,\nu)} \subset \mathcal{EXT}(L)_{(\mu,\nu)}$ we get $\mathcal{M}_{(\mu,\nu)} \cong L_{(\mu,\nu)} \cong \mathcal{EXT}(L)_{(\mu,\nu)}$. This is true for any (μ, ν) such that $\nu \in \operatorname{Supp}_{\mathrm{ess}}(L, \mu)$. Let $u \in (U_q)_0$ and $\mu \in Y$. Then we see that $\operatorname{Tr} u|_{\mathcal{EXT}(L)_{(\mu,\nu)}} = \operatorname{Tr} u|_{L_{(\mu,\nu)}} =$

$\operatorname{Tr} u|_{\mathcal{M}_{(\mu,\nu)}}$ for any $\nu \in \operatorname{Supp}_{\mathrm{ess}}(L,\mu)$. Since $\operatorname{Supp}_{\mathrm{ess}}(L,\mu)$ is Zariski dense this implies $\operatorname{Tr} u|_{\mathcal{EXT}(L)_{(\mu,\nu)}} = \operatorname{Tr} u|_{\mathcal{M}_{(\mu,\nu)}}$ for all $\nu \in \mathfrak{h}^*$. So by Lemma 6.1 $\mathcal{EXT}(L)_{(\mu,\nu)} \cong \mathcal{M}_{(\mu,\nu)}$ as $(U_q)_0$-modules for any $(\mu,\nu) \in \operatorname{wt} \mathcal{EXT}(L)$.

Then by Theorem 2.21 we get that $\mathcal{M} \cong \mathcal{EXT}(L) \oplus \mathcal{N}$ for some coherent family $\mathcal{N}$ with the property that $\mathcal{N}_{(\mu,\nu)} = 0$ for any $(\mu,\nu) \in X$ such that $\operatorname{Supp}(L,\mu) \neq \emptyset$. Since $\mathcal{M}$ is irreducible there exists a $t \in T^*$ such that the U_q-module $\mathcal{M}[t]$ is simple. We have $\mathcal{M}[t] \cong \mathcal{EXT}(L)[t] \oplus \mathcal{N}[t]$. Since $\mathcal{M}[t]$ is simple and $\mathcal{EXT}(L)[t] \neq 0$ we get that $\mathcal{N}[t] = 0$. Since $\mathcal{N}$ is a coherent family this implies that $\mathcal{N} = 0$. So $\mathcal{M} \cong \mathcal{EXT}(L)$.

So we have left to show that $\mathcal{EXT}(L)$ is irreducible. Let $F_{\beta_1}, \dots, F_{\beta_n}$ be the root vectors corresponding to $\Sigma = \{\beta_1, \dots, \beta_n\}$ and $E_{\beta_1}, \dots, E_{\beta_n}$ the corresponding E-root vectors. Let $\mu \in Y$. As above we choose a $\lambda_\mu \in \operatorname{Supp}_{\mathrm{ess}}({}^wL)$. The elements $F_{\beta_i}^{(l)} E_{\beta_i}^{(l)}$, $i = 1, \dots, n$ act on $\psi_{F_\Sigma,\nu}.(({}^wL)_{F_\Sigma})_{(\mu,\lambda_\mu)}$ by $\sum_{j=1}^s p_{i,j}^\mu(\nu) u_{i,j}$ for some $u_{i,j}^\mu \in U_{q(F_\Sigma)}$ and some polynomials $p_{i,j}^\mu : \mathfrak{h}^* \to \mathbb{C}$ so

$$p_\mu := \prod_{i=1}^n \det F_{\beta_i}^{(l)} E_{\beta_i}^{(l)}|_{\psi_{F_\Sigma,\nu}.(({}^wL)_{F_\Sigma})_{(\mu,\lambda_\mu)}}$$

is a nonzero polynomial in ν by (the proof of) Lemma 5.2. Set $p = \prod_{\mu \in Y} p_\mu$. Let Ω be the set of non-zero points for p. By [Mat00, Lemma 5.2 i)] the set $T(\Omega) := \bigcap_{\mu \in Q} (\mu + \Omega)$ is non-empty. So there exists a $\nu \in \mathfrak{h}^*$ such that $p(\nu + \mu^1) \neq 0$ for any $\mu^1 \in Q$. For such a ν we see that $F_{\beta_i}^{(l)} E_{\beta_i}^{(l)}$ act bijectively on

$$\bigoplus_{\mu \in Y} \bigoplus_{\mu^1 \in Q} \psi_{F_\Sigma,\nu}.(({}^wL)_{F_\Sigma})_{(\mu,\lambda_\mu+\mu^1)} = \psi_{F_\Sigma,\nu}.({}^wL)_{F_\Sigma}.$$

Since $F_{\beta_i}^{(l)}$ act injectively on $\psi_{F_\Sigma,\nu}.({}^wL)_{F_\Sigma}$ this implies that $E_{\beta_i}^{(l)}$ act injectively on $\psi_{F_\Sigma,\nu}.({}^wL)_{F_\Sigma}$. Let $L_1 \subset \psi_{F_\Sigma,\nu}.({}^wL)_{F_\Sigma}$ be a simple U_q-submodule of $\psi_{F_\Sigma,\nu}.({}^wL)_{F_\Sigma}$. By the above we have $\pm\Sigma \subset T_{L_1}$. So by Proposition 3.7 we get $T_{L_1} = \Phi$. Define $\mathcal{EXT}(L_1) = \left(\bigoplus_{t \in T^*} (\psi_{F_\Sigma,t}.(L_1)_{F_\Sigma})\right)^{ss}$. Then as above this is a coherent family. Let $\lambda' \in \operatorname{wt} L_1$. Then $\mathcal{EXT}[\lambda' + Q] = (L_1)_{F_\Sigma} = L_1$ by Lemma 5.19 so $\mathcal{EXT}(L_1)$ is an irreducible coherent family.

Let $\mu \in \Lambda_l$ be such that $\operatorname{Supp}(L_1,\mu) \neq \emptyset$. $\operatorname{Supp}_{\mathrm{ess}}(L_1,\mu)$ is Zariski dense in $\mathfrak{h}^*$ so $\operatorname{Supp}_{\mathrm{ess}}(L_1,\mu) \cap \Omega_1 \neq \emptyset$. Let $\nu' \in \Omega_1 \cap \operatorname{Supp}_{\mathrm{ess}}(L_1,\mu)$. Then $(L_1)_{(\mu,\nu')} \cong (\psi_{F_\Sigma,\nu}.({}^wL)_{F_\Sigma})_{(\mu,\nu')}$. Then as above (with $\mathcal{M} = \mathcal{EXT}(L)$ and L replaced by L_1) we get $\mathcal{EXT}(L) \cong \mathcal{EXT}(L_1) \oplus \mathcal{N}$ for some semisimple coherent family $\mathcal{N}$ with $\mathcal{N}_{(\mu,\nu)} = 0$ for any $(\mu,\nu) \in X$ such that $\operatorname{Supp}(L_1,\mu) \neq \emptyset$. Since $\mathcal{EXT}(L)$ contains L we get that $L = M' \oplus M''$ for some U_q-modules $M' \subset \mathcal{EXT}(L_1)$ and $M'' \subset \mathcal{N}$. Since L is simple and since there exists a $\mu \in \Lambda_l$ such that $\operatorname{Supp}(L,\mu) \neq \emptyset$ and $\operatorname{Supp}(L_1,\mu) \neq \emptyset$ we must have $M'' = 0$ and $L = M'$. But then we have proved that the irreducible coherent family $\mathcal{EXT}(L_1)$ contains L. Hence $\mathcal{EXT}(L) \cong \mathcal{EXT}(L_1)$ by the above and $\mathcal{EXT}(L)$ is irreducible. $\square$

Theorem 6.8 *Let L be an admissible infinite dimensional simple module. Then there exists a $w \in W$ and a $\lambda \in X$ such that ${}^w\mathcal{EXT}(L)$ contains an infinite dimensional simple highest weight module $L(\lambda)$ and ${}^w\mathcal{EXT}(L) \cong \mathcal{EXT}(L(\lambda))$.*

Proof. Let $w \in W$ be such that $w(F_L \backslash F_L^s) \subset \Phi^+$ and $w(T_L \backslash T_L^s) \subset \Phi^-$ and let Σ be a set of commuting roots that is a basis of Q such that $-\Sigma \subset w(T_L)$ (Exists by Lemma 5.22 and Lemma 5.23). Let F_Σ be a corresponding Ore subset. Then

$$\mathcal{EXT}(L) = \left(\bigoplus_{t \in T^*} \overline{w} \left(\psi_{F_\Sigma, t}.({}^w L)_{F_\Sigma} \right) \right)^{ss}$$

so

$${}^w\mathcal{EXT}(L) = \left(\bigoplus_{t \in T^*} (\psi_{F_\Sigma, t}.({}^w L)_{F_\Sigma}) \right)^{ss} = \mathcal{EXT}({}^w L).$$

Set $L' = {}^w L$. We will show by induction on $|T_{L'}^+|$ that there exists a $\lambda \in X$ such that $L(\lambda)$ is infinite dimensional and $\mathcal{EXT}(L') \cong \mathcal{EXT}(L(\lambda))$:

If $|T_{L'}^+| = 0$ then L' is itself an infinite dimensional highest weight module. Assume $|T_{L'}^+| > 0$. Then $T_{L'}^+ \cap \Pi \neq \emptyset$ because if this was not the case then $\Phi^+ \subset F_{L'}$ since $F_{L'}$ is closed. But $\Phi^+ \subset F_{L'}$ implies $|T_{L'}^+| = 0$.

Let $\alpha \in T_{L'}^+ \cap \Pi$. Then $\alpha \in T_{L'}^s$ since $T_{L'} \backslash T_{L'}^s \subset \Phi^-$. So $-\alpha \in T_{L'}$. Then by Lemma 5.21 there exists a $b \in \mathbb{C}$ such that $\psi_{F_\alpha, b}.L'_{F_\alpha}$ contains a simple U_q-submodule L'' with $T_{L''} \subset T_{L'}$ and $\alpha \notin T_{L''}$. By Lemma 5.23 there exists a set of commuting roots Σ that is a basis of Q such that $\alpha \in \Sigma$ and $-\Sigma \subset T_{L'}$. Then by the above there exists a $\nu = b\alpha$ such that $\psi_{F_\Sigma, \nu}.L'_{F_\Sigma}$ contains a simple U_q-submodule L'' with $T_{L''} \subset T_{L'}$ and $\alpha \notin T_{L''}$. L'' is infinite dimensional since $-\Sigma \subset T_{L''}$ and $\mathcal{EXT}(L'') \cong \mathcal{EXT}(L')$ by Proposition 6.7.

By induction there exists a $\lambda \in X$ such that $L(\lambda)$ is infinite dimensional and $\mathcal{EXT}(L'') \cong \mathcal{EXT}(L(\lambda))$. □

The twists we have defined for quantum group modules are analogues of the twists that can be made of normal Lie algebra modules as described in [Mat00]. In the next proposition we will use these Lie algebra module twists denoted by f_Σ^ν given a set of commuting roots Σ and a $\nu \in T^*$ (see Section 4 in [Mat00]). For $\lambda^1 \in \mathfrak{h}^*$ let $L_\mathbb{C}(\lambda^1)$ denote the simple highest weight Lie algebra $\mathfrak{g}$-module with highest weight λ^1. Let e_β, f_β denote root vectors in $\mathfrak{g}$ such that $[e_\beta, f_\beta] = h_\beta$.

Proposition 6.9 *Let $\lambda^1 \in \mathfrak{h}^*$ be such that $L_\mathbb{C}(\lambda^1)$ is admissible. Let Σ be a set of commuting roots that is a basis of Q with f_β acting injectively on $L_\mathbb{C}(\lambda^1)$ for each $\beta \in \Sigma$. Let $\lambda^0 \in \Lambda_l$. Define $\mathcal{M} = \left(\bigoplus_{\nu \in T^*} f_\Sigma^\nu.L_\mathbb{C}(\lambda^1)_{f_\Sigma}\right)^{[l]} \otimes L((\lambda^0, 0))$. Then $\mathcal{M}$ is an irreducible coherent family containing the simple highest weight module $L((\lambda^0, \lambda^1))$.*

Proof. $\mathcal{M}$ contains $L((\lambda^0, \lambda^1))$ by Proposition 5.17.

Set $\mathcal{M}_\mathbb{C} = \bigoplus_{\nu \in T^*} f_\Sigma^\nu.L_\mathbb{C}(\lambda^1)_{f_\Sigma}$. So $\mathcal{M} = (\mathcal{M}_\mathbb{C})^{[l]} \otimes L((\lambda^0, 0))$. Let $\mu \in \Lambda_l$ and $u \in (U_q)_0$. We need to show that the map $\nu \mapsto \operatorname{Tr} u|_{\mathcal{M}_{(\mu,\nu)}}$ is polynomial.

$$\begin{aligned}
\mathcal{M}_{(\mu,\nu)} &= \bigoplus_{\eta \in \Lambda} \left((\mathcal{M}_\mathbb{C})^{[l]} \right)_{q^\eta(0,\nu)} \otimes L((\lambda^0, 0))_{q^{-\eta}(\mu,0)} \\
&= \bigoplus_{\eta \in l\Lambda} \left((\mathcal{M}_\mathbb{C})_{\nu + \frac{\eta}{l}} \right)^{[l]} \otimes L((\lambda^0, 0))_{q^{-\eta}(\mu,0)} \\
&= \bigoplus_{\eta \in l\Lambda} \left(f_\Sigma^{\nu + \frac{\eta}{l}}.(\mathcal{M}_\mathbb{C})_0 \right)^{[l]} \otimes L((\lambda^0, 0))_{q^{-\eta}(\mu,0)}.
\end{aligned}$$

The action on $(f_\Sigma^\nu.(\mathcal{M}_\mathbb{C})_0)^{[l]} \otimes L(\lambda^0)$ is just the action on $((\mathcal{M}_\mathbb{C})_0)^{[l]} \otimes L(\lambda^0)$ twisted with the automorphism $u' \mapsto f_\Sigma^\nu u' f_\Sigma^{-\nu}$ on the first tensor factor where $u' = \mathrm{Fr}(u)$ (Fr is the Frobenius twist defined in [KL02, Theorem 1.1]). The map $u' \mapsto f_\Sigma^\nu u' f_\Sigma^{-\nu}$ is of the form $\sum_i p_i(\nu) u_i$ for some polynomials p_i and some $u_i \in (U_\mathbb{C})_0$ where $U_\mathbb{C} := U(\mathfrak{g})$ is the classical universal enveloping algebra of $\mathfrak{g}$. Composing a polynomial map with the map $\lambda \mapsto \lambda + \frac{\eta}{l}$ is still polynomial. So the trace is a finite sum of polynomials in λ which is still polynomial.

Let u_q be the small quantum group as defined in [AM15] i.e. the subalgebra of U_q generated by $E_\alpha, K_\alpha^{\pm 1}, F_\alpha$, $\alpha \in \Pi$. Then $L((\lambda^0, 0))$ restricted to u_q is a simple u_q-module by [AM15, Section 3.2].

By [Mat00, Lemma 5.3 i)] and [Mat00, Proposition 5.4] there exists a $t \in T^*$ such that $\mathcal{M}_\mathbb{C}[t]$ is simple. Then $\mathcal{M}[t] = (\mathcal{M}_\mathbb{C}[t])^{[l]} \otimes L((\lambda^0, 0))$ is simple: Let $0 \neq v_0 \otimes v_1 \in L((\lambda^0, 0)) \otimes (\mathcal{M}_\mathbb{C}[t])^{[l]}$. Then

$$\begin{aligned} U_q(v_0 \otimes v_1) =& U_q u_q (v_0 \otimes v_1) \\ =& U_q(L((\lambda^0, 0)) \otimes v_1) \\ =& L((\lambda^0, 0)) \otimes U_q v_1 \\ =& L((\lambda^0, 0)) \otimes (U_\mathbb{C} v_1)^{[l]} \\ =& L((\lambda^0, 0)) \otimes (\mathcal{M}_\mathbb{C}[t])^{[l]} \end{aligned}$$

since $L((\lambda^0, 0))$ is a simple u_q-module and since $\mathcal{M}_\mathbb{C}[t]$ is a simple $U_\mathbb{C}$-module.□

Corollary 6.10 $\left(\bigoplus_{\nu \in T^*} \left(f_\Sigma^\nu . L(\lambda^1)_{f_\Sigma}\right)^{[l]} \otimes L((\lambda^0, 0))\right)^{ss} \cong \mathcal{EXT}(L((\lambda^0, \lambda^1)))$.

Proof. This follows by the uniqueness of $\mathcal{EXT}(L(\lambda))$. □

Corollary 6.11 *Let L be an infinite dimensional admissible simple module. Then $\mathcal{EXT}(L)$ is of the form $\left((\mathcal{M})^{[l]} \otimes L((\lambda^0, 0))\right)^{ss}$ for some $\mathfrak{g}$ coherent family $\mathcal{M}$ (in the sense of [Mat00]).*

Proof. By Theorem 6.8 there exists a $w \in W$ and a $\lambda \in X$ such that ${}^w\mathcal{EXT}(L) \cong \mathcal{EXT}(L(\lambda))$. By Corollary 6.10 $\mathcal{EXT}(L(\lambda)) \cong \left(\mathcal{M} \otimes L((\lambda^0, 0))\right)^{ss}$ for some $\mathfrak{g}$ coherent family $\mathcal{M}$. By [Mat00, Proposition 6.2] and the fact that $L((\lambda^0, 0))$ is finite dimensional we see that ${}^w\left(\mathcal{M} \otimes L((\lambda^0, 0))\right)^{ss} \cong \left(\mathcal{M} \otimes L((\lambda^0, 0))\right)^{ss}$ for all $w \in W$. □

So in the root of unity case the classification of torsion free modules reduces to the classification of classical torsion free modules. By Proposition 6.7 a torsion free module is a submodule of a semisimple irreducible coherent family so the problem reduces to classifying semisimple irreducible coherent families. By Corollary 6.11 the classification of these coherent families reduces to the classification in the classical case.

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Irreducible quantum group modules with finite dimensional weight spaces. II

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Abstract

We classify the simple quantum group modules with finite dimensional weight spaces when the quantum parameter q is transcendental and the Lie algebra is not of type G_2. This is part 2 of the story. The first part being [Ped15a]. In [Ped15a] the classification is reduced to the classification of torsion free simple modules. In this paper we follow the procedures of [Mat00] to reduce the classification to the classification of infinite dimensional admissible simple highest weight modules. We then classify the infinite dimensional admissible simple highest weight modules and show among other things that they only exist for types A and C. Finally we complete the classification of simple torsion free modules for types A and C completing the classification of the simple torsion free modules.

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1 Introduction

This is part 2 of the classification of simple quantum group modules with finite dimensional weight spaces. In this paper we focus on the non root of unity case. Let $\mathfrak{g}$ be a simple Lie algebra. Let $q \in \mathbb{C}$ be a non root of unity and let U_q be the quantized enveloping algebra over $\mathbb{C}$ with q as the quantum parameter (defined below). We want to classify all simple weight modules for U_q with finite dimensional weight spaces. In the papers [Fer90] and [Mat00] this is done for $\mathfrak{g}$-modules. Fernando proves in [Fer90] that the classification of simple $\mathfrak{g}$ weight modules with finite dimensional weight spaces essentially boils down to classifying two classes of simple modules: Finite dimensional simple modules over a reductive Lie algebra and so called 'torsion free' simple modules over a simple Lie algebra. The classification of finite dimensional modules is well known in the classical case (as well as the quantum group case) so the remaining problem is to classify the torsion free simple modules. O. Mathieu classifies all torsion free $\mathfrak{g}$-modules in [Mat00]. The classification uses the concept of a $\mathfrak{g}$ coherent family which are huge $\mathfrak{g}$ modules with weight vectors for every possible weight, see [Mat00, Section 4]. Mathieu shows that every torsion free simple module is a submodule of a unique irreducible semisimple coherent family [Mat00, Proposition 4.8] and each of these irreducible semisimple coherent families contains an admissible simple highest weight module as well [Mat00, Proposition 6.2 ii)]. This reduces the classification to the classification of admissible simple highest weight modules. In this paper we will follow closely the methods described in [Mat00]. We will focus only on the case when q is not a root of unity. The root of unity case is studied in [Ped15a]. Some of the results of [Mat00] translate directly to the quantum group case but in several cases there are obstructions that need to be handled differently. In particular the case by case classification in types A and C is done differently. This is because our analog of $\mathcal{EXT}(L)$ given an admissible simple infinite dimensional module L is slightly different from the classical case see e.g. Section 7. The proof when reducing to types A and C in [Fer90] and [Mat00] uses some algebraic geometry to show that torsion free modules can only exist in types A and C. In this paper we show that infinite dimensional admissible simple highest weight modules only exist in types A and C and use this fact to show that torsion free modules can not exist for modules other than types A and C. For this we have to restrict to transcendental q. Specifically we use Theorem 8.1. If this theorem is true for a general non-root-of-unity q we can remove this restriction. The author is not aware of such a result in the litterature.

1.1 Main results

To classify simple weight modules with finite dimensional modules we follow the procedures of S. Fernando and O. Mathieu in [Fer90] and [Mat00]. The analog of the reduction done in [Fer90] is taken care of in the quantum group case in [Ped15a] so what remains is to classify the torsion free modules. We will first recall some results from [Ped15a] and [Ped15b] concerning the reduction and some formulas for commutating root vectors. This is recalled in Section 2 and Section 3. In Section 4 we do some prelimary calculations concerning Ore localization and certain 'twists' of modules necessary to define the 'Coherent families' of Section 5. Here we don't define the concept of a general coherent family but instead directly define the analog of coherent irreducible semisimple extensions $\mathcal{EXT}(L)$ of an admissible simple infinite dimensional module L. In analog with the classical case we show that for any admissible simple infinite dimensional module L, $\mathcal{EXT}(L)$ contains a submodule isomorphic to a simple highest weight module, see Theorem 5.12. We also prove a result in the other direction: If $\mathfrak{g}$ is such that there exists a simple infinite dimensional admissible module L then there exists a torsion free $U_q(\mathfrak{g})$-module, see Theorem 5.8. So the existence of torsion free modules over the quantized enveloping algebra of a specific $\mathfrak{g}$ is equivalent to the existence of an admissible infinite dimensional highest weight simple module over $U_q(\mathfrak{g})$. Using this we show that torsion free modules exist only for types A and C in the Sections 8.1, 8.2, 8.3, 8.4 and 8.5 where we also classify the admissible simple highest weight modules which are infinite dimensional. Finally in Section 9 and Section 10 we complete the classification in types A and C, respectively, by showing exactly which submodules of $\mathcal{EXT}(L(\lambda))$ are torsion free for a λ of a specific form see Theorem 9.8 and Theorem 10.7.

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1.3 Notation

We will fix some notation: We denote by $\mathfrak{g}$ a fixed simple Lie algebra over the complex numbers $\mathbb{C}$. We assume $\mathfrak{g}$ is not of type G_2 to avoid unpleasant computations.

Fix a triangular decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$: Let $\mathfrak{h}$ be a maximal toral subalgebra and let $\Phi \subset \mathfrak{h}^*$ be the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. Choose a simple system of roots $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$. Let Φ^+ (resp. Φ^-) be the positive (resp. negative) roots. Let $\mathfrak{g}^\pm$ be the positive and negative part of $\mathfrak{g}$ corresponding to the simple system Π. Let W be the Weyl group generated by the simple reflections $s_i := s_{\alpha_i}$. For a $w \in W$ let $l(w)$ be the length of W i.e. the smallest amount of simple reflections such that $w = s_{i_1} \cdots s_{i_{l(w)}}$. Let $(\cdot|\cdot)$ be a standard W-invariant bilinear form on $\mathfrak{h}^*$ and $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha|\beta)}{(\beta|\beta)}$. Since $(\cdot|\cdot)$ is standard we have $(\alpha|\alpha) = 2$ for any short root $\alpha \in \Phi$ and since $\mathfrak{g}$ is not of type G_2 we have $(\beta|\beta) = 4$ for any long root $\beta \in \Phi$. Let $Q = \operatorname{span}_{\mathbb{Z}} \{\alpha_1, \dots, \alpha_n\}$ denote the root lattice and $\Lambda = \operatorname{span}_{\mathbb{Z}} \{\omega_1, \dots, \omega_n\} \subset \mathfrak{h}^*$ the integral lattice where $\omega_i \in \mathfrak{h}^*$ is the fundamental weights defined by $(\omega_i|\alpha_j) = \delta_{ij}$.

Let $U_v = U_v(\mathfrak{g})$ be the corresponding quantized enveloping algebra defined over $\mathbb{Q}(v)$, see e.g. [Jan96] with generators $E_\alpha, F_\alpha, K_\alpha^{\pm 1}$, $\alpha \in \Pi$ and certain relations which can be found in chapter 4 of [Jan96]. We define $v_\alpha = v^{(\alpha|\alpha)/2}$ (i.e. $v_\alpha = v$ if α is a short root and $v_\alpha = v^2$ if α is a long root) and for $n \in \mathbb{Z}$, $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$. Let $[n]_\alpha := [n]_{v_\alpha} = \frac{v_\alpha^n - v_\alpha^{-n}}{v_\alpha - v_\alpha^{-1}}$. We omit the subscripts when it is clear from the context. For later use we also define the quantum binomial coefficients: For $r \in \mathbb{N}$ and $a \in \mathbb{Z}$:

$$\begin{bmatrix} a \\ r \end{bmatrix}_v = \frac{[a][a-1]\cdots[a-r+1]}{[r]!}$$

where $[r]! := [r][r-1]\cdots[2][1]$. Let $A = \mathbb{Z}[v, v^{-1}]$ and let U_A be Lusztigs A-form, i.e. the A subalgebra generated by the divided powers $E_\alpha^{(n)} := \frac{1}{[n]_\alpha!}E_\alpha^n$, $F_\alpha^{(n)} := \frac{1}{[n]_\alpha!}F_\alpha^n$ and $K_\alpha^{\pm 1}$, $\alpha \in \Pi$.

Let $q \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ be a nonzero complex number that is not a root of unity and set $U_q = U_A \otimes_A \mathbb{C}_q$ where $\mathbb{C}_q$ is the A-algebra $\mathbb{C}$ where v is sent to q.

We have a triangular decomposition of Lusztigs A-form $U_A = U_A^- \otimes U_A^0 \otimes U_A^+$ with $U_A^- = \left\langle F_\alpha^{(n)} | \alpha \in \Pi, n \in \mathbb{N} \right\rangle \in U_A$, $U_A^+ = \left\langle E_\alpha^{(n)} | \alpha \in \Pi, n \in \mathbb{N} \right\rangle \in U_A$ and $U_A^0 = \left\langle K_\alpha^{\pm 1}, \begin{bmatrix} K_\alpha; c \\ r \end{bmatrix} | \alpha \in \Pi, c \in \mathbb{Z}, r \in \mathbb{N} \right\rangle$ where

$$\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix} := \prod_{j=1}^{r} \frac{K_\alpha v_\alpha^{c-j+1} - K_\alpha^{-1} v_\alpha^{-c+j-1}}{v_\alpha^j - v_\alpha^{-j}}.$$

We have the corresponding triangular decomposition of U_q: $U_q = U_q^- \otimes U_q^0 \otimes U_q^+$ with $U_q^\pm = U_A^\pm \otimes_A \mathbb{C}_q$ and $U_q^0 = U_A^0 \otimes_A \mathbb{C}_q$.

For a $q \in \mathbb{C}^*$ define $\begin{bmatrix} a \\ r \end{bmatrix}_q$ as the image of $\begin{bmatrix} a \\ r \end{bmatrix}_v$ in $\mathbb{C}_q$. We will omit the subscript from the notation when it is clear from the context. $q_\beta \in \mathbb{C}$ and $[n]_\beta \in \mathbb{C}$ are defined as the image of $v_\beta \in A$ and $[n]_\beta \in A$, respectively abusing notation. Similarly, we will abuse notation and write $\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix}$ also for the image of $\begin{bmatrix} K_\alpha; c \\ r \end{bmatrix} \in U_A$ in U_q. Define for $\mu \in Q$, $K_\mu = \prod_{i=1}^n K_{\alpha_i}^{a_i}$ if $\mu = \sum_{i=1}^n a_i \alpha_i$ with $a_i \in \mathbb{Z}$.

There is a braid group action on U_v which we will describe now. We use the definition from [Jan96, Chapter 8]. The definition is slightly different from the original in [Lus90, Theorem 3.1] (see [Jan96, Warning 8.14]). For each simple reflection s_i there is a braid operator that we will denote by T_{s_i} satisfying the following: $T_{s_i} : U_v \to U_v$ is a $\mathbb{Q}(v)$ automorphism. For $i \neq j \in \{1, \ldots, n\}$

$$\begin{aligned}
T_{s_i}(K_\mu) =& K_{s_i(\mu)} \\
T_{s_i}(E_{\alpha_i}) =& - F_{\alpha_i} K_{\alpha_i} \\
T_{s_i}(F_{\alpha_i}) =& - K_{\alpha_i}^{-1} E_{\alpha_i} \\
T_{s_i}(E_{\alpha_j}) =& \sum_{i=0}^{-\langle \alpha_j, \alpha_i^\vee \rangle} (-1)^i v_{\alpha_i}^{-i} E_{\alpha_i}^{(r-i)} E_{\alpha_j} E_{\alpha_i}^{(i)} \\
T_{s_i}(F_{\alpha_j}) =& \sum_{i=0}^{-\langle \alpha_j, \alpha_i^\vee \rangle} (-1)^i v_{\alpha_i}^{i} E_{\alpha_i}^{(i)} E_{\alpha_j} E_{\alpha_i}^{(r-i)}.
\end{aligned}$$

The inverse $T_{s_i}^{-1}$ is given by conjugating with the $\mathbb{Q}$-algebra anti-automorphism Ψ from [Lus90, section 1.1] defined as follows:

$$\Psi(E_{\alpha_i}) = E_{\alpha_i}, \quad \Psi(F_{\alpha_i}) = F_{\alpha_i}, \quad \Psi(K_{\alpha_i}) = K_{\alpha_i}^{-1}, \quad \Psi(q) = q.$$

The braid operators T_{s_i} satisfy braid relations so we can define T_w for any $w \in W$: Choose a reduced expression of w: $w = s_{i_1} \cdots s_{i_n}$. Then $T_w = T_{s_{i_1}} \cdots T_{s_{i_n}}$ and T_w is independent of the chosen reduced expression, see e.g. [Lus90, Theorem 3.2]. We have $T_w(K_\mu) = K_{w(\mu)}$. The braid group operators restrict to automorphisms $U_A \to U_A$ and extend to automorphisms $U_q \to U_q$.

Let M be a U_q-module and $\lambda : U_q^0 \to \mathbb{C}$ a character (i.e. an algebra homomorphism into $\mathbb{C}$). Then

$$M_\lambda = \{m \in M | \forall u \in U_q^0, um = \lambda(u)m\}.$$

Let X denote the set of characters $U_q^0 \to \mathbb{C}$. Since $U_q^0 \cong \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ we can identify X with $(\mathbb{C}^*)^n$ by $U_q^0 \ni \lambda \mapsto (\lambda(K_{\alpha_1}), \dots, \lambda(K_{\alpha_n})) \in (\mathbb{C}^*)^n$.

1.4 Basic definitions

Definition 1.1 *Let* $\operatorname{wt} M$ *denote all the weights of* M*, i.e.* $\operatorname{wt} M = \{\lambda \in X | M_\lambda \neq 0\}$.

For $\mu \in \Lambda$ *and* $b \in \mathbb{C}^*$ *define the character* b^μ *by* $b^\mu(K_\alpha) = b^{(\mu|\alpha)}$, $\alpha \in \Pi$. *In particular for* $b = q$ *we get* $q^\mu(K_\alpha) = q^{(\mu|\alpha)}$. *We say that* M *only has integral weights if* $\lambda(K_\alpha) \in \pm q_\alpha^{\mathbb{Z}}$ *for all* $\lambda \in \operatorname{wt} M$, $\alpha \in \Pi$.

There is an action of W on X. For $\lambda \in X$ define $w\lambda$ by

$$(w\lambda)(u) = \lambda(T_{w^{-1}}(u))$$

Note that $wq^\mu = q^{w(\mu)}$.

Definition 1.2 *Let* M *be a* U_q*-module and* $w \in W$. *Define the twisted module* ${}^w M$ *by the following:*

As a vector space ${}^w M = M$ *but the action is given by twisting with* w^{-1}*: For* $m \in {}^w M$ *and* $u \in U_q$*:*

$$u \cdot m = T_{w^{-1}}(u)m.$$

We also define ${}^{\overline{w}}M$ *to be the inverse twist, i.e. for* $m \in {}^{\overline{w}}M$, $u \in U_q$*:*

$$u \cdot m = T_{w^{-1}}^{-1}(u)m.$$

Hence for any U_q*-module* ${}^{\overline{w}}({}^w M) = M = {}^w({}^{\overline{w}}M)$.

Note that $\operatorname{wt} {}^w M = w(\operatorname{wt} M)$ and that ${}^w({}^{w'}M) \cong {}^{ww'}M$ for $w, w' \in W$ with $l(ww') = l(w) + l(w')$ because the braid operators T_w satisfy braid relations. Also ${}^{\overline{w}}({}^{\overline{w'}}M) \cong {}^{\overline{w'w}}M$

Definition 1.3 *We define the category* $\mathcal{F} = \mathcal{F}(\mathfrak{g})$ *as the full subcategory of* $U_q - \mathrm{Mod}$ *such that for every* $M \in \mathcal{F}$ *we have*

1. M *is finitely generated as a* U_q*-module.*
2. $M = \bigoplus_{\lambda \in X} M_\lambda$ *and* $\dim M_\lambda < \infty$.

Note that the assignment $M \mapsto {}^w M$ is an endofunctor on $\mathcal{F}$ (in fact an auto-equivalence).

Let w_0 be the longest element in W and let $s_{i_1} \cdots s_{i_N}$ be a reduced expression of w_0. We define root vectors E_β and F_β for any $\beta \in \Phi^+$ by the following:

First of all set

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), \text{ for } i = 1, \dots, N$$

Then $\Phi^+ = \{\beta_1, \dots, \beta_N\}$. Set

$$E_{\beta_j} = T_{s_{i_1}} \cdots T_{s_{i_{j-1}}}(E_{\alpha_{i_j}})$$

and

$$F_{\beta_j} = T_{s_{i_1}} \cdots T_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$$

In this way we have defined root vectors for each $\beta \in \Phi^+$. These root vectors depend on the reduced expression chosen for w_0 above. For a different reduced expression we might get different root vectors. It is a fact that if $\beta \in \Pi$ then the root vectors E_β and F_β defined above are the same as the generators with the same notation (cf. e.g. [Jan96, Proposition 8.20]) so the notation is not ambigious in this case. By "Let E_β be a root vector" we will just mean a root vector constructed as above for some reduced expression of w_0.

2 Reductions

We recall the following results from [Ped15a].

Proposition 2.1 *Let β be a positive root and E_β, F_β root vectors corresponding to β. Let $M \in \mathcal{F}$. The sets $M^{[\beta]} = \{m \in M | \dim \langle E_\beta \rangle m < \infty\}$ and $M^{[-\beta]} = \{m \in M | \dim \langle F_\beta \rangle m < \infty\}$ are submodules of M and independent of the chosen root vectors E_β, F_β.*

Proof. This is shown for E_β in Proposition 2.3 and Lemma 2.4 in [Ped15a] and the proofs are the same for F_β. □

Definition 2.2 *Let $M \in \mathcal{F}$. Let $\beta \in \Phi$. M is called β-free if $M^{[\beta]} = 0$ and β-finite if $M^{[\beta]} = M$.*

Suppose $L \in \mathcal{F}$ is a simple module and β a root. Then by Proposition 2.1 L is either β-finite or β-free.

Definition 2.3 *Let $M \in \mathcal{F}$. Define $F_M = \{\beta \in \Phi | M \text{ is } \beta\text{-finite}\}$ and $T_M = \{\beta \in \Phi | M \text{ is } \beta\text{-free}\}$. For later use we also define $F_M^s := F_M \cap (-F_M)$ and $T_M^s := T_M \cap (-T_M)$ to be the symmetrical parts of F_M and T_M.*

Note that $\Phi = F_L \cup T_L$ for a simple module L and this is a disjoint union.

Definition 2.4 *A module M is called torsion free if $T_M = \Phi$.*

Proposition 2.5 *Let $L \in \mathcal{F}$ be a simple module and β a root. L is β-free if and only if $q^{\mathbb{N}\beta} \operatorname{wt} L \subset \operatorname{wt} L$.*

Proof. Proposition 2.9 in [Ped15a]. □

Proposition 2.6 *Let $L \in \mathcal{F}$ be a simple module. T_L and F_L are closed subsets of Q. That is, if $\beta, \gamma \in F_L$ (resp. $\beta, \gamma \in T_L$) and $\beta + \gamma \in \Phi$ then $\beta + \gamma \in F_L$ (resp. $\beta + \gamma \in T_L$).*

Proof. Proposition 2.10 and Proposition 2.11 in [Ped15a]. □

Theorem 2.7 *Let $\lambda \in X$. There is a $1-1$ correspondence between simple U_q-modules with weight λ and simple $(U_q)_0$-modules with weight λ given by: For V a U_q-module, V_λ is the corresponding simple $(U_q)_0$-module.*

Proof. Theorem 2.21 in [Ped15a]. □

Theorem 2.8 *Let $L \in \mathcal{F}$ be a simple $U_q(\mathfrak{g})$-module. Then there exists a $w \in W$, subalgebras $U_q(\mathfrak{p}), U_q(\mathfrak{l}), U_q(\mathfrak{u}), U_q(\mathfrak{u}^-)$ of U_q with $U_q = U_q(\mathfrak{u}^-)U_q(\mathfrak{p})$, $U_q(\mathfrak{p}) = U_q(\mathfrak{l})U_q(\mathfrak{u})$ and a simple $U_q(\mathfrak{l})$-module N such that ${}^wL$ is the unique simple quotient of $U_q \otimes_{U_q(\mathfrak{p})} N$ where N is considered a $U_q(\mathfrak{p})$-module with $U_q(\mathfrak{u})$ acting trivially.*

Furthermore there exists subalgebras U_{fr}, U_{fin} of $U_q(\mathfrak{l})$ such that $U_q(\mathfrak{l}) \cong U_{fr} \otimes U_{fin}$ and simple U_{fr} and U_{fin} modules X_{fr} and X_{fin} where X_{fin} is finite dimensional and X_{fr} is torsion free such that $N \cong X_{fin} \otimes X_{fr}$ as a $U_{fr} \otimes U_{fin}$-module.

