Quantum Invariants and Chern-Simons Theory

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Abstract

This thesis is about the topological quantum field theory (TQFT) invented by Reshetikhin-Turaev [RT90, RT91, Tur10] and motivated by Atiyah's work [Ati88], and Witten's work [Wit89] on quantum Chern-Simons theory and the Jones polynomial [Jon85, Jon87]. The main theme is the connection between this TQFT and Chern-Simons theory [CS74, CS85]. This thesis contains the results which were obtained during my PhD studies at Centre for Quantum Geometry of Moduli Spaces (QGM) at Aarhus University with Professor Jørgen Ellegaard Andersen as supervisor, most of which are published in [AP17a, AP18a, AP18b].

In [AP18a] we prove an asymptotic expansion of quantum invariants of mapping tori in terms of Chern-Simons invariants of flat connections. This is done using by the quantum representations obtained by quantizing the moduli space of flat connections, following Hitchin [Hit90] and Axelrod-Della Pietra-Witten [ADPW91], and results from Toeplitz operator theory due to Andersen [And06] and Karabegov-Schlichenmaier [KS01].

In [AP18b] we provide a resurgence analysis of quantum invariants of Seifert fibered integral homology three spheres. We show that the quantum invariants admit an asymptotic expansion in terms of a resurgent series whose Borel transform has poles which match exactly with the Chern-Simons invariants of complex flat connections. Furthermore, we show that the exact quantum invariant can be reconstructed from the Borel transform. Our work [AP18a] is inspired by and generalises work of Gukov-Marino-Putrov [GMP16]. Our results are in agreement with the ideas introduced by Witten in [Wit11], and with the complexification paradigm introduced by Dunne-Unsal [DU15] which introduces into quantum field theory the use of a combination of Écalle's theory of resurgence [É81a, É81b] and Pham-Picard-Lefschetz theory [Mal74, Pha83, PS71, Lef50]. Our results also support conjectures posed by Garoufalidis in [Gar08].

In [AP17a] we show that the modular functor constructed from a unitary modular tensor category (MTC) admits duality and unitarity structures compatible with each other with orientation reversal and with glueing surfaces along boundary components. We define the dual of the fundamental group of an MTC as well as symplectic characters. We show that if a symplectic character exists (which we prove for the MTC relevant to Chern-Simons theory), then all compatibility constants can be set to unity. We call this strict compatibility.

Resumé

Denne afhandling angår forbindelsen mellem Reshetikhin-Turaev's topologiske kvantefeltteori [Ati88, RT90, RT91] og Chern-Simons teori [CS74, CS85]. Denne forbindelse er motiveret af Witten's studie [Wit89] af Chern-Simons kvantefeltteori og Jones polynomiet [Jon85, Jon87]. Vi fokuserer især på kvanteinvarianter. Afhandlingen udgøres af de resultater, som er opnået opnået under mit PhD studie under vejledning af Professor Jørgen Ellegaard Andersen. Mange af disse er udgivet i artiklerne [AP17a, AP18a, AP18b].

I [AP17a] beviser vi eksistensen af en asymptotisk ekspansion af kvanteinvarianter for afbildningstori, i termer af Chern-Simons invarianter af flade konnektioner. Dette gøres ved brug af kvanterepræsentationerne konstrueret af Hitchin [Hit90] og Axelrod-Della Pietra-Witten [ADPW91], ved hjælp af kvantisering af moduli rummet af flade konnektioner.

I [AP18b] udfører vi en såkaldt resurgence analyse af kvanteinvarianter af Seifert fibrerede integrale holomogi sfærer. Vi viser, at disse kvanteinvarianter tillader en asymptotisk ekspansion i form af en resurgent potensrække, hvis Borel transform har singulariteter svarende præcis til Chern-Simons invarianter af komplekse flade konnektioner. Ydermere viser vi, at disse kvanteinvarianter kan genskabes eksakt ud fra Borel-Laplace resummation. Denne artikel er inspireret af artiklen af Gukov-Marino-Putrov [GMP16]. Vores resultater stemmer overens med ideer fra Witten's arbejde [Wit11] og kompleksificerings paradigmet introduceret af Dunne-Unsal [DU15], der går ud på at introducere i kvantefeltteori en kombination af Écalle's teori om resurgence [É81a, É81b] og Pham-Picard-Lefschetz teori [Mal74, Pha83, PS71, Lef50]. Vores resultater understøtter også formodninger udformet af Garoufalidis i [Gar08].

I [AP17a] viser vi, at en modular functor konstrueret fra en unitær modular tensor kategori kan udstyres med såkaldte dualitets og unitaritets strukturer, og at disse er kompatible med hinanden, med orienteringsskift, og med topologiske operationer givet ved at skære to-mangfoldigheder langs simple løkker. Vi introducerer definitionen af den duale gruppe til første fundamental gruppe af en modular tensor kategori, samt definitionen af symplektiske karakterer. Vi viser, at hvis en symplektisk karakter eksisterer, så kan alle kompatibilitets konstanter sættes til enhed. Vi viser, at symplektiske karakterer eksisterer for kategorierne associeret med de kvantegrupper, der er relevante i Chern-Simons teori.

Contents

CHAPTER 1

Introduction

1.1 Quantum topology, Chern-Simons theory and resurgence

Let $G = \mathrm{SU}(n)$, let $\tau_{G,k}$ be the Reshetikhin-Turaev TQFT [RT90, RT91, Tur10] and let $V_{G,k}$ be the associated Walker modular functor [AP17a]. Let M be a closed oriented three-manifold containing a framed oriented link L, whose components L_i are labelled by irreducible G-representations R_i of level at most $k \in \mathbb{N}$. The TQFT associates to (M, L, R) an invariant

$$\tau_{G,k}(M, L, R) \in \mathbb{C}, \tag{1.1}$$

called the *quantum invariant*. The invention of $\tau_{G,k}$ was motivated by [Ati88] and Witten's work [Wit89] on quantum Chern-Simons theory and the Jones polynomial [Jon85, Jon87]. Let $\mathcal{A}/\mathcal{G}$ be the space of G-connections modulo gauge equivalence. For $[A] \in \mathcal{A}/\mathcal{G}$, we have the Chern-Simons action

$$\mathrm{S}([A]) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A\mathrm{d}A + \frac{2}{3}A^3) \mod \mathbb{Z}. \tag{1.2}$$

The space of classical solutions $\mathrm{d}\,\mathrm{S}_{[A]} = 0$ is equal to the moduli space $\mathcal{M}(G, M)$ of flat connections. Witten argued that the path integrals (which are mathematically ill-defined)

$$\mathrm{Z}^{\mathrm{phys}}_{G,k}(M, L, R) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k \mathrm{S}(A)} \prod_{L_i \in \pi_0(L)} \mathrm{tr}\left(R_i \circ \mathrm{Hol}_A(L_i)\right) \mathcal{D}A \tag{1.3}$$

are rich topological invariants subject to surgery formulae. The TQFT $\tau_{G,k}$ is believed to be a mathematical model for $\mathrm{Z}^{\mathrm{phys}}_{G,k}$. In connection to $\tau_{G,k}$ we have the *quantum representation* $\mathrm{Z}_{G,k}$ of the mapping class group $\Gamma(\Sigma)$ of a surface Σ, constructed by Hitchin [Hit90] and Axelrod-Della Pietra-Witten [ADPW91] by quantizing the moduli space $\mathcal{M}(G, \Sigma)$. The Andersen-Ueno isomorphism [AU15] shows that $\mathrm{Z}_{G,k}$ is equivalent to the representations induced by the modular functor $V_{G,k}$. This builds on several works including [TUY89, Las98].

Recent years have seen a fruitful interplay [Gar08, Wit11, DU15, GMP16, Kon15] between quantum field theory and a combination of Écalle's theory of resurgence [É81a, É81b], Pham-Picard-Lefschetz theory [Pha65, Pha67] and the theory of Laplace integrals [Pha83, BH91, Mal74]. This paradigm is called *complexification*. In accordance with this paradigm, the action (1.2) extends to a holomorphic action $\mathrm{S}_{\mathbb{C}}$ on the space of $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$-connections [CS85] and the classical solutions are given by the moduli space $\mathcal{M}(G_{\mathbb{C}}, M)$. Thus complex Chern-Simons theory plays an important role in relation to $\tau_{G,k}$.

1.2 Results

We now present the results contained in this thesis. Many of our results are about Poincaré asymptotic expansions of quantum invariants (1.1) in terms of Chern-Simons invariants (1.2) and are motivated by the path integral formula (1.3).

1.2.1 The asymptotic expansion conjecture for mapping tori

Let C_G denote the set of conjugacy classes of G. For (M, L) as above and $C : \pi_0(L) \to C_G$, we define $\mathrm{CS}(M, L, C) = \mathrm{S}(\mathcal{M}(G, M, L, C))$, where $\mathcal{M}(G, M, L, C)$ denotes the moduli space of flat G-connections on $M \setminus L$ with meridional holonomy C around L.

Let $\Sigma = \Sigma_{g,d}$ be a closed oriented surface of genus g and with a subset P of d marked points. Let V denote a choice of $v \in (T_p\Sigma \setminus \{0\})/\mathbb{R}_{>0}$ for each $p \in P$. For $C : P \to C_G$, let $\mathcal{M} = \mathcal{M}(G, \Sigma, P, C)$ denote the moduli space of flat G-connections on $\Sigma \setminus P$ with holonomy around each $p \in P$ contained in $C(p)$. Let $\Gamma = \pi_0(\mathrm{Diff}(\Sigma, P, V)^C)$ be the group of mapping classes $[\varphi]$ where φ preserves P and V as sets, and satisfy $\varphi^*(C) = C$. Recall that $[\varphi] \in \Gamma$ acts on $\mathcal{M}$ and let $\mathcal{M}^\varphi$ be the fixed point set. Let $T_\varphi = \Sigma \times I/[(x, 0) \sim (\varphi(x), 1)]$ be the mapping torus with the framed oriented link $L \subset T_\varphi$ traced out by (P, V). The inclusion $\iota : \Sigma \hookrightarrow T_\varphi$ induces a map $\iota^* : \mathcal{M}(G, T_\varphi, L, C) \to \mathcal{M}^\varphi$. Set $r = k + n$.

The coprime case: non-degenerate fixed point set

Let $\Sigma = \Sigma_{g,1}$ with $g \geq 2$. Let $m \in \mathbb{N}$ with $(m, n) = 1$, and set $C_m = [\exp(\frac{2\pi i m}{n})\,\mathrm{I}] \in C_G$. Recall that $\mathcal{M} = \mathcal{M}(G, \Sigma, p, C_m)$ is a compact, symplectic manifold with a prequantum bundle $\mathcal{L}_{\mathrm{CS}} \to \mathcal{M}$. Set $2n_0 = 2(n^2 - 1)(g - 1) = \dim(\mathcal{M})$. Let $\varphi \in \Gamma$ and set $\mathrm{CS} = \mathrm{CS}(T_\varphi, L, C_m)$. For each $\theta \in \mathrm{CS}$, let $2m_\theta = \max(\dim(\mathrm{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})) : \iota^{*-1}(z) \subset \mathrm{S}^{-1}(\theta))$.

Let $\mathcal{T} = \mathcal{T}_\Sigma$ be Teichmüller space. Recall that Γ act on $\mathcal{T}$. Each $\sigma \in \mathcal{T}$ induces a Kähler structure $\mathcal{M}_\sigma$ on $\mathcal{M}$ [NS64] and the *Verlinde bundle* $\mathrm{H}_k \to \mathcal{T}$ is the bundle with fibre at σ given by the level k quantization of $\mathcal{M}_\sigma$, i.e. $\mathrm{H}_k(\sigma) = \mathrm{H}^0(\mathcal{M}_\sigma, \mathcal{L}_{\mathrm{CS}}^{\otimes k})$. There exists a lift $\varphi_k^* : \mathrm{H}_k \to \varphi^*(\mathrm{H}_k)$ and a projectively flat connection ∇ on H_k that is preserved by φ_k^* [Hit90]. By fixing σ and composing φ_k^* with parallel transport of ∇ we obtain the quantum action:

$$Z_{G,k} : \Gamma \to \mathrm{PGL}\left(\mathrm{H}_k(\sigma)\right). \tag{1.4}$$

Theorem 1 ([AP18a]). *If every component of $\mathcal{M}^\varphi$ is an integral manifold of $\mathrm{Ker}(\mathrm{d}\,\varphi - \mathrm{Id})$ then there exists for each $\theta \in \mathrm{CS}$ smooth densities $\{\Omega_\alpha(\theta)\}_{\alpha=0}^\infty$ on $\mathcal{M}^\varphi$ giving an asymptotic expansion*

$$\mathrm{tr}\left(Z_{G,k}(\varphi)\right) \sim_{k\to\infty} \sum_{\theta \in \mathrm{CS}} e^{2\pi i r \theta} r^{m_\theta} \sum_{\alpha=0}^{\infty} r^{-\frac{\alpha}{2}} \int_{\mathcal{M}^\varphi} \Omega_\alpha(\theta). \tag{1.5}$$

Theorem 1 is based on stationary phase approximation and on Theorem 2 below, which holds for every $\varphi \in \Gamma$. Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\}$.

Theorem 2 ([AP18a]). *There exists $\widehat{\varphi} \in C^\infty(\mathcal{M}, \mathbb{H}/2\pi i \mathbb{Z})$ and $\{\Omega_n^\varphi\}_{n=0}^\infty \subset \Omega^{2n_0}(\mathcal{M})$ with the following properties. We have that $\widehat{\varphi} \circ \iota^* = 2\pi i\,\mathrm{S}$, $\widehat{\varphi}$ is real analytic near $\mathcal{M}^\varphi$ (in the analytic structure given by σ) and $\mathcal{M}^\varphi = \{\mathrm{d}\,\widehat{\varphi} = 0\} \cap \mathrm{Re}(\widehat{\varphi})^{-1}(0)$. For every $\tilde{m} \in \mathbb{N}$ we have that*

$$\mathrm{tr}\left(Z_{G,k}(\varphi)\right) = r^{n_0} \sum_{n=0}^{\tilde{m}} r^{-n} \int_{\mathcal{M}} e^{r\widehat{\varphi}}\, \Omega_n^\varphi + O(k^{n_0 - \tilde{m} - 1}). \tag{1.6}$$

If the condition of Theorem 1 holds, we say that $\mathcal{M}^\varphi$ is *non-degenerate*.[1] Theorem 1 generalizes a result of Charles [Cha16], which is valid when in addition $\dim(\mathcal{M}^\varphi) = 0$.

The coprime case: possibly degenerate fixed point set

The real analyticity of $\widehat{\varphi}$ guarantees that for any $z \in \mathcal{M}^\varphi$ there exists a coordinate neighborhood U of z, and $\widehat{\varphi}_{\mathbb{C}} \in \mathcal{O}(U + \sqrt{-1}U)$ which is an extension of $\widehat{\varphi}_{|U}$. Thus we can apply to (1.6) Malgrange's version of saddle point analysis for Laplace integrals with holomorphic phase [Mal74]. This version imposes no non-degeneracy condition on $\mathrm{Hess}(\widehat{\varphi}_{\mathbb{C}})$. We prove:

Theorem 3 ([AP18a]). *Assume that all $z \in \mathcal{M}^\varphi$ satisfy one of the following three conditions:*

- *z is a smooth point with $T_z\mathcal{M}^\varphi = \mathrm{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})$,*
- *$\dim(\mathrm{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})) \leq 1$, or*
- *z is an isolated saddle point of the germ of $\widehat{\varphi}_{\mathbb{C}}$ at z.*

Then there exists for each $\theta \in \mathrm{CS}$ an unbounded subset $A_\theta \subset \mathbb{Q}_{\leq 0}$, $n_\theta \in \mathbb{Q}_{\geq 0}$, $d_\theta \in \mathbb{N}$ and $\{c_{\alpha,\beta}(\theta)\}_{\alpha\in A_\theta,\, 0\leq\beta\leq d_\theta} \subset \mathbb{C}$ giving an asymptotic expansion

$$\mathrm{tr}\left(\mathrm{Z}_{G,k}(\varphi)\right) \sim_{k\to\infty} \sum_{\theta\in\mathrm{CS}} e^{2\pi i r\theta} r^{n_\theta} \sum_{\alpha\in A_\theta} \sum_{\beta=0}^{d_\theta} c_{\alpha,\beta}(\theta) r^\alpha \log(r)^\beta. \tag{1.7}$$

If the first or second condition holds for all $z \in \mathcal{M}^\varphi \cap \widehat{\varphi}^{-1}(2\pi i\theta)$ then $d_\theta = 0$ and $n_\theta = m_\theta$.

The punctured torus case

Let $\Sigma = \Sigma_{1,1}$. For $l \in J = (-2,2)$ let $\mathcal{M}_l = \mathcal{M}(\mathrm{SU}(2), \Sigma, p, C_l)$ where $\mathrm{tr}(C_l) = l$. Then $\mathcal{M}_l$ is a symplectic manifold [Jef94] and for l in a dense subset $\mathcal{Q} \subset J$ it is quantizable for some levels k. For these k, we can construct a projective representation $\mathrm{Z}_{l,k}$ of Γ, which is analogous to (1.4). The analogs of Theorem 2 and 3 both hold. Fix $\varphi \in \Gamma$ and set $\mathrm{CS} = \mathrm{CS}(T_\varphi, L, C_l)$.

