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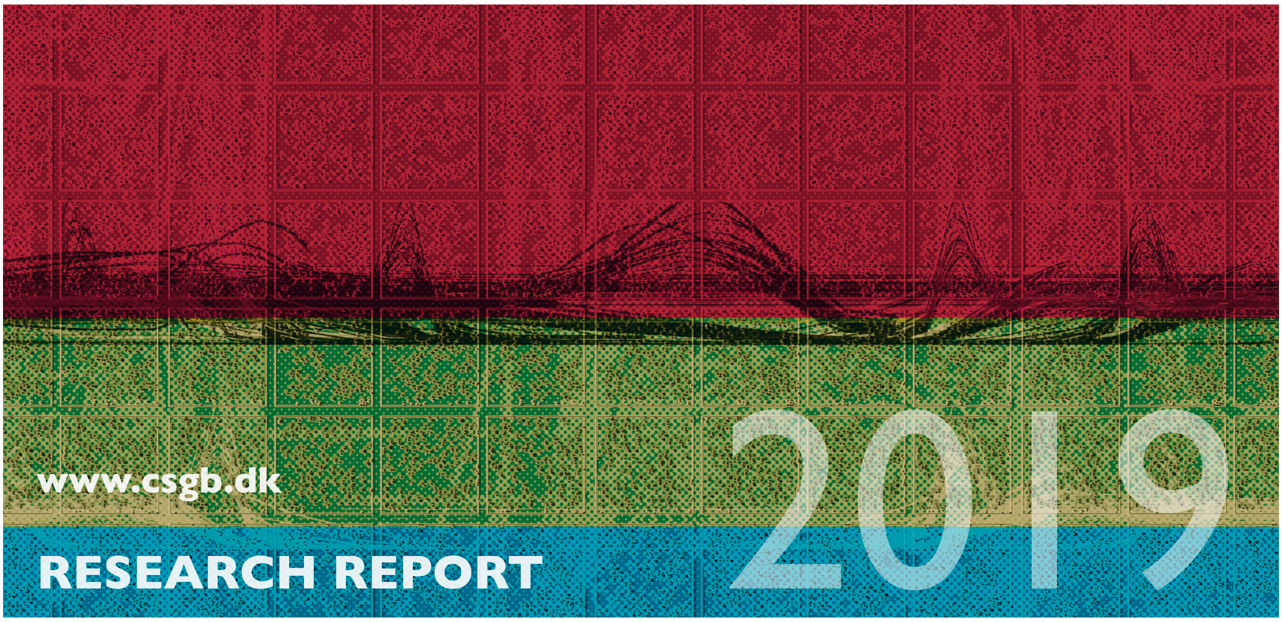

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Abstract

We consider a space-time random field on $\mathbb{R}^d \times \mathbb{R}$ given as an integral of a kernel function with respect to a Lévy basis with a convolution equivalent Lévy measure. The field obeys causality in time and is thereby not continuous along the time-axis. For a large class of such random fields we study the tail behaviour of certain functionals of the field. It turns out that the tail is asymptotically equivalent to the right tail of the underlying Lévy measure. Particular examples are the asymptotic probability that there is a time-point and a rotation of a spatial object with fixed radius, in which the field exceeds the level x, and that there is a time-interval and a rotation of a spatial object with fixed radius, in which the average of the field exceeds the level x.

Keywords: Convolution equivalence; infinite divisibility; Lévy-based modelling; asymptotic equivalence; sample paths for random fields

1 Introduction

In the present paper we investigate the extremal behaviour of a space-time random field $(X_{v,t})_{(v,t)\in B\times[0,T]}$ defined by

$$X_{v,t} = \int_{\mathbb{R}^d\times(-\infty,t]} f(|v-u|, t-s)\, M(\mathrm{d}u, \mathrm{d}s), \tag{1.1}$$

where M is an infinitely divisible, independently scattered random measure on $\mathbb{R}^{d+1}$, $d \in \mathbb{N}$, f is some kernel function, and B and $[0,T]$ are compact index sets. We think of v and t as the position in space and time, respectively. Similarly, the first d coordinates of M refers to the spatial position, while the last coordinate is interpreted as time. The random field defined in (1.1) is a causal model in the sense that $X_{v,t}$ only depends on the noise, accounted for by M, up to time t, i.e. the restriction of M to $\mathbb{R}^d \times (-\infty, t]$. We shall make continuity assumptions on f ensuring that X is continuous in the space-direction. Discontinuities in the time-direction will however be possible, and we therefore have to pay particular attention to the assumptions on f to obtain sample paths that are both continuous in space and càdlàg in time; see Definition 2.2, Assumption 2.3, and Theorem 5.7 below.

Lévy-driven moving average models, where a kernel function is integrated with respect to a Lévy basis, provide a flexible and tractable modelling framework and have been used for a variety of modelling purposes. Recent applications that, similarly to (1.1), include both time and space are modelling of turbulent flows ([2]) and growth processes ([10]). Spatial models without an additional time axis have e.g. been applied to define Cox point processes ([8]) and have served as a modelling framework for brain imaging data ([9, 19]). Lévy-based models for a stochastic process in time have gained recent popularity in finance. A simple example is a Lévy-driven Ornstein-Uhlenbeck process, with $f(t) = e^{-\lambda t}$, that has e.g. been used as a model for option pricing as illustrated in [3]. In [18] estimators for the mean and variogram in Lévy-driven moving average models are proposed, and central limit theorems for these estimators are derived.

In this paper, we will assume that the Lévy measure ρ of the random measure M has a convolution equivalent right tail ([5, 6, 13]). Note that convolution equivalent distributions, as studied in the present paper, have heavier tails than Gaussian distributions and lighter tails than those of regularly varying distributions. We derive that certain functionals of the field will have a right tail that is equivalent to the tail of the underlying Lévy measure. More precisely, we show that for a functional Ψ satisfying Assumptions 3.1, 3.5 and 3.8 given below, there exist known constants C and c such that

$$\mathbb{P}(\Psi(X) > x) \sim C\rho((x/c, \infty)) \qquad \text{as } x \to \infty.$$

Measures with a convolution equivalent tail cover the important cases of an inverse Gaussian and a normal inverse Gaussian (NIG) basis, respectively; see [16] and references therein.

We give three important examples of the functional Ψ to illustrate the generality of the setting. The simplest is $\Psi(X) = \sup_{v,t} X_{v,t}$, where it is concluded that under appropriate assumptions on f it holds that $\sup_{v,t} X_{v,t}$ asymptotically has the same right tail as ρ. A second example, see Example 3.3, involves the spatial excursion set at level x and time t

$$A_{x,t} = \{v \in B \,:\, X_{v,t} > x\}.$$

Under some further regularity conditions we show that the asymptotic probability that there exists a t for which the excursion set at level x contains some rotation of an object with a fixed radius has a tail that is equivalent with the tail of ρ. In the last example, see Example 3.4, we show a similar result for the probability that there is a time-interval and a translation and rotation of some fixed spatial object such that the field in average, over both the time-interval and the resulting spatial object, exceeds the level x.

In [4] sub–additive functionals of similar random fields, also with convolution equivalent tails, are studied. Here it is shown that under appropriate regularity conditions there exists constants $C_1 < C_2$ and a constant c such that

$$C_1\rho((x/c,\infty)) \le \mathbb{P}(\Psi(X) > x) \le C_2\rho((x/c,\infty)).$$

Note that the functional Ψ in the present paper is not necessarily required to be sub–additive. In particular, the functional corresponding to the excursion set framework is indeed not sub–additive.

In [16] and [17] the extremal behaviour of spatial random fields on the form

$$X_v = \int_{\mathbb{R}^d} f(|v-u|)\, M(\mathrm{d}u) \qquad (v \in B)$$

is studied, when M is assumed to have a convolution equivalent Lévy measure. Here assumptions are imposed on the kernel function f to ensure that $v \mapsto X_v$ is continuous. Under some further regularity conditions it is shown in [16] that $\sup_{v\in B} X_v$ has a tail that is asymptotically equivalent with the tail of the underlying Lévy measure. In [17] this result is extended to the asymptotic probability that there exists a rotation of fixed spatial object that is contained in the excursion set $A_x = \{v \in B \,:\, X_v > x\}$.

In [7], results for a moving average process on $\mathbb{R}$, obtained as an integral with respect to a Lévy process with convolution equivalent tail, are derived. Here the process $(X_t)_{t\in[0,T]}$ is given by

$$X_t = \int_{-\infty}^{t} f(t-s)M(\mathrm{d}s)\,,$$

where, again, M has a convolution equivalent Lévy measure. In agreement with the similar but more general result of the present paper for the field defined in (1.1) it is derived in [7] that $\sup_t X_t$ has a tail that asymptotically is equivalent with this.

The paper is organised as follows. In Section 2 we formally define the random field (1.1) and introduce some necessary assumptions for the field to be well defined and to have sample paths that are continuous in space and càdlàg in time. In Section 3 we state and prove the main result for a general functional Ψ and introduce two specific examples of the functional. Some of the proofs in this section will apply many of the same techniques as in [16] and [17] and are therefore deferred to the Appendix (page 25). In Section 4 we state conditions for each of the two examples under which we afterwards show that the main result can be obtained. Section 5 is devoted to showing that under appropriate regularity conditions, the field defined in (1.1) is continuous in space and càdlàg in time.

2 Preliminaries and initial assumptions

We define a Lévy basis to be an infinitely divisible and independently scattered random measure. Then the random measure M on $\mathbb{R}^{d+1}$ is independently scattered, such that for all disjoint Borelsets $(A_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{d+1}$, the random variables $(M(A_n))_{n\in\mathbb{N}}$ are independent and furthermore satisfy $M(\cup_{n\in\mathbb{N}}A_n) = \sum_{n\in\mathbb{N}} M(A_n)$. Furthermore, $M(A)$ is infinitely divisible for all Borelsets $A \subseteq \mathbb{R}^{d+1}$.

Moreover, in this paper we assume M to be a stationary and isotropic Lévy basis on $\mathbb{R}^{d+1}$. With $m(\cdot)$ denoting the Lebesgue measure, and $C(\lambda \dagger Y) = \log \mathbb{E}\mathrm{e}^{i\lambda Y}$ the cumulant function for a random variable Y, this means that the random variable $M(A)$ has Lévy-Khintchine representation

$$
\begin{aligned}
C(\lambda \dagger M(A)) = i\lambda a m(A) - \tfrac{1}{2}\lambda^2 \theta m(A)& \\
+ \int_{A\times\mathbb{R}} \big(\mathrm{e}^{i\lambda x} - 1 - i\lambda x\, \mathbf{1}_{[-1,1]}(x)\big) F(\mathrm{d}u, \mathrm{d}x)&,
\end{aligned}
\tag{2.1}
$$

where $a \in \mathbb{R}$, $\theta \geq 0$ and F is the product measure $m \otimes \rho$ of the Lebesgue measure and a Lévy measure ρ. The notion of the so-called *spot variable* M' will be useful. It is a random variable equivalent in distribution to $M(A)$ when $m(A) = 1$.

We assume that the Lévy basis M has a convolution equivalent Lévy measure ρ with index $\beta > 0$: ρ has an exponential tail with index β, i.e.

$$
\frac{\rho((x-y,\infty))}{\rho((x,\infty))} \to \mathrm{e}^{\beta y} \qquad \text{as } x \to \infty, \tag{2.2}
$$

for all $y \in \mathbb{R}$, and it furthermore satisfies the convolution property

$$
\frac{(\rho_1 * \rho_1)((x,\infty))}{\rho_1((x,\infty))} \to 2\int_{\mathbb{R}} \mathrm{e}^{\beta y} \rho_1(\mathrm{d}y) < \infty \qquad \text{as } x \to \infty, \tag{2.3}
$$

where ρ_1 is the normalized restriction of ρ to $(1,\infty)$ and $*$ denotes convolution.

For later reference, we list the mentioned properties as part of the following assumption.

Assumption 2.1. *The Lévy basis M on $\mathbb{R}^{d+1}$ is stationary and isotropic with a convolution equivalent Lévy measure ρ with index $\beta > 0$, that is, M satisfies* (2.1) *to* (2.3)*. Moreover, ρ satisfies*

$$
\int_{|y|>1} y^k \rho(\mathrm{d}y) < \infty \qquad \textit{for all } k \in \mathbb{N}. \tag{2.4}
$$

Note that the integrability along the right tail is already given from the exponential tail property, and since ρ is a Lévy measure it also satisfies $\int_{[-1,1]} y^2 \rho(\mathrm{d}y) < \infty$. Also, by [21, Theorem 25.3], (2.4) is equivalent to finite moments $\mathbb{E}|M'|^k < \infty$ of the spot variable. In Sections 2 to 4, it is assumed that Assumption 2.1 is satisfied.

We write the tail of ρ as $\rho((x,\infty)) = L(x)\exp(-\beta x)$, so for all $y \in \mathbb{R}$, (2.2) implies that

$$
\frac{L(x-y)}{L(x)} \to 1 \qquad \text{as } x \to \infty. \tag{2.5}
$$

Equation (2.5) implies that the mapping $x \mapsto L(\log(x))$ is slowly varying. A consequence is (see formula (3.6) in [17]) that for all $\gamma > 0$ there exist $x_0 > 0$ and $C_0 > 0$ such that

$$
\frac{L(\alpha x)}{L(x)} \leq C_0 \exp((\alpha - 1)\gamma x) \qquad \text{for all } x \geq x_0,\ \alpha \geq 1. \tag{2.6}
$$

Before proceeding to defining the kernel function f and consequently the field X, we introduce a continuity property called *t-càdlàg* which is essential in this paper.

Definition 2.2 (t-càdlàg)**.** *A field* $(y_{v,t})_{(v,t)}$ *is t-càdlàg if it for all* (v,t) *satisfies*

$$\lim_{(u,s)\to(v,t^-)} y_{u,s} \text{ exists in } \mathbb{R}, \qquad \text{and} \qquad \lim_{(u,s)\to(v,t^+)} y_{u,s} = y_{v,t}. \tag{2.7}$$

In defining the field $X = (X_{v,t})_{(v,t)\in B'\times T'}$ below, we make the following assumption on the integration kernel f. This assumption, together with Assumption 2.1, is sufficient to ensure the existence of the integral (2.9) defining $X_{v,t}$; see [15, Theorem 2.7]. Furthermore, as shown in Theorem 5.7, these assumptions guarantee the existence of a t-càdlàg version of X.

Assumption 2.3. *The kernel* $f : [0,\infty)\times\mathbb{R} \to [0,\infty)$ *is bounded, it satisfies* $f(x,y) = 0$ *for all* $x \in [0,\infty)$ *and* $y < 0$*, it is integrable in the sense that*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} f(|u|,s)\mathrm{d}s\mathrm{d}u < \infty,$$

and it is Lipschitz continuous on $[0,\infty)\times[0,\infty)$*, that is, there is* $C_L \in (0,\infty)$ *such that*

$$|f(x_1,y_1) - f(x_2,y_2)| \le C_L|(x_1,y_1) - (x_2,y_2)| \tag{2.8}$$

for all $(x_1,y_1),(x_2,y_2) \in [0,\infty)\times[0,\infty)$.

Let $B \subseteq \mathbb{R}^d$ be a compact set with strictly positive Lebesgue measure, and consider $[0,T]$ for deterministic $0 < T < \infty$. For $r,\ell \ge 0$ fixed, define the expanded sets $B' = B \oplus C_r(0) = \{x + y : x \in B, |y| \le r\}$ and $T' = [0, T+\ell]$. Here $C_r(u) \subseteq \mathbb{R}^d$ is the d-dimensional closed ball with radius r and center in $u \in \mathbb{R}^d$. Under Assumptions 2.1 and 2.3 we define the random field $X = (X_{v,t})_{(v,t)\in B'\times T'}$ by

$$X_{v,t} = \int_{\mathbb{R}^d\times\mathbb{R}} f(|v-u|, t-s)M(\mathrm{d}u,\mathrm{d}s). \tag{2.9}$$

Note that alternatively we can write

$$X_{v,t} = \int_{\mathbb{R}^d\times(-\infty,t]} f(|v-u|, t-s)M(\mathrm{d}u,\mathrm{d}s)$$

due to the assumptions on f. Thus X has a causal structure in the time direction in the sense that $X_{v,t}$ only depends on M restricted to the subset $\mathbb{R}^d\times(-\infty,t]$.