U_{fr} is the quantized enveloping algebra of a semisimple Lie algebra $\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_r$ where $\mathfrak{t}_1, \dots, \mathfrak{t}_r$ are some simple Lie algebras. There exists simple torsion free $U_q(\mathfrak{t}_i)$-modules X_i, $i = 1, \dots, r$ such that $X_{fr} \cong X_1 \otimes \cdots \otimes X_r$ as $U_q(\mathfrak{t}_1) \otimes \cdots \otimes U_q(\mathfrak{t}_r)$-modules.

Proof. Theorem 2.23 in [Ped15a]. □

So the problem of classifying simple modules in $\mathcal{F}$ is reduced to the problem of classifying finite dimensional simple modules and classifying simple torsion free modules of $U_q(\mathfrak{t})$ where $\mathfrak{t}$ is a simple Lie algebra.

3 U_A calculations

In this section we recall from [Ped15b] some formulas for commuting root vectors with each other that will be used later on. Recall that $A = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate and U_A is the A-subspace of U_v generated by the divided powers $E_\alpha^{(n)}$ and $F_\alpha^{(n)}$, $n \in \mathbb{N}$.

Definition 3.1 *Let $x \in (U_q)_\mu$ and $y \in (U_q)_\gamma$ then we define*

$$[x, y]_q := xy - q^{-(\mu|\gamma)}yx$$

Theorem 3.2 *Suppose we have a reduced expression of $w_0 = s_{i_1} \cdots s_{i_N}$ and define root vectors $F_{\beta_1}, \dots, F_{\beta_N}$. Let $i < j$. Let $A = \mathbb{Z}[q, q^{-1}]$ and let A' be the localization of A in [2] if the Lie algebra contains any B_n, C_n or F_4 part. Then*

$$[F_{\beta_j}, F_{\beta_i}]_q = F_{\beta_j}F_{\beta_i} - q^{-(\beta_i|\beta_j)}F_{\beta_i}F_{\beta_j} \in \operatorname{span}_{A'}\left\{F_{\beta_{j-1}}^{a_{j-1}} \cdots F_{\beta_{i+1}}^{a_{i+1}}\right\}$$

Proof. [LS91, Proposition 5.5.2]. A proof following [DP93, Theorem 9.3] can also be found in [Ped15b, Theorem 2.9]. □

Definition 3.3 *Let $u \in U_A$ and $\beta \in \Phi^+$. Define $\operatorname{ad}(F_\beta^i)(u) := [[\dots[u, F_\beta]_q \dots]_q, F_\beta]_q$ and $\widetilde{\operatorname{ad}}(F_\beta^i)(u) := [F_\beta, [\dots, [F_\beta, u]_q \dots]]_q$ where the commutator is taken i times from the left and right respectively.*

Proposition 3.4 *Let $a \in \mathbb{N}$, $u \in (U_A)_\mu$ and $r = \langle \mu, \beta^\vee \rangle$. In U_A we have the identities*

$$\begin{aligned} uF_\beta^a &= \sum_{i=0}^{a} v_\beta^{(i-a)(r+i)} \begin{bmatrix} a \\ i \end{bmatrix}_\beta F_\beta^{a-i} \operatorname{ad}(F_\beta^i)(u) \\ &= \sum_{i=0}^{a} (-1)^i v_\beta^{a(r+i)-i} \begin{bmatrix} a \\ i \end{bmatrix}_\beta F_\beta^{a-i} \widetilde{\operatorname{ad}}(F_\beta^i)(u) \end{aligned}$$

Proof. Proposition 2.13 in [Ped15b]. $\square$

Let $s_{i_1} \dots s_{i_N}$ be a reduced expression of w_0 and construct root vectors F_{β_i}, $i = 1, \dots, N$. In the next lemma F_{β_i} refers to the root vectors constructed as such. In particular we have an ordering of the root vectors.

Lemma 3.5 *Let $n \in \mathbb{N}$. Let $1 \leq j < k \leq N$.*
$\operatorname{ad}(F_{\beta_j}^i)(F_{\beta_k}^n) = 0$ *and* $\widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j}^n) = 0$ *for* $i \gg 0$.

Proof. Lemma 2.16 in [Ped15b]. $\square$

4 Ore localization and twists of localized modules

In this section we present some results towards classifying simple torsion free modules following [Mat00].

We need the equivalent of Lemma 3.3 in [Mat00]. The proofs are essentially the same but for completeness we include most of the proofs here.

Definition 4.1 *A cone C is a finitely generated submonoid of the root lattice Q containing 0. If L is a simple module define the cone of L, $C(L)$, to be the submonoid of Q generated by T_L.*

Lemma 4.2 *Let $L \in \mathcal{F}$ be an infinite dimensional simple module. Then the group generated by the submonoid $C(L)$ is Q.*

Compare [Mat00] Lemma 3.1

Proof. First consider the case where $T_L \cap (-F_L) = \emptyset$. Then in this case we have $\Phi = T_L^s \cup F_L^s$. We claim that T_L^s and F_L^s correspond to different connected components of the Dynkin diagram: Suppose $\alpha \in F_L^s$ is a simple root and suppose $\alpha' \in \Pi$ is a simple root that is connected to α in the Dynkin diagram. So $\alpha + \alpha'$ is a root. There are two possibilities. Either $\alpha + \alpha' \in F_L$ or $\alpha + \alpha' \in T_L$. If $\alpha + \alpha' \in F_L$: Since F_L^s is symmetric we have $-\alpha \in F_L^s$ and since F_L is closed (Proposition 2.6) $\alpha' = \alpha + \alpha' + (-\alpha) \in F_L$. If $\alpha + \alpha' \in T_L$ and $\alpha' \in T_L$ then we get similarly $\alpha \in T_L$ which is a contradiction. So $\alpha' \in F_L$. We have shown that if $\alpha \in F_L$ then any simple root connected to α is in F_L also. So F_L and T_L contains different connected components of the Dynkin diagram. Since L is simple and infinite we must have $\Phi = T_L^s$ and therefore $C(L) = Q$.

Next assume $T_L \cap (-F_L) \neq \emptyset$. By Lemma 4.16 in [Fer90] $P_L = T_L^s \cup F_L$ and $P_L^- = T_L \cup F_L^s$ are two opposite parabolic subsystems of Φ. So we have that $T_L \cap (-F_L)$ and $(-T_L) \cap F_L$ must be the roots corresponding to the nilradicals $\mathfrak{v}^\pm$ of two opposite parabolic subalgebras $\mathfrak{p}^\pm$ of $\mathfrak{g}$. Since we have $\mathfrak{g} = \mathfrak{v}^+ + \mathfrak{v}^- + [\mathfrak{v}^+, \mathfrak{v}^-]$ we get that $T_L \cap (-F_L)$ generates Q. Since $C(L)$ contains $T_L \cap (-F_L)$ it generates Q. $\square$

We define ρ and δ like in [Mat00, Section 3]:

Definition 4.3 *Let $x \geq 0$ be a real number. Define $\rho(x) = \operatorname{Card} B(x)$ where $B(x) = \{\mu \in Q | \sqrt{(\mu|\mu)} \leq x\}$*

Let M be a weight module with support lying in a single Q-coset, say $q^Q\lambda := \{q^\mu\lambda | \mu \in Q\}$. The density of M is $\delta(M) = \liminf_{x\to\infty} \rho(x)^{-1} \sum_{\mu \in B(x)} \dim M_{q^\mu \lambda}$

For a cone C we define $\delta(C) = \liminf_{x\to\infty} \rho(x)^{-1} \operatorname{Card}(C \cap B(x))$

Lemma 4.4 *There exists a real number $\varepsilon > 0$ such that $\delta(L) > \varepsilon$ for all infinite dimensional simple modules L.*

Proof. Note that since $q^{C(L)}\lambda \subset \operatorname{wt} L$ for all $\lambda \in \operatorname{wt} L$ we have $\delta(L) \geq \delta(C(L))$.

Since $C(L)$ is the cone generated by T_L and $T_L \subset \Phi$ (a finite set) there can only be finitely many different cones.

Since there are only finitely many different cones attached to infinite simple dimensional modules and since any cone C that generates Q has $\delta(C) > 0$ we conclude via Lemma 4.2 that there exists an $\varepsilon > 0$ such that $\delta(L) > \varepsilon$ for all infinite dimensional simple modules. □

Definition 4.5 *A module $M \in \mathcal{F}$ is called admissible if its weights are contained in a single coset of X/q^Q and if the dimensions of the weight spaces are uniformly bounded. M is called admissible of degree d if d is the maximal dimension of the weight spaces in M.*

Of course all finite dimensional simple modules are admissible but the interesting admissible modules are the infinite dimensional simple ones. In particular simple torsion free modules are admissible. We show later that each infinite dimensional admissible simple module L gives rise to a 'coherent family' $\mathcal{EXT}(L)$ containing at least one torsion free module and at least one simple highest weight module that is admissible of the same degree.

Lemma 4.6 *Let $M \in \mathcal{F}$ be an admissible module. Then M has finite Jordan-Hölder length.*

Proof. The length of M is bounded by $A + \delta(M)/\varepsilon$ where $A = \sum_{\lambda \in Y} \dim M_\lambda$ and $Y = \{\nu \in X | \nu = \sigma q^\mu, | \langle \mu, \alpha^\vee \rangle | \leq 1, \sigma(K_\alpha) \in \{\pm 1\} \text{ for all } \alpha \in \Pi\}$. Check [Mat00, Lemma 3.3] for details. Here we use the fact that finite dimensional simple quantum group modules have the same character as their corresponding Lie algebra simple modules. This is proved for transcendental q in [Jan96, Theorem 5.15] and for general non-roots-of-unity in [APW91, Corollary 7.7]. □

Lemma 4.7 *Let β be a positive root and let F_β be a corresponding root vector. The set $\{F_\beta^n | n \in \mathbb{N}\}$ is an Ore subset of U_q.*

Proof. A proof can be found in [And03] for β a simple root. If β is not simple then F_β is defined as $T_w(F_\alpha)$ for some $w \in W$ and some $\alpha \in \Pi$. Since $S := \{F_\alpha^n | n \in \mathbb{N}\}$ is an Ore subset of U_q we get for any $n \in \mathbb{N}$ and $u \in U_q$ that

$$F_\alpha^n U_q \cap uS \neq \emptyset.$$

Let $u' \in U_q$ and set $u = T_w^{-1}(u')$, then from the above

$$\emptyset \neq T_w(F_\alpha^n) T_w(U_q) \cap T_w(u) T_w(S) = F_\beta^n U_q \cap u' T_w(S).$$

Since $T_w(S) = \{F_\beta^n | n \in \mathbb{N}\}$ we have proved the lemma. □

We denote the Ore localization of U_q in the above set by $U_{q(F_\beta)}$.

Lemma 4.8 *Let p be Laurent polynomial. If*

$$p(q^{r_1},\dots,q^{r_n})=0$$

for all $r_1,\dots,r_n\in\mathbb{N}$ then $p=0$.

Proof. If $n=1$ we have a Laurent polynomial of one variable with infinitely many zero-points so $p=0$. Let $n>1$, then for constant $r_1\in\mathbb{N}$, $p(q^{r_1},-,\dots,-)$ is a Laurent polynomial in $n-1$ variables equal to zero in $(q^{r_2},\dots,q^{r_n})$ for all $r_2,\dots,r_n\in\mathbb{N}$ so by induction $p(q^{r_1},c_2,\dots,c_n)=0$ for all $c_2,\dots,c_n$. Now for arbitrary $c_2,\dots,c_n\in\mathbb{C}^*$ we get $p(-,c_2,\dots,c_n)$ is a Laurent polynomial in one variable that is zero for all q^{r_1}, $r_1\in\mathbb{N}$ hence $p(c_1,\dots,c_n)=0$ for all $c_1\in\mathbb{C}^*$.$\square$

The next lemma is crucial for the rest of the results in this paper. We will use this result again and again.

Lemma 4.9 *Let $\beta\in\Phi^+$ and let F_β be a corresponding root vector. There exists automorphisms $\varphi_{F_\beta,b}:U_{q(F_\beta)}\to U_{q(F_\beta)}$ for each $b\in\mathbb{C}^*$ such that $\varphi_{F_\beta,q^i}(u)=F_\beta^{-i}uF_\beta^i$ for $i\in\mathbb{Z}$ and such that for $u\in U_{q(F_\beta)}$ the map $\mathbb{C}^*\to U_{q(F_\beta)}$, $b\mapsto\varphi_{F_\beta,b}(u)$ is of the form $b\mapsto p(b)$ for some Laurent polynomial $p\in U_{q(F_\beta)}[X,X^{-1}]$. Furthermore for $b,b'\in\mathbb{C}^*$, $\varphi_{F_\beta,b}\circ\varphi_{F_\beta,b'}=\varphi_{F_\beta,bb'}$.*

Proof. We can assume β is simple since if $F_\beta=T_w(F_{\alpha'})$ for some $\alpha'\in\Pi$ then we can just define the homomorphism on $T_w(E_\alpha),T_w(K_\alpha^{\pm1}),T_w(F_\alpha)$ for $\alpha\in\Pi$ i.e. in this case we define $\varphi_{F_\beta,b}(u)=T_w(\varphi_{\alpha',b}(T_w^{-1}(u)))$ where we extend T_w to a homomorphism $T_w:U_{q(F_{\alpha'})}\to U_{q(F_\beta)}$ by $T_w(F_{\alpha'}^{-1})=F_\beta^{-1}$.

So β is assumed simple. For $b\in\mathbb{C}^*$ define $b_\beta=b^{(\beta|\beta)/2}$ i.e. $b_\beta=b$ if β is short and $b_\beta=b^2$ when β is long. We will define the map on the generators $E_\alpha,K_\alpha,F_\alpha$ for $\alpha\in\Pi$. If $\alpha=\beta$ the map is defined as follows:

$$\begin{aligned}\varphi_{F_\beta,b}(F_\beta^{\pm1})=&F_\beta^{\pm1}\\ \varphi_{F_\beta,b}(K_\beta^{\pm1})=&b_\beta^{\mp2}K_\beta^{\pm1}\\ \varphi_{F_\beta,b}(E_\beta)=&E_\beta+F_\beta^{-1}\frac{(b_\beta-b_\beta^{-1})(q_\beta b_\beta^{-1}K_\beta-q_\beta^{-1}b_\beta K_\beta^{-1})}{(q_\beta-q_\beta^{-1})^2}.\end{aligned}$$

Assume $\alpha\neq\beta$. Let $r=\langle\alpha,\beta^\vee\rangle$. Note that $\mathrm{ad}(F_\beta^{-r+1})(F_\alpha)=0$ because this is one of the defining relations of U_q. We define the map as follows:

$$\begin{aligned}\varphi_{F_\beta,b}(F_\alpha)=&\sum_{i=0}^{-r}b_\beta^{-r-i}q_\beta^{i(i+r)}\prod_{t=1}^{i}\frac{b_\beta q_\beta^{1-t}-b_\beta^{-1}q_\beta^{t-1}}{q_\beta^t-q_\beta^{-t}}F_\beta^{-i}\,\mathrm{ad}(F_\beta^i)(F_\alpha)\\ \varphi_{F_\beta,b}(K_\alpha)=&b_\beta^{-r}K_\alpha=b^{-(\alpha|\beta)}K_\alpha\\ \varphi_{F_\beta,b}(E_\alpha)=&E_\alpha.\end{aligned}$$

Note that if $b=q^j$ for some $j\in\mathbb{Z}$ then $\prod_{t=1}^i\frac{b_\beta q_\beta^{1-t}-b_\beta^{-1}q_\beta^{t-1}}{q_\beta^t-q_\beta^{-t}}=\begin{bmatrix}j\\i\end{bmatrix}_\beta$. Since the map $b\mapsto\varphi_{F_\beta,b}(u)$ is of the form $b\mapsto\sum_{i=1}^r p_i(b)u_i$ with p_i Laurent polynomial in b for each generator of U_q it is of this form for all $u\in U_q$. It's easy to check that $\varphi_{F_\beta,b}(u) = F_\beta^{-i} u F_\beta^i$ when $b = q^i$, $i \in \mathbb{N}$. So $\varphi_{F_\beta,b}$ satisfies the generating relations of U_q for $b = q^i$, $i \in \mathbb{N}$. By Lemma 4.8 $\varphi_{F_\beta,b}$ must satisfy the generating relations for all $b \in \mathbb{C}$.

Consider the last claim of the lemma: Let $u \in U_q$, then by the above $b \mapsto \varphi_{F_\beta,b}(u)$ is a Laurent polynomial and so $b \mapsto \varphi_{F_\beta,bb'}(u)$ and $\varphi_{F_\beta,b}(\varphi_{F_\beta,b'}(u))$ for a constant $b' \in \mathbb{C}^*$ is a Laurent polynomial as well. Now we know from above that for $b' = q^j$ for some $j \in \mathbb{Z}$ and $i \in \mathbb{Z}$:

$$\begin{aligned}\varphi_{F_\beta,q^i} \circ \varphi_{F_\beta,b'}(u) =& F_\beta^{-i} F_\beta^{-j} u F_\beta^j F_\beta^i \\ =& F_\beta^{-i-j} u F_\beta^{i+j} \\ =& \varphi_{F_\beta,q^i q^j}(u)\end{aligned}$$

So $\varphi_{F_\beta,b}(\varphi_{F_\beta,q^j}(u)) = \varphi_{F_\beta,bq^j}(u)$ for all $b \in \mathbb{C}^*$ since both sides are Laurent polynomials in b and they are equal in infinitely many points. In the same way we get the result for all $b' \in \mathbb{C}$. □

Note that if β is long then the above automorphism is a Laurent polynomial in b^2. So if $b_1^2 = b_2^2$ for $b_1, b_2 \in \mathbb{C}^*$ then $\varphi_{F_\beta,b_1} = \varphi_{F_\beta,b_2}$. We could have defined another automorphism $\varphi'_{F_\beta,b} := \varphi_{F_\beta,b^{2/(\beta|\beta)}}$ and proved the lemma above with the modification that $\varphi'_{F_\beta,q_\beta^i}(u) = F_\beta^{-i} u F_\beta^i$. The author has chosen the first option to avoid having to write the β in q_β all the time in results like Lemma 4.12 and Corollary 4.20. On the other hand this choice means that we have to take some squareroots sometimes when doing concrete calculations involving long roots see e.g the proof of Lemma 5.11. The choice of squareroot doesn't matter by the above.

We can use the formulas in Section 3 to find the value of $\varphi_{F_\beta,b}(F_{\beta'})$ and $\varphi_{F_\beta,b}(E_{\beta'})$ for general root vectors F_β, $F_{\beta'}$ and $E_{\beta'}$, $\beta, \beta' \in \Phi^+$.

Proposition 4.10 *Let $s_{i_1} \dots s_{i_N}$ be a reduced expression of w_0 and define root vectors $F_{\beta_1}, \dots, F_{\beta_N}$ and $E_{\beta_1}, \dots, E_{\beta_N}$ using this expression (i.e. $F_{\beta_j} = T_{s_{i_1}} \dots T_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$ and $E_{\beta_j} = T_{s_{i_1}} \dots T_{s_{i_{j-1}}}(E_{\alpha_{i_j}})$). Let $1 \leq j < k \leq N$ and set $r = \langle \beta_k, \beta_j^\vee \rangle$.*

$$\begin{aligned}\varphi_{F_{\beta_j},b}(F_{\beta_k}^n) =& \sum_{i \geq 0} q_{\beta_j}^{i(nr+i)} b_{\beta_j}^{-nr-i} \prod_{t=1}^{i} \frac{q_{\beta_j}^{1-t} b_{\beta_j} - q_{\beta_j}^{t-1} b_{\beta_j}^{-1}}{q_{\beta_j}^t - q_{\beta_j}^{-t}} F_{\beta_j}^{-i} \operatorname{ad}(F_{\beta_j}^i)(F_{\beta_k}^n) \\ \varphi_{F_{\beta_k},b}(F_{\beta_j}^n) =& \sum_{i \geq 0} (-1)^i q_{\beta_k}^{-i} b_{\beta_k}^{nr+i} \prod_{t=1}^{i} \frac{q_{\beta_k}^{1-t} b_{\beta_k} - q_{\beta_k}^{t-1} b_{\beta_k}^{-1}}{q_{\beta_k}^t - q_{\beta_k}^{-t}} F_{\beta_k}^{-i} \widetilde{\operatorname{ad}}(F_{\beta_k}^i)(F_{\beta_j}^n) \\ \varphi_{F_{\beta_j},b}(E_{\beta_k}) =& \sum_{i \geq 0} b_{\beta_j}^{-i} \prod_{t=1}^{i} \frac{q_{\beta_j}^{1-t} b_{\beta_j} - q_{\beta_j}^{t-1} b_{\beta_j}^{-1}}{q_{\beta_j}^t - q_{\beta_j}^{-t}} F_{\beta_j}^{-i} u_i \\ \varphi_{F_{\beta_k},b}(E_{\beta_j}) =& \sum_{i \geq 0} b_{\beta_k}^{i} \prod_{t=1}^{i} \frac{q_{\beta_k}^{1-t} b_{\beta_k} - q_{\beta_k}^{t-1} b_{\beta_k}^{-1}}{q_{\beta_k}^t - q_{\beta_k}^{-t}} F_{\beta_k}^{-i} \widetilde{u}_i\end{aligned}$$

for some $u_i, \widetilde{u}_i \in U_q$ (independent of b) such that $u_i = \widetilde{u}_i = 0$ for $i \gg 0$. In particular for any $j, k \in \{1, \dots, N\}$:

$$\begin{aligned}\varphi_{F_{\beta_j},-1}(F_{\beta_k}) = (-1)^{(\beta_j|\beta_k)} F_{\beta_k} \\ \varphi_{F_{\beta_j},-1}(E_{\beta_k}) = E_{\beta_k}.\end{aligned}$$

Note that the sums are finite because of Lemma 3.5.

Proof. By Proposition 3.4 we have for any $a \in \mathbb{N}$

$$\begin{aligned} F_{\beta_k}^n F_{\beta_j}^a =& \sum_{i=0}^{a} q_{\beta_j}^{(i-a)(nr+i)} \begin{bmatrix} a \\ i \end{bmatrix}_{\beta_j} F_{\beta_j}^{a-i} \operatorname{ad}(F_{\beta_j}^i)(F_{\beta_k}^n) \\ =& \sum_{i=0}^{\infty} q_{\beta_j}^{i(nr+i)} q_{\beta_j}^{-a(nr+i)} \prod_{t=1}^{i} \frac{q_{\beta_j}^{1-t} q_{\beta_j}^a - q_{\beta_j}^{t-1} q_{\beta_j}^{-a}}{q_{\beta_j}^t - q_{\beta_j}^{-t}} F_{\beta_j}^{a-i} \operatorname{ad}(F_{\beta_j}^i)(F_{\beta_k}^n). \end{aligned}$$

Here we use the fact that $\begin{bmatrix} a \\ i \end{bmatrix}_{\beta_j} = 0$ for $i > a$. So

$$F_{\beta_j}^{-a} F_{\beta_k}^n F_{\beta_j}^a = \sum_{i\geq 0} q_{\beta_j}^{i(nr+i)} q_{\beta_j}^{-a(nr+i)} \prod_{t=1}^{i} \frac{q_{\beta_j}^{1-t} q_{\beta_j}^a - q_{\beta_j}^{t-1} q_{\beta_j}^{-a}}{q_{\beta_j}^t - q_{\beta_j}^{-t}} F_{\beta_j}^{-i} \operatorname{ad}(F_{\beta_j}^i)(F_{\beta_k}^n).$$

Now using the fact that $\varphi_{F_{\beta_j},q^a}(F_{\beta_k}^n) = F_{\beta_j}^{-a} F_{\beta_k}^n F_{\beta_j}^a$, the fact that $\varphi_{F_{\beta_j},b}(F_{\beta_k}^n)$ is Laurent polynomial and Lemma 4.8 we get the first identity. The second identity is shown similarly by using the second identity in Proposition 3.4.

To prove the last two identities we need to calculate $F_{\beta_j}^{-a} E_{\beta_k}^n F_{\beta_j}^a$ (resp. $F_{\beta_k}^{-a} E_{\beta_j}^n F_{\beta_k}^a$) for any $a \in \mathbb{N}$. Let $w = s_{i_1} \cdots s_{i_{j-1}}$ and $w' = s_{i_{j+1}} \cdots s_{i_{k-1}}$. Then $E_{\beta_j} = T_w(E_{\alpha_{i_j}})$ and $F_{\beta_k} = T_w T_{s_{i_j}} T_{w'}(F_{\alpha_{i_k}})$.

$$\begin{aligned} E_{\beta_j} F_{\beta_k}^a =& T_w \left(E_{\alpha_{i_j}} T_{s_{i_j}} T_{w'}(F_{\alpha_{i_k}}^a) \right) \\ =& T_w T_{s_{i_j}} \left(-K_{\alpha_{i_j}}^{-1} F_{\alpha_{i_j}} T_{w'}(F_{\alpha_{i_k}}^a) \right). \end{aligned}$$

Expand $s_{i_j} \cdots s_{i_N}$ from the right to a reduced expression $s_{i_j} \cdots s_{i_N} s_{m_1} \cdots s_{m_{j-1}}$ of w_0. Do the same with $s_{i_{j+1}} \cdots s_{i_N} s_{m_1} \cdots s_{m_{j-1}}$ to get a reduced expression $s_{i_{j+1}} \cdots s_{i_N} s_{m_1} \cdots s_{m_j}$. We claim that if we use the reduced expression $s_{i_{j+1}} \cdots s_{i_N} s_{m_1} \cdots s_{m_j}$ to construct roots $\beta_1' \ldots, \beta_N'$ and root vectors $F_{\beta_j'}'$ then $F_{\beta_N'}' = T_{s_{i_{j+1}}} \cdots T_{s_{i_N}} T_{s_{m_1}} \cdots T_{s_{m_{j-1}}}(F_{\alpha_{m_j}}) = F_{\alpha_{i_j}}$. This is easy to see since β_N' is positive but $s_{i_j} \beta_N' = w_0(\alpha_{m_j}) < 0$. We have $T_{w'}(F_{\alpha_{i_k}}^a) = F_{\beta_{k-j}'}^a$. Since $k - j < N$ we can use what we just calculated above: (set $d = k - j$)

$$F_{\beta_d'}'^{-a} F_{\beta_N'}' F_{\beta_d'}'^{a} = \sum_{i\geq 0} q_{\beta_d'}^{i(r+i)} q_{\beta_d'}^{-a(r+i)} \prod_{t=1}^{i} \frac{q_{\beta_d'}^{1-t} q_{\beta_d'}^a - q_{\beta_d'}^{t-1} q_{\beta_d'}^{-a}}{q_{\beta_d'}^t - q_{\beta_d'}^{-t}} F_{\beta_d'}'^{-i} \operatorname{ad}(F_{\beta_d'}'^{i})(F_{\beta_N'}').$$

so

$$F_{\beta_k}^{-a} E_{\beta_j} F_{\beta_k}^a = K_{\beta_j} T_w T_{s_{i_j}} \left(\sum_{i\geq 0} q_{\beta_d'}^{i(r+i)} q_{\beta_d'}^{-ai} \prod_{t=1}^{i} \frac{q_{\beta_d'}^{1-t} q_{\beta_d'}^a - q_{\beta_d'}^{t-1} q_{\beta_d'}^{-a}}{q_{\beta_d'}^t - q_{\beta_d'}^{-t}} F_{\beta_d'}'^{-i} \operatorname{ad}(F_{\beta_d'}'^{i})(F_{\beta_N'}') \right).$$

This shows the third identity. The fourth is shown similarly.

Setting $b = -1$ in the above formulas we get the last claim of the proposition.□

Definition 4.11 *Let M be a $U_{q(F_\beta)}$-module. We define a new module $\varphi_{F_\beta,b}.M$ (with elements $\varphi_{F_\beta,b}.m$, $m \in M$) where the module structure is given by composing with the above automorphism $\varphi_{F_\beta,b}$. – i.e. $u\varphi_{F_\beta,b}.m = \varphi_{F_\beta,b}.\varphi_{F_\beta,b}(u)m$ for all $u \in U_{q(F_\beta)}$, $m \in M$.*

Note that $\operatorname{wt}\varphi_{F_\beta,b}.M = b^{-\beta}\operatorname{wt} M$ where $b^{-\beta}$ is the character such that $b^{-\beta}(K_\alpha) = b^{-(\alpha|\beta)}$ for $\alpha \in \Pi$.

The homomorphisms from Lemma 4.9 preserve degree so we can restrict to $(U_{q(F_\beta)})_0$ which we will do in the next lemma. The twist of a $(U_{q(F_\beta)})_0$-module is defined in the same way as the definition above. It is an important fact of these twists that they do not neccesarily preserve simplicity of U_q-modules: If L is a $U_{q(F_\beta)}$-module that is simple as a U_q-module then $\varphi_{F_\beta,b}.L$ can be nonsimple as a U_q-module for some $b \in \mathbb{C}^*$, see e.g. Lemma 4.23.

Lemma 4.12 *Let M be a $U_{q(F_\beta)}$-module. Let $i \in \mathbb{Z}$. Then*

$$\varphi_{F_\beta,q^i}.M \cong M$$

as $U_{q(F_\beta)}$-modules. Furthermore for $\lambda \in \operatorname{wt} M$ we have an isomorphism of $(U_{q(F_\beta)})_0$-modules:

$$\varphi_{F_\beta,q^i}.M_\lambda \cong M_{q^{-i\beta}\lambda}.$$

Proof. The isomorphism in both cases is given by $\varphi_{F_\beta,q^i}.m \mapsto F_\beta^i m$, $\varphi_{F_\beta,q^i}.M \to M$. The inverse is given by multiplying by F_β^{-i}. By Lemma 4.9: For $u \in U_{q(F_\beta)}$, $m \in M$; $\varphi_{F_\beta,q^i}(u) = F_\beta^{-i} u F_\beta^i$ so $u\varphi_{F_\beta,q^i}.m = \varphi_{F_\beta,q^i}.F_\beta^{-i}uF_\beta^i m \mapsto F_\beta^i F_\beta^{-i} u F_\beta^i m = uF_\beta^i m$. Thus the given map is a homomorphism. □

Definition 4.13 *Let $\Sigma \subset \Phi^+$. Then Σ is called a set of commuting roots if there exists an ordering of the roots in Σ; $\Sigma = \{\beta_1, \dots, \beta_s\}$ such that for some reduced expression of w_0 and corresponding construction of the root vectors F_β we have: $[F_{\beta_j}, F_{\beta_i}]_q = 0$ for $1 \le i < j \le s$.*

For any subset $I \subset \Pi$, let Q_I be the subgroup of Q generated by I, Φ_I the root system generated by I , $\Phi_I^+ = \Phi^+ \cap \Phi_I$ and $\Phi_I^- = -\Phi_I^+$.

The following three lemmas have exactly the same proofs as their counterparts ([Ped15a, Lemma 5.6], [Ped15a, Lemma 5.22] and [Ped15a, Lemma 5.23]) in the root of unity case in [Ped15a]. We include the proofs here as well for completeness.

We have the following equivalent of Lemma 4.1 in [Mat00]:

Lemma 4.14
1. *Let $I \subset \Pi$ and let $\alpha \in I$. There exists a set of commuting roots $\Sigma' \subset \Phi_I^+$ with $\alpha \in \Sigma'$ such that Σ' is a basis of Q_I.*
2. *Let J, F be subsets of Π with $F \neq \Pi$. Let $\Sigma' \subset \Phi_J^+ \backslash \Phi_{J\cap F}^+$ be a set of commuting roots which is a basis of Q_J. There exists a set of commuting roots Σ which is a basis of Q such that $\Sigma' \subset \Sigma \subset \Phi^+ \backslash \Phi_F^+$*

Proof. The first part of the proof is just combinatorics of the root system so it is identical to the first part of the proof of Lemma 4.1 in [Mat00]: Let us first prove assertion 2.: If J is empty we can choose $\alpha \in \Pi\backslash F$ and replace J and Σ' by $\{\alpha\}$. So assume from now on that $J \neq \emptyset$. Set $J' = J\backslash F$, $p = |J'|$, $q = |J|$. Let $J_1, \dots, J_k$ be the connected components of J and set $J_i' = J' \cap J_i$, $F_i = F \cap J_i$, and $\Sigma_i' = \Sigma \cap \Phi_{J_i}$, for any $1 \le i \le k$. Since $\Sigma' \subset \Phi_J$ is a basis of Q_J, each Σ_i' is a basis of Q_{J_i}. Since Σ_i' lies in $\Phi_{J_i}^+ \backslash \Phi_{F_i}^+$, the set $J_i' = J_i \backslash F_i$ is not empty. Hence J' meets every connected component of J. Therefore we can write $J = \{\alpha_1, \dots, \alpha_q\}$ in such a way that $J' = \{\alpha_1, \dots, \alpha_p\}$ and, for any s

with $p+1 \leq s \leq q$, α_s is connected to α_i for some $i < s$. Since Π is connected we can write $\Pi \backslash J = \{\alpha_{q+1}, \dots, \alpha_n\}$ in such a way that for any $s \geq q+1$, α_s is connected to α_i for some i with $1 \leq i < s$. So $\Pi = \{\alpha_1, \dots, \alpha_n\}$ such that for $s > p$ we have that α_s is connected to some α_i with $1 \leq i < s$.

Let $\Sigma' = \{\beta_1, \dots, \beta_q\}$. We will define $\beta_{q+1}, \dots, \beta_l$ inductively such that for each $s \geq q$, $\{\beta_1, \dots, \beta_s\}$ is a commuting set of roots which is a basis of $\Phi_{\{\alpha_1, \dots, \alpha_s\}}$. So assume we have defined $\beta_1, \dots, \beta_s$. Let w_s be the longest word in $s_{\alpha_1}, \dots, s_{\alpha_s}$ and let w_{s+1} be the longest word in $s_{\alpha_1}, \dots, s_{\alpha_{s+1}}$. Choose a reduced expression of w_s such that the corresponding root vectors $\{F_{\beta_k}\}_{k=1}^s$ satisfies $[F_{\beta_j}, F_{\beta_i}]_q = 0$ for $i < j$. Choose a reduced expression of $w_{s+1} = w_s w'$ starting with the above reduced expression of w_s. Let N_s be the length of w_s and N_{s+1} be the length of w_{s+1}. So we get an ordering of the roots generated by $\{\alpha_1, \dots, \alpha_{s+1}\}$: $\Phi^+_{\{\alpha_1, \dots, \alpha_{s+1}\}} = \{\gamma_1, \dots, \gamma_{N_s}, \gamma_{N_s+1}, \dots, \gamma_{N_{s+1}}\}$ with $\Phi^+_{\{\alpha_1, \dots, \alpha_s\}} = \{\gamma_1, \dots, \gamma_{N_s}\}$. Consider $\gamma_{N_s+1} = w_s(\alpha_{s+1})$. Since w_s only consists of the simple reflections corresponding to $\alpha_1, \dots, \alpha_s$ we must have that $\gamma_{N_s+1} = \alpha_{s+1} + \sum_{i=1}^s m_i \alpha_i$ for some coefficients $m_i \in \mathbb{N}$. So $\{\beta_1, \dots, \beta_s, \gamma_{N_s+1}\}$ is a basis of $\Phi_{\{\alpha_1, \dots, \alpha_{s+1}\}}$. From Theorem 3.2 we get for $1 \leq i \leq s$

$$[F_{\gamma_{N_s+1}}, F_{\beta_i}]_q \in \operatorname{span}_{\mathbb{C}} \left\{ F_{\gamma_{N_s}}^{a_{N_s}} \cdots F_{\gamma_2}^{a_2} | a_i \in \mathbb{N} \right\}$$

But since $\{\gamma_1, \dots, \gamma_{N_s}\} = \Phi^+_{\{\alpha_1, \dots, \alpha_s\}}$ and since $\gamma_{N_s+1} = \alpha_{s+1} + \sum_{i=1}^s m_i \alpha_i$ we get $[F_{\gamma_{N_s+1}}, F_{\beta_i}]_q = 0$.

All that is left is to show that $\gamma_{N_s+1} \notin \Phi_F$. By the above we must have that α_{s+1} is connected to some $\alpha_i \in J'$. We will show that the coefficient of α_i in γ_{N_s+1} is nonzero. Otherwise $(\gamma_{N_s+1}|\alpha_i) < 0$ and so $\gamma_{N_s+1} + \alpha_i \in \Phi_{\{\alpha_1, \dots, \alpha_{s+1}\}}$ and by Theorem 1 in [Pap94], $\gamma_{N_s+1} + \alpha_i = \gamma_j$ for some $1 < j \leq s$. This is impossible since $\gamma_{N_s+1} + \alpha_i \notin \Phi_{\{\alpha_1, \dots, \alpha_s\}}$. So we can set $\beta_{s+1} = \gamma_{N_s+1}$ and the induction step is finished.

To prove assertion 1. it can be assumed that $I = \Pi$. Thus assertion 1. follows from assertion 2. with $J = \{\alpha\}$ and $F = \emptyset$. $\square$

Lemma 4.15 *Let $L \in \mathcal{F}$ be a simple module. Then there exists a $w \in W$ such that $w(F_L \backslash F_L^s) \subset \Phi^+$ and $w(T_L \backslash T_L^s) \subset \Phi^-$.*

Proof. Since L is simple we have $\Phi = F_L \cup T_L$. By Proposition 2.6 F_L and T_L are closed subsets. Then Lemma 4.16 in [Fer90] tells us that there exists a basis B of the root system Φ such that the antisymmetrical part of F_L is contained in the positive roots Φ_B^+ corresponding to the basis B and the antisymmetrical part of T_L is contained in the negative roots Φ_B^- corresponding to the basis. Since all bases of a root system are W-conjugate the claim follows. $\square$

Lemma 4.16 *Let L be an infinite dimensional admissible simple module. Let $w \in W$ be such that $w(F_L \backslash F_L^s) \subset \Phi^+$. Let $\alpha \in \Pi$ be such that $-\alpha \in w(T_L)$ (such an α always exists). Then there exists a commuting set of roots Σ with $\alpha \in \Sigma$ which is a basis of Q such that $-\Sigma \subset w(T_L)$.*

Proof. Set $L' = {}^w L$. Since $w(T_L) = T_{^w L} = T_{L'}$ we will just work with L'. Then $F_{L'} \backslash F_{L'}^s \subset \Phi^+$.