Theorem 4 ([AP18a]). *If φ is Anosov then $\mathcal{M}_l^\varphi$ is non-degenerate*[2] *for l in an open dense subset $A_\varphi \subset J$, and for $l \in A_\varphi \cap \mathcal{Q}$ there exists for each $\theta \in \mathrm{CS}$ a sequence $\{c_\alpha(\theta)\}_{\alpha=0}^\infty \subset \mathbb{C}$ such that*

$$\mathrm{tr}\left(\mathrm{Z}_{l,k}(\varphi)\right) \sim_{k\to\infty} \sum_{\theta\in\mathrm{CS}} e^{2\pi i r\theta} \sum_{\alpha=0}^{\infty} c_\alpha(\theta) r^{-\alpha}. \tag{1.8}$$

Via the Andersen-Ueno isomorphism [AU15], all of the above expansions (1.5), (1.7) and (1.8) are special cases of the *asymptotic expansion conjecture* due to Witten [Wit89].

From semi-classical expansions to resurgence and complexification

The ideas from [Wit11] suggests that the asymptotic expansions of quantum invariants should have interesting resurgence properties. Formally this follows from the application of Pham-Picard-Lefschetz theory to the holomorphic action $S_{\mathbb{C}}$. In Section 1.2.2 we present a (mathematical) generalization of this framework together with key definitions from resurgence, that are central in complexification [DU15, BDS+15] and used in [AP18b].

[1] In [AP18a] a non-degenerate fixed point set is called a *transversely cut out* fixed point set.
[2] This result concerning $\mathcal{M}_l^\varphi$ is due to Brown [Bro98].

1.2.2 Resurgence phases, Picard-Lefschetz theory and Laplace integrals

To generalize results from [Mal74, How97, Pha83] we introduce the following definition.

Definition. Let Y be a complex manifold with $\dim_{\mathbb{C}}(Y) = d$. Let $f \in \mathcal{O}(Y)$. Let S be the set of saddle points, and let Ω be the set of critical vaules. We call f a *resurgence phase* if S is discrete and $f_{|Y \setminus f^{-1}(\Omega)}$ is a locally trivial fibration over the Riemann surface $C = f(Y) \setminus \Omega$.

Fix a resurgence phase f. Consider the homological bundle $\mathrm{H} = \mathrm{H}_{d-1}(f^{-1}(\,\cdot\,)) \to C$ with the *Gauss-Manin* connection. If B_z is a small ball centered at $z \in S \cap f^{-1}(\eta)$, then $f_{|\mathrm{B}_z \setminus f^{-1}(\eta)}$ is a *Milnor fibration* [Mil68] over a punctured disc $\mathrm{D} \setminus \{\eta\}$ with a basepoint c. The fibres are homotopy equivalent to $\vee_{j=1}^{\mu_z} \mathrm{S}_j^{d-1}$, μ_z being the *Milnor number*. Denote by E_z the homological bundle

$$\mathrm{H}_{d-1}(f^{-1}(\,\cdot\,) \cap \mathrm{B}_z) \to \mathrm{D} \setminus \{\eta\} \tag{1.9}$$

and by $\mathrm{M}_z \in \mathrm{Aut}(\mathrm{E}_z(c))$ the *monodromy*. A *vanishing cycle* σ is a parallel section of (1.9). Such σ extends to a multivalued parallel section $C \to \mathrm{H}$ and shrink to a point near z.

Definition. Let $\lambda \in \mathbb{C}^*$. Let (σ, γ) be a pair consisting of a path $\gamma : (\mathbb{R}_{\geq 0}, 0) \to (C \cup \{\eta\}, \eta)$ with $\mathrm{Re}(\lambda(\gamma - \eta))$ being strictly increasing and a vanishing cycle σ near $z \in S \cap f^{-1}(\eta)$ and defined on a neighbourhood of $\gamma(\mathbb{R}_{\geq 0}) \cap D \setminus \{\eta\}$. The λ*-Picard-Lefschetz thimble* $\Delta(\sigma, \gamma)$ is the formal sum of isotopy classes of maps $\mathrm{S}^{d-1} \times \mathbb{R}_{\geq 0} \to Y$ given by $\Delta(\sigma, \gamma)(t) = \sigma(\gamma(t))$.

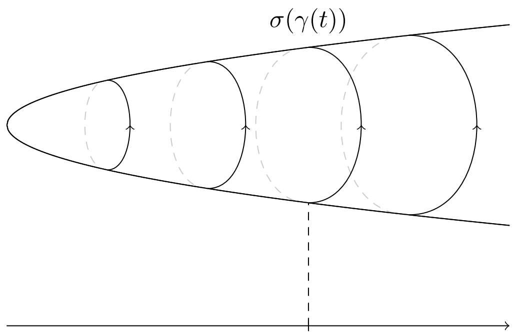


Figure 1.1: Picard-Lefschetz thimble $\Delta(\sigma, \gamma)$ in dimension $d = 2$

Saddle point analysis of resurgence phases

Theorem 5 ([AP18b]). *Let $z \in S \cap f^{-1}(\eta)$, let $\Delta(\sigma, \gamma)$ be a λ-Picard-Lefschetz thimble at z and let $\omega \in \Omega_{Hol}^{d,0}(Y)$. There exists an unbounded set $\mathcal{A} \subset \mathbb{Q}_{<0}$, $\{d_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathbb{N}$ and $\{c_{\alpha,\beta}^{\omega}\}_{\alpha \in \mathcal{A},\, 0 \leq \beta \leq d_\alpha} \subset \mathbb{C}$ giving for fixed $\arg(\lambda)$ an asymptotic expansion (provided the integral is absolutely convergent)*

$$\int_{\Delta(\sigma,\gamma)} e^{-\lambda f}\, \omega \sim_{\lambda \to \infty} e^{-\lambda \eta} \sum_{\alpha \in \mathcal{A}} \sum_{\beta=0}^{d_\alpha} c_{\alpha,\beta}^{\omega} \lambda^{\alpha} \log(\lambda)^{\beta}. \tag{1.10}$$

The set $\exp(-2\pi i \mathcal{A})$ is a subset of the set of eigenvalues of M_z and for each $\alpha \in \mathcal{A}$ the number $d_\alpha + 1$ is less than or equal to the maximal dimension of any Jordan block associated with $\exp(-2\pi i \alpha)$.

Resurgence properties of the Borel transform

We briefly present general definitions from resurgence, which we afterwards apply to (1.10).

Definition. For a Riemann surface C with universal covering space $\widetilde{C} \to C$, the algebra $\mathcal{R}(C)$ of *resurgent functions* is $\mathcal{O}(\widetilde{C})$.[3] For $\delta \in \pi_1(C)$ let $\psi_\delta \in \mathrm{Aut}(\widetilde{C})$ be the induced deck transformation and define $\mathrm{Var}_\delta = \psi_\delta^* - \mathrm{Id} \in \mathrm{End}(\mathcal{R}(C))$.

We now introduce the Borel transform.

Definition. Let $\mathrm{S} \subset \mathbb{P} = \mathbb{C} \cup \{\infty\}$ be a subset with a limit point $s \in \overline{\mathrm{S}} \setminus \mathrm{S}$. Given a sequence of functions $\{g_j\}_{j=0}^\infty \subset \mathbb{C}^{\mathrm{S}}$ the formal series $\sum_{j=0}^\infty g_j$ is said to be *s-asymptotic* if for every $j \in \mathbb{N}$ we have that $g_{j+1} = o_s(g_j)$, i.e. g_{j+1} is small o of g_j with respect to the limit s. Given a strictly increasing sequence $\{\alpha_j\}_{j=0}^\infty \subset \mathbb{R}_{>0}$ and a sequence $\{(\beta_j, c_j)\}_{j=0}^\infty \subset \mathbb{N} \times \mathbb{C}$ the *Borel transform* of the ∞-asymptotic series $\tilde{\varphi}(\lambda) = \sum_{j=0}^\infty c_j \lambda^{-\alpha_j} \log(\lambda)^{\beta_j}$ is the 0-asymptotic series given by

$$\mathcal{B}(\tilde{\varphi})(\zeta) = \sum_{j=0}^{\infty} c_j (-1)^{\beta_j} \frac{\partial^{\beta_j}}{\partial {\alpha_j}^{\beta_j}} \left(\frac{\zeta^{\alpha_j - 1}}{\Gamma(\alpha_j)} \right).$$

Consider now a resurgence phase $f \in \mathcal{O}(Y)$ and set $C = f(Y) \setminus \Omega$ where Ω is the set of critical values. For $\delta \in \pi_1(C, \hat{c})$ define $\mathrm{var}_\delta = \mathrm{P}_\delta - \mathrm{Id} \in \mathrm{End}\,(\mathrm{H}(\hat{c}))$ where P_δ denotes parallel transport of the flat bundle $\mathrm{H} \to C$ introduced above. For each $\phi \in \Omega$ we let $\mathrm{D}'(\phi) = \mathrm{D} \setminus \{\phi\}$ be a small punctured disc (with a basepoint). We denote the Borel transform of the expansion (1.10) by

$$\mathcal{B}_{\sigma,\omega} = \sum_{\alpha,\beta} c^{\omega}_{\alpha,\beta} \mathcal{B}(\lambda^\alpha \log(\lambda)^\beta).$$

Theorem 6 ([AP18b]). *The Borel transform $\mathcal{B}_{\sigma,\omega}$ is convergent for small ζ, and extends to $\mathcal{B}_{\sigma,\omega} \in \mathcal{R}(C - \eta)$ by the formula*

$$\mathcal{B}_{\sigma,\omega}(\zeta) = \int_{\sigma(\zeta+\eta)} \frac{\omega}{\mathrm{d} f}, \tag{1.11}$$

where $\frac{\omega}{\mathrm{d} f}$ is the Gelfand-Leray transform defined[4] *by $\mathrm{d} f \wedge \frac{\omega}{\mathrm{d} f} = \omega$. Further, we have that*

$$\int_{\Delta(\sigma,\gamma)} e^{-\lambda f}\, \omega = \int_\gamma e^{-\lambda \zeta} \mathcal{B}_{\sigma,\omega}(\zeta - \eta)\ \mathrm{d}\zeta. \tag{1.12}$$

For every $\phi \in \Omega$, the cycle $\chi = \mathrm{var}_{\partial \mathrm{D}'(\phi)}(\sigma)$ is a sum of vanishing cycles above ϕ and we have that[5]

$$\mathrm{Var}_{\partial \mathrm{D}'(\phi) - \eta} \left(\mathcal{B}_{\sigma,\omega}\right)(\zeta) = \mathcal{B}_{\chi,\omega}(\zeta + \eta - \phi). \tag{1.13}$$

We now compare our results with the works [Mal74, BH91, How97, Pha83]. The work [Mal74] is purely local in the sense that it deals with the case of a single isolated saddle point in $\mathbb{C}^d$, whereas [BH91, How97, Pha83] deals with the case $Y = \mathbb{C}^d$ and mostly with a polynomial phase function f with Morse singularities. We stress that the formulas (1.11) and (1.12) essentially reduces our proofs to the proofs from the cited works.

[3] Our definition of resurgent functions is slighly different from the standard definition [Sau07, MS16] but is is suitable for the study of Laplace integrals.

[4] Though the Gelfand-Leray transfom $\frac{\omega}{\mathrm{d} f}$ is not globally well-defined, its restriction to level sets of f is.

[5] For (1.13) to hold, we assume a localization principle described below in Definition 5.3.1

1.2.3 Resurgence analysis of quantum invariants

We now turn to the application of resurgence to quantum invariants [AP18b]. In Section 1.2.3 we assume $G = \mathrm{SU}(2)$ and write $\tau_{G,k} = \tau_k$. Set $\mathrm{CS}_{\mathbb{C}}(M) = \mathrm{S}_{\mathbb{C}}(\mathcal{M}(\mathrm{SL}(2,\mathbb{C}), M))$.

Seifert fibered homology spheres: the Borel transform and complex Chern-Simons

Let $n \in \mathbb{N}$ and choose $p_j, q_j \in \mathbb{Z}, j = 1, ..., n$ with $(p_j, q_j) = 1$ and $(p_j, p_l) = 1$ for $l \neq j$. Consider the Seifert fibered three-manifold $X = \Sigma((p_1/q_1), ..., (p_n/q_n))$ with n exceptional fibers and surgery link depicted below. We can and will assumme $p_j > 0$, for all j. Let

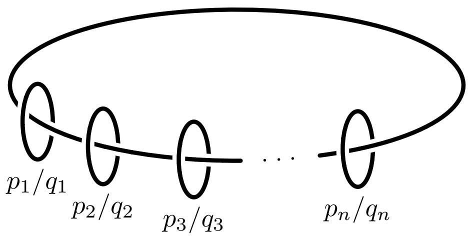


Figure 1.2: Surgery link for X.

$P = \prod_{i=1}^n p_i$ and $H = P\sum_{j=1}^n \frac{q_j}{p_j}$. We require $H_1(X,\mathbb{Z}) = 0$ which is equivalent to $H = \pm 1$. Let $S(\,\cdot\,,\,\cdot\,)$ be the Dedekind sum. Let $\mathrm{CS}^*_{\mathbb{C}} = \mathrm{CS}^*_{\mathbb{C}}(X)$ be the range of $\mathrm{S}_{\mathbb{C}}$ on the irreducible flat $\mathrm{SL}(2,\mathbb{C})$-connections. Consider the normalization

$$\widetilde{\mathrm{Z}}_k(X) = \frac{\tau_k(X)}{\tau_k(S^2 \times S^1)} \sqrt{P} \exp\left(\left(3 - \frac{H}{P} + 12\sum_{j=1}^n S(q_j, p_j)\right)\frac{i\pi}{2k} - \frac{\pi i 3H}{4}\right). \tag{1.14}$$

Theorem 7 ([AP18b]). *There exists a set of polynomials $\{\mathrm{Z}_\theta(x) \in \mathbb{C}[x]\}_{\theta \in \mathrm{CS}^*_{\mathbb{C}}}$ of degree at most $n-3$ and a formal power series $\mathrm{Z}_\infty(x) \in x^{-\frac{1}{2}}\mathbb{C}[[x^{-1}]]$ which give an asymptotic expansion*

$$\widetilde{\mathrm{Z}}_k(X) \sim_{k\to\infty} \sum_{\theta \in \mathrm{CS}^*_{\mathbb{C}}} e^{2\pi i k\theta}\,\mathrm{Z}_\theta(k) + \mathrm{Z}_\infty(k). \tag{1.15}$$

The Borel transform $\mathcal{B}(\mathrm{Z}_\infty)$ is the resurgent function given by

$$\mathcal{B}(\mathrm{Z}_\infty)(\zeta) = -\sqrt{\frac{P2}{i\zeta\pi H}}\left(\sinh\left(\sqrt{\frac{i2P\pi\zeta}{H}}\right)\right)^{2-n}\prod_{j=1}^n \sinh\left(\sqrt{\frac{i2P\pi\zeta}{H}}\frac{1}{p_j}\right). \tag{1.16}$$

Let Ω be the set of poles of $\mathcal{B}(\mathrm{Z}_\infty)$. Then $\frac{i}{2\pi}\Omega$ consists of all numbers of the form $\frac{-m^2H}{4P}$ where $m \in \mathbb{Z}$ and m is divisible by at most $n-3$ of the p_j, and we have the following identiy

$$\mathrm{CS}^*_{\mathbb{C}} = \frac{i}{2\pi}\Omega \mod \mathbb{Z}. \tag{1.17}$$

The existence of an expansion $\widetilde{\mathrm{Z}}_k(X) \sim_{k\to\infty} \sum_{\theta\in R(X)} e^{2\pi i k\theta}\,\mathrm{Z}_\theta(k) + \mathrm{Z}_\infty(k)$ where $R(X) \subset \mathbb{Q}/\mathbb{Z}$ is a finite set was proven in [LR99]. Our contribution in regard to (1.15) is to show $R(X) \subset \mathrm{CS}^*_{\mathbb{C}}$. It is of course expected that $R(X) = \mathrm{CS}(X) \setminus \{0\}$, i.e. the asymptotic expansion (1.15) should be a sum over Chern-Simons invariants of flat G-connections, rather than flat $G_{\mathbb{C}}$-connections. This is known to be true for $n = 3$ exceptional fibers [Hik05c] and in some cases for $n = 4$ [Hik05b].

Seifert fibered homology spheres: resummation of the quantum invariant

We now turn to the question of resummation of $\widetilde{Z}_k$. Introduce for $\mu \in \mathbb{Q}/\mathbb{Z}$ the set

$$\mathcal{T}(\mu) = \{m = 1, ..., 2P-1 : -m^2H/4P = \mu \mod \mathbb{Z}\}. \tag{1.18}$$

Introduce the integral operators $\mathcal{L}_\mu$ defined by

$$\mathcal{L}_\mu(\hat{\varphi})(\xi) = \frac{1}{2\pi i} \sum_{x \in \mathcal{T}(\mu)} \oint_{y=2\pi i x} \frac{e^{\xi \frac{Hiy^2}{8\pi P}}}{(1-e^{-\xi y})} \frac{yH}{P4} \hat{\varphi}\left(\frac{y^2}{i8\pi P}\right) \, \mathrm{d}\, y. \tag{1.19}$$

Observe that by definition $\mathcal{T}(\mu)$ is empty for all but finitely many $\mu \in \mathbb{Q}/\mathbb{Z}$ and therefore $\mathcal{L}_\mu$ is 0 for all but these finitely many μ.

Theorem 8 ([AP18b]). *The polynomials $\{Z_\theta(x) \in \mathbb{C}[x]\}_{\theta \in CS^*_{\mathbb{C}}}$ are determined by $\mathcal{B}(Z_\infty)$ as follows:*

$$e^{2\pi i k\theta}\, Z_\theta(k) = \mathcal{L}_\theta\left(\mathcal{B}(Z_\infty)\right)(k). \tag{1.20}$$

The quantum invariant $\widetilde{Z}_k(X)$ is determined by $\mathcal{B}(Z_\infty)$ as follows:

$$\widetilde{Z}_k(X) = \int_0^\infty e^{-k\xi}\mathcal{B}(Z_\infty)(\xi) \, \mathrm{d}\,\xi + \sum_{\theta \in \frac{i}{2\pi}\Omega \mod \mathbb{Z}} \mathcal{L}_\theta\left(\mathcal{B}(Z_\infty)\right)(k). \tag{1.21}$$

The above reconstruction from the Borel transform is reminiscent of the typical resummation process arising when applying resurgence in physics [Dor14, ABS18].

