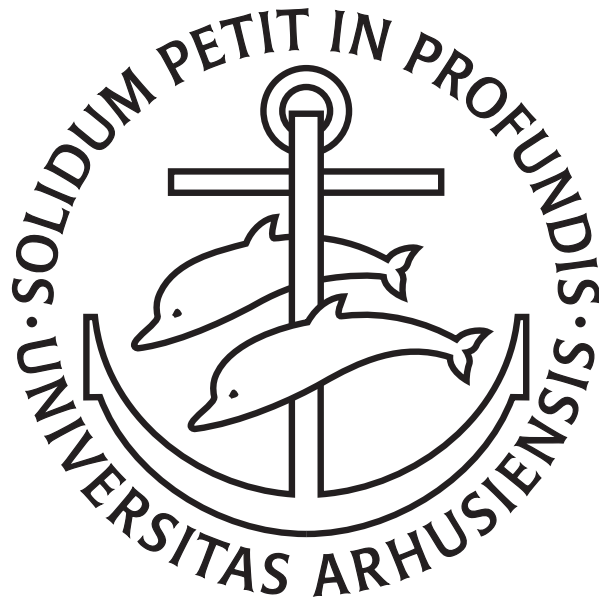


# SOME RESULTS IN DIOPHANTINE APPROXIMATION

AND SOME APPROACHES THAT DO NOT WORK



PHD THESIS  
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JULY 2016

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# Résumé på dansk

Denne afhandling består af tre artikler om Diofantisk approksimation, en undergren af talteori. Før disse artikler er der en introduktion til forskellige aspekter af Diofantisk approksimation og formelle Laurent rækker over  $\mathbb{F}_q$ , samt en opsummering af hver af de tre artikler.

Introduktionen indfører de grundlæggende begreber som artiklerne bygger på. Blandt andet indføres metrisk Diofantisk approksimation, Mahlers tilgang til algebraisk approksimation, Hausdorff målet og egenskaber ved de formelle Laurent rækker over  $\mathbb{F}_q$ . Introduktionen afsluttes med en diskussion af Mahlers problem betragtet i de formelle Laurent rækker over  $\mathbb{F}_3$ .

Den første artikel omhandler intrinsisk Diofantisk approksimation i Cantor mængden i de formelle Laurent rækker over  $\mathbb{F}_3$ . Opsummeringen indeholder en kort motivation, resultaterne fra artiklen og skitser af beviserne, hovedsageligt med fokus på de anvendte ideer. Bevisernes detaljer er i artiklen.

Den anden artikel omhandler højere dimensionel Mahler approksimation. Opsummeringen følger den samme struktur som i tilfældet med den første artikel.

Den tredje artikel omhandler forvrænget inhomogen Diofantisk approksimation i de formelle Laurent rækker over  $\mathbb{F}_q$ . Opsummeringen består af to forskellige dele. Den første del omhandler et mislykket forsøg på at anvende dynamiske metoder til at opnå resultater og er ikke en del af artiklen. Den forklarer hvordan det reelle tilfælde virker og hvad der går galt i tilfældet med formelle Laurent rækker. Den anden del indeholder artiklens resultater og skitser af beviserne.



# Abstract in English

This thesis consists of three papers in Diophantine approximation, a subbranch of number theory. Preceding these papers is an introduction to various aspects of Diophantine approximation and formal Laurent series over  $\mathbb{F}_q$  and a summary of each of the three papers.

The introduction introduces the basic concepts on which the papers build. Among other it introduces metric Diophantine approximation, Mahler's approach on algebraic approximation, the Hausdorff measure, and properties of the formal Laurent series over  $\mathbb{F}_q$ . The introduction ends with a discussion on Mahler's problem when considered in the formal Laurent series over  $\mathbb{F}_3$ .

The first paper is on intrinsic Diophantine approximation in the Cantor set in the formal Laurent series over  $\mathbb{F}_3$ . The summary contains a short motivation, the results of the paper and sketches of the proofs, mainly focusing on the ideas involved. The details of the proofs are in the paper.

The second paper is on higher dimensional Mahler approximation. The summary follows the same structure as in the case of the first paper.

The third paper is on twisted inhomogeneous Diophantine approximation in the formal Laurent series over  $\mathbb{F}_q$ . The summary consists of two distinct parts. The first part is about a failed attempt of applying dynamical methods to obtain results and is not part of the paper. It explains the ideas of how the real case works and what goes wrong in the case of the formal Laurent series. The second part contains the results of the paper and sketches of the proofs.



# Preface

*“Try a hard problem. You may not solve it, but you will prove something else.”*

— J. E. Littlewood

Does the Cantor set contain any irrational algebraic numbers? In [20] Mahler proposed this problem, and we expect the answer to be no.

In number theory we have an unofficial way of making conjectures about algebraic numbers: An algebraic number behave like almost all real numbers, unless it has a good reason not to. Since almost all numbers are normal, and rational numbers are clearly not normal, we get the conjecture, that all irrational algebraic numbers are normal, which would imply a no to Mahler’s problem.

Mahler’s problem is a hard problem! I started out my PhD trying to solve it, and as will be apparent once you have read this thesis, I did not succeed, but I did find something else along the way!

The thesis is divided into three parts: The first part is a general introduction to some main concepts in Diophantine approximation and formal Laurent series over  $\mathbb{F}_q$ . The second part is summaries of the three papers resulting from my PhD, and the final part is these three papers.

There are some people I would like to thank, people who have had a great impact on my PhD: My family, who have provided an abundance of moral support and interest in my project. My co-authors Simon Kristensen, Barak Weiss, Efrat Bank and Erez Nesharim, whose various insights and expertises have complemented my insight and expertise, producing results better than I could have obtained on my own. Finally, and again, my supervisor Simon Kristensen who have not only been a guidance and a support, but a colleague and a friend.

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# 1 Introduction

The introduction has two sections: Diophantine approximation and formal Laurent series. Diophantine approximation is the study of how well real numbers can be approximated by rational numbers, and other similar questions. My PhD is in Diophantine approximation, so the first section will be a short overview of some of the relevant concepts in Diophantine approximation, concepts that build the foundation for the research I have done.

The second section is about formal Laurent series over  $\mathbb{F}_q$ , denoted  $\mathbb{F}_q((X^{-1}))$ .  $\mathbb{F}_q((X^{-1}))$  is an analytic, algebraic construction, that have enough properties in common with  $\mathbb{R}$ , that the concepts from Diophantine approximation over  $\mathbb{R}$  can be translated to this setting. On the other hand,  $\mathbb{R}$  and  $\mathbb{F}_q((X^{-1}))$  behave differently enough, that  $\mathbb{F}_q((X^{-1}))$  is interesting enough to study in its own right, at least from the Diophantine approximation point of view. A big part of my PhD have been about doing exactly this.

At the end of the introduction I will discuss Mahler's problem when studied in  $\mathbb{F}_3((X^{-1}))$ .

## 1.1 Diophantine approximation

### Beginnings

Most expositions on Diophantine approximation begin with the following theorem of Dirichlet [9], which is the first application of the Pigeonhole Principle.

**Theorem 1** (Dirichlet's Theorem). *For any  $\alpha \in \mathbb{R}$ ,  $N \in \mathbb{N}$ , there exist  $p, q \in \mathbb{Z}$ ,  $0 < q \leq N$ , such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{Nq}.$$

Dirichlet's Theorem is the start of Diophantine approximation and improves on the statement, that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . More precisely, for any  $\alpha \in \mathbb{R}$  it guarantees a rational approximation  $\frac{p}{q}$  sufficiently close to  $\alpha$  in terms of the complexity of  $\frac{p}{q}$ , in this case  $0 < q \leq N$ .

As a corollary of Dirichlet's Theorem, or from the theory of continued fractions, we get the following:

**Corollary 2.** *For any  $\alpha \in \mathbb{R}$ , there exist infinitely many  $\frac{p}{q} \in \mathbb{Q}$ , such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

The question of whether there exist infinitely many (from now on abbreviated i.m.) numbers satisfying a certain property is one of the main topics in Diophantine approximation, and hence a large part of Diophantine approximation deals with variations of the corollary.

One way this is done, is by interpreting the right hand side of the inequality as a function in  $q$ , and then ask what happens if we substitute with another function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , that is whether

$$\left| \alpha - \frac{p}{q} \right| < \psi(|q|)$$

for i.m.  $\frac{p}{q} \in \mathbb{Q}$ . The most studied example of this is with the functions  $\psi_\tau(q) = q^{-\tau}$  where  $\tau \in [2, \infty)$ .

Another variation is by approximating with other numbers than  $\mathbb{Q}$ . Let  $A \subseteq \mathbb{R}$  and  $H : A \rightarrow \mathbb{R}$  a so called height function. Consider whether

$$|\alpha - a| < \psi(H(a))$$

for i.m.  $a \in A$ . Classical examples of  $A$  is subsets of  $\mathbb{Q}$  with the usual height function  $H(\frac{p}{q}) = |q|$  or  $\mathbb{A}_n$ , algebraic numbers of degree at most  $n$ , with  $H(a) = H(P) = \max\{|a_m|, \dots, |a_1|, |a_0|\}$ , where  $P(X) = a_m X^m + \dots + a_1 X + a_0 \in \mathbb{Z}[X]$  is the minimal polynomial of  $a$ .

Finally, a variation is done by looking only at specific  $\alpha$  like  $e$ ,  $\pi$  or algebraic irrational numbers.

Where Dirichlet's Theorem is seen as the beginning of Diophantine approximation, the following theorem by Khintchine [11] is seen as the start of metric Diophantine approximation. We let  $\lambda$  be the Lebesgue measure.

**Theorem 3** (Khintchine's Theorem). *Let  $\Psi : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{>0}$  be a continuous function, such that  $x \mapsto x^2 \Psi(x)$  is non-increasing. Let*

$$W(\Psi) = \left\{ \xi \in [0, 1] : \left| \xi - \frac{p}{q} \right| < \Psi(q), \text{ for i.m. } \frac{p}{q} \in \mathbb{Q} \right\}.$$

Then

$$\lambda(W(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} n \Psi(n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} n \Psi(n) = \infty. \end{cases}$$

In metric Diophantine approximation we attempt to determine the generic or almost all behaviour of numbers in regard to the questions from Diophantine approximation. This is among other achieved by invoking the methods and results from measure theory, ergodic theory and measurable dynamical systems. Even though metrical methods are quite powerful, the results are often quite unsatisfactory. My favourite example of this comes from normality of numbers.

**Example.** Let  $b \geq 2$  be an integer. Every  $\xi \in \mathbb{R}$  has an unique base  $b$  expansion

$$\xi = [\xi] + \sum_{i=1}^{\infty} \frac{a_i}{b^i},$$

where  $a_i \in \{0, \dots, b-1\}$  and for each  $N \in \mathbb{N}$ , there exists an  $i \geq N$ , such that  $a_i \neq b-1$ .  $\xi$  is called simply normal to base  $b$ , if

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq i \leq N : a_i = d\}}{N} = \frac{1}{b}$$

for every  $d \in \{0, \dots, b-1\}$ . It is called normal, if it is simply normal to base  $b, b^2, b^3, \dots$ .

An alternative characterisation of normality to base  $b$  is that the sequence  $\{b^n \xi\}_{n=0}^\infty$  is uniformly distributed modulo one. Since the map  $T_b : \mathbb{T} \rightarrow \mathbb{T} : x \mapsto bx \bmod 1$  is ergodic, it follows from the Pointwise Ergodic Theorem, that  $\{b^n \xi\}_{n=0}^\infty$  is uniformly distributed modulo one for almost all  $\xi \in \mathbb{R}$ . In particular almost all  $\xi \in \mathbb{R}$  is simultaneously normal to all bases  $b \geq 2$ . In spite of this fact, we have no concrete examples of numbers satisfying this.

## The irrationality measure and algebraic approximation

For a  $\xi \in \mathbb{R}$  we define the irrationality measure of  $\xi$  by

$$\mu(\xi) = \sup \left\{ \mu \in \mathbb{R} : 0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^\mu}, \text{ for i.m. } p, q \in \mathbb{Z}, q \neq 0 \right\}. \quad (1.1)$$

$\mu(\xi)$  measures how much we can improve on Corollary 2, i.e. how much we can increase the exponent 2 and still have i.m. rational approximations satisfying the stronger inequality. For  $\tau > 2$  we let  $B_\tau = \{\xi \in \mathbb{R} : \mu(\xi) \geq \tau\}$ . Since  $\sum_{n=1}^\infty n^{-\frac{1}{\tau}} < \infty$ , it follows from Khintchine's Theorem, that  $\lambda(B_\tau) = 0$ . In particular,  $\mu(\xi) = 2$  for almost all  $\xi \in \mathbb{R}$ .

Two questions about the irrationality measure are often studied: The first question is for a given  $\xi \in \mathbb{R}$  to determine  $\mu(\xi)$ . Even though a generic  $\xi$  has  $\mu(\xi) = 2$ , it is usually hard to determine  $\mu(\xi)$ , but it turns out, that if we have full control of the simple continued fraction expansion of  $\xi$ , this is often doable. In fact, if  $\xi$  has convergents  $\left\{ \frac{p_n}{q_n} \right\}_{n=0}^\infty$ , we can compute the irrationality measure by

$$\mu(\xi) = 1 + \limsup_{n \rightarrow \infty} \frac{\log(q_{n+1})}{\log(q_n)}.$$

Since we know the simple continued fraction expansion of  $e$ , we can show that  $\mu(e) = 2$ , and hence  $e$  is generic in this respect. The second question is for a subset  $A \subseteq \mathbb{R}$  to determine the spectrum  $\{\mu(a) : a \in A\}$ . For  $\xi \in \mathbb{Q}$  we have, that  $\mu(\xi) = 1$ , and the spectrum of  $\mathbb{R} \setminus \mathbb{Q}$  is  $[2, \infty]$ . The first question is just a special case of the second.

So far we have only been concerned with rational approximation, but the irrationality measure is a good way of illustrating the two different ways of generalising to algebraic approximation. First, let us slightly rewrite the irrationality measure as

$$\mu(\xi) = \sup \left\{ \mu \in \mathbb{R} : 0 < |P(\xi)| < \tilde{H}(P)^{-(\mu-1)}, \text{ for i.m. } P \in \mathbb{Z}[X], \deg P = 1 \right\}, \quad (1.2)$$

where for polynomials  $P(X) = a_n X^n + \dots + a_1 X + a_0$ ,  $\tilde{H}(P) = \max\{|a_n|, \dots, |a_1|\}$  is a non-standard height.

This shows that rational approximation can either be seen as a question about distance to rational numbers, or a question about evaluating the number in degree

one integer polynomials. In this way we can generalise to algebraic approximation in two different ways.

The polynomial approach to algebraic approximation was done by Mahler in [18]. For  $\xi \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , we define

$$\omega_n(\xi) = \sup \{ \omega \in \mathbb{R} : 0 < |P(\xi)| < H(P)^{-\omega}, \text{ for i.m. } P \in \mathbb{Z}[X], \deg P \leq n \},$$

where for a polynomial  $P(X) = a_n X^n + \dots + a_1 X + a_0$ , we let  $H(P) = \max \{|a_n|, \dots, |a_0|\}$  be the height of  $P$ .  $\omega_n(\xi)$  naturally generalises (1.2), and in particular  $\omega_1(\xi) = \mu(\xi) - 1$  for all  $\xi \in \mathbb{R}$ .

The distance approach to algebraic approximation was done by Koksma in [15]. For  $\xi \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , we define

$$\omega_n^*(\xi) = \sup \{ \omega \in \mathbb{R} : 0 < |\xi - \alpha| < H(\alpha)^{-\omega-1}, \text{ for i.m. } \alpha \in \mathbb{A}_n \},$$

where the height of  $\alpha$ ,  $H(\alpha) = H(P)$ , where  $P$  is the minimal polynomial of  $\alpha$ .  $\omega_n^*(\xi)$  generalises (1.1), and again  $\omega_1^*(\xi) = \mu(\xi) - 1 = \omega_1(\xi)$ .

In general  $\omega_n(\xi)$  and  $\omega_n^*(\xi)$  do not agree, but for a generic  $\xi$  we have  $\omega_n(\xi) = \omega_n^*(\xi) = n$ , a result of Sprindžuk [29]. Just as in the case of the irrationality measure, it turns out that  $e$  is generic, that is  $\omega_n(e) = \omega_n^*(e) = n$  for any  $n \in \mathbb{N}$ . This follows from a result by Popken [24].

In connection with discussing algebraic approximation, we will also discuss approximation of algebraic numbers. As mentioned in the preface, we expect algebraic numbers to behave like almost all numbers, unless they have a good reason not to. In particular, since  $\mu(\xi) = 2$  for almost all  $\xi \in \mathbb{R}$ , and  $\mu(\xi) = 1$  for  $\xi \in \mathbb{Q}$ , we expect algebraic irrational numbers to have irrationality measure 2. This is a celebrated result of Roth [25].

**Theorem 4** (Roth's Theorem). *Let  $\alpha \in \mathbb{R}$  be an algebraic irrational number. Then*

$$\mu(\alpha) = 2.$$

When doing algebraic approximation of algebraic numbers, it turns out, that they are generic when approximating with algebraic numbers of lower degree, but not when approximating with the same or higher degree. For  $\alpha \in \mathbb{R}$  algebraic of degree  $d$ , we have that  $\omega_n(\alpha) = \omega_n^*(\alpha) = \min \{n, d - 1\}$ . This is a huge improvement on Roth's Theorem, and essentially follows from Schmidt's Subspace Theorem [26].

## Hausdorff measure and Hausdorff dimension

Sometimes, in metric Diophantine approximation, we need a more refined notion on the size of a set, as sets of zero measure can still be rather big. One way this is done is by introducing the Hausdorff measure and the Hausdorff dimension. The construction closely follows Carathéodory's construction of the Lebesgue measure  $\lambda_d$ . For  $B \subseteq \mathbb{R}^d$  we define the diameter of  $B$  by

$$\text{diam } B = \sup \{ \|x - y\| : x, y \in B \}.$$

Furthermore, for  $\delta > 0$  we define a  $\delta$ -cover of  $B$  to be a countable collection of sets  $\{U_n\}_{n=1}^\infty$ , such that  $B \subseteq \bigcup_{n=1}^\infty U_n$ , and for each  $n \in \mathbb{N}$ ,  $\text{diam}(U_n) \leq \delta$ .

For  $s \geq 0$  we define

$$\mathcal{H}_\delta^s(B) = \inf \left\{ \sum_{n=1}^{\infty} (\text{diam } U_n)^s : \{U_n\}_{n=1}^{\infty} \text{ is a } \delta\text{-cover of } B \right\},$$

and the Hausdorff  $s$ -measure by

$$\mathcal{H}^s(B) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(B).$$

$\mathcal{H}^s$  is an outer measure, and restricted to  $\mathcal{B}(\mathbb{R}^d)$  it is a measure. Furthermore, it has the right scaling factor in the sense, that for  $\xi > 0$ ,  $B \subseteq \mathbb{R}^d$ , if we define  $\xi B = \{\xi b : b \in B\}$ , then

$$\mathcal{H}^s(\xi B) = \xi^s \mathcal{H}^s(B).$$

It turns out, that for each set  $B$ , there is only one value of  $s$  where  $\mathcal{H}^s$  is interesting. To be more precise, there exist a  $s_0$ , such that for  $r < s_0$  and  $s_0 < t$ , we have respectively  $\mathcal{H}^r(B) = \infty$  and  $\mathcal{H}^t(B) = 0$ . At  $s_0$  we can have  $\mathcal{H}^{s_0}(B) = 0$ ,  $\mathcal{H}^{s_0}(B) = \infty$  or anything in between. The point  $s_0$  is called the Hausdorff dimension of  $B$ , and we write that as  $\dim_H B$ .

The Hausdorff  $d$ -measure is comparable with  $\lambda_d$ . In particular, for  $B \in \mathcal{B}(\mathbb{R}^d)$  with  $\lambda_d(B) > 0$  we have  $\dim_H B = d$ . We can now use the Hausdorff dimension as a refined way of understanding the size of Lebesgue null sets. A good example of this is the following theorem of Jarník [10]. Remember, that for each  $\tau > 2$  the set  $B_\tau = \{\xi \in \mathbb{R} : \mu(\xi) \geq \tau\}$  is a Lebesgue null set. Jarník's Theorem tells the size of  $B_\tau$  in terms of Hausdorff dimension.

**Theorem 5** (Jarník's Theorem). *Let  $\tau > 2$  and let  $B_\tau = \{\xi \in \mathbb{R} : \mu(\xi) \geq \tau\}$ . Then  $\dim_H B_\tau = \frac{2}{\tau}$ .*

Sometimes, we need an even more refined way of understanding the size of a set. Let  $f$  be a dimension function, that is  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is continuous, increasing and satisfying  $f(0) = 0$ . The standard example of such a function is for  $s > 0$  the function  $f_s$  given by  $f_s(x) = x^s$ . We can now modify the construction of the Hausdorff  $s$ -measure to a larger class of measures. For  $\delta > 0$  we define

$$\mathcal{H}_\delta^f(B) = \inf \left\{ \sum_{n=1}^{\infty} f(\text{diam } U_n) : \{U_n\}_{n=1}^{\infty} \text{ is a } \delta\text{-cover of } B \right\},$$

and the Hausdorff  $f$ -measure by

$$\mathcal{H}^f(B) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^f(B).$$

**Example.** Let  $\mathbb{L}$  be the set of Liouville numbers, that is the set

$$\mathbb{L} = \{\xi \in \mathbb{R} : \mu(\xi) = \infty\}.$$

From Jarník's Theorem we get that  $\dim_H \mathbb{L} = 0$ . For a dimension function  $f$  we define the function  $\Gamma_f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  by

$$\Gamma_f(r) = \inf_{0 < s \leq r} r \frac{f(s)}{s}.$$

A result of Olsen and Renfro [21] then tells, that

$$\mathcal{H}^f(\mathbb{L}) = \begin{cases} 0 & \text{if } \limsup_{r \searrow 0} \frac{\Gamma_f(r)}{r^t} = 0 \text{ for some } t > 0, \\ \infty & \text{if } \limsup_{r \searrow 0} \frac{\Gamma_f(r)}{r^t} > 0 \text{ for all } t > 0. \end{cases}$$

From this we obtain a much more precise understanding of the size of  $\mathbb{L}$  than just that  $\dim_H \mathbb{L} = 0$ . If we use the result on the function  $f_s$  for  $s \in (0, 1)$ , we have that  $\Gamma_{f_s}(r) = f_s(r)$  and

$$\frac{\Gamma_{f_s}(r)}{r^{s/2}} = r^{s/2} \rightarrow 0 \text{ as } r \searrow 0,$$

so  $\mathcal{H}^s(\mathbb{L}) = 0$ . From this we get  $\dim_H \mathbb{L} = 0$ , but the strength of the result is, that for any dimension function  $f$  we get the Hausdorff  $f$ -measure, and hence the result tells the exact “cut point” where the measure drops from  $\infty$  to 0.

## 1.2 The formal Laurent series over $\mathbb{F}_q$

### Something that looks like $\mathbb{R}$

Let  $q$  be a power of the prime  $p$ , and let  $\mathbb{F}_q$  be the finite field with  $q$  elements. The formal Laurent series over  $\mathbb{F}_q$  behave similarly enough to  $\mathbb{R}$ , that the questions arising in Diophantine approximation can be asked, but it also behave different enough, that it is interesting to study in its own right. It is constructed in the following way:

Let  $\mathbb{F}_q[X]$  be the polynomial ring over  $\mathbb{F}_q$ .  $\mathbb{F}_q[X]$  and  $\mathbb{Z}$  share a lot of structure, and  $\mathbb{F}_q[X]$  can be seen as an analogue of  $\mathbb{Z}$ . They are both Euclidian rings, and where  $\mathbb{Z}$  has the normal absolute value, we can equip  $\mathbb{F}_q[X]$  with an absolute value, letting  $|0| = 0$  and for non-zero polynomials  $f$ , we let

$$|f| = q^{\deg f}.$$

From  $\mathbb{F}_q[X]$  we can, like in the case of  $\mathbb{Z}$ , construct the field of fractions, giving the rational functions  $\mathbb{F}_q(X)$  as an analogue of  $\mathbb{Q}$ . Furthermore, we can extend the absolute value on  $\mathbb{F}_q[X]$  to  $\mathbb{F}_q(X)$ , by

$$\left| \frac{f}{g} \right| = q^{\deg f - \deg g}.$$

When we have a field with an absolute value, we can make the completion. In the case of  $\mathbb{Q}$  with the normal absolute value, we get  $\mathbb{R}$ . When we take the completion of  $\mathbb{F}_q(X)$  with respect to the above constructed absolute value, we get the formal Laurent series over  $\mathbb{F}_q$ , denoted  $\mathbb{F}_q((X^{-1}))$ . This turns out to be the set

$$\left\{ \sum_{i=-N}^{\infty} a_i X^{-i} : a_i \in \mathbb{F}_q, a_{-N} \neq 0 \right\} \cup \{0\}.$$

The absolute value for non-zero elements turn out to be

$$\left| \sum_{i=-N}^{\infty} a_i X^{-i} \right| = q^N.$$

$\mathbb{F}_q((X^{-1}))$  is a locally compact field, which implies that we have a Haar measure. We normalise it to be 1 on the unit ball around 0,

$$\mathbb{I} = \{x \in \mathbb{F}_q((X^{-1})) : |x| < 1\} = \left\{ \sum_{i=1}^{\infty} a_i X^{-i} : a_i \in \mathbb{F}_q \right\},$$

using the standard convention, that if all the  $a_i = 0$  we have the zero element.