Note that it is always possible to choose a simple root $\alpha \in -T_{L'}$ since L' is infinite dimensional: If this was not possible we would have $\Phi^- \subset F_{L'}$. But since $F_{L'} \backslash F_{L'}^s \subset \Phi^+$ this would imply $F_L = \Phi$.

Set $F = F_{L'}^s \cap \Pi$. Since L' is infinite dimensional $F \neq \Pi$. By Lemma 4.14 2. applied with $J = \{\alpha\} = \Sigma'$ there exists a commuting set of roots Σ that is a basis of Q such that $\Sigma \subset \Phi^+ \backslash \Phi_F^+$. Since $F_{L'} \backslash F_{L'}^s \subset \Phi^+$ we have $\Phi^- = T_{L'}^- \cup (F_{L'}^s)^-$. To show $-\Sigma \subset T_{L'}$ we show $\left(\Phi^- \backslash \Phi_F^-\right) \cap F_{L'}^s = \emptyset$ or equivalently $(F_{L'}^s)^- \subset \Phi_F^-$.

Assume $\beta \in F_{L'}^s \cap \Phi^+$, $\beta = \sum_{\alpha \in \Pi} a_\alpha \alpha$, $a_\alpha \in \mathbb{N}$. The height of β is the sum $\sum_{\alpha \in \Pi} a_\alpha$. We will show by induction on the height of β that $-\beta \in \Phi_F^-$. If the height of β is 1 then β is a simple root and so $\beta \in F$. Clearly $-\beta \in \Phi_F^-$ in this case. Assume the height of β is greater than 1. Let $\alpha' \in \Pi$ be a simple root such that $\beta - \alpha'$ is a root. There are two possibilities: $-\alpha' \in T_{L'}$ or $\pm\alpha' \in F_{L'}^s$.

In the first case where $-\alpha' \in T_{L'}$ we must have $-\beta + \alpha' \in F_{L'}^s$ since if $-\beta + \alpha' \in T_L$ then $-\beta = (-\beta + \alpha') - \alpha' \in T_{L'}$. So $\beta - \alpha' \in F_{L'}^s$ and $\beta \in F_{L'}^s$. Since $F_{L'}$ is closed (Proposition 2.6) we get $-\alpha' = (\beta - \alpha') - \beta \in F_L$ which is a contradiction. So the first case $(-\alpha' \in T_{L'})$ is impossible.

In the second case since $F_{L'}$ is closed we get $\pm(\beta - \alpha') \in F_{L'}$ i.e. $\beta - \alpha' \in F_{L'}^s$. By the induction $-(\beta - \alpha') \in \Phi_F^-$ and since $-\beta = -(\beta - \alpha') - \alpha'$ we are done.□

Proposition 4.17 *Let $\Sigma = \{\beta_1, \dots, \beta_r\}$ be a set of commuting roots. The set $\{q^a F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} | a_i \in \mathbb{N}, a \in \mathbb{Z}\}$ is an Ore subset of U_q.*

Proof. We will prove it by induction over r. $r = 1$ is Lemma 4.7.

Let $S_r = \{q^a F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} | a_i \in \mathbb{N}, a \in \mathbb{Z}\}$. Let $a_1, \dots, a_r \in \mathbb{N}$, $a \in \mathbb{Z}$ and $u \in U_q$, then we need to show that

$$q^a F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} U_q \cap u S_r \neq \emptyset. \tag{1}$$

and

$$U_q q^a F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} \cap S_r u \neq \emptyset. \tag{2}$$

By Lemma 4.7 there exists $\widetilde{u} \in U_q$ and $b \in \mathbb{N}$ such that

$$F_{\beta_r}^{a_r} \widetilde{u} = u F_{\beta_r}^b. \tag{3}$$

By induction

$$q^a F_{\beta_1}^{a_1} \cdots F_{\beta_{r-1}}^{a_{r-1}} U_q \cap \widetilde{u} S_{r-1} \neq \emptyset$$

so

$$q^a F_{\beta_r}^{a_r} F_{\beta_1}^{a_1} \cdots F_{\beta_{r-1}}^{a_{r-1}} U_q \cap F_{\beta_r}^{a_r} \widetilde{u} S_{r-1} \neq \emptyset$$

Since Σ is a set of commuting roots $F_{\beta_r}^{a_r} F_{\beta_1}^{a_1} \cdots F_{\beta_{r-1}}^{a_{r-1}} = q^k F_{\beta_1}^{a_1} \cdots F_{\beta_{r-1}}^{a_{r-1}} F_{\beta_r}^{a_r}$ for some $k \in \mathbb{Z}$. Using this and (3) we get

$$\emptyset \neq q^{a+k} F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} U_q \cap u F_{\beta_r}^b S_{r-1} \subset q^a F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} U_q \cap u S_r$$

where $F_{\beta_r}^b S_{r-1} \subset S_r$ because F_{β_r} q-commutes with all the other root vectors.

(2) is shown similarly. □

Lemma 4.18 *Let $\nu \in X$ and let $\Sigma = \{\beta_1, \dots, \beta_n\}$ be a basis of Q. Then there exists $\mathbf{b} = (b_1, \dots, b_n) \in (\mathbb{C}^*)^n$ such that*

$$\nu = b_1^{\beta_1} b_2^{\beta_2} \cdots b_n^{\beta_n}$$

and there are only finitely many different $\mathbf{b} \in (\mathbb{C}^*)^n$ satisfying this.

Proof. If $\gamma_1, \gamma_2 \in X$ satisfy $\gamma_1(K_{\beta_i}) = \gamma_2(K_{\beta_i})$ for $i = 1, \dots, n$ then $\gamma_1 = \gamma_2$ because $\{\beta_1, \dots, \beta_n\}$ is a basis of Q. Since for $a_1, \dots, a_n \in \mathbb{C}^*$, $a_1^{\beta_1} a_2^{\beta_2} \cdots a_n^{\beta_n}(K_{\beta_i}) = a_1^{(\beta_i|\beta_1)} a_2^{(\beta_i|\beta_2)} \cdots a_n^{(\beta_i|\beta_n)}$ we have to solve the system in n unknown variables $x_1, \dots, x_n$:

$$\begin{aligned} x_1^{(\beta_1|\beta_1)} x_2^{(\beta_1|\beta_2)} \cdots x_n^{(\beta_1|\beta_n)} =& \nu(K_{\beta_1}) \\ x_1^{(\beta_2|\beta_1)} x_2^{(\beta_2|\beta_2)} \cdots x_n^{(\beta_2|\beta_n)} =& \nu(K_{\beta_2}) \\ & \vdots \\ x_1^{(\beta_n|\beta_1)} x_2^{(\beta_n|\beta_2)} \cdots x_n^{(\beta_n|\beta_n)} =& \nu(K_{\beta_n}). \end{aligned}$$

Let $c_j \in \mathbb{C}$, $j = 1, \dots, n$ be such that $\nu(K_{\beta_j}) = e^{c_j}$. There is a choice here since any $c_j + 2k\pi i$, $k \in \mathbb{Z}$ could be chosen instead. Consider the linear system in n unknowns $X_1, \dots, X_n$

$$\begin{aligned} (\beta_1|\beta_1) X_1 + (\beta_1|\beta_2) X_2 \cdots (\beta_1|\beta_n) X_n =& c_1 \\ (\beta_2|\beta_1) X_1 + (\beta_2|\beta_2) X_2 \cdots (\beta_2|\beta_n) X_n =& c_2 \\ & \vdots \\ (\beta_n|\beta_1) X_1 + (\beta_n|\beta_2) X_2 \cdots (\beta_n|\beta_n) X_n =& c_n. \end{aligned}$$

This system has a unique solution $a_1, \dots, a_n \in \mathbb{C}$ since the matrix $((\beta_i|\beta_j))_{i,j}$ is invertible. So $x_i = e^{a_i}$ is a solution to the above system. Any other solution to the original system corresponds to making a different choice when taking the logarithm of $\nu(K_{\beta_i})$. So another solution would be of the form $x_i = e^{a_i + a_i'}$ where a_i', $i = 1, \dots, n$ is a solution to a system of the form:

$$\begin{aligned} (\beta_1|\beta_1) X_1 + (\beta_1|\beta_2) X_2 \cdots (\beta_1|\beta_n) X_n =& 2k_1\pi i \\ (\beta_2|\beta_1) X_1 + (\beta_2|\beta_2) X_2 \cdots (\beta_2|\beta_n) X_n =& 2k_2\pi i \\ & \vdots \\ (\beta_n|\beta_1) X_1 + (\beta_n|\beta_2) X_2 \cdots (\beta_n|\beta_n) X_n =& 2k_n\pi i. \end{aligned}$$

for some $k_1, \dots, k_n \in \mathbb{Z}$. Since $A = ((\beta_i|\beta_j))_{i,j}$ is a matrix with only integer coefficients we have $A^{-1} = \frac{1}{\det A}\widetilde{A}$ for some $\widetilde{A}$ with only integer coefficients. So the solution to the system above is integer linear combinations in $\frac{2k_i\pi i}{\det A}$, $i = 1, \dots, n$ hence $\{(e^{a_1'}, \dots, e^{a_n'}) | (a_1', \dots, a_n') \text{ is a solution to the above system}\}$ has fewer than $n \det A$ elements so it is a finite set. □

In the next definition we would like to compose the φ's for different β. In particular let $\Sigma = \{\beta_1, \dots, \beta_n\}$ be a set of commuting roots and $F_{\beta_1}, \dots, F_{\beta_n}$ corresponding root vectors. Let $F_\Sigma := \{q^a F_{\beta_1}^{a_1} \cdots F_{\beta_n}^{a_n} | a_i \in \mathbb{N}, a \in \mathbb{Z}\}$ and let $U_{q(F_\Sigma)}$ be the Ore localization in F_Σ. For $i < j$ we have

$$F_{\beta_i}^{-k} F_{\beta_j} F_{\beta_i}^{k} = q^{-k(\beta_i|\beta_j)} F_{\beta_j}$$

or equivalently $\varphi_{F_{\beta_i}, q^k}(F_{\beta_j}) = \left(q^k\right)^{-(\beta_i|\beta_j)} F_{\beta_j}$. This implies $\varphi_{F_{\beta_i}, b}(F_{\beta_j}) = b^{-(\beta_i|\beta_j)} F_{\beta_j}$ for $b \in \mathbb{C}^*$ because $b \mapsto \varphi_{F_{\beta_i}, b}(F_{\beta_j})$ is Laurent polynomial. Similarly $\varphi_{F_{\beta_j}, b}(F_{\beta_i}) = b^{(\beta_i|\beta_j)} F_{\beta_i}$. This shows that we can define $\varphi_{F_\beta, b}(F_{\beta'}^{-1}) =$

$\varphi_{F_\beta,b}(F_{\beta'})^{-1}$ for $\beta,\beta' \in \Sigma$ extending $\varphi_{F_\beta,b}$ to a homomorphism $U_{q(F_\Sigma)} \to U_{q(F_\Sigma)}$. Also note that the φ's commute because

$$\begin{aligned} F_{\beta_i}^{-k_1} F_{\beta_j}^{-k_2} u F_{\beta_j}^{k_2} F_{\beta_i}^{k_1} =& q^{k_1 k_2 (\beta_i|\beta_j)} F_{\beta_j}^{-k_2} F_{\beta_i}^{-k_1} u q^{-k_1 k_2 (\beta_i|\beta_j)} F_{\beta_i}^{k_1} F_{\beta_j}^{k_2} \\ =& F_{\beta_j}^{-k_2} F_{\beta_i}^{-k_1} u F_{\beta_i}^{k_1} F_{\beta_j}^{k_2} \end{aligned}$$

Definition 4.19 *Let $\Sigma = \{\beta_1, \dots, \beta_r\}$ be a set of commuting roots and let $F_{\beta_1}, \dots, F_{\beta_r}$ be corresponding root vectors such that $[F_{\beta_j}, F_{\beta_i}]_q = 0$ for $i < j$. Let $U_{q(F_\Sigma)}$ denote the Ore localization of U_q in the Ore set $F_\Sigma := \{q^a F_{\beta_1}^{a_1} \cdots F_{\beta_n}^{a_r} | a_i \in \mathbb{N}, a \in \mathbb{Z}\}$. Said in words we invert F_β for all $\beta \in \Sigma$.*

Let M be a U_q-module. We define M_{F_Σ} to be the $U_{q(F_\Sigma)}$-module $U_{q(F_\Sigma)} \otimes_{U_q} M$. Let $\mathbf{b} = (b_1, \dots, b_r) \in (\mathbb{C}^*)^r$. Then for a $U_{q(F_\Sigma)}$-module N we define $\varphi_{F_\Sigma,\mathbf{b}}.N$ to be the twist of the module by $\varphi_{F_{\beta_1},b_1} \circ \cdots \circ \varphi_{F_{\beta_r},b_r}$.

For $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$ define $q^{\mathbf{i}} = (q^{i_1}, \dots, q^{i_r}) \in (\mathbb{C}^*)^r$ and $q^{\mathbb{Z}^r} = \{q^{\mathbf{i}} | \mathbf{i} \in \mathbb{Z}^r\} \subset (\mathbb{C}^*)^r$.

For $\mathbf{b} = (b_1, \dots, b_r) \in (\mathbb{C}^*)^r$ we set $\mathbf{b}^\Sigma := b_1^{\beta_1} \cdots b_r^{\beta_r} \in X$. If Σ is a basis of Q then the map $\mathbf{b} \mapsto \mathbf{b}^\Sigma$ is surjective by Lemma 4.18 but not neccesarily injective.

Corollary 4.20 (to Lemma 4.12) *Let Σ be a set of commuting roots that is a $\mathbb{Z}$ basis of Q, let F_Σ be an Ore subset corresponding to Σ, let M be a $U_{q(F_\Sigma)}$-module and let $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$. Then*

$$\varphi_{F_\Sigma, q^{\mathbf{i}}}.M \cong M$$

as $U_{q(F_\Sigma)}$-modules. Furthermore for $\lambda \in \operatorname{wt} M$ we have an isomorphism of $(U_{q(F_\Sigma)})_0$-modules:

$$\varphi_{F_\Sigma, q^{\mathbf{i}}}.M_\lambda \cong M_{(q^{-\mathbf{i}})^\Sigma \lambda} = M_{q^{-\mu}\lambda}$$

where $\mu = \sum_{j=1}^n i_j \beta_j$.

Proof. The corollary follows from Lemma 4.12 because Σ is a $\mathbb{Z}$ basis of Q. □

Definition 4.21 *Let L be an admissible module of degree d. The essential support of L is defined as*

$$\operatorname{Supp}_{\text{ess}}(L) := \{\lambda \in \operatorname{wt} L | \dim L_\lambda = d\}$$

Lemma 4.22 *Let M be an admissible module. Let $\Sigma \subset \Phi^+$ be a set of commuting roots and F_Σ a corresponding Ore subset. Assume $-\Sigma \subset T_M$. Then for $\lambda \in X$:*

$$\dim (M_{F_\Sigma})_\lambda = \max_{\mu \in \mathbb{Z}\Sigma} \{\dim M_{q^\mu \lambda}\}$$

and if $\dim M_\lambda = \max_{\mu \in \mathbb{Z}\Sigma} \{\dim M_{q^\mu \lambda}\}$ then $(M_{F_\Sigma})_\lambda \cong M_\lambda$ as $(U_q)_0$-modules.

In particular if $\Sigma \subset T_M$ as well then $M_{F_\Sigma} \cong M$ as U_q-modules.

Compare to Lemma 4.4(ii) in [Mat00].

Proof. We have $\Sigma = \{\beta_1, \dots, \beta_r\}$ for some $\beta_1, \dots, \beta_r \in \Phi^+$ and corresponding root vectors $F_{\beta_1}, \dots, F_{\beta_r}$. Let $\lambda \in X$ and set $d = \max_{\mu \in \mathbb{Z}\Sigma} \{\dim M_{q^\mu \lambda}\}$. Let V be a finite dimensional subspace of $(M_{F_\Sigma})_\lambda$. Then there exists a homogenous element $s \in F_\Sigma$ such that $sV \subset M$. Let $\nu \in \mathbb{Z}\Sigma$ be the degree of s. So $sV \subset M_{q^\nu \lambda}$

hence $\dim sV \leq d$. Since s acts injectively on M_{F_Σ} we have $\dim V \leq d$. Now the first claim follows because $F_\beta^{\pm 1}$ acts injectively on M_{F_Σ} for all $\beta \in \Sigma$.

We have an injective U_q-homomorphism from M to M_{F_Σ} sending $m \in M$ to $1 \otimes m \in M_{F_\Sigma}$ that restricts to a $(U_q)_0$-homomorphism from M_λ to $(M_{F_\Sigma})_\lambda$. If $\dim M_\lambda = d$ then this is surjective as well. So it is an isomorphism. The last claim follow because $\pm\Sigma \subset T_M$ implies $\dim M_\lambda = \dim M_{q^\mu \lambda}$ for any $\mu \in \mathbb{Z}\Sigma$; so $M_\lambda \cong (M_{F_\Sigma})_\lambda$ for any $\lambda \in X$. Since M is a weight module this implies that $M \cong M_{F_\Sigma}$ as U_q-modules. $\square$

Lemma 4.23 *Let L be a simple infinite dimensional admissible module. Let $\beta \in (T_L^s)^+$. Then there exists a $b \in \mathbb{C}^*$ such that $\varphi_{F_\beta,b}.L_{F_\beta}$ contains a simple admissible U_q-submodule L' with $T_{L'} \subset T_L$ and $\beta \notin T_L$.*

Proof. Since $\beta \in T_L^s$ we have $L \cong L_{F_\beta}$ as U_q-modules by Lemma 4.22. So we will consider L as a $U_{q(F_\beta)}$-module via this isomorphism when taking twist etc.

Let E_β and F_β be root vectors corresponding to β. Let $\lambda \in \operatorname{wt} L$. Consider $F_\beta E_\beta$ as a linear operator on L_λ. Since $\mathbb{C}$ is algebraically closed $F_\beta E_\beta$ must have an eigenvalue c_β and an eigenvector $v \in L_\lambda$. By (the proof of) Lemma 4.9

$$F_\beta E_\beta \varphi_{F_\beta,b}.v = \varphi_{F_\beta,b}.(c_\beta - (q_\beta - q_\beta^{-1})^{-2}(b_\beta - b_\beta^{-1})(q_\beta b_\beta^{-1}\lambda(K_\beta) - q_\beta^{-1} b_\beta \lambda(K_\beta)^{-1})v.$$

The Laurent polynomial, in b, $c_\beta - (q - q^{-1})^{-2}(b_\beta - b_\beta^{-1})(b_\beta \lambda(K_\beta) - b_\beta^{-1}\lambda(K_\beta^{-1}))$ has a zero point $c \in \mathbb{C}^*$.

Thus $\varphi_{F_\beta,c}.L$ contains an element v' such that $F_\beta E_\beta v' = 0$ and since F_β acts injectively on $\varphi_{F_\beta,c}.L$, we have $E_\beta v' = 0$. Set $V = \{m \in \varphi_{F_\beta,c}.L | E_\beta^N m = 0, N >> 0\} = (\varphi_{F_\beta,c}.L)^{[\beta]}$. By Proposition 2.1 this is a U_q-submodule of the U_q-module $\varphi_{F_\beta,c}.L$. It is nonzero since $v' \in V$. By Lemma 4.6 V has a simple U_q-submodule L'.

We want to show that $T_{L'} \subset T_L$. Assume $\gamma \in T_{L'}$. Then $q^{\mathbb{N}\gamma} \operatorname{wt} L' \subset \operatorname{wt} L'$. But since $\operatorname{wt} L' \subset c^{-\beta} \operatorname{wt} L$ we get for some $\nu \in \operatorname{wt} L$, $q^{\mathbb{N}\gamma} c^{-\beta} \nu \subset c^{-\beta} \operatorname{wt} L$ or equivalently $q^{\mathbb{N}\gamma}\nu \subset \operatorname{wt} L$. But this shows that $\gamma \notin F_L$ and since L is a simple U_q-module this implies that $\gamma \in T_L$. By construction we have $\beta \notin T_{L'}$. $\square$

5 Coherent families

For a U_q-module $M \in \mathcal{F}$ define $\operatorname{Tr}^M : X \times (U_q)_0 \to \mathbb{C}$ by $\operatorname{Tr}^M(\lambda, u) = \operatorname{Tr} u|_{M_\lambda}$.

Lemma 5.1 *Let $M, N \in \mathcal{F}$ be semisimple U_q-modules. If $\operatorname{Tr}^M = \operatorname{Tr}^N$ then $M \cong N$.*

Proof. Theorem 7.19 in [Lam01] states that this is true for modules over a *finite dimensional* algebra. So we will reduce to the case of modules over a finite dimensional algebra. Let L be a composition factor of M and λ a weight of L. Then the multiplicity of the U_q-composition factor L in M is the multiplicity of the $(U_q)_0$-composition factor L_λ in M_λ by Theorem 2.7. M_λ is a finite dimensional $(U_q)_0$-module. Let I be the kernel of the homomorphism $(U_q)_0 \to \operatorname{End}_{\mathbb{C}}(M_\lambda)$ given by the action of $(U_q)_0$. Then $(U_q)_0/I$ is a finite dimensional $\mathbb{C}$ algebra and M_λ is a module over $(U_q)_0/I$. Furthermore since $\operatorname{Tr}^M(\lambda, u) = 0$ for all $u \in I$ the trace of an element $u \in (U_q)_0$ is the same as the trace of $u + I \in (U_q)_0/I$ on M_λ as a $(U_q)_0/I$-module. So if $\operatorname{Tr}^M = \operatorname{Tr}^N$ the multiplicity of L_λ in M_λ and N_λ are the same and hence the multiplicity of L in M is the same as in N. $\square$

We will use the Zariski topology on $(\mathbb{C}^*)^n$: V is a closed set if it is the zero-points of a Laurent polynomial $p \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

Proposition 5.2 *Let L be an infinite dimensional admissible simple module of degree d. Let Σ be a set of commuting roots that is a basis of Q and $w \in W$ such that $-\Sigma \subset w(T_L)$. Let F_Σ be a corresponding Ore subset. Let $\lambda \in \mathrm{Supp}_{\mathrm{ess}}(L)$. The set*

$$\{\mathbf{b} \in (\mathbb{C}^*)^n | \, {}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{b}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right) \text{ is a simple } (U_q)_0\text{-module}\}$$

is a Zariski open set of $(\mathbb{C}^*)^n$.

Proof. The $(U_q)_0$-module $V := {}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{b}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)$ is simple if and only if the bilinear map $B_{\mathbf{b}}(u,v) \in (U_q)_0 \times (U_q)_0 \mapsto \mathrm{Tr}\left(uv|_{{}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{b}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)}\right)$ has maximal rank d^2: The map factors through $\mathrm{End}_{\mathbb{C}}(V) \times \mathrm{End}_{\mathbb{C}}(V)$ given by the representation $(U_q)_0 \to \mathrm{End}_{\mathbb{C}}(V)$ on V. $B_{\mathbf{b}}$ has maximal rank d^2 if and only if the representation is surjective onto $\mathrm{End}_{\mathbb{C}}(V)$ which is equivalent to V being simple.

For any finite dimensional subspace $E \subset (U_q)_0$, the set Ω_E of all $\mathbf{b}$ such that $B_{\mathbf{b}}|_E$ has rank d^2 is either empty or the non-zero points of the Laurent polynomial $\det M$ for some $d^2 \times d^2$ minor M of the matrix $(B_{\mathbf{b}}(e_i,e_j))_{i,j}$ where $\{e_i\}$ is a basis of E. Therefore $\Omega = \cup_E \Omega_E$ is open. $\square$

For a module M that is a direct sum of modules of finite length we define M^{ss} to be the unique (up to isomorphism) semisimple module with the same composition factors as M.

Lemma 5.3 *Let L be an infinite dimensional simple admissible U_q-module of degree d, $w \in W$ and $\Sigma = \{\beta_1, \dots, \beta_n\} \subset \Phi^+$ a set of commuting roots that is a basis of Q such that $-\Sigma \subset w(T_L)$. Let F_Σ be a corresponding Ore subset to Σ. Let $\mathbf{c} \in (\mathbb{C}^*)^n$ and let L' be another infinite dimensional U_q-module such that L' is contained in ${}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{c}}.({}^wL)_{F_\Sigma}\right)^{ss}$ (i.e. L' is a composition factor of ${}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{c}}.({}^wL)_{F_\Sigma}\right)$). Assume that $\Sigma' = \{\beta_1', \dots, \beta_n'\} \subset \Phi^+$ is another set of commuting roots that is a basis of Q and $w' \in W$ is such that $-\Sigma' \subset w'(T_{L'})$. Let $F_{\Sigma'}$ be a corresponding Ore subset.*

Define $a_{i,j} \in \mathbb{Z}$ by $w(w')^{-1}(\beta_i') = \sum_{j=1}^n a_{i,j}\beta_j$ and define $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ by

$$f(b_1, \dots, b_n) = \left(\prod_{i=1}^n b_i^{a_{i,1}}, \dots, \prod_{i=1}^n b_i^{a_{i,n}}\right).$$

Then L' is admissible of degree d and

$${}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.({}^{w'}L')_{F_{\Sigma'}}\right)^{ss} \cong {}^{\overline{w}}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.({}^wL)_{F_\Sigma}\right)^{ss}$$

Proof. We will show that $\mathrm{Tr}^{{}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.({}^{w'}L')_{F_{\Sigma'}}\right)^{ss}} = \mathrm{Tr}^{{}^{\overline{w}}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.({}^wL)_{F_\Sigma}\right)^{ss}}$.

Let $\lambda \in \mathrm{Supp}_{\mathrm{ess}}(L)$. Then $w(\lambda) \in \mathrm{Supp}_{\mathrm{ess}}({}^wL)$. As a $(U_q)_0$-module we have $\left({}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{c}}.({}^wL)_{F_\Sigma}\right)\right)^{ss} \cong \bigoplus_{\mathbf{i}\in\mathbb{Z}^n} {}^{\overline{w}}\left(\varphi_{F_\Sigma,q^{\mathbf{i}}\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}$ (Corollary 4.20). Let $\lambda' \in \mathrm{Supp}_{\mathrm{ess}}(L')$. Then $L'_{\lambda'}$ is a $(U_q)_0$-submodule of ${}^{\overline{w}}\left(\varphi_{F_\Sigma,q^{\mathbf{j}}\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}$ for some $\mathbf{j} \in \mathbb{Z}^n$. We can assume $\mathbf{j} = 0$ by replacing $\mathbf{c}$ with $q^{\mathbf{j}}\mathbf{c}$ (note that we have then $(\mathbf{c}^{-1})^\Sigma = w\left(\lambda'\lambda^{-1}\right)$). So $L'_{\lambda'}$ is a $(U_q)_0$-submodule of ${}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}$. For any other $\mu \in \mathrm{Supp}_{\mathrm{ess}}(L')$ there is a unique $\mathbf{j}'_\mu \in \mathbb{Z}^n$ such that $\mu = (w')^{-1}\left(\left(q^{-\mathbf{j}'_\mu}\right)^{\Sigma'}\right)\lambda'$ and a unique $\mathbf{j}_\mu \in \mathbb{Z}^n$ such that $w^{-1}\left(\left(q^{-\mathbf{j}_\mu}\mathbf{c}^{-1}\right)^\Sigma\right)\lambda = \mu$. For such $\mathbf{j}_\mu$, L'_μ is a submodule of ${}^{\overline{w}}\left(\varphi_{F_\Sigma,q^{\mathbf{j}_\mu}\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}$.

f is bijective, $f(q^{\mathbb{Z}^n}) = q^{\mathbb{Z}^n}$, $f(\mathbf{b})^\Sigma = w(w')^{-1}\left(\mathbf{b}^{\Sigma'}\right)$ for all $\mathbf{b} \in (\mathbb{C}^*)^n$ and for any $\mu \in \mathrm{Supp}_{\mathrm{ess}}(L')$, $f(q^{\mathbf{j}'_\mu}) = q^{\mathbf{j}_\mu}$. For a Laurent polynomial p, $p \circ f$ is Laurent polynomial as well. Since $q^{\mathbb{N}^n}$ is Zariski dense in $(\mathbb{C}^*)^n$ (Lemma 4.8) and f is a Laurent polynomial the set $D = \{q^{\mathbf{j}_\mu}\mathbf{c} \in (\mathbb{C}^*)^n | \mu \in \mathrm{Supp}_{\mathrm{ess}}(L')\}$ is Zariski dense. By Proposition 5.2 the $(U_q)_0$-module ${}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{b}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)$ is simple for all $\mathbf{b} \in \Omega$ for some Zariski open set Ω of $(\mathbb{C}^*)^n$. Since D is dense and Ω is open $D \cap \Omega$ is nonempty. So there exists a $\mu_0 \in \mathrm{Supp}_{\mathrm{ess}}(L')$ such that ${}^{\overline{w}}\left(\varphi_{F_\Sigma,q^{\mathbf{j}_{\mu_0}}\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)$ is simple and contains the nonzero simple $(U_q)_0$-module L'_{μ_0} as a submodule. Thus $L'_{\mu_0} \cong {}^{\overline{w}}\left(\varphi_{F_\Sigma,q^{\mathbf{j}_{\mu_0}}\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)$. We get now from Lemma 4.22 that L' is admissible of degree d and that for every $\mu \in \mathrm{Supp}_{\mathrm{ess}}(L')$,

$$\begin{aligned} L'_\mu &\cong {}^{\overline{w}}\left(\varphi_{F_\Sigma,q^{\mathbf{j}_\mu}\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right) \\ &\cong {}^{\overline{w}}\left(\varphi_{F_\Sigma,f(q^{\mathbf{j}'_\mu})\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right). \end{aligned}$$

By Lemma 4.22, Corollary 4.20 and the definition of $\mathbf{j}'_\mu$ we have for any $\mu \in \mathrm{Supp}_{\mathrm{ess}}(L')$

$${}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},q^{\mathbf{j}'_\mu}}.\left(({}^{w'}L')_{F_{\Sigma'}}\right)_{w'(\lambda')}\right) \cong L'_\mu.$$

Let $u \in (U_q)_0$. We see that for $\mathbf{b} = q^{\mathbf{j}'_\mu}$

$$\mathrm{Tr}\, u|_{{}^{\overline{w}}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)} = \mathrm{Tr}\, u|_{L'_\mu} = \mathrm{Tr}\, u|_{{}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.\left(({}^{w'}L')_{F_{\Sigma'}}\right)_{w'(\lambda')}\right)}.$$

Since $\mathbf{b} \mapsto \mathrm{Tr}\, u|_{{}^{\overline{w}}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}}$ and $\mathbf{b} \mapsto \mathrm{Tr}\, u|_{{}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.\left(({}^{w'}L')_{F_{\Sigma'}}\right)_{w'(\lambda')}\right)^{ss}}$ are both Laurent polynomials and equal on the Zariski dense subset $\{q^{\mathbf{j}'_\mu} | \mu \in \mathrm{Supp}_{\mathrm{ess}}(L')\}$ they are equal for all $\mathbf{b} \in (\mathbb{C}^*)^n$. Thus by Lemma 5.1

$${}^{\overline{w}}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss} \cong {}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.\left(({}^{w'}L')_{F_{\Sigma'}}\right)_{w'(\lambda')}\right)^{ss}$$

as $(U_q)_0$-modules. Since (by Corollary 4.20)

$${}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.({}^{w'}L')_{F_{\Sigma'}}\right)^{ss} \cong \bigoplus_{\mathbf{i}\in\mathbb{Z}^n} {}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},q^{\mathbf{i}}\mathbf{b}}.\left(({}^{w'}L')_{F_{\Sigma'}}\right)_{w'(\lambda')}\right)^{ss}$$

and

$$\overline{w}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.({}^wL)_{F_\Sigma}\right)^{ss}\cong\bigoplus_{\mathbf{i}\in\mathbb{Z}^n}\overline{w}\left(\varphi_{F_\Sigma,q^{\mathbf{i}}f(\mathbf{b})\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}$$

we get

$$\begin{aligned}\overline{w'}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.({}^{w'}L')_{F_{\Sigma'}}\right)^{ss}&\cong\bigoplus_{\mathbf{i}\in\mathbb{Z}^n}\overline{w'}\left(\varphi_{F_{\Sigma'},q^{\mathbf{i}}\mathbf{b}}.\left(({}^{w'}L')_{F_{\Sigma'}}\right)_{w'(\lambda')}\right)^{ss}\\&\cong\bigoplus_{\mathbf{i}\in\mathbb{Z}^n}\overline{w}\left(\varphi_{F_\Sigma,f(q^{\mathbf{i}}\mathbf{b})\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}\\&\cong\bigoplus_{\mathbf{i}\in\mathbb{Z}^n}\overline{w}\left(\varphi_{F_\Sigma,q^{\mathbf{i}}f(\mathbf{b})\mathbf{c}}.\left(({}^wL)_{F_\Sigma}\right)_{w(\lambda)}\right)^{ss}\\&\cong\overline{w}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.({}^wL)_{F_\Sigma}\right)^{ss}\end{aligned}$$

as $(U_q)_0$-modules. By Theorem 2.7 this implies they are isomorphic as U_q-modules as well. □

Corollary 4.20 tells us that twisting with an element of the form $q^{\mathbf{i}}$ gives us a module isomorphic to the original module. Thus it makes sense to write $\varphi_{F_\Sigma,t}.M$ for a $t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}$ and a $U_{q(F_\Sigma)}$-module M. Just choose a representative for t. Any representative gives the same $U_{q(F_\Sigma)}$-module up to isomorphism.

Let L be an admissible simple module. Assume for a $w\in W$ that $\Sigma\subset -w(T_L)$ is a set of commuting roots that is a basis of Q (it is always possible to find such w and Σ by Lemma 4.15 and Lemma 4.16) and let F_Σ be a corresponding Ore subset. Let $\nu\in X$. The U_q-module

$$\overline{w}\left(\bigoplus_{\mathbf{b}\in(\mathbb{C}^*)^n:\,\mathbf{b}^\Sigma=\nu}\varphi_{F_\Sigma,\mathbf{b}}.\left({}^wL\right)_{F_\Sigma}\right)$$

has finite length by Lemma 4.18, Lemma 4.22 and Lemma 4.6.

We define

$$\mathcal{EXT}(L)=\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}\overline{w}\left(\varphi_{F_\Sigma,t}.\left({}^wL\right)_{F_\Sigma}\right)\right)^{ss}.$$

The definition is independent (up to isomorphism) of the chosen w, Σ and F_Σ as suggested by the notation:

Lemma 5.4 *Let L be a simple admissible module. Let $w,w'\in W$ and assume $\Sigma\subset -w(T_L),\Sigma'\subset -w'(T_{L'})$ are sets of commuting roots that are both a basis of Q. Let $F_\Sigma,F'_{\Sigma'}$ be corresponding Ore subsets. Then*

$$\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}\overline{w}\left(\varphi_{F_\Sigma,t}.\left({}^wL\right)_{F_\Sigma}\right)\right)^{ss}\cong\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}\overline{w'}\left(\varphi_{F'_{\Sigma'},t}.\left({}^{w'}L\right)_{F'_{\Sigma'}}\right)\right)^{ss}$$

as U_q-modules.

Proof. Obviously L is a submodule of $\left({}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{1}}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}$ where $\mathbf{1}=(1,\dots,1)$. By Lemma 5.3 this implies that for $\mathbf{b}\in(\mathbb{C}^*)^n$

$$\left({}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.\left({}^{w'}L\right)_{F_{\Sigma'}}\right)\right)^{ss}\cong\left({}^{\overline{w}}\left(\varphi_{F_\Sigma,f(\mathbf{b})}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}$$

for some f with the property that $f(q^{\mathbb{Z}^n})=q^{\mathbb{Z}^n}$. So it makes sense to write $f(t)$ for $t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}$. Thus

$$\begin{aligned}\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}{}^{\overline{w'}}\left(\varphi_{F'_{\Sigma'},t}.\left({}^{w'}L\right)_{F'_{\Sigma'}}\right)\right)^{ss}&\cong\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}{}^{\overline{w}}\left(\varphi_{F_\Sigma,f(t)}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}\\&\cong\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}{}^{\overline{w}}\left(\varphi_{F_\Sigma,t}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}\end{aligned}$$

since f is bijective. ☐

Proposition 5.5 *Let L be a simple infinite dimensional admissible module. For $x\in W$:*

$$\mathcal{EXT}({}^{x}L)\cong{}^{x}\left(\mathcal{EXT}(L)\right)$$

and

$$\mathcal{EXT}({}^{\overline{x}}L)\cong{}^{\overline{x}}\left(\mathcal{EXT}(L)\right).$$

Proof. Let $w\in W$ be such that $w(F_L\backslash F_L^s)\subset\Phi^+$ (exists by Lemma 4.15). Let Σ be a set of commuting roots that is a basis of Q such that $-\Sigma\subset w(T_L)$ (exists by Lemma 4.16) and let F_Σ be a corresponding Ore subset. First we will define $\mathcal{EXT}'(L)=\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}{}^{w^{-1}}\left(\varphi_{F_\Sigma,t}.({}^{\overline{w^{-1}}}L)_{F_\Sigma}\right)\right)^{ss}$ and show that $\mathcal{EXT}'(L)\cong\mathcal{EXT}(L)$ as U_q-modules: Going through the proof of Lemma 5.3 and Lemma 5.4 and and replacing $T_{w^{-1}}$ and $T_{w^{-1}}^{-1}$ with T_w^{-1} and T_w respectively we get that $\mathrm{Tr}^{\mathcal{EXT}'(L)}=\mathrm{Tr}^{\mathcal{EXT}(L)}$ so they are isomorphic by Lemma 5.1.

We will show for any $\alpha\in\Pi$ that

$$\mathcal{EXT}({}^{s_\alpha}L)\cong{}^{s_\alpha}\left(\mathcal{EXT}(L)\right)$$

which implies the claim by induction over the length $l(x)$ of x (where $l(x)$ is the smallest number of simple reflections need to write x, i.e. there is a reduced expression $x=s_{i_1}\cdots s_{i_{l(x)}}$).