We are ultimately interested in extremal probabilities of the form

$$\mathbb{P}\big(\Psi((X_{v,t})_{(v,t)\in B'\times T'}) > x\big), \tag{2.10}$$

where $\Psi : \mathbb{R}^{B'\times T'} \to \mathbb{R}$ is a functional satisfying some assumptions that will be given in Section 3. For notational convenience, we usually write $\Psi(y_{v,t})$, when applying Ψ to a field $(y_{v,t})_{(v,t)\in B'\times T'}$, however, when it is necessary to clarify the indices of the field, we write it fully. For the type of functionals Ψ we shall consider, it will be convenient to make some further assumptions on the kernel. The following Assumption 2.4 clearly implies Assumption 2.3 above. In Sections 2 to 4, Assumption 2.4 is assumed satisfied.

Assumption 2.4. *The kernel $f : [0, \infty) \times \mathbb{R} \to [0, \infty)$ satisfies $f(0, 0) = 1$ and $f(x, y) = 0$ for all $x \in [0, \infty)$ and $y < 0$. Moreover,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}} \sup_{v \in B'} \sup_{t \in T'} f(|v-u|, t-s) \mathrm{d}s \mathrm{d}u < \infty,$$

and f is Lipschitz on $[0, \infty) \times [0, \infty)$, i.e. it satisfies (2.8). Lastly, there is a constant C_1 such that

$$f(x, y) \le \frac{C_1}{|(x, y)| + 1} \qquad \text{for all } (x, y) \in [0, \infty) \times \mathbb{R}. \tag{2.11}$$

It turns out that the infinite divisibility of M is inherited to the field X. We shall spend the remainder of this section establishing this property and use it to obtain a useful representation of the field as an independent sum of a compound Poisson term and a term with a lighter tail than exponentials. The procedure is inspired by a similar technique used in [16], [17] and [20]. Here, we present the procedure fully to introduce all relevant notation.

The cumulant function of $X_{v,t}$ takes the form, cf. [15, Theorem 2.7],

$$\begin{aligned} C(\lambda \dagger X_{v,t}) &= i\lambda a \int_{\mathbb{R}^d} \int_{\mathbb{R}} f(|v-u|, t-s) \mathrm{d}s \mathrm{d}u \\ &- \tfrac{1}{2}\theta\lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}} f(|v-u|, t-s)^2 \mathrm{d}s \mathrm{d}u \\ &+ \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{R}} \Big(e^{if(|v-u|, t-s)\lambda z} - 1 - if(|v-u|, t-s)\lambda z 1_{[-1,1]}(z) \Big) \rho(\mathrm{d}z) \mathrm{d}s \mathrm{d}u. \end{aligned}$$

A similar expression can be obtained for any finite linear combination of $X_{v,t}$'s by substitution f with a relevant linear combination of f's. Thus, all finite-dimensional distributions of $(X_{v,t})_{(v,t) \in B' \times T'}$ are infinitely divisible, and consequently any countably indexed field $(X_{v,t})$ is infinitely divisible. Define the countable set $\mathbb{K} = (B' \times T') \cap \mathbb{Q}^{d+1}$, and let $\nu = (m \otimes m \otimes \rho) \circ H^{-1}$ be the measure on $(\mathbb{R}^{\mathbb{K}}, \mathcal{B}(\mathbb{R}^{\mathbb{K}}))$ defined as the image-measure of H on $m \otimes m \otimes \rho$, where $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\mathbb{K}}$ is given by

$$H(u, s, z) = (zf(|v-u|, t-s))_{(v,t) \in \mathbb{K}}.$$

Then direct manipulations show that ν is the Lévy measure of $(X_{v,t})_{(v,t) \in \mathbb{K}}$, and furthermore the Lévy-Khintchine representation is

$$\begin{aligned} &C(\beta \dagger (X_{v,t})_{(v,t) \in \mathbb{K}}) \\ &\quad = i \sum_{(v,t)} \beta_{v,t} a_{v,t} - \tfrac{1}{2}\theta \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Big(\sum_{(v,t)} \beta_{v,t} f(|v-u|, t-s) \Big)^2 \mathrm{d}s \mathrm{d}u \\ &\qquad + \int_{\mathbb{R}^{\mathbb{K}}} \Big(\mathrm{e}^{i \sum_{(v,t)} \beta_{v,t} z_{v,t}} - 1 - i \sum_{(v,t)} \beta_{v,t} z_{v,t} 1_{[-1,1]^{\mathbb{K}}}(z) \Big) \nu(\mathrm{d}z) \end{aligned}$$

for suitable $(a_{v,t})_{(v,t) \in \mathbb{K}} \in \mathbb{R}^{\mathbb{K}}$. Here $\beta \in \mathbb{R}^{\mathbb{K}}$ with $\beta_{v,t} \neq 0$ for at most finitely many $(v, t) \in \mathbb{K}$. From the infinite divisibility, $(X_{v,t})_{(v,t) \in \mathbb{K}}$ can be represented as the independent sum

$$X_{v,t} = X^1_{v,t} + X^2_{v,t}.$$

The field $(X^1_{v,t})_{(v,t)\in\mathbb{K}}$ is a compound Poisson sum

$$X^1_{v,t} = \sum_{n=1}^{N} V^n_{v,t},$$

where N is Poisson distributed with intensity $\nu(A) < \infty$ and

$$A = \bigl\{ z \in \mathbb{R}^{\mathbb{K}} : \sup_{(v,t)\in\mathbb{K}} z_{v,t} > 1 \bigr\}.$$

The finiteness of $\nu(A)$ follows from arguments similar to those of [16, Lemma A.1]. The fields $((V^n_{v,t})_{(v,t)\in\mathbb{K}})_{n\in\mathbb{N}}$ are i.i.d. with common distribution $\nu_1 = \nu_A/\nu(A)$, that is, the normalization of the restriction of ν to A. Also $(X^2_{v,t})_{(v,t)\in\mathbb{K}}$ is infinitely divisible and has Lévy measure ν_{A^c}, the restriction of ν to A^c.

It will be essential that there exist extensions of the fields $(X^1_{v,t})$ and $(X^2_{v,t})$ to $B' \times T'$ with t-càdlàg sample paths. In law, each of the fields $(V^n_{v,t})$ can be represented by $(Zf(|v-U|, t-S))_{(v,t)\in\mathbb{K}}$, where $(U, S, Z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ has distribution F_1, the normalized restriction of F to the set

$$H^{-1}(A) = \bigl\{ (u, s, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : \sup_{(v,t)\in\mathbb{K}} zf(|v-u|, t-s) > 1 \bigr\}.$$

Hence, clearly a t-càdlàg extension $(V_{v,t})_{(v,t)\in B'\times T'}$ exists, and it is represented by $(Zf(|v-U|, t-S))_{(v,t)\in B'\times T'}$. As X^1 is a finite sum of such fields it also has an extension to $B' \times T'$ which is t-càdlàg. As mentioned above and shown in Theorem 5.7, the entire field $(X_{v,t})_{(v,t)\in B'\times T'}$ has a version with t-càdlàg sample paths, and hence also X^2 has en extension with such paths.

3 Functional assumptions and main theorem

In this section we introduce assumptions on Ψ and related functionals, and we derive the main theorem on the asymptotic behaviour of the extremal probability $\mathbb{P}(\Psi(X_{v,t}) > x)$ as $x \to \infty$. As the proofs of some of the results follow the same ideas as in [16] and [17], we refer to the Appendix (page 25) for these.

Throughout this section we shall assume the following.

Assumption 3.1. *The functional* $\Psi : \mathbb{R}^{B'\times T'} \to \mathbb{R}$ *satisfies*

(i) For all deterministic fields $(y_{v,t})_{(v,t)\in B'\times T'}$ *and all* $a \geq 0$ *and* $b \in \mathbb{R}$ *it holds that*

$$\Psi(a\, y_{v,t} + b) = a\, \Psi(y_{v,t}) + b.$$

(ii) Ψ *is increasing, i.e.*

$$\Psi(y_{v,t} + z_{v,t}) \geq \Psi(y_{v,t})$$

whenever the field $(z_{v,t})_{(v,t)\in B'\times T'}$ *satisfies that* $z_{v,t} \geq 0$ *for all* $(v,t) \in B' \times T'$.

(iii) For all $x > 0$, $u \in \mathbb{R}^d$ *and* $s \in \mathbb{R}$, *there is a functional* $\psi_{x,u,s} : \mathbb{R}^{B'\times T'} \to \mathbb{R}$ *such that*

$$\Psi\bigl(a\, f(|v-u|, t-s) + y_{v,t}\bigr) > x \qquad \textit{if and only if} \qquad \psi_{x,u,s}(y_{v,t}) < a$$

for all $a \geq 0$ *and all fields* $(y_{v,t})$.

Proposition 3.2. *The functionals* Ψ *and* $\psi_{x,u,s}$ *satisfy*

(i) $\psi_{x,u,s}$ *is decreasing, that is, for all* $x > 0$, $u \in \mathbb{R}^d$ *and* $s \in \mathbb{R}$ *and all fields* $(y_{v,t})$

$$\psi_{x,u,s}(y_{v,t}) \geq \psi_{x,u,s}(y_{v,t} + z_{v,t})$$

if $z_{v,t} \geq 0$ *for all* $(v,t) \in B' \times T'$.

(ii) For all fields $(y_{v,t})$ *and any constant* $y \in \mathbb{R}$,

$$\psi_{x,u,s}(y_{v,t} + y) = \psi_{x-y,u,s}(y_{v,t}).$$

(iii) For all $x > 0$, $u \in \mathbb{R}^d$ *and* $s \in \mathbb{R}$ *and all fields* $(y_{v,t})$,

$$\psi_{x,u,s}(y_{v,t}) \geq \psi_{x,u,s}(y^*) = \frac{x - y^*}{\Psi((f(|v-u|, t-s))_{(v,t)})},$$

where $y^* = \sup_{(v,t) \in B' \times T'} y_{v,t}$.

Proof. Statement (i) is seen as follows: Let x, u, s be fixed, and assume for contradiction the existence of $\epsilon > 0$ such that $\psi_{x,u,s}(y_{v,t}) + \epsilon = \psi_{x,u,s}(y_{v,t} + z_{v,t})$. Now choose a such that $a - \epsilon \leq \psi_{x,u,s}(y_{v,t}) < a$, and therefore $\psi_{x,u,s}(y_{v,t} + z_{v,t}) \geq a$. However, appealing to Assumption 3.1(ii) and Assumption 3.1(iii) we also conclude that

$$x < \Psi(af(|v-u|, t-s) + y_{v,t}) \leq \Psi(af(|v-u|, t-s) + y_{v,t} + z_{v,t}),$$

so also $\psi_{x,u,s}(y_{v,t} + z_{v,t}) < a$; a contradiction.

Part (ii) and (iii) are seen using Assumption 3.1(i) and Assumption 3.1(iii). □

Before giving two examples of functionals easily seen to satisfy Assumption 3.1, we introduce some notation. Let $D \subseteq C_r(0) \subseteq \mathbb{R}^d$ be a fixed spatial object, and for all rotations $R \in \mathrm{SO}(d)$ and translations $v \in \mathbb{R}^d$, define $D^R(v) = RD + v$. Similarly, let $D(v) = D + v$. Furthermore, let $I(t) = [t, t + \ell]$ for all $t \geq 0$. In Example 3.3 below we assume that the set D in fact has radius $r/2 \geq 0$, by which we mean there is $\alpha \in \mathbb{S}^{d-1}$ such that $\{-\alpha r/2, \alpha r/2\} \subseteq D \subseteq C_{r/2}(0)$.

Example 3.3. Suppose we are interested in the probability that there exist a time-point t, a translation v_0 and a rotation R of a given set D such that the field exceeds the level x on the entire set $\{t\} \times D^R(v_0)$. More formally, we assume that $D \subseteq C_{r/2}(0) \subseteq \mathbb{R}^d$ has radius $r/2$ and study the probability

$$\mathbb{P}\big(\text{there exist } t \in [0,T], v_0 \in B, R \in \mathrm{SO}(d) \,:\, X_{v,t} > x \text{ for all } v \in D^R(v_0)\big).$$

To put this within the more general framework introduced in (2.10), we define Ψ by

$$\Psi(y_{v,t}) = \sup_{t \in [0,T]} \sup_{v_0 \in B} \sup_{R \in \mathrm{SO}(d)} \inf_{v \in D^R(v_0)} y_{v,t}.$$

Consequently (obtained by straightforward manipulations),

$$\psi_{x,u,s}(y_{v,t}) = \inf_{t \in [0,T]} \inf_{v_0 \in B} \inf_{R \in \mathrm{SO}(d)} \sup_{v \in D^R(v_0)} \frac{x - y_{v,t}}{f(|v-u|, t-s)}.$$

Example 3.4. Suppose we are interested in the probability that there is a time-interval and a location and rotation of the fixed spatial object D, in which the average of the field exceeds the level x. For this, let $D \subseteq C_r(0) \subseteq \mathbb{R}^d$ be given and consider the probability

$$\mathbb{P}\big(\text{there exist } t_0 \in [0,T], v_0 \in B, R \in \mathrm{SO}(d) \,:\, \tfrac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} X_{v,t}\mathrm{d}t\mathrm{d}v > x\big),$$

where $K = \int_D \int_0^\ell 1\, \mathrm{d}t\mathrm{d}v$. The set D can both be of full dimension in $\mathbb{R}^d$ and a subset of some lower dimensional subspace. In either case, $\mathrm{d}v$ refers to the relevant version of the Lebesgue measure. The special cases of

$$\mathbb{P}\big(\text{there exist } t \in [0,T], v_0 \in B, R \in \mathrm{SO}(d) \,:\, \tfrac{1}{K}\int_{D^R(v_0)} X_{v,t}\mathrm{d}v > x\big),$$

with a time-point instead of an interval (and K defined appropriately), and

$$\mathbb{P}\big(\text{there exist } t_0 \in [0,T], v \in B \,:\, \tfrac{1}{K}\int_{I(t_0)} X_{v,t}\mathrm{d}t > x\big),$$

with a single spatial point, will be covered by the general formulation of the example, simply be defining $\int_{I(t_0)} X_{v,t}\mathrm{d}t = X_{t_0,v}$ when $\ell = 0$, and $\int_{D^R(v_0)} X_{v,t}\mathrm{d}v = X_{t,v_0}$ when $D = \{0\}$ and hence $D^R(v_0) = \{v_0\}$. In the same spirit, the special case of

$$\mathbb{P}\big(\text{there exist } t \in [0,T], v \in B \,:\, X_{v,t} > x\big)$$

corresponds to letting $\ell = 0$ and $D = \{0\}$. To put this example in the framework of functionals, we define

$$\Psi(y_{v,t}) = \sup_{t_0\in[0,T]} \sup_{v_0\in B} \sup_{R\in\mathrm{SO}(d)} \tfrac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} y_{v,t}\,\mathrm{d}t\mathrm{d}v,$$

leading to

$$\psi_{x,u,s}(y_{v,t}) = \inf_{t_0\in[0,T]} \inf_{v_0\in B} \inf_{R\in\mathrm{SO}(d)} \frac{x - \frac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{\frac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} f(|v-u|, t-s)\,\mathrm{d}t\mathrm{d}v}.$$

For the further arguments to hold it will be important that $\psi_{x,u,s}$ converges in a particular way as $x \to \infty$. The following assumption is satisfied under further case specific assumptions on the kernel f in each of the Examples 3.3 and 3.4 as illustrated in Section 4.