We now compare with [GMP16]. In [GMP16] the authors analyse $\tau_k(X)$ for some examples with $n = 3$ exceptional fibers. The identity (1.17) was verified for these examples. They connect their results to q-series with integer coefficients, the so-called $\hat{Z}_a(q)$ invariants and propose to interpret the integrality of $\hat{Z}_a(q)$ as a categorification of τ_k, and this was further developed in [CCF+18]. One can obtain $\hat{Z}_q$ from $\mathcal{B}(Z_\infty)$ as discussed in Section 7.1.5.

Surgeries on the figure eight knot

We now turn to the hyperbolic three-manifolds $M_{r/s}$ with surgery link giving by the figure eight knot with framing r/s. In [AH12], Andersen-Hansen gives an expression for $\tau_k(M_{r/s})$

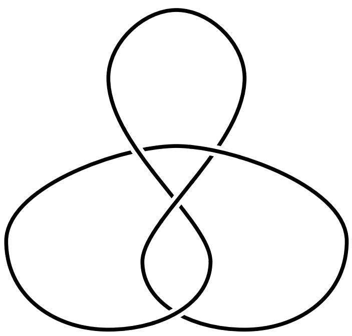


Figure 1.3: Figure eight knot

involving Fadeev's quantum dilogarithm S [Fad95, FK94]. As S can be semi-classically approximated by the dilogarithm Li_2, they are led to the following conjecture.

Conjecture 1 ([AH06] conjecture 2). *Choose $c, d \in \mathbb{Z}$ with $rd - cs = 1$. Introduce for $\alpha, \beta \in \{0,1\}$ and $n \in \mathbb{Z}/|s|\mathbb{Z}$ the function*

$$\Phi^n_{\alpha,\beta}(x,y) = \frac{Li_2(e^{2\pi i(x+y)}) - Li_2(e^{2\pi i(x-y)})}{4\pi^2} - \frac{dn^2}{s} + (-\frac{r}{4s}x + \frac{n}{s} + y + \alpha + \beta)x + y(\alpha - \beta).$$

There exists chains $\Gamma^n_{\alpha,\beta} \subset \mathbb{C}^2$ of real dimensions 2 meeting only non-degenerate stationary points of $\Phi^n_{\alpha,\beta}$ in $\{(x,y) \in \mathbb{R} \times \mathbb{C} : e^{2\pi i y} \in]-\infty, 0[\}$, and holomorphic 2-forms $\chi^n_{\alpha,\beta}$ such that[6] *for some $m_0 \in \mathbb{N}$ and every $m \in \mathbb{N}$ we have:*

$$\tau_k(M_{r/s}) = k \sum_n \sum_{\alpha,\beta} \int_{\Gamma^n_{\alpha,\beta}} e^{2\pi i k \Phi^{\alpha,\beta}_n} \chi^n_{\alpha,\beta} + \mathcal{O}(k^{m_0 - m}). \tag{1.22}$$

Set $\mathrm{CS}_{\mathbb{C}} = \mathrm{CS}_{\mathbb{C}}(M_{r/s})$ and set $\mathrm{CS} = \mathrm{CS}(M_{r/s})$. Andersen-Hansen show ([AH12] Theorem 2) that the $\Phi^n_{\alpha,\beta}$ have isolated saddle points and that the set of relevant critical values (modulo the integers) is equal to $\mathrm{CS}_{\mathbb{C}}$. By applying Theorem 5, we obtain the following result (see Remark 7.2.5).

Theorem 9 ([AP18b]). *Assume conjecture 2 in [AH12] is true, and assume the integration chains $\Gamma^n_{\alpha,\beta}$ can be decomposed into Picard-Lefschetz thimbles. Then there exists a set of formal power series $\{Z_\theta(x) \in x^{-1}\mathbb{C}[[x^{-1}]]\}_{\theta \in \mathrm{CS}}$ giving an asymptotic expansion of the form*

$$\tau_k(M_{r/s}) \sim_{k\to\infty} k \sum_{\theta \in \mathrm{CS}} e^{2\pi i k \theta} Z_\theta(k). \tag{1.23}$$

For each $\theta \in \mathrm{CS}$ we have$\mathcal{B}(Z_\theta) \in \mathcal{R}(U(\theta) \setminus \Omega(\theta))$ where $U(\theta) \subset \mathbb{C}$ is open and $\Omega(\theta)$ is a closed discrete subset satisfying

$$\mathrm{CS}_{\mathbb{C}} - \theta \supset \frac{i}{2\pi}\Omega(\theta) \mod \mathbb{Z}. \tag{1.24}$$

Each $\mathcal{B}(Z_\theta)$ can be decomposed into a finite sum of resurgent functions

$$\mathcal{B}(Z_\theta) = \sum_{\lambda \in \Lambda(\theta)} \check{Z}_\lambda(\theta), \tag{1.25}$$

with the following property. For any two Chern-Simons values θ, θ' and $\lambda \in \Lambda(\theta)$, there exists $\{n_{\lambda,\mu}\}_{\mu \in \Lambda(\theta')} \subset \mathbb{C}$ such that up to a change of parameter we have that

$$\mathrm{Var}_{\gamma_{2\pi i(\theta-\theta')}}(\check{Z}_\lambda(\theta)) = \sum_{\mu \in \Lambda(\theta')} n_{\lambda,\mu} \check{Z}_\mu(\theta'), \tag{1.26}$$

where $\gamma_{2\pi i(\theta-\theta')}$ is a small circle which encircles $2\pi i(\theta - \theta')$.

The quantum dilogarithm and analytic extensions

The use of the quantum dilogarithm in connection with quantum invariants is also central to an ongoing collaboration with Andersen concerning analytic extensions of quantum invariants which we describe in Section 1.3.

[6] We have strengthened the conjecture here. The original conjecture only considers the leading order asymptotic.

1.2.4 Modular functors with duality and unitarity

To state our results on modular functors [AP18b] we recall the properties of the modular functor $\mathrm{V}_{G,k}$. Let $\Lambda(k)$ be the label set of isomorphisms classes of irreducible G-representations of level at most k. Let $\Sigma = (\Sigma, P, V, \lambda, L)$ be a closed oriented surface with a finite subset P, a choice V of $v \in (T_p\Sigma \setminus \{0\})/\mathbb{R}_{>0}$ for each $p \in P$, a labelling $\lambda \in \Lambda(k)^P$ and a Lagrangian subspace $L \subset H_1(\Sigma, \mathbb{Q})$. To such data $\mathrm{V}_{G,k}$ associates a finite dimensional vector space

$$\mathrm{V}_{G,k}(\Sigma) \in \mathrm{Vect}(\mathbb{C}).$$

We let $\Gamma(\Sigma) = \pi_0(\mathrm{Diff}^+(\Sigma, P, V)^\lambda)$, i.e. $\varphi \in \mathrm{Diff}^+(\Sigma)$ represents a class in $\Gamma(\Sigma)$ if it satisfies $\varphi(P) = P, \mathrm{d}\,\varphi(V) = V$ and $\varphi^*(\lambda) = \lambda$. There is a projective action

$$\mathrm{V}_{G,k} : \Gamma(\Sigma) \to \mathrm{PGL}(\mathrm{V}_{G,k}(\Sigma))$$

and mapping class invariant isomorphisms $\mathrm{V}_{G,k}(\Sigma_1 \sqcup \Sigma_2) \simeq \mathrm{V}_{G,k}(\Sigma_1) \otimes \mathrm{V}_{G,k}(\Sigma_2)$. We have a factorization axiom: given an oriented simple loop γ in Σ and a label λ we obtain a surface Σ_γ^λ by cutting out γ and then collapse the two resulting boundaries and label these by λ and the dual representation $\lambda^\dagger$ respectively. There is a decomposition

$$P_\gamma : \mathrm{V}_{G,k}(\Sigma) \xrightarrow{\sim} \oplus_{\lambda \in \Lambda(k)} \mathrm{V}_{G,k}(\Sigma_\gamma^\lambda). \tag{1.27}$$

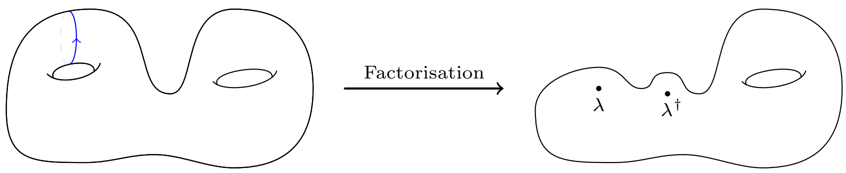


Figure 1.4: Factorization

For $W \in \mathrm{Vect}(\mathbb{C})$ let $\overline{W}$ have the conjugated scalar multiplication. For a surface Σ we let$-\Sigma$ denote the surface with reversed the orientation, and with each label λ replaced with the dual representation $\lambda^\dagger$. There exists mapping class invariant perfect pairings

$$D_\Sigma : \mathrm{V}_{G,k}(\Sigma) \otimes \mathrm{V}_{G,k}(-\Sigma) \to \mathbb{C}, \quad U_\Sigma : \mathrm{V}_{G,k}(\Sigma) \otimes \overline{\mathrm{V}_{G,k}(\Sigma)} \to \mathbb{C}.$$

The pairing D_Σ is called the duality and the pairing U_Σ is called the unitarity. As we have $-(-\Sigma) = \Sigma$, we can ask if duality and orientation reversal is compatible in the sense that $D_\Sigma = D_{-\Sigma} \circ P$ where $P(v \otimes w) = v \otimes w$. We can also ask if the duality and unitary are compatible, in the sense that they induce a commutative diagram

$$\begin{tikzcd}
\mathrm{V}_{G,k}(\Sigma) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{V}_{G,k}(-\Sigma)^* \arrow[d, "\simeq"] \\
\overline{\mathrm{V}_{G,k}(\Sigma)^*} \arrow[r, "\simeq"] & \overline{\mathrm{V}_{G,k}(-\Sigma)}.
\end{tikzcd}$$

Finally, we can ask if D_Σ, U_Σ are compatible with factorization, i.e. that they are orthogonal with respect to (1.27) for all γ. If all three answers are affirmative we say that duality, unitarity and factorization are strictly compatible.

In [Tur10] Turaev constructs a TQFT $\tau_{\mathcal{V}}$ for every modular category $(\mathcal{V}, \{V_i\}_{i\in I})$ (see Definition 3.1.6 below). Here $\{V_i\}_{i\in I}$ is a finite set of simple objects. To each $V \in \mathrm{Obj}(\mathcal{V})$ there exists $V^* \in \mathrm{Obj}(\mathcal{V})$ and we have an involution $i \mapsto i^\dagger$ defined by $V_{i^\dagger} \simeq (V_i)^*$. There is a preferred $0 \in I$ with $0^\dagger = 0$, and $K = \mathrm{End}_{\mathcal{V}}(V_0, V_0)$ is a commutative ring.

The strict compatibility theorem

As mentioned above we work in this thesis with Walker's axioms for a modular functor. The precise definitions and axioms are given in Chapter 4. We stress that these differ slightly from Turaev's axioms [Tur10] but we prove the following.

Theorem 10 ([AP17a]). *Let $(\mathcal{V}, \{V_i\}_{i\in I}$ be a modular category with ground ring K.*

- *For any choice of isomorphisms*

$$q = \{q_i : V_{i^*} \to (V_i)^* : i \in I\} \tag{1.28}$$

  there exists a modular functor $\mathrm{V}_{\mathcal{V}}(q)$ based on (I, K) as defined in Definition 4.1.6 and $\mathrm{V}_{\mathcal{V}}(q)$ can be given a duality $D_{q'}$ as defined in Definition 4.2.1, depending on a (possibly different) choice q' of isomorphisms. If $\mathcal{V}$ is has a unitarity structure then $\mathrm{V}_{\mathcal{V}}(q)$ can be given a unitarity U as defined in Definition 4.2.2 which is compatible with the duality.

- *Assume that $(\mathcal{V}, \{V_i\}_{i\in I})$ has a fundamental symplectic character as defined in Definition 4.3.2, and has a unitarity structure (this implies $K = \mathbb{C}$). Then $\mathrm{V}_{\mathcal{V}}$ can be given a duality D and a unitarity U satisfying the strict compatibility condition defined in Definition 8.2.1. All the modular tensor categories associated with quantum groups at roots of unity admits fundamental symplectic structures.*

1.3 Future perspectives

Analytic extensions of quantum invariants

Let $q_k = \exp(2\pi i/k)$ and let $\Lambda(k) = \{1, ..., k-1\}$ be the label set in SU(2) theory. Let $L \subset S^3$ be a framed oriented link. Let M be the three-manifold obtained by surgery on L. For $\lambda \in \mathrm{Col}(L, k) = \Lambda(k)^{\pi_0(L)}$ let $J_\lambda(L, q)$ denote the *Colored Jones polynomial*. Using the R matrix approach to $J_\lambda(L, q)$ and the quantum dilogarithm, we prove the following.

Theorem 11. *Fix a a coloring $\lambda \in \mathrm{Col}(L, k_0)$ The meromorphic function $\mathcal{J}_\lambda(L) \in \mathcal{M}(\mathbb{C})$ defined in Definition 8.1.4 satisfies for every integer $k \geq k_0$ the equation*

$$\mathcal{J}_\lambda(L)(k) = J_\lambda(L, q).$$

Conjecture 2. *Let Φ be the semiclassical approximation of $\mathcal{J}$, as defined in Definition 8.2.1. Let $\Omega(\Phi)$ denote the set of critical values of Φ. We have that*

$$\mathrm{CS}_{\mathbb{C}}(M) = \Omega(\Phi) \mod \mathbb{Z}. \tag{1.29}$$

Conjecture 2 is based upon a comparison with the potential function $\mathbb{W}_0$ defined by Yoon [Yoo18]. The potential is a function defined from the surgery link L, which is strongly related to the action $\mathrm{S}_{\mathbb{C}}$ and $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}), M)$ (see Theorem 2.2.6 below).

Complexification, brane quantization and the moduli spaces of Higgs bundles

Let C be a Riemann surface with underlying smooth manifold Σ. In light of the importance in TQFT of complexification and the quantum geometry of the moduli space $\mathcal{M}(\mathrm{SU}(2),\Sigma)$, it is natural to ask if resurgence analysis of τ_k is connected to the moduli space $\mathcal{M}_{\mathrm{Higgs}}(C)$ of Higgs bundles [Hit87] since this moduli space can be seen as a complexification of $\mathcal{M}(\mathrm{SU}(2),\Sigma)$. In relation to this, it is interesting to adress the possible relation to Gukov and Witten's idea of brane quantization [GW09, Guk10] in which complexification of the phase space plays a central role. In fact the application of brane quantization (and mirror symmetry) to Chern-Simons theory is a key topic in both of the works [GW09, Guk10].

Resurgence and recursion

Consider again the asymptotic expansion (1.10). This is almost always divergent, and this divergence is a source of hidden information, which can be extracted by resurgence techniques. This idea is central to the remarkable *resurgence formula* (1.30) [BH91, How97]. Let f be a resurgence phase with $f(Y)=\mathbb{C}$, and with finitely many saddle points $S=\{z_j\}$, all of which are Morse singularities. Assume $f_{|S}$ is injective. For $j\neq l$ let $f_{j,l}=f(z_l)-f(z_j)$ and assume $\arg(\lambda)\neq -\arg(f_{j,l})$ for all pairs $z_j\neq z_l$. For each z_j let $\gamma_j=e^{-i\arg(\lambda)}\mathbb{R}_{\geq 0}+f(z_j)$ and let σ_j be the vanishing cycle emanating from z_j (this is unique up to a choice of orientation). Choose for each j a regular value c_j near $f(z_j)$. Introduce

$$T^{(j)}(\lambda)=\lambda^{\frac{d}{2}}e^{\lambda f(z_j)}\int_{\Delta(\sigma_j,\gamma_j)}e^{-\lambda f(x)}\omega(x).$$

Theorem 12. *If $\zeta\mapsto\int_{\sigma_j(\zeta)}\omega/\,\mathrm{d}\,f$ decays fast enough at infinity*[7] *the expansion* (1.10) *is exact in the sense that there exists for each z_j a sequence $\{c(j)_r\}_{r\in\mathbb{N}}\subset\mathbb{C}$ such that for every $M\in\mathbb{N}$ the followhing holds, where $\langle\cdot,\cdot\rangle$ is the intersection pairing and*

$$T^{(j)}(\lambda)=\sum_{r=0}^{M-1}\frac{c(j)_r}{\lambda^r}+\frac{1}{2\pi i}\sum_{l\neq j}\frac{(-1)^{\frac{d(d-1)}{2}}\langle\sigma_j(c_l),\sigma_l(c_l)\rangle}{(\lambda f_{j,l})^M}\int_0^\infty\frac{v^{M-1}e^{-v}}{(1-v/(\lambda f_{j,l}))}T^{(l)}\left(\frac{v}{f_{j,l}}\right)\mathrm{d}\,v. \tag{1.30}$$

One can iterate (1.30) and this leads to a recursion known as hyperasymptotics which asymptotically gives the high order coefficients $c(j)_r, r>>0$ in terms of lower order coefficients $c(l)'_r, r'<r$ at distant saddles – see [How97] and [DH02]. In principle, the proof of (1.30) in the context of resurgence phases can be reduced to the proof given in [How97] by using (1.12). a suitable version of the Pham-Picard-Lefschetz theorem [Pha65, Pha67, PS71, Lef50] must be applied. See the appendix in [DH02] for more details.