So let $\alpha\in\Pi$ and let w and Σ be defined as above. Let $w'=ws_\alpha$. Note that $w'(F_{{}^{s_\alpha}L}\backslash F^s_{{}^{s_\alpha}L})\subset\Phi^+$ and $-\Sigma\subset T_{{}^{s_\alpha}L}$. We split into two cases: If $l(w')<l(w)$ then

$$\begin{aligned}{}^{s_\alpha}\left(\mathcal{EXT}(L)\right)=&{}^{s_\alpha}\left(\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}{}^{\overline{w's_\alpha}}\left(\varphi_{F_\Sigma,t}.({}^{w's_\alpha}L)_{F_\Sigma}\right)\right)^{ss}\right)\\\cong&{}^{s_\alpha}\left(\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}{}^{\overline{s_\alpha}}\left({}^{\overline{w'}}\left(\varphi_{F_\Sigma,t}.({}^{w'}({}^{s_\alpha}L))_{F_\Sigma}\right)\right)\right)^{ss}\right)\\\cong&\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}}{}^{\overline{w'}}\left(\varphi_{F_\Sigma,t}.({}^{w'}({}^{s_\alpha}L))_{F_\Sigma}\right)\right)^{ss}\\=&\mathcal{EXT}({}^{s_\alpha}L).\end{aligned}$$

If $l(w') > l(w)$ we get

$$\begin{aligned}
{}^{s_\alpha}\left(\mathcal{EXT}(L)\right) \cong& {}^{s_\alpha}\left(\mathcal{EXT}'(L)\right)\\
=& {}^{s_\alpha}\left(\left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}} {}^{w^{-1}}\left(\varphi_{F_\Sigma,t}.({}^{\overline{w^{-1}}}L)_{F_\Sigma}\right)\right)^{ss}\right)\\
\cong& \left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}} {}^{(w')^{-1}}\left(\varphi_{F_\Sigma,t}.({}^{\overline{(w')^{-1}}}({}^{s_\alpha}L))_{F_\Sigma}\right)\right)^{ss}\\
=& \mathcal{EXT}({}^{s_\alpha}L).
\end{aligned}$$

The second claim is shown similarly. □

Proposition 5.6 *Let L be an infinite dimensional admissible simple module of degree d. If L' is an infinite dimensional simple submodule of $\mathcal{EXT}(L)$ then L' is admissible of degree d and $\mathcal{EXT}(L) \cong \mathcal{EXT}(L')$.*

Proof. Let $w \in W$ and let Σ be a set of commuting roots that is a basis of Q such that $\Sigma \subset -w(T_L)$ (possible by Lemma 4.15 and Lemma 4.16). Then by definition

$$\mathcal{EXT}(L) = \left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}} {}^{\overline{w}}\left(\varphi_{F_\Sigma,t}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}.$$

L' being a submodule of $\mathcal{EXT}(L)$ implies that L' must be a submodule of

$$\left({}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{c}}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}$$

for some $\mathbf{c} \in (\mathbb{C}^*)^n$. Let $w' \in W$ and let Σ' be a set of commuting roots that is a basis of Q such that $\Sigma' \subset -w'(T_{L'})$. By Lemma 5.3 L' is admissible of degree d and there exists a bijective map $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ such that $f(q^{\mathbb{Z}^n}) = q^{\mathbb{Z}^n}$ and

$$\left({}^{\overline{w'}}\left(\varphi_{F_{\Sigma'},\mathbf{b}}.\left({}^{w'}L'\right)_{F_{\Sigma'}}\right)\right)^{ss} \cong \left({}^{\overline{w}}\left(\varphi_{F_\Sigma,f(\mathbf{b})\mathbf{c}}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}.$$

Since $f(q^{\mathbb{Z}^n}) = q^{\mathbb{Z}^n}$ it makes sense to write $f(t)$ for $t \in (\mathbb{C}^*)^n/q^{\mathbb{Z}^n}$. So writing $t_{\mathbf{c}} = q^{\mathbb{Z}^n}\mathbf{c} \in (\mathbb{C}^*)^n/q^{\mathbb{Z}^n}$ we get

$$\begin{aligned}
\mathcal{EXT}(L') =& \left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}} {}^{\overline{w'}}\left(\varphi_{F_\Sigma,t}.\left({}^{w'}(L')\right)_{F_{\Sigma'}}\right)\right)^{ss}\\
\cong& \left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}} {}^{\overline{w}}\left(\varphi_{F_\Sigma,f(t)t_{\mathbf{c}}}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}\\
\cong& \left(\bigoplus_{t\in(\mathbb{C}^*)^n/q^{\mathbb{Z}^n}} {}^{\overline{w}}\left(\varphi_{F_\Sigma,t}.\left({}^{w}L\right)_{F_\Sigma}\right)\right)^{ss}\\
=& \mathcal{EXT}(L)
\end{aligned}$$

since the assignment $t \mapsto f(t)t_{\mathbf{c}}$ is bijective. □

Lemma 5.7 *Let $f \in \mathbb{C}[X_1^{\pm 1},\dots,X_n^{\pm 1}]$ be a nonzero Laurent polynomial. There exists $b_1,\dots,b_n \in \mathbb{C}^*$ such that for all $i_1,\dots,i_n \in \mathbb{Z}$*

$$f(q^{i_1}b_1,\dots,q^{i_n}b_n) \neq 0.$$

Proof. Assume $f = X_1^{-N_1}\cdots X_n^{-N_n} g$ with $g \in \mathbb{C}[X_1,\dots,X_n]$. g has coefficients in some finitely generated (over $\mathbb{Q}$) subfield k of $\mathbb{C}$. Let $b_1,\dots b_n$ be generators of n disjoint extensions of k of degree $> \deg g$. The monomials $b_1^{m_1}\cdots b_n^{m_n}$, $0 \leq m_i \leq \deg g$ are all linearly independent over k. Since $q^i \neq 0$ for $i \in \mathbb{Z}$ the same is true for the monomials $(q^{i_1}b_1)^{m_1}\cdots(q^{i_n}b_n)^{m_n}$. So $g(q^{i_1}b_1,\dots,q^{i_n}b_n) \neq 0$, hence $f(q^{i_1}b_1,\dots,q^{i_n}b_n) \neq 0$. □

Theorem 5.8 *Let L be an infinite dimensional admissible simple modules of degree d. Then $\mathcal{EXT}(L)$ contains at least one simple torsion free module.*

Proof. Let $\lambda \in w(\operatorname{wt} L)$. Then as a $(U_q)_0$-module

$$\mathcal{EXT}(L) = \left({}^{\overline{w}}\left(\bigoplus_{\mathbf{b}\in(\mathbb{C}^*)^n} \varphi_{F_\Sigma,\mathbf{b}}.\left(({}^wL)_{F_\Sigma}\right)_\lambda \right)\right)^{ss}$$

for some $w \in W$ and some Ore subset F_Σ corresponding to a set of commuting roots Σ that is a basis of Q. Let $u \in (U_q)_0$. Then the map $\mathbf{b} \mapsto \det u|_{{}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{b}}.\left(({}^wL)_{F_\Sigma}\right)_\lambda\right)} = \det \varphi_{F_\Sigma,\mathbf{b}}(T_w^{-1}(u))|_{\left(({}^wL)_{F_\Sigma}\right)_\lambda}$ is Laurent polynomial. Let $p(\mathbf{b}) = \prod_{\beta\in\Sigma}\det E_\beta F_\beta|_{{}^{\overline{w}}\left(\varphi_{F_\Sigma,\mathbf{b}}.\left(({}^wL)_{F_\Sigma}\right)_\lambda\right)}$. p is a Laurent polynomial by the above. By Lemma 5.7 there exists a $\mathbf{c} \in (\mathbb{C}^*)^n$ such that $p(\mathbf{b}) \neq 0$ for all $\mathbf{b} \in q^{\mathbb{Z}^n}\mathbf{c}$ which implies that $E_\beta F_\beta$ acts injectively on the module $L' := {}^{\overline{w}}\left(\varphi_{F_\beta,\mathbf{c}}.({}^wL)_{F_\Sigma}\right)$ for all $\beta \in \Sigma$. Since F_β acts injectively on the module by construction this implies that E_β acts injectively as well. So we have $\pm\Sigma \subset T_{L'}$. Any simple submodule V of L' is admissible of degree d by Lemma 5.3 and since F_β and E_β act injectively we get $\dim V_\lambda = d = \dim L'_\lambda$ for any $\lambda \in \operatorname{wt} L'$ thus $V = L'$. So L' is a simple module. Using Proposition 2.5 it is easy to see that L' is torsion free since $\pm\Sigma \subset T_{L'}$ and Σ is a basis of Q. □

Proposition 5.9 *Let L be an infinite dimensional admissible simple module. Let $\beta \in \Phi^+$. If $-\beta \in T_L$ then $\mathcal{EXT}(L)$ contains $\left(\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}}\varphi_{F_\beta,t}.L_{F_\beta}\right)^{ss}$ as a U_q-submodule.*

Proof. Let $w \in W$ and $\Sigma = \{\beta_1,\dots,\beta_n\}$ be such that Σ is a set of commuting roots that is a basis of Q and $-\Sigma \subset w(T_L)$ and F_Σ a corresponding Ore subset (always possible by Lemma 4.15 and Lemma 4.16).

We have $w(\beta) = \sum_{i=1}^n a_i\beta_i$ for some $a_i \in \mathbb{Z}$. Set $x = F_{\beta_1}^{a_1}\cdots F_{\beta_n}^{a_n} \in U_{q(F_\Sigma)}$. Let $U_{q(x)}$ be the U_q-subalgebra generated by x in $U_{q(F_\Sigma)}$. x is playing the role of F_β and that is why the notation resembles the notation for Ohre localization. The Ohre localization of U_q in x does not neccesarily make sense though because x is not neccesarily an element of U_q.

Let V be the $U_{q(x)}$-submodule of $({}^wL)_{F_\Sigma}$ generated by $1 \otimes {}^wL$. For any $t \in \mathbb{C}^*/q^{\mathbb{Z}}$

$${}^{\overline{w}}\left(\varphi_{F_\Sigma,(t^{a_1},\dots,t^{a_n})}.V\right) := \left\{\varphi_{F_\Sigma,(t^{a_1},\dots,t^{a_n})}.v \in {}^{\overline{w}}\left(\varphi_{F_\Sigma,(t^{a_1},\dots,t^{a_n})}.({}^wL)_{F_\Sigma}\right) | v \in V\right\}$$

is a $U_{q(x)}$-submodule of ${}^{\overline{w}}\left(\varphi_{F_\Sigma,(t^{a_1},\dots,t^{a_n})}.({}^wL)_{F_\Sigma}\right)$: To show this we show that for $u \in U_{q(x)}$ and $c \in \mathbb{C}^*$, $\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}(u) \in U_{q(x)}$. We know that $\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}(u) \in U_{q(F_\Sigma)}[c^{\pm 1}]$ and we also see by construction that for $c = q^i$, $i \in \mathbb{Z}$, we have $\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}(u) = x^{-i}ux^i \in U_{q(x)}$. Choose a vector space basis of $U_{q(x)}$, $\{u_i\}_{i\in I}$ and extend to a basis $\{u_i, u_j'\}_{i\in I, j\in J}$ of $U_{q(F_\Sigma)}$ where I and J are some index sets. Then for $u \in U_{q(x)}$ we have $\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}(u) = \sum_{i\in I'} u_i p_i(c) + \sum_{j\in J'} u_j' p_j'(c)$ for some finite $I' \subset I$ and $J' \subset J$ and some $p_i, p_j' \in \mathbb{C}[X^{\pm 1}]$. We see that for $j \in J'$, $p_j'(q^i) = 0$ for all $i \in \mathbb{Z}$ so $p_j' = 0$. Hence $\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}(u) = \sum_{i\in I'} u_i p_i(c) \in U_{q(x)}$. This shows that ${}^{\overline{w}}\left(\varphi_{F_\Sigma,(t^{a_1},\dots,t^{a_n})}.V\right)$ is a submodule of ${}^{\overline{w}}\left(\varphi_{F_\Sigma,(t^{a_1},\dots,t^{a_n})}.({}^wL)_{F_\Sigma}\right)$. Set

$$\mathcal{V} = \left(\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}} {}^{\overline{w}}\left(\varphi_{F_\Sigma,(t^{a_1},\dots,t^{a_n})}.V\right)\right)^{ss}.$$

Clearly $\mathcal{V}$ is a U_q-submodule of $\mathcal{EXT}(L)$. We claim that $\mathcal{V} \cong \left(\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}} \varphi_{F_\beta,t}.L_{F_\beta}\right)^{ss}$ as U_q-modules. We will show this using Lemma 5.1.

Note that for $\lambda \in \operatorname{wt} V$ and $i \in \mathbb{Z}$ we have

$${}^{\overline{w}}\left(\varphi_{F_\Sigma,((q^i)^{a_1},\dots,(q^i)^{a_n})}.V_\lambda\right) \cong {}^{\overline{w}}\left(V_{q^{-i\sum_{k=1}^n a_k\beta_k}\lambda}\right)$$

as a $(U_q)_0$-module by Corollary 4.20.

We have $\operatorname{wt}\mathcal{V} = (\mathbb{C}^*)^\beta \operatorname{wt} L = \operatorname{wt}\left(\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}} \varphi_{F_\beta,t}.L_{F_\beta}\right)^{ss}$. Let $\lambda \in \operatorname{wt} L$ be such that $\dim L_\lambda = \max_{i\in\mathbb{Z}}\{\dim L_{q^{i\beta}\lambda}\}$ then $V_{w(\lambda)} \cong ({}^wL)_{w(\lambda)} \cong {}^w(L_\lambda)$ as a $(U_q)_0$-module by Lemma 4.22 and we have for $\nu \in (\mathbb{C}^*)^\beta\lambda$:

$$\mathcal{V}_\nu = \left(\bigoplus_{c\in\mathbb{C}^*: c^{w(\beta)} = w(\nu^{-1}\lambda)} {}^{\overline{w}}\left(\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}.V_{w(\lambda)}\right)\right)^{ss}$$

so for $u \in (U_q)_0$:

$$\operatorname{Tr} u|_{\mathcal{V}_\nu} = \sum_{c\in\mathbb{C}^*: c^\beta = \nu^{-1}\lambda} \operatorname{Tr}\left(\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}(T_w^{-1}(u))\right)|_{V_{w(\lambda)}}$$

(note that $c^{w(\beta)} = w(\nu^{-1}\lambda)$ if and only if $c^\beta = \nu^{-1}\lambda$ since $c^{w(\beta)} = w(c^\beta)$).

Set $p(c) = \operatorname{Tr}\left(\varphi_{F_\Sigma,(c^{a_1},\dots,c^{a_n})}(T_w^{-1}(u))\right)|_{V_{w(\lambda)}}$. p is Laurent polynomial in c and $p(q^i) = \operatorname{Tr} u|_{L_{q^{-i\beta}\lambda}}$ for $i \in \mathbb{N}$.

On the other hand we can show similarly that

$$\begin{aligned}&\operatorname{Tr} u|_{\left(\left(\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}} \varphi_{F_\beta,t}.L_{F_\beta}\right)^{ss}\right)_\nu} \\ &= \sum_{c\in\mathbb{C}^*: c^\beta = \nu^{-1}\lambda} \operatorname{Tr}\left(\varphi_{F_\beta,c}(u)\right)|_{(L_{F_\beta})_\lambda}.\end{aligned}$$

Similarly $\operatorname{Tr}\left(\varphi_{F_\beta,c}(u)\right)|_{(L_{F_\beta})_\lambda}$ is Laurent polynomial in c and equal to $\operatorname{Tr} u|_{L_{q^{-i\beta}\lambda}}$ for $c = q^i$, $i \in \mathbb{N}$. So $\operatorname{Tr}\left(\varphi_{F_\beta,c}(u)\right)|_{(L_{F_\beta})_\lambda} = p(c)$. We conclude that $\operatorname{Tr}^{\mathcal{V}} = \operatorname{Tr}^{\left(\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}} \varphi_{F_\beta,t}.L_{F_\beta}\right)^{ss}}$ so $\mathcal{V} \cong \left(\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}} \varphi_{F_\beta,t}.L_{F_\beta}\right)^{ss}$ as U_q-modules by Lemma 5.1.$\square$

For any $\lambda \in X$ there is a unique simple highest weight module which we call $L(\lambda)$. It is the unique simple quotient of the Verma module $M(\lambda) := U_q \otimes_{U_q^{\geq 0}} \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$ is the 1-dimensional $U_q^{\geq 0}$-module with U_q^+ acting trivially and U_q^0 acting like λ. Let $\rho = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta$. In the following we use the dot action on X. For $w \in W$, $w.\lambda := q^{-\rho}w(q^\rho\lambda)$.

Proposition 5.10 *Let $\lambda \in X$ be such that $L(\lambda)$ is admissible. Let $\alpha \in \Pi$. Assume $\lambda(K_\alpha) \notin \pm q^{\mathbb{N}}$. Let $a = \frac{2}{(\alpha|\alpha)}$. If $a = \frac{1}{2}$ choose a squareroot $\lambda(K_\alpha)^{\frac{1}{2}}$ of $\lambda(K_\alpha)$. Then*

- $-\alpha \in T_{L(\lambda)}$.
- *$L(s_\alpha.\lambda)$ is admissible.*
- *${}^{s_\alpha}L(s_\alpha.\lambda)$ is a subquotient of the U_q-module $L(\lambda)_{F_\alpha}$.*
- *$L(s_\alpha.\lambda)$ and ${}^{s_\alpha}L(\lambda)$ are subquotients of the U_q-module. $\varphi_{F_\alpha,\lambda(K_\alpha)^a}.L(\lambda)_{F_\alpha}$.*

Proof. $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$ implies that $-\alpha \subset T_{L(\lambda)}$ since for $i \in \mathbb{N}$:

$$E_\alpha^{(i)}F_\alpha^{(i)}v_\lambda = \prod_{j=1}^{i}\frac{q_\alpha^{j-1}\lambda(K_\alpha) - q_\alpha^{1-j}\lambda(K_\alpha)^{-1}}{q_\alpha^{j} - q_\alpha^{-j}}v_\lambda.$$

This is only zero for an $i \in \mathbb{N}$ when $\lambda(K_\alpha) \in \pm q_\alpha^{\mathbb{N}}$.

Let $v_\lambda \in L(\lambda)$ be a highest weight vector. Denote the vector $\varphi_{F_\alpha,\lambda(K_\alpha)^a}.F_\alpha v_\lambda \in \varphi_{F_\alpha,\lambda(K_\alpha)}.L(\lambda)_{F_\Sigma}$ as $v_{s_\alpha.\lambda}$. This is a highest weight vector of weight $s_\alpha.\lambda$: For $\mu \in Q$:

$$\begin{aligned}
K_\mu v_{s_\alpha.\lambda} =& K_\mu \varphi_{F_\alpha,\lambda(K_\alpha)^a}.F_\alpha v_\lambda \\
=& \varphi_{F_\alpha,q\lambda(K_\alpha)^a}.\left(\left(q\lambda(K_\alpha)^a\right)^{-(\mu|\alpha)}\lambda(K_\mu)F_\alpha v_\lambda\right) \\
=& q^{-(\mu|\alpha)}\lambda\left(K_\alpha^{-\langle\mu,\alpha^\vee\rangle}K_\mu\right)\varphi_{F_\alpha,q_\alpha\lambda(K_\alpha)}.F_\alpha v_\lambda \\
=& q^{-(\mu|\alpha)}(s_\alpha\lambda)(K_\mu)v_{s_\alpha.\lambda} \\
=& s_\alpha.\lambda(K_\mu)v_{s_\alpha.\lambda}.
\end{aligned}$$

For $\alpha' \in \Pi\backslash\{\alpha\}$

$$E_{\alpha'}\varphi_{F_\alpha,\lambda(K_\alpha)^a}.v_\lambda = \varphi_{F_\alpha,\lambda(K_\alpha)^a}.E_{\alpha'}v_\lambda$$

and for $\alpha' = \alpha$ we have by the formula in the proof of Lemma 4.9

$$\begin{aligned}
&E_\alpha\varphi_{F_\alpha,\lambda(K_\alpha)^a}.F_\alpha v_\lambda \\
&\quad= \varphi_{F_\alpha,\lambda(K_\alpha)^a}.F_\alpha\varphi_{F_\alpha,q\lambda(K_\alpha)^a}(E_\alpha)v_\lambda \\
&\quad= \varphi_{F_\alpha,\lambda(K_\alpha)^a}.F_\alpha\left(E_\alpha + F_\alpha^{-1}\frac{q_\alpha(q_\alpha\lambda(K_\alpha))^{-1}K_\alpha - q_\alpha^{-1}q_\alpha\lambda(K_\alpha)K_\alpha^{-1}}{(q_\alpha - q_\alpha^{-1})^2}\right)v_\lambda \\
&\quad= 0.
\end{aligned}$$

So $v_{s_\alpha.\lambda}$ is a highest weight vector of weight $s_\alpha.\lambda$ hence $L(s_\alpha.\lambda)$ is a subquotient of $\varphi_{F_\alpha,\lambda(K_\alpha)^a}.L(\lambda)_{F_\alpha}$. Since $L(s_\alpha.\lambda)$ is a subquotient of $\varphi_{F_\alpha,\lambda(K_\alpha)^a}.L(\lambda)_{F_\alpha}$ it is admissible by Lemma 4.22.

Consider ${}^{\overline{s_\alpha}}\left(\varphi_{F_\alpha,\lambda(K_\alpha)^a}.L(\lambda)_{F_\alpha}/(U_q v_{s_\alpha.\lambda})\right)$ and the vector

$$v' = F_\alpha^{-1} v_{s_\alpha.\lambda} + U_q v_{s_\alpha.\lambda} \in {}^{\overline{s_\alpha}}\left(\varphi_{F_\alpha,\lambda(K_\alpha)^a}.L(\lambda)_{F_\alpha}/(U_q v_{s_\alpha.\lambda})\right).$$

Then $E_\beta v' = 0$ for all $\beta \in \Pi$: First of all

$$\begin{aligned} E_\alpha \cdot v' =& T_{s_\alpha}^{-1}(E_\alpha) v' \\ =& - K_\alpha F_\alpha v' \\ =& - K_\alpha v_{s_\alpha.\lambda} + U_q v_{s_\alpha.\lambda} \\ =& 0. \end{aligned}$$

For $\beta \in \Pi \backslash \{\alpha\}$

$$\begin{aligned} E_\beta \cdot v' =& T_{s_\alpha}^{-1}(E_\beta) v'. \\ =& \sum_{i=0}^{-\langle \beta,\alpha^\vee \rangle} (-1)^i q_\alpha^{-i} E_\alpha^{(i)} E_\beta E_\alpha^{(-\langle \beta,\alpha^\vee\rangle - i)} v' \\ =& (-1)^{\langle \beta,\alpha^\vee\rangle} q_\alpha^{\langle \beta,\alpha^\vee\rangle} E_\alpha^{\left(-\langle \beta,\alpha^\vee\rangle\right)} E_\beta v' \\ =& (-1)^{\langle \beta,\alpha^\vee\rangle} q_\alpha^{\langle \beta,\alpha^\vee\rangle} E_\alpha^{\left(-\langle \beta,\alpha^\vee\rangle\right)} F_\alpha^{-1} E_\beta v_{s_\alpha.\lambda} + U_q v_{s_\alpha.\lambda} \\ =& 0 \end{aligned}$$

since $E_\alpha v' = 0$ and $E_\beta v_{s_\alpha.\lambda} = 0$ by the above.

So v' is a highest weight vector and v' has weight λ: For $\mu \in Q$:

$$\begin{aligned} K_\mu \cdot v' =& K_{s_\alpha \mu} v' \\ =& K_{s_\alpha \mu} F_\alpha^{-1} v_{s_\alpha.\lambda} + U_q v_{s_\alpha.\lambda} \\ =& q^{(s_\alpha(\mu)|\alpha)} s_\alpha.\lambda(K_{s_\alpha \mu}) F_\alpha^{-1} v_{s_\alpha.\lambda} + U_q v_{s_\alpha.\lambda} \\ =& \lambda(K_\mu) F_\alpha^{-1} v_{s_\alpha.\lambda} + U_q v_{s_\alpha.\lambda}. \end{aligned}$$

So $L(\lambda)$ is a subquotient of ${}^{\overline{s_\alpha}}(\varphi_{F_\alpha,\lambda(K_\alpha)^a}.L(\lambda)_{F_\alpha})$ hence ${}^{s_\alpha}L(\lambda)$ is a subquotient of $\varphi_{F_\alpha,\lambda(K_\alpha)^a}.L(\lambda)_{F_\alpha}$. Consider the vector

$$v'' = F_\alpha^{-1} v_\lambda + U_q v_\lambda \in {}^{\overline{s_\alpha}}\left(L(\lambda)_{F_\alpha}/(U_q v_\lambda)\right).$$

By an argument analog to above we get $E_\beta \cdot v'' = 0$ for all $\beta \in \Pi \backslash \{\alpha\}$ since E_β and F_α^{-1} commutes and v_λ is a highest weight vector. We get $E_\alpha \cdot v'' = 0$ by the following:

$$\begin{aligned} E_\alpha \cdot v'' =& T_{s_\alpha}^{-1}(E_\alpha) v'' \\ =& - K_\alpha F_\alpha v'' \\ =& - q^{-2} F_\alpha K_\alpha F_\alpha^{-1} v_\lambda + U_q v_\lambda \\ =& 0. \end{aligned}$$

So v'' is a highest weight vector in ${}^{\overline{s_\alpha}}\left(L(\lambda)_{F_\alpha}/(U_q v_\lambda)\right)$. v'' has weight $s_\alpha.\lambda$: For $\mu \in Q$:

$$\begin{aligned} K_\mu \cdot v'' =& K_{s_\alpha \mu} v'' \\ =& K_{s_\alpha \mu} F_\alpha^{-1} v_\lambda + U_q v_\lambda \\ =& q^{(s_\alpha(\mu)|\alpha)} \lambda(K_{s_\alpha \mu}) v'' \\ =& (q^{-\alpha} s_\alpha \lambda)(K_\mu) v''. \end{aligned}$$

Hence $L(s_\alpha.\lambda)$ is a subquotient of $\overline{^{s_\alpha}}L(\lambda)_{F_\Sigma}$ and therefore $^{s_\alpha}L(s_\alpha.\lambda)$ is a subquotient of $L(\lambda)_{F_\Sigma}$. $\square$

Lemma 5.11 *Let $\lambda \in X$ be such that $L(\lambda)$ is an infinite dimensional admissible module of degree d. Let $\alpha \in \Pi$. Then*

$$\mathcal{EXT}(L(\lambda)) \cong \mathcal{EXT}(^{s_\alpha}L(\lambda))$$

and if $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$ then $\mathcal{EXT}(L(\lambda))$ contains $L(s_\alpha.\lambda)$ and $^{s_\alpha}L(s_\alpha.\lambda)$ as U_q-submodules, where $s_\alpha.\lambda := q^{-\rho}s_\alpha(q^\rho\lambda) = q^{-\alpha}s_\alpha\lambda$.

Proof. Assume first that $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$. By Proposition 5.10 the U_q-module $\bigoplus_{t\in\mathbb{C}^*/q^{\mathbb{Z}}} \varphi_{F_\alpha,t}.L(\lambda)_{F_\alpha}$ contains $L(s_\alpha.\lambda)$, $^{s_\alpha}L(\lambda)$ and $^{s_\alpha}L(s_\alpha.\lambda)$ as subquotients. By Proposition 5.9 and Proposition 5.6 this finishes the proof of the claim when $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$.

Assume now that $\lambda(K_\alpha) = \pm q_\alpha^k$ for some $k \in \mathbb{N}$: If $\lambda(K_\alpha) = q_\alpha^k$ it is easy to prove that $L(\lambda) \cong {^{s_\alpha}L(\lambda)}$. Assume from now on that $\lambda(K_\alpha) = -q_\alpha^k$. We have

$$\mathcal{EXT}(L(\lambda)) = \left(\bigoplus_{t\in(\mathbb{C}^*)/q^{\mathbb{Z}^n}} \varphi_{F_\Sigma,t}.L(\lambda)_{F_\Sigma}\right)^{ss}$$

for some set of commuting roots $\Sigma = \{\beta_1,\dots,\beta_n\}$ that is a basis of Q with $-\Sigma \subset T_{L(\lambda)}$. Since Σ is a basis of Q there exists $a_1,\dots,a_n \in \mathbb{Z}$ such that $\alpha = \sum_{i=1}^n a_i\beta_i$. Let v_λ be a highest weight vector in $L(\lambda)$. We will show that $v_0 := \varphi_{F_\Sigma,((-1)^{a'_1},\dots,(-1)^{a'_n})}.F_\alpha^i v_\lambda \in \overline{^{s_\alpha}}\mathcal{EXT}(L(\lambda))$ is a highest weight vector of weight λ where $a'_i = \frac{2a_i}{(\alpha|\alpha)}$. This will imply $\mathcal{EXT}(^{s_\alpha}L(\lambda)) \cong \mathcal{EXT}(L(\lambda))$ by Proposition 5.6. The weight of v_0: Let $\mu \in Q$:

$$\begin{aligned}
K_\mu \cdot v_0 =& K_{s_\alpha(\mu)}\varphi_{F_\Sigma,((-1)^{a'_1},\dots,(-1)^{a'_n})}.F_\alpha^i v_\lambda\\
=&(-1)^{\left(\sum_{i=1}^n a'_i\beta_i|\mu\right)}q^{i(\alpha|\mu)}\lambda(K_\mu K_\alpha^{-\langle\mu,\alpha^\vee\rangle})\varphi_{F_\Sigma,((-1)^{a'_1},\dots,(-1)^{a'_n})}.F_\alpha^i v_\lambda\\
=&(-1)^{\langle\mu,\alpha^\vee\rangle}q_\alpha^{i\langle\mu,\alpha^\vee\rangle}(-q_\alpha^i)^{-\langle\mu,\alpha^\vee\rangle}\lambda(K_\mu)v_0\\
=&\lambda(K_\mu)v_0.
\end{aligned}$$

By Proposition 4.10 $\varphi_{F_\beta,(-1)^{\frac{2}{(\beta|\beta)}}}(E_{\alpha'}) = E_{\alpha'}$ and $\varphi_{F_\beta,(-1)^{\frac{2}{(\beta|\beta)}}}(F_{\alpha'}) = \pm F_{\alpha'}$ for any $\alpha' \in \Pi$ and any $\beta \in \Phi^+$. So $\varphi_{F_\Sigma,((-1)^{a'_1},\dots,(-1)^{a'_n})}(E_\beta)$, $\beta \in \Pi\backslash\{\alpha\}$ and $\varphi_{F_\Sigma,((-1)^{a'_1},\dots,(-1)^{a'_n})}(F_\alpha)$ kills $F_\alpha^i v_\lambda \in L(\lambda)$ because E_β and F_α does. Hence E_β, $\beta \in \Pi$ kills v_0 by the same argument as in the proof of Proposition 5.10 when proving that v' is a highest weight vector. $\square$

Theorem 5.12 *Let L be an infinite dimensional admissible simple module of degree d. Then the U_q-module $\mathcal{EXT}(L)$ contains an infinite dimensional admissible simple highest weight module $L(\lambda)$ of degree d for some weight $\lambda \in X$. Furthermore for any $x \in W$:*

$$^{x}\mathcal{EXT}(L) \cong \mathcal{EXT}(L).$$

Proof. Let $w \in W$ be such that $w(F_L\backslash F_L^s) \subset \Phi^+$ and $w(T_L\backslash T_L^s) \subset \Phi^-$. Set $L' = \overline{^{w^{-1}}}L$ (then $^{w^{-1}}L' = L$). We will show the result first for L' by induction on $|T_{L'}^+|$. If $|T_{L'}^+| = 0$ then L' is itself a highest weight module. Assume $|T_{L'}^+| > 0$. Let $\beta \in T_{L'}^+$. Then $\beta \in T_{L'}^s$ since $T_{L'} \setminus T_{L'}^s \subset \Phi^-$. So $-\beta \in T_{L'}$. Then by Lemma 4.23 there exists a $b \in \mathbb{C}^*$ such that $\varphi_{F_\beta,b}.L'_{F_\beta}$ contains a U_q-submodule L'' with $T_{L''} \subset T_{L'}$ and $\beta \notin T_{L''}$. By Proposition 5.9 and Proposition 5.6 $\mathcal{EXT}(L') \cong \mathcal{EXT}(L'')$ as U_q-modules. By induction $\mathcal{EXT}(L'')$ contains an infinite dimensional admissible simple highest weight module $L(\lambda)$ for some λ. So $\mathcal{EXT}(L') \cong \mathcal{EXT}(L(\lambda))$ by Proposition 5.6. Choose a reduced expression $s_{i_r} \cdots s_{i_1}$ for w^{-1}. By Proposition 5.5 and Lemma 5.11

$$\begin{aligned}\mathcal{EXT}(L) \cong & \mathcal{EXT}({}^{w^{-1}}L') \\ \cong & {}^{w^{-1}}\mathcal{EXT}(L') \\ \cong & {}^{w^{-1}}\mathcal{EXT}(L(\lambda)) \\ \cong & {}^{s_{i_r}\cdots s_{i_2}}\mathcal{EXT}({}^{s_{i_1}}L(\lambda)) \\ \cong & {}^{s_{i_r}\cdots s_{i_2}}\mathcal{EXT}(L(\lambda)) \\ & \vdots \\ \cong & \mathcal{EXT}(L(\lambda)).\end{aligned}$$

So $\mathcal{EXT}(L)$ contains a simple highest weight module $L(\lambda)$. For any $x \in W$ we can do as above to show ${}^x\mathcal{EXT}(L) \cong \mathcal{EXT}({}^xL(\lambda)) \cong \mathcal{EXT}(L(\lambda)) \cong \mathcal{EXT}(L)$.□

Corollary 5.13 *Let L be a simple torsion free module. Then there exists a set of commuting roots Σ that is a basis of Q with corresponding Ore subset F_Σ, a $\lambda \in X$ and $\mathbf{b} \in (\mathbb{C}^*)^n$ such that $-\Sigma \subset T_{L(\lambda)}$ and $L \cong \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$*

Proof. By Theorem 5.12 $\mathcal{EXT}(L) \cong \mathcal{EXT}(L(\lambda))$ for some $\lambda \in X$. So L is a U_q-submodule of $\mathcal{EXT}(L(\lambda))$. Let Σ be a set of commuting roots such that $-\Sigma \subset L(\lambda)$ (exists by Lemma 4.16 by setting $w = e$, the neutral element in W) then

$$\mathcal{EXT}(L(\lambda)) = \left(\bigoplus_{t \in (\mathbb{C}^*)^n / q^{\mathbb{Z}^n}} \varphi_{F_\Sigma,t}.L(\lambda)_{F_\Sigma} \right)^{ss}.$$

Since L is simple we must have that L is a submodule of $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ for some $\mathbf{b} \in (\mathbb{C}^*)^n$. By Proposition 5.6 and Lemma 4.22 $\dim \left(\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}\right)_\lambda = \dim L_\lambda$ for all $\lambda \in \operatorname{wt} L$ so we have $L \cong \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. □

So to classify torsion free simple modules we need to classify the admissible infinite dimensional simple highest weight modules $L(\lambda)$ and then we need to determine the $t \in (\mathbb{C}^*)^n / q^{\mathbb{Z}^n}$ such that $\varphi_{F_\Sigma,t}.L(\lambda)_{F_\Sigma}$ is simple. Furthermore we have that if there exists an admissible infinite dimensional simple module then there exists a torsion free simple module. In the classical case torsion free modules only exists if $\mathfrak{g}$ is of type A or C so we expect the same to be true in the quantum group case. We show this in Section 8.5.

6 Classification of simple torsion free $U_q(\mathfrak{sl}_2)$-modules

In this section let $\mathfrak{g} = \mathfrak{sl}_2$. In this case there is a single simple root α. It is natural to identify X with $\mathbb{C}^*$ via $\lambda \mapsto \lambda(K_\alpha)$. We define $F = F_\alpha$, $E = E_\alpha$ and

$K^{\pm 1} = K_\alpha^{\pm 1}$. Let $\lambda \in \mathbb{C}^* \backslash \{\pm q^{\mathbb{N}}\}$ and consider the simple highest weight module $L(\lambda)$. Let $0 \neq v_0 \in L(\lambda)_\lambda$. $\operatorname{wt} L = q^{-2\mathbb{N}}\lambda$ so $L(\lambda)$ is an admissible infinite dimensional highest weight module. Thus $\mathcal{EXT}(L(\lambda))$ contains a torsion free module by Theorem 5.8. Let $b \in \mathbb{C}^*$. We will describe the action on the module $\varphi_{F,b}.L(\lambda)_{(F)}$ and determine exactly for which b's $\varphi_{F,b}.L(\lambda)_{(F)}$ is torsion free.

Let $v_i = F^i \varphi_{F,b}.v_0$ for all $i \in \mathbb{Z}$. Then we have for $i \in \mathbb{Z}$

$$\begin{aligned} Fv_i =& v_{i+1} \\ K^{\pm 1} v_i =& q^{-2i} b^{\mp 2} \lambda v_i \\ Ev_i =& \frac{(q^i b - q^{-i} b^{-1})(q^{1-i} b^{-1}\lambda - q^{i-1} b \lambda^{-1})}{(q-q^{-1})^2} v_{i-1}. \end{aligned}$$

We see that unless $b = \pm q^i$ or $b = \pm q^i \lambda$ for some $i \in \mathbb{Z}$ then $\varphi_{F,b}.L(\lambda)_{(F)}$ is torsion free. In this case we see that $\varphi_{F,-b} = \varphi_{F,b}$ since for all $u \in U_q(\mathfrak{sl}_2)$, $\varphi_{F,b}(u)$ is Laurent polynomial in b^2.

So in this case $\mathcal{EXT}(L(\lambda))$ contains a maximum of four different simple submodules which are *not* torsion free: We have $(\varphi_{F,\pm q^i}.L(\lambda)_{(F)})^{ss} \cong (L(\lambda)_{(F)})^{ss} \cong L(\lambda) \oplus {}^{s_\alpha}L(s_\alpha.\lambda)$ (which can be seen directly from the calculations but also follows from Corollary 4.20 and the fact that $\varphi_{F,-b} = \varphi_{F,b}$) and $(\varphi_{F,\pm q^i\lambda}.L(\lambda)_{(F)})^{ss} \cong (L(s_\alpha.\lambda)_{(F)})^{ss} \cong L(s_\alpha.\lambda) \oplus {}^{s_\alpha}L(\lambda)$ if $\lambda \notin \pm q^{\mathbb{Z}}$.