Assumption 3.5. *With the functionals Ψ and $\psi_{x,u,s}$ as in Assumption 3.1, there exists c such that*

$$c = \Psi((f(|v-u|, t-s))_{(v,t)}) \tag{3.1}$$

for all $(u,s) \in B \times [0,T]$, and $\Psi((f(|v-u|, t-s))_{(v,t)}) < c$ for all $(u,s) \notin B \times [0,T]$. Furthermore, for all $(u,s) \in B \times [0,T]$ there is a functional $\lambda_{u,s} : \mathbb{R}^{B'\times T'} \to \mathbb{R}$, such that

$$\psi_{x,u,s}(y_{v,t}) - \frac{x}{c} + \lambda_{u,s}(y_{v,t}) \to 0 \tag{3.2}$$

as $x \to \infty$, holds for all t-càdlàg fields $(y_{v,t})_{(v,t)\in B'\times T'}$.

The following proposition is easily seen from Assumption 3.1 and Proposition 3.2.

Proposition 3.6. *With c and $\lambda_{u,s}$ as in Assumption 3.5 it holds that*

(i) If the field $(y_{v,t})_{(v,t)\in B'\times T'}$ is constantly equal to $y \in \mathbb{R}$, then

$$\lambda_{u,s}(y_{v,t}) = \lambda_{u,s}(y) = \frac{y}{c}.$$

(ii) For all constants $y \in \mathbb{R}$ and fields $(y_{v,t})$,

$$\lambda_{u,s}(y_{v,t} + y) = \lambda_{u,s}(y_{v,t}) + \frac{y}{c}.$$

(iii) $\lambda_{u,s}$ is increasing.

In the remainder of this section, it as assumed that also Assumption 3.5 is satisfied.

The first step in proving the asymptotic behaviour of the extremal probability $\mathbb{P}(\Psi(X_{v,t}) > x)$ is to consider the asymptotic behaviour of extremal sets of a single jump-field $V = (V_{v,t})_{(v,t)\in B'\times T'}$ with distribution ν_1.

Theorem 3.7. *Let $(V_{v,t})_{(v,t)\in B'\times T'}$ have distribution ν_1 and let $(y_{v,t})_{(v,t)\in B'\times T'}$ be t-càdlàg. As $x \to \infty$, it holds that*

$$\frac{\mathbb{P}(\Psi(V_{v,t} + y_{v,t}) > x)}{L(x/c)\exp(-\beta x/c)} \to \frac{1}{\nu(A)} \int_B \int_0^T \exp\left(\beta \lambda_{u,s}(y_{v,t})\right) \mathrm{d}s\mathrm{d}u. \tag{3.3}$$

Proof. For sufficiently large $x > 0$ we find

$$\begin{aligned}
&\nu(A)\,\mathbb{P}(\Psi(V_{v,t} + y_{v,t}) > x) \\
&\quad = F\left(\left\{(u,s,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_+ \,:\, \Psi\big(zf(|v-u|, t-s) + y_{v,t}\big) > x\right\}\right) \\
&\quad = F\left(\left\{(u,s,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_+ \,:\, \psi_{x,u,s}(y_{v,t}) < z\right\}\right) \\
&\quad = \int_{B\times[0,T]} L\left(\psi_{x,u,s}(y_{v,t})\right) \exp\left(-\beta \psi_{x,u,s}(y_{v,t})\right) m(\mathrm{d}u, \mathrm{d}s) \\
&\qquad\quad + \int_{(B\times[0,T])^c} L\left(\psi_{x,u,s}(y_{v,t})\right) \exp\left(-\beta \psi_{x,u,s}(y_{v,t})\right) m(\mathrm{d}u, \mathrm{d}s).
\end{aligned} \tag{3.4}$$

First we show that the latter integral is of order $o(L(x/c)\exp(-\beta x/c))$ as $x \to \infty$. Let $y^* = \sup_{(v,t)\in B'\times T'} y_{v,t}$. Using Proposition 3.2(iii) and that $x \mapsto L(x)\exp(-\beta x)$ is decreasing, we obtain that the second integral in (3.4) is bounded from above by

$$\int_{(B\times[0,T])^c} L\left(\frac{x - y^*}{\Psi(f(|v-u|, t-s))}\right) \exp\left(-\beta \frac{x - y^*}{\Psi(f(|v-u|, t-s))}\right) m(\mathrm{d}u, \mathrm{d}s).$$

Let $h(u,s;x)$ denote the integrand. For all $(u,s) \in (B \times [0,T])^c$ we have $\Psi(f(|v-u|, t-s)) < c$. In combination with (2.5) and (2.6), this implies the existence of $\gamma > 0$ and $C > 0$ such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)} \le C\exp(-\gamma x)$$

for sufficiently large x. Thus, $h(u,s;x)$ is of order $o(L(x/c)\exp(-\beta x/c))$ at infinity. By dominated convergence, also the integral is of order $o(L(x/c)\exp(-\beta x/c))$ if we can find an integrable function $g:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)}\leq g(u,s)$$

for all $(u,s)\in\mathbb{R}^d\times\mathbb{R}$. Returning to (2.6) we see that for all $0<\gamma<\beta/c$ there is $C>0$ and $x_0>y^*$ such that

$$\begin{aligned}&\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)}\\ &\qquad\leq C\exp\Big(-(x_0-y^*)(\beta-\gamma c)\Big(\frac{1}{\Psi(f(|v-u|,t-s))}-\frac{1}{c}\Big)\Big)\end{aligned}\tag{3.5}$$

for all $x\geq x_0$. Since $B'\times T'$ is bounded we can choose $b\in(0,\infty)$ such that $B'\times T'\subseteq C_b(0)\subseteq\mathbb{R}^{d+1}$, where $C_b(0)$ is the $(d+1)$-dimensional ball with radius b and center $0\in\mathbb{R}^{d+1}$. For all $(u,s)\notin C_b(0)$ we obtain from (2.11) that there is a constant C such that

$$\begin{aligned}\Psi(f(|v-u|,t-s))&\leq\sup_{(v,t)\in B'\times T'}f(|v-u|,t-s)\\ &\leq\sup_{(v,t)\in C_b(0)}f(|v-u|,t-s)\leq\frac{C}{(|(u,s)|-b+1)},\end{aligned}$$

where we in the first inequality used Assumption 3.1(i) and (ii). It follows that (3.5) is integrable.

It remains to show that the first integral in (3.4) has the desired mode of convergence. For this, we have from (3.2), the representation of L, and the fact that ρ has an exponential tail, that for any $(u,s)\in B\times[0,T]$,

$$\frac{L\left(\psi_{x,u,s}(y_{v,t})\right)\exp\left(-\beta\psi_{x,u,s}(y_{v,t})\right)}{L\left(\frac{x}{c}\right)\exp\left(-\beta\frac{x}{c}\right)}\to\exp\left(\beta\lambda_{u,s}(y_{v,t})\right)$$

as $x\to\infty$. Since $x\mapsto L(x)\exp(-\beta x)$ is decreasing, we find using Proposition 3.2(iii) that for sufficiently large x,

$$\begin{aligned}&\frac{L\left(\psi_{x,u,s}(y_{v,t})\right)\exp\left(-\beta\psi_{x,u,s}(y_{v,t})\right)}{L\left(\frac{x}{c}\right)\exp\left(-\beta\frac{x}{c}\right)}\\ &\qquad\leq\frac{L\left(\frac{x-y^*}{c}\right)\exp\left(-\beta\frac{x-y^*}{c}\right)}{L\left(\frac{x}{c}\right)\exp\left(-\beta\frac{x}{c}\right)}\leq C\exp(\beta y^*/c),\end{aligned}$$

for any $(u,s)\in B\times[0,T]$, where, according to (2.5), C is such that $L\big(\frac{x-y^*}{c}\big)/L\big(\frac{x}{c}\big)\leq C$. As $B\times[0,T]$ is compact, the upper bound is integrable over $B\times[0,T]$ and (3.3) then follows by dominated convergence. □

The next step is to extend the relation (3.3) to an asymptotic result for $\mathbb{P}(\Psi(V^1_{v,t}+\cdots+V^n_{v,t}+y_{v,t})>x)$, where, for $i=1,\ldots,n$, V^i are independent and identically distributed with common distribution ν_1. Here it will be useful to recall that each V^i can be represented by $(Z^i f(|v - U^i|, t - S^i))_{(v,t)\in B'\times T'}$, where (U^i, S^i, Z^i) has distribution F_1. Before being able to extend (3.3), we need a final assumption on the existence of a function ϕ ensuring sufficient integrability properties.

For the assumption we need some notation representing a deterministic version of the sum $V^1_{v,t} + \cdots + V^n_{v,t}$. Thus let for each $i = 1, \ldots, n$ the field $(y^i_{v,t})_{(v,t)\in B'\times T'}$ be given by

$$y^i_{v,t} = z^i f(|v - u^i|, t - s^i),$$

where all $z^i \geq 0$, $u^i \in \mathbb{R}^d$ and $s^i \in \mathbb{R}$.

Assumption 3.8. *There exists a function* $\phi : \mathbb{R}^d \times \mathbb{R} \to [0, \infty)$ *such that*

$$\phi(u,s) \begin{cases} = c & \text{for} \quad (u,s) \in B' \times T' \\ < c & \text{for} \quad (u,s) \notin B' \times T', \end{cases} \tag{3.6}$$

where $c > 0$ *is the constant defined in* (3.1)*, and there is* $b > 0$ *and a constant* C_2 *such that*

$$\phi(u,s) \leq \frac{C_2}{|(u,s)| - b + 1} \tag{3.7}$$

for all (u,s) *with* $|(u,s)| > b$.

The function ϕ *satisfies*

$$\Psi\Big(\sum_{i=1}^n y^i_{v,t}\Big) \leq \sum_{i=1}^n z^i \phi(u^i, s^i), \tag{3.8}$$

and

$$\sup_{s\in[0,T]} \sup_{u\in B} \lambda_{u,s}\Big(\sum_{i=1}^n y^i_{v,t}\Big) \leq \frac{1}{c} \sum_{i=1}^n z^i \phi(u^i, s^i). \tag{3.9}$$

The definition of ϕ along with (3.7) ensures that the tail of $Z\phi(U,S)$ is asymptotically equivalent to $\rho((x/c, \infty))$ and hence, $Z\phi(U,S)$ is convolution equivalent with index β/c; see Lemma 3.9 below. Equation (3.8) then provides a convolution equivalent upper bound of the extremal probability for the functional Ψ of a sum of jump-fields. Finally, finiteness of relevant exponential moments of $\lambda_{u,s}$ applied to jump-fields is ensured by (3.9). This result is seen in Theorem 3.10 below.

In the remainder of this section it is also assumed that Assumption 3.8 is satisfied. The proof of Lemma 3.9 below follows by similar arguments as the proof of Theorem 3.7 above, however, for completeness the proof can be found in the Appendix.

Lemma 3.9. *Let* (U, S, Z) *have distribution* F_1*. Then, as* $x \to \infty$*,*

$$\frac{\mathbb{P}(Z\phi(U,S) > x)}{L(x/c)\exp(-\beta x/c)} \to \frac{1}{\nu(A)} m(B' \times T'). \tag{3.10}$$

In particular, the distribution of $Z\phi(U,S)$ *is convolution equivalent with index* β/c *and*

$$\mathbb{E}\Big[\exp\Big(\frac{\beta}{c} Z\phi(U,S)\Big)\Big] < \infty. \tag{3.11}$$

As mentioned, the convolution equivalence of $Z\phi(U,S)$ is translated into a convolution equivalent upper bound for the extremal probability of a sum of jump-fields. Here (3.8) is applied together with the relation $\overline{F^{*n}}(x) \sim n\overline{F}(x)\big(\int \mathrm{e}^{\beta y} F(\mathrm{d}y)\big)^{n-1}$, $x \to \infty$, when F is a convolution equivalent distribution with index β, F^{*n} is its n-fold convolution, and $\overline{F}$ is its tail. For this relation see e.g. [6, Corollary 2.11].

In Theorem 3.10 below, a similar convolution equivalence for the sum of jump-fields is obtained.

Theorem 3.10. *Let $V^1, V^2, \ldots$ be i.i.d. fields with common distribution ν_1, and assume that $(y_{v,t})_{(v,t)\in B'\times T'}$ is t-càdlàg. For all $n \in \mathbb{N}$ it holds that*

$$\frac{\mathbb{P}(\Psi(V^1_{v,t} + \cdots + V^n_{v,t} + y_{v,t}) > x)}{\mathbb{P}(\Psi(V^1_{v,t}) > x)} \to \frac{n}{m(B\times[0,T])} \int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(V^1_{v,t} + \cdots + V^{n-1}_{v,t} + y_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u$$

as $x \to \infty$.

Recall that the field X^1 is defined as the compound Poisson sum with i.i.d. jump-fields $V^1, V^2, \ldots$ and an independent Poisson distributed variable N with intensity $\nu(A) < \infty$. The following result on the extremal behaviour of X^1 follows from Theorem 3.10 by conditioning on the value of N.

Theorem 3.11. *For each $(u,s) \in B\times[0,T]$, $\mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X^1_{v,t})\right)\right] < \infty$. For a field $(y_{v,t})_{(v,t)\in B'\times T'}$ satisfying (2.7),*

$$\frac{\mathbb{P}(\Psi(X^1_{v,t} + y_{v,t}) > x)}{L(x/c)\exp(-\beta x/c)} \to \int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X^1_{v,t} + y_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u$$

as $x \to \infty$.

Now recall that we write the field $X = (X_{v,t})_{(v,t)\in B'\times T'}$ defined in (2.9) as the independent sum $X = X^1 + X^2$, where X^1 is the compound Poisson sum of fields with distribution ν_1. Also, the fields in the decomposition can be assumed to be t-càdlàg. One can show that the tail of $\sup_{(v,t)} X^2_{v,t}$ is lighter than that of $\Psi(X^1_{v,t})$, which is equivalent to the tail of ρ by Theorem 3.11. Combining this fact with [13, Lemma 2.1], an argument based on independence and dominated convergence can be used to conclude Theorem 3.13 below from Theorem 3.11.

Lemma 3.12. *For all $(u,s) \in B\times[0,T]$ it holds that $\mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X_{v,t})\right)\right] < \infty$.*

Theorem 3.13. *Let the field X be given by (2.9), where the Lévy basis M satisfies Assumption 2.1 and the kernel function f satisfies Assumption 2.4. Let the functionals Ψ and $\lambda_{u,s}$ satisfy Assumptions 3.1 and 3.5, respectively. Then*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\Psi(X_{v,t}) > x)}{\rho((x/c,\infty))} = \int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u.$$

4 Example results

In this section we return to Examples 3.3 and 3.4 to show versions of Theorem 3.13 when Ψ is specifically given as in the examples. We make further assumptions on the kernel f that guarantee Assumptions 3.5 and 3.8.

In the setting of Example 3.3 we assume the following.

Assumption 4.1. *The kernel $f : [0, \infty) \times \mathbb{R} \to [0, \infty)$ is decreasing in both coordinates on $[0, \infty) \times [0, \infty)$, and it is strictly decreasing in the point $(r/2, 0)$ in the sense that*

$$f(x, y) < f(r/2, 0) \qquad \text{for all } (x, y) \in \big([r/2, \infty) \times [0, \infty)\big) \setminus \big\{(r/2, 0)\big\}. \tag{4.1}$$

Moreover, the derivative $f_1(x) = \frac{\partial f}{\partial x}(x, 0)$ exists for all $x \geq 0$, and there is a function g such that

$$g(x) = f_1(r/2)(x - r/2) + f(r/2, 0) \tag{4.2}$$

for all $x \in [0, r]$, where also $f(x, 0) \leq g(x)$ for all $x \in [0, r]$.