The fact that we have a measure enables us to do metric Diophantine approximation. Furthermore, the construction of the Hausdorff measure and Hausdorff dimension only requires an absolute value, and hence carries through to the setting of formal Laurent series over  $\mathbb{F}_q$ . The theory of continued fraction also carries over to this setting since  $\mathbb{F}_q[X]$  is an Euclidean ring. This was done by Artin [2] and the theory become slightly simpler.

An introduction to formal Laurent series over  $\mathbb{F}_q$  would not be complete without also explaining how it behave differently from  $\mathbb{R}$ , and hence is interesting in its own right. There are essentially two properties that make it behave significantly different from  $\mathbb{R}$ . First, the absolute value, and hence the induced metric, is ultrametric, meaning that it satisfies the strong triangle inequality

$$|x + y| \leq \max\{|x|, |y|\}$$

for all  $x, y \in \mathbb{F}_q((X^{-1}))$ . This inequality is the reason that the geometry of  $\mathbb{F}_q((X^{-1}))$  behave different from the geometry of  $\mathbb{R}$ . For instance two balls have either non-empty intersection, or one is completely contained in the other. Another consequence is that every triangle is isosceles.

When doing Diophantine approximation in the real case, we often get crucial estimates by doing geometric considerations. In the formal Laurent series case, the fact that the geometry is different from usual, disenables us to get the estimates in this way. Instead, a way to get these estimates, is to use that the ultrametric property gives a tree like structure on the open balls, and hence estimates can be obtained by counting arguments.

Second, the fact that  $\mathbb{F}_q$  has characteristic  $p$ , is carried over to  $\mathbb{F}_q((X^{-1}))$ , and as always, there are differences between working in positive characteristic and working in characteristic zero. In particular when taking powers,  $\mathbb{F}_q((X^{-1}))$  behaves different from  $\mathbb{R}$ , in the sense that Freshman's Dream hold true

$$(x + y)^p = x^p + y^p$$

for all  $x, y \in \mathbb{F}_q((X^{-1}))$ . This plays a central role when discussing algebraicity of formal Laurent series over  $\mathbb{F}_q$ .

## Mahler's problem in $\mathbb{F}_3((X^{-1}))$

Let us try to tackle Mahler's problem, but in the formal Laurent series over  $\mathbb{F}_3$ : Let  $\alpha$  be an algebraic formal Laurent series in the Cantor set. Is  $\alpha$  a rational function?

For the question to make sense, we need to define both what we mean by algebraic formal Laurent series and by the Cantor set.

An  $\alpha \in \mathbb{R}$  is algebraic, if there exists a polynomial  $P \in \mathbb{Z}[Y]$ , such that  $P(\alpha) = 0$ . Using that  $\mathbb{F}_q[X]$  is an analogy of  $\mathbb{Z}$ , we call an  $\alpha \in \mathbb{F}_q((X^{-1}))$  algebraic, if there exists

a polynomial  $P \in \mathbb{F}_q[X][Y]$ , such that  $P(\alpha) = 0$ . As in the real case, the degree of  $\alpha$  is  $\min \{\deg_Y(P) : P \in \mathbb{F}_q[X][Y], P(\alpha) = 0\}$ , and there is no restriction in considering polynomials  $P \in \mathbb{F}_q(X)[Y]$ .

The Cantor set in the real case is

$$C = \left\{ \sum_{i=1}^{\infty} a_i 3^{-i} \in \mathbb{R} : a_i \in \{0, 2\} \right\},$$

so if we loosely speaking move to the world of formal Laurent series over  $\mathbb{F}_3$ , we let  $3 = X$ , and get an analogue of the Cantor set by

$$\mathcal{C} = \left\{ \sum_{i=1}^{\infty} a_i X^{-i} \in \mathbb{F}_3((X^{-1})) : a_i \in \{0, 2\} \right\}.$$

We can now formulate Mahler's problem: Let  $\alpha \in \mathcal{C}$  be algebraic. Does this imply that  $\alpha \in \mathbb{F}_3(X)$ ?

The answer is no as the following example show:

**Example.** Let

$$\alpha = \sum_{n=0}^{\infty} X^{-3^n} = X^{-1} + X^{-3} + X^{-9} + \dots$$

From Freshman's Dream we get

$$\alpha^3 = \sum_{n=0}^{\infty} X^{-3^{n+1}} = \alpha - X^{-1},$$

and hence

$$\alpha^3 - \alpha + X^{-1} = 0.$$

It can be checked, that  $Y^3 - Y + X^{-1}$  is irreducible, and hence  $\alpha$  is algebraic of degree 3. Now,  $2\alpha \in \mathcal{C}$  is also algebraic of degree 3 giving the counter example.

The existence of  $\alpha$  was known to Mahler, as he used it as a counter example to Roth's Theorem in the formal Laurent series over finite fields [19].

By modifying the argument, we can construct algebraic formal Laurent series in  $\mathcal{C}$  of degree 9, 27, and so forth. But can we find algebraic formal Laurent series in  $\mathcal{C}$  of degrees not a power of the characteristic 3, or are these degrees the only ones producing counterexamples to Mahler's problem? The answer to this is also no, as the following example constructs a degree 2 algebraic formal Laurent series in  $\mathcal{C}$ :

**Example.** For  $i \in \mathbb{N}_0$  define  $c_i$  to be 0 if  $i$  written to base 3 contains an 1, and 1 otherwise. Note, that  $c_{3i} = c_i$ ,  $c_{3i+1} = 0$  and  $c_{3i+2} = c_i$ . Let

$$\mathfrak{C} = \sum_{i=0}^{\infty} c_i X^{-i},$$

and compute

$$\begin{aligned} \mathfrak{C} &= \sum_{i=0}^{\infty} c_{3i} X^{-3i} + \sum_{i=0}^{\infty} c_{3i+1} X^{-(3i+1)} + \sum_{i=0}^{\infty} c_{3i+2} X^{-(3i+2)} \\ &= \sum_{i=0}^{\infty} c_i X^{-3i} + X^{-2} \sum_{i=0}^{\infty} c_i X^{-3i} \\ &= \left( \sum_{i=0}^{\infty} c_i X^{-i} \right)^3 + X^{-2} \left( \sum_{i=0}^{\infty} c_i X^{-i} \right)^3 \\ &= \mathfrak{C}^3 + X^{-2} \mathfrak{C}^3. \end{aligned}$$



So we have

$$(1 + X^{-2})\mathfrak{C}^3 - \mathfrak{C} = 0,$$

and since  $\mathfrak{C} \neq 0$ , we have

$$(1 + X^{-2})\mathfrak{C}^2 - 1 = 0.$$

Again, it can be shown, that  $(1 + X^{-2})Y^2 - 1$  is irreducible, and hence  $\mathfrak{C}$  is algebraic of degree 2. Finally,  $2X^{-1}\mathfrak{C} \in \mathcal{C}$  is algebraic of degree 2.

$\mathfrak{C}$  is called the Cantor Laurent series, and the construction is taken from the book of Allouche and Shallit [1].

The two examples lead to the following natural and still open question:

**Question.** *Does there for any  $d \in \mathbb{N}$ , exists an  $\alpha \in \mathcal{C}$  algebraic of degree  $d$ ?*

In the literature there are several results connecting the coefficients of an  $\alpha \in \mathbb{F}_q((X^{-1}))$ , to whether  $\alpha$  is algebraic, most famously the following theorem by Christol [8].

**Theorem 6** (Christol's Theorem). *Let*

$$\alpha = \sum_{i=0}^{\infty} a_i X^{-i}$$

*be a formal Laurent series over  $\mathbb{F}_q$ . Then  $\alpha$  is algebraic if and only if  $\{a_i\}_{i=0}^{\infty}$  is an automatic sequence.*

A sequence  $\{a_i\}_{i=0}^{\infty}$  is automatic, or more precisely  $q$ -automatic, if there exists a finite automaton with states  $\mathbb{F}_q$ , such that for each  $n$ ,  $a_n$  is the output, when given  $n$  written to base  $q$  as input. For more information see the book [1].

All of these results have the restriction, that they only tell whether  $\alpha$  is algebraic, but not what the degree is. By analysing the proofs we can sometimes get an upper bound on the degree of  $\alpha$ , but for answering the question, these results are insufficient.

I see two ways of approach for answering the question. Either we explicitly construct algebraic formal Laurent series having the desired degree, or we develop theory to get sufficient control of both the degree of the algebraic formal Laurent series, as well as the coefficients.



## 2 Summaries of the papers

There are three papers resulting from my PhD. The paper “A Cantor set type result in the field of formal Laurent series” is published in *Functiones et Approximatio Commentarii Mathematici* [22], the paper “Some remarks on Mahler’s classification in higher dimension” is to appear in *Moscow Journal of Combinatorics and Number Theory* [16], and the paper “Solution of Cassels’ Problem on a Diophantine Constant over Function Fields” is to appear in *International Mathematics Research Notices* [3].

The paper “Some remarks on Mahler’s classification in higher dimension” is written together with Simon Kristensen and Barak Weiss and the paper “Solution of Cassels’ Problem on a Diophantine Constant over Function Fields” is written together with Efrat Bank and Erez Nesharim. My contribution to these papers is proportional.

### 2.1 A Cantor set type result in the field of formal Laurent series

As in the previous chapter, we let  $C$  be the Cantor set in  $\mathbb{R}$  and  $\mathcal{C}$  the Cantor set in  $\mathbb{F}_3((X^{-1}))$ . We let  $\gamma = \frac{\log(2)}{\log(3)}$ , which turns out to be the Hausdorff dimension of both  $C$  and  $\mathcal{C}$ . Furthermore,  $\mathcal{H}^\gamma(C) = \mathcal{H}^\gamma(\mathcal{C}) = 1$ .

This paper is about understanding  $\mathcal{C}$  with respect to rational approximation. More precisely it is about intrinsic well approximation in  $\mathcal{C}$  i.e. how well can elements in  $\mathcal{C}$  be approximated by rational functions in  $\mathcal{C}$ .

In the real case, we would for an approximation function  $\Psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be interested in the set

$$W(\Psi) = \left\{ \alpha \in C : \left| \alpha - \frac{p}{q} \right| < \Psi(q) \text{ for i.m. } \frac{p}{q} \in C \right\}.$$

Understanding  $W(\Psi)$  is hard, because we need to simultaneously control both the base 3 expansion of a fraction to check whether it is in  $C$ , as well as the height of the fraction. In an ideal world all fractions in  $C$  would be endpoints of the intervals in the construction, and hence on the form  $\frac{p}{3^n}$ , in which case control would be easier. This is not the case as  $\frac{1}{4} \in C$ .

Instead we consider the set

$$W_C(\Psi) = \left\{ \alpha \in C : \left| \alpha - \frac{p}{3^n} \right| < \Psi(3^n) \text{ for i.m. } \frac{p}{3^n} \in C \right\},$$

where we only approximate by the endpoints from the construction of  $C$ , and we can to some extent control the approximations.

Studying  $W_C(\Psi)$  was done by Levesley, Salp and Velani in [17], where they established the following:

**Theorem 7.** *Let  $f$  be a dimension function such that  $r^{-\gamma}f(r)$  is monotonic. Then*

$$\mathcal{H}^f(W_C(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} f(\Psi(3^n)) \times (3^n)^\gamma < \infty, \\ \mathcal{H}^f(C) & \text{if } \sum_{n=1}^{\infty} f(\Psi(3^n)) \times (3^n)^\gamma = \infty. \end{cases}$$

In particular for  $f(x) = x^\gamma$ , we get

$$\mathcal{H}^\gamma(W_C(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} (\Psi(3^n) \times 3^n)^\gamma < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} (\Psi(3^n) \times 3^n)^\gamma = \infty. \end{cases}$$

We let  $\psi : \{3^n : n \in \mathbb{N}\} \rightarrow \{3^r : r \in \mathbb{Z}\}$  be an approximation function. The problem of controlling the rational numbers in  $C$  also appear in  $\mathcal{C}$ . In particular,  $\frac{2}{X^2-1} \in \mathcal{C}$  is not coming from the endpoints of the construction of  $\mathcal{C}$ . So for the same reason as in the real case, we are going to study

$$W_{\mathcal{C}}(\psi) = \left\{ h \in \mathcal{C} : \left| h - \frac{g}{X^N} \right| < \psi(3^N), \text{ for i.m. } N \in \mathbb{N}, \text{ where } g \in \mathcal{F}(N) \right\},$$

where

$$\mathcal{F}(N) = \{ f \in \mathbb{F}[X] : \text{Coeff}(f) \subseteq \{0, 2\}, \deg f < N \},$$

which is the analogue of  $W_C(\Psi)$ .

The first part of this paper establishes the analogue of Theorem 7 in  $\mathcal{C}$ .

**Theorem 8.** *Let  $f$  be a dimension function such that  $r^{-\gamma}f(r)$  is monotonic. Then*

$$\mathcal{H}^f(W_{\mathcal{C}}(\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^\gamma < \infty, \\ \mathcal{H}^f(\mathcal{C}) & \text{if } \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^\gamma = \infty. \end{cases}$$

Again, in the special case  $f(x) = x^\gamma$ , we get

$$\mathcal{H}^\gamma(W_{\mathcal{C}}(\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} (\psi(3^n) \times 3^n)^\gamma < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} (\psi(3^n) \times 3^n)^\gamma = \infty. \end{cases} \quad (2.1)$$

The result generalise easily to the case of a finite field  $\mathbb{F}_q$  in stead of  $\mathbb{F}_3$  and a general missing digit set.

The structure of the proof follows that of [17], but with some changes and simplifications coming from the fact, that  $\mathbb{F}_3((X^{-1}))$  is an ultrametric field. As is often the case in such Khintchine type theorems, the part where the series converges is easy, and is essentially a replication of the proof of the convergent part of the Borel–Cantelli Lemma.

For the case where the series diverge, we first prove the divergent part of (2.1), and then a standard application of the Mass Transference Principle from [5] implies the result. To prove the divergent part of (2.1), we look at the subset of  $W_{\mathcal{C}}(\psi)$ , where we only approximate with reduced rational functions.

$$W_{\mathcal{C}}^*(\psi) = \left\{ h \in \mathcal{C} : \left| h - \frac{g}{X^N} \right| < \psi(3^N), \text{ for i.m. } N \in \mathbb{N}, \text{ where } g \in \mathcal{F}^*(N) \right\},$$

where

$$\mathcal{F}^*(N) = \{f \in \mathbb{F}[X] : \text{Coeff}(f) \subseteq \{0, 2\}, \deg f < N \text{ and } f(0) = 2\}.$$

Since  $W_{\mathcal{C}}^*(\psi) \subseteq W_{\mathcal{C}}(\psi) \subseteq \mathcal{C}$ , it is sufficient to prove, that if  $\sum_{n=1}^{\infty} (\psi(3^n) \times 3^n)^{\gamma} = \infty$ , then  $\mathcal{H}^{\gamma}(W_{\mathcal{C}}^*(\psi)) = 1$ . We rewrite  $W_{\mathcal{C}}^*(\psi)$  as a lim sup set

$$W_{\mathcal{C}}^*(\psi) = \limsup_{N \rightarrow \infty} A_N^*,$$

where

$$A_N^* = \bigcup_{g \in \mathcal{F}^*(N)} B\left(\frac{g}{X^N}, \psi(3^N)\right) \cap \mathcal{C}.$$

Then we use a version of the divergent Borel–Cantelli Lemma to get, that if  $\sum_{n=1}^{\infty} \mathcal{H}^{\gamma}(A_n^*) = \infty$ , and  $\{A_n^*\}_{n=1}^{\infty}$  is pairwise quasi-independent, meaning that

$$\mathcal{H}^{\gamma}(A_m^* \cap A_n^*) \leq \mathcal{H}^{\gamma}(A_m^*) \mathcal{H}^{\gamma}(A_n^*)$$

for distinct  $m$  and  $n$ , then  $\mathcal{H}^{\gamma}(W_{\mathcal{C}}^*(\psi)) = 1$ .

By careful counting, and using the tree structure of balls in ultrametric spaces, we get both the quasi-independence, and that

$$\sum_{n=1}^{\infty} \mathcal{H}^{\gamma}(A_n^*) = \sum_{n=1}^{\infty} (\psi(3^n) \times 3^n)^{\gamma} \times 3^{-\gamma} = \infty.$$

In the real case, we use geometric arguments, and are only able to prove a weaker version of pairwise quasi-independence, that there exists a  $K > 1$ , such that

$$\mathcal{H}^{\gamma}(A_m^* \cap A_n^*) \leq K \mathcal{H}^{\gamma}(A_m^*) \mathcal{H}^{\gamma}(A_n^*)$$

for distinct  $m$  and  $n$ , and hence we only have  $\mathcal{H}^{\gamma}(W_{\mathcal{C}}^*(\psi)) > \frac{1}{K}$ , and we need to do some tricks to blow up to measure 1. So, as mentioned in the introduction, we have two different ways of getting the crucial estimate.

The second part of this paper is concerned with the irrationality measure of elements in  $\mathcal{C}$ . Whereas the first part is about the generic properties of elements in  $\mathcal{C}$ , the second part is about the existence of elements in  $\mathcal{C}$  with certain properties. In analogy to the real case, we define the irrationality measure of an element  $\xi$  by

$$\mu(\xi) = \sup \left\{ \mu \in \mathbb{R} : \left| \xi - \frac{g}{h} \right| < |h|^{-\mu} \text{ for i.m. } \frac{g}{h} \in \mathbb{F}(X) \right\}.$$

We prove, that for any  $\tau \geq 2$ , there exists an element  $\xi \in \mathcal{C}$  with  $\mu(\xi) = \tau$ . The key ingredient is the Folding Lemma from [23], that enables us to build a  $\xi$ , where we can simultaneously control both the Laurent series, and hence that  $\xi \in \mathcal{C}$ , and the simple continued fraction expansion, and hence the irrationality measure. The strategy of proof follows that of Bugeaud [7], where he proves the real case counterpart of the theorem.

## 2.2 Some remarks on Mahler's classification in higher dimension

Whereas the previous paper was about understanding the Cantor set, this paper leans more towards understanding algebraicity and transcendence. It is concerned with higher dimensional Mahler approximation, a natural generalisation of Mahler approximation. Instead of considering polynomials  $P \in \mathbb{Z}[X]$  when approximating a  $x \in \mathbb{R}$ , we consider polynomials in  $d$  variables  $P \in \mathbb{Z}[X_1, \dots, X_d]$  when approximating a  $\mathbf{x} \in \mathbb{R}^d$ .

We let  $k \in \mathbb{N}$ ,  $\mathbf{x} \in \mathbb{R}^d$ , and as in the one dimensional case we define the exponent

$$\omega_k(\mathbf{x}) = \sup \{ \omega \in \mathbb{R} : |P(\mathbf{x})| \leq H(P)^{-\omega} \text{ for i.m. } P \in \mathbb{Z}[X_1, \dots, X_d], \deg_{\text{total}}(P) \leq k \}.$$

Furthermore, throughout this summary we let  $n = \binom{k+d}{d} - 1$  be the number of non-constant monomials in the variables  $X_1, \dots, X_d$  of total degree at most  $k$ . From Yu [30] we get, that  $\omega_k(\mathbf{x}) \geq n$  for all  $\mathbf{x} \in \mathbb{R}^d$ . The proof follows closely the proof, that  $\omega_k(x) \geq k$  for all  $x \in \mathbb{R}$ .

The first result of the paper deals with how many points have better approximation than  $n$ . We say, that a point  $\mathbf{x} \in \mathbb{R}^d$  is  $k$ -very well approximable, if there exists an  $\varepsilon > 0$  and infinitely many polynomials  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$ , such that

$$|P(\mathbf{x})| \leq H(P)^{-(n+\varepsilon)}.$$

It turns out, that the set of  $k$ -very well approximable points is small.

**Theorem 9.** *Lebesgue almost all  $\mathbf{x} \in \mathbb{R}^d$  is not  $k$ -very well approximable. In particular,  $\omega_k(\mathbf{x}) = n$  for Lebesgue almost all  $\mathbf{x} \in \mathbb{R}^d$ .*

In fact, we prove something a bit stronger, as we show it not only for the Lebesgue measure, but for any absolutely decaying Federer measure. For the definition of absolutely decaying Federer measure see the paper.

The next result goes in the other direction and deals with  $k$ -badly approximable points. We call a point  $\mathbf{x} \in \mathbb{R}^d$   $k$ -badly approximable, if there exists an  $C = C(k, \mathbf{x})$ , such that

$$|P(\mathbf{x})| \geq CH(P)^{-n},$$

for all non-zero polynomials  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$ . If we let  $B_k$  be the set of  $k$ -badly approximable points, then from the work of Beresnevich, Bernik, Kleinbock and Margulis [6] it follows, that  $\lambda_d(B_k) = 0$ . On the other hand we can show, that it has not only full dimension, but it is even thick.

**Theorem 10.** *Let  $B \subseteq \mathbb{R}^d$  be an open ball and let  $M \in \mathbb{N}$ . Then*

$$\dim_H B \cap \bigcap_{k=1}^M B_k = d.$$

The next result in the paper is about  $(k, \varepsilon)$ -Dirichlet improvable points and  $k$ -singular points. We call a point  $\mathbf{x} \in \mathbb{R}^d$   $(k, \varepsilon)$ -Dirichlet improvable if there exists a  $Q_0 \in \mathbb{N}$ , such that for any  $Q \geq Q_0$  there exists a polynomial  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$ , such that

$$\tilde{H}(P) \leq \varepsilon Q \text{ and } |P(\mathbf{x})| \leq \varepsilon Q^{-n}.$$

From [30] it follows, that for  $\varepsilon \geq 1$  any point satisfies the definition, and hence the set of  $(k, \varepsilon)$ -Dirichlet improvable points is only interesting for  $\varepsilon < 1$ . If a point is  $(k, \varepsilon)$ -Dirichlet improvable for any  $\varepsilon > 0$ , we call it  $k$ -singular.

For a  $k$ -friendly measure, for the definition see the paper, but for instance  $\lambda_d$ , we show that for  $\varepsilon$  small enough, the set of  $(k, \varepsilon)$ -Dirichlet improvable points is a null set.

**Theorem 11.** *Let  $\mu$  be a  $k$ -friendly measure on  $\mathbb{R}^d$ . Then there is an  $\varepsilon_0 = \varepsilon_0(d, \mu)$ , such that the set of  $(k, \varepsilon)$ -Dirichlet improvable points has measure zero for any  $\varepsilon < \varepsilon_0$ . In particular, the set of  $k$ -singular vectors has measure zero.*

Finally, we say that a point  $\mathbf{x} \in \mathbb{R}^d$  is  $k$ -algebraic, if there exists a polynomial  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$ , such that  $P(\mathbf{x}) = 0$ . Furthermore, we say that a point  $\mathbf{x} \in \mathbb{R}^d$  is algebraic of total degree  $k$ , if it is  $k$ -algebraic, and  $\mathbf{x}$  does not vanish at any polynomial of total degree less than  $k$ .

If a point is  $k$ -algebraic, then it is also  $k$ -singular, so a relevant question is, whether there exists a  $k$ -singular point that is not  $k$ -algebraic? At least for  $d \geq 2$  this turns out to be the case. For  $d = 1$  it is unknown.

**Theorem 12.** *For  $d \geq 2$ , for any  $k \geq 1$ , there exists a  $k$ -singular point in  $\mathbb{R}^d$  which is not  $k$ -algebraic.*

We also improve on Theorem 11 in the special case of the Lebesgue measure  $\lambda_d$ .

**Theorem 13.** *For any  $d$ , the set of  $\mathbf{x}$  which are  $(k, \varepsilon)$ -Dirichlet improvable for some  $\varepsilon < 1$  and some  $k$ , has Lebesgue measure zero.*

Theorem 13 tells that in the case of the Lebesgue measure, the set

$$\bigcup_{0 < \varepsilon < 1} \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} \text{ is } (k, \varepsilon)\text{-Dirichlet improvable} \},$$

is a null set, whereas Theorem 11 only tells that

$$\bigcup_{0 < \varepsilon < \varepsilon_0} \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} \text{ is } (k, \varepsilon)\text{-Dirichlet improvable} \},$$

is a null set for some  $\varepsilon_0$  dependent on the measure. So Theorem 13 in some sense says, that in the case of the Lebesgue measure, we can let  $\varepsilon$  grow to 1, and still have a null set.

The final result of the paper is a higher dimensional version of Roth's Theorem.

**Theorem 14.** *Let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$  be an algebraic vector of total degree more than  $k$ . Then for any  $\varepsilon > 0$  there are only finitely many non-zero polynomials  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$  with*

$$|P(\boldsymbol{\alpha})| < H(P)^{-(n+\varepsilon)}.$$

The proofs of Theorems 9, 10, 11 and 14 follow the same underlying pattern. First, the proof is reduced to the existence of non-zero integer linear combinations of the monomials  $X_1, X_2, \dots, X_{d-1}X_d^{k-1}, X_d^k$ , satisfying some criteria dependent on the number of monomials,  $n$ . Second, we realise that the one dimensional version of the

theorem is shown by using some powerful theory to establish the existence of non-zero integer linear combinations of the monomials  $X, X^2, \dots, X^k$  satisfying the same criteria, but this time dependent on  $k$ , the number of monomials in this case. Third, we use the powerful theory on the monomials  $X_1, X_2, \dots, X_{d-1}X_d^{k-1}, X_d^k$ . The cost of using it on these, is that the criteria is now dependent on  $n$ , which is exactly what we want.