The weights of $\varphi_{F,b}.L(\lambda)_{(F)}$ are $b^{-\alpha} \operatorname{wt} L(\lambda)_{(F)} = q^{2\mathbb{Z}} b^{-2}\lambda$. Suppose we want to find a torsion free $U_q(\mathfrak{sl}_2)$-modules with integral weights. Then we just need to find $\lambda, b \in \mathbb{C}^*$ such that $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$, $b \notin \pm q^{\mathbb{Z}}$ and $b \notin \pm q^{\mathbb{Z}}\lambda$ such that $b^{-2}\lambda \in q^{\mathbb{Z}}$. For example choose a square root $q^{1/2}$ of q and set $\lambda = q^{-1}$ and $b = q^{1/2}$. Then we have a torsion free module $L = \operatorname{span}_{\mathbb{C}} \{v_i | i \in \mathbb{Z}\}$ with action given by:

$$\begin{aligned} Fv_i =& v_{i+1} \\ Kv_i =& q^{-2i-2} v_i \\ Ev_i =& \frac{(q^{1/2+i} - q^{-1/2-i})(q^{-1/2-i} - q^{i+1/2})}{(q-q^{-1})^2} v_{i-1} \\ =& \frac{q(q^{-i-1} - q^i)^2}{(q-q^{-1})^2} v_{i-1}. \end{aligned}$$

In this paper we only focus on quantized enveloping algebras over $\mathbb{C}$ but note that we can define, for a general field $\mathbb{F}$ with $q \in \mathbb{F}\backslash\{0\}$ a non-root of unity, a simple torsion free $U_{\mathbb{F}}(\mathfrak{sl}_2)$-module with integral weights by the above formulas (here $U_{\mathbb{F}}(\mathfrak{sl}_2) = U_A \otimes_A \mathbb{F}$ where $\mathbb{F}$ is considered an A-algebra by sending v to q).

7 An example for $U_q(\mathfrak{sl}_3)$

In this section we will show how we can construct a specific torsion free simple module for $U_q(\mathfrak{sl}_3)$. In Section 9 we classify all torsion free $U_q(\mathfrak{sl}_n)$-modules with $n \geq 3$ so this example is of course included there. If you are only interested in the general classification you can skip this section but the calculations in this section gives a taste of the calculations needed in the general case in Section 9 and they show a phenomona that does not happen in the classical case.

Let α_1 and α_2 be the two simple roots of the root system. We will consider the set of commuting roots $\Sigma = \{\beta_1, \beta_2\}$ where $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_1 + \alpha_2$. Set $F_{\beta_1} :=$

F_{α_1} and $F_{\beta_2} := T_{s_1}(F_{\alpha_2}) = F_{\alpha_2}F_{\alpha_1} - qF_{\alpha_1}F_{\alpha_2} = [F_{\alpha_2}, F_{\alpha_1}]_q$. We have $(\beta_1|\beta_2) = 1$ and $0 = [F_{\beta_2}, F_{\beta_1}]_q = F_{\beta_2}F_{\beta_1} - q^{-1}F_{\beta_1}F_{\beta_2}$ or equivalently $F_{\beta_1}F_{\beta_2} = qF_{\beta_2}F_{\beta_1}$. Let $\lambda \in X$ be determined by $\lambda(K_{\alpha_1}) = q^{-1}$ and $\lambda(K_{\alpha_2}) = 1$. Then $M(s_{\alpha_1}.\lambda)$ is a submodule of $M(\lambda)$ and $L(\lambda) = M(\lambda)/M(s_{\alpha_2}.\lambda) = M(\lambda)/M(q^{-\alpha_2}\lambda)$ is admissible of degree 1. Let $\xi = e^{2\pi i/3}$. We will show that $\varphi_{F_\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$ is a torsion free module. We have here a phenomona that does not happen in the classical case: $\operatorname{wt} L(\lambda)_{F_\Sigma} = \operatorname{wt} \varphi_{F_\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$ but $L(\lambda)_{F_\Sigma} \not\cong \varphi_{F_\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$ as U_q-modules since one is simple and torsion free and the other isn't (compare to [Mat00, Section 10] where Mathieu classifies the torsion free simple modules by determining for a coherent family $\mathcal{M}$ for which cosets $t \in \mathfrak{h}^*/Q$, $\mathcal{M}[t]$ is torsion free).

We will show that E_{α_1} and E_{α_2} act injectively on the module $\varphi_{F_\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$. So we need to calculate $\varphi_{F_\Sigma,(\xi,\xi)}(E_{\alpha_1})$ and $\varphi_{F_\Sigma,(\xi,\xi)}(E_{\alpha_2})$. $\varphi_{F_\Sigma,(\xi,\xi)} = \varphi_{F_{\beta_1},\xi} \circ \varphi_{F_{\beta_2},\xi}$. We have

$$\begin{aligned}[E_{\alpha_1}, F_{\beta_2}] =& F_{\alpha_2}[E_{\alpha_1}, F_{\alpha_1}] - q[E_{\alpha_1}, F_{\alpha_1}]F_{\alpha_2} \\ =& F_{\alpha_2}\frac{K_{\alpha_1} - K_{\alpha_1}^{-1}}{q-q^{-1}} - qF_{\alpha_2}\frac{qK_{\alpha_1} - q^{-1}K_{\alpha_1}^{-1}}{q-q^{-1}} \\ =& F_{\alpha_2}\frac{K_{\alpha_1} - q^2K_{\alpha_1}}{q-q^{-1}} \\ =& -F_{\alpha_2}q\frac{q-q^{-1}}{q-q^{-1}}K_{\alpha_1} \\ =& -qF_{\alpha_2}K_{\alpha_1}.\end{aligned}$$

We can show by induction that

$$[E_{\alpha_1}, F_{\beta_2}^j] = -q^{2-j}[j]F_{\beta_2}^{j-1}F_{\alpha_2}K_{\alpha_1}$$

for any $j \in \mathbb{N}$. Using that $\varphi_{F_{\beta_2},b}(E_{\alpha_1})$ is Laurent polynomial and equal to $F_{\beta_2}^{-j}E_{\alpha_1}F_{\beta_2}^j$ for $b = q^j$ we get

$$\varphi_{F_{\beta_2},b}(E_{\alpha_1}) = E_{\alpha_1} - q^2b^{-1}\frac{b-b^{-1}}{q-q^{-1}}F_{\beta_2}^{-1}F_{\alpha_2}K_{\alpha_1}.$$

We have $F_{\beta_2}F_{\beta_1} = q^{-1}F_{\beta_1}F_{\beta_2}$ so $F_{\beta_1}^{-i}F_{\beta_2}F_{\beta_1}^i = q^{-i}F_{\beta_2}$ thus $\varphi_{F_{\beta_1},b}(F_{\beta_2}^{-1}) = bF_{\beta_2}^{-1}$. We have

$$\begin{aligned}\varphi_{F_{\alpha_1},b}(F_{\alpha_2}) =& bF_{\alpha_2} - \frac{b-b^{-1}}{q-q^{-1}}F_{\alpha_1}^{-1}(qF_{\alpha_1}F_{\alpha_2} - F_{\alpha_2}F_{\alpha_1}) \\ =& bF_{\alpha_2} + \frac{b-b^{-1}}{q-q^{-1}}F_{\alpha_1}^{-1}F_{\beta_2}\end{aligned}$$

and

$$\begin{aligned}
&\varphi_{F_{\beta_1},b_1}(\varphi_{F_{\beta_2},b_2}(E_{\alpha_1}))\\
&\quad=\varphi_{F_{\alpha_1},b_1}\left(E_{\alpha_1}-q^2b_2^{-1}\frac{b_2-b_2^{-1}}{q-q^{-1}}F_{\beta_2}^{-1}F_{\alpha_2}K_{\alpha_1}\right)\\
&\quad=E_{\alpha_1}+F_{\alpha_1}^{-1}\frac{(b_1-b_1^{-1})(qb_1^{-1}K_{\alpha_1}-q^{-1}b_1K_{\alpha_1}^{-1})}{(q-q^{-1})^2}\\
&\qquad-q^2b_2^{-1}\frac{b_2-b_2^{-1}}{q-q^{-1}}b_1F_{\beta_2}^{-1}\left(b_1F_{\alpha_2}+\frac{b_1-b_1^{-1}}{q-q^{-1}}F_{\alpha_1}^{-1}F_{\beta_2}\right)b_1^{-2}K_{\alpha_1}\\
&\quad=E_{\alpha_1}+F_{\alpha_1}^{-1}\frac{(b_1-b_1^{-1})(qb_1^{-1}K_{\alpha_1}-q^{-1}b_1K_{\alpha_1}^{-1})}{(q-q^{-1})^2}\\
&\qquad-q^2b_2^{-1}\frac{b_2-b_2^{-1}}{q-q^{-1}}F_{\beta_2}^{-1}F_{\alpha_2}K_{\alpha_1}\\
&\qquad-qb_1^{-1}b_2^{-1}\frac{(b_2-b_2^{-1})(b_1-b_1^{-1})}{(q-q^{-1})^2}F_{\alpha_1}^{-1}K_{\alpha_1}\\
&\quad=E_{\alpha_1}+b_2^{-1}F_{\alpha_1}^{-1}\frac{(b_1-b_1^{-1})(qb_1^{-1}b_2^{-1}K_{\alpha_1}-q^{-1}b_1b_2K_{\alpha_1}^{-1})}{(q-q^{-1})^2}\\
&\qquad-q^2b_2^{-1}\frac{b_2-b_2^{-1}}{q-q^{-1}}F_{\beta_2}^{-1}F_{\alpha_2}K_{\alpha_1}.
\end{aligned}$$

Let v'_λ be a highest weight vector in $L(\lambda)$ and set $v_\lambda=1\otimes v'_\lambda\in L(\lambda)_{F_\Sigma}$. We have $F_{\alpha_2}v_\lambda=0$ by construction so we have

$$\varphi_{F_\Sigma,(b_1,b_2)}(E_{\alpha_1})v_\lambda=b_2^{-1}\frac{(b_1-b_1^{-1})(b_1^{-1}b_2^{-1}-b_1b_2)}{(q-q^{-1})^2}F_{\alpha_1}^{-1}v_\lambda.$$

$\varphi_{F_\Sigma,(c_1,c_2)}.L(\lambda)_{F_\Sigma}$ is spanned by the elements $F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.v_\lambda$, $i,j\in\mathbb{Z}$ because every weight space is one-dimensional and $F_{\beta_1}^iF_{\beta_2}^j$ acts injectively. Since

$$\begin{aligned}
F_{\beta_2}^{-j}F_{\beta_1}^{-i}E_{\alpha_1}F_{\beta_1}^iF_{\beta_2}^j&=F_{\beta_1}^{-i}F_{\beta_2}^{-j}E_{\alpha_1}F_{\beta_2}^jF_{\beta_1}^i\\
&=\varphi_{F_{\beta_1},q^i}(\varphi_{F_{\beta_2},q^j}(E_{\alpha_1}))\\
&=\varphi_{F_\Sigma,(q^i,q^j)}(E_{\alpha_1})
\end{aligned}$$

we have

$$\begin{aligned}
&E_{\alpha_1}F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.v_\lambda\\
&\quad=F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(q^i,q^j)}(E_{\alpha_1})\varphi_{F_\Sigma,(c_1,c_2)}.v_\lambda\\
&\quad=F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.\varphi_{F_\Sigma,(q^ic_1,q^jc_2)}(E_{\alpha_1})v_\lambda\\
&\quad=q^{-j}c_2^{-1}\frac{(q^ic_1-q^{-i}c_1^{-1})(q^{-i-j}c_1^{-1}c_2^{-1}-q^{i+j}c_1c_2)}{(q-q^{-1})^2}F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.F_{\alpha_1}^{-1}v_\lambda\\
&\quad=\frac{(q^ic_1-q^{-i}c_1^{-1})(q^{-i-j}c_1^{-1}c_2^{-1}-q^{i+j}c_1c_2)}{(q-q^{-1})^2}F_{\beta_1}^{i-1}F_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.v_\lambda.
\end{aligned}$$

This is only zero when $c_1=\pm q^{-i}$ or $c_1c_2=\pm q^{-i-j}$. Set $c_1=c_2=e^{2\pi i/3}=:\xi$. Then we have shown that E_{α_1} acts injectively on $\varphi_{F_\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$.

Now we will show that E_{α_2} acts injectively on $F_{\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$. We can show by induction that

$$[E_{\alpha_2}, F_{\beta_2}^j] =[j]F_{\alpha_1}F_{\beta_2}^{j-1}K_{\alpha_2}^{-1}$$

so $\varphi_{F_{\beta_2},b}(E_{\alpha_2}) = E_{\alpha_2} + b\frac{b-b^{-1}}{q-q^{-1}}F_{\alpha_1}F_{\beta_2}^{-1}K_{\alpha_2}^{-1}$ and

$$\begin{aligned}\varphi_{F_\Sigma,(b_1,b_2)}(E_{\alpha_2}) =&\varphi_{F_{\beta_1},b_1}(\varphi_{F_{\beta_2},b_2}(E_{\alpha_2}))\\ =&E_{\alpha_2} + b_2\frac{b_2 - b_2^{-1}}{q-q^{-1}}F_{\alpha_1}F_{\beta_2}^{-1}K_{\alpha_2}^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}E_{\alpha_2}F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.v_\lambda =&F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(q^i,q^j)}(E_{\alpha_2})\varphi_{F_\Sigma,(c_1,c_2)}.v_\lambda\\ =&F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.\varphi_{F_\Sigma,(q^ic_1,q^jc_2)}(E_{\alpha_2})v_\lambda\\ =&F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(c_1,c_2)}.c_2\frac{q^jc_2 - q^{-j}c_2^{-1}}{q-q^{-1}}F_{\alpha_1}F_{\beta_2}^{-1}K_{\alpha_2}^{-1}v_\lambda\\ =&q^{-j-1}c_2\frac{q^jc_2 - q^{-j}c_2^{-1}}{q-q^{-1}}F_{\beta_1}^{i+1}F_{\beta_2}^{j-1}\varphi_{F_\Sigma,(c_1,c_2)}.v_\lambda.\end{aligned}$$

We see that this is nonzero only if $c_2 = \pm q^{-j}$ so again setting $c_1 = c_2 = \xi$ ensures that this is nonzero.

We have shown that the U_q-module $\varphi_{F_\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$ has a basis $F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda$, $i,j \in \mathbb{Z}$ and we have

$$\begin{aligned}F_{\beta_1}F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda =&F_{\beta_1}^{i+1}F_{\beta_2}^{j-1}\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda\\ F_{\beta_2}F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda =&q^{-j}F_{\beta_1}^iF_{\beta_2}^{j+1}\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda\\ E_{\alpha_1}F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda =&C_1F_{\beta_1}^{i-1}F_{\beta_2}^j\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda\\ E_{\alpha_1}E_{\alpha_2}F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda =&C_2F_{\beta_1}^iF_{\beta_2}^{j-1}\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda\end{aligned}$$

for some nonzero constants $C_1, C_2 \in \mathbb{C}^*$. We see that any of the basis vectors $F_{\beta_1}^iF_{\beta_2}^j\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda$ can be mapped injectively to any other basis vector $F_{\beta_1}^{i'}F_{\beta_2}^{j'}\varphi_{F_\Sigma,(\xi,\xi)}.v_\lambda$ by elements of U_q so $\varphi_{F_\Sigma,(\xi,\xi)}.L(\lambda)_{F_\Sigma}$ is a simple module. The module is torsion free by Proposition 2.5.

8 Classification of admissible simple highest weight modules

8.1 Preliminaries

In this section we prove some preliminary results with the goal to classify all admissible simple highest weight modules. We will only focus on non-integral weights since we have the following theorem from [AM15]:

Theorem 8.1 *Assume $q \in \mathbb{C}\backslash\{0\}$ is transcendental. Let $\lambda : U_q^0 \to \mathbb{C}$ be a weight such that $\lambda(K_\alpha) = q_\beta^i$ for some $i \in \mathbb{Z}$ for every $\alpha \in \Pi$ - i.e. $\lambda \in q^Q$. Say $\lambda = q^\mu$, $\mu \in Q$. Let $L_\mathbb{C}(\mu)$ denote the simple highest weight $\mathfrak{g}$-module of highest weight μ. Then the character of $L(\lambda)$ and $L_\mathbb{C}(\mu)$ are equal - i.e. for any $\nu \in Q$, $\dim L(\lambda)_{q^\nu\lambda} = \dim L_\mathbb{C}(\mu)_{\nu+\mu}$.*

Proof. [AM15, Corollary 6.3]. □

Extending to modules which are not of type 1 is done in the usual way (cf. e.g. [Jan96, Section 5.1–5.2]). The above theorem implies that the integral admissible simple highest weight modules can be classified from the classification of the classical admissible simple highest weight modules when q is transcendental. Hence we need only to consider weights $\lambda \in X$ such that $\lambda(K_\alpha) \notin \pm q^{\mathbb{Z}}$ for at least one $\alpha \in \Pi$ in this case. *So in the rest of the paper we will restrict our attention to the case when q is transcendental.* If a similar theorem is true for any non-root-of-unity q then the results in this paper extend to all non-root-of-unities but the author is not aware of any such result.

Theorem 8.2 *Let $\lambda \in X$. Then there exists a filtration of $M(\lambda)$, $M(\lambda) \supset M_1 \supset \cdots \supset M_r$ such that M_1 is the unique maximal submodule of $M(\lambda)$ and*

$$\sum_{i=1}^{r} \operatorname{ch} M_i = \sum_{\substack{\beta \in \Phi^+ \\ q^\rho \lambda(K_\beta) \in \pm q_\beta^{\mathbb{Z}_{>0}}}} \operatorname{ch} M(s_\beta.\lambda)$$

The filtration is called the Jantzen filtration and the formula is called the Jantzen sum formula.

Proof. This is proved in [Jos95, Section 4.1.2-4.1.3]. A proof using twisting functors can also be found in [Ped15b, Theorem 6.3]. □

Definition 8.3 *Let $\lambda \in X$.*

$$A(\lambda) = \{\alpha \in \Pi | \lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}\}.$$

Let $\gamma \in \Pi$.

$$D(\gamma) = \{\beta \in \Phi^+ | \beta = \sum_{\alpha \in \Pi} m_\alpha \alpha,\, m_\gamma > 0\}.$$

Lemma 8.4 *Let $\lambda \in X$. Let $\gamma \in \Pi$ be such that $\gamma \in A(\lambda)$. Then $-D(\gamma) \subset T_{L(\lambda)}$.*

Proof. Let $\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha \in D(\gamma)$. We prove by induction over $\operatorname{ht} \beta = \sum_{\alpha \in \Pi} m_\alpha$ that $-\beta \in T_{L(\lambda)}$. If $\operatorname{ht} \beta = 1$ then $\beta = \gamma$ and $-\gamma \in T_{L(\lambda)}$ by Proposition 5.10.

Assume $\operatorname{ht} \beta > 1$. Then $\beta - \alpha \in \Phi^+$ for some $\alpha \in \Pi$. We have either $\alpha = \gamma$ or $\beta - \alpha \in D(\gamma)$. In either case we get $\beta = \beta' + \beta''$ for some $\beta', \beta'' \in \Phi^+$ with $\beta' \in D(\gamma)$ and $\operatorname{ht} \beta' < \operatorname{ht} \beta$. By induction $-\beta' \in T_{L(\lambda)}$. If $-\beta \in F_{L(\lambda)}$ then $-\beta' = -\beta + \beta'' \in F_{L(\lambda)}$ since $\Phi^+ \subset F_{L(\lambda)}$ and $F_{L(\lambda)}$ is closed (Proposition 2.6). A contradiction. So $-\beta \in T_{L(\lambda)}$. □

Lemma 8.5 *Let $\gamma \in \Pi$. $D(\gamma)$ generates Q.*

Proof. Let $\langle D(\gamma) \rangle$ be the subgroup of Q generated by $D(\gamma)$. Assume $\Pi \cap \langle D(\gamma) \rangle \neq \Pi$. Let $\alpha \notin \langle D(\gamma) \rangle$ be a simple root that is connected to an $\alpha' \in \langle D(\gamma) \rangle$ (possible since the Dynkin diagram of a simple Lie algebra is connected). Then $\alpha + \alpha' \in \langle D(\gamma) \rangle$. But then $\alpha = \alpha + \alpha' - \alpha' \in \langle D(\gamma) \rangle$. A contradiction. So $\langle D(\gamma) \rangle = Q$. □

Lemma 8.6 *Let $\lambda \in X$ be a non-integral weight. Assume that $L(\lambda)$ is admissible. Then $A(\lambda)$ is connected and $|A(\lambda)| \leq 2$.*

Proof. Assume $|A(\lambda)| \geq 2$. Let $\alpha, \alpha' \in A(\lambda)$ be two distinct elements. We will show that α and α' are connected. So assume $(\alpha|\alpha') = 0$ to reach a contradiction. $L(\lambda)$ is admissible of some degree d. By Lemma 5.11 and Proposition 5.6 ${}^{s_\alpha}L(s_\alpha.\lambda)$ is admissible of the same degree d ($L(s_\alpha.\lambda)$ is infinite dimensional since $s_\alpha.\lambda(K_{\alpha'}) = \lambda(K_{\alpha'}) \notin \pm q_{\alpha'}^{\mathbb{N}}$). Let Σ be a set of commuting roots that is a basis of Q such that $\alpha \in \Sigma$ and $-\Sigma \subset T_{L(\lambda)}$ (Lemma 4.16). By Proposition 5.10 ${}^{s_\alpha}L(s_\alpha.\lambda)$ is a subquotient of $L(\lambda)_{F_\Sigma}$. We claim that $\mathrm{Supp}_{\mathrm{ess}}(L(\lambda)) \cap \mathrm{Supp}_{\mathrm{ess}}({}^{s_\alpha}L(s_\alpha.\lambda)) \neq \emptyset$. If this is true then we have for $\nu \in \mathrm{Supp}_{\mathrm{ess}}(L(\lambda)) \cap \mathrm{Supp}_{\mathrm{ess}}({}^{s_\alpha}L(s_\alpha.\lambda))$, $L(\lambda)_\nu \cong (L(\lambda)_{F_\Sigma})_\nu \cong ({}^{s_\alpha}L(s_\alpha.\lambda))_\nu$ as $(U_q)_0$-modules by Lemma 4.22. But then by Theorem 2.7 $L(\lambda) \cong {}^{s_\alpha}L(s_\alpha.\lambda)$ which is clearly a contradiction by looking at the weights of the modules. So we will prove the claim that $\mathrm{Supp}_{\mathrm{ess}}(L(\lambda)) \cap \mathrm{Supp}_{\mathrm{ess}}({}^{s_\alpha}L(s_\alpha.\lambda)) \neq \emptyset$:

We have $-D(\alpha') \subset T_{L(\lambda)}$ and $-D(\alpha') \subset T_{{}^{s_\alpha}L(s_\alpha.\lambda)} = s_\alpha(T_{L(s_\alpha.\lambda)})$ by Lemma 8.4 and the fact that $(\alpha|\alpha') = 0$. So $-D(\alpha') \subset C(L(\lambda)) \cap C({}^{s_\alpha}L(s_\alpha.\lambda))$ thus $C(L(\lambda)) \cap C({}^{s_\alpha}L(s_\alpha.\lambda))$ generate Q by Lemma 8.5. This implies that $C(L(\lambda)) - C({}^{s_\alpha}L(s_\alpha.\lambda)) = Q$. The weights of $L(\lambda)$ and ${}^{s_\alpha}L(s_\alpha.\lambda)$ are contained in $q^Q\lambda$ so a weight in the essential support of $L(\lambda)$ (resp. ${}^{s_\alpha}L(s_\alpha.\lambda)$) is of the form $q^{\mu_1}\lambda$ (resp. $q^{\mu_2}\lambda$) for some $\mu_1, \mu_2 \in Q$. By the above $q^{C(L(\lambda))+\mu_1}\lambda \cap q^{C({}^{s_\alpha}L(s_\alpha.\lambda))+\mu_2}\lambda \neq \emptyset$. Since $q^{C(L(\lambda))+\mu_1}\lambda \subset \mathrm{Supp}_{\mathrm{ess}}(L(\lambda))$ and $q^{C({}^{s_\alpha}L(s_\alpha.\lambda))+\mu_2}\lambda \subset \mathrm{Supp}_{\mathrm{ess}}({}^{s_\alpha}L(s_\alpha.\lambda))$ we have proved the claim.

So we have proved that any two roots of $A(\lambda)$ are connected. Since there are no cycles in the Dynkin diagram of a simple Lie algebra we get $A(\lambda) = 2$. $\square$

8.2 Rank 2 calculations

Following the procedure in [Mat00, Section 7] we classify admissible simple highest weight modules in rank 2 in order to classify the modules in higher ranks. We only consider non-integral weights because of Theorem 8.1. We assume that q is transcendental over $\mathbb{Q}$.

Lemma 8.7 *Assume $\mathfrak{g} = \mathfrak{sl}_3$. Let $\lambda \in X$ be a non-integral weight. The module $L(\lambda)$ is admissible if and only if $q^\rho\lambda(K_\beta) \in \pm q^{\mathbb{Z}_{>0}}$ for at least one root $\beta \in \Phi^+$.*

Proof. It is easy to show that the Verma module $M(\lambda)$ is not admissible. So $q^\rho\lambda(K_\beta) \in \pm q^{\mathbb{Z}_{>0}}$ for at least one root $\beta \in \Phi^+$ by Theorem 8.2. On the other hand suppose $q^\rho\lambda(K_\beta) \in \pm q^{\mathbb{Z}_{>0}}$ for at least one root $\beta \in \Phi^+$. If $q^\rho\lambda(K_\alpha) \in \pm q^{\mathbb{Z}_{>0}}$ for a simple root $\alpha \in \Pi$ then by easy calculations we see that $M(s_\alpha.\lambda)$ is a submodule of $M(\lambda)$. If $q^\rho\lambda(K_\alpha) \notin \pm q^{\mathbb{Z}_{>0}}$ for both simple roots $\alpha \in \Pi$ then we get that $M(s_\beta.\lambda)$ is a submodule by Theorem 8.2. So in both cases we have a submodule $M(s_\beta.\lambda)$ of $M(\lambda)$. Since $L(\lambda)$ is the unique simple quotient of $M(\lambda)$, $L(\lambda)$ is a subquotient of $M(\lambda)/M(s_\beta.\lambda)$. Since $M(\lambda)/M(s_\beta.\lambda)$ is admissible we see that $L(\lambda)$ is admissible as well. $\square$

Lemma 8.8 *Assume $\mathfrak{g}$ is of type C_2 (i.e. $\mathfrak{g} = \mathfrak{sp}(4)$). Let $\Pi = \{\alpha_1, \alpha_2\}$ where α_1 is short and α_2 is long. Let $\lambda \in X$ be a non-integral weight. The module $L(\lambda)$ is infinite dimensional and admissible if and only if $q^\rho\lambda(K_{\alpha_1}), q^\rho\lambda(K_{\alpha_1+\alpha_2}) \in \pm q^{\mathbb{Z}_{>0}}$ and $\lambda(K_{\alpha_2}), \lambda(K_{2\alpha_1+\alpha_2}) \in \pm q^{1+2\mathbb{Z}} (= \pm q_{\alpha_2}^{1/2+\mathbb{Z}} = \pm q_{2\alpha_1+\alpha_2}^{1/2+\mathbb{Z}})$.*

Proof. Theorem 8.2 implies that $q^\rho\lambda(K_\beta) \in q_\beta^{\mathbb{Z}_{>0}}$ for at least two $\beta \in \Phi^+$ because otherwise $L(\lambda) = M(\lambda)/M(s_\beta.\lambda)$ for some $\beta \in \Phi^+$. But $M(\lambda)/M(s_\beta.\lambda)$ is not admissible. Since λ is not integral we know $q^\rho\lambda(K_\alpha) \notin q_\alpha^{\mathbb{Z}}$ for some $\alpha \in \Pi$. Suppose $\lambda(K_{\alpha_1}) \notin \pm q_{\alpha_1}^{\mathbb{Z}}$. We split into cases and arrive at a contradiction in both cases: If $\lambda(K_{\alpha_2}) \notin \pm q_{\alpha_2}^{\mathbb{Z}_{>0}}$ then by the above $q^\rho\lambda(K_{\alpha_1+\alpha_2}) \in \pm q_{\alpha_1+\alpha_2}^{\mathbb{Z}_{>0}} = \pm q^{\mathbb{Z}_{>0}}$ and $q^\rho\lambda(K_{2\alpha_1+\alpha_2}) \in \pm q_{2\alpha_1+\alpha_2}^{\mathbb{Z}_{>0}} = \pm q^{2\mathbb{Z}_{>0}}$ which implies that $q^\rho\lambda(K_{\alpha_1}) = q^\rho\lambda(K_{2\alpha_1+\alpha_2}K_{\alpha_1+\alpha_2}^{-1}) \in \pm q^{\mathbb{Z}} = \pm q_{\alpha_1}^{\mathbb{Z}}$. A contradiction.

The other case is $q^\rho\lambda(K_{\alpha_2}) \in \pm q_{\alpha_2}^{\mathbb{Z}_{>0}} = \pm q^{2\mathbb{Z}_{>0}}$: In this case we get $\lambda(K_{\alpha_1+\alpha_2}) \notin \pm q^{\mathbb{Z}} = \pm q_{\alpha_1+\alpha_2}^{\mathbb{Z}}$ so the last root, $2\alpha_1+\alpha_2$, must satisfy that $q^\rho\lambda(K_{2\alpha_1+\alpha_2}) \in \pm q_{2\alpha_1+\alpha_2}^{\mathbb{Z}_{>0}} = \pm q^{2\mathbb{Z}_{>0}}$. But this implies that $\lambda(K_{\alpha_1})^2 = \lambda(K_{2\alpha_1+\alpha_2}K_{\alpha_2}^{-1}) \in \pm q^{2\mathbb{Z}}$ which implies that $\lambda(K_{\alpha_1}) \in \pm q^{\mathbb{Z}}$. A contradiction.

So $\lambda(K_{\alpha_1}) \in \pm q^{\mathbb{Z}}$. Since λ is not integral we get $\lambda(K_{\alpha_2}) \notin \pm q_{\alpha_2}^{\mathbb{Z}} = \pm q^{2\mathbb{Z}}$. This implies that $\lambda(K_{2\alpha_1+\alpha_2}) \notin \pm q^{2\mathbb{Z}} = \pm q_{2\alpha_1+\alpha_2}^{\mathbb{Z}}$. Since $q^\rho\lambda(K_\beta) \in \pm q_\beta^{\mathbb{Z}_{>0}}$ for at least two $\beta \in \Phi^+$ we get $q^\rho\lambda(K_{\alpha_1}) \in \pm q^{\mathbb{Z}_{>0}}$ and $q^\rho\lambda(K_{\alpha_1+\alpha_2}) \in \pm q^{\mathbb{Z}_{>0}}$. This in turn implies that $\lambda(K_{\alpha_2}) = \lambda(K_{\alpha_1+\alpha_2}K_{\alpha_1}^{-1}) \in \pm q^{\mathbb{Z}}$. Since $\lambda(K_{\alpha_2}) \notin \pm q^{2\mathbb{Z}}$ we get $\lambda(K_{\alpha_2}) \in \pm q^{1+2\mathbb{Z}}$. Similarly $\lambda(K_{2\alpha_1+\alpha_2}) = \lambda(K_{\alpha_1+\alpha_2}K_{\alpha_1}) \in \pm q^{1+2\mathbb{Z}}$. So we have shown the only if part.

Assume λ is as required in the lemma. We will show that $L(\lambda)$ is admissible. By Theorem 8.2 we see that the composition factors of $M(s_{\alpha_1}.\lambda)$ are $L(s_{\alpha_1}.\lambda)$ and $L(s_{\alpha_1+\alpha_2}s_{\alpha_1}.\lambda) = M(w_0.\lambda)$ and the composition factors of $M(s_{\alpha_1+\alpha_2})$ are $L(s_{\alpha_1+\alpha_2})$ and $L(s_{\alpha_1}s_{\alpha_1+\alpha_2}) = M(w_0.\lambda)$. So

$$\sum_{\substack{\beta\in\Phi^+ \\ q^\rho\lambda(K_\beta)\in\pm q_\beta^{\mathbb{Z}_{>0}}}} \operatorname{ch} M(s_\beta.\lambda) = \operatorname{ch} L(s_{\alpha_1}.\lambda) + \operatorname{ch} L(s_{\alpha_1+\alpha_2}.\lambda) + 2\operatorname{ch} L(w_0.\lambda).$$

So the composition factors of the maximal submodule of $M(\lambda)$ are $L(s_{\alpha_1}.\lambda)$, $L(s_{\alpha_1+\alpha_2}.\lambda)$ and $L(w_0.\lambda)$. The worst case scenario being multiplicity one. In this case the character of $L(\lambda)$ is

$$\begin{aligned}&\operatorname{ch} M(\lambda) - \operatorname{ch} L(s_{\alpha_1}.\lambda) - \operatorname{ch} L(s_{\alpha_1+\alpha_2}) - \operatorname{ch} L(w_0.\lambda) = \\ &\qquad = \operatorname{ch} M(\lambda) - \operatorname{ch} M(s_{\alpha_1}.\lambda) - \operatorname{ch} M(s_{\alpha_1+\alpha_2}.\lambda) + \operatorname{ch} M(w_0.\lambda)\end{aligned}$$

The character of Verma modules are known and by an easy calculation it is seen that this would imply $L(\lambda)$ is admissible (cf. the proof of Lemma 7.2 in [Mat00]). □

8.3 Type A, D, E

In this section we complete the classification of all simple admissible highest weight modules when the Dynkin diagram of $\mathfrak{g}$ is simply laced. In particular we show that $\mathfrak{g}$ does not admit infinite dimensional simple admissible modules when $\mathfrak{g}$ is of type D and E. In Section 8.5 we show that the same is the case when $\mathfrak{g}$ is of type B or F. Combining this and Section 8.5 we get that $\mathfrak{g}$ admits infinite dimensional simple admissible modules if and only if $\mathfrak{g}$ is of type A or C. Remember that we restrict our attention to transcendental q and to non-integral weights because of Theorem 8.1.

Definition 8.9 *Let $\lambda : U_q^0 \to \mathbb{C}$ be a weight. In the Dynkin diagram of $\mathfrak{g}$ let any node corresponding to $\alpha \in \Pi \cap A(\lambda)$ be written as $\circ$ and every other as $\bullet$. e.g. if $\mathfrak{g} = \mathfrak{sl}_3$ and $|A(\lambda)| = 1$ then the graph corresponding to λ would look like this:*

$$\bullet \text{———} \circ$$

We call this the colored Dynkin diagram corresponding to λ.

In this way we get a 'coloring' of the Dynkin diagram for every λ.

Lemma 8.10 *Let $\lambda \in X$ be a non-integral weight such that $L(\lambda)$ is admissible. If the colored Dynkin diagram of λ contains*

$$\overset{\alpha'}{\circ} \text{———} \overset{\alpha}{\circ}$$

as a subdiagram then $q^\rho \lambda(K_{\alpha+\alpha'}) \in \pm q_{\alpha'+\alpha}^{\mathbb{Z}_{>0}}$.

Proof. Let v_λ be a highest weight vector of $L(\lambda)$. Let $\mathfrak{s}$ be the Lie algebra $\mathfrak{sl}_3$ with α and α' as simple roots. Let U be the subalgebra of U_q generated by $F_\alpha, F_{\alpha'}, K_\alpha^{\pm 1}, K_{\alpha'}^{\pm 1}, E_\alpha, E_{\alpha'}$. Then $U \cong U_{q_\alpha}(\mathfrak{s})$ as algebras and Uv_λ contains the simple highest weight $U_{q_\alpha}(\mathfrak{s})$-module $L(\lambda, \mathfrak{s})$ of highest weight λ (restricted to $U_{q_\alpha}^0(\mathfrak{s})$) as a subquotient. Since $L(\lambda)$ is admissible so is Uv_λ hence $L(\lambda, \mathfrak{s})$ is admissible. Then Lemma 8.7 implies that $q^\rho \lambda(K_{\alpha+\alpha'}) \in \pm q_\alpha^{\mathbb{Z}_{>0}}$. $\square$

Lemma 8.11 *Let $\lambda \in X$ be a non-integral weight such that $L(\lambda)$ is admissible. If the colored Dynkin diagram of λ contains*

$$\overset{\alpha'}{\circ} \text{———} \overset{\alpha}{\circ} \text{———} \overset{\alpha''}{\bullet}$$

as a subdiagram then $L(s_\alpha.\lambda)$ is admissible and the colored Dynkin diagram corresponding to $s_\alpha.\lambda$ contains

$$\overset{\alpha'}{\bullet} \text{———} \overset{\alpha}{\circ} \text{———} \overset{\alpha''}{\circ}$$

i.e. we can 'move' $\circ \text{———} \circ$ and still get an admissible module.

Proof. $L(s_\alpha.\lambda)$ is admissible by Proposition 5.10. It is easy to see that $q^\rho s_\alpha.\lambda(K_\alpha) \notin \pm q^{\mathbb{Z}}$ (follows by Lemma 8.10 since λ is non-integral), that $q^\rho s_\alpha.\lambda(K_{\alpha''}) \notin \pm q^{\mathbb{Z}}$ and that $q^\rho s_\alpha.\lambda(K_{\alpha'}) \in \pm q^{\mathbb{Z}_{>0}}$ (by Lemma 8.10) $\square$

Lemma 8.12 *Assume $\mathfrak{g} \neq \mathfrak{sl}_2$. Let $\lambda \in X$ be a non-integral weight such that $L(\lambda)$ is admissible.*

If $A(\lambda) = \{\alpha\}$ then α is only connected to one other simple root α', $L(s_\alpha.\lambda)$ is admissible and the corresponding colored Dynkin diagram of $s_\alpha.\lambda$ contains

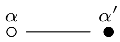
$$\overset{\alpha}{\circ} \text{———} \overset{\alpha'}{\circ}$$

as a subdiagram.