Much recent research in mathematical physics focuses on combining resurgence and recursion techniques – see for instance [CD19]. It is an important task, and beyond the scope of this thesis, to further investigate the connection between resurgence and topological and geometric recursion [EO09, ABO17]. These theories have been used to great succes in many settings, including in Chern-Simons theory, see for example [BEMn12].

[7] See [DH02] or Howls [How97] for the precise condition.

1.4 Organization, notations and conventions

We let $\mathcal{M}\mathrm{an}(\mathbb{R})$ (resp. $\mathcal{M}\mathrm{an}_d(\mathbb{R})$) be the category whose objects are smooth manifolds (of dimension d) without boundary, and whose morphisms are smooth maps. We let $\mathcal{M}\mathrm{an}_d(\mathbb{C})$ be the category of complex manifolds of complex dimension d, with morphisms given by holomorphic maps. The category of smooth manifolds (of dimension d) with (possibly empty) boundary, and morphisms given by smooth maps, is denoted by $\mathcal{M}\mathrm{an}^{\partial}$ (resp. $\mathcal{M}\mathrm{an}^{\partial}(d)$).

Organization

This thesis is organised as follows.

- **Chapter** 2 presents classical Chern-Simons theory following [Fre95].
- **Chapter** 3 presents the construction of quantum invariants from modular tensor categories following [Tur10]. We give an explicit construction of the modular tensor category relevant for $\mathrm{SU}(2)$.
- **Chapter** 4 presents the relevant definitions from [AP17a]. In the interest of brevity, we have not included the proof of Theorem 10. The proof is given in full detail in [AP17a].
- **Chapter** 5 presents classical results concerning stationary phase approximation and proofs of Theorem 5 and Theorem 6. We also prove that the rapid decay homology groups introduced by Pham in [Pha83] can be defined for resurgence phases, and are generated by Picard-Lefschetz thimbles.
- **Chapter** 6 presents geometric quantization following [Sch12] and Hitchin connections following [And12]. We present the quantum representations and prove the Theorems 1, 2, 3 and 4.
- **Chapter** 7 proves Theorems 7 , 8 and 9. We also present the parts of the computations from [LR99] which are relevant for our purposes.
- **Chapter** 8 Presents some details on work in progress concerning analytic extensions of quantum invariants. In particular, we prove Theorem 11 and present some computations in support of Conjecture 2.

CHAPTER 2

Chern-Simons theory

We will assume familiarity with basic notions and foundational results in gauge theory.

2.1 Classical theory

Chern-Simons theory was originally introduced in [CS74]. We will closely follow [Fre95]. Fix for now a compact, simple, simply connected Lie group G with Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$. Fix a bilinear, symmetric, Ad-invariant, non-degenerate form $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{g}$, which is normalized such that the closed three form $\langle \omega_{MC} \wedge [\omega_{MC} \wedge \omega_{MC}] \rangle \in H^3(G, \mathbb{R})$, represents an integral class. A principal G bundle $G \hookrightarrow P \overset{\pi}{\twoheadrightarrow} M$ will be denoted by (P, π, M) and the space of connections will be denoted by $\mathcal{A}(P)$. For a connection ω, we denote its curvature by $F_\omega = \mathrm{d}\,\omega + \frac{1}{2}[\omega, \omega]$. Recall that ω is said to be flat if $F_\omega = 0$, and if so (P, ω), is referred to as a flat bundle. Recall that a connection on a principal G bundle P is *reducible*, if for the induced connection ∇ on Ad_P, we have a splitting $(\mathrm{Ad}_P, \nabla) \simeq \oplus_i (A_i, \nabla_i)$. A connection is *irreducible* if it is not reducible.

Definition 2.1.1. Let (P, π, M) be a principal G bundle. The *Chern-Simons* form associated with $\omega \in \mathcal{A}(P)$ is

$$\alpha(\omega) = \langle \omega \wedge F_\omega \rangle - \frac{1}{6} \langle \omega \wedge [\omega \wedge \omega] \rangle \in \Omega^3(P).$$

Let $M \in \mathcal{M}\mathrm{an}_d(\mathbb{R}), d \leq 3$. It follows from results in homotopy theory that any principal G bundle (P, π, M) admits a global section $\sigma \in \Gamma^\infty(M, P)$.

Definition 2.1.2. Let $M \in \mathcal{M}\mathrm{an}^\partial(3)$ be compact and oriented. Let (P, π, M) be a principal G bundle. For any pair $(\omega, \psi) \in \mathcal{A}(P) \times \Gamma^\infty(M, P)$, the *Chern-Simons action* is defined by

$$\mathrm{S}(\omega, \psi) = \int_M \psi^*(\alpha(\omega)).$$

If we wish to stress the depence of the base space M, we write S_M. We define $\mathrm{S}_\varnothing = 0$. Given $Y \in \mathcal{M}\mathrm{an}_2(\mathbb{R})$ which is compact and oriented we can define the *Wess-Zumino-Witten action* $W_Y : C^\infty(Y, G) \to \mathbb{R}/\mathbb{Z}$ as follows. Choose a compact oriented three-manifold B with

$\partial B = Y$, extend $\psi \in C^\infty(Y, G)$ to $\tilde{\psi} \in C^\infty(B, G)$ and define

$$W_Y(\psi) = -\int_B \langle \tilde{\psi}^*(\omega_{MC}) \wedge [\tilde{\psi}^*(\omega_{MC}) \wedge \tilde{\psi}^*(\omega_{MC})] \rangle \bmod \mathbb{Z}. \tag{2.1}$$

The Wess-Zumino-Witten action is in fact independent of the choice of extension. See Lemma 2.12 in [Fre95]. The following result summarizes Proposition 2.3, Proposition 2.7, and Proposition 2.10 in [Fre95].

Proposition 2.1.1. *Let $M \in \mathcal{M}an^\partial(3)$ be compact and oriented, with boundary $Y = \partial M \overset{\iota}{\hookrightarrow} M$. Let (P, π, M) be a principal G bundle. For any pair $(\omega, \psi) \in \mathcal{A}(P) \times \Gamma^\infty(M, P)$ and any $\eta \in \mathcal{G}(P)$ if we let $g = g_\eta \circ \psi \in C^\infty(X, G)$ we have*

$$\mathsf{S}(\omega, \eta \circ \psi) = \mathsf{S}(\eta^*(\omega), \psi) = \mathsf{S}(\omega, \psi) + \int_Y \iota^*(\langle Ad_{g^{-1}}(\psi^*(\omega)) \wedge g^*(\omega_{MC}) \rangle) + W_Y(\iota^*(g)).$$

If $\partial M = \emptyset$ we have

$$\mathsf{S}(\omega, \eta \circ \psi) = \mathsf{S}(\eta^*(\omega), \psi) = \mathsf{S}(\omega, \psi) - \int_X \langle g^*(\omega_{MC}) \wedge [g^*(\omega_{MC}) \wedge g^*(\omega_{MC})] \rangle.$$

In particular, in the case $\partial M = \emptyset$, the Chern-Simons action descends to well-defined map

$$\mathsf{S} : \mathcal{A}(P)/\mathcal{G}(P) \to \mathbb{R}/\mathbb{Z}.$$

If $(F, f) : (P, \pi, M, \omega) \to (P', \pi', M', \omega')$ is a gauge equivalence, and M' is oriented, we have $\mathsf{S}(\omega) = \deg(f)\, \mathsf{S}(\omega')$. If $M = M_1 \sqcup M_2$ and $\omega = \omega_1 \sqcup \omega_2$ we have $\mathsf{S}(\omega) = \mathsf{S}(\omega_1) + \mathsf{S}(\omega_2)$.

As a result, we can make the following definition.

Definition 2.1.3. The *Chern-Simons invariant* of a gauge equivalence class of a principal G bundle with connection (P, π, M, ω), with $M \in \mathcal{M}\mathrm{an}_3(\mathbb{R})$ compact and oriented, is by definition $\mathsf{S}(\omega, \psi) \in \mathbb{R}/\mathbb{Z}$, where ψ is any element of $\Gamma^\infty(M, P)$.

Let us now focus on the case of $G = \mathrm{SU}(2)$. We will use the following ad invariant, non-degenerate, symmetric bilinear form

$$\langle \cdot, \cdot \rangle = \frac{1}{8\pi^2} \mathrm{tr}(\cdot, \cdot).$$

Let M be a closed oriented three manifold equipped with the trivial $\mathrm{SU}(2)$ bundle $P = M \times \mathrm{SU}(2)$. The Chern-Simons action takes the following (perhaps more familiar) form (where $\wedge$ denotes matrix multiplication)

$$\mathsf{S}([A]) = \frac{1}{8\pi^2} \int_M \mathrm{tr}\left(A \wedge \mathrm{d}\, A + \frac{2}{3} A \wedge A \wedge A \right) \in \mathbb{R}/\mathbb{Z}.$$

2.1.1 The Chern-Simons lines

Let $\mathcal{L}$ be the groupoid with objects $\mathbb{C}$ vector spaces of dimension one, equipped with a Hermitian metric, and with morphisms given by unitary isomorphisms. Given a groupoid $\mathcal{C}$ and a functor $T : \mathcal{C} \to \mathcal{L}$ we define $\mathcal{L}_T$ to be the set of coherent sequences, $\{V(c) \in T(c)\}_{c \in \mathcal{C}}$, such that for every $q \in \mathcal{C}(c,c')$ we have $T(q)(V(c)) = V(c')$. Assume that there exists a morphism between every pair of objects in $\mathcal{C}$ and that $\mathcal{C}(c,c) = \mathrm{Id}_c$ for every object c. We say that $\mathcal{C}$ is a *connected groupoid with no non-trivial automorphisms*. This ensures that $\mathcal{L}_T$ is naturally a $\mathbb{C}$ vector space of dimension one, equipped with a Hermitian metric.

Now consider a principal G bundle (Q,π,Y) where $Y \in \mathcal{M}\mathrm{an}_2(\mathbb{R})$ is compact and oriented. Let $\mathcal{C}(Q) = \Gamma^\infty(Y,Q)$. Given any pair of global sections $\sigma_1, \sigma_2 \in \mathcal{C}(Q)$ there is a unique gauge transform $f \in \mathcal{G}(Q)$ with $f^*(\sigma_1) = \sigma_2$. Thus $\mathcal{C}(Q)$ is naturally a connected groupoid with no non-trivial automorphisms. Given $\omega \in \mathcal{A}(Q)$, we define a functor $T_\omega : \mathcal{C}(Q) \to \mathcal{L}$ as follows. It is constantly equal to $\mathbb{C}$ on the level of objects. For any gauge transformation $\eta : \sigma_1 \to \sigma_2$ we let $T_\omega(\eta)$ be multiplication by $c_Y(\sigma_1^*(\eta), g_\eta \circ \sigma_1)$, where $c_Y : \Omega^1(Y,\mathfrak{g}) \times C^\infty(Y,G) \to U(1)$ is defined by

$$c_Y(a,\psi) = \exp(2\pi i \int_Y \langle \mathrm{Ad}_{\psi^{-1}}(a) \wedge \psi^*(\omega_{MC})\rangle + W_Y(\psi)).$$

Here W_y is the Wezz-Zumino-Witten action (2.1). Functoriality of T_ω follows from the cocycle condition $C_Y(a, \psi\cdot\psi') = c_Y(a\psi)c_Y(\mathrm{Ad}_{\psi^{-1}}(a) + \psi^*(\omega_{MC}), \psi')$.

Definition 2.1.4. Let (Q,π,Y) be a principal G bundle, with $Y \in \mathcal{M}\mathrm{an}_2(\mathbb{R})$, compact and oriented. The *Chern-Simons line* associated with $\omega \in \mathcal{A}(Q)$ is by defintion $\mathcal{L}_\omega := \mathcal{L}_{T_\omega}$. If we wish to stress the depence of the base space Y we write $\mathcal{L}_Y$. We define $\mathcal{L}_\emptyset = \mathbb{C}$.

Let (P,π,M) be a principal G bundle, with $M \in \mathcal{M}\mathrm{an}_3(\mathbb{R})$ compact, oriented and with boundary $\partial M = Y$. Let (Q,π,Y) be the restriction of P to Y. Let $\omega \in \mathcal{A}(P)$. Proposition 2.1.1 implies that the assignment

$$\Gamma^\infty(M,P) \ni \psi \mapsto e^{2\pi i S(\omega,\psi)} \in \mathbb{C},$$

naturally defines an element of the Chern-Simons line $\mathcal{L}_{\omega_{|Y}}$. We implicitly use, that any $\mu \in \Gamma^\infty(Y,Q)$ extends to $\eta_\mu \in \Gamma^\infty(M,P)$.

Theorem 2.1.2 (Theorem 2.19, [Fre95])**.** *Let (P,π,M) be a principal G bundle, with $M \in \mathcal{M}an_3^\partial(\mathbb{R})$ compact, oriented and with (possibly empty) boundary $\partial M = Y \overset{\iota}{\hookrightarrow} M$, equipped with the Stokes orientation. Let (Q,π,Y) be the restriction of P to Y. Then the following holds. The Chern-Simons lines form a smooth hermitian line bundle $\mathcal{L} \to \mathcal{A}(Q)$. The map $e^{2\pi i \mathrm{S}} : \mathcal{A}(P) \to \mathcal{L}$ determines a smooth unit norm section of the pullback line bundle $\iota^*(\mathcal{L}) \to \mathcal{A}(P)$. Moreover, we have that:*

- *Functoriality: If $(G,g) : (Q',\pi',Y') \to (Q,\pi,Y)$ is an isomorphism, and g is an orientation preserving diffeomorphism, then there exists an induced isometry*

$$G^* : \mathcal{L}_Y \to G^*(\mathcal{L}_{Y'}),$$

and this defines a contravariant functor. If (G,g) is the restriction of an isomorphism $(F,f) : (P',\pi',M') \to (P,\pi,M)$ and f is orientation preserving then

$$G^* \circ e^{2\pi i \mathrm{S}} = e^{2\pi i \mathrm{S} \circ F^*}.$$

- *Orientation compatibility: We have a natural isometry $\mathcal{L}_{-Y} \simeq \overline{\mathcal{L}_Y}$ and with respect to this we have*
$$e^{2\pi i \mathrm{S}_M} = \overline{e^{2\pi i \mathrm{S}_{-M}}}.$$

- *Multiplicativity: If $Y = Y_1 \sqcup Y_2$ we have a natural isomorphism $\mathcal{L}_Y \simeq \mathcal{L}_{Y_1} \boxtimes \mathcal{L}_{Y_2}$. If the decomposition of Y results from a decomposition $M = M_1 \sqcup M_2$, we have*
$$e^{2\pi i \mathrm{S}_M} = e^{2\pi i \mathrm{S}_{M_1}} \boxtimes e^{2\pi i \mathrm{S}_{M_2}}.$$

- *Locality: Suppose $N \hookrightarrow M$ is a closed, oriented, embedded submanifold of dimension 2 and M^{cut} is the compact, oriented manifold obtained by cutting M along N. Then $\partial M^{cut} = \partial M \sqcup N \sqcup (-N)$. There is an induced map $j : \mathcal{A}(P) \to \mathcal{A}(P^{cut})$, and we have*
$$e^{2\pi i \mathrm{S}_M(\omega)} = Tr\left(e^{2\pi i \mathrm{S}_{M^{cut}}(j(\omega))}\right)$$
where Tr is the contraction
$$Tr : \mathcal{L}_{j(\omega)} \simeq \mathcal{L}_{j(\omega)_{|\partial M}} \otimes \mathcal{L}_{j(\omega)_{|N}} \otimes \mathcal{L}_{j(\omega)_{|-N}} \to \mathcal{L}_{j(\omega)_{|\partial M}},$$
which is induced using the multiplicative property and the Hermitian metric.

Remark 2.1.3. Concerning the locality property: There is a natural map $i \in C^\infty(M^{\mathrm{cut}}, M)$ and $P^{\mathrm{cut}} = i^*(P)$ and $j = i^*$.

Though we shall not present it here, one can also consider the case where the surface Y has boundary components (or punctures), and define Chern-Simons lines over G bundles over Y. This is done by Freed in Chapter 4 of [Fre95]. It is also thoroughly discussed in [AHJ+17].

2.1.2 The Euler-Lagrange equation

As the space of connections is a Frechét manifold, it makes sense to differentiate the Chern-Simons action. According to the prinicple of least action, we think of the critical points of the Chern-Simons action, as the space of classical solution in Chern-Simons.

Proposition 2.1.4 (Proposition 3.1, [Fre95])**.** *Let (P, π, M) be a principal G bundle, with $M \in \mathcal{M}an_3(\mathbb{R})$ oriented and closed. For any $\omega \in \mathcal{A}(P)$ and any $\psi \in \Gamma^\infty(M, P)$ we have for every $\eta \in T_\omega \mathcal{A}(P) \simeq \Omega^1(M, \mathfrak{g})$*
$$\mathrm{d}\,\mathrm{S}_{\omega,\psi}(\eta) = 2\int_X \psi^*(\langle F_\omega \wedge \eta \rangle).$$
In particular we have $\mathrm{d}\,\mathrm{S}_{\omega,\psi} = 0$, if and only if $F_\omega = 0$.

This proposition shows the importance of the moduli spaces $\mathcal{M}(G, M)$ of flat connections introduced in Section 1.1.

2.2 Complex Chern-Simons theory

One can define Chern-Simons invariants for $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{PSL}(n,\mathbb{C})$ connections as well, by simply extending the formula for $\mathrm{SU}(2)$ connections. This was done (in higher generality) in the following article by Cheeger and Simons [CS85]. We shall focus on $n = 2$.