Such a g exists in particular when f is concave on $[0, r]$. The following lemma shows that Assumption 3.5 is satisfied when the kernel satisfies Assumption 4.1.

Lemma 4.2. *If Ψ and $\psi_{x,u,s}$ are given as in Example 3.3 and f satisfies Assumption 4.1, then Assumption 3.5 is satisfied with $c = f(r/2, 0)$. Furthermore, for a t-càdlàg field $y = (y_{v,t})_{(v,t) \in B' \times [0,T]}$, the functional $\lambda_{u,s}$ takes the form*

$$\lambda_{u,s}\big((y_{v,t})_{(v,t) \in B' \times [0,T]}\big) = \lambda_u\big((y_{v,s})_{v \in B'}\big)$$

for a functional $\lambda_u : \mathbb{R}^{B'} \to \mathbb{R}$.

Proof. From [17, Lemma 3.1] we have, for fixed $s \in [0, T]$ and for all $u \in B$, a functional λ_u such that

$$\inf_{v_0 \in B} \inf_{R \in \mathrm{SO}(d)} \sup_{v \in D^R(v_0)} \frac{x - y_{v,s}}{f(|v - u|, 0)} - \frac{x}{f(r/2, 0)} + \lambda_u((y_{v,s})_{v \in B'}) \to 0 \tag{4.3}$$

as $x \to \infty$. with $\lambda_{u,s}$ defined by $\lambda_{u,s}((y_{v,t})_{(v,t) \in B' \times [0,T]}) = \lambda_u((y_{v,s})_{v \in B'})$, we claim that Assumption 3.5 is satisfied. For notational convenience, we write $C = -\lambda_u(y_{v,s})$.

For all sufficiently large x, we can choose $t_x \in [s, T]$, $v_x \in B$ and $R_x \in \mathrm{SO}(d)$ such that

$$\sup_{v \in D^{R_x}(v_x)} \frac{x - y_{v,t_x}}{f(|v - u|, t_x - s)} = \inf_{t \in [0,T]} \inf_{v_0 \in B} \inf_{R \in \mathrm{SO}(d)} \sup_{v \in D^R(v_0)} \frac{x - y_{v,t}}{f(|v - u|, t - s)}.$$

With $y^* = \sup_{(v,t) \in B' \times [0,T]} y_{v,t}$ and $y_* = \inf_{(v,t) \in B' \times [0,T]} y_{v,t}$, we then find

$$\begin{aligned} \frac{x - y^*}{\inf_{v \in D^{R_x}(v_x)} f(|v - u|, t_x - s)} &\leq \sup_{v \in D^{R_x}(v_x)} \frac{x - y_{v,t_x}}{f(|v - u|, t_x - s)} \\ &\leq \sup_{v \in D^{R_x}(u)} \frac{x - y_{v,s}}{f(|v - u|, 0)} \leq \frac{x - y_*}{f(r/2, 0)}. \end{aligned}$$

Going to the limit $x \to \infty$ and using that $\inf_{v \in D^R(v_0)} f(|v-u|, t-s) \le f(r/2, 0)$, shows $\inf_{v \in D^{R_x}(v_x)} f(|v-u|, t_x - s) \to f(r/2,0)$ as $x \to \infty$. Since in fact the inequality $\inf_{v \in D^R(v_0)} f(|v-u|, t-s) < f(r/2, 0)$ is true for all $(v_0, t) \neq (u, s)$ and $R \in \mathrm{SO}(d)$, the convergence implies that also $v_x \to u$ and $t_x \to s$. We will show the desired convergence

$$\sup_{v \in D^{R_x}(v_x)} \frac{x - y_{v,t_x}}{f(|v-u|, t_x - s)} - \frac{x}{f(r/2,0)} \to C \qquad (x \to \infty)$$

by contradiction. Since

$$\sup_{v \in D^{R_x}(v_x)} \frac{x - y_{v,t_x}}{f(|v-u|, t_x - s)} \le \inf_{v_0 \in B} \inf_{R \in \mathrm{SO}(d)} \sup_{v \in D^R(v_0)} \frac{x - y_{v,s}}{f(|v-u|, 0)},$$

we assume the existence of $\epsilon > 0$ and a sequence (x_n), $x_n \to \infty$, such that

$$\sup_{v \in D^{R_n}(v_n)} \frac{x_n - y_{v,t_n}}{f(|v-u|, t_n - s)} - \frac{x_n}{f(r/2,0)} \le C - \epsilon \tag{4.4}$$

for all n, where $t_n = t_{x_n}$, $v_n = v_{x_n}$ and $R_n = R_{x_n}$. By t-càdlàg properties of the y-field, we can find n_0 such that

$$\sup_{v \in B'} \left| \frac{y_{v,t_n} - y_{v,s}}{f(|v-u|, 0)} \right| \le \frac{\epsilon}{2}$$

for all $n \ge n_0$. Consequently and using that f is decreasing and (4.4)

$$\begin{aligned}
&\sup_{v \in D^{R_n}(v_n)} \frac{x_n - y_{v,s}}{f(|v-u|, 0)} - \frac{x_n}{f(r/2,0)} \\
&\quad \le \sup_{v \in D^{R_n}(v_n)} \frac{x_n - y_{v,t_n}}{f(|v-u|, 0)} - \frac{x_n}{f(r/2,0)} + \frac{\epsilon}{2} \\
&\quad \le \sup_{v \in D^{R_n}(v_n)} \frac{x_n - y_{v,t_n}}{f(|v-u|, t_n - s)} - \frac{x_n}{f(r/2,0)} + \frac{\epsilon}{2} \le C - \frac{\epsilon}{2},
\end{aligned}$$

which contradicts the limit relation (4.3). □

Theorem 4.3. *Let the field X be given by* (2.9)*, where the Lévy basis M satisfies Assumption 2.1 and the kernel function f satisfies Assumptions 2.4 and 4.1. Let $D \subseteq C_{r/2}(0)$ have radius $r/2 > 0$ and let Ψ be defined by*

$$\Psi(y_{v,t}) = \sup_{t \in [0,T]} \sup_{v_0 \in B} \sup_{R \in \mathrm{SO}(d)} \inf_{v \in D^R(v_0)} y_{v,t}.$$

Furthermore, let $\lambda_{u,s}$ be the functional given in Lemma 4.2 and write $c = f(r/2, 0)$. Then

$$\lim_{x \to \infty} \frac{\mathbb{P}(\Psi(X_{v,t}) > x)}{\rho((x/c, \infty))} = \int_B \int_0^T \mathbb{E}\left[\exp\left(\beta \lambda_{u,s}(X_{v,t})\right)\right] \mathrm{d}s \mathrm{d}u.$$

Proof. The result follows from Theorem 3.13 and Lemma 4.2 once we show the existence of a function ϕ satisfying Assumption 3.1. Now define ϕ as

$$\begin{aligned}\phi(u,s) = f(r/2,0)\ \mathbf{1}_{B'\times[0,T]}(u,s) \\ + \sup_{t\in[0,T]}\sup_{v\in B\oplus C_{r/2}} f(|v-u|,t-s)\ \mathbf{1}_{(B'\times[0,T])^c}(u,s).\end{aligned}$$

By (4.1), ϕ satisfies (3.6). Choosing $b>0$ such that $B'\times[0,T]\subseteq C_b(0)$, we use (2.11) to find a constant C such that

$$\phi(u,s) \leq \sup_{(v,t)\in C_b(0)} f(|v-u|,t-s) \leq \frac{C}{|(u,s)|-b+1}$$

whenever $|(u,s)|>b$. Hence (3.7) is also satisfied.

Appealing to Lemma 4.2 and [17, Lemma 3.2],

$$\lambda_{u,s}(y_{v,t}) = \frac{1}{2f(r/2,0)} \sup_{\alpha\in\mathbb{S}^{d-1}} (y_{u+\alpha r/2,s} + y_{u-\alpha r/2,s}) \tag{4.5}$$

for all $(u,s)\in B\times[0,T]$, if $D=\{-\alpha r/2, \alpha r/2\}$ for some $\alpha\in\mathbb{S}^{d-1}$. Adapting the proof of [17, Lemma 3.3] to this time-dependant setting, it is seen using (4.5) that (3.8) and (3.9) follow when it is shown that

$$\begin{aligned}\tfrac{1}{2}(y^i_{u+\alpha r/2,s} + y^i_{u-\alpha r/2,s}) \leq z^i\phi(u^i,s^i) \\ \text{for all } (u,s)\in B\times[0,T],\ \alpha\in\mathbb{S}^{d-1},\end{aligned} \tag{4.6}$$

with $y^i_{v,t}$ defined as just before Assumption 3.8. Since f is decreasing in both coordinates,

$$\frac{1}{2}\big(y^i_{u+\alpha r/2,s} + y^i_{u-\alpha r/2,s}\big) \leq \frac{z^i}{2}\big(f(|u+\alpha r/2-u^i|,0) + f(|u-\alpha r/2-u^i|,0)\big).$$

Using the upper bound g assumed by (4.2), arguments as in [17, Lemma 3.3] show that (4.6) is satisfied when $(u^i,s^i)\in B'\times[0,T]$. When $(u^i,s^i)\in(B'\times[0,T])^c$ it is immediately seen that

$$\tfrac{1}{2}\big(y^i_{u+\alpha r/2,s} + y^i_{u-\alpha r/2,s}\big) \leq z^i \sup_{t\in[0,T]}\sup_{v\in B\oplus C_{r/2}} f(|v-u^i|,t-s^i) = z^i\phi(u^i,s^i).$$

This concludes the proof. ∎

In the setting of Example 3.4 the following is assumed.

Assumption 4.4. *For the set $D\subseteq C_r(0)\subseteq\mathbb{R}^d$ the kernel function f satisfies*

$$\int_{D(v_0)}\int_{I(t_0)} f(|v-u|,t-s)\mathrm{d}t\mathrm{d}v < \int_D\int_0^\ell f(|v|,t)\mathrm{d}t\mathrm{d}v \tag{4.7}$$

for all $(v_0,t_0)\neq(u,s)\in\mathbb{R}^d\times\mathbb{R}$.

Lemma 4.5. *Let $y=(y_{v,t})_{(v,t)\in B'\times T'}$ be a t-càdlàg field. For all $(u,s)\in B\times[0,T]$ it holds that*

$$\begin{aligned}\inf_{t_0\in[0,T]}\inf_{v_0\in B}\inf_{R\in \mathrm{SO}(d)} &\frac{x-\frac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{\frac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v}\\ &-\frac{x}{c}+\sup_{R\in\mathrm{SO}(d)}\frac{1}{c}\frac{1}{K}\int_{D^R(u)}\int_{I(s)} y_{v,t}\mathrm{d}t\mathrm{d}v\to 0\end{aligned}\tag{4.8}$$

as $x\to\infty$, where $c=\frac{1}{K}\int_D\int_0^\ell f(|v|,t)\mathrm{d}t\mathrm{d}v$. That is, with Ψ and $\psi_{x,u,s}$ as in Example 3.4, and with $\lambda_{u,s}(y_{v,t})=\sup_R\frac{1}{c}\frac{1}{K}\int_{D^R(u)}\int_{I(s)} y_{v,t}\mathrm{d}t\mathrm{d}v$, Assumption 3.5 is satisfied.

Proof. For all sufficiently large $x>0$, choose $t_x\in[s,T]$, $v_x\in B$ and $R_x\in\mathrm{SO}(d)$ with

$$\begin{aligned}\inf_{t_0,v_0,R}&\frac{x-\frac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{\frac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v}\\ &=\frac{x-\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v}.\end{aligned}$$

By definition of t_x and v_x we find that

$$\begin{aligned}&\frac{x-y^*}{\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v}\\ &\quad\le\frac{x-\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v}\\ &\quad\le\inf_{R\in\mathrm{SO}(d)}\frac{x-\frac{1}{K}\int_{D^R(u)}\int_{I(s)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{\frac{1}{K}\int_{D(u)}\int_{I(s)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v}\\ &\quad=\frac{x-\sup_{R\in\mathrm{SO}(d)}\frac{1}{K}\int_{D^R(u)}\int_{I(s)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{c},\end{aligned}$$

where $y^*=\sup y_{v,t}$. Since $\frac{1}{K}\int_{D^R(v_0)}\int_{I(t_0)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v<c$ for all $(v_0,t_0)\neq(u,s)$ and any $R\in\mathrm{SO}(d)$, we conclude that $\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v\to c$ and consequently $v_x\to u$ and $t_x\to s$ as $x\to\infty$. Since the field $(y_{v,t})$ is t-càdlàg, we furthermore find that, as $x\to\infty$,

$$\sup_{R\in\mathrm{SO}(d)}\int_{D^R(v_x)}\int_{I(t_x)} y_{v,t}\,\mathrm{d}t\mathrm{d}v\to\sup_{R\in\mathrm{SO}(d)}\int_{D^R(u)}\int_{I(s)} y_{v,t}\,\mathrm{d}t\mathrm{d}v.$$

Recalling that $\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v\le c$ for all x, and turning to the inequalities above, we conclude (4.8) by

$$\begin{aligned}0&\le\frac{x-\sup_R\frac{1}{K}\int_{D^R(u)}\int_{I(s)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{c}-\frac{x-\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} f(|v-u|,t-s)\,\mathrm{d}t\mathrm{d}v}\\ &\le\frac{\frac{1}{K}\int_{D^{R_x}(v_x)}\int_{I(t_x)} y_{v,t}\,\mathrm{d}t\mathrm{d}v-\sup_R\frac{1}{K}\int_{D^R(u)}\int_{I(s)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{c}\\ &\le\frac{\sup_R\frac{1}{K}\int_{D^R(v_x)}\int_{I(t_x)} y_{v,t}\,\mathrm{d}t\mathrm{d}v-\sup_R\frac{1}{K}\int_{D^R(u)}\int_{I(s)} y_{v,t}\,\mathrm{d}t\mathrm{d}v}{c}\to 0\end{aligned}$$

as $x \to \infty$. $\square$

Theorem 4.6. *Let the field X be given by* (2.9)*, where the Lévy basis M satisfies Assumption 2.1 and the kernel function f satisfies Assumptions 2.4 and 4.4. Let $D \subseteq C_r(0) \subseteq \mathbb{R}^d$ for $r \geq 0$ be given, and let Ψ be defined by*

$$\Psi(y_{v,t}) = \sup_{t_0 \in [0,T]} \sup_{v_0 \in B} \sup_{R \in \mathrm{SO}(d)} \tfrac{1}{K} \int_{D^R(v_0)} \int_{I(t_0)} y_{v,t} \,\mathrm{d}t\mathrm{d}v,$$

where $K = \int_D \int_0^\ell 1 \,\mathrm{d}t\mathrm{d}v$. Furthermore, let $c = \frac{1}{K} \int_D \int_0^\ell f(|v|, t)\mathrm{d}t\mathrm{d}v$. Then

$$\begin{aligned}
&\lim_{x \to \infty} \frac{\mathbb{P}(\Psi(X_{v,t}) > x)}{\rho((x/c, \infty))} \\
&\qquad = m(B \times [0,T]) \mathbb{E}\Big[\exp\Big(\beta \sup_{R \in \mathrm{SO}(d)} \tfrac{1}{c}\tfrac{1}{K} \int_{D^R(u)} \int_{I(s)} X_{v,t} \mathrm{d}t\mathrm{d}v\Big)\Big],
\end{aligned}$$

where $(u, s) \in B \times [0, T]$ is chosen arbitrarily.