On the other hand if the colored Dynkin diagram of λ contains

$$\overset{\alpha}{\circ} \text{———} \overset{\alpha'}{\circ}$$

and α' is the only root connected to α then the colored Dynkin diagram of $s_\alpha.\lambda$ contains



as a subdiagram.

Proof. Since $\alpha \in A(\lambda)$, $L(s_\alpha.\lambda)$ is admissible by Proposition 5.10. First assume $A(\lambda) = \{\alpha\}$. If α is connected to two distinct roots α' and α'' then it is easily seen that $\alpha', \alpha'' \in A(s_\alpha.\lambda)$ contradicting the fact that $A(s_\alpha.\lambda)$ is connected (Lemma 8.6). It is easily seen that $q^\rho s_\alpha.\lambda(K_\alpha) \notin \pm q^{\mathbb{Z}_{>0}}$ (since λ is non integral) and $q^\rho s_\alpha.\lambda(K_{\alpha'}) \notin q^{\mathbb{Z}_{>0}}$.

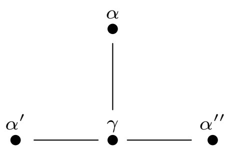
On the other hand if $A(\lambda) = \{\alpha, \alpha'\}$ then $q^\rho s_\alpha.\lambda(K_{\alpha'}) = q^\rho \lambda(K_{\alpha+\alpha'}) \in \pm q^{\mathbb{Z}_{>0}}$ by Lemma 8.10. □

Now we can eliminate the types that are not type A by the following theorem:

Theorem 8.13 *Assume $\mathfrak{g}$ is a simple Lie algebra of simply laced type. If there exists an infinite dimensional admissible simple module then $\mathfrak{g}$ is of type A.*

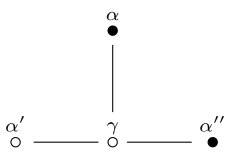
Proof. Suppose there exists an infinite dimensional admissible simple module then by Theorem 5.12 there exists a $\lambda \in X$ such that $L(\lambda)$ is an infinite admissible simple highest weight module. By Theorem 8.1 and the classification in [Mat00] there exists no highest weight simple admissible modules with integral weights unless $\mathfrak{g}$ is of type A. We need to show the same for non-integral weights.

If the Dynkin diagram is simply laced and not of type A then the Dynkin diagram contains

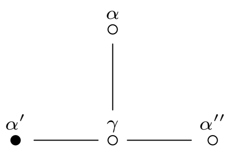


as a subdiagram.

By Lemma 8.12 we can assume without loss of generality that $|A(\lambda)| = 2$ and by Lemma 8.11 we can assume that the colored Dynkin diagram corresponding to λ contains the following:



But then $L(s_\gamma.\lambda)$ is admissible as well by Proposition 5.10 and the colored Dynkin diagram for $s_\gamma.\lambda$ contains

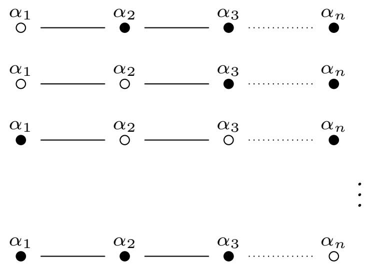


contradicting the fact that $A(\lambda)$ is connected. □

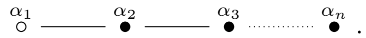
Combining all the above results we get

Theorem 8.14 *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $n \geq 2$ with simple roots $\alpha_1, \dots, \alpha_n$ such that $(\alpha_i|\alpha_{i+1}) = -1$, $i = 1, \dots, n$. Let $\lambda \in X$ be a non-integral weight.*

$L(\lambda)$ is admissible if and only if the colored Dynkin diagram of λ is of one of the following types:



Proof. By the above results these are the only possibilites. To show that $L(\lambda)$ is admissible when the colored Dynkin diagram is of the above form use the fact that by Lemma 8.11 and Lemma 8.12 we can assume λ has colored Dynkin diagram as follows:



Let $\beta_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$, $i = 1, \dots, n$. We see easily that $T_{L(\lambda)} = -\{\beta_1, \beta_2, \dots, \beta_n\}$ and $F_L = \Phi^+ \cup \Phi_{\{\alpha_2, \dots, \alpha_n\}}$. Let $\mathfrak{l}$, $\mathfrak{u}$, $\mathfrak{p}$ etc. be defined as in Section 2 of [Ped15a]. By [Ped15a, Theorem 2.23] $N := L(\lambda)^{\mathfrak{u}}$ is a simple finite dimensional $U_q(\mathfrak{l})$-module and $L(\lambda)$ is the unique simple quotient of $\mathcal{M}(N) = U_q \otimes_{U_q(\mathfrak{p})} N$. Since the vectors $\beta_1, \dots, \beta_n$ are linearly independent $\mathcal{M}(N)$ is admissible. This implies that $L(\lambda)$ is admissible since it is a quotient of $\mathcal{M}(N)$. $\square$

We can now make Corollary 5.13 more specific in type A:

Corollary 8.15 *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $n \geq 2$ with simple roots $\alpha_1, \dots, \alpha_n$ such that $(\alpha_i|\alpha_{i+1}) = -1$, $i = 1, \dots, n$. Let $\beta_j = \alpha_1 + \cdots + \alpha_j$, $j = 1, \dots, n$ and $\Sigma = \{\beta_1, \dots, \beta_n\}$. Let $F_{\beta_j} = T_{s_1} \cdots T_{s_{j-1}}(F_{\alpha_j})$ and let $F_\Sigma = \{q^a F_{\beta_1}^{a_1} \cdots F_{\beta_n}^{a_n} | a_i \in \mathbb{N}, a \in \mathbb{Z}\}$ be the corresponding Ore subset. Then Σ is a set of commuting roots that is a basis of Q with corresponding Ore subset F_Σ.*

Let $\beta'_j = \alpha_n + \cdots + \alpha_{n-j}$, $j = 1, \dots, n$ and $\Sigma = \{\beta'_1, \dots, \beta'_n\}$. Let $F'_{\beta'_j} = T_{s_n} \cdots T_{s_{n-j+1}}(F_{\alpha_{n-j}})$ and let $F_{\Sigma'} = \{q^a (F'_{\beta'_1})^{a_1} \cdots (F'_{\beta'_n})^{a_n} | a_i \in \mathbb{N}, a \in \mathbb{Z}\}$ be the corresponding Ore subset. Then Σ' is a set of commuting roots that is a basis of Q with corresponding Ore subset $F_{\Sigma'}$.

Let L be a simple torsion free module then one of the two following claims hold

- *There exists a $\lambda \in X$ with $\lambda(K_{\alpha_1}) \notin \pm q^{\mathbb{N}}$, $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$, $i = 2, \dots, n$ and $\mathbf{b} \in (\mathbb{C}^*)^n$ such that*
$$L \cong \varphi_{F_\Sigma, \mathbf{b}}.L(\lambda)_{F_\Sigma}.$$
- *There exists a $\lambda \in X$ with $\lambda(K_{\alpha_n}) \notin \pm q^{\mathbb{N}}$, $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$, $i = 1, \dots, n-1$ and $\mathbf{b} \in (\mathbb{C}^*)^n$ such that*
$$L \cong \varphi_{F_{\Sigma'}, \mathbf{b}}.L(\lambda)_{F_{\Sigma'}}.$$

Proof. By Theorem 5.12 $\mathcal{EXT}(L) \cong \mathcal{EXT}(L(\lambda'))$ for some $\lambda' \in X$. If λ' is non-integral then by Theorem 8.14, Lemma 8.11, Lemma 5.11 and Proposition 5.6 there exists a λ such that $\lambda(K_{\alpha_1}) \notin \pm q^{\mathbb{N}}$, $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$, $i = 2, \dots, n$ and such that $\mathcal{EXT}(L(\lambda')) \cong \mathcal{EXT}(L(\lambda))$. By Lemma 5.4 we can choose Σ as the commuting set of roots that is used in the definition of $\mathcal{EXT}(L(\lambda))$.

If λ' is integral we see by Theorem 8.1, Lemma 5.11, Proposition 5.6 and the classification in [Mat00, Section 8] that $\mathcal{EXT}(L(\lambda')) \cong \mathcal{EXT}(L(\lambda))$ for a λ such that $A(\lambda) = \{\alpha_1\}$ or $A(\lambda) = \{\alpha_n\}$ (cf. e.g. [Mat00, Proposition 8.5]).

Now the result follows just like in the proof of Corollary 5.13. □

In Section 9 we determine all $\mathbf{b} \in (\mathbb{C}^*)^n$ such that $\varphi_{F_\Sigma, \mathbf{b}}.L(\lambda)_{F_\Sigma}$ is torsion free with Σ as above in Corollary 8.15 and λ such that $\lambda(K_{\alpha_1}) \notin \pm q^{\mathbb{N}}$, $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$, $i = 2, \dots, n$. By symmetry of the Dynkin diagram and Corollary 8.15 this classifies all simple torsion free modules.

8.4 Quantum Shale-Weil representation

In this section we assume $\mathfrak{g}$ is of type C_n. Let $\alpha_1, \dots, \alpha_n$ be the simple roots such that α_i is connected to α_{i+1} and α_1 is long. We will describe a specific admissible module V and show that $V = L(\omega^+) \oplus L(\omega^-)$ for some weights $\omega^\pm$ with the purpose of classifying the admissible simple highest weight modules, see Theorem 8.17. Let $V = \mathbb{C}[X_1, \dots, X_n]$. We describe an action of the simple root vectors on V: For $i \in \{2, \dots, n\}$

$$\begin{aligned}
E_{\alpha_1} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} =& -\frac{[a_1][a_1-1]}{[2]} X_1^{a_1-2} X_2^{a_2} \cdots X_n^{a_n} \\
F_{\alpha_1} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} =& \frac{1}{[2]} X_1^{a_1+2} X_2^{a_2} \cdots X_n^{a_n} \\
E_{\alpha_i} X_1^{a_1} \cdots X_n^{a_n} =& [a_i] X_1^{a_1} \cdots X_{i-1}^{a_{i-1}+1} X_i^{a_i-1} \cdots X_n^{a_n} \\
F_{\alpha_i} X_1^{a_1} \cdots X_n^{a_n} =& [a_{i-1}] X_1^{a_1} \cdots X_{i-1}^{a_{i-1}-1} X_i^{a_i+1} \cdots X_n^{a_n} \\
K_{\alpha_1}^{\pm 1} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} =& q^{\mp(2a_1+1)} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} \\
K_{\alpha_i}^{\pm 1} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} =& q^{\pm(a_{i-1}-a_i)} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}.
\end{aligned}$$

We check that this is an action of U_q by checking the generating relations. These are tedious and kind of long calculations but just direct calculations. We refer to the generating relations as (R1) to (R6) like in [Jan96, Section 4.3].

(R1) is clear. (R2) and (R3): Let $j \in \{1, \dots, n\}$

$$\begin{aligned}
K_{\alpha_j} E_{\alpha_1} X_1^{a_1} \cdots X_n^{a_n} =& \begin{cases} -q^{-2a_1+3} \frac{[a_1][a_1-1]}{[2]} X_1^{a_1-2} X_2^{a_2} \cdots X_n^{a_n} & \text{if } j = 1 \\ -q^{a_1-2-a_2} \frac{[a_1][a_1-1]}{[2]} X_1^{a_1-2} X_2^{a_2} \cdots X_n^{a_n} & \text{if } j = 2 \\ -q^{a_{j-1}-a_j} \frac{[a_1][a_1-1]}{[2]} X_1^{a_1-2} X_2^{a_2} \cdots X_n^{a_n} & \text{if } j > 2 \end{cases} \\
=& q^{(\alpha_1|\alpha_j)} E_{\alpha_1} K_{\alpha_j} X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}.
\end{aligned}$$

Similar for $K_{\alpha_j}F_{\alpha_1}$. For $i \in \{2,\dots,n\}$

$$K_{\alpha_j}E_{\alpha_i}X_1^{a_1}\cdots X_n^{a_n} = \begin{cases} q^{a_{j-1}-a_j}[a_i]X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-1}\cdots X_n^{a_n} & \text{if } |j-i|>1 \\ q^{a_{j-1}-a_j-1}[a_i]X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-1}\cdots X_n^{a_n} & \text{if } j=i-1 \\ q^{a_{j-1}+1-a_j+1}[a_i]X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-1}\cdots X_n^{a_n} & \text{if } j=i \\ q^{a_{j-1}-1-a_j}[a_i]X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-1}\cdots X_n^{a_n} & \text{if } j=i+1 \end{cases}$$
$$=q^{(\alpha_i|\alpha_j)}E_{\alpha_1}K_{\alpha_j}X_1^{a_1}X_2^{a_2}\cdots X_n^{a_n}.$$

Similarly for $K_{\alpha_j}F_{\alpha_i}$.

(R4):

$$\begin{aligned}[E_{\alpha_1},F_{\alpha_1}]X_1^{a_1}X_2^{a_2}\cdots X_n^{a_n} =& E_{\alpha_1}\frac{1}{[2]}X_1^{a_1+2}X_2^{a_2}\cdots X_n^{a_n} + F_{\alpha_1}\frac{[a_1][a_1-1]}{[2]}X_1^{a_1-2}X_2^{a_2}\cdots X_n^{a_n} \\ =& \left(-\frac{[a_1+2][a_1+1]}{[2][2]}+\frac{[a_1][a_1-1]}{[2][2]}\right)X_1^{a_1}\cdots X_n^{a_n} \\ =& \frac{q^{-2a_1-1}-q^{2a_1+1}}{q^2-q^{-2}}X_1^{a_1}\cdots X_n^{a_n} \\ =& \frac{K_{\alpha_1}-K_{\alpha_1}^{-1}}{q^2-q^{-2}}X_1^{a_1}\cdots X_n^{a_n}.\end{aligned}$$

$$\begin{aligned}[E_{\alpha_1},F_{\alpha_2}]X_1^{a_1}\cdots X_n^{a_n} =& [a_1]E_{\alpha_1}X_1^{a_1-1}X_2^{a_2+1}\cdots X_n^{a_n} + \frac{[a_1][a_1-1]}{[2]}F_{\alpha_2}X_1^{a_1-2}X_2^{a_2}\cdots X_n^{a_n} \\ =& -\frac{[a_1][a_1-1][a_1-2]}{[2]}X_1^{a_1-3}X_2^{a_2+1}\cdots X_n^{a_n} \\ &+\frac{[a_1][a_1-1][a_1-2]}{[2]}X_1^{a_1-3}X_2^{a_2+1}\cdots X_n^{a_n} \\ =& 0.\end{aligned}$$

For $i>2$ clearly $[E_{\alpha_1},F_{\alpha_i}]X_1^{a_1}\cdots X_n^{a_n}=0$. For $i,j\in\{2,\dots,n\}$: If $|i-j|>1$ clearly $[E_{\alpha_i},F_{\alpha_j}]X_1^{a_1}\cdots X_n^{a_n}=0$.

$$\begin{aligned}[E_{\alpha_i},F_{\alpha_{i+1}}]X_1^{a_n}\cdots X_n^{a_n} =& [a_i]E_{\alpha_i}X_1^{a_1}\cdots X_i^{a_i-1}X_{i+1}^{a_{i+1}+1}\cdots X_n^{a_n} \\ &-[a_i]F_{\alpha_{i+1}}X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-1}\cdots X_n^{a_n} \\ =& [a_i][a_i-1]X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-2}X_{i+1}^{a_{i+1}+1}\cdots X_n^{a_n} \\ &-[a_i][a_i-1]X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-2}X_{i+1}^{a_{i+1}+1}\cdots X_n^{a_n} \\ =& 0.\end{aligned}$$

$$\begin{aligned}[E_{\alpha_2},F_{\alpha_1}]X_1^{a_1}\cdots X_n^{a_n} =& E_{\alpha_2}\frac{1}{[2]}X_1^{a_1+2}X_2^{a_2}\cdots X_n^{a_n} - [a_2]F_{\alpha_1}X_1^{a_1+1}X_2^{a_2-1}\cdots X_n^{a_n} \\ =& \frac{[a_2]}{[2]}X_1^{a_1+3}X_2^{a_2-1}\cdots X_n^{a_n} - \frac{[a_2]}{[2]}X_1^{a_1+3}X_2^{a_2-1}\cdots X_n^{a_n} \\ =& 0.\end{aligned}$$

For $i > 2$:

$$\begin{aligned}[E_{\alpha_i}, F_{\alpha_{i-1}}]X_1^{a_n} \cdots X_n^{a_n} =&[a_{i-2}]E_{\alpha_i}X_1^{a_1} \cdots X_{i-2}^{a_{i-2}-1}X_{i-1}^{a_{i-1}+1} \cdots X_n^{a_n} \\ &- [a_i]F_{\alpha_{i-1}}X_1^{a_1} \cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-1} \cdots X_n^{a_n} \\ =&[a_{i-2}][a_i]X_1^{a_1} \cdots X_{i-2}^{a_{i-2}-1}X_{i-1}^{a_{i-1}+2}X_i^{a_i-1} \cdots X_n^{a_n} \\ &- [a_i][a_{i-2}]F_{\alpha_{i-1}}X_1^{a_1} \cdots X_{i-2}^{a_{i-2}-1}X_{i-1}^{a_{i-1}+2}X_i^{a_i-1} \cdots X_n^{a_n} \\ =&0.\end{aligned}$$

For $i > 1$:

$$\begin{aligned}[E_{\alpha_i}, F_{\alpha_i}]X_1^{a_1} \cdots X_n^{a_n} =&[a_{i-1}]E_{\alpha_i}X_1^{a_1} \cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i-1} \cdots X_n^{a_n} \\ &- [a_i]F_{\alpha_i}X_1^{a_1} \cdots X_{i-1}^{a_{i-1}-1}X_i^{a_i+1} \cdots X_n^{a_n} \\ =&([a_{i-1}][a_i-1] - [a_i][a_{i-1}-1])X_1^{a_1} \cdots X_{i-1}^{a_{i-1}}X_i^{a_i} \cdots X_n^{a_n} \\ =&[a_{i-1} - a_i]X_1^{a_1} \cdots X_n^{a_n} \\ =&\frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q - q^{-1}}X_1^{a_1} \cdots X_n^{a_n}.\end{aligned}$$

Finally we have the relations (R5) and (R6): Clearly $[E_{\alpha_i}, E_{\alpha_j}]X_1^{a_1} \cdots X_n^{a_n} = 0$ and $[F_{\alpha_i}, F_{\alpha_j}]X_1^{a_1} \cdots X_n^{a_n} = 0$ when $|j - i| > 1$.

$$\begin{aligned}&(E_{\alpha_2}^3E_{\alpha_1} - [3]E_{\alpha_2}^2E_{\alpha_1}E_{\alpha_2} + [3]E_{\alpha_2}E_{\alpha_1}E_{\alpha_2}^2 - E_{\alpha_1}E_{\alpha_2}^3)X_1^{a_1} \cdots X_n^{a_n} \\ &=\frac{1}{[2]}\Big(- [a_1][a_1-1][a_2][a_2-1][a_2-2] \\ &\quad + [3][a_1+1][a_1][a_2][a_2-1][a_2-2] \\ &\quad - [3][a_1+2][a_1+1][a_2][a_2-1][a_2-2] \\ &\quad + [a_1+3][a_1+2][a_2][a_2-1][a_2-2]\Big)X_1^{a_1+1}X_2^{a_2-3} \cdots X_n^{a_n} \\ &=\frac{[a_2][a_2-1][a_2-2]}{[2]}\Big(- [a_1][a_1-1] + [3][a_1+1][a_1] \\ &\quad - [3][a_1+2][a_1+1] + [a_1+3][a_1+2]\Big)X_1^{a_1+1}X_2^{a_2-3} \cdots X_n^{a_n} \\ &=0.\end{aligned}$$

$$\begin{aligned}(E_{\alpha_1}^2E_{\alpha_2} - [2]_{\alpha_1}E_{\alpha_1}E_{\alpha_2}E_{\alpha_1} + E_{\alpha_2}E_{\alpha_1}^2)X_1^{a_1} \cdots X_n^{a_n}& \\ =&\frac{[a_2]}{[2][2]}\big([a_1+1][a_1][a_1-1][a_1-2] \\ &- [2]_{\alpha_1}[a_1][a_1-1][a_1-1][a_1-2] \\ &+ [a_1][a_1-1][a_1-2][a_1-3]\Big)X_1^{a_1+3}X_2^{a_2-1} \cdots X_n^{a_n} \\ =&\frac{[a_2][a_1][a_1-1][a_1-2]}{[2][2]}\Big([a_1+1] - [2]_{\alpha_1}[a_1-1] \\ &+ [a_1-3]\Big)X_1^{a_1+3}X_2^{a_2-1} \cdots X_n^{a_n} \\ =&0.\end{aligned}$$

For $i > 1$:

$$\begin{aligned}(E_{\alpha_i}^2 E_{\alpha_{i+1}} - &[2]E_{\alpha_i}E_{\alpha_{i+1}}E_{\alpha_i} + E_{\alpha_{i+1}}E_{\alpha_i}^2)X_1^{a_1}\cdots X_n^{a_n}\\ =&[a_{i+1}][a_i]([a_i+1]-[2][a_i]+[a_i-1])X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+2}X_i^{a_i-1}X_{i+1}^{a_{i+1}-1}\cdots X_n^{a_n}\\ =&0.\end{aligned}$$

$$\begin{aligned}(E_{\alpha_{i+1}}^2 E_{\alpha_i} - &[2]E_{\alpha_{i+1}}E_{\alpha_i}E_{\alpha_{i+1}} + E_{\alpha_i}E_{\alpha_{i+1}}^2)X_1^{a_1}\cdots X_n^{a_n}\\ =&[a_{i+1}][a_{i+1}-1]([a_i]-[2][a_i+1]+[a_i+2])X_1^{a_1}\cdots X_{i-1}^{a_{i-1}+1}X_i^{a_i+1}X_{i+1}^{a_{i+1}-2}\cdots X_n^{a_n}\\ =&0.\end{aligned}$$

$$\begin{aligned}(F_{\alpha_1}^2F_{\alpha_2} - [2]_{\alpha_1}&F_{\alpha_1}F_{\alpha_2}F_{\alpha_1} + F_{\alpha_2}F_{\alpha_1}^2)X_1^{a_1}\cdots X_n^{a_n}\\ =&\frac{1}{[2][2]}([a_1]-[2]_{\alpha_1}[a_1+2]+[a+4])X_1^{a_1+3}X_2^{a_2+1}\cdots X_n^{a_n}\\ =&0.\end{aligned}$$

$$\begin{aligned}(F_{\alpha_2}^3F_{\alpha_1} - [3]F_{\alpha_2}^2&F_{\alpha_1}F_{\alpha_2} + [3]F_{\alpha_2}F_{\alpha_1}F_{\alpha_2}^2 - F_{\alpha_1}F_{\alpha_2}^3)X_1^{a_1}\cdots X_n^{a_n}\\ =&\frac{1}{[2]}\Big([a_1+2][a_1+1][a_1]-[3][a_1][a_1+1][a_1]\\ &+[3][a_1][a_1-1][a_1]\\ &-[a_1][a_1-1][a_1-2]\Big)X_1^{a_1-1}X_2^{a_2+3}\cdots X_n^{a_n}\\ =&\frac{[a_1]}{[2]}\Big([a_1+2][a_1+1]-[3][a_1+1][a_1]\\ &+[3][a_1][a_1-1]\\ &-[a_1-1][a_1-2]\Big)X_1^{a_1-1}X_2^{a_2+3}\cdots X_n^{a_n}\\ =&0.\end{aligned}$$

For $i > 1$:

$$\begin{aligned}(F_{\alpha_i}^2 F_{\alpha_{i+1}} - &[2]F_{\alpha_i}F_{\alpha_{i+1}}F_{\alpha_i} + F_{\alpha_{i+1}}F_{\alpha_i}^2)X_1^{a_1}\cdots X_n^{a_n}\\ =&[a_{i-1}][a_{i-1}-1]([a_i]-[2][a_i+1]+[a_i+2])X_1^{a_1}\cdots X_{i-1}^{a_{i-1}-2}X_i^{a_i+1}X_{i+1}^{a_{i+1}+1}\cdots X_n^{a_n}\\ =&0.\end{aligned}$$

$$\begin{aligned}(F_{\alpha_{i+1}}^2 F_{\alpha_i} - &[2]F_{\alpha_{i+1}}F_{\alpha_i}F_{\alpha_{i+1}} + F_{\alpha_i}F_{\alpha_{i+1}}^2)X_1^{a_1}\cdots X_n^{a_n}\\ =&[a_{i-1}][a_i]([a_i+1]-[2][a_i]+[a_i-1])X_1^{a_1}\cdots X_{i-1}^{a_{i-1}-1}X_i^{a_i-1}X_{i+1}^{a_{i+1}+2}\cdots X_n^{a_n}\\ =&0.\end{aligned}$$

So we have shown that V is a $U_q(\mathfrak{g})$-module. Note that V is admissible of degree 1 and $V = V^{even} \oplus V^{odd}$ where V^{even} are even degree polynomials and V^{odd} are odd degree polynomials. Furthermore we see that $V^{even} = L(\omega^+)$ and $V^{odd} = L(\omega^-)$ where $\omega^\pm$ are the weights defined by $\omega^+(K_{\alpha_1}) = q^{-1}$, $\omega^+(K_{\alpha_i}) = 1$, $i > 1$ and $\omega^-(K_{\alpha_1}) = q^{-3}$, $\omega^-(K_{\alpha_2}) = q^{-1}$, $\omega^-(K_{\alpha_i}) = 1$, $i > 2$. V^{even} is generated by 1 and V^{odd} is generated by X_1. We will use the fact that $L(\omega^+)$ is admissible in Theorem 8.17 in the next section.

8.5 Type B, C, F

In this section we classify the simple highest weight admissible modules when $\mathfrak{g}$ is of type B, C or F. Remember that we have assumed that q is transcendental.

Theorem 8.16 *Let $\mathfrak{g}$ be a simple Lie algebra not of type G_2. Suppose there exists an infinite dimensional admissible simple $U_q(\mathfrak{g})$-module. Then $\mathfrak{g}$ is of type A or C.*

Proof. If $\mathfrak{g}$ is simply laced then Theorem 8.14 gives that $\mathfrak{g}$ is of type A. So assume $\mathfrak{g}$ is not of simply laced type. Theorem 8.1 and the classification in the classical case tells us that no admissible infinite dimensional simple highest weight modules exists with integral weights when $\mathfrak{g}$ is not simply laced (cf. [Mat00, Lemma 9.1]).

We have assumed that $\mathfrak{g}$ is not of type G_2 so the remaining non-simply laced types are B, C or F. We will show that the Dynkin diagram of $\mathfrak{g}$ can't contain the subdiagram

$$\overset{\alpha_1}{\bullet} \Longleftarrow \overset{\alpha_2}{\bullet} \longrightarrow \overset{\alpha_3}{\bullet}.$$

Assume the Dynkin diagram contains the above as a subdiagram. If there exists a simple admissible infinite dimensional module L then there exists a non-integral $\lambda \in X$ such that $L(\lambda)$ is infinite dimensional and admissible (Theorem 5.12). Let $\lambda \in X$ be a non-integral weight such that $L(\lambda)$ is admissible. Then by Lemma 8.8, $q^\rho\lambda(K_{\alpha_1}) \in \pm q_{\alpha_1}^{\mathbb{Z}} = \pm q^{\mathbb{Z}}$. By Lemma 8.11 and Lemma 8.12 we can assume without loss of generality that the colored Dynkin diagram of λ is of the form

$$\overset{\alpha_1}{\bullet} \Longleftarrow \overset{\alpha_2}{\circ} \longrightarrow \overset{\alpha_3}{\circ}.$$

Let $\mathfrak{s}$ be the simple rank 3 Lie algebra of type B_3. Let U be the subalgebra of U_q generated by $E_{\alpha_i}, F_{\alpha_i}, K_{\alpha_i}^{\pm 1}$, $i = 1, 2, 3$. Then $U \cong U_q(\mathfrak{s})$. Let $Q_{\mathfrak{s}} := \mathbb{Z}\{\alpha_1, \alpha_2, \alpha_3\} \subset Q$. Let v_λ be a highest weight vector of $L(\lambda)$. Then Uv_λ contains the simple highest weight $U_q(\mathfrak{s})$-module $L(\lambda, \mathfrak{s})$ of highest weight λ (restricted to $U_q^0(\mathfrak{s})$) as a subquotient. Since $L(\lambda)$ is admissible so is $L(\lambda, \mathfrak{s})$.

Like in the proof of Lemma 8.6 we get a contradiction if we can show that $T_{L(\lambda,\mathfrak{s})} \cap T_{{}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})}$ generates $Q_{\mathfrak{s}}$. It is easily seen that $\{-\alpha_1-\alpha_2, -\alpha_3, -2\alpha_1-\alpha_2\} \subset T_{L(\lambda,\mathfrak{s})} \cap T_{{}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})}$, so $T_{L(\lambda,\mathfrak{s})} \cap T_{{}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})}$ generates $Q_{\mathfrak{s}}$. So $C(L(\lambda,\mathfrak{s})) \cap C({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s}))$ generates $Q_{\mathfrak{s}}$. Therefore $C(L(\lambda,\mathfrak{s})) - C({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})) = Q_{\mathfrak{s}}$. The weights of $L(\lambda,\mathfrak{s})$ and ${}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})$ are contained in $q^{Q_{\mathfrak{s}}}\lambda$ so a weight in the essential support of $L(\lambda,\mathfrak{s})$ (resp. ${}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})$) is of the form $q^{\mu_1}\lambda$ (resp. $q^{\mu_2}\lambda$) for some $\mu_1, \mu_2 \in Q_{\mathfrak{s}}$. By the above $q^{C(L(\lambda,\mathfrak{s}))+\mu_1}\lambda \cap q^{C({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s}))+\mu_2}\lambda \neq \emptyset$. Since $q^{C(L(\lambda,\mathfrak{s}))+\mu_1}\lambda \subset \mathrm{Supp}_{\mathrm{ess}}(L(\lambda))$ and $q^{C({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s}))+\mu_2}\lambda \subset \mathrm{Supp}_{\mathrm{ess}}({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s}))$ we have proved that $\mathrm{Supp}_{\mathrm{ess}}({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})) \cap \mathrm{Supp}_{\mathrm{ess}}(L(\lambda,\mathfrak{s})) \neq \emptyset$. By Proposition 5.10 $L(\lambda,\mathfrak{s})$ and ${}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})$ are subquotients of $L(\lambda,\mathfrak{s})_{F_{\alpha_2}}$. Let $\nu \in \mathrm{Supp}_{\mathrm{ess}}({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})) \cap \mathrm{Supp}_{\mathrm{ess}}(L(\lambda,\mathfrak{s}))$. Then by Lemma 4.22 $L(\lambda,\mathfrak{s})_\nu \cong (L(\lambda,\mathfrak{s})_{F_{\alpha_2}})_\nu \cong ({}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s}))_\nu$ so by Theorem 2.7 $L(\lambda,\mathfrak{s}) \cong {}^{s_{\alpha_2}}L(s_{\alpha_2}.\lambda,\mathfrak{s})$. This is a contradiction by looking at weights of the modules. $\square$

Theorem 8.17 *Let $\mathfrak{g}$ be a simple Lie algebra of type C_n (i.e. $\mathfrak{g} = \mathfrak{sp}(2n)$). Let $\alpha_1, \dots, \alpha_n$ be the simple roots such that α_i is connected to α_{i+1} and α_1 is long – i.e. the Dynkin diagram of C_n is*

$$\overset{\alpha_1}{\bullet} \Longrightarrow \overset{\alpha_2}{\bullet} \cdots\cdots \overset{\alpha_{n-1}}{\bullet} \longrightarrow \overset{\alpha_n}{\bullet} .$$

Let $\lambda \in X$. $L(\lambda)$ is infinite dimensional and admissible if and only if

- $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$ *for* $1 < i \leq n$
- $\lambda(K_{\alpha_1}) \in \pm q_{\alpha_1}^{1/2+\mathbb{Z}} = \pm q^{1+2\mathbb{Z}}$
- $\lambda(K_{\alpha_1+\alpha_2}) \in \pm q^{\mathbb{Z}_{\geq -2}}$

or equivalently $q^\rho \lambda(K_\beta) \in \pm q^{\mathbb{Z}_{>0}}$ for every short root $\beta \in \Phi^+$ and $\lambda(K_{\beta'}) \in \pm q^{1+2\mathbb{Z}}$ for every long root $\beta' \in \Phi^+$.

Proof. Assume $\lambda(K_{\alpha_i}) \notin \pm q^{\mathbb{N}}$ for some $i > 1$. Then by Lemma 8.11 there exists a λ' such that $L(\lambda')$ is admissible and such that $\lambda'(K_{\alpha_2}) \notin q^{\mathbb{N}}$. Let $\mathfrak{s}$ be the Lie algebra $\mathfrak{sp}(4)$ with simple roots α_2 and α_1. Let U be the subalgebra of U_q generated by $F_{\alpha_1}, F_{\alpha_2}, K_{\alpha_1}, K_{\alpha_2}, E_{\alpha_1}, E_{\alpha_2}$. Then $U \cong U_q(\mathfrak{s})$ as algebras and $Uv_{\lambda'}$ contains the simple highest weight $U_q(\mathfrak{s})$-module $L(\lambda', \mathfrak{s})$ of highest weight λ' (restricted to $U^0_{q_\alpha}(\mathfrak{s})$) as a subquotient. Since $L(\lambda')$ is admissible so is $Uv_{\lambda'}$ hence $L(\lambda', \mathfrak{s})$ is admissible. So $\lambda'(K_{\alpha_2}) \in \pm q^{\mathbb{N}}$ by Lemma 8.8. A contradiction. So we have proven that $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$ for $1 < i \leq n$ is a neccesary condition. We get also from Lemma 8.8 that $\lambda(K_{\alpha_1}) \in q^{1+2\mathbb{Z}}$ and $q^3 \lambda(K_{\alpha_1+\alpha_2}) = q^\rho \lambda(K_{\alpha_1+\alpha_2}) \in \pm q^{\mathbb{Z}_{>0}}$ which shows that the two other conditions are neccesary.

Now assume we have a weight $\lambda \in X$ that satisfies the above. So $\lambda(K_{\alpha_1}) = q^{-1+r}$ for some $r \in 2\mathbb{Z}$. We can assume $r \in \mathbb{N}$ by Lemma 5.11 and Proposition 5.6 (if $r < 0$ replace λ with $s_1.\lambda$, $L(\lambda)$ is admissible if and only if $L(s_1.\lambda)$ is). We have $\lambda = \omega^+ \lambda_0$ for some dominant integral weight λ_0 and $L(\lambda)$ is a subquotient of $L(\omega^+) \otimes L(\lambda_0)$. Since $L(\omega^+)$ is admissible and $L(\lambda_0)$ is finite dimensional $L(\omega^+) \otimes L(\lambda_0)$ is admissible and since $L(\lambda)$ is a subquotient of $L(\omega^+) \otimes L(\lambda_0)$, $L(\lambda)$ is admissible as well. □

Corollary 8.18 *Let $\mathfrak{g}$ be a simple Lie algebra of type C_n (i.e. $\mathfrak{g} = \mathfrak{sp}(2n)$). Let $\alpha_1, \dots, \alpha_n$ be the simple roots such that α_i is connected to α_{i+1} and α_1 is long.*

Let $\beta_j = \alpha_1 + \cdots + \alpha_j$, $j = 1, \dots, n$ and $\Sigma = \{\beta_1, \dots, \beta_n\}$. Let $F_{\beta_j} = T_{s_1} \cdots T_{s_{j-1}}(F_{\alpha_j})$ and let $F_\Sigma = \{q^a F_{\beta_1}^{a_1} \cdots F_{\beta_n}^{a_n} | a_i \in \mathbb{N}, a \in \mathbb{Z}\}$ be the corresponding Ore subset. Then Σ is a set of commuting roots that is a basis of Q with corresponding Ore subset F_Σ.

Let L be a simple torsion free module. Then there exists a $\lambda \in X$ with $\lambda(K_\beta) \in \pm q^{\mathbb{N}}$ for all short $\beta \in \Phi^+$ and $\lambda(K_\gamma) \in \pm q^{1+2\mathbb{Z}}$ for all long $\gamma \in \Phi^+$ and a $\mathbf{b} \in (\mathbb{C}^*)^n$ such that

$$L \cong \varphi_{F_\Sigma, \mathbf{b}}.L(\lambda)_{F_\Sigma}$$

Proof. By Theorem 5.12 there exists a $\lambda \in X$ such that $\mathcal{EXT}(L) \cong \mathcal{EXT}(L(\lambda))$. By Proposition 5.6 $L(\lambda)$ is admissible and by Theorem 8.17 λ is as described in the statement of the corollary. Now the result follows just like in the proof of Corollary 5.13. □

In Section 10 we determine all $\mathbf{b} \in (\mathbb{C}^*)^n$ such that $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is torsion free (with Σ and λ as above in Corollary 8.18). By the corollary this classifies all simple torsion free modules for type C.

9 Classification of simple torsion free modules. Type A.

In this section we assume $\mathfrak{g} = \mathfrak{sl}_{n+1}$ with $n \geq 2$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots such that $(\alpha_i|\alpha_{i+1}) = -1$, $i = 1, \dots, n-1$. Set $\beta_j = s_1 \cdots s_{j-1}(\alpha_j) = \alpha_1 + \cdots + \alpha_j$, then $\Sigma = \{\beta_1, \dots, \beta_n\}$ is a set of commuting roots with corresponding root vectors $F_{\beta_j} = T_{s_1} \cdots T_{s_{j-1}}(F_{\alpha_j})$. We will show some commutation formulas and use these to calculate $\varphi_{F_\Sigma,\mathbf{b}}$ on all simple root vectors. This will allow us to determine exactly for which $\mathbf{b} \in (\mathbb{C}^*)^n$, $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is torsion free, see Theorem 9.8.

Choose a reduced expression of w_0 starting with $s_1 \cdots s_n$ and define roots $\gamma_1, \dots, \gamma_N$ and root vectors $F_{\gamma_1}, \dots, F_{\gamma_N}$ from this expression. Note that $F_{\beta_i} = F_{\gamma_i}$ for $i = 1, \dots, n$.