Definition 2.2.1. Let $H \in \{\mathrm{SL}(2,\mathbb{C}), \mathrm{PSL}(2,\mathbb{C})\}$ with Lie algebra $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})$. The *Cheeger-Chern-Simons* invariant of a gauge equivalence class $[A] \in \Omega^1(M,\mathfrak{h})/C^\infty(M,H)$ is given by

$$\mathrm{S}_{\mathbb{C}}([A]) = \frac{1}{8\pi^2}\int_M \mathrm{tr}\left(A \wedge \mathrm{d}\,A + \frac{2}{3} A \wedge A \wedge A\right) \in \mathbb{C}/\mathbb{Z}.$$

As in the case with compact gauge group, the classical solutions correspond to flat connections. These Cheeger-Chern-Simons invariants are closely connected to hyperbolic geometry.

Recall that by the *Mostow-rigidity theorem* [Mos68] the *hyperbolic volume* $\int_M \mathrm{Vol}(g)$ of an oriented closed hyperbolic three-manifold (M, g) is a topological invariant and will be denoted simply by $\mathrm{Vol}(M)$.

Theorem 2.2.1 ([Yos85])**.** *Let M be a closed oriented hyperbolic three-manifold and let A be the gauge equivalence class of the flat $\mathrm{PSL}(2,\mathbb{C})$ connection associated with the geometric representation. We have that*

$$4\pi^2\,\mathrm{Im}(\mathrm{S}([A]) = \mathrm{Vol}(M).$$

This was conjectured by Thurston [Thu82], and Neumann and Zagier [NZ85]. See [GTZ15] for further discussion of these aspects.

2.2.1 Computing Cheeger-Chern-Simons invariants

We shall present a formula for the difference of (Cheeger)-Chern-Simons invariants due to Kirk and Klassen.

Theorem 2.2.2 ([KK90])**.** *Let M be a closed oriented three-manifold containing a knot K. Let Y be the complement of a tubular neighborhood of K in M. With respect to an identification $M \setminus Y \simeq D^2 \times S^1$, choose simple closed curves μ, λ on ∂Y intersecting in a single point such that μ bounds a disc of the form $\partial D^2 \times \{1\}$. Let $\rho_t : \pi_1(Y) \to SU(2)$ be a path of representations such that $\rho_0(\mu) = \rho_1(\mu) = 1$, and for which there exists piecewise continuous differentiable functions $\alpha, \beta : I \to \mathbb{R}$ with*

$$\rho_t(\mu) = \begin{pmatrix} e^{2\pi i\alpha(t)} & 0 \\ 0 & e^{-2\pi i\alpha(t)} \end{pmatrix},\ \rho_t(\lambda) = \begin{pmatrix} e^{2\pi i\beta(t)} & 0 \\ 0 & e^{-2\pi i\beta(t)} \end{pmatrix}.$$

Thinking of ρ_1, ρ_0 as flat connections on M we have

$$\mathrm{S}(\rho_0) - \mathrm{S}(\rho_1) = -2\int_0^1 \beta(t)\alpha'(t)\ \mathrm{d}\,t \mod \mathbb{Z}.$$

Notice that the formula (2.2.2) differs from the corresponding formula in [KK90] by a sign. This discrepancy was already discussed by Freed and Gompf in [FG91] and is due to a sign convention. See the footnote on page 98 in [FG91].

We make the following remark, which explains how the formula (2.2.2) also works for some paths of $\mathrm{SL}(2,\mathbb{C})$ representations.

Remark 2.2.3. Kirk and Klassen remarks in [KK90] that (2.2.2) is also valid for a path of $\mathrm{SL}(2,\mathbb{C})$ connections, as long as the path ρ_t stays away from parabolic representations. This is to ensure that ρ_t is conjugate to a path which maps λ, μ to the maximal $\mathbb{C}^*$ torus of diagonal matrices.

Theorem 2.2.2 was further developed in [KK93], in a work which puts special emphasis on the $\mathrm{SL}(2,\mathbb{C})$ case. In [Auc94] Theorem 2.2.2 was used by Auckley to compute $\mathrm{SU}(2)$ Chern-Simons for a large class of oriented closed three-manifolds. It is conjectured that all $\mathrm{SU}(2)$ Chern-Simons invariants of flat connections are rational numbers.

Recent work by Garoufalidis-Thurston-Zickert [GTZ15] (see also [DZ06]) provides an algorithm for computing the Chern-Simons invariants, which makes use of the so-called Ptolemy coordinates on the character variety $\mathcal{M}(\mathrm{SL}(2,\mathbb{C}), M)$. This was (even more recently) used by Yoon [Yoo18], to give a computation of Cheeger-Chern-Simons invariants from a surgery presentation of M. We shall now present the main result of this work.

Yoon: the potential of a surgery link

Let M be a closed oriented three-manifold. Let $L \subset S^3$ be a surgery link. Let $D = D(L)$ be an oriented diagram of L, with at least one over crossing and one under crossing in each component. Let $C = C(D)$ be the set of crossings of L. Let $R = R(D)$ be the set of regions, which are the connected components of the complement of D. Let $n = |R|$. We will introduce a a holomorphic function $\mathbb{W} \in \mathcal{O}(X_D)$, where X_D is the universal cover of the complement of a divisor D in $\mathbb{C}^{\pi_0(L)} \times \mathbb{C}^R$. First we recall the dilogarithm

Definition 2.2.2. The *dilogarithm* Li_2 is the holomorphic function on the universal cover of $\mathbb{C} \setminus \{1\}$ given by

$$\mathrm{Li}_2(z) = -\int_{\gamma_z} \frac{\log(1-u)}{u}\, \mathrm{d}\, u,$$

where γ_z is the homotopy class of a path from 0 to z in $\mathbb{C} \setminus \{1\}$.

Let us record two of the well-known functional equations of the dilogarithm.

Theorem 2.2.4. *We have*

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}(\log(-z))^2,$$
$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) + \frac{\pi^2}{6} - \log(z)\log(1-z).$$

We next introduce some divisors.

Definition 2.2.3. On $\mathbb{C}^2 \times \mathbb{C}^4$ we consider coordinates $(m_l, m_r, w_l, w_r, w_o, w_d)$. We consider the following divisors

$$D^+ = D_1^+ \cup D_2^+ \cup D_3^+ \cup D_4^+ \cup D_5^+, \quad D^- = D_1^- \cup D_2^- \cup D_3^- \cup D_4^- \cup D_5^-$$

where

$$D_1^+ = \{w_l - w_d - m_r \in \mathbb{Z}\}, \ D_2^+ = \{w_r - w_d - m_l \in \mathbb{Z}\}, \ D_3^+ = \{w_o - w_r - m_r \in \mathbb{Z}\}$$

and

$$D_4^+ = \{w_o - w_l - m_l \in \mathbb{Z}\}, \ D_5^+ = \{w_d - w_l + w_o - w_r \in \mathbb{Z}\}.$$

Further

$$D_1^- = \{m_r + w_o - w_r \in \mathbb{Z}\}, \ D_2^- = \{m_l + w_o - w_l \in \mathbb{Z}\}, \ D_3^- = \{m_r + w_l - w_d \in \mathbb{Z}\}$$

and

$$D_4^- = \{m_l + w_r - w_d \in \mathbb{Z}\}, \ D_5^- = \{w_d - w_l + w_o - w_r \in \mathbb{Z}\}.$$

For each crossing $c \in C$ we get a divisor D_c inside $\mathbb{C}^{\pi_0(L)} \times \mathbb{C}^R$ by pulling back $D^{s(c)}$ with respect to the projection

$$\tilde{\pi}_c : \mathbb{C}^{\pi_0(L)} \times \mathbb{C}^R \to \mathbb{C}^2 \times \mathbb{C}^4.$$

We then let $D' = \bigcup_{c \in C} D_c$. We define the projection maps from $X_{D'}$ to $X^{s(c)}$ for each c, which we also denote $\tilde{\pi}_c$.

We can now introduce the potential

Definition 2.2.4. Let $X^\pm$ be the universal cover of $\mathbb{C}^2 \times \mathbb{C}^4 - D^\pm$ and define $\mathbb{W}_\pm \in \mathcal{O}(X^\pm)$ by

$$\begin{aligned}
\mathbb{W}_+(m_l, m_r, w_l, w_r, w_o, w_d) &= \mathrm{Li}_2(e^{2\pi i(w_l - w_d - m_r)}) + \mathrm{Li}_2(e^{2\pi i(w_r - w_d - m_l)}) \\
&\quad - \mathrm{Li}_2(e^{2\pi i(w_o - w_r - m_r)}) - \mathrm{Li}_2(e^{2\pi i(w_o - w_l - m_l)}) - \frac{\pi^2}{6} \\
&\quad + \mathrm{Li}_2(e^{2\pi i(w_d - w_l + w_o - w_r)}) + 4\pi^2(w_l - w_d - m_r)(w_r - w_d - m_l),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{W}_-(m_l, m_r, w_l, w_r, w_o, w_d) &= \mathrm{Li}_2(e^{2\pi i(w_o - w_r + m_r)}) + \mathrm{Li}_2(e^{2\pi i(w_o - w_l + m_l)}) \\
&\quad - \mathrm{Li}_2(e^{2\pi i(w_l - w_d + m_r)}) - \mathrm{Li}_2(e^{2\pi i(w_r - w_d + m_l)}) + \frac{\pi^2}{6} \\
&\quad + 4\pi^2(w_l - w_d + m_r)(w_r - w_d + m_l) - \mathrm{Li}_2(e^{2\pi i(w_d - w_l + w_o - w_r)}).
\end{aligned}$$

We define the *pre-potential function* $\mathbb{W} \in \mathcal{O}(X_{D'})$ by

$$\mathbb{W} = \sum_{c \in C} \mathbb{W}_{s(c)} \circ \tilde{\pi}_c.$$

The *potential function* of the link L is

$$\mathbb{W}_0(m, w) = \mathbb{W}(m, w) - \sum_{r \in R} \frac{\partial \mathbb{W}}{\partial w_r}(m, w) w_r - \sum_{\gamma \in \pi_0(L)} \frac{\partial \mathbb{W}}{\partial m_\gamma}(m, w) m_\gamma.$$

Remark 2.2.5. We work with logarithmic coordinates compared to [Yoo18] and we will use slightly different notation. That is, if his coordinates are denoted by $(\tilde{m}, \tilde{w})$ and (m, w) are the coordinates we introduced above, we have

$$\tilde{m} = \exp(2\pi i m), \quad \tilde{w} = \exp(2\pi i w)$$

Although we use coordinate expressions, each expression in the arguments of the dilogarithm symbolizes the corresponding map from $X^{\pm}$ to the universal cover of $\mathbb{C} - \{1\}$. Moreover, in comparison with Yoon, we have switched the over and under crossings. This is to take into account a different sign convention for the Chern-Simons action, and the fact that if L is a surgery link for M, then the mirror image of L is a surgery link for $-M$.

Let us now introduce the following set

Definition 2.2.5. Let

$$\mathcal{S}' = \{(m, w) \in X_D \mid \frac{\partial \mathbb{W}}{\partial w_r} \in 4\pi^2\mathbb{Z}, r \in R; \rho_{(m,w)}(\lambda_\gamma) = \mathrm{Id}, m_\gamma \notin \mathbb{Z}, \gamma \in \pi_0(L)\},$$

and let

$$\mathcal{M}' = \{\rho \in \hom(\pi_1(M), \mathrm{PSL}(2, \mathbb{C})) \mid \rho(\mu_\gamma) \neq \pm\mathrm{Id}, \gamma \in \pi_0(L)\} / \mathrm{PSL}(2, \mathbb{C}).$$

Theorem 2.2.6 ([Yoo18])**.** *There is a surjective holomorphic map $\rho : \mathcal{S}' \to \mathcal{M}'$, and we have*

$$\pi^2 \, \mathrm{S}_{\mathbb{C}} \circ \rho = \mathbb{W}_0 \mod \pi^2\mathbb{Z}.$$

CHAPTER 3

Quantum invariants

There is a bijection between closed oriented three-manifolds considered up to orientation preserving diffeomorphism and framed links in S^3 considered up to so-called Kirby equivalence (see Theorem 3.1.4 below) and the process of associating a three-manifold to a link via this correspondence is called surgery. The Reshetikhin-Turaev TQFT is essentially constructed using these surgery presentations of three-manifolds and certain categories which can be used to construct invariants of links that are invariant under Kirby equivalence. These categories are the so-called modular tensor categories (MTC) or modular categories, as they are called in Turaev's book [Tur10]. The MTCs connected to Chern-Simons theory arise as certain representation categories of quantum groups. The discussion of modular tensor categories given below closely follows Turaev's book [Tur10].

3.1 Modular tensor categories

We begin by introducing the notion of a monoidal category. We will assume familiarity with basic category theory. A great introduction which more than suffices is the classic monograph by S. MacLane [ML98].

Definition 3.1.1. A *unital strict monoidal category* is a triple $(\mathcal{V}, \otimes, \mathbb{1})$, consisting of the the following data. A category $\mathcal{V}$, an associative covariant functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and an object $\mathbb{1} \in \mathrm{Obj}(\mathcal{V})$ which satisfies the following. The covariant functors $(\,\cdot\,) \otimes \mathbb{1} : \mathcal{V} \to \mathcal{V}$ and $\mathbb{1} \otimes (\,\cdot\,) : \mathcal{V} \to \mathcal{V}$ are both equal to the identity functor $\mathrm{Id}_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$. The object $\mathbb{1} \in \mathrm{Obj}(\mathcal{V})$ is called the *unit*. A *strict monoidal functor* is a covariant functor between unital strict monoidal categories $F : (\mathcal{V}, \otimes, \mathbb{1}) \to \mathcal{V}', \otimes', \mathbb{1})'$ with $(F, F) \circ \otimes = \otimes' \circ (F, F) : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \times \mathcal{V}$.

We observe that the definition imply that for any morphism $f \in \mathcal{V}(V, W)$ we have $f \otimes \mathrm{Id}_{\mathbb{1}} = \mathrm{Id}_{\mathbb{1}} \otimes f = f$.

Many authors relax the condition that $\otimes$ is strictly associative, and instead require the existence of certain natural transformations known as left and right associaters, which are required to (together with $\otimes$) satisfy identities known as the pentagon and the triangle identity. See [ML98]. From a certain viewpoint, nothing is lost by considering instead strict monoidal categories because of MacLane's coherence Theorem. We shall return to this point below. Next we introduce Ribbon categories.

Definition 3.1.2. A *Ribbon category* is a tuple $(\mathcal{V}, \otimes, \mathbb{1}, C, \cdot^*, \theta, b, d)$ which consists of the following data. The triple $(\mathcal{V}, \otimes, \mathbb{1})$ is a strict unital monoidal category. We have a natural isomorphism $C : \otimes \Rightarrow \otimes \circ P$, called the *braiding*, where $P : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \times \mathcal{V}$ is the permutation functor, i.e. $P(V, W) = (W, V)$ for all pairs $V, W \in \mathrm{Obj}(\mathcal{V})$. For any triple of objects $U, V, W \in \mathrm{Obj}(\mathcal{V})$ we have

$$C_{U,V\otimes W} = (\mathrm{Id}_V \otimes C_{U,W}) \circ (C_{U,V} \otimes \mathrm{Id}_W), \quad C_{U\otimes V,W} = (C_{U,W} \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_U \otimes C_{V,W}).$$

We have a natural isomorphism $\theta : \mathrm{Id}_{\mathcal{V}} \to \mathrm{Id}_{\mathcal{V}}$, called the *twist*, such that for every pair of objects $V, W \in \mathrm{Obj}(\mathcal{V})$ we have

$$\theta_{V\otimes W} = C_{W,V} \circ C_{V,W} \circ (\theta_V \otimes \theta_W).$$

We have a map $\cdot^* : \mathrm{Obj}(\mathcal{V}) \to \mathrm{Obj}(\mathcal{V})$, together with an assignment for each $V \in \mathrm{Obj}(\mathcal{V})$ of morphisms $b_V \in \mathcal{V}(\mathbb{1}, V \otimes V^*), d_V \in \mathcal{V}(V^* \otimes V, \mathbb{1})$ which satisfy

$$(\mathrm{Id}_V \otimes d_V) \circ (b_V \otimes \mathrm{Id}_V) = \mathrm{Id}_V, \quad (d_v \otimes \mathrm{Id}_{V^*}) \circ (\mathrm{Id}_{V^*} \otimes b_V) = \mathrm{Id}_{V^*},$$

and

$$(\theta_V \otimes \mathrm{Id}_{V^*}) \circ b_V = (\mathrm{Id}_V \otimes \theta_{V^*}) \circ b_v.$$

The triple $(\cdot^*, b, d)$ is called the *duality*, d is called the *evaluation* whereas b is called the *co-evaluation*.

It is easy to verify that for any object $V \in \mathrm{Obj}(\mathcal{V})$ in a ribbon category the braiding must satisfy $C_{V,\mathbb{1}} = C_{\mathbb{1},V} = \mathrm{Id}_V$. Moreover, the twist must satisfy $\theta_{\mathbb{1}} = \mathrm{Id}_{\mathbb{1}}$.

Next we introduce the notions of trace and dimension.

Definition 3.1.3. Let $(\mathcal{V}, \otimes, \mathbb{1}, C, \cdot^*, \theta, b, d)$ be a ribbon category. For any object $V \in \mathrm{Obj}(\mathcal{V})$ define $\mathrm{tr} : \mathcal{V}(V, V) \to \mathcal{V}(\mathbb{1}, \mathbb{1})$ by

$$\mathrm{tr}(f) = d_v \circ C_{V,V^*} \circ ((\theta_V \circ f) \otimes \mathrm{Id}_{V^*}) \circ b_V.$$

Define $\dim(V) = \mathrm{tr}(\mathrm{Id}_V)$.

Next we introduce abelian categories (in the sense of [Tur10]) and some algebraic notions pertaining to ribbon categories which are also abelian categories.