Proof of Theorem 4.6. The result follows from Theorem 3.13 and Lemma 4.5 once we show the existence of a function ϕ satisfying Assumption 3.1. Note that the integrand in the limit in Theorem 3.13 is constant due to the stationarity of X and $\lambda_{u,s}$. Define

$$\begin{aligned}
\phi(u, s) &= c\, \mathbf{1}_{B' \times T'}(u, s) \\
&\quad + \sup_{t_0 \in [0,T]} \sup_{v_0 \in B} \tfrac{1}{K} \int_{D(v_0)} \int_{I(t_0)} f(|v - u|, t - s)\mathrm{d}t\mathrm{d}v \, \mathbf{1}_{(B' \times T')^c}(u, s).
\end{aligned}$$

By (4.7), ϕ satisfies (3.6). Choosing $b > 0$ such that $B' \times T' \subseteq C_b(0)$, we use (2.11) to find a constant C such that

$$\phi(u, s) \leq \sup_{(v,t) \in C_b(0)} f(|v - u|, t - s) \leq \frac{C}{|(u, s)| - b + 1}$$

whenever $|(u, s)| > b$. Hence (3.7) is also satisfied. Now let $n \in \mathbb{N}$ be fixed, and let $(y^i_{v,t})_{(v,t) \in B' \times T'}$ for $i = 1, \ldots, n$ be t-càdlàg fields. Then

$$\begin{aligned}
\Psi\Big(\sum_{i=1}^n y^i_{v,t}\Big) &= \sup_{t_0 \in [0,T]} \sup_{v_0 \in B} \sup_{R \in \mathrm{SO}(d)} \tfrac{1}{K} \int_{D^R(v_0)} \int_{I(t_0)} \sum_{i=1}^n y^i_{v,t} \,\mathrm{d}t\mathrm{d}v \\
&\leq \sum_{i=1}^n \sup_{t_0 \in [0,T]} \sup_{v_0 \in B} \sup_{R \in \mathrm{SO}(d)} \tfrac{1}{K} \int_{D^R(v_0)} \int_{I(t_0)} y^i_{v,t} \,\mathrm{d}t\mathrm{d}v \\
&= \sum_{i=1}^n \Psi(y^i_{v,t}).
\end{aligned}$$

Furthermore, if $y^i_{v,t} = z^i f(|v - u^i|, t - s^i)$, it is easily seen that $\Psi(y^i_{v,t}) \leq z^i \phi(u^i, s^i)$, and hence, (3.8) is satisfied. Since

$$\sup_{s \in [0,T]} \sup_{u \in B} \lambda_{u,s}(y^i_{v,t}) = \frac{1}{c} \Psi(y^i_{v,t}),$$

(3.9) is also satisfied, which concludes the proof. $\square$

As mentioned in Example 3.4, the case of $\mathbb{P}(\sup_{t\in[0,T]}\sup_{v\in B} X_{v,t} > x)$ follows from Theorem 4.6 by letting $\ell = 0$ and $D = \{0\}$. In this case, the constant $c = f(0,0) = 1$ and (4.7) translates into $f(|v_0-u|, t_0-s) < f(0,0)$ for all $(v_0,t_0) \neq (u,s)$, or equivalently $f(x,y) < f(0,0)$ for all $(x,y) \neq (0,0)$.

Theorem 4.7. *Let the field X be given by* (2.9)*, where the Lévy basis M satisfies Assumption 2.1 and the kernel function f satisfies Assumption 2.4 and $f(x,y) < f(0,0)$ for all $(x,y) \neq (0,0)$. Then*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\sup_{t\in[0,T]}\sup_{v\in B} X_{v,t} > x)}{\rho((x,\infty))} = m(B\times[0,T])\mathbb{E}\left[\exp\left(\beta X_{u,s}\right)\right],$$

where $(u,s) \in B\times[0,T]$ is chosen arbitrarily.

5 Continuity properties

The main purpose of this section is to show that the field defined in (2.9) has a version with t-càdlàg sample paths. This result will be obtained in Theorem 5.7 below. However, the proof involves showing two other results on continuity properties of related random fields of independent value. Therefore these results are formulated as separate theorems; see Theorems 5.2 and 5.4 below. Only the main results, Theorems 5.2, 5.4 and 5.7, are stated fully with all assumptions included in the statement. The rest are to be understood in relation to the context. As stated in Section 2, Assumption 2.1 on the Lévy basis is partly used to guarantee that the field is t-càdlàg. However, if the aim is solely to obtain the t-càdlàg property, we can relax the assumption. In this section we therefore consider Assumption 5.1 below. It will both be referred to with the dimension of the Lévy basis being d and $d+1$. Thus, both the assumption and the subsequent Theorem 5.2 will be formulated with $m \in \mathbb{N}$ indicating the dimension.

Assumption 5.1. *The Lévy basis M on $\mathbb{R}^m$ is stationary and isotropic satisfying* (2.1)*. Moreover, the Lévy measure, denoted ρ, satisfies*

$$\int_{|y|>1} y^k \rho(\mathrm{d}y) < \infty \qquad \text{for all } k \in \mathbb{N}. \tag{5.1}$$

For the first result in this section, consider a compact set $K \subseteq \mathbb{R}^m$ and define the random field $Y = (Y_v)_{v\in K}$ by

$$Y_v = \int_{\mathbb{R}^m} h(|v-u|)M(\mathrm{d}u), \tag{5.2}$$

where M is a Lévy basis on $\mathbb{R}^m$ satisfying Assumption 5.1. It is shown in [16, Theorem A.1] that such a field has a continuous version when $h : [0,\infty) \to \mathbb{R}$ satisfies certain properties including being differentiable. Under much less restrictive assumptions on the kernel function h, we show that this is still the case. We only assume that h is bounded and integrable

$$\int_{\mathbb{R}^m} h(|u|)\mathrm{d}u < \infty, \tag{5.3}$$

and that h is Lipschitz continuous. That is, there exist $C_L > 0$ such that

$$|h(x) - h(y)| \leq C_L |x - y| \tag{5.4}$$

for all $x, y \geq 0$. Having Assumption 5.1 satisfied for the basis M and (5.3) and (5.4) satisfied for the bounded kernel function ensures in particular that the integral (5.2) exists; see [15, Theorem 2.7].

To show continuity, we appeal to a result in [1], in which finite moments and cumulants of the spot variable M' of the basis M are needed. As already mentioned, (5.1) is equivalent to saying that M' has finite moments and thus cumulants of any order; see [14, Corollary 3.2.2] for the relation between moments and cumulants.

Theorem 5.2. *If the field Y is given by* (5.2) *with the Lévy basis M on $\mathbb{R}^m$ satisfying Assumption 5.1, and if the kernel is bounded and satisfies* (5.3) *and* (5.4)*, then the field has a continuous version.*

Proof. For $r \in \mathbb{R}$ and $n \in \mathbb{N}$ we shall consider moments of the form $\mathbb{E}[(Y_{v+r} - Y_v)^n]$. Note that only indices in K are relevant, so in particular, $0 \leq |r| \leq \operatorname{diam}(K)$. By (5.4) and the triangle inequality, there is a finite C such that

$$\big|h(|v + r - u|) - h(|v - u|)\big| \leq C|r|.$$

Now let $\kappa_n[\,\cdot\,]$ denote the nth cumulant of a random variable; for a brief overview of the relation between cumulants and moments we refer to [14, Chapter 3] and in particular [14, Corollary 3.2.2]. The cumulants κ_n of the difference $Y_{v+r} - Y_v$ satisfy $\kappa_1[Y_{v+r} - Y_v] = 0$ and, for $n > 1$,

$$\begin{aligned}
|\kappa_n[Y_{v+r} - Y_v]| &\leq \big|\kappa_n[M']\big| \int_{\mathbb{R}^m} \big|h(|v + r - u|) - h(|v - u|)\big|^n \mathrm{d}u \\
&\leq \big|\kappa_n[M']\big| C^{n-1} |r|^{n-1} \int_{\mathbb{R}^m} \big|h(|v + r - u|) - h(|v - u|)\big| \mathrm{d}u \leq C_n |r|^{n-1},
\end{aligned}$$

where $C_n \geq 0$ is a finite constant, chosen independently of r and $v \in K$ by

$$C_n = \big|\kappa_n[M']\big| C^{n-1} 2 \int_{\mathbb{R}^m} h(|u|) \mathrm{d}u < \infty,$$

see e.g. [18, Appendix A] for the cumulant formulas. Consequently, for all $n \in \mathbb{N}$, there exist finite constants C_n' and natural numbers $n' \geq n/2$ such that

$$\mathbb{E}[(Y_{v+r} - Y_v)^n] \leq C_n' |r|^{n'}$$

with the equality $n' = n/2$ whenever n is even; see [14, Corollary 3.2.2]. Using the fact that $|r| \leq \operatorname{diam}(K)$, we find finite $C' \geq 0$ and $\eta > 4(m + 1)$ such that

$$\mathbb{E}|Y_{v+r} - Y_v|^{4(m+1)} \leq C'_{4(m+1)} |r|^{2m} |r|^2 \leq \frac{C' |r|^{2m}}{\big|\log|r|\big|^{1+\eta}}$$

for all $v \in K$. From a corollary to [1, Theorem 3.2.5] we conclude that $(Y_v)_{v \in K}$ has a continuous version on K. $\square$

Next, we consider a field indexed by $\mathbb{R}^d \times \mathbb{R}$ allowing for discontinuities in time, and we show that it has a t-càdlàg version. For compact sets $K \subseteq \mathbb{R}^d$ and $[0, S]$, $S > 0$, we let the random field $Z = (Z_{v,t})_{(v,t)\in K\times[0,S]}$ be given by

$$Z_{v,t} = \int_{\mathbb{R}^d} \int_{[0,t]} g(|v-u|) M(\mathrm{d}s, \mathrm{d}u), \tag{5.5}$$

where M is a Lévy basis satisfying Assumption 5.1 with $m = d + 1$, and the integration kernel $g : [0, \infty) \to \mathbb{R}$ is assumed to be bounded, integrable and Lipschitz continuous, i.e. it satisfies (5.3) and (5.4); again with $m = d + 1$.

Choose $0 = t_0 < \cdots < t_n$ in $[0, S]$ and $v \in K$. Arguing as in Section 2, the cumulant function for $(Z_{v,t_1}, Z_{v,t_2} - Z_{v,t_1}, \dots, Z_{v,t_n} - Z_{v,t_{n-1}})$ can be found to be

$$\begin{aligned} &C(\boldsymbol{\lambda} \dagger (Z_{v,t_1}, Z_{v,t_2} - Z_{v,t_1}, \dots, Z_{v,t_n} - Z_{v,t_{n-1}})) \\ &\quad = \sum_{j=1}^{n} (t_j - t_{j-1}) \Big(i\lambda_j a \int_{\mathbb{R}^d} g(|v-u|)\mathrm{d}u - \tfrac{1}{2}\theta\lambda_j^2 \int_{\mathbb{R}^d} g(|v-u|)^2 \mathrm{d}u \\ &\qquad + \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathrm{e}^{ig(|v-u|)\lambda_j z} - 1 - ig(|v-u|)\lambda_j z 1_{[-1,1]}(z)\rho(\mathrm{d}z)\mathrm{d}u \Big) \end{aligned}$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. By a change of measure, we see for fixed $v \in \mathbb{K}$ that $(Z_{v,t})_{t\in[0,S]}$ is a one-dimensional Lévy process in law. In the following we shall extend this to a result concerning the process of random fields indexed by time.

In this section we will often consider the field as being a collection of real-valued functions defined on space K or $\tilde{K} = K \cap \mathbb{Q}^d$, with the functions indexed by time in $[0, S]$ or $\tilde{S} = [0, S] \cap \mathbb{Q}$. As such we introduce the notation $\boldsymbol{Z}_t = (Z_{v,t})_{v\in K}$, with the entire field denoted by $\boldsymbol{Z} = (\boldsymbol{Z}_t)_{t\in[0,S]}$ when considered as a collection of random functions. We use the same notation when space and time are indexed by $\tilde{K}$ and $\tilde{S}$, respectively, although when it is unclear which is meant and it is necessary to distinguish the cases, we explicitly state it.

Let $t \in [0, S]$ be fixed and choose $v_1, \dots, v_n \in K$. Then $(Z_{v_1,t}, \dots, Z_{v_n,t})$ has cumulant function given by

$$\begin{aligned} C(\boldsymbol{\lambda} \dagger (Z_{v_1,t}, \dots, Z_{v_n,t})) &= tia \int_{\mathbb{R}^d} \sum_{j=1}^{n} \lambda_j g(|v_j - u|)\mathrm{d}u \\ &- t\tfrac{1}{2}\theta \int_{\mathbb{R}^d} \Big(\sum_{j=1}^{n} \lambda_j g(|v_j - u|) \Big)^2 \mathrm{d}u \\ &+ t \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathrm{e}^{i\sum_{j=1}^{n} \lambda_j g(|v_j - u|) z} - 1 - i \sum_{j=1}^{n} \lambda_j g(|v_j - u|) z 1_{[-1,1]}(z)\rho(\mathrm{d}z)\mathrm{d}u. \end{aligned} \tag{5.6}$$

Replacing (a, θ, ρ) by $(ta, t\theta, t\rho)$, we see from (5.6) that $\boldsymbol{Z}_t$ is the type of field defined in (5.2). Thus, by Theorem 5.2, $(Z_{v,t})_{v\in\tilde{K}}$ is almost surely uniformly continuous. This holds jointly for all rational time points $t \in \tilde{S}$, and therefore a version of $(\boldsymbol{Z}_t)_{t\in\tilde{S}}$ can be chosen with $\boldsymbol{Z}_t$ being continuous for all $t \in \tilde{S}$, i.e. it has values in the space of real-valued functions on the compact set K. It will be useful in the following that this space equipped with the uniform norm, here denoted $(\mathcal{C}(K, \mathbb{R}), \|\cdot\|_\infty)$, is a separable Banach space; see [11, Theorem 4.19]. The following lemma concerns this specific version of $(\boldsymbol{Z})_{t\in\tilde{S}}$ taking its values in $(\mathcal{C}(K,\mathbb{R}),\|\cdot\|_\infty)$.

Lemma 5.3. *The process $(\boldsymbol{Z}_t)_{t\in\tilde{S}}$ is a Lévy process in law, i.e. each $\boldsymbol{Z}_t$ has an infinitely divisible distribution, the process has stationary and independent increments, and it is stochastically continuous with respect to the uniform norm.*

Proof. As $\boldsymbol{Z}_t$ is a version of the field studied in (5.6) for each $t\in\tilde{S}$, also the cumulant function for $(Z_{v_1,t},\dots,Z_{v_n,t})$ will be as in (5.6). With similar considerations, but heavier notation, it can be realised that for $v_1,\dots,v_n\in K$ and $0=t_0<t_1<\dots<t_m\in\tilde{S}$, and defining $Z^n_{t_j}=(Z_{v_1,t_j},\dots,Z_{v_n,t_j})$ for $j=1,\dots,m$, it holds that

$$\begin{aligned}C(\boldsymbol{\lambda}\dagger(Z^n_{t_1},Z^n_{t_2}-Z^n_{t_1},\dots,Z^n_{t_m}-Z^n_{t_{m-1}}))\\ =\sum_{j=1}^m C(\lambda_j\dagger Z^n_{t_j}-Z^n_{t_{j-1}})=\sum_{j=1}^m C(\lambda_j\dagger Z^n_{t_j-t_{j-1}}),\end{aligned}$$

where $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_n)$ and each $\lambda_j\in\mathbb{R}^n$, and the natural convention $Z^n_0=(0,\dots,0)$ is applied. This shows that $(\boldsymbol{Z}_t)_{t\in\tilde{S}}$ has stationary and independent increments.