Proposition 9.1 *Let $i \in \{2, \dots, n\}$ and $j \in \{1, \dots, n\}$.*

$$[F_{\alpha_i}, F_{\beta_j}]_q = \begin{cases} F_{\beta_i}, & \text{if } j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$[E_{\alpha_i}, F_{\beta_j}] = \begin{cases} F_{\beta_{i-1}} K_{\alpha_i}^{-1}, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We will show the proposition for the F's first and then for the E's.

Assume first that $j < i-1$. Then clearly $[F_{\alpha_i}, F_{\beta_j}]_q = [F_{\alpha_i}, F_{\beta_j}] = 0$ since α_i is not connected to any of the simple roots $\alpha_1, \dots, \alpha_j$ appearing in β_j.

Then assume $j \geq i$. We must have $\alpha_i = \gamma_k$ for some $k > n$ since $\{\gamma_1, \dots, \gamma_N\} = \Phi^+$. By Theorem 3.2 $[F_{\alpha_i}, F_{\beta_j}]_q$ is a linear combination of monomials of the form $F_{\gamma_{j+1}}^{a_{j+1}} \cdots F_{\gamma_{k-1}}^{a_{k-1}}$. For a monomial of this form to appear with nonzero coefficient we must have

$$\sum_{h=j+1}^{k-1} a_h \gamma_h = \alpha_i + \beta_j = \alpha_1 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \dots \alpha_j.$$

For this to be possible one of the positive roots γ_s, $j < s < k$ must be equal to $\alpha_1 + \alpha_2 + \cdots + \alpha_m$ for some $m \leq j$ but $\alpha_1 + \alpha_2 + \cdots + \alpha_m = \gamma_m$ by construction and $m \leq j < s$ so $m \neq s$. We conclude that this is not possible.

Finally we investigate the case when $j = i-1$. We have

$$\begin{aligned} [F_{\alpha_i}, F_{\beta_{i-1}}]_q =& [T_{s_1} \cdots T_{s_{i-2}}(F_{\alpha_i}), T_{s_1} \cdots T_{s_{i-2}}(F_{\alpha_{i-1}})]_q \\ =& T_{s_1} \cdots T_{s_{i-2}} \left([F_{\alpha_i}, F_{\alpha_{i-1}}]_q \right) \\ =& T_{s_1} \cdots T_{s_{i-2}} T_{s_{i-1}}(F_{\alpha_i}) \\ =& F_{\beta_i}. \end{aligned}$$

For the E's: Assume first $j < i$: Since F_{β_j} is a polynomial in $F_{\alpha_1}, \dots, F_{\alpha_j}$, E_{α_i} commutes with F_{β_j} when $j < i$.

Assume then $j = i$: We have by the above

$$F_{\beta_i} = [F_{\alpha_i}, F_{\beta_{i-1}}]_q$$

so

$$\begin{aligned}[E_{\alpha_i}, F_{\beta_i}] =&[E_{\alpha_i}, (F_{\alpha_i}F_{\beta_{i-1}} - q^{-(\beta_{i-1}|\alpha_i)}F_{\beta_{i-1}}F_{\alpha_i})] \\ =&[E_{\alpha_i}, F_{\alpha_i}]F_{\beta_{i-1}} - qF_{\beta_{i-1}}[E_{\alpha_i}, F_{\alpha_i}] \\ =&\frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q - q^{-1}}F_{\beta_{i-1}} - qF_{\beta_{i-1}}\frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q - q^{-1}} \\ =&F_{\beta_{i-1}}\frac{qK_{\alpha_i} - q^{-1}K_{\alpha_i}^{-1} - qK_{\alpha_i} + qK_{\alpha_i}^{-1}}{q - q^{-1}} \\ =&F_{\beta_{i-1}}K_{\alpha_i}^{-1}.\end{aligned}$$

Finally assume $j > i$: Observe first that we have

$$T_{s_{i+1}} \cdots T_{s_{j-1}}F_{\alpha_j} = \sum_{s=1}^{m} u_s F_{\alpha_{i+1}} u_s'$$

for some $m \in \mathbb{N}$ and some u_s, u_s' that are polynomials in $F_{\alpha_{i+2}}, \dots F_{\alpha_j}$. Note that $T_{s_i}(u_s) = u_s$ and $T_{s_i}(u_s') = u_s'$ for all s since α_i is not connected to any of the simple roots $\alpha_{i+2}, \dots \alpha_j$. So

$$\begin{aligned}T_{s_i}T_{s_{i+1}} \cdots T_{s_{j-1}}F_{\alpha_j} =&T_{s_i}\left(\sum_{s=1}^{m} u_s F_{\alpha_{i+1}} u_s'\right) \\ =&\sum_{s=1}^{m} u_s T_{s_i}(F_{\alpha_{i+1}})u_s' \\ =&\sum_{s=1}^{m} u_s(F_{\alpha_{i+1}}F_{\alpha_i} - qF_{\alpha_i}F_{\alpha_{i+1}})u_s' \\ =&\sum_{s=1}^{m} u_s F_{\alpha_{i+1}} u_s' F_{\alpha_i} - qF_{\alpha_i}\sum_{s=1}^{m} u_s F_{\alpha_{i+1}} u_s' \\ =&T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})F_{\alpha_i} - qF_{\alpha_i}T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j}).\end{aligned}$$

Thus we see that

$$\begin{aligned}F_{\beta_j} =&T_{s_1} \dots T_{s_i} \cdots T_{s_{j-1}}(F_{\alpha_j}) \\ =&T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})T_{s_1} \cdots T_{s_{i-1}}(F_{\alpha_i}) - qT_{s_1} \cdots T_{s_{i-1}}(F_{\alpha_i})T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j}) \\ =&T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})F_{\beta_i} - qF_{\beta_i}T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})\end{aligned}$$

and therefore

$$\begin{aligned}[E_{\alpha_i}, F_{\beta_j}] =&T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})[E_{\alpha_i}, F_{\beta_i}] - q[E_{\alpha_i}, F_{\beta_i}]T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j}) \\ =&T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})F_{\beta_{i-1}}K_{\alpha_i}^{-1} - qF_{\beta_{i-1}}K_{\alpha_i}^{-1}T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j}) \\ =&F_{\beta_{i-1}}T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_i}^{-1} - F_{\beta_{i-1}}T_{s_{i+1}} \cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_i}^{-1} \\ =&0.\end{aligned}$$

□

Proposition 9.2 *Let $i \in \{2, \dots, n\}$. Let $a \in \mathbb{Z}_{>0}$. Then*

$$[F_{\alpha_i}, F_{\beta_{i-1}}^a]_q = [a] F_{\beta_{i-1}}^{a-1} F_{\beta_i}$$

and for $b \in \mathbb{C}^*$

$$\varphi_{F_{\beta_{i-1}},b}(F_{\alpha_i}) = bF_{\alpha_i} + \frac{b-b^{-1}}{q-q^{-1}} F_{\beta_{i-1}}^{-1} F_{\beta_i}.$$

Proof. The first claim is proved by induction over a. $a = 1$ is shown in Proposition 9.1. The induction step:

$$\begin{aligned} F_{\alpha_i} F_{\beta_{i-1}}^{a+1} =& \left(q^a F_{\beta_{i-1}}^a F_{\alpha_i} + [a] F_{\beta_{i-1}}^{a-1} F_{\beta_i} \right) F_{\beta_{i-1}} \\ =& q^{a+1} F_{\beta_{i-1}}^{a+1} F_{\alpha_i} + q^a F_{\beta_{i-1}}^a F_{\beta_i} + q^{-1}[a] F_{\beta_{i-1}}^a F_{\beta_i} \\ =& q^{a+1} F_{\beta_{i-1}}^{a+1} F_{\alpha_i} + [a+1] F_{\beta_{i-1}}^a F_{\beta_i}. \end{aligned}$$

So we have proved the first claim. We get then for $a \in \mathbb{Z}_{>0}$

$$\varphi_{F_{\beta_{i-1}},q^a}(F_{\alpha_i}) = F_{\beta_{i-1}}^{-a} F_{\alpha_i} F_{\beta_{i-1}}^a = q^a F_{\alpha_i} + \frac{q^a - q^{-a}}{q-q^{-1}} F_{\beta_{i-1}}^{-1} F_{\beta_i}.$$

Using the fact that $\varphi_{F_{\beta_{i-1}},b}(F_{\alpha_i})$ is Laurent polynomial in b we get the second claim of the proposition. □

Proposition 9.3 *Let $i \in \{2, \dots, n\}$. Let $a \in \mathbb{Z}_{>0}$. Then*

$$[E_{\alpha_i}, F_{\beta_i}^a] = q^{a-1}[a] F_{\beta_i}^{a-1} F_{\beta_{i-1}} K_{\alpha_i}^{-1}$$

and for $b \in \mathbb{C}^*$

$$\varphi_{F_{\beta_i},b}(E_{\alpha_i}) = E_{\alpha_i} + q^{-1} b \frac{b-b^{-1}}{q-q^{-1}} F_{\beta_i}^{-1} F_{\beta_{i-1}} K_{\alpha_i}^{-1}.$$

Proof. The first claim is proved by induction over a. $a = 1$ is shown in Proposition 9.1. The induction step:

$$\begin{aligned} E_{\alpha_i} F_{\beta_i}^{a+1} =& \left(F_{\beta_i}^a E_{\alpha_i} + q^{a-1}[a] F_{\beta_i}^{a-1} F_{\beta_{i-1}} K_{\alpha_i}^{-1} \right) F_{\beta_i} \\ =& F_{\beta_i}^{a+1} E_{\alpha_i} + F_{\beta_i}^a F_{\beta_{i-1}} K_{\alpha_i}^{-1} + q^{a+1}[a] F_{\beta_i}^a F_{\beta_{i-1}} K_{\alpha_i}^{-1} \\ =& F_{\beta_i}^{a+1} E_{\alpha_i} + q^a (q^{-a} + q[a]) F_{\beta_i}^a F_{\beta_{i-1}} K_{\alpha_i}^{-1} \\ =& F_{\beta_i}^{a+1} E_{\alpha_i} + q^a [a+1] F_{\beta_i}^a F_{\beta_{i-1}} K_{\alpha_i}^{-1}. \end{aligned}$$

This proves the first claim. We get then for $a \in \mathbb{Z}_{>0}$

$$\varphi_{F_{\beta_i},q^a}(E_{\alpha_i}) = F_{\beta_i}^{-a} E_{\alpha_i} F_{\beta_i}^a = E_{\alpha_i} + q^{-1} q^a \frac{q^a - q^{-a}}{q-q^{-1}} F_{\beta_i}^{-1} F_{\beta_{i-1}} K_{\alpha_i}^{-1}.$$

Using the fact that $\varphi_{F_{\beta_i},b}(E_{\alpha_i})$ is Laurent polynomial in b we get the second claim of the proposition. □

In our classification we don't need to calculate $\varphi_{F_\Sigma,\mathbf{b}}(E_{\alpha_1})$ but for completeness we show what it is in this case in Proposition 9.5. To do this we need the following proposition:

Proposition 9.4 *Let $j \in \{2, \dots, n\}$. Then*

$$[E_{\alpha_1}, F_{\beta_j}] = -qT_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_1},$$

for $a \in \mathbb{Z}_{>0}$:

$$[E_{\alpha_1}, F_{\beta_j}^a] = -q^{2-a}[a]F_{\beta_j}^{a-1}T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_1}$$

and for $b \in \mathbb{C}^*$:

$$\varphi_{F_{\beta_j},b}(E_{\alpha_1}) = E_{\alpha_1} - q^2 b\frac{b - b^{-1}}{q - q^{-1}}F_{\beta_j}^{-1}T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_1}.$$

Proof. Like in the proof of Proposition 9.1 we see that

$$T_{s_2} \cdots T_{s_{j-1}}F_{\alpha_j} = \sum_{s=1}^{m} u_s F_{\alpha_2} u_s'$$

for some $m \in \mathbb{N}$ and some u_s, u_s' that are polynomials in $F_{\alpha_3}, \dots F_{\alpha_j}$. Note that $T_{s_1}(u_s) = u_s$ and $T_{s_1}(u_s') = u_s'$ for all s since α_1 is not connected to any of the simple roots $\alpha_3, \dots \alpha_j$. So

$$\begin{aligned}
T_{s_1}T_{s_2} \cdots T_{s_{j-1}}F_{\alpha_j} =& T_{s_1}\left(\sum_{s=1}^{m} u_s F_{\alpha_2} u_s'\right) \\
=& \sum_{s=1}^{m} u_s T_{s_1}(F_{\alpha_2}) u_s' \\
=& \sum_{s=1}^{m} u_s (F_{\alpha_2}F_{\alpha_1} - qF_{\alpha_1}F_{\alpha_2}) u_s' \\
=& \sum_{s=1}^{m} u_s F_{\alpha_2} u_s' F_{\alpha_1} - qF_{\alpha_1}\sum_{s=1}^{m} u_s F_{\alpha_2} u_s' \\
=& T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})F_{\alpha_1} - qF_{\alpha_1}T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}).
\end{aligned}$$

Thus

$$\begin{aligned}
[E_{\alpha_1}, F_{\beta_j}] =& T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})[E_{\alpha_1}, F_{\alpha_1}] - q[E_{\alpha_1}, F_{\alpha_1}]T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) \\
=& T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})\frac{K_{\alpha_1} - K_{\alpha_1}^{-1}}{q - q^{-1}} - q\frac{K_{\alpha_1} - K_{\alpha_1}^{-1}}{q - q^{-1}}T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) \\
=& T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})\frac{K_{\alpha_1} - K_{\alpha_1}^{-1} - q^2K_{\alpha_1} + K_{\alpha_1}^{-1}}{q - q^{-1}} \\
=& - qT_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_1}.
\end{aligned}$$

Note that $T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})$ is a polynomial in $F_{\alpha_2}, \dots, F_{\alpha_j}$. By Proposition 9.1 $[F_{\alpha_i}, F_{\beta_j}]_q = [F_{\alpha_i}, F_{\beta_j}] = 0$ for $1 < i < j$ and $[F_{\alpha_j}, F_{\beta_j}]_q = F_{\alpha_j}F_{\beta_j} - q^{-1}F_{\beta_j}F_{\alpha_j} = 0$ so

$$\begin{aligned}
T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j})F_{\beta_j} - q^{-1}F_{\beta_j}T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) =& [T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}), F_{\beta_j}]_q \\
=& 0.
\end{aligned}$$

The second claim is by induction on a:

$$\begin{aligned}
E_{\alpha_1} F_{\beta_j}^{a+1} =& \left(F_{\beta_j}^a E_{\alpha_1} - q^{2-a}[a] F_{\beta_j}^{a-1} T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) K_{\alpha_1}\right) F_{\beta_j} \\
=& F_{\beta_j}^{a+1} E_{\alpha_1} - q F_{\beta_j}^a T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) K_{\alpha_1} \\
& - q^{-a}[a] F_{\beta_j}^a T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) K_{\alpha_1} \\
=& F_{\beta_j}^{a+1} E_{\alpha_1} - q^{1-a}\left(q^a + q^{-1}[a]\right) F_{\beta_j}^a T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) K_{\alpha_1} \\
=& F_{\beta_j}^{a+1} E_{\alpha_1} - q^{1-a}[a+1] F_{\beta_j}^a T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) K_{\alpha_1}.
\end{aligned}$$

So we get for $a \in \mathbb{Z}_{>0}$:

$$\varphi_{F_{\beta_j}, q^a}(E_{\alpha_1}) = F_{\beta_j}^{-a} E_{\alpha_1} F_{\beta_j}^a = E_{\alpha_1} - q^2 q^{-a} \frac{q^a - q^{-a}}{q - q^{-1}} T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) K_{\alpha_1}.$$

Using the fact that $\varphi_{F_{\beta_j}, b}(E_{\alpha_1})$ is Laurent polynomial in b we get the third claim of the proposition. □

We can combine the above propositions in the following proposition

Proposition 9.5 *Let $i \in \{2, \ldots, n\}$. For $\mathbf{b} = (b_1, \ldots, b_n) \in (\mathbb{C}^*)^n$*

$$\begin{aligned}
\varphi_{F_\Sigma, \mathbf{b}}(F_{\alpha_i}) =& b_i^{-1} b_{i+1}^{-1} \cdots b_n^{-1} \varphi_{F_{\beta_{i-1}}, b_{i-1}}(F_{\alpha_i}) \\
=& b_i^{-1} b_{i+1}^{-1} \cdots b_n^{-1} (b_{i-1} F_{\alpha_i} + \frac{b_{i-1} - b_{i-1}^{-1}}{q - q^{-1}} F_{\beta_{i-1}}^{-1} F_{\beta_i}) \\
\varphi_{F_\Sigma, \mathbf{b}}(E_{\alpha_i}) =& \varphi_{F_{\beta_i}, b_i}(E_{\alpha_i}) = E_{\alpha_i} + q^{-1} b_i \frac{b_i - b_i^{-1}}{q - q^{-1}} F_{\beta_i}^{-1} F_{\beta_{i-1}} K_{\alpha_i}^{-1}.
\end{aligned}$$

Furthermore

$$\varphi_{F_\Sigma, \mathbf{b}}(F_{\alpha_1}) = b_2 \cdots b_n F_{\alpha_1}$$

and

$$\begin{aligned}
\varphi_{F_\Sigma, \mathbf{b}}(E_{\alpha_1}) =& E_{\alpha_1} - q^2 \sum_{j=2}^{n} b_j b_{j+1}^{-1} \cdots b_n^{-1} \frac{b_j - b_j^{-1}}{q - q^{-1}} F_{\beta_j}^{-1} T_{s_2} \cdots T_{s_{j-1}}(F_{\alpha_j}) K_{\alpha_1} \\
& + b_2^{-1} \cdots b_n^{-1} F_{\beta_1}^{-1} \frac{(b_1 - b_1^{-1})(q b_1^{-1} \cdots b_n^{-1} K_{\alpha_1} - q^{-1} b_1 \cdots b_n K_{\alpha_1}^{-1})}{(q - q^{-1})^2}.
\end{aligned}$$

Proof. The first two equations follow from Proposition 9.1, Proposition 9.2 and Proposition 9.3. The third follows because $F_{\alpha_1} = F_{\beta_1}$ q-commutes with all the other root vectors $F_{\beta_2}, \ldots, F_{\beta_n}$ (see also the discussion before Definition 4.19).

For the last equation we use Proposition 9.4:

$$\begin{aligned}
\varphi_{F_\Sigma,\mathbf{b}}(E_{\alpha_1}) =&\varphi_{F_{\beta_n},b_n}\circ\cdots\circ\varphi_{F_{\beta_1},b_1}(E_{\alpha_1})\\
=&\varphi_{F_{\beta_n},b_n}\circ\cdots\circ\varphi_{F_{\beta_2},b_2}\left(E_{\alpha_1}-F_{\beta_1}^{-1}\frac{(b_1-b_1^{-1})(qb_1^{-1}K_{\alpha_1}-q^{-1}b_1K_{\alpha_1}^{-1})}{(q-q^{-1})^2}\right)\\
=&\varphi_{F_{\beta_n},b_n}\circ\cdots\circ\varphi_{F_{\beta_3},b_3}(E_{\alpha_1}-q^2b_2\frac{b_2-b_2^{-1}}{q-q^{-1}}F_{\beta_2}^{-1}F_{\alpha_2}K_{\alpha_1}\\
&-b_2^{-1}F_{\beta_1}^{-1}\frac{(b_1-b_1^{-1})(qb_1^{-1}b_2^{-1}K_{\alpha_1}-q^{-1}b_1b_2K_{\alpha_1}^{-1})}{(q-q^{-1})^2})\\
&\vdots\\
=&E_{\alpha_1}-q^2\sum_{j=2}^n b_jb_{j+1}^{-1}\cdots b_n^{-1}\frac{b_j-b_j^{-1}}{q-q^{-1}}F_{\beta_j}^{-1}T_{s_2}\cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_1}\\
&-b_2^{-1}\cdots b_n^{-1}F_{\beta_1}^{-1}\frac{(b_1-b_1^{-1})(qb_1^{-1}\cdots b_n^{-1}K_{\alpha_1}-q^{-1}b_1\cdots b_nK_{\alpha_1}^{-1})}{(q-q^{-1})^2}\square
\end{aligned}$$

Proposition 9.6 *Let λ be a weight such that $\lambda(K_{\alpha_i})\in\pm q^{\mathbb{N}}$ for $i=2,\dots,n$ and $\lambda(K_{\alpha_1})\notin\pm q^{\mathbb{N}}$. Let $\mathbf{b}=(b_1,\dots,b_n)\in(\mathbb{C}^*)^n$. Let $i\in\{2,\dots,n\}$. Then E_{α_i} acts injectively on the U_q-module $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ if and only if $b_i\notin\pm q^{\mathbb{Z}}$ and F_{α_i} acts injectively on $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ if and only if $b_{i-1}\notin\pm q^{\mathbb{Z}}$.*

Proof. By Proposition 4.10 and Corollary 4.20 a root vector acts injectively on the U_q-module

$$\varphi_{F_\Sigma,(b_1,\dots,b_n)}.L(\lambda)_{F_\Sigma}$$

if and only if it acts injectively on

$$\varphi_{F_\Sigma,(\varepsilon_1q^{i_1}b_1,\dots,\varepsilon_nq^{i_n}b_n)}.L(\lambda)_{F_\Sigma}$$

for any $i_1,\dots,i_n\in\mathbb{Z}$ and $\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}$.

Assume there exists a $0\neq v\in\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ such that $E_{\alpha_i}v=0$. We have $v=F_{\beta_1}^{a_1}\cdots F_{\beta_n}^{a_n}\otimes v'$ for some $a_1,\dots,a_n\in\mathbb{Z}_{\leq0}$ and some $v'\in L(\lambda)$. So $E_{\alpha_i}v=0$ implies

$$0=\varphi_{F_\Sigma,\mathbf{b}}(E_{\alpha_i})F_{\beta_1}^{a_1}\cdots F_{\beta_n}^{a_n}\otimes v'=F_{\beta_1}^{a_1}\cdots F_{\beta_n}^{a_n}\otimes\varphi_{F_\Sigma,\mathbf{c}}(E_{\alpha_i})v'$$

where $\mathbf{c}=(q^{a_1}b_1,\dots,q^{a_n}b_n)$. So there exists a $v'\in L(\lambda)$ such that $\varphi_{F_\Sigma,\mathbf{c}}(E_{\alpha_i})v'=0$. That is

$$\left(E_{\alpha_i}+q^{-1}c_i\frac{c_i-c_i^{-1}}{q-q^{-1}}F_{\beta_i}^{-1}F_{\beta_{i-1}}K_{\alpha_i}^{-1}\right)v'=0$$

or equivalently

$$F_{\beta_i}E_{\alpha_i}v'=q^{-1}c_i\frac{c_i^{-1}-c_i}{q-q^{-1}}F_{\beta_{i-1}}K_{\alpha_i}^{-1}v'.$$

Since $L(\lambda)$ is a highest weight module we have some $r\in\mathbb{N}$ such that $E_{\alpha_i}^rv'\neq0$ and $E_{\alpha_i}^{r+1}v'=0$. Fix this r. We get

$$E_{\alpha_i}^{(r)}F_{\beta_i}E_{\alpha_i}v'=E_{\alpha_i}^{(r)}q^{-1}c_i\frac{c_i^{-1}-c_i}{q-q^{-1}}F_{\beta_{i-1}}K_{\alpha_i}^{-1}v'$$

and calculating the right hand side and left hand side we get

$$q^{r-1}[r]F_{\beta_{i-1}}K_{\alpha_i}^{-1}E_{\alpha_i}^{(r)}v' = q^{-1+2r}c_i\frac{c_i^{-1}-c_i}{q-q^{-1}}F_{\beta_{i-1}}K_{\alpha_i}^{-1}E_{\alpha_i}^{(r)}v'.$$

So we must have

$$q^{r-1}[r] = q^{-1+2r}c_i\frac{c_i^{-1}-c_i}{q-q^{-1}}$$

or equivalently $c_i = \pm q^{-r}$. Since $c_i \in q^{\mathbb{Z}}b_i$ we have proved the first claim.

The other claim is shown similarly (see e.g. the calculations done in the proof of Proposition 10.5. The calculations will be the same in this case). □

Proposition 9.7 *Let M be a weight U_q-module of finite Jordan-Hölder length with finite dimensional weight spaces. Let $\alpha \in \Pi$. If E_α and F_α both act injectively on M then E_α and F_α act injectively on every composition factor of M.*

Proof. Let V be a simple U_q-submodule of M. Let μ be a weight of V. Then V_μ is a simple $(U_q)_0$-module by Theorem 2.7 and $E_\alpha F_\alpha$ and $F_\alpha E_\alpha$ act injectively on V_μ by assumption. Since $\dim M_\mu < \infty$ this implies that $F_\alpha E_\alpha$ and $E_\alpha F_\alpha$ act injectively on the $(U_q)_0$ module $(M/V)_\mu \cong M_\mu/V_\mu$. Since M/V is the sum of its weight spaces this implies that $E_\alpha F_\alpha$ and $F_\alpha E_\alpha$ act injectively on M/V. This in turn implies that E_α and F_α act injectively on M/V. Doing induction on the Jordan-Hölder length of M finishes the proof. □

The above proposition is true for a general simple Lie algebra $\mathfrak{g}$ and we will use it in the next section as well.

Theorem 9.8 *Let λ be a weight such that $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$ for $i = 2,\dots,n$ and $\lambda(K_{\alpha_1}) \notin \pm q^{\mathbb{N}}$. Let $\mathbf{b} = (b_1,\dots,b_n) \in (\mathbb{C}^*)^n$. Then $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is simple and torsion free if and only if $b_i \notin \pm q^{\mathbb{Z}}$, $i = 1,\dots,n$ and $\lambda(K_{\alpha_1})^{-1}b_1\cdots b_n \notin \pm q^{\mathbb{Z}}$.*

Proof. By Proposition 5.10 $L(\lambda)$ is a subquotient of

$$^{\overline{s_1}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1}),1,\dots,1)}.L(\lambda)_{F_\Sigma}\right).$$

So by Lemma 5.3 we get (using that $L(\lambda) = {}^{s_1}\left({}^{\overline{s_1}}L(\lambda)\right)$) for any $\mathbf{c} = (c_1,\dots,c_n) \in (\mathbb{C}^*)^n$

$$\left(\varphi_{F_\Sigma,\mathbf{c}}.L(\lambda)_{F_\Sigma}\right)^{ss} \cong {}^{\overline{s_1}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})c_1^{-1}\cdots c_n^{-1},c_2,\dots,c_n)}.L(\lambda)_{F_\Sigma}\right)^{ss}.$$

We have $\lambda(K_{\alpha_2}) = \varepsilon q^r$ for some $r \in \mathbb{N}$ and some $\varepsilon \in \{\pm 1\}$. We see in the proof of Lemma 5.11 that $L(\lambda)$ is a subqoutient of

$$^{\overline{s_2}}\left(\varphi_{F_\Sigma,(\varepsilon,\varepsilon,1,\dots,1)}.L(\lambda)_{F_\Sigma}\right).$$

We get by Lemma 5.3 (using that $L(\lambda) = {}^{s_2}\left({}^{\overline{s_2}}L(\lambda)\right)$) for any $\mathbf{c} = (c_1,\dots,c_n) \in (\mathbb{C}^*)^n$

$$\left(\varphi_{F_\Sigma,\mathbf{c}}.L(\lambda)_{F_\Sigma}\right)^{ss} \cong {}^{\overline{s_2}}\left(\varphi_{F_\Sigma,(\varepsilon c_2,\varepsilon c_1,c_3,\dots,c_n)}.L(\lambda)_{F_\Sigma}\right)^{ss}.$$

Combining the above we get

$$\begin{aligned}\left(\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}\right)^{ss} &\cong \overline{^{s_2}}\left(\varphi_{F_\Sigma,(\varepsilon b_2,\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right)^{ss} \\ &\cong \overline{^{s_2}}\left(\overline{^{s_1}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})b_1^{-1}\cdots b_n^{-1},\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right)\right)^{ss} \\ &\cong \overline{^{s_1 s_2}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})b_1^{-1}\cdots b_n^{-1},\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right)^{ss}.\end{aligned}$$

Since $T_{s_1}^{-1}T_{s_2}^{-1}(E_{\alpha_1}) = E_{\alpha_2}$ and $T_{s_1}^{-1}T_{s_2}^{-1}(F_{\alpha_1}) = F_{\alpha_2}$ we get by Proposition 9.6 that E_{α_1} acts injectively on $\overline{^{s_1 s_2}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})b_1^{-1}\cdots b_n^{-1},\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right)$ if and only if $b_1 \notin \pm q^{\mathbb{Z}}$ and F_{α_1} acts injectively on $\overline{^{s_1 s_2}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})b_1^{-1}\cdots b_n^{-1},\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right)$ if and only if $\lambda(K_{\alpha_1})^{-1}b_1\cdots b_n \notin \pm q^{\mathbb{Z}}$.

Assume $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is torsion free. Then all root vectors act injectively on $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. We claim $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is simple: Let $V \subset \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ be a simple module. Then V is admissible of the same degree d as $L(\lambda)$ by Proposition 5.6 and because all root vectors act injectively $\dim V_{q^\mu\lambda} = d$ for all $\mu \in Q$. So $V = \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. Thus $\left(\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}\right)^{ss} = \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. Then by the above

$$\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma} \cong \overline{^{s_1 s_2}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})b_1^{-1}\cdots b_n^{-1},\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right).$$

This shows that when $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is torsion free we must have $\lambda(K_{\alpha_1})^{-1}b_1\cdots b_n \notin \pm q^{\mathbb{Z}}$. By Proposition 9.6 $b_i \notin \pm q^{\mathbb{Z}}$, $i = 1,\dots,n$.

Assume on the other hand that $b_i \notin \pm q^{\mathbb{Z}}$ for $i \in \{1,\dots,n\}$ and $\lambda(K_{\alpha_1})^{-1}b_1\cdots b_n \notin \pm q^{\mathbb{Z}}$. By Proposition 9.6 we get that the simple root vectors $E_{\alpha_2},\dots,E_{\alpha_n}$ and $F_{\alpha_1},\dots,F_{\alpha_n}$ all act injectively on $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. We need to show that E_{α_1} acts injectively on the module. By the above

$$\left(\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}\right)^{ss} \cong \overline{^{s_1 s_2}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})b_1^{-1}\cdots b_n^{-1},\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right)^{ss}$$

and the root vectors $E_{\alpha_1}, F_{\alpha_1}$ act injectively on

$$\overline{^{s_1 s_2}}\left(\varphi_{F_\Sigma,(\lambda(K_{\alpha_1})b_1^{-1}\cdots b_n^{-1},\varepsilon b_1,b_3,\dots,b_n)}.L(\lambda)_{F_\Sigma}\right).$$

Then by Proposition 9.7 E_{α_1} act injectively on all composition factors of $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$.

Let V be a simple U_q-submodule of $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. By the above all simple root vectors act injectively on V and then like above this implies $V = \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ i.e. $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is simple and torsion free. □

By the comments after Corollary 8.15 the above Theorem completes the classification of simple torsion free modules in type A.

10 Classification of simple torsion free modules. Type C.

In this section we assume $\mathfrak{g}$ is of type C_n (i.e. $\mathfrak{g} = \mathfrak{sp}_{2n}$) with $n \geq 2$. Let $\Pi = \{\alpha_1,\dots,\alpha_n\}$ denote the simple roots such that $(\alpha_i|\alpha_{i+1}) = -1$, $i = 2,\dots,n-1$, $\langle \alpha_2, \alpha_1^\vee \rangle = -1$ and $\langle \alpha_1, \alpha_2^\vee \rangle = -2$ i.e. α_1 is long and $\alpha_2,\dots,\alpha_n$ are short.

Set $\beta_j = s_1 \cdots s_{j-1}(\alpha_j) = \alpha_1 + \cdots + \alpha_j$, then $\Sigma = \{\beta_1, \ldots, \beta_n\}$ is a set of commuting roots with corresponding root vectors $F_{\beta_j} = T_{s_1} \cdots T_{s_{j-1}}(F_{\alpha_j})$. We will show some commutation formulas and use these to calculate $\varphi_{F_\Sigma, \mathbf{b}}$ on most of the simple root vectors.

Choose a reduced expression of w_0 starting with $s_1 \cdots s_n s_1 \cdots s_{n-1}$ and define root vectors $F_{\gamma_1}, \ldots, F_{\gamma_N}$ from this expression. Note that $F_{\beta_i} = F_{\gamma_i}$ for $i = 1, \ldots, n$. Note for use in the proposition below that for $j \in \{1, \ldots, n-1\}$,

$$\gamma_{n+j} = s_1 \cdots s_n s_1 \cdots s_{j-1}(\alpha_j) = \alpha_1 + 2\alpha_2 + \alpha_3 + \cdots \alpha_{j+1}$$

and

$$\begin{aligned} F_{\gamma_{n+j}} =& T_{s_1} \cdots T_{s_n} T_{s_1} \cdots T_{s_{j-1}}(F_{\alpha_j}) \\ =& T_{s_1} \cdots T_{s_{j+1}} T_{s_1} \cdots T_{s_{j-1}}(F_{\alpha_j}). \end{aligned}$$

In particular $F_{\alpha_1 + 2\alpha_2} = T_{s_1} T_{s_2}(F_{\alpha_1})$.

Proposition 10.1 *Let $i \in \{2, \ldots, n\}$ and $j \in \{1, \ldots, n\}$*

$$[F_{\alpha_i}, F_{\beta_j}]_q = \begin{cases} [2] F_{\alpha_1 + 2\alpha_2}, & \text{if } j = i = 2 \\ F_{\alpha_1 + 2\alpha_2 + \alpha_3 + \cdots + \alpha_j}, & \text{if } i = 2 \text{ and } j > 2 \\ F_{\beta_i}, & \text{if } j = i - 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$[E_{\alpha_i}, F_{\beta_j}] = \begin{cases} [2] F_{\beta_1} K_{\alpha_2}^{-1}, & \text{if } j = 2 = i \\ F_{\beta_{i-1}} K_{\alpha_i}^{-1}, & \text{if } j = i > 2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We will show the proposition for the F's first and then for the E's. Assume first that $j < i - 1$. Then clearly $[F_{\alpha_i}, F_{\beta_j}]_q = [F_{\alpha_i}, F_{\beta_j}] = 0$ since α_i is not connected to any of the simple roots $\alpha_1, \ldots, \alpha_j$ appearing in β_j.

Then assume $j \geq i > 2$. We must have $\alpha_i = \gamma_k$ for some $k > n$ since $\{\gamma_1, \ldots, \gamma_N\} = \Phi^+$. By Theorem 3.2 $[F_{\alpha_i}, F_{\beta_j}]_q$ is a linear combination of monomials of the form $F_{\gamma_{j+1}}^{a_{j+1}} \cdots F_{\gamma_{k-1}}^{a_{k-1}}$. For a monomial of this form to appear with nonzero coefficient we must have

$$\sum_{h=j+1}^{k-1} a_h \gamma_h = \alpha_i + \beta_j = \alpha_1 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \ldots \alpha_j.$$

For this to be possible one of the positive roots γ_s, $j < s < k$ must be equal to $\alpha_1 + \alpha_2 + \cdots + \alpha_m$ for some $m \leq j$ but $\alpha_1 + \alpha_2 + \cdots + \alpha_m = \gamma_m$ by construction and $m \leq j < s$ so $m \neq s$. We conclude that this is not possible.

Assume $j = i - 1$. We have

$$\begin{aligned} [F_{\alpha_i}, F_{\beta_{i-1}}]_q =& [T_{s_1} \cdots T_{s_{i-2}}(F_{\alpha_i}), T_{s_1} \cdots T_{s_{i-2}}(F_{\alpha_{i-1}})]_q \\ =& T_{s_1} \cdots T_{s_{i-2}} \left([F_{\alpha_i}, F_{\alpha_{i-1}}]_q \right) \\ =& T_{s_1} \cdots T_{s_{i-2}} T_{s_{i-1}}(F_{\alpha_i}) \\ =& F_{\beta_i}. \end{aligned}$$

Assume $j = 2 = i$. Then

$$\begin{aligned}[F_{\alpha_2}, F_{\beta_2}]_q =& F_{\alpha_2}F_{\beta_2} - F_{\beta_2}F_{\alpha_2}\\ =& F_{\alpha_2}(F_{\alpha_2}F_{\alpha_1} - q^2F_{\alpha_1}F_{\alpha_2}) - (F_{\alpha_2}F_{\alpha_1} - q^2F_{\alpha_1}F_{\alpha_2})F_{\alpha_2}\\ =& (q^2F_{\alpha_1}F_{\alpha_2}^2 - q[2]F_{\alpha_2}F_{\alpha_1}F_{\alpha_2} + F_{\alpha_2}^2F_{\alpha_1})\\ =& [2]T_{s_2}^{-1}(F_{\alpha_1})\\ =& [2]T_{s_2}^{-1}T_{s_2}T_{s_1}T_{s_2}(F_{\alpha_1})\\ =& [2]T_{s_1}T_{s_2}(F_{\alpha_1})\\ =& [2]F_{\alpha_1+2\alpha_2}.\end{aligned}$$

Assume $i = 2$ and $j = 3$. Then

$$\begin{aligned}F_{\alpha_1+2\alpha_2+\alpha_3} =& F_{\gamma_{n+2}}\\ =& T_{s_1}T_{s_2}T_{s_3}T_{s_1}(F_{\alpha_2})\\ =& T_{s_1}T_{s_2}T_{s_1}T_{s_3}(F_{\alpha_2})\\ =& T_{s_1}T_{s_2}T_{s_1}(F_{\alpha_2}F_{\alpha_3} - qF_{\alpha_3}F_{\alpha_2})\\ =& F_{\alpha_2}F_{\beta_3} - qF_{\beta_3}F_{\alpha_2}.\end{aligned}$$

Finally assume $i = 2$ and $j > 3$. We have

$$\begin{aligned}F_{\alpha_1+2\alpha_2+\alpha_3+\cdots+\alpha_j} =& F_{\gamma_{n+j-1}}\\ =& T_{s_1}\cdots T_{s_{j-2}}T_{s_{j-1}}T_{s_j}T_{s_1}\cdots T_{s_{j-3}}T_{s_{j-2}}(F_{\alpha_{j-1}})\\ =& T_{s_1}\cdots T_{s_{j-2}}T_{s_1}\cdots T_{s_{j-3}}T_{s_{j-1}}T_{s_{j-2}}T_{s_j}(F_{\alpha_{j-1}})\\ =& T_{s_1}\cdots T_{s_{j-2}}T_{s_1}\cdots T_{s_{j-3}}T_{s_{j-1}}T_{s_{j-2}}(F_{\alpha_{j-1}}F_{\alpha_j} - qF_{\alpha_j}F_{\alpha_{j-1}})\\ =& T_{s_1}\cdots T_{s_{j-2}}T_{s_1}\cdots T_{s_{j-3}}(F_{\alpha_{j-2}}T_{s_{j-1}}(F_{\alpha_j}) - qT_{s_{j-1}}(F_{\alpha_j})F_{\alpha_{j-2}})\\ =& F_{\alpha_2}F_{\beta_j} - qF_{\beta_j}F_{\alpha_2}\\ =& [F_{\alpha_2}, F_{\beta_j}]_q\end{aligned}$$

using the facts that $T_{s_{j-1}}T_{s_{j-2}}(F_{\alpha_{j-1}}) = F_{\alpha_{j-2}}$ and $T_{s_1}\cdots T_{s_{j-2}}T_{s_1}\cdots T_{s_{j-3}}(F_{\alpha_{j-2}}) = F_{\alpha_2}$ by Proposition 8.20 in [Jan96] (The proposition is about the E root vectors but the proposition is true for the F's as well).