Definition 3.1.4. An *abelian* category is a category $\mathcal{C}$, such that for every pair $V, W \in \mathrm{Obj}(\mathcal{C})$, the set $\mathcal{C}(V, W)$ is aquipped with the structure of an abelian group, and for every $Z \in \mathrm{Obj}(\mathcal{C})$, the composition $\circ : \mathcal{C}(V, W) \times \mathcal{C}(W, Z) \to \mathcal{C}(V, Z)$ is linear.

Proposition 3.1.1. *Let $(\mathcal{V}, \otimes, \mathbb{1}, C, \cdot^*, \theta, b, d)$ be a ribbon category which is also an abelian category. The set $K = \mathcal{V}(\mathbb{1}, \mathbb{1})$ is a commutative ring, with multiplication given by composition. For any pair of objects $V, W \in Obj(\mathcal{V})$, the group $\mathcal{V}(V, W)$ is canonically a left K-module with the action $(k, f) \mapsto k \otimes f, k \in K, f \in \mathcal{V}(V, W)$. For any triple of objects $V, W, Z \in Obj(\mathcal{V})$, the composition $\circ : \mathcal{V}(V, W) \times \mathcal{V}(W, Z) \to \mathcal{V}(V, Z)$ is K-bilinear. Similarly, the tensor product $\otimes$ is K-bilinear.*

Motivated by this proposition, we make the following definition.

Definition 3.1.5. A ribbon category, which is also an abelian category, is called an *abelian ribbon category*. The *ground ring* of an abelian ribbon cateogory $(\mathcal{V}, \otimes, \mathbb{1}, C, \cdot^*, \theta, b, d)$, is defined to be $K = \mathcal{V}(\mathbb{1}, \mathbb{1})$. An object $V \in \mathrm{Obj}(\mathcal{V})$ is called *simple* if $\mathcal{V}(V, V)$ is a free K-module of rank 1. Let $\{V_i\}_{i \in I} \subset \mathrm{Obj}(\mathcal{V})$. An object $V \in \mathrm{Obj}(\mathcal{V})$ is said to be *dominated* by $\{V_i\}_{i \in I}$ if the following composition induces a surjective K-linear map

$$\circ : \bigoplus_{i \in I} \mathcal{V}(V, V_i) \otimes_K \mathcal{V}(V_i, V) \twoheadrightarrow \mathcal{V}(V, V).$$

We are now finally in position to define modular tensor categories.

Definition 3.1.6. A *modular tensor category* is an abelian ribbon category $(\mathcal{V}, \otimes, \mathbb{1}, C, \cdot^*, \theta, b, d)$ together with a finite family of simple objects $\{V_i\}_{i \in I} \subset \mathrm{Obj}(\mathcal{V})$ which satisfies the following conditions. There exists $0 \in I$ such that $V_0 = \mathbb{1}$. For any $i \in I$ there exists $i^* \in I$ such that V_{i^*} is isomorphic to $(V_i)^*$. Any object $V \in \mathrm{Obj}(\mathcal{V})$ is dominated by $\{V_i\}_{i \in I}$. The matrix S with entries $S = (S_{i,j})_{i,j \in I} \in K$ is invertible over K.

We shall often abbreviate a modular tensor category by $(\mathcal{V}, \{V_i\}_{i \in I})$. We now introduce some concepts related to a modular tensor category.

Definition 3.1.7. Let $(\mathcal{V}, \{V_i\}_{i \in I})$ be a modular tensor category. For every $i \in I$ we introduce $\dim(i) = \dim(V_i)$. As V_i is simple, there is a uniquely defined invertible element $k_i \in K$, which satisfies $k_i \otimes \mathrm{Id}_{V_i} = \phi_{V_i}$. A *rank of* $(\mathcal{V}, \{V_i\}_{i \in I})$ is an element $\mathcal{D} \in K$ which satisfies

$$\mathcal{D}^2 = \sum_{i \in I} (\dim(i))^2.$$

Introduce

$$\Delta = \sum_{i \in I} k_i^{-1} (\dim(i))^2 \in K.$$

The following remark is very important.

Remark 3.1.2. A rank $\mathcal{D}$ may not always exist, and when it does it is not necessarily unique. However, a rank can always be formally added to a slighly refined modular tensor category $\tilde{\mathcal{V}}$. See the discussion in Section 1.6 in [Tur10].

3.1.1 The graphical calculus

Fix throughout this section a ribbon category $(\mathcal{V}, \otimes, \mathbb{1}, C, \cdot^*, \theta, b, d)$. We simply denote this by $\mathcal{V}$. We shall adopt the following convention. For an object $V \in \mathcal{V}$ and a sign $\epsilon \in \{\pm 1\}$ we let $V^\epsilon = V$ if $\epsilon = 1$, and $V^\epsilon = V^*$ if $\epsilon = -1$.

We now introduce ribbon graphs in three manifolds.

Definition 3.1.8. Let $M \in \mathcal{M}\mathrm{an}^\partial(3)$. Assume M is oriented and is equipped with a decomposition $\partial M = (-\partial_-(M)) \sqcup \partial_+(M)$ and that ∂M is equipped with a finite set of oriented embedded intervals I with a sign I_ϵ. These are called *marked arcs*. A *band* is the square $[0,1] \times [0,1]$ or a homeomorphic image of it in M. The images of the intervals $[0,1] \times \{0\}$

and $[0,1] \times \{1\}$ are called the *bases* of the band,whereas $\{1/2\} \times [0,1]$ is called the *core*. An *annulus* is the cylinder $S^1 \times [0,1]$ or a homeomorphic image in M. The image of the circle $S^1 \times \{1/2\}$ is called the *core*. A band or an annulis is *directed* if its core is oriented, and the orientation is called the *direction*. A *coupon* is a band with a distinguished base called the *bottom*, the opposite is called the *top*. A a *ribbon graph in M* is an oriented two manifold Ω (with corners) embedded in M and decomposed into a union of a finite number of annuli, bands and coupons such that

- The intersection $\Omega \cap \partial M$ is transversal and equal to the marked arcs, meeting the bases of Ω which are not incident to coupons. If an arc has sign 1, the base is directed inside M, and if the arc has sign -1 the base is directed outside. The orientation of Ω induces on each arc the orientation opposite to the given one. The intervals in $\partial_-(M)$ are called the *bottom boundary intervals*, wheres the intervals in $\partial_+(M)$ are called the *top boundary intervals*.

- Other bases of bands lie on bases of coupons, otherwise, the bands, coupons, and annuli are disjoint.

- The bands and annuli are directed.

A *Ribbon graph isotopy* is an ambient isotopy of M which is constant on the boundary intervals, preserves the splitting into bands, annuli and coupons, and preserves directions of bands and annuli as well as the orientation of the ribbon graph. A *tangle* in M is a ribbon graph with no coupons. The pair (M, Ω) is called a *three-manifold with a ribbon graph*. A *homeomorphism of three-manifolds with ribbon graphs* $f : (M_1, \Omega_1) \to (M_2, \Omega_2)$ is an orientation preserving homeomorphism $f : M_1 \to M_2$ which satisfies the following. We have $f(\partial_\pm(M_1) = \partial_\pm(M_2)$, and f induces an orientation preserving homeomorphism $f : \Omega_1 \to \Omega_2$, which respects the splitting into bands, coupons and annuli.

Let us introduce the notion of a (k,l) ribbon graph.

Definition 3.1.9. For non-negative integers (k,l) a (k,l) *Ribbon graph* is a ribbon graph inside $M = \mathbb{R}^2 \times [0,1]$ with $\partial_-(M) = \mathbb{R}^2 \times \{0\}, \partial_+(M) = \mathbb{R}^2 \times \{1\}$, and where the marked arcs are the embedded intervals

$$\{[i-(1/10), i+(1/10)] \times \{0\} \times \{0\} \mid i = 1, ..., k\},$$
$$\{[j-(1/10), j+(1/10)] \times \{0\} \times \{1\} \mid j = 1, ..., l\}.$$

The orientation of a ribbon graph Ω is equivalent to a choice of preffered side, or equivalently a framing in the form of a unit normal $\nu(\Omega)$. As explained in [Tur10], the isotopy class of a ribbon graph (in $\mathbb{R}^2 \times \mathbb{R}$) can be recovered from a ribbon graph diagram.

Definition 3.1.10. A *diagram of a ribbon graph* Ω *in* $R^2 \times [0,1]$ is a projection of Ω in the $\mathbb{R} \times \{0\} \times \mathbb{R}$ plane with the following properties. The projections of the coupons, and cores of annuli and bands are disjoint except for possibly finitely many transversal double points between two bands, two annuli, or an annulus and a band. Any coupon of Ω is point-wise fixed under the projection, bases of coupons are parallel to $\mathbb{R} \times \{0\} \times \{0\}$, and the top base is higher than the botton base. The unit normal $\nu(\Omega)$ is pointing upwards (towards the reader) everywhere. Any band or annuli is replaced by their (directed) cores with an appropriate number of directed kinks inserted which accounts for the winding number of the unit normal $\nu(\Omega)$ along the core.

An example of a diagram of $(4,3)$ ribbon graph is depicted in Figure~3.1.

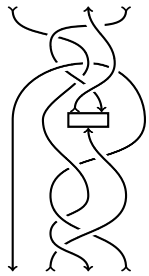


Figure 3.1: Ribbon graph.

We now discuss colored Ribbon graphs

Definition 3.1.11. Let $M \in \mathcal{M}\mathrm{an}^{\partial}(3)$. Assume M is oriented and is equipped with a decomposition $\partial M = (-\partial_-(M)) \sqcup \partial_+(M)$. A ribbon graph Ω in M is said to be *$\mathcal{V}$-colored* if the following holds. Each band and each annuli is assigned an object of $\mathcal{V}$, called its *color*. The coupons are assigned a morphism in $\mathcal{V}$, called their color. This has to be in accordance with the following convention. Let Q be a coupon of Ω. Let $V_1, .., V_m$ be the colors of the bands of Ω which are incident to the bottom base of Q and numbered counterclockwise according to the orientation of Q induced from Ω. Let $W_1, .., W_n$ be the colors of the bands of Ω which are incident to the bottom base of Q, and numbered counterclockwise according to the orientation of Q induced from Ω. Let $\epsilon_1, ..., \epsilon_m \in \{\pm 1\}$ (resp. $\nu_1, ..., \nu_n \in \{\pm 1\}$) be the signs determined by the directions of these bands: $\epsilon_i = 1$ (resp. $\nu_j = -1$) if the band is directed out of the coupon, and $\epsilon_i = -1$ (resp. $\nu_j = 1$) if the band is directed towards the coupon. A color of Q is an arbitrary morphism $f \in \mathcal{V}(V_1^{\epsilon_1} \otimes \cdots \otimes V_m^{\epsilon_m}, W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n})$. By isotopy of $\mathcal{V}$-colored ribbon graphs, we mean an isotopy of ribbon graphs which preserve the coloring. The pair (M, Ω) is called a *three-manifold with a $\mathcal{V}$-colored ribbon graph*. A *$\mathcal{V}$-colored homeomorphism* $f : (M_1, \Omega_1) \to (M_2, \Omega_2)$ between pairs of three-manifolds with $\mathcal{V}$-colored ribbon graphs is an orientation preserving homeomorphism $f : M_1 \to M_2$ which satisfies the following. We have $f(\partial_\pm(M_1)) = \partial_\pm(M_2)$, and f induces a color and orientation preserving homeomorphism $f : \Omega_1 \to \Omega_2$ (which therefore have to respect the splitting into bands, coupons and annuli).

Definition 3.1.12. For $V_1, .., V_m, W_1, ..., W_n \in \mathrm{Obj}(\mathcal{V}), \epsilon_1, ..., \epsilon_m, \nu_1, ..., \nu_n \in \{\pm 1\}$, and $f \in \mathcal{V}(V_1^{\epsilon_1} \otimes \cdots V_m^{\epsilon_m}, W_1^{\nu_1} \otimes W_n^{\epsilon_n})$ introduce $\Omega(f)$ as depicted in figure~3.2. The directions of the bands are determined by the signs ϵ, ν. For $V, W \in \mathrm{Obj}(\mathcal{V})$ introduce the pair $X^+_{V,W}$ and $X^-_{V,W}$ as depicted in figure~3.3 ($X^+_{V,W}$ is to the left). Define $\cup_V, \cup_V^-, \cap_V, \cap_V^-, \phi_V, \phi_V^-$ as depicted in figure~3.4. When neglecting the color, we denote these tangles by $X^\pm, \cap^\pm, \cup^\pm, \phi^\pm$.

Next we introduce the monoidal category of $\mathcal{V}$-colored ribbon graphs (in $\mathbb{R}^2 \times [0,1]$).

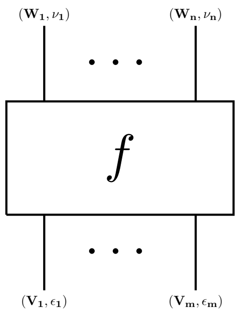


Figure 3.2: Elementary coupons

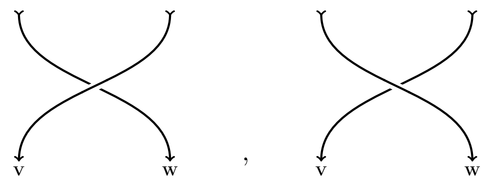


Figure 3.3: Colored crossings

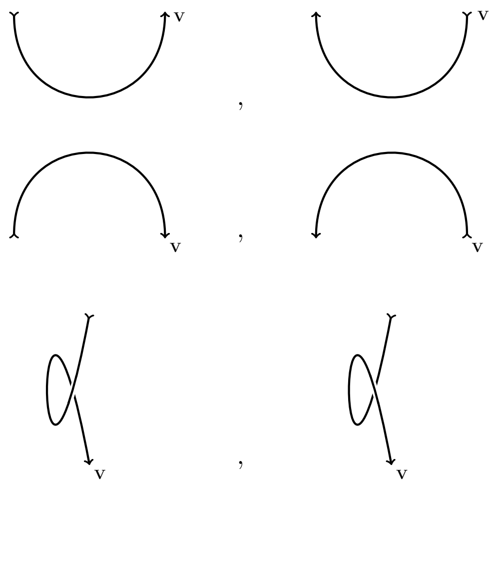


Figure 3.4: Tangles $\cup_V, \cup_V^-, \cap_V, \cap_V^-, \phi_V$, and ϕ_V^- listed from top to bottom, left to right.

Definition 3.1.13. We define the unital strict monoidal category $\mathrm{Rib}_{\mathcal{V}}$ of $\mathcal{V}$-colored ribbon graphs as follows. The objects of $\mathrm{Rib}_{\mathcal{V}}$ are (possibly empty) finite sequences of the fom $((V_1, \epsilon_1), ..., (V_m, \epsilon_m))$ where $V_1, ..., V_m$ are objects of $\mathcal{V}$, and $\epsilon_1, ..., \epsilon_m \in \{\pm 1\}$. A morphism $f : \eta \to \eta'$ in $\mathrm{Rib}_{\mathcal{V}}$ is an isotopy type of a $\mathcal{V}$-colored ribbon graph in $\mathbb{R}^2 \times [0, 1]$ satisfying the following. The sequence η is the sequence of colors and signs of the parts of the tangle, which meets the bottom boundary intervals where $\epsilon = 1$ if the direction is downward and $\epsilon = -1$ otherwise. The sequence η' is the sequence of colors and signs of the part of the tangle which meet the top boundary intervals where $\epsilon = 1$, if the direction is downward, and $\epsilon = -1$ otherwise. Composition of composable morphisms (f, g) is done by glueing the bottom of (a representative of) f to the top of (a representative of) g, and compress the result into a $\mathcal{V}$-colored ribbon graph in $\mathbb{R}^2 \times [0, 1]$. The composition $f \circ g$ is the isotopy type of the resulting $\mathcal{V}$-colored ribbon graph. Identity morphisms are represented by $\mathcal{V}$-colored ribbon graphs with no annuli or coupons, and vertical unlinked bands. The tensor product of objects $\eta = ((V_1, \epsilon_1), ..., (V_m, \epsilon_n))$ and $\eta' = ((W_1, \nu_1), ..., (W_n, \nu_n))$ is defined by horizontal juxtaposition $\eta \otimes \eta' = ((V_1, \epsilon_1), ..., (V_m, \epsilon_n), (W_1, \nu_1), ..., (W_n, \nu_n))$. The tensor products of morphisms f and g is also defined by horizontal juxtaposition, that is, a representative of f is placed to the left of a representative of g with no mutual linking or intersection and compressed into $\mathbb{R}^2 \times [0, 1]$. The $f \otimes g$ is the isotopy type of the resulting $\mathcal{V}$-colored ribbon graph. The unit object is the empty string $\emptyset$.

The following theorem is of foundational importance for this thesis.

Theorem 3.1.3 ([RT90], [RT91], [Tur10])**.** *There exists a uniquely defined strict monoidal functor $F : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ satisfying the following properties. For any object $V \in \mathrm{Obj}(\mathcal{V})$ we have the equalities $F(V, +1) = V$ and $F(V, -1) = V^*$. For any pair of objects $V, W \in \mathrm{Obj}(\mathcal{V})$, and for any elementary coupon $\Omega(f)$ associated with a morphism f in $\mathcal{V}$ and we have*

$$F(X^+_{V,W}) = C_{V,W},\ F(\phi_V) = \theta_V,\ F(\cup_V) = b_V,\ F(\cap_V) = d_V,\ F(\Omega(f)) = f.$$

3.1.2 Surgery

Definition 3.1.14. The (k, m)-*handle* is the manifold $\mathrm{H}(k, m) = \mathrm{B}^k \times \mathrm{B}^{m-k}$. Let $Y \in \mathcal{M}\mathrm{an}^{\partial}_m(\mathbb{R})$. The *attachment of handles* takes as input an embedding $\phi : \sqcup_{j=1}^n \mathrm{S}_j^{k_j - 1} \times \mathrm{B}_j^m \hookrightarrow \partial Y$ and produces the manifold

$$Y(\phi) = \left(\sqcup_{j=1}^n \mathrm{H}(k_j, m)_j \right) \cup_\phi Y.$$

If Y is oriented, then $Y(\phi)$ is naturally oriented.