The powerful theory that is used in each of the cases is the work of Kleinbock, Lindenstrauss and Weiss [12], Beresnevich [4], Kleinbock and Weiss [14] and Schmidt's Subspace Theorem [26].

The proofs of Theorems 12 and 13 is more or less direct consequences of the work of Shah [27] and of Kleinbock and Weiss [13].

## 2.3 Solution of Cassels' Problem on a Diophantine Constant over Function Fields

This paper takes its starting point in a failed attempt to get results in twisted inhomogeneous Diophantine approximation over the formal Laurent series over  $\mathbb{F}_q$ , by the help of homogeneous dynamics.

Shapira showed in [28] the following theorem on twisted inhomogeneous Diophantine approximation.

**Theorem 15.** *For any  $\gamma, \delta \in \mathbb{R}$ , Lebesgue almost all  $(\alpha, \beta) \in \mathbb{R}^2$  satisfy*

$$\liminf_{n \rightarrow \infty} n |\langle n\alpha - \gamma \rangle| |\langle n\beta - \delta \rangle| = 0. \quad (2.2)$$

Here  $|\langle \cdot \rangle|$  means the distance to the nearest integer. The proof of the theorem builds heavily on the connection between Diophantine approximation and dynamical systems, and is proved by means of homogeneous dynamics.

Our attempt was to prove the analogue of Theorem 15 over the formal Laurent series, which would require us to understand homogeneous dynamics over the formal Laurent series. My personal goal was to understand how well methods of homogeneous dynamics can be used in Diophantine approximation over formal Laurent series over  $\mathbb{F}_q$ , and even though the attempt was a failure, I gained valuable information towards my goal. Understanding the limitations and strenghts of a method is extremely valuable.

The idea of the proof of Theorem 15 is fairly clear. It can be split into the following steps. First, we find a more general question about unimodular lattices, or more precisely translated unimodular lattices called grids. We let  $X_3$  be the space of unimodular lattices in  $\mathbb{R}^3$ , and let  $Y_3$  be the space of grids in  $\mathbb{R}^3$ . We define the product function  $N : \mathbb{R}^3 \rightarrow \mathbb{R}$  by  $N((a, b, c)^t) = abc$ . Furthermore, we define the product function on subsets  $P : \mathcal{P}(\mathbb{R}^3) \rightarrow \mathcal{P}(\mathbb{R})$  by  $P(B) = \{N(b) : b \in B\}$  for any  $B \in \mathcal{P}(\mathbb{R}^3)$ . For a grid  $y \in Y_3$ , we say it is dense product, DP, if

$$\overline{P(y)} = \mathbb{R}.$$

Furthermore, for a lattice  $x \in X_3$ , we say it is grid dense product, GDP, if

$$\overline{P(x + v)} = \mathbb{R}$$



for all  $v \in \mathbb{R}^3$ .

For  $u = (\alpha, \beta)^t \in \mathbb{R}^2$  we define the lattice

$$h_u = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^3.$$

Theorem 15 is now a consequence of the following more general theorem.

**Theorem 16.** *For Lebesgue almost all  $u \in \mathbb{R}^2$ ,  $h_u$  is GDP.*

The second step is to realise, that the diagonal action on  $X_3$  plays nicely together with being GDP. We let  $A = \{\text{diag}(e^{t_1}, e^{t_2}, e^{t_3}) : t_1, t_2, t_3 \in \mathbb{R}, t_1 + t_2 + t_3 = 0\}$  be the set of diagonal matrices in  $\text{SL}_3(\mathbb{R})$  with positive diagonal entries.  $A$  acts on  $X_3$  and  $Y_3$  in the following way: For  $a \in A, b \in \text{SL}_3(\mathbb{R})$  and  $v \in \mathbb{R}^3$ , we have  $a(b\mathbb{Z}^3) = (ab)\mathbb{Z}^3$  and  $a(b\mathbb{Z}^3 + v) = (ab)\mathbb{Z}^3 + av$ .

We let  $a \in A, y \in Y_3$ , and note that  $P(ay) = P(y)$ . Furthermore, for  $y_0 \in \overline{Ay}$ , we have

$$\overline{P(y_0)} \subseteq \overline{P(y)}, \quad (2.3)$$

and hence in particular if  $y_0$  is DP, then so is  $y$ . This implies in turn, that for  $x_0, x \in X_3$ ,

$$\text{if } x_0 \in \overline{Ax}, x_0 \text{ is GDP, then } x \text{ is GDP.} \quad (2.4)$$

The third step is to show the existence of a  $x_0 \in X_3$  which is GDP. It is in this step, that most of the hard work is done, so we will postpone the discussion until after the forth step.

The forth step is to show, that Lebesgue almost all  $u \in \mathbb{R}^2$  have  $\overline{Ah_u} = X_3$ . This is a well known fact from homogeneous dynamics. It is a mixed consequence of the fact, that for  $t > 0$ ,  $a_t = \text{diag}(e^t, e^t, e^{-2t})$ , the unstable horospherical subgroup of  $a_t$  is

$$U_0^+(a_t) = \{h_u : u \in \mathbb{R}^2\},$$

that the action of  $a_t$  is ergodic, and that the Haar measure on  $X_3$  restricts nicely to  $U_0^+(a_t)$ . For more details see [28, Lemma 4.8].

From step three there exist a  $x_0 \in X_3$  which is GDP. From step four it follows, that for almost all  $u \in \mathbb{R}^2$ ,  $x_0 \in \overline{Ah_u}$ , and hence from (2.4) Theorem 16 follows.

Going back to step three, the fact, that such a  $x_0$  exists is by no means clear, but follows from the following surprising theorem, together with the fact that there exist lattices with compact  $A$ -orbits.

**Theorem 17.** *If  $x_0, \tilde{x} \in X_3$ ,  $A\tilde{x}$  is compact and  $\tilde{x} \in \overline{Ax_0} \setminus Ax_0$ , then  $x_0$  is GDP.*

The proof of Theorem 17 follows the following line: Let  $w \in \mathbb{R}^3$ , and consider the orbit closure  $F = \overline{A(x_0 + w)}$ . Furthermore, we consider grids in  $F$  of a special form  $\tilde{F} = \{\tilde{y} \in F : \tilde{y} = \tilde{x} + v, \text{ for some } v \in \mathbb{R}^3\}$ . It can be shown, that  $\tilde{F} \neq \emptyset$ , so we let  $\tilde{y} \in \tilde{F}$ .

Now one of two cases arise: Either  $\tilde{y} \in \tilde{F}$  has a non-compact  $A$  orbit, in which case it can be shown, that  $\overline{A\tilde{y}}$  contains  $\tilde{x} + v$  for all  $v \in \mathbb{R}^3$ , which then implies  $\tilde{y}$  is DP, and hence  $x_0 + w$  is DP.

The other case is, that  $A\tilde{y}$  is compact. In this case we want to show, that  $F$  is sufficiently big.

For each pair  $1 \leq i, j \leq 3, i \neq j$  and  $t \in \mathbb{R}$ , let  $u_{i,j}(t)$  be the  $3 \times 3$  matrix with 1 on the diagonal,  $t$  on the  $(i, j)$  place and 0 elsewhere.

Now for each  $\tilde{y} \in \tilde{F} \subseteq F$ , we show that there exists a non-zero  $t \in \mathbb{R}$ , and a pair  $(i, j)$ , such that  $u_{i,j}(t)\tilde{y} \in F$ . Next we realise, that not only does this extra point lie in  $F$ , but in fact for any  $s \in \mathbb{R}_+t$ , we have that  $u_{i,j}(s)\tilde{y} \in F$ . Now for each  $\xi \in \mathbb{R}$  there is a  $w \in \tilde{y}$  with all coordinates different from zero, and a  $s \in \mathbb{R}_+t$ , such that

$$N(u_{i,j}(s)w) = N(w)\left(\frac{w_j}{w_i}s + 1\right) = \xi,$$

and hence it follows from 2.3, that

$$\xi \in P(u_{i,j}(s)\tilde{y}) \subseteq \overline{P(u_{i,j}(s)\tilde{y})} \subseteq \overline{P(x_0 + w)},$$

which implies that  $x_0 + w$  is DP. So for all  $w \in \mathbb{R}^3$ , we have, that  $x_0 + w$  is DP, so  $x_0$  is GDP.

We are now ready to explain the deeper reason why the proof does not work over formal Laurent series over  $\mathbb{F}_q$ .

First, most of it works, and even become slightly prettier. We want  $A$  to be a two parameter group of diagonal matrices in  $SL_3(\mathbb{F}_q((X^{-1})))$ , so we let

$$A = \left\{ \text{diag}(X^k, X^\ell, X^m) : k, \ell, m \in \mathbb{Z}, k + \ell + m = 0 \right\}.$$

Furthermore,  $A$  acts on the space of grids in  $\mathbb{F}_q((X^{-1}))^3$ . Now step one and two still works, and a lot of Theorem 17, that is step three, still work. In fact due to the ultrametric property only the second of the cases arise. The thing that goes wrong is when we try to move from the existence of a  $t \in \mathbb{R}$ , such that  $u_{i,j}(t)\tilde{y} \in F$ , to it being true for any  $s \in \mathbb{R}_+t$ .

In the real case, we have, that

$$\left\{ \frac{a_i}{a_j} : \text{diag}(a_1, a_2, a_3) \in \text{Stab}_A(\tilde{y}) \right\} \quad (2.5)$$

is dense in  $\mathbb{R}_+$ . Now for  $a \in \text{Stab}_A(\tilde{y})$ , we have that

$$au_{i,j}(t)\tilde{y} = au_{i,j}(t)a^{-1}a\tilde{y} = u_{i,j}\left(\frac{a_i}{a_j}t\right)a\tilde{y} = u_{i,j}\left(\frac{a_i}{a_j}t\right)\tilde{y}$$

lies in  $F$ , and combined with (2.5) being dense in  $\mathbb{R}_+$ , we get that  $u_{i,j}(s)\tilde{y} \in F$  for every  $s \in \mathbb{R}_+t$ .

Now this is where it goes wrong, because in the formal Laurent series case, we have no hope of having the analogue of (2.5) being dense in the monic formal Laurent series. Even in the unlikely case, that  $\text{Stab}_A(\tilde{y}) = A$ , we only have, that (2.5) is the set of monic monomials, which is not dense in the set of monic formal Laurent series.

Since step three failed, we never got around to check the details of whether step four holds true.

The lecture to be learned from the attempt is the following: If we want to apply the methods of homogeneous dynamics to do Diophantine approximation in the field of formal Laurent series, we might not be able to use the methods, if they require underlying denseness results.

Now, if you are really not interested in dynamics, you could have started the summary here, but then you would have missed out on the fun.

Discouraged, we turned to another approach on inhomogeneous Diophantine approximation over formal Laurent series, the approach of linear algebra.

For  $\theta, \gamma \in \mathbb{F}_q((X^{-1}))$ , we define

$$c(\theta, \gamma) = \inf_{N \neq 0} |N| |\langle N\theta - \gamma \rangle|,$$

where the infimum is taken over the non-zero  $N \in \mathbb{F}_q[[X]]$ . Furthermore, we define

$$\text{BA}_\theta = \{\gamma \in \mathbb{F}_q((X^{-1})) : c(\theta, \gamma) > 0\},$$

$$c(\theta) = \sup_{\gamma} c(\theta, \gamma)$$

and

$$c = \inf_{\theta} c(\theta).$$

By using linear algebra, we are able to prove the following theorems:

**Theorem 18.** *For every  $\theta \in \mathbb{F}_q((X^{-1}))$  the set  $\text{BA}_\theta$  has full hausdorff dimension.*

**Theorem 19.**

$$c = q^{-2}.$$

The most surprising is, that at present the value of  $c$  in the real case is only known to satisfy

$$\frac{3}{32} \leq c \leq \frac{68}{483},$$

so we are determining a constant whose value is unknown in the real case.

We also show higher dimensional version of these theorems, using what we call generalised weights. A function  $\mathbf{g} = (g_1, \dots, g_d) : \mathbb{N}_0 \rightarrow \mathbb{N}_0^d$  is called a generalised weight, if for all  $1 \leq s \leq d$  the function  $g_s$  is non-decreasing, and

$$\sum_{s=1}^d g_s(h) = h$$

for every  $h \in \mathbb{N}_0$ . One can think of a generalised weight as sending 0 to  $\mathbf{0}$ , and for each subsequent value, it increases one of the coordinates by one.

Given a generalised weight  $\mathbf{g}$  and  $\boldsymbol{\theta}, \boldsymbol{\gamma} \in \mathbb{F}_q((X^{-1}))^d$ , we define

$$c_{\mathbf{g}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) = \inf_{N \neq 0} \max_{1 \leq s \leq d} |N|^{\frac{g_s(\deg N)}{\deg N}} |\langle N\theta_s - \gamma_s \rangle|.$$

Furthermore, we define

$$\text{BA}_{\boldsymbol{\theta}}(\mathbf{g}) = \{\boldsymbol{\gamma} \in \mathbb{F}_q((X^{-1}))^d : c_{\mathbf{g}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) > 0\},$$

$$c_{\mathbf{g}}(\boldsymbol{\theta}) = \sup_{\boldsymbol{\gamma}} c_{\mathbf{g}}(\boldsymbol{\theta}, \boldsymbol{\gamma})$$

and

$$c_{\mathbf{g}} = \inf_{\boldsymbol{\theta}} c_{\mathbf{g}}(\boldsymbol{\theta}).$$

**Theorem 20.** For every generalized weight  $\mathbf{g}$  and  $\boldsymbol{\theta} \in \mathbb{F}_q((X^{-1}))^d$ ,  $\text{BA}_{\boldsymbol{\theta}}(\mathbf{g})$  is non-empty. Moreover, if

$$\inf_{h \in \mathbb{N}} \frac{\min \mathbf{g}(h)}{h} > 0,$$

then  $\text{BA}_{\boldsymbol{\theta}}(\mathbf{g})$  has full Hausdorff dimension.

**Theorem 21.** Any generalised weight  $\mathbf{g}$  satisfy

$$c_{\mathbf{g}} = q^{-2}.$$

For the sake of clarity we will only sketch the ideas of the proofs in the one dimensional theorems. The higher dimensional theorems follow the same idea, but with added technical notation disrupting the picture of what is going on.

For a  $\theta = \sum_{i=-\deg \theta}^{\infty} \theta_i X^{-i} \in \mathbb{F}_q((X^{-1}))$ , we can for each  $i, j \in \mathbb{N}$  associate its Hankel matrix

$$\Delta[i, j] = \begin{pmatrix} \theta_1 & \cdots & \theta_j \\ \vdots & & \vdots \\ \theta_i & \cdots & \theta_{i+j-1} \end{pmatrix}.$$

Now, for a non-zero polynomial  $N$  of degree  $h$ , a  $\gamma \in \mathbb{F}_q((X^{-1}))$  and an integer  $\ell \geq 0$ , we have that

$$|N| |\langle N\theta - \gamma \rangle| < q^{-(1+\ell)} \iff \Delta[h+1+\ell, h+1] \cdot \mathbf{n} = \pi_{h+1+\ell}(\gamma),$$

where  $\mathbf{n}$  is the coefficient vector of  $N$ , and  $\pi_k(\gamma)$  is the projection that maps  $\gamma$  to  $(\gamma_1, \dots, \gamma_k)^t$ . In this way, the questions of interest is translated into questions about non-zero solutions to systems of linear equations. In particular, we have, that  $\gamma \in \text{BA}_{\boldsymbol{\theta}}$  if and only if there exist  $\ell \geq 0$ , such that

$$\Delta[h+1+\ell, h+1] \cdot \mathbf{n} = \pi_{h+1+\ell}(\gamma) \tag{2.6}$$

has no solution for  $h \geq 0$  and  $\mathbf{n} \in \mathbb{F}_q^{h+1}$ .

The next step is to get control of the rank of  $\Delta[i, j]$ . To do this, we construct for each  $\ell > 0$  two sequences  $\mathcal{I}_{\ell} = \{i_m\}_{m=0}^{\infty}$ ,  $\mathcal{J}_{\ell} = \{j_m\}_{m=0}^{\infty}$  in an inductive fashion, by

1.  $j_0 = 0, i_0 = \ell$ .
2.  $j_{m+1} = \min_j \{j : \text{rank}(\Delta[i_m, j]) = i_m\}$ . If this minimum is not obtained, we let  $j_{m+1} = \infty$ .
3. If  $j_{m+1} = \infty$ , let  $i_{m+1} = i_m$ . Else let  $i_{m+1} = \min_i \{i : \text{rank}(\Delta[i, j_{m+1}]) = i - \ell\}$ .

Along this sequences we have sufficiently control of solutions to (2.6). For  $\ell > 0$  we define  $\Gamma_{\ell}$  to be the set of  $\gamma \in \mathbb{F}_q((X^{-1}))$  which for each  $m \geq 0, 0 < j < j_{m+1}$ , the equation

$$\Delta[i_m, j] \mathbf{n} = \pi_{i_m}(\gamma)$$

has no solutions.

Now, this  $\Gamma_{\ell}$  contains as a subset a Cantor like set  $C_{\infty}$ , created by starting with  $\mathbb{I}$ , cutting it into a number of subintervals of the same length, then throwing some of these away, and for each of the kept subintervals, we repeat the procedure. How

many subintervals we cut into, and how many subintervals we throw away, depends on the sequence  $\mathcal{I}_\ell$ . The limit set is what we call  $C_\infty$ , and using methods from fractal geometry, we get a lower bound on the dimension of  $C_\infty$ , and hence of  $\Gamma_\ell$ . We get that

$$\dim_H \Gamma_\ell \geq 1 - \frac{\xi}{\ell}$$

for some explicit constant  $0 < \xi < 1$  depending only on  $q$ .

Next, we get that for each  $\ell > 0$ , the set  $\Gamma_\ell$  is contained in  $\text{BA}_\theta$ , and hence

$$\dim_H \text{BA}_\theta \geq 1 - \frac{\xi}{\ell},$$

but since this is true for each  $\ell > 0$ , we get Theorem 18.

For Theorem 19, we have that  $\dim_H \Gamma_1 > 0$ , and hence  $\Gamma_1 \neq \emptyset$ . For each  $\gamma \in \Gamma_1$ , we have, that  $c(\theta, \gamma) \geq q^{-2}$ , and hence for each  $\theta \in \mathbb{F}_q((X^{-1}))$  we have that  $c(\theta) \geq q^{-2}$ .

In order to get the result, we show that if  $c(\theta) \geq q^{-1}$ , then there exist a  $m_0 \in \mathbb{N} \cup \{\infty\}$ , such that  $\Delta[m, m]$  is invertible exactly for  $0 < m < m_0$ . It is then a simple matter to find a  $\theta$  not satisfying this, and hence  $c = q^{-2}$ .

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# Papers



*Functiones et Approximatio*  
 53.1 (2015), 7–21  
 doi: 10.7169/facm/2015.53.1.1

## A CANTOR SET TYPE RESULT IN THE FIELD OF FORMAL LAURENT SERIES

STEFFEN H. PEDERSEN

**Abstract:** We prove a Khintchine type theorem for approximation of elements in the Cantor set, as a subset of the formal Laurent series over  $\mathbb{F}_3$ , by rational functions of a specific type.

Furthermore we construct elements in the Cantor set with any prescribed irrationality exponent  $\geq 2$ .

**Keywords:** formal Laurent series, diophantine approximation, metric theory.

### 1. Introduction

In [6] Khintchine proved, that for  $\psi : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{>0}$  a continuous function with  $x \mapsto x^2\psi(x)$  non-increasing, the set

$$W(\psi) = \left\{ \xi \in \mathbb{R} : \left| \xi - \frac{p}{q} \right| < \psi(q) \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}$$

of  $\psi$ -well approximable numbers has Lebesgue measure 0 if the series

$$\sum_{q=1}^{\infty} q\psi(q)$$

converges, and full Lebesgue measure if the series diverge. The analogous statement in the field of formal Laurent series over finite fields was shown by de Mathan in [4].

In [7] Levesly, Salp and Velani established a Khintchine type theorem for  $\psi$ -well approximable numbers in the Cantor set by rational numbers of the form  $\frac{p}{3^n}$ ,  $p \in \mathbb{N}$ .

The first part of this paper will establish the analogous statement in the field of formal Laurent series over  $\mathbb{F}_3$ , where the Cantor set consists of those formal Laurent series in the unit ball around 0 having only the coefficients 0 and 2.

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**2010 Mathematics Subject Classification:** primary: 11J61; secondary: 11J83, 11K55

The second part of the paper will construct elements of the Cantor set with any prescribed irrationality exponent  $\geq 2$ . This is the analogue of the result in [3] by Bugeaud.

The proofs follow the approach from [7] and [3], but with the simplifications and complications of working over an ultrametric field.

## 2. Preliminaries

Let  $\mathbb{F}_3$  be the field with 3 elements and let  $\mathbb{F}_3[X]$  be the polynomial ring over  $\mathbb{F}_3$ . We can introduce an absolute value on  $\mathbb{F}_3[X]$ , by letting  $|P| = 3^{\deg P}$  for  $P \in \mathbb{F}_3[X] \setminus \{0\}$ , and  $|0| = 0$ . This in turn gives an absolute value on the rational functions  $\mathbb{F}_3(X)$ , and by completing with respect to this absolute value, we get the field of formal Laurent series over  $\mathbb{F}_3$ , that is the set

$$\mathbb{F}_3((X^{-1})) = \left\{ \sum_{n=-N}^{\infty} a_{-n} X^{-n} : a_{-n} \in \mathbb{F}_3, a_N \neq 0 \right\} \cup \{0\},$$

where we have the absolute value

$$\left| \sum_{n=-N}^{\infty} a_{-n} X^{-n} \right| = 3^N$$

for the nonzero elements, and still  $|0| = 0$ .  $\mathbb{F}_3((X^{-1}))$  with the given absolute value is an ultrametric space. We will restrict our attention to the unit ball in  $\mathbb{F}_3((X^{-1}))$  around 0, that is the set

$$\mathbb{I} = \{h \in \mathbb{F}_3((X^{-1})) : |h| < 1\}.$$

$\mathbb{I}$  is the set of formal Laurent series on the form

$$\sum_{n=1}^{\infty} a_{-n} X^{-n}$$

where  $a_{-n} \in \mathbb{F}_3$ , and where 0 is the element with all the coefficients  $a_{-n} = 0$ . We can write the absolute value on  $\mathbb{I}$  as

$$|h| = \begin{cases} 0, & \text{if } h = 0, \\ 3^{-N}, & \text{if } h \neq 0, N = \min \{n : a_{-n} \neq 0\}. \end{cases}$$

For  $x \in \mathbb{F}_3((X^{-1}))$  we let  $B(x, 3^n)$  be the ball around  $x$  with radius  $3^n$ , and for  $a_{-1}, \dots, a_{-\ell} \in \mathbb{F}_3$  we let

$$B[a_{-1}, \dots, a_{-\ell}] = B(a_{-1}X^{-1} + \dots + a_{-\ell}X^{-\ell}, r^{-\ell}) \subseteq \mathbb{I}.$$

This ball consists of those elements in  $\mathbb{I}$  with the first  $\ell$  coefficients given by  $a_{-1}, \dots, a_{-\ell}$ .

It follows from the definition of the absolute value that every ball have radius  $3^{-n}$  for some  $n$ . In particular every ball inside  $\mathbb{I}$  is of the given form. We denote the radius of the ball  $B$  by  $r(B)$ .

In this paper we will look at the Cantor set, but in the setting of formal Laurent series. We define the 'Cantor set' as

$$\mathcal{C} = \{h \in \mathbb{I} : a_{-n} \in \{0, 2\}\}.$$

We let  $\psi : \{3^n : n \in \mathbb{N}\} \rightarrow \{3^{-r} : r \in \mathbb{Z}\}$  be a function, and are going to study the set

$$W_{\mathcal{C}}(\psi) = \left\{ h \in \mathcal{C} : \left| h - \frac{g}{X^N} \right| < \psi(3^N), \right. \\ \left. \text{for infinitely many } N \in \mathbb{N}, \text{ where } g \in \mathcal{F}(N) \right\},$$

where

$$\mathcal{F}(N) = \{f \in \mathbb{F}_3[X] : \text{Coeff}(f) \subseteq \{0, 2\}, \deg f < N\},$$

of  $\psi$ -well approximable elements in the Cantor set, by rational functions contained in the Cantor set of a specific form. In this respect, we are concerned with intrinsic Diophantine approximation.