To show stochastic continuity it suffices to show that

$$\lim_{n\to\infty}\mathbb{P}(\|\boldsymbol{Z}_{t_n}\|_\infty\geq\epsilon)=0$$

for any rational sequence (t_n) satisfying $t_n\downarrow 0$. As $(\mathcal{C}(K,\mathbb{R}),\|\cdot\|_\infty)$ is a separable Banach space, this is equivalent to showing that $\boldsymbol{Z}_{t_n}$ converges to $\delta_{\mathbf{0}}$ in law in the uniform norm, where $\delta_{\mathbf{0}}$ is the degenerate probability measure concentrated at $\mathbf{0}$. For $t\in\tilde{S}$, let ν_t denote the distribution of $\boldsymbol{Z}_t$ and let $\hat{\nu}_{t_n}$ be its characteristic function defined on the dual space of $\mathcal{C}(K,\mathbb{R})$, see [12, Section 1.7]. Since $(\mathcal{C}(K,\mathbb{R}),\|\cdot\|_\infty)$ is separable, ν_t is a Radon measure [12, Proposition 1.1.3] and the results in [12, Chapters 2 & 5] apply. Due to the infinite divisibility, $\nu_1=\nu_{t_n}*\nu_{1-t_n}$ for any $n\in\mathbb{N}$ (assuming $t_n\leq 1$), and we conclude that $\{\nu_{t_n}\}$ is relatively shift compact [12, Theorem 2.3.1]. Following the proofs of [12, Propositions 5.1.4 & 5.1.5] we obtain that $\lim_{n\to\infty}\hat{\nu}_{t_n}\to 1$ uniformly on bounded sets of the dual space. Combining [12, Propositions 2.3.9 & 1.8.2] shows that $\boldsymbol{Z}_{t_n}$ converges in law to $\delta_{\mathbf{0}}$ as claimed. □

The next theorem states that the field Z defined in (5.5) indeed has a t-càdlàg version.

Theorem 5.4. *Let the field Z be given by* (5.5) *such that the Lévy basis M on $\mathbb{R}^{d+1}$ satisfies Assumption 5.1 with $m=d+1$, and the bounded kernel g satisfies* (5.3) *and* (5.4)*, with $m=d$. There is a field $Z'=(Z'_{v,t})_{(v,t)\in K\times[0,S]}$ that is a version of Z, i.e. $\mathbb{P}(Z'_{v,t}=Z_{v,t})=1$ for all $(v,t)\in K\times[0,S]$, and such that $\lim_{s\downarrow t}\boldsymbol{Z}'_s(\omega)=\boldsymbol{Z}'_t(\omega)$ and $\lim_{s\uparrow t}\boldsymbol{Z}'_s(\omega)$ exists with respect to $\|\cdot\|_\infty$ for all ω. Furthermore, the map $v\mapsto Z'_{v,t}$ from K into $\mathbb{R}$ is continuous for all $t\in[0,S]$. In particular, Z' has t-càdlàg sample paths.*

The desired t-càdlàg version will be an extension of the field $(\boldsymbol{Z}_t)_{t\in\tilde{S}}$ studied in Lemma 5.3. Thus, $(\boldsymbol{Z}_t)_{t\in\tilde{S}}$ will still be a version chosen such that each $\boldsymbol{Z}_t$ is a continuous random field. The result relies on a sequence of lemmas that are shown using an adaption of the ideas of [21, Theorems 11.1 & 11.5] and [21, Lemmas 11.2-11.4] for Lévy processes on $\mathbb{R}$. Lemmas 5.5 and 5.6 are shown using similar techniques for the Lévy process $(\boldsymbol{Z}_t)_{t\in\tilde{S}}$, and therefore we omit the proofs here, and refer to the Appendix (page 25) for completeness.

For the statement and proof of these lemmas, the following notation will be useful. We say that $\boldsymbol{Z}(\omega)$ has ϵ-oscillation n times in a set $M \subseteq \mathbb{Q}\cap[0,\infty)$ if there exist $t_0 < t_1 < \cdots < t_n \in M$ such that

$$\|\boldsymbol{Z}_{t_j}(\omega) - \boldsymbol{Z}_{t_{j-1}}(\omega)\|_\infty = \sup_{v\in K} |Z_{v,t_j}(\omega) - Z_{v,t_{j-1}}(\omega)| > \epsilon$$

for all $j = 1,\dots,n$. We say that $\boldsymbol{Z}(\omega)$ has ϵ-oscillation infinitely often in M if it has ϵ-oscillation n times in M for any $n \in \mathbb{N}$. Consider Ω_1 given by

$$\begin{aligned}\Omega_1 = \{\omega \in \Omega \mid & \lim_{s\in\mathbb{Q}, s\downarrow t} \boldsymbol{Z}_s(\omega) \text{ exists with respect to } \|\cdot\|_\infty \text{ for all } t\in[0,S] \text{ and} \\ & \lim_{s\in\mathbb{Q}, s\uparrow t} \boldsymbol{Z}_s(\omega) \text{ exists with respect to } \|\cdot\|_\infty \text{ for all } t\in[0,S]\}.\end{aligned}$$

Furthermore define the sets

$$A_k = \{\omega\in\Omega \mid \boldsymbol{Z}(\omega) \text{ does not have } \tfrac{1}{k}\text{-oscillation infinitely often in } \tilde{S}\},$$

and from these define $\Omega_1' = \cap_{k\in\mathbb{N}} A_k$. Each A_k is measurable as each $\boldsymbol{Z}_t$ is continuous on K for $t\in\tilde{S}$, such that $\|\cdot\|_\infty = \sup_{v\in K}|\cdot| = \sup_{v\in\tilde{K}}|\cdot|$.

Lemma 5.5. $\Omega_1' \subseteq \Omega_1$.

Lemma 5.6. $\mathbb{P}(\Omega_1') = 1$.

Having established that the $\lim_{s\in\mathbb{Q},s\downarrow t}\boldsymbol{Z}_s$ and $\lim_{s\in\mathbb{Q},s\uparrow t}\boldsymbol{Z}_s$ exist almost surely, we now prove the main result Theorem 5.4 on the existence of a t-càdlàg version of Z.

Proof of Theorem 5.4. we have $\mathbb{P}(\Omega_1') = 1$ by Lemma 5.6. For all $t\in[0,S]$, define $\boldsymbol{Z}_t'(\omega) = \mathbf{1}_{\Omega_1'}(\omega)(\lim_{s\in\mathbb{Q},s\downarrow t}\boldsymbol{Z}_s(\omega))$, where the limit is with respect to $\|\cdot\|_\infty$, and exists according to Lemma 5.5. The càdlàg-assertion is trivially true for $\omega\notin\Omega_1'$. Now consider $\omega\in\Omega_1'$ but suppress ω in ease of notation. By definition of $\boldsymbol{Z}_t'$,

$$\forall\epsilon>0\ \exists N\ \forall s\in(t,t+\tfrac{1}{N})\cap\mathbb{Q}:\ \|\boldsymbol{Z}_t' - \boldsymbol{Z}_s\|_\infty < \epsilon. \tag{5.7}$$

Let (t_n) be any sequence satisfying $t_n\downarrow t$. Fix $\epsilon>0$, and let $N\in\mathbb{N}$ satisfy (5.7) with the bound $\frac{\epsilon}{2}$. There is $n_0\in\mathbb{N}$ such that $|t_n - t| < \frac{1}{N}$ for all $n\geq n_0$. Now fix such n. By another application of (5.7) there exist N_n such that $t_n + \frac{1}{N_n} \leq t + \frac{1}{N}$ and $\|\boldsymbol{Z}_{t_n}' - \boldsymbol{Z}_s\|_\infty < \frac{\epsilon}{2}$ for all $s\in(t_n, t_n+\frac{1}{N_n})\cap\mathbb{Q}$. For any of those s we in particular find that

$$\|\boldsymbol{Z}_{t_n}' - \boldsymbol{Z}_t'\|_\infty \leq \|\boldsymbol{Z}_{t_n}' - \boldsymbol{Z}_s\|_\infty + \|\boldsymbol{Z}_t' - \boldsymbol{Z}_s\|_\infty < \epsilon.$$

As this is true for all $n\geq n_0$ we conclude that $\boldsymbol{Z}' = (\boldsymbol{Z}_t')_{t\in[0,S]}$ is right-continuous with respect to $\|\cdot\|_\infty$. Similar arguments show that $\boldsymbol{Z}'$ has limits from the left and that the limits are unique. The mapping $v\mapsto Z_{v,t}'$ is continuous because the space $(\mathcal{C}(K,\mathbb{R}), \|\cdot\|_\infty)$ is complete, and $\boldsymbol{Z}_t'$ is defined as the limit of such functions.

We now argue that $\boldsymbol{Z}'$ is indeed a version $\boldsymbol{Z}$. If $(t_n) \subset \tilde{S}$ with $t_n \downarrow t$ then $Z_{v,t_n} \xrightarrow{\mathbb{P}} Z_{v,t}$ for all $v \in K$ as $(Z_{v,t})_{t\in[0,S]}$ is a Lévy process in law and thus especially stochastically continuous. Since $\mathbb{P}(\Omega_1') = 1$ we have $Z_{v,t_n} \to Z'_{v,t}$ almost surely, and by uniqueness of limits we conclude that $\mathbb{P}(Z'_{v,t} = Z_{v,t}) = 1$ for all $(v,t) \in K \times [0,S]$.

It remains to show that Z' is t-càdlàg. Since we for given $(v,t) \in K \times [0,S]$ can write

$$|Z'_{v,t} - Z'_{u,s}| \le |Z'_{v,t} - Z'_{u,t}| + \|\boldsymbol{Z}'_t - \boldsymbol{Z}'_s\|_\infty$$

for any choice of $(u,s) \in K \times [0,S]$, we conclude that $\lim_{(u,s)\to(v,t^+)} Z'_{u,s} = Z'_{v,t}$, from the continuity of $\boldsymbol{Z}'_t$ and the uniform càdlàg property of $(\boldsymbol{Z}'_t)_{t\in[0,S]}$. Similar arguments give that the limit $\lim_{(u,s)\to(v,t^-)} Z'_{u,s}$ exists in $\mathbb{R}$ and that it is unique. □

Theorem 5.7 below is stated under Assumption 2.1. However, in order to establish t-càdlàg sample paths, the milder Assumption 5.1 would have been sufficient.

Theorem 5.7. *Let the field $X = (X_{v,t})_{(v,t)\in B'\times T'}$ be given by (2.9), where the Lévy basis M satisfies Assumption 2.1 and the kernel function f satisfies Assumption 2.3. Then X has a version with t-càdlàg sample paths.*

Proof. We decompose the field $(X_{v,t})$ as

$$\begin{aligned} X_{v,t} = &\int_{\mathbb{R}^d\times[0,t]} f(|v-u|, t-s) - f(|v-u|, 0) M(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_{\mathbb{R}^d\times(-\infty,0)} f(|v-u|, t-s) M(\mathrm{d}s, \mathrm{d}u) + \int_{\mathbb{R}^d\times[0,t]} f(|v-u|, 0) M(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

By Theorem 5.4, choosing $g(\cdot) = f(\cdot, 0)$, the third term has a t-càdlàg version. Due to continuity of the integrands, the first and second terms have continuous versions by arguments similar to those in the proof of Theorem 5.2: Defining the continuous function $\phi : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$\phi(x,y) = \mathbf{1}_{[0,\infty)}(y)\big(f(x,y) - f(x,0)\big),$$

the first term above reads

$$Y'_{v,t} = \int_{\mathbb{R}^d\times[0,\infty)} \phi(|v-u|, t-s) M(\mathrm{d}u, \mathrm{d}s) = Y_{v,t} + y_{v,t},$$

where $y_{v,t} = \mathbb{E}Y'_{v,t}$ and $Y_{v,t} = Y'_{v,t} - y_{v,t}$. The field $(Y_{v,t})$ is continuous by previous arguments replacing the assumptions (5.3) and (5.4) by the conditions

$$\int_{\mathbb{R}^d} \int_0^\infty \big|\phi(|u|, T+\ell-s)\big| \mathrm{d}s\mathrm{d}u < \infty \tag{5.8}$$

and the Lipschitz continuity of ϕ

$$\big|\phi(|u_1|, t_1 - s) - \phi(|u_2|, t_2 - s)\big| \le C\big|(u_1 - u_2, t_1 - t_2)\big|$$

for all $u_1, u_2 \in \mathbb{R}^d$ and $t_1, t_2 \in T'$. These conditions are easily seen to be satisfied under Assumption 2.3. As

$$y_{v,t} = y_{0,t} = \mathbb{E}[M'] \int_{\mathbb{R}^d} \int_0^\infty \phi(|u|, t-s) \mathrm{d}s\mathrm{d}u < \infty$$

the deterministic field $(y_{v,t})$ is continuous by a dominated convergence argument using (5.8). The continuity of the second term follows similarly. □

Appendix: Supplement

A Proofs of Section 3

Proof of Lemma 3.9. For sufficiently large x we find that

$$\begin{aligned}\mathbb{P}(Z\phi(U,S) > x) &= \frac{1}{\nu(A)} F\Big(\Big\{(u,s,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_+ \,:\, z\phi(u,s) > x\Big\}\Big) \\ &= \frac{1}{\nu(A)} \int_{B'\times T'} L\Big(\frac{x}{c}\Big) \exp\Big(-\beta \frac{x}{c}\Big) m(\mathrm{d}u, \mathrm{d}s) \\ &\quad + \frac{1}{\nu(A)} \int_{(B'\times T')^c} L\Big(\frac{x}{\phi(u,s)}\Big) \exp\Big(-\beta \frac{x}{\phi(u,s)}\Big) m(\mathrm{d}u, \mathrm{d}s),\end{aligned}$$

where the first term equals $L(x/c)\exp(-\beta x/c)$ times the desired limit. The result follows when the latter integral is shown to be of order $o(L(x/c)\exp(-\beta x/c))$, as $x \to \infty$. Let $h(u,s;x)$ denote the integrand. For all $(u,s) \notin B' \times T'$ we have $\phi(u,s) < c$. Combined with (2.5), this implies the existence of $\gamma > 0$ and $C > 0$ such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)} \leq C \exp(-\gamma x)$$

for sufficiently large x. Thus, the integrand $h(u,s;x)$ is $o(L(x/c)\exp(-\beta x/c))$ at infinity. By dominated convergence, the integral is of order $o(L(x/c)\exp(-\beta x/c))$ if we can find an integrable function $g : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)} \leq g(u,s)$$

for all $(u,s) \in \mathbb{R}^d \times \mathbb{R}$. Returning to (2.6) we see that for all $0 < \gamma < \beta/c$ there is $C > 0$ and x_0 such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)} \leq C \exp\Big(-x_0(\beta - \gamma c)\Big(\frac{1}{\phi(u,s)} - \frac{1}{c}\Big)\Big) \tag{A.1}$$

for all $x \geq x_0$. Since $B' \times T'$ is bounded, we can choose $b \in (0,\infty)$ such that $B' \times T' \subseteq C_b(0)$, where $C_b(0)$ is the $(d+1)$-dimensional ball with radius b and center $0 \in \mathbb{R}^d \times \mathbb{R}$. Turning to (3.7) and choosing b sufficiently large, we conclude that the right hand side of (A.1) is integrable over the complement of $C_b(0)$. This shows the desired order of convergence.

From [13, Lemma 2.4(i)] the distribution of $Z\phi(U,S)$ is convolution equivalent with index β/c. The integrability result follows from [13, Corollary 2.1(ii)]. $\square$

Corollary A.1. *If $V^1, V^2, \ldots$ are i.i.d. fields with distribution ν_1, then*

$$\mathbb{E}\Big[\exp\Big(\beta \sup_{u\in B} \sup_{s\in[0,T]} \lambda_{u,s}((V^1_{v,t} + \cdots + V^n_{v,t})_{(v,t)})\Big)\Big] < \infty$$

for all $n \in \mathbb{N}$.