Recall that a *link* L in $M \in \mathcal{M}\mathrm{an}_3(\mathbb{R})$ is an isotopy class of an embedded $\sqcup_{j=1}^m \mathrm{S}^1_j$, and a framing ϕ is an isotopy class of an embedding $\phi : \sqcup_{j=}^m \mathrm{S}^1_j \times \mathrm{B}^2_j \hookrightarrow M$ with the property $\phi(\sqcup_{j=1}^m \mathrm{S}^1 \times \{0\}) = L$. This determines an embedding of a ribbon graph consisting of annuli.

Definition 3.1.15. Let $L = (L, \phi)$ be a framed link in S^3 and let $\Omega \subset \mathbb{R}^2 \times [0, 1]$ be a ribbon graph which is disjoint from L. Denote by (M_L, Ω) the three-manifold with a ribbon graph obtained as follows. We have $M_L = \partial \left(B^4(\phi) \right)$ and the ribbon graph Ω is naturally induced. We say (L, Ω) is a *surgery link for* (M_L, Ω).

The framing of $L \subset \mathrm{S}^3$ is equivalent to a choice $\gamma = (\gamma_j)$ of simple closed curve on each component ∂T_j of ∂T where T is a tubular neighbourhood of L. In the literature such γ_j is often called a longitude or a parallel. This choice in fact determines a diffemorphism $\varphi_\gamma : \sqcup_{j=1}^m \mathrm{S}^1_j \times \mathrm{S}^1_j \to \partial T$ which maps the meridian $\mathrm{S}^1_j \times \{1_j\}$ to γ_j, and we have

$$M_L = \left(\mathrm{S}^3 \setminus \mathrm{Interior}\,(T)\right) \cup_{\varphi_\gamma} \left(\sqcup_{j=1}^m \mathrm{B}^2_j \times \mathrm{S}^1_j\right).$$

In particular the topological type of M_L only depends on the choice γ. For a proof, see for instance Lemma 10.1.2 in [Mar16]. This description is useful for computing $\pi_1(M_L)$. One can compute $\pi_1(\mathrm{S}^3 \setminus T)$ using the Wirtinger presentation and then use (Proposition 10.1.3 [Mar16])

$$\pi_1(M_L) \simeq \frac{\pi_1(\mathrm{S}^3 \setminus T)}{\mathrm{N}(\gamma)}$$

where $\mathrm{N}(\gamma)$ is the normal subgroup generated by the simple closed curves γ_j.

Definition 3.1.16. For $i = 1, 2$ let (L_i, Ω_i) be a pair consisting of a framed link L_i in S^3 with a disjoint ribbon graph Ω_i. We say that (L_1, Ω_1) and (L_2, Ω_2) are *Kirby equivalent* if they are equivalent under the equivalence relation generated by the relation depicted in figure~3.5. This should be understood as follows. A pair $\mathbf{L} = (L, \Omega)$ which in a small neighbourhood

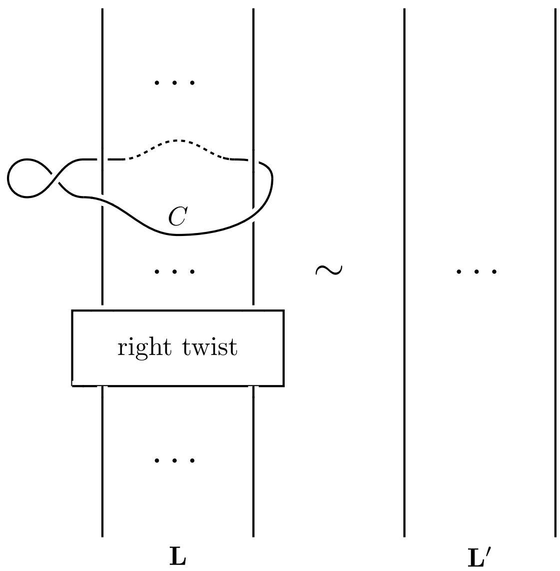


Figure 3.5: Generalized Kirby equivalence

B disjoint from any coupon of Ω looks like the part on the left of figure~3.5, where C is a component of L, is equivalent to the pair $\mathbf{L}' = (L', \Omega')$, which is identical to it outside of B, and inside B it looks like the part on the right. On the right hand side, we have a system of unlinked untwisted vertical strands represented by dots. On the left hand we have the same, except that 1) these strands are now encircled by an unknot C with a kink, and 2) below C we have used the isotopy given by a full clockwise rotation of a plane perpendicular to the strands.

It follows from the works of Lickorish and Wallace [Lic62, Wal60], that any closed oriented three-manifold can be obtained through Dehn surgery, and the Kirby calculus [Kir78] tells us exactly which links give the same three-manifold (up to homeomorphism). The Kirby calculus was subsequencely simplifed by Fenn-Rourke [FR79]. The following refinement of Kirby calculus, which works for three-manifolds with ribbon graphs, is due to Reshetikhin and Turaev. See Section 7.3 in [RT91].

Theorem 3.1.4 ([RT91])**.** *For any connected closed three-manifold with a ribbon graph (M, Ω_M) there exists a link with a ribbon graph (L, Ω) and a homeomorphism of three-manifolds with ribbon graphs $f : (M_L, \Omega) \to (M, \Omega_M)$. For $i = 1, 2$, let (M_i, Ω_i) be a closed three-manifold with a ribbon graph, obtained from Dehn surgery on a link with a ribbon graph (L_i, Ω_i). There exists a homeomorphism of three-manifolds with ribbon graphs $f : (M_1, \Omega_1) \to (M_2, \Omega_2)$ if and only if (L_1, Ω_2) and (L_2, Ω_2) are Kirby equivalent.*

3.1.3 The quantum invariant

Fix a modular tensor category $(\mathcal{V}, \{V_i\}_{i\in I})$ together with a rank $\mathcal{D}$.

Definition 3.1.17. Let $L = (L, \varphi) \subset \mathrm{S}^3$ be a framed oriented link. Let $\mathrm{lk}(L)$ be the linking matrix of L. Let $\sigma(L)$ be the number of positive eigenvalues of $\mathrm{lk}(L)$ minus the number of negative eigenvalues of $\mathrm{lk}(L)$. Choose an orientation of L and let $\mathrm{Col}(L) = \{V_i \mid i \in I\}^{\pi_0(L)}$. Choose an ordering $\pi_0(L) = \{L_1, ..., L_m\}$. For any $\lambda = (\lambda_1, ..., \lambda_m) \in \mathrm{Col}(L)$ we let $\Gamma(L, \lambda)$ be the associated colored ribbon graph. Let also $\dim(\lambda) = \prod_{j=1}^{m} \dim(\lambda_j)$.

Remark 3.1.5. The signature $\sigma(L)$ of $\mathrm{lk}(L)$ is equal to the signature of the intersection form on $H_2(B^4(\varphi))$, and in fact Turaev uses this description in [Tur10].

We can now state the main theorem of this chapter.

Theorem 3.1.6 ([RT90, RT91, Tur10])**.** *Let Ω be colored ribbon graph in S^3, and let $L \subset \mathrm{S}^3$ be a framed oriented link with $L \cap \Omega = \emptyset$. The following quantity is independent of the choice of orientation on L and invariant under color preserving generalized kirby moves on (L, Ω).*

$$\tau_{\mathcal{V}}(L, \Omega) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{\lambda \in \mathrm{Col}(L)} dim(\lambda) F(\Gamma(L, \lambda) \cup \Omega).$$

Definition 3.1.18. The *quantum invariant of* (M, Ω) is by definition

$$\tau_{\mathcal{V}}(M, \Omega) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{\lambda \in \mathrm{Col}(L)} \dim(\lambda) F(\Gamma(L, \lambda) \cup \Omega)$$

where (L, Ω) is any surgery link for (M, Ω).

The quantum invariant have the following properties. We have

$$\tau_{\mathcal{V}}(\mathrm{S}^1 \times \mathrm{S}^2) = 1. \tag{3.1}$$

For any colored ribbon graph $\Omega \subset \mathrm{S}^3$ we have

$$\tau_{\mathcal{V}}(\mathrm{S}^3, \Omega) = \mathcal{D}^{-1} F(\Omega). \tag{3.2}$$

The equations (3.1), (3.2) are normalization properties. Let $(M_i, \Omega_i), i = 1, 2$ be two three-manifolds with colored ribbon graphs. By choosing embedded three balls (disjoint from the ribbon graphs), we can form a new three-manifold with a colored ribbon graph in the form of their connected sum $(M_1 \# M_2, \Omega_1 \sqcup \Omega_2)$. We have

$$\tau_{\mathcal{V}}(M_1 \# M_2, \Omega_1 \sqcup \Omega_2) = \mathcal{D}\tau_{\mathcal{V}}(M_1, \Omega_1)\tau_{\mathcal{V}}(M_2, \Omega_2). \tag{3.3}$$

The formula (3.3) is referred to as the multiplicative property. It follows that the normalized invariant $(M, \Omega) \mapsto \mathcal{D}\tau_{\mathcal{V}}(M, \Omega)$ is strictly multiplicative with respect to connected sum.

3.2 Modular Hopf algebras, quantum groups and knot polynomials

In this section, we sketch how to construct modular tensor categories from algebraic objects called modular Hopf algebras. We then introduce the modular Hopf algebra relevant for Chern-Simons theory with gauge group SU(2) and give an explicit formula for the quantum invariants. We also introduce the colored Jones polynomial.

We now commence a presentation of modular Hopf algebras. We follow Chapter 11 in [Tur10] closely. Fix a commutative ring K with unit.

Definition 3.2.1. A *Hopf algebra* A over K is a unital algebra over K, provided with K-linear homomorphisms $\Delta : A \to A \otimes_K A$ and $\epsilon : A \to K$ called the *comultiplication* and the *counit*, and a K-linear homomorphism $s : A \to A$ called the *antipode.* If 1_A is the unit, we have $\Delta(1_a) = 1_A \otimes 1_A$ and $\epsilon(1_A) = 1$. Let $m : A \times A \to A$ be the multiplication. We have

$$\begin{aligned}
(\mathrm{Id}_A \otimes \Delta) \circ \Delta &= (\Delta \otimes \mathrm{Id}_A) \circ \Delta, \\
m \circ (s \otimes \mathrm{Id}_A) \circ \Delta &= m \circ (\mathrm{Id}_A \otimes s) \circ \Delta = \epsilon \cdot 1_A, \\
(\epsilon \otimes \mathrm{Id}_A) \circ \Delta &= (\mathrm{Id}_A \otimes \epsilon) = \mathrm{Id}_A.
\end{aligned}$$

Here the natural isomorphism $A \otimes_K (A \otimes_K A) \simeq (A \otimes_K A) \otimes_K A$ is neglected from the notation. Let $\mathrm{Rep}(A)$ be the category of left A-modules whose underlying K-module is projective (of finite type).

Observe that the axioms imply that the antipode s is in fact an anti-automorphism of the algebra $s(m(a, b)) = m(s(b), s(a)), \forall a, b \in A$. For a Hopf algebra A, let $P : A \otimes_K A \to A \otimes_K A$ be the flip homomorphism $P(a \otimes b) = b \otimes a$. The axioms imply $\Delta \circ s = P \circ s \otimes s \circ \Delta, s(1_A) = 1_A$ and $\epsilon \circ s = \epsilon$.

Definition 3.2.2. A *quasi-triangular Hopf algebra* is a pair (A, R) consisting of a Hopf algebra A and an invertible element $R \in A \otimes_K A$ called the *universal R matrix*. The pair must satisfy the following conditions. We have for all $a \in A$

$$P \circ \Delta(a) = R\Delta(a)R^{-1}, \ (\mathrm{Id}_A \otimes \Delta)(R) = R_{1,3}R_{1,2}, \ (\Delta \otimes \mathrm{Id}_A)(R) = R_{1,3}R_{2,3},$$

where, for any $x \in A \otimes_K A$, we define $x_{1,2} = x \otimes 1_A \in A^{\otimes 3}$, $x_{2,3} = 1_A \otimes x \in A^{\otimes 3}$ and $x_{1,3} = (\mathrm{Id}_A \otimes P)(x_{1,2})$. A *ribbon Hopf algebra* is a triple (A, R, v) consisting of a quasi-triangular Hopf algebra (A, R) and an invertible element $v \in A$ called the *universal twist* which belong to the center of A. The universal twist must satisfy

$$\Delta(v) = P(R)R(v \otimes v), \ s(v) = v.$$

We have the following theorem (which is Theorem 3.2 in Chapter 11 in [Tur10]).

Theorem 3.2.1. *Let (A, R, v) be a ribbon Hopf algebra. The category Rep(A) can be given the structure of an abelian ribbon category as follows. For $V, W \in Obj(Rep(A))$ the tensor product is $V \otimes_k W$ where A acts from the left via the comultiplication, i.e. $a.(v \otimes w) = \Delta(a).(v \otimes w), \forall a \in A, v \in V, w \in W$. The unit is K, where the action is given by the counit $a.k = \epsilon(a)k, \forall a \in A, k \in K$. The braiding $C_{V,W} : V \otimes W \to W \otimes V$ is defined by $u \mapsto P(R.u)$ where $P : V \otimes W \to W \otimes V$ is the flip map $P(v \otimes w) = w \otimes v, \forall v \in V, w \in W$. The dual of V is $V^* = Hom_K(V, K)$, with action given by $(a.\rho)(v) = \rho(s(a).v)$ for all $a \in A, \rho \in V^*, v \in V$. The pairing d_V is simply the evaluation map, whereas $b_V = d_V^*$. The twists $\theta_V : V \to V$ is given by $\theta_V(v) = u.v, \forall v \in V$.*

It follows that we can make the following definition.

Definition 3.2.3. Let (A, R, v) be a ribbon Hopf algebra. We write tr_Q for the trace map introduced in Definition 3.1.3. For any $V \in \mathrm{Obj}(\mathrm{Rep}(A))$ we define its *quantum dimension* to be $\dim_Q(V) = \mathrm{tr}_Q(\mathrm{Id}_V) \in K$.

Next we discuss the process of purification.

Definition 3.2.4. Let $\mathcal{C}$ be an abelian ribbon category. A morphism $f : V \to W$ in $\mathcal{C}$ is said to be *neglible* if $\mathrm{tr}(g \circ f) = 0$, for every $g : W \to V$. Denote the subgroup of negligble morphisms by $\mathrm{Negl}(V, W) \subset \mathcal{C}(V, W)$. The *purified category* $\mathcal{C}_p$ is the category with the same objects as $\mathcal{C}$, and for any pair of objects V, W we have $\mathcal{C}_p(V, W) = \mathcal{C}(V, W)/\mathrm{Negl}(V, W)$. An object $V \in \mathrm{Obj}(\mathcal{C})$ is said to be *quasidominated* by a finite family $\{V_i\}_{i\in I}$ of objects if there exists two finite families of morphisms $\{f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)}\}_{r=1}^m$ (here $i(r) \in I$ for $r = 1, ..., m$), such that

$$\mathrm{Id}_V - \sum_{r=1}^{m} f_r \circ g_r \in \mathrm{Negl}(V, V).$$

A *quasimodular category* is a pair $(\mathcal{C}, \{V_i\}_{i\in I})$ consisting of an abelian ribbon category $\mathcal{C}$ and a finite family of simple ojects satisfying the following. There exists $0 \in I$ such that $V_0 = \mathbb{1}$. For any $i \in I$, there exists $i^* \in I$ such that V_{i^*} is isomorphic to $(V_i)^*$. The matrix S with entries $S = (S_{i,j})_{i,j\in I} \in K$ is invertible over K. Moreover, any object V is quasidominated by $\{V_i\}_{i\in I}$.

We have the following result, which is Lemma 4.3.2 in Chapter 11 of [Tur10].

Theorem 3.2.2. *Let $\mathcal{C}$ be an abelian ribbon category. The category $\mathcal{C}_p$ is naturally an abelian ribbon category. If $(\mathcal{C}, \{V_i\}_{i\in I})$ is quasi-modular, then $(\mathcal{C}_p, \{V_i\}_{i\in I})$ is a modular tensor category.*

Now to some preliminaries on modules. Here and below, finite rank means over the ground ring K.

Definition 3.2.5. Let A be an algebra over K. An A-module V is said to be *simple* if every A-linear endomorphism of V is multiplication by an element of K. If A is a ribbon Hopf algebra, we say that an A-module of finite rank V is Negligble if $\mathrm{tr}_Q(f) = 0$ for any A-linear endomorpism $f : V \to V$.

We can now finally define a modular Hopf algebra.

Definition 3.2.6. A *modular Hopf algebra* is a pair $(A, \{V_i\}_{i\in I})$ consisting of a ribbon Hopf algebra $A = (A, R, v)$ and a finite family of simple A-modules of finite rank $\{V_i\}_{i\in I}$ satisfying the following. For some $0 \in I$, we have $V_0 = K$ (where A act via the counit as described above). For each $i \in I$ there exists $i^* \in I$ such that V_{i^*} is isomorphic to $(V_i)^*$. For any $i, j \in I$ the tensor product $V_i \otimes_K V_j$ splits as a direct sum of certain $V_i, i \in I$ (possibly with multiplicities) and a negligible A-module. For each i, j define $W_{i,j} : V_i \otimes V_j \to V_j \otimes V_i$ by $W_{i,j}(x) = P(R)R.x$, and let $S_{i,j} = \mathrm{tr}_Q(W_{i,j})$. The matrix S with entries $S_{i,j} \in K$ is invertible over K.