For later we note that

$$\#\mathcal{F}(N) = 2^N, \tag{1}$$

and that

$$W_{\mathcal{C}}(\psi) = \left\{ h \in \mathcal{C} : h \in \bigcup_{g \in \mathcal{F}(N)} B\left(\frac{g}{X^N}, \psi(3^N)\right) \text{ for infinitely many } N \in \mathbb{N} \right\},$$

when expressed in terms of balls instead of approximation. So

$$W_{\mathcal{C}}(\psi) = \limsup_{N \rightarrow \infty} A_N = \{f \in \mathcal{C} : f \in A_N \text{ for infinitely many } N \in \mathbb{N}\},$$

where

$$A_N = \bigcup_{g \in \mathcal{F}(N)} B\left(\frac{g}{X^N}, \psi(3^N)\right).$$

Just as with every metric, locally compact space we can introduce the notion of Hausdorff measure, and Hausdorff dimension. We let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a dimension function i.e.  $f$  is continuous, non-decreasing and satisfy  $f(0) = 0$ . We can now define the Hausdorff  $f$ -measure in the following manner. For  $A \subseteq \mathbb{F}_3((X^{-1}))$  and  $\rho > 0$ , we let  $\mathcal{B}_\rho$  be the family of countable open covers of  $A$ , by balls  $B$  of radius  $r(B) \leq \rho$ . We can now define the Hausdorff  $f$ -measure by

$$\mathcal{H}^f(A) = \lim_{\rho \rightarrow 0} \inf_{\mathcal{B} \in \mathcal{B}_\rho} \sum_{B_i \in \mathcal{B}} f(r(B_i)).$$

If  $f$  is the dimension function given by  $f(x) = x^s$  for a  $s > 0$ , we call it the Hausdorff  $s$ -measure, and denote it by  $\mathcal{H}^s$ . We define the Hausdorff dimension by

$$\dim_H(A) = \inf \{s > 0 : \mathcal{H}^s(A) = 0\}.$$

Using standard techniques we can determine the Hausdorff dimension of  $\mathcal{C}$ , in fact we have the following result.

**Proposition 1.** *Let  $\gamma = \frac{\log(2)}{\log(3)}$ . For any ball  $B$  with  $r(B) \leq 1$  and  $B \cap \mathcal{C} \neq \emptyset$  we have*

$$\mathcal{H}^\gamma(B \cap \mathcal{C}) = r(B)^\gamma,$$

and in particular for  $B = \mathbb{I}$  we have

$$\mathcal{H}^\gamma(\mathcal{C}) = 1 \quad \text{and} \quad \dim_H(\mathcal{C}) = \gamma.$$

**Proof.** Throughout the proof let  $B$  be a ball with  $B \cap \mathcal{C} \neq \emptyset$  and  $r(B) = 3^{-\ell_0} \leq 1$ . Then  $B = B[a_{-1}, \dots, a_{-\ell_0}]$  for some  $a_{-1}, \dots, a_{-\ell_0} \in \mathbb{F}_3$ .

For the upper bound, let  $\rho = 3^{-j} \leq 3^{-\ell_0}$ . Then  $B \cup \mathcal{C}$  can be covered by the collection of  $2^{j-\ell_0}$  balls

$$\mathcal{B}' = \{B[a_{-1}, \dots, a_{-\ell_0}, a_{-(\ell_0+1)}, \dots, a_{-j}] : a_{-(\ell_0+1)}, \dots, a_{-j} \in \{0, 2\}\}$$

of radius  $3^{-j}$ . We then have

$$\inf_{\mathcal{B} \in \mathcal{B}_\rho} \sum_{B_i \in \mathcal{B}} (r(B_i))^\gamma \leq \sum_{B_i \in \mathcal{B}'} (r(B_i))^\gamma = 2^{j-\ell_0} (3^{-j})^\gamma = 2^{-\ell_0} = (3^{-\ell_0})^\gamma = r(B)^\gamma,$$

since  $3^\gamma = 2$ . By letting  $j \rightarrow \infty$ , we get that  $\mathcal{H}^\gamma(B \cap \mathcal{C}) \leq r(B)^\gamma$ , which gives the upper bound.

For the lower bound let  $\mathcal{B}$  be a cover of  $B \cap \mathcal{C}$  by balls. Then we want to show that

$$r(B)^\gamma \leq \sum_{B_i \in \mathcal{B}} (r(B_i))^\gamma.$$

First we may restrict the balls to lie in  $B$ , potentially decreasing the sum. If the inequality holds true when summing over a subset of  $\mathcal{B}$ , then it holds true when summing over  $\mathcal{B}$ . Since  $B \cap \mathcal{C}$  is compact, due to balls in  $\mathbb{F}_3((X^{-1}))$  being clopen, we can cover  $B \cap \mathcal{C}$  by a finite subset of  $\mathcal{B}$ . Furthermore if a ball  $\tilde{B} = B[a_{-1}, \dots, a_{-\ell}]$  of radius  $3^{-\ell}$  have one of  $a_{-1}, \dots, a_{-\ell}$  equal to 1, then  $\tilde{B} \cap \mathcal{C} = \emptyset$ , and we remove it from the finite subcover. The remaining balls we denote by  $\mathcal{B}'$ , and note that  $\mathcal{B}'$  is a finite cover of  $B \cap \mathcal{C}$ , each ball having nonempty intersection with  $\mathcal{C}$ .

Let the smallest ball in  $\mathcal{B}'$  have radius  $3^{-k}$ . For balls  $\tilde{B} = B[a_{-1}, \dots, a_{-\ell}]$  of radius  $3^{-\ell} > 3^{-k}$ ,  $\tilde{B}$  can disjointly be split into three balls  $A_0, A_1, A_2$  of radius  $3^{-(\ell+1)}$  by  $A_i = B[a_{-1}, \dots, a_{-\ell}, i]$  for  $i = 0, 1, 2$ . Now  $A_1 \cap \mathcal{C} = \emptyset$  and

$$r(\tilde{B})^\gamma = (3^{-\ell})^\gamma = 3^\gamma (3^{-(\ell+1)})^\gamma = 2(3^{-(\ell+1)})^\gamma = r(A_0)^\gamma + r(A_2)^\gamma,$$

so replacing the  $B$  by  $A_0$  and  $A_2$  does not change the sum, and we still have a cover of  $B \cap \mathcal{C}$ .

By iterating the procedure we end up with a cover of  $B \cap \mathcal{C}$  by balls of radius  $3^{-k}$ . Since it is a cover we must have at least  $2^{k-\ell_0}$  such balls, and hence

$$\sum_{B_i \in \mathcal{B}} r(B_i)^\gamma \geq \sum_{B_i \in \mathcal{B}'} r(B_i)^\gamma \geq 2^{k-\ell_0} 3^{-k\gamma} = 2^{-\ell_0} = r(B)^\gamma,$$

which is the lower bound. ■

We are now ready to state the analogue of the main result of [7] in the setting of formal Laurent series.

**Theorem 2.** *Let  $f$  be a dimension function such that  $r^{-\gamma} f(r)$  is monotonic. Then*

$$\mathcal{H}^f(W_{\mathcal{C}}(\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^\gamma < \infty \\ \mathcal{H}^f(\mathcal{C}) & \text{if } \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^\gamma = \infty \end{cases}$$

### 3. Toolbox

In this section we collect a lot of results which we will use in the rest of the paper.

We will need the following version of the diverging part of the Borel–Cantelli lemma, Lemma 2.3 in [5].

**Lemma 3.** *Let  $(X, \mu)$  be a finite measure space. Let  $\mathcal{A}_n$  be a sequence of measurable subsets of  $X$ . If*

$$\sum_{n=1}^{\infty} \mu(\mathcal{A}_n) = \infty,$$

*then*

$$\mu(\limsup_{n \rightarrow \infty} \mathcal{A}_n) \geq \limsup_{N \rightarrow \infty} \frac{\left( \sum_{k=1}^N \mu(\mathcal{A}_k) \right)^2}{\sum_{n,m=1}^N \mu(\mathcal{A}_n \cap \mathcal{A}_m)}.$$

Furthermore we need the following generalisation of the Mass Transference Principle, Theorem 3 in [2], but slightly simplified to the current setting.

For a dimension function  $f$  and a ball  $B$  inside  $\mathcal{C}$ , that is a ball in the relative topology, of the form  $B = B(x, r)$ , we can define the transformation of  $B$  by  $f$  as the ball

$$B^f = B(x, f(r)^{1/\gamma}).$$

If the dimension function is just  $r \mapsto r^s$  for some  $s > 0$ , we just write the transformed ball as  $B^s$ . In particular we have that  $B^\gamma = B$ .

**Theorem 4 (The Generalised Mass Transference Principle).** *Let  $\{B_i\}_{i \in \mathbb{N}}$  be a sequence of balls in  $\mathcal{C}$  with  $r(B_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Let  $f$  be a dimension function such that  $x \mapsto x^{-\gamma} f(x)$  is monotonic. Suppose that any ball  $B \subseteq \mathcal{C}$  satisfy*

$$\mathcal{H}^\gamma \left( B \cap \limsup_{i \rightarrow \infty} B_i^f \right) = \mathcal{H}^\gamma(B).$$

Then any ball  $B \subseteq \mathcal{C}$  satisfy

$$\mathcal{H}^f \left( B \cap \limsup_{i \rightarrow \infty} B_i^\gamma \right) = \mathcal{H}^f(B).$$

We will also need the theory of continued fractions over formal Laurent series as first studied by Artin in [1]. Every rational function  $\frac{g}{h}$  can be written uniquely as a finite continued fraction

$$\frac{g}{h} = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}$$

with  $a_0, a_1, \dots, a_n \in \mathbb{F}_3[X]$  and  $\deg(a_1), \dots, \deg(a_n) \geq 1$ . In a similar way every  $x \in \mathbb{F}_3((X^{-1})) \setminus \mathbb{F}_3(X)$  can uniquely be written as an infinite continued fraction

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

with  $a_i \in \mathbb{F}_3[X]$  for all  $i \geq 0$ , and  $\deg(a_i) \geq 1$  for  $i \geq 1$ . We call the polynomials  $a_i$  the partial quotients of  $x$ , and the rational functions

$$\frac{P_j}{Q_j} = [a_0; a_1, \dots, a_j]$$

the convergents to  $x$ .

Furthermore, from the ultrametric property on  $\mathbb{F}_3((X^{-1}))$ , we have that

$$\left| x - \frac{P_j}{Q_j} \right| = \frac{1}{|a_{j+1}| |Q_j|^2}$$

for all the convergents.

We will also need the Folding Lemma, Proposition 2 in [8].

**Lemma 5 (Folding Lemma).** *If  $\frac{g}{h} = [a_0; a_1, \dots, a_n]$  is a rational function, and  $t$  is a polynomial with  $\deg(t) \geq 1$ , then*

$$\frac{g}{h} + \frac{(-1)^n}{th^2} = [a_0; a_1, \dots, a_n, t, -a_n, \dots, -a_1].$$

## 4. Proof of Theorem 2

### Convergent case

Since

$$\sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^\gamma < \infty,$$

and since  $f$  is a dimension function, we have that  $\psi(3^n) \rightarrow 0$  as  $n \rightarrow \infty$ .



Let  $\rho > 0$  be given. Then there exists an integer  $N_\rho$ , such that

$$\psi(3^n) \leq \rho \quad \text{for all } n \geq N_\rho. \quad (2)$$

Furthermore we may choose  $N_\rho$  such that  $N_\rho \rightarrow \infty$  as  $\rho \rightarrow 0$ .

We can now cover  $W_{\mathcal{C}}(\psi)$  by the countable collection of balls

$$W_{\mathcal{C}}(\psi) \subseteq \bigcup_{N \geq N_\rho} A_N = \bigcup_{N \geq N_\rho} \bigcup_{g \in \mathcal{F}(N)} B\left(\frac{g}{X^N}, \psi(3^N)\right),$$

each having radius  $\leq \rho$  by (2). Hence

$$\begin{aligned} \inf_{\mathcal{B} \in \mathcal{B}_\rho} \sum_{B_i \in \mathcal{B}} f(r(B_i)) &\leq \sum_{N \geq N_\rho} \sum_{g \in \mathcal{F}(N)} f(\psi(3^N)) \\ &= \sum_{N \geq N_\rho} f(\psi(3^N)) \times \#\mathcal{F}(N) \\ &\stackrel{(1)}{=} \sum_{N \geq N_\rho} f(\psi(3^N)) \times (3^N)^\gamma \rightarrow 0 \end{aligned}$$

as  $\rho \rightarrow 0$ . So we have that

$$\mathcal{H}^f(W_{\mathcal{C}}(\psi)) = 0$$

in this case.

### Divergent case

To simplify the notation we let  $\mu$  be the Hausdorff  $\gamma$ -measure restricted to  $\mathcal{C}$ , that is

$$\mu(A) = \mathcal{H}^\gamma(A \cap \mathcal{C})$$

for every Borel set  $A$ .

Furthermore we define

$$W_{\mathcal{C}}^*(\psi) = \left\{ h \in \mathcal{C} : \left| h - \frac{g}{X^N} \right| < \psi(3^N), \right. \\ \left. \text{for infinitely many } N \in \mathbb{N}, \text{ where } g \in \mathcal{F}^*(N) \right\},$$

where

$$\mathcal{F}^*(N) = \{f \in \mathbb{F}_3[X] : \text{Coeff}(f) \subseteq \{0, 2\}, \deg f < N \text{ and } f(0) = 2\}.$$

We note that

$$\#\mathcal{F}^*(N) = 2^{N-1}, \quad (3)$$

and that just like before

$$W_{\mathcal{C}}^*(\psi) = \left\{ h \in \mathcal{C} : h \in \bigcup_{g \in \mathcal{F}^*(N)} B\left(\frac{g}{X^N}, \psi(3^N)\right) \text{ for infinitely many } N \in \mathbb{N} \right\},$$

when expressed in terms of balls instead of approximation, and

$$W_{\mathcal{C}}^*(\psi) = \limsup_{N \rightarrow \infty} A_N^* = \{f \in \mathcal{C} : f \in A_N^* \text{ for infinitely many } N \in \mathbb{N}\},$$

where

$$A_N^* = \bigcup_{g \in \mathcal{F}^*(N)} B\left(\frac{g}{X^N}, \psi(3^N)\right).$$

Proving the divergent part of the theorem, but with  $W_{\mathcal{C}}^*(\psi)$  instead of  $W_{\mathcal{C}}(\psi)$ , proves the result since

$$W_{\mathcal{C}}^*(\psi) \subseteq W_{\mathcal{C}}(\psi) \subseteq \mathcal{C},$$

so we do that.

First, we prove the divergent part of the theorem in the special case when the dimension function  $f$  is just the function  $r \mapsto r^\gamma$ , that is the following theorem:

**Theorem 6.**  $\mu(W_{\mathcal{C}}^*(\psi)) = \mu(\mathcal{C}) = 1$  if  $\sum_{n=1}^{\infty} (\psi(3^n) \times 3^n)^\gamma = \infty$ .

**Proof.** The proof is divided into six steps.

i) Without loss of generality we may assume that

$$\psi(3^n) \leq 3^{-n} \text{ for all } n \in \mathbb{N}. \quad (4)$$

If that was not the case, the function  $\Psi$ , defined by  $\Psi(r) = \min\{r^{-1}, \psi(r)\}$ , would satisfy (4). Furthermore if  $\Psi(3^n) = 3^{-n}$  infinitely often, we have that

$$\sum_{n=1}^{\infty} (\Psi(3^n) \times 3^n)^\gamma = \infty.$$

On the other hand if  $\Psi(3^n) = 3^{-n}$  only a finite number of times,

$$\sum_{n=1}^{\infty} (\Psi(3^n) \times 3^n)^\gamma \geq \sum_{n=N}^{\infty} (\Psi(3^n) \times 3^n)^\gamma = \sum_{n=N}^{\infty} (\psi(3^n) \times 3^n)^\gamma = \infty,$$

for  $N$  sufficiently large. Since  $W_{\mathcal{C}}^*(\Psi) \subseteq W_{\mathcal{C}}^*(\psi)$ , we could just prove the theorem with  $\Psi$  instead of  $\psi$ .

ii) Let  $g, h \in \mathcal{F}^*(n)$  be different. Then

$$\left| \frac{g}{X^n} - \frac{h}{X^n} \right| = \left| \frac{g-h}{X^n} \right| \geq 3^{-n} \geq \psi(3^n),$$

and hence

$$B\left(\frac{g}{X^n}, \psi(3^n)\right) \cap B\left(\frac{h}{X^n}, \psi(3^n)\right) = \emptyset$$

due to the ultrametric property. This implies that  $A_n^*$  is a disjoint union

$$A_n^* = \bigsqcup_{g \in \mathcal{F}^*(n)} B\left(\frac{g}{X^n}, \psi(3^n)\right)$$

for every  $n \in \mathbb{N}$ .

iii) For any ball  $B$  with  $r(B) = 3^{-\ell} \leq 1$  and  $B \cap \mathcal{C} \neq \emptyset$ , if  $n > \ell$ , then

$$\#\left\{g \in \mathcal{F}^*(n) : B\left(\frac{g}{X^n}, \psi(3^n)\right) \subseteq B\right\} = 2^{n-\ell-1}. \quad (5)$$

This follows since any polynomial  $g \in \mathcal{F}^*(n)$  has the coefficient  $a_0 = 2$  and coefficients  $a_{n-1}, \dots, a_1$  either 0 or 2. The requirement that the ball  $B\left(\frac{g}{X^n}, \psi(3^n)\right)$  is contained in  $B$  fixes the coefficients  $a_{n-1}, \dots, a_{n-\ell}$ . The remaining  $n - \ell - 1$  coefficients can be either 0 or 2 giving  $2^{n-\ell-1}$  elements in the set.

iv) We can now, under the assumptions of iii), compute

$$\begin{aligned} \mu(B \cap A_n^*) &= \mu\left(\bigsqcup_{g \in \mathcal{F}^*(n)} B \cap B\left(\frac{g}{X^n}, \psi(3^n)\right)\right) \\ &= \sum_{g \in \mathcal{F}^*(n)} \mu\left(B \cap B\left(\frac{g}{X^n}, \psi(3^n)\right)\right) \\ &= \sum_{\substack{g \in \mathcal{F}^*(n) \\ B\left(\frac{g}{X^n}, \psi(3^n)\right) \subseteq B}} \mu\left(B\left(\frac{g}{X^n}, \psi(3^n)\right)\right) \\ &= \sum_{\substack{g \in \mathcal{F}^*(n) \\ B\left(\frac{g}{X^n}, \psi(3^n)\right) \subseteq B}} \psi(3^n)^\gamma \\ &\stackrel{(5)}{=} 2^{n-\ell-1} \psi(3^n)^\gamma \end{aligned}$$

where we have used Proposition 1. Since

$$2^{n-\ell-1} = (3^n)^\gamma \times (3^{-\ell})^\gamma \times 3^{-\gamma}$$

we have that

$$\mu(B \cap A_n^*) = r(B)^\gamma \times (\psi(3^n) \times 3^n)^\gamma \times 3^{-\gamma}. \quad (6)$$

For  $B = \mathbb{I}$  it follows that

$$\mu(A_n^*) = (\psi(3^n) \times 3^n)^\gamma \times 3^{-\gamma} \quad (7)$$

for all  $n \in \mathbb{N}$ , and hence

$$\sum_{n=1}^{\infty} \mu(A_n^*) = \infty. \quad (8)$$

v) We have the following quasi-independence result

**Proposition 7.** *For  $n > m$  we have*

$$\mu(A_m^* \cap A_n^*) \leq \mu(A_m^*) \mu(A_n^*) \quad (9)$$

**Proof.** Let  $\psi(3^m) = 3^{-\ell}$ . If  $n \leq \ell$ , then  $3^{-n} \geq 3^{-\ell} = \psi(3^m)$  and from (4) we have  $3^{-n} \geq \psi(3^n)$ . Let  $g \in \mathcal{F}^*(n), h \in \mathcal{F}^*(m)$ . Since  $g - hX^{n-m}$  evaluated in 0 is 2, we have that  $g - hX^{n-m} \neq 0$ , and hence

$$\left| \frac{g}{X^n} - \frac{h}{X^m} \right| = \left| \frac{g - hX^{n-m}}{X^n} \right| \geq 3^{-n}.$$

From this we get that

$$B\left(\frac{g}{X^n}, \psi(3^n)\right) \cap B\left(\frac{h}{X^m}, \psi(3^m)\right) = \emptyset,$$

and by definition of  $A_n^*$  and  $A_m^*$  we then have

$$A_n^* \cap A_m^* = \emptyset,$$

which implies that

$$\mu(A_n^* \cap A_m^*) = 0$$

and the quasi-independence is trivially satisfied.

If  $n > \ell$  we have

$$\begin{aligned} \mu(A_m^* \cap A_n^*) &= \mu\left(\bigsqcup_{g \in \mathcal{F}^*(m)} \left(B\left(\frac{g}{X^m}, \psi(3^m)\right) \cap A_n^*\right)\right) \\ &= \sum_{g \in \mathcal{F}^*(m)} \mu\left(B\left(\frac{g}{X^m}, \psi(3^m)\right) \cap A_n^*\right) \\ &\stackrel{(6)}{=} \sum_{g \in \mathcal{F}^*(m)} (\psi(3^m))^\gamma \times (\psi(3^n) \times 3^n)^\gamma \times 3^{-\gamma} \\ &\stackrel{(3)}{=} 2^{m-1} \times (\psi(3^m))^\gamma \times (\psi(3^n) \times 3^n)^\gamma \times 3^{-\gamma} \\ &= \left(3^{-\gamma} \times (\psi(3^m) \times 3^m)^\gamma\right) \times \left(3^{-\gamma} \times (\psi(3^n) \times 3^n)^\gamma\right) \\ &\stackrel{(7)}{=} \mu(A_m^*) \times \mu(A_n^*). \end{aligned}$$

This concludes the proof of the quasi-independence. ■

vi) From (8) we can use Lemma 3 to get

$$\mu(W_{\mathcal{C}}^*(\psi)) \geq \limsup_{N \rightarrow \infty} \frac{\left(\sum_{k=1}^N \mu(A_k^*)\right)^2}{\sum_{n,m=1}^N \mu(A_n^* \cap A_m^*)} \geq \limsup_{N \rightarrow \infty} 1 = 1,$$

where the last inequality comes from the quasi-independence. Since we trivially have

$$1 = \mu(\mathcal{C}) \geq \mu(W_{\mathcal{C}}^*(\psi)),$$

the result follows. ■

We will now deduce the diverging part of Theorem 2 from Theorem 6 by a standard application of the Mass Transference Principle.

**Proof.** Without loss of generality we will assume that  $\psi(3^n) \rightarrow 0$  when  $n \rightarrow \infty$ , since else  $W_{\mathcal{C}}^*(\psi) = \mathcal{C}$  and the result is clear. By assumption we have that

$$\sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^{\gamma} = \infty$$

and  $r \mapsto r^{-\gamma} f(r)$  is monotonic. Define  $\theta$  by  $\theta(r) = \lceil f(\psi(r))^{1/\gamma} \rceil_3$ , where  $\lceil \cdot \rceil_3$  is the function that rounds up to the nearest power of 3.

Then

$$\sum_{n=1}^{\infty} (\theta(3^n) \times 3^n)^{\gamma} = \infty,$$

and from Theorem 6 we get that  $\mu(W_{\mathcal{C}}^*(\theta)) = \mu(\mathcal{C}) = 1$ . This in turn implies that

$$\mu(B \cap W_{\mathcal{C}}^*(\theta)) = \mu(B \cap \mathcal{C})$$

for any ball  $B \subseteq \mathcal{C}$ . Now Theorem 4 gives

$$\mathcal{H}^f(B \cap W_{\mathcal{C}}^*(\psi)) = \mathcal{H}^f(B \cap \mathcal{C})$$

for any ball  $B \subseteq \mathcal{C}$ . In particular for  $B = \mathcal{C}$  we get the desired result. ■

## 5. Irrationality exponent

For an element  $\xi \in \mathbb{F}_3((X^{-1}))$  we define the irrationality exponent of  $\xi$  as

$$\tau(\xi) = \sup \left\{ \tau : \left| \xi - \frac{g}{h} \right| < |h|^{-\tau} \text{ for infinitely many } \frac{g}{h} \in \mathbb{F}_3(X) \right\}.$$

From Dirichlet's theorem in the field of formal Laurent series we get that  $\tau(\xi) \geq 2$  for all  $\xi \in \mathbb{F}_3((X^{-1}))$ .

Furthermore for  $\psi : \{3^n : n \in \mathbb{N}\} \rightarrow \mathbb{R}_+$  a non-increasing function we define

$$\mathcal{K}(\psi) = \left\{ \xi \in \mathbb{I} : \left| \xi - \frac{g}{h} \right| < \psi(|h|), \text{ for infinitely many } \frac{g}{h} \in \mathbb{F}_3(X) \right\}.$$

We now have the following theorem.

**Theorem 8.** *Let  $\psi : \{3^n : n \in \mathbb{N}\} \rightarrow \mathbb{R}_+$  be a non-increasing function such that  $x \mapsto x^2 \psi(x)$  is non-increasing and tends to 0 as  $3^n$  tends to infinity. For any  $c \in (0, \frac{1}{3})$  the set*

$$\mathcal{K}(\psi) \setminus \mathcal{K}(c\psi) \cap \mathcal{C}$$

*is uncountable.*

From Theorem 8 we get the following result.