Proof. Because each V^i can be represented by $(Z^i f(|v-U^i|, t-S^i))_{(v,t)\in B'\times T'}$, the result follows from (3.9) and (3.11). $\square$

Proof of Theorem 3.10. We will show the claim by induction over n: We note that the case $n=1$ follows easily from Theorem 3.7. Now assume that the result holds true for some $n\in\mathbb{N}$ and let for convenience $V^{*n} = V^1+\cdots+V^n$. Also, let $y^* = \sup_{(v,t)\in B'\times T'} y_{v,t}$. Using (3.8) and the representation $V^i = Z^i f(|v-U^i|, t-S^i)$, we find

$$
\begin{aligned}
&\mathbb{P}(\Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x) \\
&\quad \le \mathbb{P}\Big(\sum_{i=1}^n Z^i\phi(U^i,S^i) > \frac{x-y^*}{2}, Z^{n+1}\phi(U^{n+1},S^{n+1}) > \frac{x-y^*}{2}, \\
&\qquad\quad \Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x\Big) \\
&\qquad + \mathbb{P}\Big(\sum_{i=1}^n Z^i\phi(U^i,S^i) \le \frac{x-y^*}{2}, \Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x\Big) \\
&\qquad + \mathbb{P}\Big(Z^{n+1}\phi(U^{n+1},S^{n+1}) \le \frac{x-y^*}{2}, \Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x\Big).
\end{aligned}
\tag{A.2}
$$

The first term in (A.2) is bounded from above by

$$
\mathbb{P}\Big(\sum_{i=1}^n Z^i\phi(U^i,S^i) > \frac{x-y^*}{2}\Big)\mathbb{P}\Big(Z^{n+1}\phi(U^{n+1},S^{n+1}) > \frac{x-y^*}{2}\Big).
$$

In Lemma 3.9 we showed that the distribution of $Z^i\phi(U^i,S^i)$ is convolution equivalent with index β/c, and hence, from [6, Corollary 2.11] and (3.10), both factors are asymptotically equivalent to $\rho_1((x/(2c),\infty))$ as $x\to\infty$. Following the proof of [5, Lemma 2] we see that the product is $o((\rho_1 * \rho_1)((x/c,\infty)))$, and as such the first term in (A.2) is $o(\rho_1((x/c,\infty)))$ due to the convolution equivalence of ρ_1. By Theorem 3.7 it is of order $o(\mathbb{P}(\Psi(V^1_{v,t}) > x))$ as $x\to\infty$.

By independence, the two remaining terms in (A.2) divided by $\mathbb{P}(\Psi(V^1_{v,t}) > x)$ are

$$
\begin{aligned}
&\int_{C_x} \frac{\mathbb{P}(\Psi(\sum_{i=1}^n z^i f(|v-u^i|, t-s^i) + V^{n+1}_{v,t} + y_{v,t}) > x)}{\mathbb{P}(\Psi(V^1_{v,t}) > x)} \\
&\qquad \times F_1^{\otimes n}(\mathrm{d}(u^1,s^1,z^1;\dots;u^n,s^n,z^n)) \\
&+ \int_{\tilde{C}_x} \frac{\mathbb{P}(\Psi(V^{*n}_{v,t} + z^1 f(|v-u^1|, t-s^1) + y_{v,t}) > x)}{\mathbb{P}(\Psi(V^1_{v,t}) > x)} F_1(\mathrm{d}(u^1,s^1,z^1)),
\end{aligned}
\tag{A.3}
$$

where $F_1^{\otimes n}$ is the n-fold product measure of F_1 and

$$
\begin{aligned}
C_x &= \Big\{(u^1,s^1,z^1;\dots;u^n,s^n,z^n) : \sum_{i=1}^n z^i\phi(u^i,s^i) \le \frac{x-y^*}{2}\Big\}, \\
\tilde{C}_x &= \Big\{(u^1,s^1,z^1) : z^1\phi(u^1,s^1) \le \frac{x-y^*}{2}\Big\}.
\end{aligned}
$$

Above we used the representation $V^i = Z^i f(|v-U^i|, t-S^i)$ again. By Theorem 3.7 and the induction assumption, the integrands of (A.3) have the following limits as $x \to \infty$,

$$
\begin{aligned}
&f_1(u^1, s^1, z^1; \dots; u^n, s^n, z^n) \\
&= \frac{\int_B \int_0^T \exp\left(\beta\lambda_{u,s}(\sum_{i=1}^n z^i f(|v-u^i|, t-s^i) + y_{v,t})\right) \mathrm{d}s\mathrm{d}u}{m(B\times[0,T])}, \\
&f_2(u^1, s^1, z^1) \\
&= \frac{n\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(V^1_{v,t} + \cdots + V^{n-1}_{v,t} + z^1 f(|v-u^1|, t-s^1) + y_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u}{m(B\times[0,T])},
\end{aligned}
$$

respectively. When integrated with respect to the relevant measures we find

$$
\begin{aligned}
&\int f_1(u^1, s^1, z^1; \dots; u^n, s^n, z^n) F_1^{\otimes n}(\mathrm{d}(u^1, s^1, z^1; \dots; u^n, s^n, z^n)) \\
&\qquad + \int f_2(u^1, s^1, z^1) F_1(\mathrm{d}(u^1, s^1, z^1)) \\
&= \frac{n+1}{m(B\times[0,T])} \int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(V^1_{v,t} + \cdots + V^n_{v,t} + y_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u,
\end{aligned}
$$

which is the desired expression. To show convergence of the integrals in (A.3), using Fatou's lemma, it suffices to find integrable functions $g_1(u^1, s^1, z^1; \dots; u^n, s^n, z^n; x)$ and $g_2(u^1, s^1, z^1; x)$ that are upper bounds of the integrands such that their limits exist when $x \to \infty$ and such that

$$
\begin{aligned}
&\int g_1(u^1, s^1, z^1; \dots; u^n, s^n, z^n; x) F_1^{\otimes n}(\mathrm{d}(u^1, s^1, z^1; \dots; u^n, s^n, z^n)) \\
&\qquad + \int g_2(u^1, s^1, z^1; x) F_1(\mathrm{d}(u^1, s^1, z^1)) \\
&\to \int \lim_{x\to\infty} g_1(u^1, s^1, z^1; \dots; u^n, s^n, z^n; x) F_1^{\otimes n}(\mathrm{d}(u^1, s^1, z^1; \dots; u^n, s^n, z^n)) \\
&\qquad + \int \lim_{x\to\infty} g_2(u^1, s^1, z^1; x) F_1(\mathrm{d}(u^1, s^1, z^1))
\end{aligned}
$$

as $x \to \infty$. Using (3.8) and properties of Ψ, we can choose the functions

$$
g_1(u^1, s^1, z^1; \dots; u^n, s^n, z^n; x) = \mathbf{1}_{C_x} \frac{\mathbb{P}(Z^1\phi(U^1, Z^1) > x - y^* - \sum_{i=1}^n z^i \phi(u^i, s^i))}{\mathbb{P}(\Psi(V^1_{v,t}) > x)}
$$

and

$$
g_2(u^1, s^1, z^1; x) = \mathbf{1}_{\tilde{C}_x} \frac{\mathbb{P}(\sum_{i=1}^n Z^i \phi(U^i, Z^i) > x - y^* - z^1\phi(u^1, s^1))}{\mathbb{P}(\Psi(V^1_{v,t}) > x)}.
$$

From Theorem 3.7 and (3.10) we find that

$$
\mathbb{P}(Z^1\phi(U^1, S^1) > x) \sim \frac{m(B'\times T')}{m(B\times[0,T])} \mathbb{P}(\Psi(V^1_{v,t}) > x) \tag{A.4}
$$

as $x \to \infty$. The fact that the distribution of $Z^1\phi(U^1, S^1)$ is convolution equivalent and in particular has an exponential tail implies

$$g_1(u^1, s^1, z^1; \ldots; u^n, s^n, z^n; x) \to \frac{m(B' \times T')}{m(B \times [0,T])} \exp\Big(\frac{\beta}{c}\Big(y^* + \sum_{i=1}^{n} z^i\phi(u^i, s^i)\Big)\Big)$$

as $x \to \infty$. Similarly, (A.4) and an application of [6, Corollary 2.11] gives

$$\begin{aligned} &g_2(u^1, s^1, z^1; x) \\ &\quad\to \frac{m(B' \times T')}{m(B \times [0,T])} n \exp\Big(\frac{\beta}{c}(y^* + z^1\phi(u^1, s^1))\Big)\Big(\mathbb{E}\exp\Big(\frac{\beta}{c} Z^1\phi(U^1, S^1)\Big)\Big)^{n-1} \end{aligned}$$

as $x \to \infty$, and we conclude that

$$\begin{aligned} &\int \lim_{x\to\infty} g_1(u^1, s^1, z^1; \ldots; u^n, s^n, z^n; x) F_1^{\otimes n}(\mathrm{d}(u^1, s^1, z^1; \ldots; u^n, s^n, z^n)) \\ &\qquad + \int \lim_{x\to\infty} g_2(u^1, s^1, z^1; x) F_1(\mathrm{d}(u^1, s^1, z^1)) \\ =&\frac{m(B' \times T')}{m(B \times [0,T])}(n+1)\exp(\beta y^*/c)\Big(\mathbb{E}\exp\Big(\frac{\beta}{c} Z^1\phi(U^1, S^1)\Big)\Big)^{n}. \end{aligned} \tag{A.5}$$

For notational convenience, we let μ denote the distribution of $Z^i\phi(U^i, S^i)$. Then, again by [6, Corollary 2.11] and (A.4), (A.5) equals

$$\lim_{x\to\infty} \frac{m(B' \times T')}{m(B \times [0,T])} \frac{\mu^{*(n+1)}((x - y^*, \infty))}{\mu((x,\infty))} = \lim_{x\to\infty} \frac{\mu^{*(n+1)}((x - y^*, \infty))}{\mathbb{P}(\Psi(V^1_{v,t}) > x)}. \tag{A.6}$$

Furthermore, we see

$$\begin{aligned} &\mathbb{P}(\Psi(V^1_{v,t}) > x)\Big(\int g_1(z^1; \ldots; z^n; x)\mu^{\otimes n}(\mathrm{d}(z^1; \ldots; z^n)) + \int g_2(z; x)\mu(\mathrm{d}z)\Big) \\ &= \int_0^{(x-y^*)/2} \mu((x - y^* - z, \infty))\mu^{*n}(\mathrm{d}z) + \int_0^{(x-y^*)/2} \mu^{*n}((x - y^* - z, \infty))\mu(\mathrm{d}z). \end{aligned}$$

Since, in particular, the tails of μ and μ^{*n} are exponential with index β/c, we see from [5, Lemma 2] that the sum of integrals is asymptotically equivalent to $\mu^{*(n+1)}((x-y^*, \infty))$. Returning to (A.6) concludes the proof. □

Before proving the theorem on the extremal behaviour of X^1, we need the following lemma for a dominated convergence argument.

Lemma A.2. *Let $V^1, V^2, \ldots$ be i.i.d. fields with distribution ν_1, and let (U, S, Z) be distributed according to F_1. There exist a constant K such that*

$$\mathbb{P}(\Psi(V^1_{v,t} + \cdots + V^n_{v,t}) > x) \leq K^n \mathbb{P}(Z\phi(U, S) > x)$$

for all $n \in \mathbb{N}$ and all $x \geq 0$.

Proof. By Lemma 3.9 the distribution of $Z\phi(U,S)$ is convolution equivalent, and it follows from [6, Lemma 2.8] that there is a constant K such that

$$\mathbb{P}\Big(\sum_{i=1}^{n} Z^i\phi(U^i,S^i) > x\Big) \le K^n \mathbb{P}(Z\phi(U,S) > x),$$

for i.i.d. variables $(U^1,S^1,Z^1),(U^2,S^2,Z^2),\dots$ with distribution F_1. The result follows directly from (3.8). □

Proof of Theorem 3.11. From (3.9) and the representation $V^i = (Z^i f(|v-U^i|, t-S^i))_{(v,t)}$, we see that

$$\mathbb{E}\left[\exp\big(\beta\lambda_{u,s}(X^1_{v,t})\big)\right] \le \exp\big(\nu(A)\big(\mathbb{E}[\exp(\tfrac{\beta}{c}Z\phi(U,S)] - 1\big)\big)\,.$$

The first claim now follows from (3.11).

For the limit result, we find by independence and Lemma A.2,

$$\begin{aligned}
&\mathbb{P}(\Psi(X^1_{v,t} + y_{v,t}) > x)\\
&= \mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^n}{n!}\mathbb{P}(\Psi(V^1_{v,t} + \cdots + V^n_{v,t} + y_{v,t}) > x)\\
&\le \mathbb{P}(Z\phi(U,S) > x - y^*)\mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^n K^n}{n!},
\end{aligned}$$

where $y^* = \sup_{(v,t)} y_{v,t}$ and $\mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^nK^n}{n!} < \infty$. With the convention that $V^1_{v,t} + \cdots + V^{n-1}_{v,t} = 0$ for $n = 1$, by dominated convergence, Theorems 3.7 and 3.10 and Lemma 3.9 yield

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\mathbb{P}(\Psi(X^1_{v,t} + y_{v,t}) > x)}{L(x/c)\exp(-\beta x/c)}\\
&= \frac{n}{\nu(A)}\mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^n}{n!}\int_B\int_0^T \mathbb{E}\left[\mathrm{e}^{\beta\lambda_{u,s}(V^1_{v,t}+\cdots+V^{n-1}_{v,t}+y_{v,t})}\right]\mathrm{d}s\mathrm{d}u\\
&= \mathrm{e}^{-\nu(A)}\sum_{n=0}^{\infty}\frac{\nu(A)^n}{n!}\int_B\int_0^T \mathbb{E}\left[\mathrm{e}^{\beta\lambda_{u,s}(V^1_{v,t}+\cdots+V^{n}_{v,t}+y_{v,t})}\right]\mathrm{d}s\mathrm{d}u\\
&= \int_B\int_0^T \mathbb{E}\left[\mathrm{e}^{\beta\lambda_{u,s}(X^1_{v,t}+y_{v,t})}\right]\mathrm{d}s\mathrm{d}u.
\end{aligned}$$

This concludes the proof. □

Proof of Lemma 3.12. First we show that

$$\mathbb{E}\exp(\gamma \sup_{(v,t)\in B'\times T'} X^2_{v,t}) < \infty \tag{A.7}$$

for all $\gamma > 0$. Applying arguments as in Section 2, we write X^2 as the independent sum $X^2_{v,t} = Y^1_{v,t} + Y^2_{v,t}$. Here Y^1 is a compound Poisson sum

$$Y^1_{v,t} = \sum_{k=1}^{M} J^k_{v,t}$$

with finite intensity $\nu(A^c \cap D) < \infty$ and jump distribution $\nu_2 = \nu_{A^c \cap D}/\nu(A^c \cap D)$, where $D = \{z \in \mathbb{R}^{\mathbb{K}} : \inf_{(v,t)\in\mathbb{K}} z_{v,t} < -1\}$. Furthermore, Y^2 is infinitely divisible with Lévy measure $\nu_{A^c\cap D^c}$, the restriction of ν to the set $A^c \cap D^c = \{z \in \mathbb{R}^{\mathbb{K}} : \sup_{(v,t)\in\mathbb{K}} |z_{v,t}| \leq 1\}$. By arguments as before, both fields have t-càdlàg extensions to $B' \times T'$. For each k, $J^k_{v,t} \leq 0$ for all $(v,t) \in B' \times T'$ almost surely, and in particular $\mathbb{E}\exp(\gamma \sup_{(v,t)\in B'\times T'} Y^1_{v,t}) < \infty$ for all $\gamma > 0$. As $(Y^2_{v,t})_{(v,t)\in B'\times T'}$ is t-càdlàg on the compact set $B' \times T'$, we find that $\mathbb{P}(\sup_{(v,t)\in B'\times T'} |Y^2_{v,t}| < \infty) = 1$. Since also $\nu_{A^c\cap D^c}(\{z \in \mathbb{R}^{\mathbb{K}} : \sup_{(v,t)\in\mathbb{K}} |z_{v,t}| > 1\}) = 0$, we obtain from [4, Lemma 2.1] that $\mathbb{E}\exp(\gamma \sup_{(v,t)\in B'\times T'} |Y^2_{v,t}|) < \infty$ for all $\gamma > 0$, which yields the claim (A.7).