For the E's: Assume first $j < i$: Since F_{β_j} is a polynomial in $F_{\alpha_1}, \ldots, F_{\alpha_j}$, E_{α_i} commutes with F_{β_j} when $j < i$.

Assume then $j = i$: We have by the above

$$F_{\beta_i} = [F_{\alpha_i}, F_{\beta_{i-1}}]_q$$

so

$$\begin{aligned}[E_{\alpha_i},F_{\beta_i}] =&[E_{\alpha_i},(F_{\alpha_i}F_{\beta_{i-1}}-q^{-(\beta_{i-1}|\alpha_i)}F_{\beta_{i-1}}F_{\alpha_i})]\\ =&[E_{\alpha_i},F_{\alpha_i}]F_{\beta_{i-1}}-q_{\alpha_{i-1}}F_{\beta_{i-1}}[E_{\alpha_i},F_{\alpha_i}]\\ =&\frac{K_{\alpha_i}-K_{\alpha_i}^{-1}}{q-q^{-1}}F_{\beta_{i-1}}-q_{\alpha_{i-1}}F_{\beta_{i-1}}\frac{K_{\alpha_i}-K_{\alpha_i}^{-1}}{q-q^{-1}}\\ =&F_{\beta_{i-1}}\frac{q_{\alpha_{i-1}}K_{\alpha_i}-q_{\alpha_{i-1}}^{-1}K_{\alpha_i}^{-1}-q_{\alpha_{i-1}}K_{\alpha_i}+q_{\alpha_{i-1}}K_{\alpha_i}^{-1}}{q-q^{-1}}\\ =&\frac{q_{\alpha_{i-1}}-q_{\alpha_{i-1}}^{-1}}{q-q^{-1}}F_{\beta_{i-1}}K_{\alpha_i}^{-1}\\ =&\begin{cases}[2]F_{\beta_{i-1}}K_{\alpha_i}^{-1}, & \text{if } i=2\\ F_{\beta_{i-1}}K_{\alpha_i}^{-1}, & \text{otherwise.}\end{cases}\end{aligned}$$

Finally assume $j>i$: Observe first that we have

$$T_{s_{i+1}}\cdots T_{s_{j-1}}F_{\alpha_j}=\sum_{s=1}^m u_sF_{\alpha_{i+1}}u_s'$$

for some $m\in\mathbb{N}$ and some u_s,u_s' that are polynomials in $F_{\alpha_{i+2}},\dots F_{\alpha_j}$. Note that $T_{s_i}(u_s)=u_s$ and $T_{s_i}(u_s')=u_s'$ for all s since α_i is not connected to any of the simple roots $\alpha_{i+2},\dots\alpha_j$. So

$$\begin{aligned}T_{s_i}T_{s_{i+1}}\cdots T_{s_{j-1}}F_{\alpha_j} =&T_{s_i}\left(\sum_{s=1}^m u_sF_{\alpha_{i+1}}u_s'\right)\\ =&\sum_{s=1}^m u_sT_{s_i}(F_{\alpha_{i+1}})u_s'\\ =&\sum_{s=1}^m u_s(F_{\alpha_{i+1}}F_{\alpha_i}-qF_{\alpha_i}F_{\alpha_{i+1}})u_s'\\ =&\sum_{s=1}^m u_sF_{\alpha_{i+1}}u_s'F_{\alpha_i}-qF_{\alpha_i}\sum_{s=1}^m u_sF_{\alpha_{i+1}}u_s'\\ =&T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})F_{\alpha_i}-qF_{\alpha_i}T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j}).\end{aligned}$$

Thus we see that

$$\begin{aligned}F_{\beta_j} =&T_{s_1}\dots T_{s_i}\cdots T_{s_{j-1}}(F_{\alpha_j})\\ =&T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})T_{s_1}\cdots T_{s_{i-1}}(F_{\alpha_i})-qF_{\alpha_i}T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})\\ =&T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})F_{\beta_i}-qF_{\beta_i}T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})\end{aligned}$$

and therefore

$$\begin{aligned}[E_{\alpha_i},F_{\beta_j}] =&T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})[E_{\alpha_i},F_{\beta_i}]-q[E_{\alpha_i},F_{\beta_i}]T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})\\ =&[r_i](T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})F_{\beta_{i-1}}K_{\alpha_i}^{-1}-qF_{\beta_{i-1}}K_{\alpha_i}^{-1}T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j}))\\ =&[r_i](F_{\beta_{i-1}}T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_i}^{-1}-F_{\beta_{i-1}}T_{s_{i+1}}\cdots T_{s_{j-1}}(F_{\alpha_j})K_{\alpha_i}^{-1})\\ =&0\end{aligned}$$

where

$$r_i = \begin{cases} 2 & \text{if } i = 2 \\ 1 & \text{otherwise.} \end{cases}$$

□

Proposition 10.2 *Let $i \in \{2, \dots, n\}$. Let $a \in \mathbb{Z}_{>0}$. Then*

$$[F_{\alpha_i}, F^a_{\beta_{i-1}}]_q = [a]_{\beta_{i-1}} F^{a-1}_{\beta_{i-1}} F_{\beta_i}$$

and for $b \in \mathbb{C}^*$

$$\varphi_{F_{\beta_{i-1}},b}(F_{\alpha_i}) = \begin{cases} b^2 F_{\alpha_2} + \frac{b^2-b^{-2}}{q^2-q^{-2}} F^{-1}_{\beta_1} F_{\beta_2}, & \text{if } i = 2 \\ bF_{\alpha_i} + \frac{b-b^{-1}}{q-q^{-1}} F^{-1}_{\beta_{i-1}} F_{\beta_i}, & \text{otherwise.} \end{cases}$$

Proof. The first claim is proved by induction over a. $a = 1$ is shown in Proposition 9.1. The induction step:

$$\begin{aligned} F_{\alpha_i} F^{a+1}_{\beta_{i-1}} =& \left(q^a_{\beta_{i-1}} F^a_{\beta_{i-1}} F_{\alpha_i} + [a]_{\beta_{i-1}} F^{a-1}_{\beta_{i-1}} F_{\beta_i} \right) F_{\beta_{i-1}} \\ =& q^{a+1}_{\beta_{i-1}} F^{a+1}_{\beta_{i-1}} F_{\alpha_i} + q^a_{\beta_{i-1}} F^a_{\beta_{i-1}} F_{\beta_i} + q^{-1}_{\beta_{i-1}} [a]_{\beta_{i-1}} F^a_{\beta_{i-1}} F_{\beta_i} \\ =& q^{a+1}_{\beta_{i-1}} F^{a+1}_{\beta_{i-1}} F_{\alpha_i} + [a+1]_{\beta_{i-1}} F^a_{\beta_{i-1}} F_{\beta_i}. \end{aligned}$$

So we have proved the first claim. We get then for $a \in \mathbb{Z}_{>0}$:

$$\varphi_{F_{\beta_{i-1}},q^a}(F_{\alpha_i}) = F^{-a}_{\beta_{i-1}} F_{\alpha_i} F^a_{\beta_{i-1}} = q^a_{\beta_{i-1}} F_{\alpha_i} + \frac{q^a_{\beta_{i-1}} - q^{-a}_{\beta_{i-1}}}{q_{\beta_{i-1}} - q^{-1}_{\beta_{i-1}}} F^{-1}_{\beta_{i-1}} F_{\beta_i}.$$

Using the fact that $\varphi_{F_{\beta_{i-1}},b}(F_{\alpha_i})$ is Laurent polynomial in b we get the second claim of the proposition. □

Proposition 10.3 *Let $i \in \{2, \dots, n\}$. Let $a \in \mathbb{Z}_{>0}$. Then*

$$[E_{\alpha_i}, F^a_{\beta_i}] = \begin{cases} q^{a-1}[2][a] F^{a-1}_{\beta_2} F_{\beta_1} K^{-1}_{\alpha_2}, & \text{if } i = 2 \\ q^{a-1}[a] F^{a-1}_{\beta_i} F_{\beta_{i-1}} K^{-1}_{\alpha_i}, & \text{otherwise.} \end{cases}$$

and for $b \in \mathbb{C}^*$

$$\varphi_{F_{\beta_i},b}(E_{\alpha_i}) = \begin{cases} E_{\alpha_2} + q^{-1}[2] b \frac{b-b^{-1}}{q-q^{-1}} F^{-1}_{\beta_2} F_{\beta_1} K^{-1}_{\alpha_2}, & \text{if } i = 2 \\ E_{\alpha_i} + q^{-1} b \frac{b-b^{-1}}{q-q^{-1}} F^{-1}_{\beta_i} F_{\beta_{i-1}} K^{-1}_{\alpha_i}, & \text{otherwise.} \end{cases}$$

Proof. The first claim is proved by induction over a. $a = 1$ is shown in Proposition 9.1. The induction step: For $i > 2$:

$$\begin{aligned} E_{\alpha_i} F^{a+1}_{\beta_i} =& \left(F^a_{\beta_i} E_{\alpha_i} + q^{a-1}[a] F^{a-1}_{\beta_i} F_{\beta_{i-1}} K^{-1}_{\alpha_i} \right) F_{\beta_i} \\ =& F^{a+1}_{\beta_i} E_{\alpha_i} + F^a_{\beta_i} F_{\beta_{i-1}} K^{-1}_{\alpha_i} + q^{a+1}[a] F^a_{\beta_i} F_{\beta_{i-1}} K^{-1}_{\alpha_i} \\ =& F^{a+1}_{\beta_i} E_{\alpha_i} + q^a (q^{-a} + q[a]) F^a_{\beta_i} F_{\beta_{i-1}} K^{-1}_{\alpha_i} \\ =& F^{a+1}_{\beta_i} E_{\alpha_i} + q^a [a+1]_{\alpha_{i-1}} F^a_{\beta_i} F_{\beta_{i-1}} K^{-1}_{\alpha_i}. \end{aligned}$$

For $i = 2$:

$$\begin{aligned}E_{\alpha_2}F_{\beta_2}^{a+1} =&\left(F_{\beta_2}^a E_{\alpha_2} + q^{a-1}[2][a]F_{\beta_2}^{a-1}F_{\beta_1}K_{\alpha_2}^{-1}\right)F_{\beta_2}\\ =&F_{\beta_2}^{a+1}E_{\alpha_2} + [2]F_{\beta_2}^{a}F_{\beta_1}K_{\alpha_2}^{-1} + q^{a+1}[2][a]F_{\beta_2}^{a}F_{\beta_1}K_{\alpha_2}^{-1}\\ =&F_{\beta_2}^{a+1}E_{\alpha_2} + q^a[2](q^{-a}+q[a])F_{\beta_2}^{a}F_{\beta_1}K_{\alpha_2}^{-1}\\ =&F_{\beta_2}^{a+1}E_{\alpha_2} + q^a[2][a+1]F_{\beta_2}^{a}F_{\beta_1}K_{\alpha_2}^{-1}.\end{aligned}$$

This proves the first claim. We get then for $a \in \mathbb{Z}_{>0}$

$$\varphi_{F_{\beta_i},q^a}(E_{\alpha_i}) = F_{\beta_i}^{-a}E_{\alpha_i}F_{\beta_i}^a = \begin{cases} E_{\alpha_2} + q^{-2}q^{2a}\frac{q^{2a}-q^{-2a}}{q-q^{-1}}F_{\beta_2}^{-1}F_{\beta_1}K_{\alpha_2}^{-1}, & \text{if } i = 2\\ E_{\alpha_i} + q^{-1}q^a\frac{q^a-q^{-a}}{q-q^{-1}}F_{\beta_i}^{-1}F_{\beta_{i-1}}K_{\alpha_i}^{-1}, & \text{otherwise.}\end{cases}$$

Using the fact that $\varphi_{F_{\beta_i},b}(E_{\alpha_i})$ is Laurent polynomial in b we get the second claim of the proposition. □

We combine the above propositions in the following proposition

Proposition 10.4 *Let $i \in \{3,\dots,n\}$. For $\mathbf{b} = (b_1,\dots,b_n) \in (\mathbb{C}^*)^n$*

$$\begin{aligned}\varphi_{F_\Sigma,\mathbf{b}}(F_{\alpha_i}) =&\varphi_{F_{\beta_{i-1}},b_{i-1}}(F_{\alpha_i})\\ =&b_{i-1}F_{\alpha_i} + \frac{b_{i-1}-b_{i-1}^{-1}}{q-q^{-1}}F_{\beta_{i-1}}^{-1}F_{\beta_i}\\ \varphi_{F_\Sigma,\mathbf{b}}(E_{\alpha_i}) =&\varphi_{F_{\beta_i},b_i}(E_{\alpha_i}) = E_{\alpha_i} + q^{-1}b_i\frac{b_i-b_i^{-1}}{q-q^{-1}}F_{\beta_i}^{-1}F_{\beta_{i-1}}K_{\alpha_i}^{-1}.\end{aligned}$$

Furthermore

$$\varphi_{F_\Sigma,\mathbf{b}}(E_{\alpha_2}) = E_{\alpha_2} + q^{-1}[2]b_2\frac{b_2-b_2^{-1}}{q-q^{-1}}F_{\beta_2}^{-1}F_{\beta_1}K_{\alpha_2}^{-1}$$

and

$$\varphi_{F_\Sigma,\mathbf{b}}(F_{\alpha_1}) = b_2\cdots b_nF_{\alpha_1}.$$

With similar proof as the proof of Proposition 9.6 we can show

Proposition 10.5 *Let λ be a weight such that $\lambda(K_\beta) \in \pm q^{\mathbb{N}}$ for all short $\beta \in \Phi^+$ and $\lambda(K_\gamma) \in \pm q^{1+2\mathbb{Z}}$ for all long $\gamma \in \Phi^+$. Let $\mathbf{b} = (b_1,\dots,b_n) \in (\mathbb{C}^*)^n$. E_{α_2} acts injectively on the U_q-module $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ if and only if $b_2 \notin \pm q^{\mathbb{Z}}$. Let $i \in \{3,\dots,n\}$. Then E_{α_i} acts injectively on the module $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ if and only if $b_i \notin \pm q^{\mathbb{Z}}$ and F_{α_i} acts injectively on $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ if and only if $b_{i-1} \notin \pm q^{\mathbb{Z}}$.*

Proof. By Proposition 4.10 and Corollary 4.20 a root vector acts injectively on the U_q-module

$$\varphi_{F_\Sigma,(b_1,\dots,b_n)}.L(\lambda)_{F_\Sigma}$$

if and only if it acts injectively on

$$\varphi_{F_\Sigma,(\varepsilon_1q^{i_1}b_1,\dots,\varepsilon_nq^{i_n}b_n)}.L(\lambda)_{F_\Sigma}$$

for any $i_1,\dots,i_n \in \mathbb{Z}$ and $\varepsilon_1,\dots,\varepsilon_n \in \{\pm 1\}$.

Assume there exists a $0 \neq v \in \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ such that $F_{\alpha_i} v = 0$. We have $v = F_{\beta_1}^{a_1} \cdots F_{\beta_n}^{a_n} \otimes v'$ for some $a_1, \dots, a_n \in \mathbb{Z}_{\leq 0}$ and some $v' \in L(\lambda)$. $F_{\alpha_i} v = 0$ implies

$$0 = \varphi_{F_\Sigma,\mathbf{b}}(F_{\alpha_i}) F_{\beta_1}^{a_1} \cdots F_{\beta_n}^{a_n} \otimes v' = F_{\beta_1}^{a_1} \cdots F_{\beta_n}^{a_n} \otimes \varphi_{F_\Sigma,\mathbf{c}}(F_{\alpha_i}) v'$$

where $\mathbf{c} = (q^{a_1} b_1, \dots, q^{a_n} b_n)$. So there exists a $v' \in L(\lambda)$ such that $\varphi_{F_\Sigma,\mathbf{c}}(F_{\alpha_i}) v' = 0$. That is

$$\left(c_{i-1} F_{\alpha_i} + \frac{c_{i-1} - c_{i-1}^{-1}}{q - q^{-1}} F_{\beta_{i-1}}^{-1} F_{\beta_i} \right) v' = 0$$

or equivalently

$$\left(F_{\beta_{i-1}} F_{\alpha_i} + c_{i-1}^{-1} \frac{c_{i-1} - c_{i-1}^{-1}}{q - q^{-1}} F_{\beta_i} \right) v' = 0.$$

Let $r \in \mathbb{N}$ be such that $F_{\alpha_i}^{(r)} v' \neq 0$ and $F_{\alpha_i}^{(r+1)} v' = 0$ (possible since $\lambda(K_{\alpha_i}) \in \pm q^{\mathbb{N}}$ so $-\alpha_i \in F_{L(\lambda)}$). So the above being equal to zero implies

$$\begin{aligned} 0 =& F_{\alpha_i}^{(r)} \left(F_{\beta_{i-1}} F_{\alpha_i} + c_{i-1}^{-1} \frac{c_{i-1} - c_{i-1}^{-1}}{q - q^{-1}} F_{\beta_i} \right) v' \\ =& \left([r] F_{\beta_i} F_{\alpha_i}^{(r)} + q^{-r} \frac{1 - c_{i-1}^{-2}}{q - q^{-1}} F_{\beta_i} F_{\alpha_i}^{(r)} \right) v' \\ =& \left([r] + q^{-r} \frac{1 - c_{i-1}^{-2}}{q - q^{-1}} \right) F_{\beta_i} F_{\alpha_i}^{(r)} v'. \end{aligned}$$

Since $F_{\beta_i} F_{\alpha_i}^{(r)} v' \neq 0$ this is equivalent to

$$0 = q^r - q^{-r} + q^{-r} - q^{-r} c_{i-1}^{-2} = q^r - q^{-r} c_{i-1}^{-2}$$

or equivalently $c_{i-1} = \pm q^{-r}$.

The other claims are shown similarly. $\square$

Proposition 10.6 *Let λ be a weight such that $\lambda(K_\beta) \in \pm q^{\mathbb{N}}$ for all short $\beta \in \Phi^+$ and $\lambda(K_\gamma) \in \pm q^{1+2\mathbb{Z}}$ for all long $\gamma \in \Phi^+$. Let $\mathbf{b} = (b_1, \dots, b_n) \in (\mathbb{C}^*)^n$. Then $F_{\alpha_1+2\alpha_2}$ acts injectively on the U_q-module $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$.*

Proof. We can show similarly to the above calculations in this section that

$$\varphi_{F_\Sigma,\mathbf{b}}(F_{\alpha_1+2\alpha_2}) = b_2^2 F_{\alpha_1+2\alpha_2} + (1 - q^2) b_2^2 b_1^{-2} \frac{b_1^2 - b_1^{-2}}{q^2 - q^{-2}} F_{\beta_1}^{-1} F_{\beta_2}^{(2)}.$$

By Proposition 4.10 and Corollary 4.20 a root vector acts injectively on the U_q-module

$$\varphi_{F_\Sigma,(b_1,\dots,b_n)}.L(\lambda)_{F_\Sigma}$$

if and only if it acts injectively on

$$\varphi_{F_\Sigma,(\varepsilon_1 q^{i_1} b_1,\dots,\varepsilon_n q^{i_n} b_n)}.L(\lambda)_{F_\Sigma}$$

for any $i_1, \dots, i_n \in \mathbb{Z}$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.

Assume there exists a $0 \neq v \in \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ such that $F_{\alpha_1+2\alpha_2}v = 0$. We have $v = F_{\beta_1}^{a_1}\cdots F_{\beta_n}^{a_n} \otimes v'$ for some $a_1,\dots,a_n \in \mathbb{Z}$ and some $v' \in L(\lambda)$. So $F_{\alpha_1+2\alpha_2}v = 0$ implies

$$0 = \varphi_{F_\Sigma,\mathbf{b}}(F_{\alpha_1+2\alpha_2})F_{\beta_1}^{a_1}\cdots F_{\beta_n}^{a_n} \otimes v' = F_{\beta_1}^{a_1}\cdots F_{\beta_n}^{a_n} \otimes \varphi_{F_\Sigma,\mathbf{c}}(F_{\alpha_1+2\alpha_2})v'$$

where $\mathbf{c} = (q^{a_1}b_1,\dots,q^{a_n}b_n)$. So there exists a $v' \in L(\lambda)$ and $a_1,\dots,a_n \in \mathbb{Z}$ such that for $\mathbf{c} = (q^{a_1}b_1,\dots,q^{a_n}b_n)$, $\varphi_{F_\Sigma,\mathbf{c}}(F_{\alpha_1+2\alpha_2})v' = 0$. That is

$$\left(c_2^2F_{\alpha_1+2\alpha_2} + (1-q^2)c_2^2c_1^{-2}\frac{c_1^2-c_1^{-2}}{q^2-q^{-2}}F_{\beta_1}^{-1}F_{\beta_2}^{(2)}\right)v' = 0$$

or equivalently

$$F_{\beta_1}F_{\alpha_1+2\alpha_2}v' + (1-q^2)c_1^{-2}\frac{c_1^2-c_1^{-2}}{q^2-q^{-2}}F_{\beta_2}^{(2)}v' = 0.$$

So to prove our claim it is enough to prove that

$$\left(F_{\beta_1}F_{\alpha_1+2\alpha_2} + (1-q^2)c_1^{-2}\frac{c_1^2-c_1^{-2}}{q^2-q^{-2}}F_{\beta_2}^{(2)}\right)v' \neq 0$$

for any $v' \in L(\lambda)$ and any $c_1 \in \mathbb{C}^*$.

So let $v' \in L(\lambda)$ and let $c_1 \in \mathbb{C}^*$. Let $r \in \mathbb{N}$ be such that $E_{\alpha_2}^{(r)}v' \neq 0$ and $E_{\alpha_2}^{(r+1)}v' = 0$ (possible since $L(\lambda)$ is a highest weight module). It is straightforward to show that for $a \in \mathbb{N}$:

$$[E_{\alpha_2}^{(a)}, F_{\alpha_1+2\alpha_2}] = q^{-a+1}[2]F_{\beta_2}E_{\alpha_2}^{(a-1)}K_{\alpha_2}^{-1} + q^{4-2a}F_{\beta_1}E_{\alpha_2}^{(a-2)}K_{\alpha_2}^{-2}$$

and

$$[E_{\alpha_2}^{(a)}, F_{\beta_2}^{(2)}] = q^{2-a}[2]F_{\beta_2}F_{\beta_1}E_{\alpha_2}^{(a-1)}K_{\alpha_2}^{-1} + q^{3-2a}[2]F_{\beta_1}^2E_{\alpha_2}^{(a-2)}K_{\alpha_2}^{-2}.$$

Using this we get

$$\begin{aligned}
&E_{\alpha_2}^{(r+2)}\left(F_{\beta_1}F_{\alpha_1+2\alpha_2} + (1-q^2)c_1^{-2}\frac{c_1^2-c_1^{-2}}{q^2-q^{-2}}F_{\beta_2}^{(2)}\right)v' \\
&=\left(q^{-2r} + q^{-1-2r}[2](1-q^2)c_1^{-2}\frac{c_1^2-c_1^{-2}}{q^2-q^{-2}}\right)F_{\beta_1}^2E_{\alpha_2}^{(r)}K_{\alpha_2}^{-2}v' \\
&=q^{-2r}c_1^{-4}F_{\beta_1}^2E_{\alpha_2}^{(r)}K_{\alpha_2}^{-2}v' \\
&\neq 0
\end{aligned}$$

since F_{β_1} acts injectively on $L(\lambda)$. Thus

$$\left(F_{\beta_1}F_{\alpha_1+2\alpha_2} + (1-q^2)c_1^{-2}\frac{c_1^2-c_1^{-2}}{q^2-q^{-2}}F_{\beta_2}^{(2)}\right)v' \neq 0. \qquad \square$$

Theorem 10.7 *Let λ be a weight such that $\lambda(K_\beta) \in \pm q^{\mathbb{N}}$ for all short $\beta \in \Phi$ and $\lambda(K_\gamma) \in \pm q^{1+2\mathbb{Z}}$ for all long $\gamma \in \Phi$. Let $\mathbf{b} = (b_1,\dots,b_n) \in (\mathbb{C}^*)^n$. Then the U_q-module $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is simple and torsion free if and only if $b_i \notin \pm q^{\mathbb{Z}}$, $i = 2,\dots,n$ and $b_1^2b_2\cdots b_n \notin \pm q^{\mathbb{Z}}$.*

Proof. Let $i \in \{2,\dots,n\}$. By Proposition 10.5, E_{α_i} acts injectively on $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ if and only if $b_i \notin \pm q^{\mathbb{Z}}$. If $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is torsion free then every root vector acts injectively. So $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ being torsion free implies $b_i \notin \pm q^{\mathbb{Z}}$.

Let $\Sigma' = \{\beta_1',\dots,\beta_n'\}$ denote the set of commuting roots with $\beta_1' = \alpha_1+\alpha_2$, $\beta_2' = \alpha_1+2\alpha_2$, $\beta_j' = \alpha_1+2\alpha_2+\alpha_3+\cdots+\alpha_j$, $j=3,\dots,n$. Let $F'_{\beta_1'} := T_{s_1}(F_{\alpha_2}) = F_{\beta_2}$, $F'_{\beta_2'} := T_{s_1}T_{s_2}(F_{\alpha_1}) = F_{\alpha_1+2\alpha_1}$, $F'_{\beta_j'} := T_{s_1}\cdots T_{s_n}T_{s_1}\cdots T_{s_{j-2}}(F_{\alpha_{j-1}}) = T_{s_2}(F_{\beta_j}) = F_{\alpha_1+2\alpha_2+\alpha_3+\cdots+\alpha_j}$, $j=3,\dots,n$ (in this case we actually have $F'_{\beta_j'} = F_{\beta_j'}$) and $F_{\Sigma'}$ the Ore subset generated by $F'_{\beta_1'},\dots,F'_{\beta_n'}$. Similarly to the above calculations in this section we can show that for $\mathbf{c} \in (\mathbb{C}^*)^n$

$$\varphi_{F_{\Sigma'},\mathbf{c}}(F_{\alpha_2}) = c_n^{-1}\cdots c_3^{-1}c_2^{-2}\left(F_{\alpha_2} + q[2]c_1^{-1}\frac{c_1-c_1^{-1}}{q-q^{-1}}(F'_{\beta_1'})^{-1}F'_{\beta_2'}\right).$$

Let $v \in L(\lambda)$ and let $r \in \mathbb{N}$ be such that $F_{\alpha_2}^{(r)}v \neq 0$ and $F_{\alpha_2}^{(r+1)}v = 0$ (possible since $\lambda(K_{\alpha_2}) \in \pm q^{\mathbb{N}}$). Then we see like in the proof of Proposition 10.5 that $\varphi_{F_{\Sigma'},\mathbf{c}}(F_{\alpha_2})v = 0$ if and only if $c_1 = \pm q^{-r}$ thus $\varphi_{F_{\Sigma'},\mathbf{c}}.L(\lambda)_{F_{\Sigma'}}$ is not torsion free whenever $c_1 \in \pm q^{\mathbb{Z}}$ by Proposition 4.10 and Corollary 4.20.

Set $f(\mathbf{b}) = (b_1^2b_2\cdots b_n, b_1^{-1}b_3^{-1}\cdots b_n, b_3,\dots,b_n)$. Then by Lemma 5.3

$$(\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma})^{ss} \cong \left(\varphi_{F_{\Sigma'},f(\mathbf{b})}.L(\lambda)_{F_{\Sigma'}}\right)^{ss}.$$

If $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is torsion free then it is simple so

$$\begin{aligned}\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma} &\cong (\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma})^{ss}\\ &\cong \left(\varphi_{F_{\Sigma'},f(\mathbf{b})}.L(\lambda)_{F_{\Sigma'}}\right)^{ss}\\ &\cong \varphi_{F_{\Sigma'},f(\mathbf{b})}.L(\lambda)_{F_{\Sigma'}}.\end{aligned}$$

We see that $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ being torsion free implies $b_1^2b_2\cdots b_n \notin \pm q^{\mathbb{Z}}$.

Now assume $b_i \notin \pm q^{\mathbb{Z}}$, $i=2,\dots,n$ and $b_1^2b_2\cdots b_n \notin \pm q^{\mathbb{Z}}$. By Proposition 10.5 and Proposition 9.7 E_{α_i} and F_{α_i}, $i=3,\dots,n$ act injectively on all composition factors of $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$.

Let L_1 be a simple submodule of $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ and let L_2 be a simple submodule of $\varphi_{F_{\Sigma'},f(\mathbf{b})}.L(\lambda)_{F_{\Sigma'}}$. By Proposition 10.6, $F_{\alpha_1+2\alpha_2}$ acts injectively on $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. Now clearly $\{-\alpha_1-\alpha_2, -\alpha_1-2\alpha_2, \alpha_3,\dots,\alpha_n\} \subset T_{L_1}\cap T_{L_2}$ so $C(L_1)\cap C(L_2)$ generates Q. This implies that $C(L_1)-C(L_2) = Q$. Since $(\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma})^{ss} \cong \left(\varphi_{F_{\Sigma'},f(\mathbf{b})}.L(\lambda)_{F_{\Sigma'}}\right)^{ss}$ we have $\operatorname{wt} L_k \subset q^Q(\mathbf{b}^{-1})^\Sigma\lambda$, $k=1,2$. Choose $\mu_1,\mu_2 \in Q$ such that $q^{\mu_1}(\mathbf{b}^{-1})^\Sigma\lambda \in \operatorname{Supp}_{\mathrm{ess}}(L_1)$ and $q^{\mu_2}(\mathbf{b}^{-1})^\Sigma\lambda \in \operatorname{Supp}_{\mathrm{ess}}(L_2)$. Then obviously $q^{C(L_1)+\mu_1}(\mathbf{b}^{-1})^\Sigma\lambda \subset \operatorname{Supp}_{\mathrm{ess}}(L_1)$ and $q^{C(L_2)+\mu_2}(\mathbf{b}^{-1})^\Sigma\lambda \subset \operatorname{Supp}_{\mathrm{ess}}(L_2)$. By the above $q^{C(L_1)+\mu_1}(\mathbf{b}^{-1})^\Sigma\lambda \cap q^{C(L_2)+\mu_2}(\mathbf{b}^{-1})^\Sigma\lambda \neq \emptyset$ so $\operatorname{Supp}_{\mathrm{ess}}(L_1)\cap\operatorname{Supp}_{\mathrm{ess}}(L_2) \neq \emptyset$. Let $\nu \in \operatorname{Supp}_{\mathrm{ess}}(L_1)\cap\operatorname{Supp}_{\mathrm{ess}}(L_2)$. By Proposition 5.6, L_1 and L_2 are admissible of the same degree as $L(\lambda)$. So we have as $(U_q)_0$-modules (using that $(L_1)_\nu$ and $(L_2)_\nu$ are simple $(U_q)_0$-modules by Theorem 2.7)

$$\begin{aligned}(L_1)_\nu = (\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma})_\nu &\cong \left((\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma})_\nu\right)^{ss}\\ &\cong \left(\left(\varphi_{F_{\Sigma'},f(\mathbf{b})}.L(\lambda)_{F_{\Sigma'}}\right)_\nu\right)^{ss} \cong \left(\varphi_{F_{\Sigma'},f(\mathbf{b})}.L(\lambda)_{F_{\Sigma'}}\right)_\nu = (L_2)_\nu.\end{aligned}$$

By Theorem 2.7 this implies $L_1 \cong L_2$.

Let $\Sigma'' = \{\beta_1'',\dots,\beta_n''\}$ denote the set of commuting roots with $\beta_1'' = \alpha_1 + 2\alpha_2$, $\beta_2'' = \alpha_2$, $\beta_j'' = \alpha_1+2\alpha_2+\alpha_3+\cdots+\alpha_j$, $j=3,\dots,n$. Let $F''_{\beta_1''} :=$

$T_{s_1}T_{s_2}(F_{\alpha_1})$, $F''_{\beta''_2} := F_{\alpha_2}$, $F''_{\beta''_j} := T_{s_2}T_{s_1}T_{s_2}T_{s_3}\cdots T_{s_{j-1}}(F_{\alpha_j}) = T_{s_1}T_{s_2}(F_{\beta_j})$, $j = 3,\dots,n$ and $F_{\Sigma''}$ the Ore subset generated by $F''_{\beta''_1},\dots,F''_{\beta''_n}$. Note that $F''_{\beta''_j} = T_{s_1}T_{s_2}(F_{\beta_j})$ for all $j \in \{1,\dots,n\}$. The root vectors $F''_{\beta''_1},\dots,F''_{\beta''_n}$ act injectively on $\overline{^{s_2s_1}}L(\lambda)$. By Theorem 5.12 and Proposition 5.5 $L(\lambda)$ is a submodule of $\left(\varphi_{F_{\Sigma''},\mathbf{d}}.(\overline{^{s_2s_1}}L(\lambda))_{F_{\Sigma''}}\right)^{ss}$ for some $\mathbf{d} \in (\mathbb{C}^*)^n$. Then by Lemma 5.3

$$\left(\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}\right)^{ss} \cong \left(\varphi_{F_{\Sigma''},g(\mathbf{b})\mathbf{d}}.(\overline{^{s_2s_1}}L(\lambda))_{F_{\Sigma''}}\right)^{ss}$$

for some $g(\mathbf{b}) \in (\mathbb{C}^*)^n$.

Observe that for $a_1,\dots,a_n \in \mathbb{N}$:

$$\begin{aligned}
\varphi_{F_{\Sigma''},(q^{a_1},\dots,q^{a_n})}&(-K_{\alpha_1}E_{\alpha_1})\\
&=\varphi_{F_{\Sigma''},(q^{a_1},\dots,q^{a_n})}(T_{s_1}T_{s_2}(F_{\alpha_1+2\alpha_2}))\\
&=\left(F''_{\beta''_1}\right)^{-a_1}\cdots\left(F''_{\beta''_n}\right)^{-a_n}T_{s_1}T_{s_2}(F_{\alpha_1+2\alpha_2})\left(F''_{\beta''_n}\right)^{a_n}\cdots\left(F''_{\beta''_1}\right)^{a_1}\\
&=T_{s_1}T_{s_2}\left(F_{\beta_1}^{-a_n}\cdots F_{\beta_n}^{-a_n}F_{\alpha_1+2\alpha_2}F_{\beta_n}^{a_n}\cdots F_{\beta_1}^{a_1}\right)\\
&=T_{s_1}T_{s_2}\left(\varphi_{F_\Sigma,(q^{a_1},\dots,q^{a_n})}(F_{\alpha_1+2\alpha_2})\right).
\end{aligned}$$

Since $\varphi_{F_{\Sigma''},\mathbf{c}}(-K_{\alpha_1}E_{\alpha_1})$ and $T_{s_1}T_{s_2}\left(\varphi_{F_\Sigma,\mathbf{c}}(F_{\alpha_1+2\alpha_2})\right)$ are both Laurent polynomial in $\mathbf{c}$ we get by Lemma 4.8 that $\varphi_{F_{\Sigma''},\mathbf{c}}(-K_{\alpha_1}E_{\alpha_1}) = T_{s_1}T_{s_2}\left(\varphi_{F_\Sigma,\mathbf{c}}(F_{\alpha_1+2\alpha_2})\right)$ for any $\mathbf{c} \in (\mathbb{C}^*)^n$. $T_{s_1}T_{s_2}\left(\varphi_{F_\Sigma,\mathbf{c}}(F_{\alpha_1+2\alpha_2})\right)$ acts injectively on $\overline{^{s_2s_1}}L(\lambda)$ for any $\mathbf{c} \in (\mathbb{C}^*)^n$ by Proposition 10.6. This implies that $-K_{\alpha_1}E_{\alpha_1}$ acts injectively on $\varphi_{F_{\Sigma''},g(\mathbf{b})\mathbf{d}}.(\overline{^{s_2s_1}}L(\lambda))_{F_{\Sigma''}}$ and this implies that E_{α_1} acts injectively.

Let L_3 be a simple submodule of $\varphi_{F_{\Sigma''},g(\mathbf{b})\mathbf{d}}.(\overline{^{s_2s_1}}L(\lambda))_{F_{\Sigma''}}$. We see that $\{-\alpha_2,-\alpha_1-2\alpha_2,\alpha_3,\dots,\alpha_n\} \subset T_{L_3}\cap T_{L_2}$ so $C(L_2)\cap C(L_3)$ generates Q ($\{\alpha_3,\dots,\alpha_n\} \subset T_{L_3}$ because of Proposition 9.7 and the fact that L_3 is a composition factor of $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$). Arguing as above this implies that $L_2 \cong L_3$. We have shown that $L_1 \cong L_2 \cong L_3$. Above we have shown that E_{α_1} acts injectively on L_3, F_{α_2} acts injectively on L_2 and $F_{\alpha_1}, E_{\alpha_2}, F_{\alpha_i}, E_{\alpha_i}$, $i = 3,\dots,n$ act injectively on L_1. In conclusion we have shown that all root vectors act injectively on the simple submodule L_1 of $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ thus $\operatorname{wt} L_1 = \operatorname{Supp}_{\mathrm{ess}}(L_1) = q^Q(\mathbf{b}^{-1})^\Sigma\lambda$ and therefore $L_1 = \varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$. This shows that $\varphi_{F_\Sigma,\mathbf{b}}.L(\lambda)_{F_\Sigma}$ is simple and torsion free with our assumptions on $\mathbf{b}$. $\square$

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