**Corollary 9.** *For any  $\tau \in [2, \infty]$  there exist uncountably many elements in  $\mathcal{C}$  with irrationality exponent  $\tau$ .*

**Proof.** For  $\tau \in (2, \infty)$  we can use Theorem 8 with  $\psi(x) = x^{-\tau}$ . For  $\tau = 2$  we can use the function  $\psi(x) = (x \log x)^{-2}$ . Finally for  $\tau = \infty$  the element

$$\sum_{n=1}^{\infty} 2X^{-n!}$$

has the desired irrationality exponent, since the proof by Liouville for the corresponding real case can be applied. In a similar way we can construct uncountably many with irrationality exponent  $\tau = \infty$ .  $\blacksquare$

**Proof of Theorem 8.** Let  $u_1, v_1 = 1$  and define recursively  $u_{i+1}$  as the integer satisfying

$$1 < 3^{u_{i+1}} 3^{2v_i} \psi(3^{v_i}) \leq 3,$$

and  $v_{i+1}$  by

$$v_{i+1} = u_{i+1} + 2v_i.$$

From the assumption on  $x \mapsto x^2 \psi(x)$  we get that the sequence  $\{u_i\}_{i \in \mathbb{N}}$  is non-decreasing and tends to infinity as  $i \rightarrow \infty$ .

From  $\{u_i\}_{i \in \mathbb{N}}$  we now construct the following sequence of rational functions:

Let

$$\begin{aligned} \xi_{\mathbf{u},1} &= [0; -X^{u_1}] = \frac{-1}{X^{u_1}} = \frac{P_1}{X^{v_1}} \\ \xi_{\mathbf{u},2} &= [0; -X^{u_1}, X^{u_2}, X^{u_1}] = \frac{-1}{X^{u_1}} + \frac{-1}{X^{u_2+2u_1}} = \frac{P_2}{X^{v_2}} \\ \xi_{\mathbf{u},3} &= [0; -X^{u_1}, X^{u_2}, X^{u_1}, X^{u_3}, -X^{u_1}, -X^{u_2}, X^{u_1}] = \frac{-1}{X^{v_1}} + \frac{-1}{X^{v_2}} + \frac{-1}{X^{v_3}} = \frac{P_3}{X^{v_3}} \\ &\vdots \end{aligned}$$

where the element  $\xi_{\mathbf{u},n+1}$  is constructed from  $\xi_{\mathbf{u},n}$  by applying the Folding Lemma. Since we are in characteristic 3, and  $\{v_i\}_{i \in \mathbb{N}}$  is strictly increasing, each of the rational functions is in  $\mathcal{C}$ . They converge to the element  $\xi_{\mathbf{u},\infty} \in \mathcal{C}$ . Furthermore by construction each of the rational functions  $\xi_{\mathbf{u},n}$  is a convergent of  $\xi_{\mathbf{u},\infty}$ .

We have that

$$|\xi_{\mathbf{u},\infty} - \xi_{\mathbf{u},n}| = \frac{1}{3^{v_{n+1}}} < \psi(3^{v_n}),$$

and hence  $\xi_{\mathbf{u},\infty} \in \mathcal{K}(\psi)$ .

For the other part it is sufficient to show that all the convergents  $\frac{P_j}{Q_j}$  satisfy

$$\left| \xi_{\mathbf{u},\infty} - \frac{P_j}{Q_j} \right| > c\psi(|Q_j|)$$

as the convergents are best approximants.

For  $2^{i-1} \leq j < 2^i$  we have that  $|a_{j+1}| \leq 3^{u_{i+1}}$ . Now

$$\left| \xi_{\mathbf{u}, \infty} - \frac{P_j}{Q_j} \right| = \frac{1}{|a_{j+1}| |Q_j|^2}$$

but since

$$|a_{j+1}| |Q_j|^2 \psi(|Q_j|) \leq 3^{u_{i+1}} |Q_j|^2 \psi(|Q_j|) \leq 3^{u_{i+1}} 3^{2v_i} \psi(3^{v_i}) \leq 3$$

we have

$$\left| \xi_{\mathbf{u}, \infty} - \frac{P_j}{Q_j} \right| \geq \frac{\psi(|Q_j|)}{3} > c\psi(|Q_j|)$$

and hence  $\xi_{\mathbf{u}, \infty} \notin \mathcal{K}(c\psi)$ .

In order to get uncountable many elements with the desired property, the sequence  $\{u_i\}_{i \in \mathbb{N}}$  can be modified in the following way. Define  $\{u'_i\}_{i \in \mathbb{N}}$  by  $u'_1 = 1$ ,  $u'_{2n} = u_n$  and  $u'_{2n+1} \in \{1, 2\}$ . By the same proof we get that each of the formal Laurent series  $\xi_{\mathbf{u}', \infty} \in \mathcal{K}(\psi) \setminus \mathcal{K}(c\psi) \cap \mathcal{C}$ , and since there is uncountably many such sequences, each giving different formal Laurent series, we have the desired result.  $\blacksquare$

## 6. Concluding remarks

Let  $p$  be a prime,  $q = p^n$  for some  $n \geq 1$ , and  $\mathbb{F}_q$  the field with  $q$  elements. We can from  $\mathbb{F}_q$  construct the polynomials  $\mathbb{F}_q[X]$  and the rational functions  $\mathbb{F}_q(X)$  with absolute value  $\left| \frac{g}{h} \right| = q^{\deg g - \deg h}$  for the non-zero rational functions, and  $|0| = 0$ . Completing with respect to this absolute value gives the formal Laurent series over  $\mathbb{F}_q$ .

Like before we restrict ourselves to the unit ball, that is elements of the form

$$\sum_{n=1}^{\infty} a_{-n} X^{-n}, \quad a_{-n} \in \mathbb{F}_q.$$

Let  $\mathcal{A} \subseteq \mathbb{F}_q$  with  $2 \leq \#\mathcal{A} < q$ , and construct the missing digit set

$$\mathbf{MDS}(\mathcal{A}) = \left\{ \sum_{n=1}^{\infty} a_{-n} X^{-n} : a_{-n} \in \mathcal{A} \right\}.$$

In the particular the case  $q = 3$  and  $\mathcal{A} = \{0, 2\}$  we just have  $\mathbf{MDS}(\mathcal{A}) = \mathcal{C}$ .

The results of this paper also holds true in the more general setting of missing digit sets, as the proofs can be modified to this situation. We have that the Hausdorff dimension of  $\mathbf{MDS}(\mathcal{A})$  is  $\gamma_{\mathcal{A}} = \frac{\log \#\mathcal{A}}{\log q}$  with  $\mathcal{H}^{\gamma_{\mathcal{A}}}(\mathbf{MDS}(\mathcal{A})) = 1$ . Furthermore for a function  $\psi : \{q^n : n \in \mathbb{N}\} \rightarrow \{q^{-r} : r \in \mathbb{Z}\}$  we define the set  $W_{\mathbf{MDS}(\mathcal{A})}(\psi)$  by

$$\left\{ h \in \mathbf{MDS}(\mathcal{A}) : \left| h - \frac{g}{X^N} \right| < \psi(q^N), \right. \\ \left. \text{for infinitely many } N \in \mathbb{N}, \text{ where } g \in \mathcal{F}_{\mathcal{A}}(N) \right\},$$

where

$$\mathcal{F}_{\mathcal{A}}(N) = \{f \in \mathbb{F}_q[X] : \text{Coeff}(f) \subseteq \mathcal{A}, \deg f < N\},$$

we have the following theorem.

**Theorem 10.** *Let  $f$  be a dimension function such that  $r^{-\gamma_{\mathcal{A}}} f(r)$  is monotonic. Then*

$$\mathcal{H}^f(W_{\mathbf{MDS}(\mathcal{A})}(\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} f(\psi(q^n)) \times (q^n)^{\gamma_{\mathcal{A}}} < \infty \\ \mathcal{H}^f(\mathbf{MDS}(\mathcal{A})) & \text{if } \sum_{n=1}^{\infty} f(\psi(q^n)) \times (q^n)^{\gamma_{\mathcal{A}}} = \infty \end{cases}$$

Finally the results about irrationality exponents also hold true in the more general setting. For an element  $\xi \in \mathbb{F}_q((X^{-1}))$  we define the irrationality exponent in the same way as before as

$$\tau(\xi) = \sup \left\{ \tau : \left| \xi - \frac{g}{h} \right| < |h|^{-\tau} \text{ for infinitely many } \frac{g}{h} \in \mathbb{F}_q(X) \right\}.$$

Furthermore for  $\psi : \{q^n : n \in \mathbb{N}\} \rightarrow \mathbb{R}_+$  a non-increasing function and  $\mathbb{I}$  the unit ball in  $\mathbb{F}_q((X^{-1}))$  we define

$$\mathcal{K}(\psi) = \left\{ \xi \in \mathbb{I} : \left| \xi - \frac{g}{h} \right| < \psi(|h|), \text{ for infinitely many } \frac{g}{h} \in \mathbb{F}_q(X) \right\},$$

and we have

**Theorem 11.** *Assume that  $x \mapsto x^2\psi(x)$  is non-increasing and tends to 0 as  $q^n$  tends to infinity. For any  $c \in (0, \frac{1}{q})$  the set*

$$\mathcal{K}(\psi) \setminus \mathcal{K}(c\psi) \cap \mathbf{MDS}(\mathcal{A})$$

*is uncountable.*

And the corresponding corollary

**Corollary 12.** *For any  $\tau \in [2, \infty]$  there exist uncountably many elements in  $\mathbf{MDS}(\mathcal{A})$  with irrationality exponent  $\tau$ .*

**Acknowledgements.** I would like to thank my Ph.D. advisor Simon Kristensen for directing me toward this problem, and for helpful suggestions along the way.

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**Received:** 31 July 2014; **revised:** 14 August 2014



# SOME REMARKS ON MAHLER'S CLASSIFICATION IN HIGHER DIMENSION

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ABSTRACT. We prove a number of results on the metric and non-metric theory of Diophantine approximation for Yu's multidimensional variant of Mahler's classification of transcendental numbers. Our results arise as applications of well known results in Diophantine approximation to the setting of Yu's classification.

## 1. INTRODUCTION

In [11], Mahler introduced a classification of transcendental numbers in terms of their approximation properties by algebraic numbers. More precisely, he introduced for each  $k \in \mathbb{N}$  and each  $\alpha \in \mathbb{R}$  the Diophantine exponent

$$\omega_k(x) = \sup\{\omega \in \mathbb{R} : |P(x)| \leq H(P)^{-\omega} \text{ for infinitely many irreducible } P \in \mathbb{Z}[X], \deg(P) \leq k\}. \quad (1)$$

Here,  $H(P)$  denotes the naive height of the polynomial  $P$ , i.e. the maximum absolute value among the coefficients of  $P$ .

Mahler defined classes of numbers according to the asymptotic behaviour of these exponents as  $k$  increases. More precisely, let

$$\omega(x) = \limsup_{k \rightarrow \infty} \frac{\omega_k(x)}{k}.$$

The number  $x$  belongs to one of the following four classes.

- $x$  is an  $A$ -number if  $\omega(x) = 0$ , so that  $x$  is algebraic over  $\mathbb{Q}$ .
- $x$  is an  $S$ -number if  $0 < \omega(x) < \infty$ .
- $x$  is a  $T$ -number if  $\omega(x) = \infty$ , but  $\omega_k(x) < \infty$  for all  $k$ .
- $x$  is a  $U$ -number if  $\omega(x) = \infty$  and  $\omega_k(x) = \infty$  for all  $k$  large enough.

All four classes are non-empty, with almost all real numbers being  $S$ -numbers. Every real number belongs to one of the classes, and the classes are invariant under algebraic operations over  $\mathbb{Q}$ .

In analogy with Mahler's classification, Koksma [10] introduced a different classification based on the exponent

$$\omega_k^*(\alpha) = \sup\{\omega^* \in \mathbb{R} : |x - \alpha| \leq H(\alpha)^{-\omega^*} \text{ for infinitely many } \alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}, \deg(\alpha) \leq k\}.$$

In this case,  $H(\alpha)$  denotes the naive height of  $\alpha$ , i.e. the naive height of the minimal integer polynomial of  $\alpha$ . In analogy with Mahler's classification, one defines  $w^*(x)$  and  $A^*$ -,  $S^*$ -,  $T^*$ - and  $U^*$ -numbers.

The reader is referred to the monograph [4] for an excellent overview of the classifications and their properties. A particular property is that the classifications coincide,

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2010 *Mathematics Subject Classification.* 11J82, 11J83.

SK and SHP supported by the Danish Natural Science Research Council.

so that  $A$ -numbers are  $A^*$ -numbers,  $S$ -numbers are  $S^*$ -numbers and so on. The individual exponents however need not coincide.

In [18], Yu introduced a classification similar to Mahler's for  $d$ -tuples of real numbers. In brief, the classification is completely similar, except that the exponents  $\omega_k(x)$  are now defined in terms of integer polynomials in  $d$  variables.

An analogue of Koksma's classification was introduced by Schmidt [16]. However, the relation between the two classifications is not at all clear, and it is conjectured that the two classifications do not agree [16].

It is the purpose of the present note to study the Diophantine approximation problems arising within Yu's classification. We recall the simple connection between the questions arising from Mahler's classification, and the problem of Diophantine approximation with dependent quantities. A classical problem in Diophantine approximation, given  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ , is to find  $\omega$  for which

$$\|\mathbf{q} \cdot \mathbf{x}\| \leq (\max_{1 \leq i \leq d} |q_i|)^{-\omega} \text{ for infinitely many } \mathbf{q} = (q_1, \dots, q_d) \in \mathbb{Z}^d, \quad (2)$$

where as usual  $\|\cdot\|$  denotes the distance to the nearest integer. Comparing (1) and (2), one sees that one can define Mahler's exponents  $\omega_k$  by restricting the classical problem to a consideration of vectors  $\mathbf{x}$  belonging to the Veronese curve

$$\Gamma = \{(x, x^2, \dots, x^k) \in \mathbb{R}^k : x \in \mathbb{R}\}.$$

Similarly, in order to understand the exponents arising in Yu's classification, one should once more consider the corresponding problem of a single linear form, but replace the Veronese curve by the variety obtained by letting the coordinates consist of the distinct non-constant monomials in  $d$  variables of total degree at most  $k$ , say. The resulting Diophantine approximation properties considered in this case would correspond to the multidimensional analogue of  $\omega_k$ , i.e.

$$\omega_k(\mathbf{x}) = \sup\{\omega \in \mathbb{R} : |P(\mathbf{x})| \leq H(P)^{-\omega} \text{ for infinitely many } P \in \mathbb{Z}[X_1, \dots, X_d], \deg(P) \leq k\}.$$

Throughout, let  $n = \binom{k+d}{d} - 1$  be the number of nonconstant monomials in  $d$  variables of total degree at most  $k$ . In addition to the usual, naive height  $H(P)$ , we will also use the following modification  $\tilde{H}(P)$ , which is the maximum absolute value of the coefficients of the non-constant terms of  $P$ . The following is a slight re-statement of [18, Theorem 1].

**Theorem 1.** *For any  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ , there exists  $c(k, \mathbf{x}) > 0$  such that for all  $Q > 1$ , there is a polynomial  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$  and height  $H(P) \leq Q$ , such that*

$$|P(\mathbf{x})| < c(k, \mathbf{x})Q^{-n}.$$

*Replacing the condition  $H(P) \leq Q$  by  $\tilde{H}(P) \leq Q$ , we may always choose  $c(k, \mathbf{x}) = 1$ .*

The proof is essentially an application of the pigeon hole principle, and is completely analogous to the classical proof of Dirichlet's approximation theorem in higher dimension. As a standard corollary, one obtains the first bounds on the exponents  $\omega_k(\mathbf{x})$ .

**Corollary 2.** *For any  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ , there exists a  $c(k, \mathbf{x}) > 0$  such that*

$$|P(\mathbf{x})| < c(k, \mathbf{x})H(P)^{-n},$$

for infinitely many  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$ . In particular,  $\omega_k(\mathbf{x}) \geq n$ .

The corollary tells us what the normalising factor in the multidimensional definition of  $\omega(\mathbf{x})$  should be, namely the number of non-constant monomials in  $d$  variables of total degree at most  $k$ .

Inspired by the above result, we will define the notions of  $k$ -very well approximable,  $k$ -badly approximable,  $k$ -singular and  $k$ -Dirichlet improvable. We will then proceed to prove that the set defined in this manner are all Lebesgue null-sets and so are indeed exceptional. In the case of  $k$ -badly approximable results, we will also show that these form a thick set, i.e. a set whose intersection with any ball has maximal Hausdorff dimension. In fact, many of our results are somewhat stronger than these statements. The properties are all consequences of other work by various authors (see below). Finally, we will deduce a Roth type theorem from Schmidt's Subspace Theorem [15].

It is not the aim of the present paper to prove deep results concerning Yu's classification, but rather to examine the extent to which already existing methods have something interesting to say about the classification.

## 2. RESULTS AND PROOFS

In each of the following subsections we introduce a property of approximation of  $d$ -tuples of real numbers by algebraic numbers, and prove a result about it which extends previous results known in case  $d = 1$ .

**2.1.  $k$ -very well approximable points.** A point  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$  is called  *$k$ -very well approximable* if there exists  $\varepsilon > 0$  and infinitely many polynomials  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$ , such that

$$|P(\mathbf{x})| \leq H(P)^{-(n+\varepsilon)}. \quad (3)$$

In other words,  $\mathbf{x}$  is  $k$ -very well approximable if the exponent  $n$  on the right hand side in Corollary 2 can be increased by a positive amount. We will prove that this property is exceptional in the sense that almost no points with respect to the  $d$ -dimensional Lebesgue measure are  $k$ -very well approximable. In fact, we will show that this property is stable under restriction to subsets supporting a measure with nice properties.

We recall some properties of measures from [7]. A measure  $\mu$  on  $\mathbb{R}^d$  is said to be *Federer* (or doubling) if there is a number  $D > 0$  such that for any  $x \in \text{supp}(\mu)$  and any  $r > 0$ , the ball  $B(x, r)$  centered at  $x$  of radius  $r$  satisfies

$$\mu(B(x, 2r)) < D\mu(B(x, r)). \quad (4)$$

The measure  $\mu$  is said to be *absolutely decaying* if for some pair of numbers  $C, \alpha > 0$

$$\mu(B(x, r) \cap \mathcal{L}^{(\varepsilon)}) \leq C \left( \frac{\varepsilon}{r} \right)^\alpha \mu(B(x, r)), \quad (5)$$

for any ball  $B(x, r)$  with  $x \in \text{supp}(\mu)$  and any affine hyperplane  $\mathcal{L}$ , where  $\mathcal{L}^{(\varepsilon)}$  denotes the  $\varepsilon$ -neighbourhood of  $\mathcal{L}$ . A weaker variant of the property of being absolutely decaying is obtained by replacing  $r$  in the denominator on the right hand side of (5) by the quantity

$$\text{sup}\{c > 0 : \mu(\{z \in B(x, r) : \text{dist}(z, \mathcal{L}) > c\}) > 0\}.$$

In this case, we say that  $\mu$  is *decaying*. If the measure  $\mu$  has the property that

$$\mu(\mathcal{L}) = 0, \quad (6)$$

for any affine hyperplane  $\mathcal{L}$ ,  $\mu$  is called *non-planar*. Note that an absolutely decaying measure is automatically non-planar, but a decaying measure need not be non-planar. Finally,  $\mu$  is called *absolutely friendly* if it is Federer and absolutely decaying, and is called *friendly* if it is Federer, decaying, and non-planar.

**Theorem 3.** *Let  $\mu$  be an absolutely decaying Federer measure on  $\mathbb{R}^d$ . For any  $k \in \mathbb{N}$ , the set of  $k$ -very well approximable points is a null set with respect to  $\mu$ . In particular, Lebesgue almost-no points are  $k$ -very well approximable.*

Our proof relies on results of [7], in which the case  $d = 1$  was proved.

*Proof.* Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}^n$  be defined by  $f(x_1, \dots, x_d) = (x_1, x_2, \dots, x_{d-1}x_d^{k-1}, x_d^k)$ , so that  $f$  maps  $(x_1, \dots, x_d)$  to the  $n$  distinct nonconstant monomials in  $d$  variables of total degree at most  $k$ . Clearly,  $f$  is smooth, and by taking partial derivatives, we easily see that  $\mathbb{R}^n$  may be spanned by the partial derivatives of  $f$  of order up to  $k$ .

From [7, Theorem 2.1(b)] we immediately see that the pushforward  $f_*\mu$  is a friendly measure on  $\mathbb{R}^n$ . We now apply [7, Theorem 1.1], which states that a friendly measure is strongly extremal, i.e. for any  $\delta > 0$ , almost no points in the support of the measure have the property that

$$\prod_{i=1}^n |qy_i - p_i| < q^{-(1+\delta)},$$

for infinitely many  $\mathbf{p} \in \mathbb{Z}^n$ ,  $q \in \mathbb{N}$ . Clearly, this implies the weaker property of extremality, i.e. that for any  $\delta' > 0$ , almost no points in the support of the measure satisfy

$$\max_{1 \leq i \leq n} |qy_i - p_i| < q^{-(\frac{1}{n} + \delta')}, \quad (7)$$

for infinitely many  $\mathbf{p} \in \mathbb{Z}^n$ ,  $q \in \mathbb{N}$ .

To get from the above to a proof of the theorem, we need to re-interpret this in terms of polynomials. We apply Khintchine's transference principle [5, Theorem V.IV] to see that (7) is satisfied infinitely often if and only if

$$|\mathbf{q} \cdot \mathbf{y} - p| < H(\mathbf{q})^{-(n+\delta'')}, \quad (8)$$

for infinitely many  $\mathbf{q} \in \mathbb{Z}^n$ ,  $p \in \mathbb{Z}$ , where  $\delta'' > 0$  can be explicitly bounded in terms of  $n$  and  $\delta'$ . Now,  $\mathbf{y}$  lies in the image of  $f$ , so that the coordinates of  $\mathbf{y}$  consist of all monomials in the variables  $(x_1, \dots, x_d)$ , whence any polynomial in these  $d$  variables may be expressed on the form  $P(\mathbf{x}) = \mathbf{q} \cdot \mathbf{y} - p$ . The coefficients of  $P$  include all the coordinates of  $\mathbf{q}$  and hence  $H(P) \geq H(\mathbf{q})$ , so that if (3) holds for infinitely many  $P$  with  $\varepsilon = \delta''$ , then (8) holds for infinitely many  $\mathbf{q}, p$ . Since the latter condition is satisfied on a set of  $\mu$ -measure zero, it follows that  $\mu$ -almost all points in  $\mathbb{R}^d$  are not  $k$ -very well approximable.

The final statement of the theorem follows immediately, as the Lebesgue measure clearly is Federer and absolutely decaying.  $\square$

Some interesting open questions present themselves at this stage. One can ask whether a vector exists which is  $k$ -very well approximable for all  $k$ . We will call such vectors  *$k$ -very very well approximable*. It is not difficult to prove that the set of  $k$ -very well approximable vectors is a dense  $G_\delta$ -set, so the question of existence can be easily answered in the affirmative. However, determining the Hausdorff dimension of the set of very very well approximable vectors is an open question. When  $d = 1$ , it is known

that the Hausdorff dimension is equal to 1 due to work of Durand [6], but the methods of that paper do not easily extend to larger values of  $d$ .

Taking the notion one step further, one can ask whether vectors  $\mathbf{x} \in \mathbb{R}^d$  exist such that for some fixed  $\varepsilon > 0$ , for any  $k \in \mathbb{N}$ , there are infinitely many integer polynomials  $P$  in  $d$  variables of total degree at most  $k$ , such that

$$|P(\mathbf{x})| \leq H(P)^{-(n+\varepsilon)},$$

where as usual  $n = \binom{k+d}{d} - 1$ , i.e. in addition to  $\mathbf{x}$  being very very well approximable, we require the very very very significant improvement in the rate of approximation to be uniform in  $k$ . We will call such vectors *very very very well approximable*. Determining the Hausdorff dimension of the set of very very very well approximable numbers is an open problem.

**2.2.  $k$ -badly approximable points.** A point  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$  is called  *$k$ -badly approximable* if there exists  $C = C(k, \mathbf{x})$  such that

$$|P(\mathbf{x})| \geq CH(P)^{-n},$$

for all non-zero polynomials  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$ . In other words, a point  $\mathbf{x} \in \mathbb{R}^d$  is  $k$ -badly approximable if the approximation rate in Corollary 2 can be improved by at most a positive constant in the denominator. Let  $B_k$  be the set of  $k$ -badly approximable points. Note that each set  $B_k$  is a null set, which is easily deduced from the work of Beresnevich, Bernik, Kleinbock and Margulis [2]. We will now show:

**Theorem 4.** *Let  $B \subseteq \mathbb{R}^d$  be an open ball and let  $M \in \mathbb{N}$ . Then*

$$\dim B \cap \bigcap_{k=1}^M B_k = d.$$

This statement is deduced from the work of Beresnevich [1], who proved the case  $d = 1$ .

*Proof.* Let  $n_k = \binom{k+d}{d} - 1$  as before, but with the dependence on  $k$  made explicit in notation. Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}^{n_M}$  be given by  $f(x_1, \dots, x_d) = (x_1, x_2, \dots, x_{d-1}x_d^{M-1}, x_d^M)$ , with the monomials ordered in blocks of increasing total degree. Let  $\mathbf{r}_k = (\frac{1}{n_k}, \dots, \frac{1}{n_k}, 0, \dots, 0) \in \mathbb{R}^{n_M}$ , where the non-zero coordinates are the first  $n_k$  coordinates, so that  $\mathbf{r}_k$  is a probability vector.