Appealing to properties of $\lambda_{u,s}$ we find that

$$\lambda_{u,s}(X_{v,t}) \leq \lambda_{u,s}\big(X^1_{v,t} + \sup_{(v,t)\in B'\times T'} X^2_{v,t}\big) = \lambda_{u,s}(X^1_{v,t}) + \frac{\sup_{(v,t)\in B'\times T'} X^2_{v,t}}{c}.$$

The assertion now follows from (A.7) and the first claim of Theorem 3.11. □

Proof of Theorem 3.13. Let π be the distribution of $(X^2_{v,t})_{(v,t)\in B'\times T'}$. Conditioning on $(X^2_{v,t})_{(v,t)} = (y_{v,t})_{(v,t)}$ we find by independence that

$$\frac{\mathbb{P}(\Psi(X_{v,t}) > x)}{\mathbb{P}(\Psi(X^1_{v,t}) > x)} = \int \frac{\mathbb{P}(\Psi(X^1_{v,t} + y_{v,t}) > x)}{\mathbb{P}(\Psi(X^1_{v,t}) > x)} \pi(\mathrm{d}y) = \int f(y;x)\pi(\mathrm{d}y)$$

with $f(y;x) = \mathbb{P}(\Psi(X^1_{v,t} + y_{v,t}) > x)/\mathbb{P}(\Psi(X^1_{v,t}) > x)$, which, according to Theorem 3.11, satisfies

$$f(y;x) \to f(y) = \frac{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X^1_{v,t} + y_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u}{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X^1_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u}$$

as $x \to \infty$. By another application of Theorem 3.11 and since

$$\int f(y)\pi(\mathrm{d}y) = \frac{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u}{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}(X^1_{v,t})\right)\right] \mathrm{d}s\mathrm{d}u},$$

the proof is completed if we can find non-negative and integrable functions $g(y;x)$ and $g(y) = \lim_{x\to\infty} g(y;x)$ such that $f(y;x) \leq g(y;x)$ and such that

$$\int g(y;x)\pi(\mathrm{d}y) \to \int g(y)\pi(\mathrm{d}y)$$

as $x \to \infty$. With $y^* = \sup_{(v,t)\in B'\times T'} y_{v,t}$ we use the function

$$g(y;x) = \mathbb{P}(\Psi(X^1_{v,t}) + y^* > x)/\mathbb{P}(\Psi(X^1_{v,t}) > x)$$

which, according to properties of $\lambda_{u,s}$ and Theorem 3.11, satisfies

$$g(y;x) \to g(y) = \exp(\beta y^*/c)$$

as $x \to \infty$. From [13, Lemma 2.4(i)] and Theorem 3.11 the distribution of $\Psi(X^1_{v,t})$ is convolution equivalent with index β/c. Now let G and H denote the distributions of $\Psi(X^1_{v,t})$ and $\sup_{(v,t)\in B'\times T'} X^2_{v,t}$, respectively. If $\overline{H}(x) = o(\overline{G}(x))$, $x\to\infty$, it follows from the integrability statement (A.7) and [13, Lemma 2.1] that

$$\int g(y;x)\pi(\mathrm{d}y) = \frac{\mathbb{P}(\Psi(X^1_{v,t}) + \sup_{(v,t)\in B'\times T'} X^2_{v,t} > x)}{\mathbb{P}(\Psi(X^1_{v,t}) > x)}$$
$$\to \mathbb{E}\exp\Big(\frac{\beta}{c}\sup_{(v,t)\in B'\times T'} X^2_{v,t}\Big) = \int g(y)\pi(\mathrm{d}y)$$

as $x\to\infty$. From (A.7) we find that $\lim_{x\to\infty} \mathrm{e}^{\gamma x}\mathbb{P}(\sup_{(v,t)\in B'\times T'} X^2_{v,t} > x) = 0$ for all $\gamma > 0$. Combined with the convolution equivalence of the distribution of $\Psi(X^1_{v,t})$, this yields $\overline{H}(x) = o(\overline{G}(x))$ and the claim follows. □

B Proofs of Section 5

Proof of Lemma 5.5. Let $\omega\in\Omega_1'$ and $(s_n)\subset\tilde{S}$ such that $s_n\downarrow t\in[0,S]$. For all $k\in\mathbb{N}$ there exists $N\in\mathbb{N}$ such that

$$\|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{s_N}(\omega)\|_\infty \le \tfrac{1}{k} \qquad \text{for all } n\ge N. \tag{B.1}$$

This is seen by contradiction as follows: Assume that for any $N\in\mathbb{N}$ there exists $n\ge N$ such that

$$\|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{s_N}(\omega)\|_\infty > \tfrac{1}{k}.$$

Now fix $p\in\mathbb{N}$. By this there exist $n_0<n_1<n_2<\cdots<n_p$ such that

$$\|\boldsymbol{Z}_{s_{n_j}}(\omega) - \boldsymbol{Z}_{s_{n_{j-1}}}(\omega)\|_\infty > \tfrac{1}{k} \qquad \text{for } j=1,\dots,p$$

and we conclude that $\boldsymbol{Z}(\omega)$ has $\frac{1}{k}$-oscillation p times in $\tilde{S}$ for any p. Hence $\omega\in A_k^c$, which is a contradiction. From (B.1) and the fact that the metric space $(\mathcal{C}(K,\mathbb{R}),\|\cdot\|_\infty)$ is complete, we know that $\lim_{n\to\infty}\boldsymbol{Z}_{s_n}(\omega)$ exists with respect to $\|\cdot\|_\infty$ as a continuous function on K. To show uniqueness of the limit, let $(t_n)\subset\tilde{S}$ be another sequence such that $t_n\downarrow t$. Then $\lim_{n\to\infty}\boldsymbol{Z}_{s_n}(\omega) = \lim_{n\to\infty}\boldsymbol{Z}_{t_n}(\omega)$: Let (r_n) be the union of (s_n) and (t_n) ordered such that $r_n\downarrow t$. Then similarly for any $\epsilon>0$ there is N' such that

$$\|\boldsymbol{Z}_{r_n}(\omega) - \boldsymbol{Z}_{r_{N'}}(\omega)\|_\infty < \tfrac{\epsilon}{2} \qquad \text{for } n\ge N'.$$

Also there is $N\in\mathbb{N}$ such that $(s_n)_{n\ge N}, (t_n)_{n\ge N}\subseteq (r_n)_{n\ge N'}$, and hence

$$\|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{t_n}(\omega)\|_\infty \le \|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{r_{N'}}(\omega)\|_\infty + \|\boldsymbol{Z}_{t_n}(\omega) - \boldsymbol{Z}_{r_{N'}}(\omega)\|_\infty < \epsilon$$

for all $n\ge N$. Thus, the limit $\lim_{s\in\mathbb{Q},s\downarrow t}\boldsymbol{Z}_s(\omega)$ exists uniquely with respect to $\|\cdot\|_\infty$. Similarly for $\lim_{s\in\mathbb{Q},s\uparrow t}\boldsymbol{Z}_s(\omega)$. □

We let

$$B(p,\epsilon,D) = \{\omega\in\Omega\ :\ \boldsymbol{Z}(\omega) \text{ has } \epsilon\text{-oscillation } p \text{ times in } D\},$$

with $D\subseteq\mathbb{Q}\cap[0,\infty)$, and

$$\alpha_\epsilon(r) = \sup\{\mathbb{P}(\|\boldsymbol{Z}_t\|_\infty\ge\epsilon)\ :|\ t\in[0,r]\cap\mathbb{Q}\}.$$

Note that a direct consequence of the stochastic continuity from Lemma 5.3 is that $\alpha_\epsilon(r)\to 0$ as $r\to 0$ for all $\epsilon>0$.

Lemma B.1. *Let p be a positive integer, $D = \{t_1, \dots, t_n\} \subseteq \mathbb{Q} \cap [0, \infty)$ and $u, r \in \mathbb{Q}$ such that $0 \le u \le t_1 < \dots < t_n \le r$. Then $\mathbb{P}(B(p, 4\epsilon, D)) \le (2\alpha_\epsilon(r-u))^p$.*

Proof. We will show the statement by induction in p. For this, define

$$\begin{aligned} C_k &= \{\|\boldsymbol{Z}_{t_j} - \boldsymbol{Z}_u\|_\infty \le 2\epsilon,\ j = 1, \dots, k-1,\ \|\boldsymbol{Z}_{t_k} - \boldsymbol{Z}_u\|_\infty > 2\epsilon\}\ , \\ D_k &= \{\|\boldsymbol{Z}_{t_k} - \boldsymbol{Z}_r\|_\infty > \epsilon\} \end{aligned}$$

and note that $C_1, \dots, C_n$ are disjoint and

$$\begin{aligned} B(1, 4\epsilon, D) &\subseteq \bigcup_{k=1}^n \{\|\boldsymbol{Z}_{t_k} - \boldsymbol{Z}_u\|_\infty > 2\epsilon\} = \bigcup_{k=1}^n C_k \\ &= \bigcup_{k=1}^n (C_k \cap D_k^c) \cup (C_k \cap D_k) \\ &\subseteq \{\|\boldsymbol{Z}_r - \boldsymbol{Z}_u\|_\infty \ge \epsilon\} \cup \bigcup_{k=1}^n (C_k \cap D_k). \end{aligned}$$

By the Lévy properties in Lemma 5.3 we have $\mathbb{P}(\|\boldsymbol{Z}_r - \boldsymbol{Z}_u\|_\infty \ge \epsilon) \le \alpha_\epsilon(r-u)$ and furthermore that $\mathbb{P}(C_k \cap D_k) = \mathbb{P}(C_k)\mathbb{P}(D_k) \le \mathbb{P}(C_k)\alpha_\epsilon(r-u)$. The fact that $C_1, \dots, C_n$ are disjoint then implies

$$\mathbb{P}(B(1, 4\epsilon, D)) \le \mathbb{P}(\|\boldsymbol{Z}_r - \boldsymbol{Z}_u\|_\infty \ge \epsilon) + \sum_{k=1}^n \mathbb{P}(C_k \cap D_k) \le 2\alpha_\epsilon(r-u),$$

which is the desired expression for $p = 1$. We now assume the result to be true for arbitrary $p \in \mathbb{N}$. We define the sets

$$\begin{aligned} F_k = \{\omega : \ &\boldsymbol{Z}(\omega) \text{ has } 4\epsilon\text{-oscillation } p \text{ times in } \{t_1, \dots, t_k\}, \\ &\text{but does not have } 4\epsilon\text{-oscillation } p \text{ times in } \{t_1, \dots, t_{k-1}\}\}\ , \\ G_k = \{\omega : \ &\boldsymbol{Z}(\omega) \text{ has } 4\epsilon\text{-oscillation one time in } \{t_k, \dots, t_n\}\}. \end{aligned}$$

Then $F_1, \dots, F_n$ are disjoint, and $\mathbb{P}(F_k \cap G_k) = \mathbb{P}(F_k)\mathbb{P}(G_k)$ for all $k = 1, \dots, n$ due to the Lévy properties. Also $B(p, 4\epsilon, D) = \cup_{k=1}^n F_k$, and furthermore

$$B(p+1, 4\epsilon, D) = \bigcup_{k=1}^n (F_k \cap G_k)$$

with the inclusion $\subseteq$ seen as follows: If $\omega \in B(p+1, 4\epsilon, D)$ then $\boldsymbol{Z}(\omega)$ has 4ϵ-oscillation $p+1$ times in some $\{t_{n_0}, \dots, t_{n_{p+1}}\} \subseteq D$ with $n_0 < n_1 < \dots < n_{p+1}$. Hence there is $k \le n_p$ such that $\omega \in F_k$. Also $\|\boldsymbol{Z}_{t_{n_{p+1}}}(\omega) - \boldsymbol{Z}_{t_{n_p}}(\omega)\|_\infty > 4\epsilon$ and as such also $\omega \in G_k$. From the induction assumption, the case $p = 1$ and the fact that $F_1, \dots, F_n$ are disjoint we find that

$$\begin{aligned} \mathbb{P}(B(p+1, 4\epsilon, D)) &= \sum_{k=1}^n \mathbb{P}(G_k)\mathbb{P}(F_k) \le 2\alpha_\epsilon(r-u)\mathbb{P}\Big(\bigcup_{k=1}^n F_k\Big) \\ &= 2\alpha_\epsilon(r-u)\mathbb{P}(B(p, 4\epsilon, M)) \le (2\alpha_\epsilon(r-u))^{p+1}\,. \end{aligned}$$

□

Proof of Lemma 5.6. To show that $\mathbb{P}(\Omega_1') = 1$ it suffices to prove $\mathbb{P}(A_k^c) = 0$ for any fixed $k \in \mathbb{N}$. Since $\alpha_\epsilon(r) \to 0$ as $r \downarrow 0$ for any $\epsilon > 0$, we can choose $\ell \in \mathbb{N}$ such that $2\alpha_{1/(4k)}(S/\ell) < 1$. Then by continuity of $\mathbb{P}$ we get

$$\begin{aligned}\mathbb{P}(A_k^c) &\leq \mathbb{P}(\boldsymbol{Z} \text{ has } \tfrac{1}{k}\text{-oscillation infinitely often in } \tilde{S}) \\ &\leq \sum_{j=1}^{\ell} \mathbb{P}(\boldsymbol{Z} \text{ has } \tfrac{1}{k}\text{-oscillation infinitely often in } [\tfrac{j-1}{\ell}S, \tfrac{j}{\ell}S] \cap \mathbb{Q}) \\ &= \sum_{j=1}^{\ell} \lim_{p\to\infty} \mathbb{P}(B(p, \tfrac{1}{k}, [\tfrac{j-1}{\ell}S, \tfrac{j}{\ell}S] \cap \mathbb{Q})).\end{aligned}$$

Now fix $j = 1, \dots, \ell$, and enumerate the elements of $[\frac{j-1}{\ell}S, \frac{j}{\ell}S] \cap \mathbb{Q}$ by $(t_m)_{m\in\mathbb{N}}$. From Lemma B.1 we know that

$$\mathbb{P}(B(p, \tfrac{1}{k}, \{t_1, \dots, t_n\})) \leq (2\alpha_{1/(4k)}(\tfrac{S}{\ell}))^p$$

for any $n \in \mathbb{N}$. By continuity of $\mathbb{P}$ we see that

$$\mathbb{P}(B(p, \tfrac{1}{k}, [\tfrac{j-1}{\ell}S, \tfrac{j}{\ell}S] \cap \mathbb{Q})) = \lim_{n\to\infty} \mathbb{P}(B(p, \tfrac{1}{k}, \{t_1, \dots, t_n\})) \leq (2\alpha_{1/(4k)}(\tfrac{S}{\ell}))^p$$

which tends to 0 as $p \to \infty$ since ℓ is chosen such that $2\alpha_{1/(4k)}(S/\ell) < 1$. As this holds for all $j = 1, \dots, \ell$ we conclude that $\mathbb{P}(A_k^c) = 0$. $\square$

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