We define as in [1] the set of  $\mathbf{r}$ -approximable points for a probability vector  $\mathbf{r}$  to be the set

$$\text{Bad}(\mathbf{r}) = \left\{ \mathbf{y} = (y_1, \dots, y_{n_M}) : \text{for some } C(\mathbf{y}) > 0, \right. \\ \left. \max_{1 \leq i \leq n_M} \|qy_i\|^{1/r_i} \geq C(\mathbf{y})q^{-1}, \text{ for any } q \in \mathbb{N} \right\}.$$

Here,  $\|z\|$  denotes the distance to the nearest integer, and we use the convention that  $z^{1/0} = 0$ .

Let  $1 \leq k \leq M$  be fixed and let  $\mathbf{x} \in \mathbb{R}^d$  satisfy that  $f(\mathbf{x}) \in \text{Bad}(\mathbf{r}_k)$ . From [1, Lemma 1]) it follows, that there exists a constant  $C = C(k, \mathbf{x})$ , such that the only integer solution  $(a_0, a_1, \dots, a_{n_k})$  to the system

$$|a_0 + a_1x_1 + a_2x_2 + \dots + a_{n_k-1}x_{d-1}x_d^{k-1} + a_{n_k}x_d^k| < CH^{-1}, \quad \max_i |a_i| < H^{1/n_k}$$

is zero. Here, the choice of  $\mathbf{r}_k$  and the ordering of the monomials in the function  $f$  ensure that the effect of belonging to  $\text{Bad}(\mathbf{r}_k)$  will only give a polynomial expression of total degree at most  $k$ . Indeed, writing out the full equivalence, we would have the first inequality unchanged, with the second being  $\max_i |a_i| < H^{r_{k,i}}$ , where the exponent is the  $i$ 'th coordinate of  $\mathbf{r}_k$ . If this coordinate is 0, we are only considering polynomials where the corresponding  $a_i$  is equal to zero.

Rewriting this in terms of polynomials, for any non-zero  $P \in \mathbb{Z}[X_1, \dots, X_d]$  with  $H(P) < H^{1/n_k}$  and total degree at most  $k$ , we must have

$$|P(\mathbf{x})| \geq CH^{-1} > CH(P)^{-n_k}.$$

It follows that  $\mathbf{x} \in B_k$ , and hence  $f^{-1}(\text{Bad}(\mathbf{r}_k)) \subseteq B_k$ . The result now follows by applying [1, Theorem 1], which implies that the Hausdorff dimension of the intersection of the sets  $f^{-1}(\text{Bad}(\mathbf{r}_k))$  is maximal.  $\square$

Again, an interesting open problem presents itself, namely the question of uniformity of the constant  $C(k, \mathbf{x})$  in  $k$ . Is it possible to construct a vector in  $B_k$  for all  $k$  with the constant being the same for all  $k$ ? And in the affirmative case, what is the Hausdorff dimension of this set? A weaker version of this question would be to ask whether there is some natural dependence of  $C(k, \mathbf{x})$  on  $k$ , i.e. whether one can choose  $C(k, \mathbf{x}) = C(\mathbf{x})^k$  or a similar dependence. We do not at present know the answer to these questions.

**2.3.  $(k, \varepsilon)$ -Dirichlet improvable vectors and  $k$ -singular vectors.** Let  $\varepsilon > 0$ . A point  $\mathbf{x}$  is called  $(k, \varepsilon)$ -Dirichlet improvable if for any  $\varepsilon$  there exists a  $Q_0 \in \mathbb{N}$ , such that for any  $Q \geq Q_0$  there exists a polynomial  $P \in \mathbb{Z}[X_1, \dots, X_d]$  with total degree at most  $k$ ,

$$\tilde{H}(P) \leq \varepsilon Q \quad \text{and} \quad |P(\mathbf{x})| \leq \varepsilon Q^{-n}.$$

Note that we are now using  $\tilde{H}$  as a measure of the complexity of our polynomials.

In view of Theorem 1, if  $\varepsilon \geq 1$ , all points clearly have this property, and so the property is only of interest when  $\varepsilon < 1$ . A vector is called  $k$ -singular if it is  $(k, \varepsilon)$ -Dirichlet improvable for every  $\varepsilon > 0$ .

We will need a few additional definitions before proceeding. For a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$ , a measure  $\mu$  on  $\mathbb{R}^d$  and a subset  $B \subseteq \mathbb{R}^d$  with  $\mu(B) > 0$ , we define

$$\|f\|_{\mu, B} = \sup_{\mathbf{x} \in B \cap \text{supp } \mu} |f(\mathbf{x})|.$$

Let  $C, \alpha > 0$  and let  $U \subseteq \mathbb{R}^d$  be open. We will say that the function  $f$  is  $(C, \alpha)$ -good with respect to  $\mu$  on  $U$  if for any ball  $B \subseteq U$  with centre in  $\text{supp } \mu$  and any  $\varepsilon > 0$ ,

$$\mu\{\mathbf{x} \in B : |f(\mathbf{x})| < \varepsilon\} \leq C \left( \frac{\varepsilon}{\|f\|_{\mu, B}} \right)^\alpha \mu(B).$$

We will say that a measure  $\mu$  on  $\mathbb{R}^d$  is  $k$ -friendly if it is Federer, non-planar and the function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^n$  given by  $f(x_1, \dots, x_d) = (x_1, x_2, \dots, x_{d-1}x_d^{k-1}, x_d^k)$  is  $(C, \alpha)$ -good with respect to  $\mu$  on  $\mathbb{R}^d$  for some  $C, \alpha > 0$ .

We have

**Theorem 5.** *Let  $\mu$  be a  $k$ -friendly measure on  $\mathbb{R}^d$ . Then there is an  $\varepsilon_0 = \varepsilon_0(d, \mu)$  such that the set of  $(k, \varepsilon)$ -Dirichlet improvable points has measure zero for any  $\varepsilon < \varepsilon_0$ . In particular, the set of  $k$ -singular vector has measure zero.*



In the case when  $d = 1$ ,  $k \geq 2$  and  $\mu$  being the Lebesgue measure on  $\mathbb{R}$ , the result is immediate from work of Bugeaud [3, Theorem 7], in which an explicit value of  $\varepsilon$  is given, namely  $\varepsilon = 2^{-3k-3}$ . Our proof is non-effective and relies on [9, Theorem 1.5].

*Proof.* Under the assumption on the measure  $\mu$ , [9, Theorem 1.5] implies the existence of an  $\varepsilon_0 > 0$  such that for all  $\tilde{\varepsilon} < \varepsilon_0$

$$f_*\mu(\text{DI}_{\tilde{\varepsilon}}(\mathcal{T})) = 0 \text{ for any unbounded } \mathcal{T} \subseteq \mathfrak{a}^+.$$

Here,  $f$  is the usual function  $f(x_1, \dots, x_d) = (x_1, x_2, \dots, x_{d-1}x_d^{k-1}, x_d^k)$ ,  $\mathfrak{a}^+$  denotes the set of  $(n+1)$ -tuples of  $(t_0, t_1, \dots, t_n)$  such that  $t_0 = \sum_{i=1}^n t_i$ ,  $t_i > 0$  for each  $i$ , and  $\text{DI}_{\tilde{\varepsilon}}(\mathcal{T})$  denotes the set of vectors  $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$  for which there is a  $T_0$  such that for any  $t \in \mathcal{T}$  with  $\|t\| \geq T_0$ , the system of inequalities

$$\begin{cases} |\mathbf{q} \cdot \mathbf{y} - p| < \tilde{\varepsilon}e^{-t_0} \\ |q_i| < \tilde{\varepsilon}e^{t_i} \end{cases} \quad i = 1, \dots, n,$$

has infinitely many non-trivial integer solutions  $(\mathbf{q}, p) = (q_1, \dots, q_n, p) \in \mathbb{Z}^{n+1} \setminus \{0\}$ .

Our result follows by specialising the above property. Indeed, we apply this to  $\varepsilon = \tilde{\varepsilon}^{n+1} < \varepsilon_0^{n+1}$  and the central ray in  $\mathfrak{a}^+$ ,

$$\mathcal{T} = \left\{ \left( t, \frac{t}{n}, \dots, \frac{t}{n} \right) : t = \log \left( \frac{Q}{\tilde{\varepsilon}} \right) n, Q \geq [\varepsilon_0] + 1, Q \in \mathbb{N} \right\}.$$

The measure  $f_*\mu$  is the pushforward under  $f$  of the  $k$ -friendly measure  $\mu$ . It follows that the set of  $\mathbf{x} \in \mathbb{R}^d$  for which their image under  $f$  is in  $\text{DI}_{\tilde{\varepsilon}}(\mathcal{T})$  is of measure zero for all  $\tilde{\varepsilon} < \varepsilon_0^{n+1}$ . From the definition of  $\text{DI}_{\tilde{\varepsilon}}$  and the choice of  $\mathfrak{a}^+$  and  $\mathcal{T}$ ,  $f(\mathbf{x}) \in \text{DI}_{\tilde{\varepsilon}}$  if and only if there is a  $Q_0 \geq \max\{[\varepsilon_0] + 1, \tilde{\varepsilon}e^{T_0/n}\}$ , such that for  $Q > Q_0$  there exists  $q_0, q_1, \dots, q_n \in \mathbb{Z}$  with  $\max_{1 \leq i \leq n} |q_i| < \tilde{\varepsilon}e^{t/n} = Q$ , such that

$$|(q_1, \dots, q_n) \cdot f(\mathbf{x}) + q_0| < \tilde{\varepsilon}e^{-t} = \varepsilon Q^{-n}.$$

Reinterpreting the right hand side of the above as a polynomial expression in  $\mathbf{x}$ , this recovers the exact definition of  $\mathbf{x}$  being  $(k, \varepsilon^{1/(n+1)})$ -Dirichlet improvable.  $\square$

Note that the proof in fact yields a stronger statement. Namely, by adjusting the choice of  $\mathfrak{a}^+$ , we could have put different weights on the coefficients of the approximating polynomials, thus obtaining the same result, but with a non-standard (weighted) height of the polynomial.

As with the preceding results, some open problems occur. We do not at present know if there exist a vector  $\mathbf{x}$ , for which there are positive numbers  $\varepsilon_k > 0$ , such that  $\mathbf{x} \in \text{DI}(k, \varepsilon_k)$ . If this is the case, determining the Hausdorff dimension of the set of such vectors is another open problem. Additionally, the same questions can be asked if we require  $\varepsilon$  to be independent of  $k$ , i.e. if we ask for the existence of a vector  $\mathbf{x} \in \text{DI}(k, \varepsilon)$  for all  $k$ .

Let us now say that  $\mathbf{x} \in \mathbb{R}^d$  is *k-algebraic* if there exists a nontrivial polynomial  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of degree at most  $k$ , such that  $P(\mathbf{x}) = 0$ . It is clear that if  $\mathbf{x}$  is *k-algebraic*, then it is *k-singular*. In light of Theorem 5, it is natural to inquire whether all *k-singular* points are *k-algebraic*. In this direction we have:

**Theorem 6.** *For  $d \geq 2$ , for any  $k \geq 1$ , there exists a  $k$ -singular point in  $\mathbb{R}^d$  which is not  $k$ -algebraic.*

The proof relies on results of [8]. For  $d = 1$ , much less appears to be known in general. For  $k = 2$ , it follows from a result of Roy [13] combined with a transference result (see [5, Theorem V.XII]) that the answer is affirmative. Roy further indicates

in [14] that he has an unpublished result for  $k = 3$ , which would imply the analogue of Theorem 6 in the case  $d = 1$ ,  $k = 3$ . Already for  $k = 2$ , the construction is rather involved and a general approach would be desirable.

*Proof.* Once more, for a fixed  $k$ , we take  $f$  as in the proof of Theorem 5. In the notation of [8], it is clear that  $\mathbf{x} \in \mathbb{R}^d$  is  $k$ -singular if  $f(\mathbf{x}) \in \text{Sing}(\mathbf{n})$ . Also  $f(\mathbf{x})$  is totally irrational in the notation of [8] if and only if  $\mathbf{x}$  is  $k$ -algebraic.

Since the image of  $f$  is a  $d$ -dimensional nondegenerate analytic submanifold of  $\mathbb{R}^n$ , for  $d \geq 2$  we can apply [8, Theorem 1.2] to conclude that the intersection of  $f(\mathbb{R}^d)$  with  $\text{Sing}(\mathbf{n})$  contains a totally irrational point.  $\square$

Theorem 5 does not give an explicit value of  $\varepsilon_0$ , and indeed the value depends on the measure  $\mu$ . However we can at least push  $\varepsilon_0$  to the limit  $\varepsilon_0 \nearrow 1$  in the case when  $\mu$  is the Lebesgue measure on  $\mathbb{R}^d$  to obtain a result on the  $k$ -singular vectors.

**Theorem 7.** *For any  $d$ , the set of  $\mathbf{x}$  which are  $(k, \varepsilon)$ -Dirichlet improvable for some  $\varepsilon < 1$  and some  $k$ , has Lebesgue measure zero.*

The proof relies on the work of Shah [17].

*Proof.* This is a direct consequence of [17, Corollary 1.4], where the set  $\mathcal{N}$  is chosen to be the diagonal  $\mathcal{N} = \{(N, \dots, N) : N \in \mathbb{N}\}$ .  $\square$

Note that once again, the result of Shah gives a stronger result in the sense that we may take a non-standard height as in the preceding case and retain the conclusion.

**2.4. Algebraic vectors.** Our final result, which is again a corollary of known results, is an analogue of Roth's Theorem [12], which states that algebraic numbers are not very well approximable. Schmidt's Subspace Theorem, see e.g. [15], provides a higher dimensional analogue of this result, and it is this theorem we will apply. We will say that a vector  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$  is *algebraic of total degree  $k$*  if there is a polynomial  $P_{\boldsymbol{\alpha}} \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree  $k$  with  $P_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = 0$  and if no polynomial of lower total degree vanishes at  $\boldsymbol{\alpha}$ .

**Theorem 8.** *Let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$  be an algebraic  $d$ -vector of total degree more than  $k$ . Then for any  $\varepsilon > 0$  there are only finitely many non-zero polynomials  $P \in \mathbb{Z}[X_1, \dots, X_d]$  of total degree at most  $k$  with*

$$|P(\boldsymbol{\alpha})| < H(P)^{-(n+\varepsilon)},$$

where  $n = \binom{k+d}{d} - 1$  as usual.

*Proof.* Since  $\boldsymbol{\alpha}$  is not algebraic of total degree at most  $k$ , by definition it follows that the numbers  $1, \alpha_1, \alpha_2, \dots, \alpha_{d-1}\alpha_d^{k-1}, \alpha_d^k$  are algebraically independent over  $\mathbb{Q}$ . From a corollary to Schmidt's Subspace Theorem, [15, Chapter VI Corollary 1E], it follows that there are only finitely many non-zero integer solutions  $(q_0, \dots, q_n)$  to

$$|q_0 + q_1\alpha_1 + q_2\alpha_2 + \dots + q_{n-1}\alpha_{d-1}\alpha_d^{k-1} + q_n\alpha_d^k| < \left(\max_{1 \leq i \leq n} |q_i|\right)^{-(n+\varepsilon)}.$$

This immediately implies the result.  $\square$

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# Solution of Cassels' Problem on a Diophantine Constant over Function Fields

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July 13, 2016

This paper deals with an analogue of Cassels' problem on inhomogeneous Diophantine approximation in function fields. The inhomogeneous approximation constant of a Laurent series  $\theta \in \mathbb{F}_q((\frac{1}{t}))$  with respect to  $\gamma \in \mathbb{F}_q((\frac{1}{t}))$  is defined to be  $c(\theta, \gamma) = \inf_{0 \neq N \in \mathbb{F}_q[t]} |N| \cdot |\langle N\theta - \gamma \rangle|$ . We show that  $\inf_{\theta \in \mathbb{F}_q((\frac{1}{t}))} \sup_{\gamma \in \mathbb{F}_q((\frac{1}{t}))} c(\theta, \gamma) = q^{-2}$ , and prove that for every  $\theta$  the set  $BA_\theta = \{\gamma \in \mathbb{F}_q((\frac{1}{t})) : c(\theta, \gamma) > 0\}$  has full Hausdorff dimension. Our methods generalize easily to the case of vectors in  $\mathbb{F}_q((\frac{1}{t}))^d$ .

## 1 Introduction

For a real number  $\theta$ , denote by  $\langle \theta \rangle = \theta - \lfloor \theta + \frac{1}{2} \rfloor$  the representative in  $[-\frac{1}{2}, \frac{1}{2})$  of  $\theta$  modulo the integers, and let  $|\theta|$  denote the absolute value of  $\theta$ . In these notation,  $|\langle \theta \rangle|$  is the distance from  $\theta$  to the integers.

A main topic in Diophantine approximation deals with the inhomogeneous approximations of a real number (see [Cas57]). Given two real numbers  $\theta$  and  $\gamma$ , define the *inhomogeneous approximation constant* of  $\theta$  with respect to  $\gamma$  as

$$c(\theta, \gamma) \stackrel{\text{def}}{=} \inf_{n \neq 0} |n| \cdot |\langle n\theta - \gamma \rangle|. \quad (1)$$

Also define the set

$$BA_\theta \stackrel{\text{def}}{=} \{\gamma \in \mathbb{R} : c(\theta, \gamma) > 0\}. \quad (2)$$

It was proved by [BW92] (cf. [Kim07] for a second proof):

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**Theorem 1.1.** *For every  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ , the set  $BA_\theta$  has zero Lebesgue measure.*

On the other hand, the following result concerning  $BA_\theta$  is proved in [Tse09] (see also [BW92, Theorem 2.3]):

**Theorem 1.2.** *For every  $\theta \in \mathbb{R}$ , the set  $BA_\theta$  has Hausdorff dimension 1.*

We mention that subsets of  $\mathbb{R}^d$  with positive Hausdorff dimension are uncountable, and that subsets with positive Lebesgue measure in  $\mathbb{R}^d$  have maximal dimension, i.e.,  $d$  (see [Fal14] for the definition of Hausdorff dimension). In view of that, Theorem 1.1 states that the set  $BA_\theta$  is small, while Theorem 1.2 states that  $BA_\theta$  is large, and in particular, not empty. Therefore, for every  $\theta$  there exists a  $\gamma$  such that  $c(\theta, \gamma) > 0$ . This leads to the definition of the following two constants:

$$c(\theta) \stackrel{\text{def}}{=} \sup_{\gamma} c(\theta, \gamma), \quad (3)$$

and

$$c \stackrel{\text{def}}{=} \inf_{\theta} c(\theta). \quad (4)$$

Khinchine [Khi26] proved that  $c > 0$ . Davenport [Dav51] was the first to give an explicit lower bound on  $c$ . The problem of finding the exact value of it was posed by Cassels [Cas57, p.86]. According to [Mos12], the best estimate of  $c$  was found in [God53]:

**Theorem 1.3.**

$$\frac{3}{32} \leq c \leq \frac{68}{483}.$$

In this work we study the analogues of these constants in the context of function fields.

*Remark 1.4.* Some authors consider a constant which is similar to the one defined in (4):

$$\tilde{c} \stackrel{\text{def}}{=} \inf_{\theta} \sup_{\gamma} \liminf_{n \rightarrow \infty} |n| \cdot |\langle n\theta - \gamma \rangle|. \quad (5)$$

By definition we have  $c \leq \tilde{c}$ , and we are not aware of any result regarding equality. However, the function fields analogues of those constants coincide (cf. Theorem 3.11).

## 1.1 Higher Dimensions

Throughout the paper, we will denote vectors by bold symbols, and their coordinates with superscripts. Assume  $d \geq 1$ . A *weight* in  $\mathbb{R}^d$  is a vector  $\mathbf{r} \in \mathbb{R}^d$  with  $r^1 + \dots + r^d = 1$ ,  $r^s \geq 0$  for any  $1 \leq s \leq d$ . Given a weight  $\mathbf{r}$  and  $\boldsymbol{\theta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ , define the *approximation constant* with weight  $\mathbf{r}$  of  $\boldsymbol{\theta}$  with respect to  $\boldsymbol{\gamma}$  by

$$c_{\mathbf{r}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) \stackrel{\text{def}}{=} \inf_{n \neq 0} \max_{1 \leq s \leq d} \left( |n|^{r^s} |\langle n\theta^s - \gamma^s \rangle| \right),$$

and let

$$BA_{\boldsymbol{\theta}}(\mathbf{r}) \stackrel{\text{def}}{=} \{ \boldsymbol{\gamma} \in \mathbb{R}^d : c_{\mathbf{r}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) > 0 \}.$$

As in the one dimensional case, define

$$c_{\mathbf{r}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \sup_{\gamma} c_{\mathbf{r}}(\boldsymbol{\theta}, \gamma),$$

and

$$c_{\mathbf{r}} \stackrel{\text{def}}{=} \inf_{\boldsymbol{\theta}} c_{\mathbf{r}}(\boldsymbol{\theta}). \quad (6)$$

A higher dimensional version of Theorem 1.1 is proved in [Sha13] by dynamical methods:

**Theorem 1.5.** *For almost every  $\boldsymbol{\theta} \in \mathbb{R}^d$  (described explicitly), the set  $BA_{\boldsymbol{\theta}}(\frac{1}{d}, \dots, \frac{1}{d})$  has measure zero.*

The higher dimensional version of Theorem 1.2 appeared only recently in [BM15], extending a result proved independently by [BHKV10] and [ET11] about the weight  $\mathbf{r} = (\frac{1}{d}, \dots, \frac{1}{d})$ :

**Theorem 1.6.** *For every weight  $\mathbf{r}$  and  $\boldsymbol{\theta} \in \mathbb{R}^d$ , the set  $BA_{\boldsymbol{\theta}}(\mathbf{r})$  has dimension  $d$ .*

As for (6); Cassels [Cas57, Theorem X] showed that  $c_{(\frac{1}{d}, \dots, \frac{1}{d})} > 0$ , and an explicit lower bound was established in [BL05]:

**Theorem 1.7.** *For every  $d \geq 1$*

$$c_{(\frac{1}{d}, \dots, \frac{1}{d})} \geq \frac{1}{72d^2 8^{\frac{1}{d}}}.$$

We know of no results regarding  $c_{\mathbf{r}}$  for a general weight  $\mathbf{r}$ .

## 1.2 The Function Fields Analogue of Diophantine Approximation

The function fields analogue of Diophantine approximation has been studied since the work of Artin [Art24]. It is sometimes referred to as Diophantine approximation in positive characteristic. Every statement in Diophantine approximation has an analogous statement in this context. Let us introduce the dictionary which is used for translating statements (and sometimes, their proofs) from one context to the other. Let  $q$  be a prime power, and let  $\mathbb{F}_q$  be the field with  $q$  elements. Define an absolute value on  $\mathbb{F}_q[t]$  by  $|N| \stackrel{\text{def}}{=} q^{\deg N}$  for  $0 \neq N \in \mathbb{F}_q[t]$ , and  $|0| = 0$ . Extend this definition to the fraction field, the field of rational functions  $\mathbb{F}_q(t)$ , by  $|\frac{M}{N}| \stackrel{\text{def}}{=} q^{\deg M - \deg N}$  where  $M, N \in \mathbb{F}_q[t]$ ,  $N \neq 0$ . The field  $\mathbb{F}_q((\frac{1}{t}))$  of formal Laurent series in  $t$  with finite number of non zero coefficients of positive powers of  $t$ , is the completion of  $\mathbb{F}_q(t)$  with respect to this absolute value. Extending the absolute value continuously to  $\mathbb{F}_q((\frac{1}{t}))$ , gives that the absolute value of a non zero  $\theta \in \mathbb{F}_q((\frac{1}{t}))$ , written as

$$\theta = \sum_{i=-\deg \theta}^{\infty} \theta_i t^{-i},$$

where  $\deg \theta \stackrel{\text{def}}{=} \max \{-i : \theta_i \neq 0\}$ , is

$$|\theta| = q^{\deg \theta}.$$

The set

$$I \stackrel{\text{def}}{=} \left\{ \theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right) : |\theta| < 1 \right\}.$$

is a natural set of representatives for elements in  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$  up to the equivalence relation of having a difference which is a polynomial. We denote  $\langle \theta \rangle = \sum_{i=1}^{\infty} \theta_i t^{-i}$  and consider it to be the representative of  $\theta$  in  $I$ . We call  $\langle \theta \rangle$  and  $\theta - \langle \theta \rangle$  the *fractional part* and the *polynomial part* of  $\theta$ , respectively. These definitions give the dictionary:

$$\begin{array}{lll} \mathbb{F}_q[t] & \rightsquigarrow & \mathbb{Z} \\ \mathbb{F}_q(t) & \rightsquigarrow & \mathbb{Q} \\ \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right) & \rightsquigarrow & \mathbb{R} \\ |\theta| = q^{\deg \theta} & \rightsquigarrow & |\theta| \\ |\langle \theta \rangle| = \text{dist}(\theta, \mathbb{F}_q[t]) & \rightsquigarrow & |\langle \theta \rangle| = \text{dist}(\theta, \mathbb{Z}). \end{array}$$

### 1.3 Previous Works in Inhomogeneous Approximation in Function Fields

The analogue of inhomogeneous approximation in function fields was studied in [Mah41]. Recently, this subject has regained interest, parallel to a significant progress in the real case [Kri11, KN11, CF12, FK15]. Let us use the dictionary described above in order to define the function fields analogues of (1), (2), (3) and (4). For  $\theta, \gamma \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ , denote

$$c(\theta, \gamma) \stackrel{\text{def}}{=} \inf_{0 \neq N} |N| \cdot |\langle N\theta - \gamma \rangle| \quad (7)$$

where  $N$  varies over the non zero polynomials in  $\mathbb{F}_q[t]$ ,

$$BA_\theta \stackrel{\text{def}}{=} \left\{ \gamma \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right) : c(\theta, \gamma) > 0 \right\}, \quad (8)$$

$$c(\theta) \stackrel{\text{def}}{=} \sup_{\gamma} \inf_{0 \neq N} |N| \cdot |\langle N\theta - \gamma \rangle|, \quad (9)$$

and

$$c \stackrel{\text{def}}{=} \inf_{\theta} c(\theta). \quad (10)$$

An analogue of Theorem 1.1 was proved in [KN11]:

**Theorem 1.8.** *For every  $\theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right) \setminus \mathbb{F}_q(t)$ , the set  $BA_\theta$  has zero measure.*

The measure mentioned here is the natural measure on  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ , which we will recall in Section 2. Before we formulate higher dimensional analogues, let us introduce a more general notion of weight which is more natural to this context. A *generalized weight*



is a function  $\mathbf{g} = (g^1, \dots, g^d) : \mathbb{N} \rightarrow \mathbb{N}^d$ , such that for every  $1 \leq s \leq d$  the function  $g^s : \mathbb{N} \rightarrow \mathbb{N}$  is non decreasing, and

$$\sum_{s=1}^d g^s(h) = h,$$

for every  $h \in \mathbb{N}$ . Define the higher dimensional versions of (7), (8), (9), and (10): For any  $\boldsymbol{\theta}, \boldsymbol{\gamma} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ , let

$$c_{\mathbf{g}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) \stackrel{\text{def}}{=} \inf_{0 \neq N} \max_{1 \leq s \leq d} |N|^{\frac{g^s(\deg N)}{\deg N}} \cdot |\langle N\boldsymbol{\theta}^s - \boldsymbol{\gamma}^s \rangle|,$$

where, by convention,  $\frac{g^s(0)}{0} = 1$ ,

$$BA_{\boldsymbol{\theta}}(\mathbf{g}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{\gamma} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d : c_{\mathbf{g}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) > 0 \right\},$$

$$c_{\mathbf{g}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \sup_{\boldsymbol{\gamma}} c_{\mathbf{g}}(\boldsymbol{\theta}, \boldsymbol{\gamma}),$$

and

$$c_{\mathbf{g}} \stackrel{\text{def}}{=} \inf_{\boldsymbol{\theta}} c_{\mathbf{g}}(\boldsymbol{\theta}).$$

While the approach of [Sha13] is likely to give a proof for the function fields analogue of Theorem 1.5, this line will not be pursued in this note. The reader is referred to [Gho07, HP02] to learn more about the dynamical approach towards Diophantine approximation in function fields.

*Remark 1.9.* Any weight  $\mathbf{r}$  in the sense of Section 1.1, induces a generalized weight  $\mathbf{g}_{\mathbf{r}}$ , by letting  $\mathbf{g}_{\mathbf{r}}(0) \stackrel{\text{def}}{=} (0, \dots, 0)$  and  $\mathbf{g}_{\mathbf{r}}(h+1) \stackrel{\text{def}}{=} \mathbf{g}_{\mathbf{r}}(h) + \mathbf{e}_s$ , where  $1 \leq s \leq d$  is any index satisfying

$$r^s \cdot (h+1) - g_{\mathbf{r}}^s(h) = \max_{1 \leq t \leq d} r^t \cdot (h+1) - g_{\mathbf{r}}^t(h) \quad (11)$$

Note that for every  $h \geq 0$ , we have

$$\sum_{1 \leq s \leq d} r^s h = \sum_{1 \leq s \leq d} g_{\mathbf{r}}^s(h) = h. \quad (12)$$

Therefore, there exists  $1 \leq s \leq d$  such that

$$r^s \cdot (h+1) - g_{\mathbf{r}}^s(h) \geq \frac{1}{d}. \quad (13)$$

By induction on  $h$ , using (11) and (13), we conclude that

$$r^s h - g_{\mathbf{r}}^s(h) \geq - \left( 1 - \frac{1}{d} \right),$$

for every  $h \geq 0$  and  $1 \leq s \leq d$ . On the other hand, by (12) and (13) we get

$$r^s h - g_{\mathbf{r}}^s(h) = - \left( \sum_{t \neq s} r^t h - g_{\mathbf{r}}^t(h) \right) \leq (d-1) \left( 1 - \frac{1}{d} \right),$$

The upshot is that for any  $\boldsymbol{\theta}, \boldsymbol{\gamma} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ , the approximation constant  $c_{\mathbf{g}_{\mathbf{r}}}(\boldsymbol{\theta}, \boldsymbol{\gamma})$  differs from

$$c_{\mathbf{r}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) \stackrel{\text{def}}{=} \inf_{0 \neq N} \max_{1 \leq s \leq d} |N|^{r^s} \cdot |\langle N\boldsymbol{\theta}^s - \boldsymbol{\gamma}^s \rangle|.$$

by a multiplicative factor smaller than  $q^d$ . In particular, for every  $\boldsymbol{\theta} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ , the set

$$BA_{\boldsymbol{\theta}}(\mathbf{r}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{\gamma} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d : c_{\mathbf{r}}(\boldsymbol{\theta}, \boldsymbol{\gamma}) > 0 \right\},$$

equals  $BA_{\boldsymbol{\theta}}(\mathbf{g}_{\mathbf{r}})$ .

## 1.4 Main Results

In this paper, we prove the function fields analogue of Theorem 1.6 and determine the value of the function fields analogue of (10). More precisely, we show:

**Theorem 1.10.**  $BA_{\boldsymbol{\theta}}(\mathbf{g}) \neq \emptyset$  for every generalized weight  $\mathbf{g}$  and  $\boldsymbol{\theta} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ . Moreover, if

$$\inf_{h \in \mathbb{N}} \frac{\min \mathbf{g}(h)}{h} > 0 \tag{14}$$

then  $\dim(BA_{\boldsymbol{\theta}}(\mathbf{g})) = d$  for every  $\boldsymbol{\theta} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ .

**Theorem 1.11.** Any generalized weight  $\mathbf{g}$  satisfies  $c_{\mathbf{g}} = q^{-2}$ .

*Remark 1.12.* It should be mentioned that [Arm57, Agg69] deal with a related question concerning products of linear forms. Assume  $F_i(x, y) = a_i x + b_i y$ ,  $i \in \{1, 2\}$ , are two linear forms with coefficients  $a_i, b_i \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ . Using the methods of [Dav51, Cas52], it was proven that:

$$\sup_{\gamma, \delta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)} \inf_{x, y \in \mathbb{F}_q[t]} |F_1(x + \gamma, y + \delta)| \cdot |F_2(x + \gamma, y + \delta)| = |a_1 b_2 - a_2 b_1| q^{-2}. \tag{15}$$

where the upper bound has been already proved in [Mah41, p. 519]. Given any  $\boldsymbol{\theta} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ , take  $F_1 = \theta x + y$  and  $F_2 = x$ , and plug them into (15) to obtain:

$$\sup_{\gamma, \delta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)} \inf_{N \in \mathbb{F}_q[t]} |N + \gamma| \cdot |\langle N\boldsymbol{\theta} + \delta + \gamma\boldsymbol{\theta} \rangle| = q^{-2}. \tag{16}$$

Note that forcing  $\gamma = 0$  and  $N \neq 0$  can a priori make the left hand side of (16) bigger or smaller, so one cannot apply (16) directly in order to estimate  $c(\boldsymbol{\theta})$ .

## 2 Measure and dimension

In order to prove  $BA_\theta$  has the same Hausdorff dimension as  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ , we will construct subsets of it by nested intersection. In this section we recall a general criterion which gives rise to a lower bound on the Hausdorff dimension of such intersections.

### 2.1 Tree-like collections

Let  $X$  be a complete metric space with a metric  $\rho$ , and let  $\mu$  be a Borel measure on  $X$ . Following the terminology of [KW10], a collection  $\mathcal{C}$  of compact subsets of  $X$  is called *tree-like* if there exists a sequence of collections  $\{\mathcal{C}_m\}_{m=0}^\infty$  such that  $\mathcal{C} = \bigcup_{m=0}^\infty \mathcal{C}_m$  which satisfy the following conditions:

1.  $\mathcal{C}_0 = \{C_0\}$ , with  $C_0 \subseteq X$  compact.
2.  $\mu(C) > 0$  for any  $C \in \mathcal{C}$ .
3. For any  $m \in \mathbb{N}$  and  $C, C' \in \mathcal{C}_m$ , either  $C = C'$  or  $\mu(C \cap C') = 0$ .
4. For any  $m \in \mathbb{N}$  and  $C \in \mathcal{C}_{m+1}$ , there exists  $C' \in \mathcal{C}_m$  such that  $C \subseteq C'$ .
5. For any  $m \in \mathbb{N}$  and  $C' \in \mathcal{C}_m$ , there exists  $C \in \mathcal{C}_{m+1}$  such that  $C \subseteq C'$ .

Given a tree-like collection  $\mathcal{C} = \bigcup_{m=0}^\infty \mathcal{C}_m$  we define its limit set to be

$$C_\infty = \bigcap_{m=0}^\infty \bigcup_{C \in \mathcal{C}_m} C.$$

For each  $m \in \mathbb{N}$  define

$$\rho_m = \sup_{C \in \mathcal{C}_m} \rho(C),$$

where  $\rho(C) = \max_{x,y \in C} \rho(x,y)$ , and

$$D_m = \inf_{C' \in \mathcal{C}_m} \frac{\mu \left( \bigcup_{C \in \mathcal{C}_{m+1}, C \subseteq C'} C \right)}{\mu(C')}.$$

A tree like collection is said to be *strongly tree-like* if, in addition:

6.  $\rho_m \xrightarrow{m \rightarrow \infty} 0$ .

The following is a specific case of [KW10, Lemma 2.5]:

**Theorem 2.1.** *Let  $X$  be a complete metric space with a metric  $\rho$ , and  $\mu$  be a Borel measure. Assume that there exist constants  $c, \alpha > 0$  such that*

$$\mu(B(x,r)) \geq cr^\alpha, \tag{17}$$

*for any  $x \in X$  and  $0 < r < 1$ . Then any strongly tree-like collection  $\mathcal{C} = \bigcup_{m=0}^\infty \mathcal{C}_m$  satisfies*

$$\dim C_\infty \geq \alpha - \limsup_{m \rightarrow \infty} \frac{\sum_{k=0}^m \log D_k}{\log \rho_m}.$$

## 2.2 A metric and a measure on $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$

We shall make use of the standard metric and measure on  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ , which will be denoted by  $\rho$  and  $\mu$  respectively. The metric  $\rho$  is defined by  $\rho(\theta, \varphi) = |\theta - \varphi|$ , for all  $\theta, \varphi \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ , where  $|\cdot|$  stands for the absolute value on  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ , as described in Section 1.2. Note that the balls of this metric are of the form

$$B(\theta, q^{-\ell}) = \theta + t^{-\ell} I,$$

for  $\ell \in \mathbb{Z}$  and  $\theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ . The measure  $\mu$  is the Haar measure on  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ , normalized by  $\mu(I) = 1$ . This measure is characterized by assigning a measure  $q^{-\ell}$  to any ball of radius  $q^{-\ell}$ , and by being invariant under addition.

The metric and the measure on  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$  are defined by

$$\rho^d(\boldsymbol{\theta}, \boldsymbol{\varphi}) = \max_{1 \leq s \leq d} \rho(\theta^s, \varphi^s),$$

for all  $\boldsymbol{\theta}, \boldsymbol{\varphi} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$  and

$$\mu^d = \mu \times \dots \times \mu,$$

$d$  many times. Note that for any  $\ell \geq 0$ ,

$$\mu^d(B(\boldsymbol{\theta}, q^{-\ell})) = q^{-d\ell},$$

and that whenever  $q^{-\ell-1} < r \leq q^{-\ell}$ , we have

$$B(\boldsymbol{\theta}, r) = B(\boldsymbol{\theta}, q^{-\ell}).$$

This proves that  $\mu^d$  satisfies (17) with  $c = 1$  and  $\alpha = d$ .

## 2.3 Cantor constructions in $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$

In this section we describe a construction of a tree-like collection in  $\mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ , which we refer to as a Cantor construction. To introduce the construction, we need some additional notation; For any vector of non negative integers  $\boldsymbol{\ell} = (\ell^1, \dots, \ell^d)$ , denote

$$\bar{\ell} = \sum_{s=1}^d \ell^s,$$

and  $\mathbb{F}_q^\ell = \mathbb{F}_q^{\bar{\ell}}$ . Let  $\pi_{\boldsymbol{\ell}} : \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d \rightarrow \mathbb{F}_q^\ell$  be the projection defined by

$$\pi_{\boldsymbol{\ell}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} (\theta_1^1, \dots, \theta_{\ell^1}^1, \dots, \theta_1^d, \dots, \theta_{\ell^d}^d)^t.$$

For convenience, we denote  $\mathbb{F}_q^0 = \{\emptyset\}$  and  $\pi_{(0, \dots, 0)}(\boldsymbol{\theta}) = \emptyset$  for any  $\boldsymbol{\theta} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ . By abuse of notation, let us use  $\pi_{\boldsymbol{\ell}}$  to denote the projection to the first  $\boldsymbol{\ell}$  coordinates from

$\mathbb{F}_q^{\ell'}$  to  $\mathbb{F}_q^\ell$ , whenever  $\ell' \geq \ell$ , where this inequality should be understood coordinatewise. For any  $\mathbf{v} \in \mathbb{F}_q^\ell$  define the cylinder of  $\mathbf{v}$  by

$$[\mathbf{v}] \stackrel{\text{def}}{=} \{\boldsymbol{\theta} \in I^d : \pi_\ell(\boldsymbol{\theta}) = \mathbf{v}\},$$

and denote  $\ell = \ell(\mathbf{v})$ . For any  $\mathbf{h} \in \mathbb{N}^d$ , denote  $q^{-\mathbf{h}} = (q^{-h^1}, \dots, q^{-h^d})$ . Given a collection of cylinders  $\mathcal{C}$ , define

$$q^{-\mathbf{h}}\mathcal{C} \stackrel{\text{def}}{=} \{[\mathbf{u}] : [\pi_\ell(\mathbf{u})] \in \mathcal{C}, \ell = \ell(\mathbf{u}) - \mathbf{h}\}.$$

Assume  $(\ell_m)_{m=0}^\infty$  is a sequence of  $d$  dimensional non negative integer vectors. Let  $(\ell'_m)_{m=0}^\infty$  be any sequence of non negative integers satisfying  $\ell'_m < \overline{\ell_m}$  for all  $m$ . Define a  $((q^{\ell_m})_{m=0}^\infty, (q^{\ell'_m})_{m=0}^\infty)$  Cantor construction as a set  $\{\mathcal{C}_m : m \geq 0\}$  satisfying  $\mathcal{C}_0 = \{I^d\}$ ,

$$\mathcal{C}_{m+1} \subseteq q^{-\ell_m}\mathcal{C}_m,$$

and

$$|q^{-\ell_m}\{C\} \setminus \mathcal{C}_{m+1}| = q^{\ell'_m},$$

for every  $m \geq 0$  and  $C \in \mathcal{C}_m$ . The *limit set* of such a construction is the set  $C_\infty = \bigcap_{m=0}^\infty \bigcup_{C \in \mathcal{C}_m} C$ , which we call a  $((q^{\ell_m})_{m=0}^\infty, (q^{\ell'_m})_{m=0}^\infty)$  Cantor set. If the sequences  $(\ell_m)_{m=0}^\infty, (\ell'_m)_{m=0}^\infty$  are constant, and equal, say, to  $\ell, \ell'$  respectively, then we shall call such a set a  $(q^\ell, q^{\ell'})$  Cantor set.

## 2.4 Measure and dimension of Cantor constructions

First note that for any  $((q^{\ell_m})_{m=0}^\infty, (q^{\ell'_m})_{m=0}^\infty)$  Cantor construction  $\{\mathcal{C}_m : m \geq 0\}$ , for any  $m \geq 0$ , we have

$$\mu\left(\bigcup_{C \in \mathcal{C}_{m+1}} C\right) = \frac{q^{\overline{\ell_m}} - q^{\ell'_m}}{q^{\overline{\ell_m}}} \mu\left(\bigcup_{C \in \mathcal{C}_m} C\right).$$

This follows from the fact that  $\mathcal{C}_{m+1}$  is composed of equal length cylinders which, therefore, have the same measure. This provides an expression for  $\mu(C_\infty)$ , and shows that if  $|\overline{\ell_m} - \ell'_m|$  is bounded then

$$\mu(C_\infty) = 0.$$

We now apply Theorem 2.1 to get a lower bound on the dimension of Cantor sets:

**Theorem 2.2.** Assume  $C_\infty$  is a  $((q^{\ell_m})_{m=0}^\infty, (q^{\ell'_m})_{m=0}^\infty)$  Cantor set. If

$$\min \sum_{k=0}^{m-1} \ell_k \xrightarrow{m \rightarrow \infty} \infty, \tag{18}$$

then

$$\dim(C_\infty) \geq d - \limsup_{m \rightarrow \infty} \frac{m+1}{\min \sum_{k=0}^{m-1} \ell_k} \frac{\log \frac{q}{q-1}}{\log q}.$$

*Proof.* Let  $\{\mathcal{C}_m : m \geq 0\}$  be the Cantor construction corresponding to  $C_\infty$ . So  $\mathcal{C} = \bigcup_{m=0}^\infty \mathcal{C}_m$  is a tree-like collection. Moreover, we have that for every  $m \geq 0$ ,

$$\rho_m = q^{-\min \sum_{k=0}^{m-1} \ell_k},$$

and

$$D_m = \frac{q^{\overline{\ell}_m} - q^{\ell'_m}}{q^{\overline{\ell}_m}} = 1 - q^{\ell'_m - \overline{\ell}_m} \geq \frac{q-1}{q}.$$

(18) implies that  $\mathcal{C}$  is strongly tree-like. By Theorem 2.1, we get

$$\begin{aligned} \dim(C_\infty) &\geq d - \limsup_{m \rightarrow \infty} \frac{\sum_{k=0}^m \log D_k}{\log \rho_m} \\ &\geq d - \limsup_{m \rightarrow \infty} \frac{m+1}{\min \sum_{k=0}^{m-1} \ell_k} \frac{\log \frac{q}{q-1}}{\log q}. \end{aligned}$$

□

### 3 The One Dimensional Case

In this section we state and prove the one dimensional versions of Theorems 1.10 and 1.11. Our method of proof is inspired by [DL63], and utilizes a characterization of the approximations of  $\theta$  by means of solutions to a certain linear system of equations.

#### 3.1 The Corresponding Matrix of an Element in $\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$

Assume  $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$  is a Laurent series, and  $N = n_h t^h + \dots + n_0 \in \mathbb{F}_q[t]$  is a polynomial of degree  $h$ . Then

$$\langle N\theta \rangle = L_1(\theta)t^{-1} + L_2(\theta)t^{-2} + \dots$$

where for any  $i \geq 1$ ,

$$L_i(\theta) = n_0 \theta_i + n_1 \theta_{i+1} + \dots + n_h \theta_{i+h}.$$

For any  $\gamma \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$  and  $\ell \geq 0$ , one has

$$\begin{aligned} |N| \cdot |\langle N\theta - \gamma \rangle| < q^{-(1+\ell)} &\iff |\langle N\theta - \gamma \rangle| < q^{-(h+1+\ell)} \\ &\iff L_i(\theta) = \gamma_i, \quad 1 \leq i \leq h+1+\ell. \end{aligned} \tag{19}$$

In order to write the above linear system of equations in a matrix form, let us define  $\Delta(\theta)$  to be the infinite matrix:

$$\Delta(\theta) = \begin{pmatrix} \theta_1 & \theta_2 & \cdots \\ \theta_2 & \theta_3 & \cdots \\ \vdots & & \ddots \end{pmatrix}.$$

Denote the  $i \times j$  sub-matrix of  $\Delta(\theta)$ :

$$\Delta[i, j] = \begin{pmatrix} \theta_1 & \cdots & \theta_j \\ \vdots & & \vdots \\ \theta_i & \cdots & \theta_{i-1+j} \end{pmatrix}.$$

In these notation, we may rewrite (19) as

$$|N| \cdot |\langle N\theta - \gamma \rangle| < q^{-(1+\ell)} \iff \Delta[h+1+\ell, h+1] \cdot \mathbf{n} = \pi_{h+1+\ell}(\gamma). \quad (20)$$

Here  $\mathbf{n}$  is the coefficients vector of the polynomial  $N$ .

Consider the same matrix equation, where  $\mathbf{n}$  is now a vector of variables. Note that the matrix  $\Delta[h+1+\ell, h+1]$  is a  $(h+1+\ell) \times (h+1)$  matrix. Therefore, for any  $\ell > 0$  and any fixed  $h$ , there exists a  $\gamma \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$  such that equation (20) has no solutions  $\mathbf{n} \in \mathbb{F}_q^{h+1}$ . This means that for any  $N$  of degree  $h$ ,  $|N| \cdot |\langle N\theta - \gamma \rangle| \geq q^{-(1+\ell)}$ . Our intent is to construct elements  $\gamma \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$  for which  $c(\theta, \gamma) \geq q^{-(1+\ell)}$ . This is equivalent to the equality on the right hand side of (20) to have no solutions for all  $h \geq 0$  at once. To this end, we carefully analyze the rank of the non square submatrices  $\Delta[i, j]$ .

*Remark 3.1.* We mention that for  $\theta$ 's such that  $\Delta[m, m]$  is invertible for all  $m > 0$ , our construction is reduced to a slightly easier one. However, it should be noted that the set of  $\theta$  for which this happens is a set of measure zero. Indeed, for  $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$  and  $m > 0$ ,

$$\det(\Delta[m+1, m+1]) = \det(\Delta[m, m]) \theta_{2m+1} + F(\theta_1, \dots, \theta_{2m}),$$

where  $F(\theta_1, \dots, \theta_{2m})$  is an explicit polynomial which only involves  $\theta_1, \dots, \theta_{2m}$  of  $\theta$  (and not  $\theta_{2m+1}$ ). Therefore, if  $\det(\Delta[m, m]) \neq 0$  for all  $m$ , then

$$\det(\Delta[m+1, m+1]) \neq 0 \iff \theta_{2m+1} \neq -\frac{F(\theta_1, \dots, \theta_{2m})}{\det(\Delta[m, m])}.$$

Hence, the set of  $\theta$ 's for which  $\Delta[m, m]$  is invertible for all  $m > 0$  is a  $(q^2, q)$  Cantor set. As discussed in Section 2.4, such sets have measure zero.

## 3.2 Indices Construction

Given any  $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$  and an integer  $\ell > 0$ , we define the sequences of indices  $\mathcal{I}_\ell = \{i_m\}_{m=0}^\infty$ ,  $\mathcal{J}_\ell = \{j_m\}_{m=0}^\infty$  as follows:

1.  $j_0 = 0, i_0 = \ell$ .
2.  $j_{m+1} = \min\{j : \text{rank}(\Delta[i_m, j]) = i_m\}$ .  
If this minimum is not obtained, we let  $j_{m+1} = \infty$ .
3. If  $j_{m+1} = \infty$ , let  $i_{m+1} = i_m$ . Otherwise, define

$$i_{m+1} = \min\{i : \text{rank}(\Delta[i, j_{m+1}]) = i - \ell\}.$$

For convenience, we write  $i_{-1} = 0$ . Note that if  $\det(\Delta[m, m]) \neq 0$  for all  $m > 0$ , then  $i_m = (m+1)\ell$  and  $j_m = m\ell$  for all  $m \geq 0$ . The following lemma summarizes some properties of these indices.

**Lemma 3.2.** *Let  $\mathcal{I}_\ell, \mathcal{J}_\ell$  be as defined above. If  $j_{m+1} < \infty$ , then  $i_{m+1}$  is defined, and the indices satisfy:*

1.  $i_{m+1} \leq j_{m+1} + \ell$ .
2.  $i_{m+1} \geq i_m + \ell$ .
3.  $j_{m+1} \geq j_m + \ell$ .

*Proof.* General facts about rank of matrices imply that

$$\text{rank}(\Delta[i, j]) \leq \min(i, j), \quad (21)$$

and

$$\text{rank}(\Delta[i, j]) \leq \text{rank}(\Delta[i+1, j]), \quad (22)$$

for every  $i, j > 0$ . By the definition of  $j_{m+1}$ , one has that  $i_m - \text{rank}(\Delta[i_m, j_{m+1}]) = 0$ . On the other hand, putting  $i = j_{m+1} + \ell$  and  $j = j_{m+1}$  in (21) gives  $(j_{m+1} + \ell) - \text{rank}(\Delta[j_{m+1} + \ell, j_{m+1}]) \geq \ell$ . By (22), any  $i, j > 0$  satisfy  $(i+1) - \text{rank}(\Delta[i+1, j]) \leq i - \text{rank}(\Delta[i, j]) + 1$ . Therefore, there exists some  $i_m \leq i \leq j_{m+1} + \ell$  for which  $i - \text{rank}(\Delta[i, j_{m+1}]) = \ell$ . We conclude that  $i_{m+1}$  is well defined.

1. By the definition of  $i_{m+1}$  and the above discussion, it satisfies  $i_{m+1} \leq j_{m+1} + \ell$ .
2. Since  $\text{rank}(\Delta[i_m, j_{m+1}]) = i_m$ , we have  $i_m \leq j_{m+1}$ . It follows that  $\text{rank}(\Delta[i, j_{m+1}]) = i$  for any  $i \leq i_m$ , while if  $i > i_m$ , one has  $\text{rank}(\Delta[i, j_{m+1}]) \geq i_m$ . Using  $\text{rank}(\Delta[i_{m+1}, j_{m+1}]) = i_{m+1} - \ell$ , we conclude that  $i_{m+1} \geq i_m + \ell$ .
3. Note that  $\text{rank}(\Delta[i_m, j_m]) = i_m - \ell$ , and that for all  $j \leq j_m$ , one has that  $\text{rank}(\Delta[i_m, j]) \leq i_m - \ell$ . By the definition of  $j_{m+1}$  as the minimal  $j$  for which  $\text{rank}(\Delta[i_m, j]) = i_m$ , it follows that  $j_{m+1} \geq j_m + \ell$ .

□

*Remark 3.3.* If  $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$  is rational, i.e.  $\theta \in \mathbb{F}_q(t)$ , then there exists an  $m$  for which  $j_m = \infty$ . Indeed, since  $\theta$  is rational, its coefficients sequence is eventually periodic, i.e., there exist  $m_0, p \in \mathbb{N}$  such that  $\theta_m = \theta_{m+p}$  for all  $m \geq m_0$ . Therefore, whenever  $j_m \geq m_0 + p$ , we must already have  $j_m = \infty$ . The implication holds in the other direction as well. Assume that  $j_{m+1} = \infty$  for some  $m \in \mathbb{N}$ . Then there exists  $0 \neq \mathbf{b} \in \mathbb{F}_q^{i_m}$  such that  $\mathbf{b}^t \cdot (\Delta[i_m, j]) = 0$  for all  $j > 0$  at once. Since the  $i$ -th row of  $\Delta(\theta)$  consists of the coefficients of  $\langle t^{i-1}\theta \rangle$ , this means that  $\sum_{s=1}^{i_m} b^s t^{s-1} \theta$  is a polynomial. Therefore  $\theta$  is rational (this argument appears in [Har21, p.438]).



### 3.3 Main Proposition

The following proposition is the key ingredient of the proofs of our main results. To prove it, we make use of the indices constructed in Section 3.2. In fact, the construction of the indices serves as a way to bypass the fact that the matrices  $\Delta[m, m]$  are not necessarily invertible.

**Proposition 3.4.** *For any  $\theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ ,  $\ell > 0$  consider the indices sequences  $\mathcal{I}_\ell, \mathcal{J}_\ell$  constructed in Section 3.2. Let  $\Gamma_\ell$  be the set of  $\gamma \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$  such that for any  $m \geq 0$  and  $0 < j < j_{m+1}$ , the equation*

$$\Delta[i_m, j] \cdot \mathbf{n} = \pi_{i_m}(\gamma) \quad (23)$$

*has no solutions. Then*

$$\dim \Gamma_\ell \geq 1 - \frac{1}{\ell} \frac{\log \frac{q}{q-1}}{\log q}.$$

*Proof.* Let  $\mathcal{C}_0 = \{I\}$ . For  $m \geq 0$ , assume that  $\mathcal{C}_m$  is already defined. By definition,  $\text{rank}(\Delta[i_m, j]) \leq i_m - 1$  for every  $j < j_{m+1}$ . Moreover, for every  $j \leq j'$  we have

$$\{\mathbf{b} \in \mathbb{F}_q^{i_m} : \mathbf{b}^t \cdot \Delta[i_m, j'] = \mathbf{0}^t\} \subseteq \{\mathbf{b} \in \mathbb{F}_q^{i_m} : \mathbf{b}^t \cdot \Delta[i_m, j] = \mathbf{0}^t\}.$$

Hence, there exists  $\mathbf{0} \neq \mathbf{b}_m \in \mathbb{F}_q^{i_m}$  such that

$$(\mathbf{b}_m)^t \cdot \Delta[i_m, j] = \mathbf{0}^t,$$

for all  $j < j_{m+1}$ . Define:

$$\mathcal{C}_{m+1} = \bigcup_{C \in \mathcal{C}_m} \{\pi_{i_m}^{-1}(v) : v \in \pi_{i_m}(C), (\mathbf{b}_m)^t \cdot v \neq 0\}.$$

Note that  $\mathcal{C}_{m+1}$  is a set of sets. Finally, define

$$C_\infty = \bigcap_{m=0}^{\infty} \bigcup_{C \in \mathcal{C}_m} C.$$

*Claim 1:*  $C_\infty \subseteq \Gamma_\ell$

Let  $\gamma \in C_\infty$ . For  $m \geq 0$  and  $0 < j < j_{m+1}$  we have that

$$(\mathbf{b}_m)^t \cdot \Delta[i_m, j] = \mathbf{0}^t \text{ and } (\mathbf{b}_m)^t \cdot \pi_{i_m}(\gamma) \neq 0.$$

Therefore, there are no solutions to (23), and hence  $\gamma \in \Gamma_\ell$ .

*Claim 2:*  $\dim(C_\infty) \geq 1 - \frac{1}{\ell} \frac{\log \frac{q}{q-1}}{\log q}$

If  $j_{m+1} \neq \infty$  for all  $m \geq 0$ , then  $C_\infty$  is a  $((q^{i_m - i_{m-1}})_{m=0}^\infty, (q^{i_m - i_{m-1} - 1})_{m=0}^\infty)$  Cantor set. Indeed, for every  $m \geq 0$ , recall that  $\text{rank}(\Delta[i_{m-1}, j_m]) = i_{m-1}$ , and hence, at least one

of the last  $i_m - i_{m-1}$  coefficients of  $\mathbf{b}_m$  is non zero. Therefore, for every  $\gamma \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$  for which  $\pi_{i_{m-1}}^{-1}(\pi_{i_{m-1}}(\gamma)) \in \mathcal{C}_m$ , there are exactly  $q^{i_m - i_{m-1} - 1}$  vectors  $u \in \mathbb{F}_q^{i_m - i_{m-1}}$  for which

$$(\mathbf{b}_m)^t \cdot \begin{pmatrix} \pi_{i_{m-1}}(\gamma) \\ u \end{pmatrix} = 0.$$

Applying Lemma 3.2(2)  $m - 1$  times, yields  $i_{m-1} \geq m\ell$ . Since  $\sum_{k=0}^{m-1} i_k - i_{k-1} = i_{m-1}$ , it follows by Theorem 2.2 that

$$\dim(C_\infty) \geq 1 - \limsup_{m \rightarrow \infty} \frac{m+1}{i_{m-1}} \frac{\log \frac{q}{q-1}}{\log q} \geq 1 - \frac{1}{\ell} \frac{\log \frac{q}{q-1}}{\log q}.$$

If there exists  $m \geq 0$  for which  $j_{m+1} = \infty$ , then  $C_\infty$  is a non empty union of cylinders of length  $i_m$ , and therefore has a positive measure, thus Hausdorff dimension one.  $\square$

### 3.4 The One Dimensional Case - Results

This section is devoted for the statements and proofs of Theorems 1.10 and 1.11 in the one dimensional case.

**Theorem 3.5.** *For every  $\theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ ,  $\dim(BA_\theta) = 1$ .*

*Proof.* Fix any  $\ell > 0$ . Consider the sequences  $\mathcal{I}_\ell, \mathcal{J}_\ell$  of indices from Section 3.2, and the set  $\Gamma_\ell$  from Proposition 3.4. Assume  $\gamma \in \Gamma_\ell$ . By Proposition 3.4, for all  $m \geq 0$  and  $0 < j < j_{m+1}$ , there are no non zero solutions to (23). For any  $h \in \mathbb{N}$ , let  $m$  be such that  $j_m \leq h + 1 < j_{m+1}$ . In particular,

$$\Delta[i_m, h + 1] \cdot \mathbf{n} = \pi_{i_m}(\gamma), \tag{24}$$

has no non zero solutions. Using Lemma 3.2(1), we get  $i_m \leq j_m + \ell \leq h + 1 + \ell$ . Therefore, the equation

$$\Delta[h + 1 + \ell, h + 1] \cdot \mathbf{n} = \pi_{h+1+\ell}(\gamma),$$

has no non zero solutions, as it is obtained from (24) by increasing the number of equations. By (20), we get that

$$|N| \cdot |\langle N\theta - \gamma \rangle| \geq q^{-(1+\ell)}$$

for any  $0 \neq N \in \mathbb{F}_q[t]$ . Therefore,  $\Gamma_\ell \subseteq BA_\theta$ . We apply Proposition 3.4 to bound the dimension of  $BA_\theta$  from below:

$$\dim BA_\theta \geq \dim(\Gamma_\ell) \geq 1 - \frac{1}{\ell} \frac{\log \frac{q}{q-1}}{\log q}.$$

Since the above holds for all  $\ell > 0$ , and since  $1 - \frac{1}{\ell} \frac{\log \frac{q}{q-1}}{\log q} \xrightarrow{\ell \rightarrow \infty} 1$ , we conclude that  $\dim(BA_\theta) = 1$ .  $\square$

**Proposition 3.6.** *For every  $\theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$  one has that  $c(\theta) \geq q^{-2}$ .*

*Proof.* The proof of Theorem 3.5 shows that any  $\gamma \in \Gamma_1$  satisfies  $c(\theta, \gamma) \geq q^{-2}$ . Proposition 3.4 implies  $\dim(\Gamma_1) > 0$ , hence in particular  $\Gamma_1 \neq \emptyset$ . Therefore,  $c(\theta) \geq q^{-2}$ .  $\square$

We now give a property of the elements  $\theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$  for which  $c(\theta) \geq q^{-1}$ .

**Proposition 3.7.** *If  $c(\theta) \geq q^{-1}$  then there exists  $m_0 \in \mathbb{N} \cup \{\infty\}$  such that*

$$\Delta[m, m] \text{ is invertible exactly for } 0 < m < m_0. \quad (25)$$

*Proof.* Assume that there is no  $m_0$  satisfying (25). Therefore, there are  $0 < m_1 < m_2 < \infty$  such that  $\Delta[m_1, m_1]$  is not invertible and  $\Delta[m_2, m_2]$  is invertible. By assumption, there exists  $\gamma$  such that

$$|N| \cdot |\langle N\theta - \gamma \rangle| < q^{-1} \quad (26)$$

has no solutions  $0 \neq N$  in  $\mathbb{F}_q[t]$ . By (20) with  $\ell = 0$ ,

$$\Delta[m, m] \cdot \mathbf{n} = \pi_m(\gamma) \quad (27)$$

has no solutions  $0 \neq \mathbf{n} \in \mathbb{F}_q^m$  for any  $m > 0$  with  $n_m \neq 0$ . Therefore, there are no non zero solutions to (27). In particular,  $\Delta[m_2, m_2] \cdot \mathbf{n} = \pi_{m_2}(\gamma)$  has no non zero solutions  $\mathbf{n} \in \mathbb{F}_q^{m_2}$ . However,  $\Delta[m_2, m_2]$  is invertible, so we must have  $\pi_{m_2}(\gamma) = \mathbf{0}$ . Since  $m_1 < m_2$ , we have  $\pi_{m_1}(\gamma) = \mathbf{0}$ . Now,  $\Delta[m_1, m_1]$  is non-invertible, therefore, the equation  $\Delta[m_1, m_1] \cdot \pi_{m_1}(\mathbf{n}) = \mathbf{0}$  has non zero solutions, contradicting (27) for  $m_1$ .  $\square$

*Remark 3.8.* In the extreme cases  $m_0 = 1$  and  $m_0 = \infty$ , the other implication also holds. To see this, note that if  $\det(\Delta[m, m]) = 0$  for all  $m > 0$  then we must have  $\theta = 0$ . Indeed, for any  $m > 0$ , assume that  $\theta_1 = \dots = \theta_{m-1} = 0$ . Then  $\Delta[m, m]$  have  $\theta_m$  on the anti diagonal, and zeroes above it. Therefore,  $0 = \det(\Delta[m, m]) = (\theta_m)^m$ , so  $\theta_m = 0$ . For  $\theta = 0$ , any  $\gamma \neq 0$  does not have solutions for (27). If  $m_0 = \infty$ , choose  $\gamma = 0$ . Since  $\Delta[m, m]$  is invertible for every  $m$ , the only solution to (27) is  $\mathbf{n} = \mathbf{0}$ .

As a corollary of Propositions 3.6 and 3.7, we get:

**Theorem 3.9.**  $c = q^{-2}$ .

*Proof.* One only needs to make sure that there exists  $\theta$  such that  $c(\theta) = q^{-2}$ . It is enough to find  $\theta$  which does not satisfy the conclusion of Proposition 3.7. Any  $\theta$  with  $\theta_1 = 0$  and  $\theta_2 \neq 0$  works.  $\square$

We complete the discussion on the one dimensional case by showing that replacing the inf by lim inf in the definition of  $c(\theta, \gamma)$  does not change the value of the constant:

**Proposition 3.10.** *Let  $\theta \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)$ . If  $\tilde{c}(\theta) \stackrel{\text{def}}{=} \sup_{\gamma} \liminf \{ |N| |\langle N\theta - \gamma \rangle| : 0 \neq N \in \mathbb{F}_q[t] \} \geq q^{-1}$  then there exists  $m_0 \in \mathbb{N} \cup \{\infty\}$  such that  $\Delta[m, m]$  is either invertible for all  $m \geq m_0$  or non invertible for all  $m \geq m_0$ .*

*Proof.* The proof here is similar to the proof of Theorem 3.7. Assume that there are infinitely many pairs  $0 < m_1 < m_2$  for which  $\Delta[m_1, m_1]$  is non invertible and  $\Delta[m_2, m_2]$  is invertible. This implies that (26) has infinitely many non zero solutions, hence,  $\tilde{c}(\theta) < q^{-1}$ , which contradicts the assumptions of the proposition.  $\square$

**Theorem 3.11.**  $\tilde{c} \stackrel{\text{def}}{=} \inf_{\theta} \tilde{c}(\theta) = q^{-2}$ .

*Proof.* By definition we have  $c \leq \tilde{c}$ , so it is enough to find  $\theta$  for which  $\tilde{c}(\theta) \leq q^{-2}$ . Define  $\theta$  by  $\theta_{m_k} = 1$  for the sequence  $m_k = 2^{k+1} - 2$ ,  $k = 1, 2, \dots$ , and  $\theta_m = 0$  for every other  $m \in \mathbb{N} \setminus \{m_k : k \in \mathbb{N}\}$ . For this  $\theta$  we have that  $\Delta[m_k, m_k]$  is invertible because the anti diagonal is full with ones, and below the anti diagonal there are only zeros. On the other hand,  $\Delta[m_k + 1, m_k + 1]$  is non invertible since the last row and column are zero. Hence, by Proposition 3.10,  $\tilde{c}(\theta) \leq q^{-2}$ .  $\square$

## 4 The General Case

We now turn to prove Theorems 1.10 and 1.11. To this end, we need to further generalize our indices construction. Fix a generalized weight  $\mathbf{g}$ , a vector  $\boldsymbol{\theta} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$  and  $\ell > 0$ , and define the matrices

$$\Delta[i, j] = \begin{pmatrix} \theta_1^1 & \cdots & \theta_j^1 \\ \vdots & & \vdots \\ \theta_{g^1(i)}^1 & \cdots & \theta_{g^1(i)+j}^1 \\ \cdots & \cdots & \cdots \\ \theta_1^d & \cdots & \theta_j^d \\ \vdots & & \vdots \\ \theta_{g^d(i)}^d & \cdots & \theta_{g^d(i)+j}^d \end{pmatrix}. \quad (28)$$

We construct the set of indices  $\mathcal{I}_{\mathbf{g}, \ell}$ ,  $\mathcal{J}_{\mathbf{g}, \ell}$  the same way as in the one dimensional case. This construction has the same properties as summarized in Lemma 3.2, as one can prove by repeating the proof of Lemma 3.2 verbatim. The same argument works since the rank is independent of the order of the rows.

Define the set

$$BA_{\boldsymbol{\theta}}(\mathbf{g}, \ell) = \left\{ \gamma \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d : \inf_{N \neq 0} \max_{1 \leq s \leq d} |N|^{\frac{g^s(\deg N + 1 + \ell)}{\deg N}} |\langle N\theta^s - \gamma^s \rangle| \geq 1 \right\}.$$

For every  $1 \leq s \leq d$ , we have  $g^s(n+1) \leq g^s(n) + 1$  for all  $n$ . It follows that

$$BA_{\boldsymbol{\theta}}(\mathbf{g}, \ell) \subseteq BA_{\boldsymbol{\theta}}(\mathbf{g}). \quad (29)$$

Note that for any polynomial  $N$  of  $\deg N = h$ , and every  $\ell > 0$ , one has:

$$\begin{aligned} \max_{1 \leq s \leq d} |N|^{\frac{g^s(h+1+\ell)}{h}} |\langle N\theta^s - \gamma^s \rangle| < 1 & \iff \\ \Delta[h+1+\ell, h+1] \mathbf{n} = \pi_{\mathbf{g}}(\gamma), & \end{aligned} \quad (30)$$

where  $\mathbf{n}$  is the coefficients vector of the polynomial  $N$ . This is the higher dimensional version of (20). The next proposition is the higher dimensional version of Proposition 3.4, and the idea of the proof is the same. Therefore, we will mainly emphasize the differences in the proof.

**Proposition 4.1.** Assume  $\boldsymbol{\theta} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$ , a generalized weight  $\mathbf{g}$ , and  $\ell > 0$ . Define  $\Gamma_{\boldsymbol{\theta}}(\mathbf{g}, \ell)$  as the set of all  $\boldsymbol{\gamma} \in \mathbb{F}_q \left( \left( \frac{1}{t} \right) \right)^d$  such that for any  $m \in \mathbb{N}$  and  $0 < j < j_{m+1}$ , the equation

$$\Delta[i_m, j] \mathbf{n} = \pi_{\mathbf{g}(i_m)}(\boldsymbol{\gamma}) \quad (31)$$

has no solutions  $\mathbf{n} \in \mathbb{F}_q^j$ . Then  $\Gamma_{\boldsymbol{\theta}}(\mathbf{g}, \ell) \neq \emptyset$ . Moreover, if

$$\min_{m \rightarrow \infty} \mathbf{g}(i_m) \longrightarrow \infty \quad (32)$$

then

$$\dim(C_{\infty}) \geq d - \limsup_{m \rightarrow \infty} \frac{m+1}{\min \mathbf{g}(i_{m-1})} \frac{\log \frac{q}{q-1}}{\log q}. \quad (33)$$

*Proof.* Let  $\mathcal{C}_0 = \{I^d\}$ . In the same way as is in the proof of Proposition 3.4 define for each  $m \geq 1$  the sets  $\mathcal{C}_m$ , vectors  $\mathbf{b}_m \in \mathbb{F}_q^{i_m}$  and the set  $\mathcal{C}_{\infty}$ , but using the matrices (28), and projections  $\pi_{\mathbf{g}(i)}$  instead of  $\pi_i$ .

*Claim 1:*  $C_{\infty} \subseteq \Gamma_{\boldsymbol{\theta}}(\mathbf{g}, \ell)$ .

The argument is the same as in Proposition 3.4, but with the two aforementioned changes.

*Claim 2:* If  $j_{m+1} \neq \infty$  for all  $m$ , then  $C_{\infty}$  is a  $\left( (q^{\mathbf{g}(i_m) - \mathbf{g}(i_{m-1})})_{m=0}^{\infty}, (q^{i_m - i_{m-1} - 1})_{m=0}^{\infty} \right)$  Cantor set.

An analysis like the one in Proposition 3.4 gives, that for each  $m \geq 0$ , and for each  $C \in \mathcal{C}_m$ , there are exactly  $q^{i_m - i_{m-1} - 1}$  vectors  $v \in \pi_{\mathbf{g}(i_m)}(C)$ , for which  $(\mathbf{b}_m)^t \cdot v = 0$ . From the construction of  $\mathcal{C}_{m+1}$  from  $\mathcal{C}_m$ , we get the desired.

*Claim 3:*  $\Gamma_{\boldsymbol{\theta}}(\mathbf{g}, \ell) \neq \emptyset$ .

If  $j_{m+1} \neq \infty$  for all  $m$ , it follows since Cantor sets are non empty, that  $C_{\infty} \neq \emptyset$ , and hence  $\Gamma_{\boldsymbol{\theta}}(\mathbf{g}, \ell) \neq \emptyset$ .

If there exists  $m \geq 0$  for which  $j_{m+1} = \infty$ , then  $C_{\infty}$  is a non empty union of cylinders of length  $\mathbf{g}(i_m)$ . Therefore it is non empty.

*Claim 4:* If

$$\min_{m \rightarrow \infty} \mathbf{g}(i_m) \longrightarrow \infty,$$

then

$$\dim(C_{\infty}) \geq d - \limsup_{m \rightarrow \infty} \frac{m+1}{\mathbf{g}(i_{m-1})} \frac{\log \frac{q}{q-1}}{\log q}.$$

If  $j_{m+1} \neq \infty$  for all  $m$ , then since  $\sum_{k=0}^{m-1} \mathbf{g}(i_k) - \mathbf{g}(i_{k-1}) = \mathbf{g}(i_{m-1})$ , the result follows from Theorem 2.2.

If there exists  $m \geq 0$  for which  $j_{m+1} = \infty$ , then  $C_\infty$  has positive measure, and hence dimension  $d$ .  $\square$

*Proof of Theorem 1.10.* Recall that we want to show that  $\dim(BA_\theta(\mathbf{g})) = d$ . Let  $\ell > 0$  be any integer. By imitating the proof of Theorem 3.5, we get that  $\Gamma_\theta(\mathbf{g}, \ell) \subseteq BA_\theta(\mathbf{g}, \ell)$ . By Proposition 4.1, we get  $\Gamma_\theta(\mathbf{g}, \ell) \neq \emptyset$ , and hence,  $BA_\theta(\mathbf{g}, \ell) \neq \emptyset$ . To conclude the second part of the theorem, we assume that (14) holds. Therefore, there exists  $r > 0$  such that  $\min \mathbf{g}(h) \geq rh$  for all  $h$ . By applying Lemma 3.2(2)  $m$  times we see that for every  $m \geq 0$ ,  $i_m \geq (m+1)\ell$ . By the monotonicity of  $g^s$  for all  $1 \leq s \leq d$ , we thus obtain that

$$\min \mathbf{g}(i_m) \geq \min \mathbf{g}((m+1)\ell) \geq r(m+1)\ell \xrightarrow{m \rightarrow \infty} \infty.$$

Hence, condition (32) is satisfied. As a consequence of Proposition 4.1, the inequality (33) also holds.

Finally,

$$\begin{aligned} \dim BA_\theta(\mathbf{g}, \ell) &\geq \dim(\Gamma_\theta(\mathbf{g}, \ell)) \\ &\geq d - \limsup_{m \rightarrow \infty} \frac{m+1}{\min \mathbf{g}(i_{m-1})} \frac{\log \frac{q}{q-1}}{\log q} \\ &\geq d - \frac{1}{r\ell}. \end{aligned}$$

As  $\ell > 0$  is arbitrary, by (29) we get

$$\dim BA_\theta(\mathbf{g}) = d.$$

$\square$

*Proof of Theorem 1.11.* We want to show that  $c_{\mathbf{g}} = q^{-2}$ . As in the proof of Proposition 3.6, we note that  $\Gamma_\theta(\mathbf{g}, 1)$  is not empty. Therefore,  $c_{\mathbf{g}}(\theta) \geq q^{-2}$  for every  $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)^d$ . To show equality, it is enough to find one  $\theta$  for which  $c_{\mathbf{g}}(\theta) = q^{-2}$ . Let  $1 \leq s_1, s_2 \leq d$  be such that  $g^{s_1}(1) \neq 0$  and  $g^{s_2}(1) \neq g^{s_2}(2)$ . If  $s_1 = s_2$ , choose any  $\theta$  with  $\theta_1^{s_1} = 0$ ,  $\theta_2^{s_1} = 1$  and  $\theta_3^{s_1} = 0$ . Otherwise, choose  $\theta_1^{s_1} = 0$ ,  $\theta_2^{s_1} = 1$ ,  $\theta_1^{s_2} = 1$  and  $\theta_2^{s_2} = 0$ .  $\square$

## Acknowledgements

We would like to thank Lior Bary-Soroker, Simon Kristensen and Barak Weiss for an abundance of remarks and support, and for suggesting that we work together on this project. We are indebt for the anonymous referee for carefully reading this note, helping us to greatly improve it. The time spent by the first and second named authors at ICERM semester program on "Dimension and Dynamics" had a significant role in the publication of this note. Part of this work was funded by ERC starter grant DLGAPS 279893, BSF grant 2010428.

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