Given a modular Hopf algebra, we introduce the following category.

Definition 3.2.7. Let $(A, \{V_i\}_{i\in I})$ be a modular Hopf algebra. Define $\mathcal{V} = \mathcal{V}(A, \{V_i\}_{i\in I})$ to be the full subcategory of $\mathrm{Rep}(A)$, whose objects are all A-modules of finite rank quasidominated by $\{V_i\}_{i\in I}$.

Finally, we can state the following result which is Theorem 5.3.2 in Chapter 11 of [Tur10].

Theorem 3.2.3. *Let $(A, \{V_i\}_{i\in I})$ be a modular Hopf algebra. The subcategory $\mathcal{V}$ given in Definition 3.2.7 is unital monoidal subcategory of $\mathrm{Rep}(A)$ and $(\mathcal{V}, \{V_i\}_{i\in I})$ is a quasimodular category. In particular $(\mathcal{V}_p, \{V_i\}_{i\in I})$ is a modular tensor category.*

We shall introduce a special element, which is convenient for computations.

Proposition 3.2.4. *Let (A, R, v) be a ribbon Hopf algebra with universal R element $R = \sum_i \alpha_i \otimes \beta_i$. Introduce $\mu = \sum_i s(\alpha_i)\beta_i v$. The element μ is invertible and satisfies for all $x \in A$*

$$\mu x \mu^{-1} = S^2(x), \quad \sum_i \alpha_i \mu^{-1} \beta = \sum_i \beta_i \mu \alpha_i, \quad S(\mu) = \mu^{-1}. \tag{3.4}$$

In [KM91], any invertible element $\mu \in A$ satisfying (3.4) is called *charmed*.

3.2.1 Quantum groups

The presentation given here closely follows Section 1 and Section 2 of Kirby and Melvin's article [KM91]. Let $k \in \mathbb{N}$. We introduce the quantities $t = \exp(i\pi/k2), s = t^2$ and $q = s^2$.

Definition 3.2.8. Let $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ be the associative unital algebra over $\mathbb{C}$ with generators $X, Y, K, \overline{K}$ and relations

$$\begin{array}{ccc} \overline{K} = K^{-1}, & KX = sXK, & KY = \bar{s}YK, \\ [X, Y] = \frac{K^2 - \overline{K}^2}{s - \bar{s}}, & X^k = Y^k = 0, & K^{4k} = 1. \end{array}$$

Define $\mathbb{C}$ linear maps $\Delta : \tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})) \to \tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})) \times \tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})), S : \tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})) \to \tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\epsilon : \tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})) \to \mathbb{C}$ by

$$\begin{array}{cccc} \Delta(X) = X \otimes K + \overline{K} \otimes X, & \Delta(Y) = Y \otimes K + \overline{K} \otimes Y, & \Delta(K) = K \otimes K & \Delta(\overline{K}) = \overline{K} \otimes \overline{K}, \\ S(X) = -sX, & S(Y) = -\bar{s}Y, & S(K) = \overline{K}, & S(\overline{K}) = K, \\ \epsilon(X) = 0, & \epsilon(Y) = 0, & \epsilon(K) = 1, & \epsilon(\overline{K}) = 1. \end{array}$$

Remark 3.2.5. If we drop the relations $X^k = Y^k = 0, K^{4k} = 1$ and let $q \neq \pm 1$ be a generic complex number, we obtain instead the algebra often denoted by $U_q(\mathfrak{sl}(2,\mathbb{C}))$. This algebra is an example of a *quantum group*. For more on quantum groups see Section 6 in Chapter 11 of [Tur10], or the references contained in [RT90, RT91].

Proposition 3.2.6. *The tuple $(\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})), \Delta, S, \epsilon)$ is a Hopf algebra over $\mathbb{C}$, and $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is finite dimensional algebra over $\mathbb{C}$.*

We now introduce the quantum integers and the quantum factorials.

Definition 3.2.9. For an integer m, introduce the *quantum integer* $[m] = [m]_q$ given by

$$[m] = \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{\sin\left(m\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)}.$$

The *quantum factorial* of a positive integer m is given by $[m]! = \prod_{j=1}^{m}[j]$.

Theorem 3.2.7. *The element*

$$R = \frac{1}{4k}\sum_{n,a,b} \frac{(s-\bar{s})^n}{[n]!}\bar{t}^{ab+(b-a)n+n} X^n K^a \otimes Y^n K^b$$

where the sum is over all $0 \leq n < k$ and $0 \leq a, b < 4k$, is a universal R-matrix for $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. If we write $R = \sum_{j=1} \alpha_j \otimes \beta_j$ and let $u = \sum_j S(\beta_j)\alpha_j$, then u is invertible and

$$v = u^{-1}K^2$$

is a universal twist for $(\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})), R)$. Thus $(\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})), R, v)$ is a Ribbon Hopf algebra.

Remark 3.2.8. The twist v introduced above is the inverse of the twist introduced in [RT91]. This stem from the fact there is a slight difference between the definition of a Ribbon hopf algebra given in [Tur10], which we follow, and the definition given in [RT91]. This is clarified in Appendix A in [KH01].

We now turn to the representation theory.

Theorem 3.2.9. *For $0 < r < k$ there exists an irreducible self-dual $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$-module $V^{(r)}$ defined as follows. Write $r = 2l_r + 1$. There is a $\mathbb{C}$ basis $e^{(r)}_{-l_r}, ..., e^{(r)}_{l_r}$ for which the left action is given by*

$$Xe^{(r)}_j = [l_r + j + 1]e^{(r)}_{j+1}, \quad Ye^{(r)}_j = [l_r - j + 1]e^{(r)}_{j-1}, \quad Ke^{(r)}_j = s^j e^{(r)}_j.$$

The pair $(\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})), \{V^{(r)}\}_{0<r<k})$ is a modular Hopf algebra. The module $V^{(2)}$ is known as the fundamental representation.

Thus we appeal to the Theorem 3.2.3 and make the following definition.

Definition 3.2.10. The SU(2) TQFT $\tau_{\mathcal{V}}$ is the TQFT constructed using the modular tensor category $\mathcal{V} = \mathcal{V}(\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})), \{V^{(r)}\}_{0<r<k})$ and with $\mathcal{D} = \sqrt{\frac{k}{2}\frac{1}{\sin(\pi/k)}}$. We write $\Lambda(k) = \{1, ..., k-1\}$ for the label set.

For the resut of this chapter we work with $\mathcal{V}(\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C})), \{V^{(r)}\}_{0<r<k})$.

Theorem 3.2.10. *For $0 < n, m < k$ we have*

$$dim(n) = [n], \quad v_n = t^{(n-1)(n+1)}, \quad S_{n,m} = \frac{\sin\left(\frac{\pi nm}{k}\right)}{\sin(\frac{\pi}{k})}.$$

Consider the braiding isomorphism $C = C_{n,m} : V^{(n)} \otimes V^{(m)} \to V^{(m)} \otimes V^{(n)}$. Write $n = 2l_n + 1$ and $m = 2l_m + 1$.With respect to the distinguished bases $e^{(n)}, e^{(m)}$ we have that

$$C\left(e_i^{(n)} \otimes e_j^{(m)}\right) = \sum_{v,w} C_{i,j}^{v,w} e_v^{(m)} \otimes e_w^{(n)}$$

where

$$C(n,m)_{i,j}^{v,w} = \frac{(s-\bar{s})^{w-i}}{[w-i]!} \frac{[l_n+w]!}{[l_n+i]!} \frac{[l_m-v]!}{[l_m-j]!} t^{4ij-2(w-i)(i-j)-(w-i)(w-i+1)} \delta_{v+w}^{i+j} \tag{3.5}$$

if (i,j,v,w) satisfy

$$\begin{aligned} l_n \geq w \geq i \geq -l_n, \\ l_m \geq j \geq v \geq -l_m. \end{aligned} \tag{3.6}$$

and $C(n,m)_{i,j}^{v,w} = 0$ otherwise. Moreover, we have

$$\Delta\mathcal{D}^{-1} = \exp\left(\frac{i\pi 3(2-k)}{4k}\right) \tag{3.7}$$

and

$$\mu = K^2. \tag{3.8}$$

All of these expressions are well known. The formula for the braiding isomorphism is the content of Corollary 2.32 in [KM91]. The identity (3.8) is Theorem 3.24 in [KM91]. Equation (3.7) is derived in [KH01].

Normalizations of the SU(2) quantum invariant

Several normalizations of the SU(2) quantum invariant are used in the literature. We shall write τ_k for invariant defined in [RT90, RT91] using the Hopf algebra $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Let $\mathcal{V}$ be the modular tensor category associated with $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. We have the following relation

$$\tau_k(M,\Omega) = (\Delta\mathcal{D}^{-1})^{b_1(M)} \mathcal{D}\tau_{\mathcal{V}}(M,\Omega). \tag{3.9}$$

Here $b_1(M) = \dim_{\mathbb{R}}(H_1(M,\mathbb{R}))$. The formula (3.9) is derived in Appendix *A* in [KH01].

3.2.2 The colored Jones polynomial

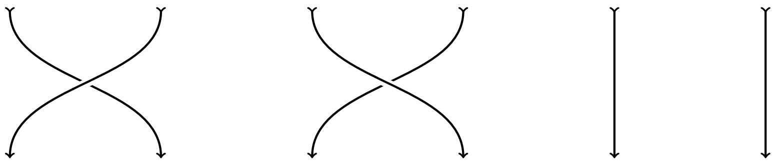
Recall the notation F for the Reshetikhin-Turaev functor.

Definition 3.2.11. Let $L \subset S^3$ be an oriented framed link. For $\lambda \in \mathrm{Col}(L)$ the *colored Jones polynomial* of (L,λ) is given by

$$J_\lambda(L,q) = F(L,\lambda).$$

If we let λ be the coloring that associate to each component the fundamental representation $V^{(2)}$ then $J_\lambda(L) = V_q(L)$, where $V(L)$ is the Jones polynomial (or a suitable normalization thereof) originally introduced by Jones [Jon85, Jon87]. For more on knot polynomials see [Mor93].

Definition 3.2.12. A *Skein triple* is a triple (L_+, L_-, L_0) consisting of three oriented blackboard framed link diagrams, which are identical except in a small rectangle where L_+ looks like X^+, L_- looks like X^-, and L_0 looks like two parallel strands as depicted in figure~3.6



$\mathbf{L_+}$, $\mathbf{L_-}$, $\mathbf{L_0}$

Figure 3.6: Skein triple

We have the following result, known as the *Skein relation*.

Theorem 3.2.11 ([Jon85, Jon87])**.** *Let (L_+, L_-, L_0) be a Skein triple. Let **2** be the coloring that associate to every component the fundamental representation $V^{(2)}$. We have*

$$tJ_{\mathbf{2}}(L_+) - \bar{t}J_{\mathbf{2}}(L_-) = (s - \bar{s})J_{\mathbf{2}}(L_0). \tag{3.10}$$

The Skein relation (3.10) together with the formula $J_{\mathbf{2}}(O^m) = [2]^m$, (where O^m is the disjoint union of m unknots given the blackboad framing) completely determines $J_{\mathbf{2}}$ as one can always modify the crossings of a link diagram to the unlink of m components. For more general colorings, one can in fact compute $J_\lambda(L)$ from $J_{\mathbf{2}}$ using cabling, as explained in Section 4 of [KM91]. One can use the theory of Skein relations to give a purely combinatorial/topological construction of modular tensor categories, as was done by Blanchet-Habegger-Masbaum-Vogel [BHMV92, BHMV95, Bla00]. The colored Jones polynomial is connected to hyperbolic geometry of cusped three-manifolds [Kas97] and satisfy rich holonomicity properties [Guk05, GL05]. From the Jones polynomial, one can recover classical knot invariants, such as the Alexander polynomial, as conjectured by Melvin-Morton-Rozansky and proven by Bar-Natan and Garoufalidis [BNG96].

Let M be a closed oriented three-manifold, with surgery link $L \subset \mathrm{S}^3$. Let $m = |\pi_0(L)|$. Let $\sigma(L)$ be the signature of the linking matrix of L. We have the following formula for the quantum invariant

$$\tau_{\mathcal{V}}(M) = \exp\left(\frac{i\pi 3(2-k)}{4k}\right)^{-\sigma(L)} \left(\sqrt{\frac{2}{k}}\sin\left(\frac{\pi}{k}\right)\right)^{m+1} \sum_{\lambda \in \mathrm{Col}(L)} \dim(\lambda) J_\lambda(L).$$

Rational surgery formulas

Definition 3.2.13. Let L be a link in S^3 together with a framing consisting in a tubuluar neighboorhood T with a parametrization $\phi : \sqcup_{j=1}^m B^2 \times S^1 \xrightarrow{\simeq} T \subset S^3$. Choose an ordering of the components $\pi_0(L) = \{L_1, ..., L_m\}$ and choose

$$U = \left\{ U_j = \begin{pmatrix} p_j & s_j \\ q_j & t_j \end{pmatrix} \right\}_{j=1}^m \subset \mathrm{SL}(2, \mathbb{Z}).$$

For each $j = 1, ..., m$, let $\psi_j : S^1 \times S^1 \to S^1 \times S^1$ be the diffeomorphism which is equal to U_j with respect to the identification $S^1 \times S^1 \simeq \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. The result of *rational surgery* on L with framing data U is

$$M_{L,U} = \left(S^3 \setminus \mathrm{Interior}\,(T)\right) \cup_{\sqcup\phi\circ\psi_j} \left(\sqcup_{j=1}^m B_j^2 \times S^1\right).$$

In case $q_j \neq 0$ for each j, we also say that $M_{L,U}$ is the result of rational surgery with coefficients p_j/q_j on the j'th component.

Let m_j and l_j be the meridian and the longitude on the component T_j of T which are determined by the framing ϕ. Then $M_{L,U}$ is obtained by gluing $S^3 \setminus T$ with solid tori $\tilde{T}_j = (B^2 \times S^1)_j$ via diffeomorphisms $\phi \circ \psi_j : \tilde{T}_j \to \partial T_j$ that maps the meridian $\tilde{m}_j = S^1 \times \{1\}$ of $\tilde{T}_j$ to the curve of T_j which is homologous to $p_j m_j + q_j l_j$.

Recall that $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\langle -1 \rangle$ admits a finite presentation with generators S and T subject to the relation $S^2 = (ST)^3 = 1$. Define the representation ρ_k by

$$\rho_k(S)_{a,b} = \sqrt{\frac{2}{k}} \sin\left(\frac{\pi ab}{k}\right), \quad \rho_k(T)_{a,b} = \exp\left(\frac{\pi i}{2k} a^2 - \frac{\pi i}{4}\right) \delta_a^b. \tag{3.11}$$

There exists a more explicit formula due to Jeffrey. To present it, we must recall the definition of the Dedekind function and the Rademacher function. Define the *Dedekind sum* for $z > 0$ by

$$S(x, z) = \frac{1}{4z} \sum_{t=1}^{z-1} \cot\left(\frac{t\pi}{z}\right) \cot\left(\frac{t\pi x}{z}\right),$$

and $S(x, -|z|) = -S(x, z), S(x, 0) = 0$. Define the *Rademacher function* Φ given by

$$\Phi \begin{pmatrix} x & y \\ z & v \end{pmatrix} = \begin{cases} \frac{x+v}{z} - 12S(x, z), \text{ if } z \neq 0, \\ \frac{y}{v}, \text{ otherwise.} \end{cases}$$

We have the following result due to Jeffrey.

Proposition 3.2.12 ([Jef92]). *For an element A, which can be represented by $\begin{pmatrix} x & y \\ z & v \end{pmatrix}$, $z \neq 0$, we have*

$$\rho_k(A)_{a,b} = \frac{sign(z)}{\sqrt{2k|z|}} \exp\left(\frac{-\pi i}{4}\Phi(A) + \frac{\pi i}{2kz} v b^2\right) \sum_{\substack{t \bmod 2kz, \\ t \equiv a \bmod 2k}} \exp\left(\frac{\pi i x t^2}{kz}\right) 2 \sin\left(\frac{\pi t b}{kz}\right). \tag{3.12}$$

We observe that for any $m \in \mathbb{Z}$ and $x' \in \mathbb{Z}$ satisfying $xx' \equiv 1 \mod z$, we have

$$S(-x,z) = -S(x,z), \quad S(x+mz,z) = S(x,z), \quad S(x,z) = S(x',z). \tag{3.13}$$

Furthermore we have that

$$\Phi(S) = 0, \quad \Phi\left(S\begin{pmatrix} x & y \\ z & v \end{pmatrix}\right) = \Phi\begin{pmatrix} x & y \\ z & v \end{pmatrix} - 3sign(xz) \tag{3.14}$$

We now present a rational surgery formula for quantum invariants.

Proposition 3.2.13 ([Jef92])**.** *Let $L \subset \mathrm{S}^3$ be a framed oriented link, and let $U \in \mathrm{SL}(2,\mathbb{Z})^{\pi_0(L)}$. Let M be the three-manifold resulting from this surgery data. Let $\sigma(L)$ be the signature of the linking matrix of L, with diagonal entries given x_c/z_c, where $U_c = \begin{pmatrix} x_c & y_z \\ z_c & v_c \end{pmatrix}$. We have*

$$\tau_k(M) = \exp\left(\frac{\pi i(k-2)}{4k}\left(\sum_{c\in\pi_0(L)} \Phi(U_c) - 3\sigma(L)\right)\right) \sum_{\lambda\in\mathrm{Col}(L)} J_\lambda(L) \prod_{c\in\pi_0(L)} \rho_k(U_c)_{\lambda(c),1}. \tag{3.15}$$

CHAPTER **4**

Modular functors with duality and unitarity

4.1 Walker's Axiom's for a Modular Functor

Definition 4.1.1. A *label set* is a triple $(\Lambda, ^\dagger, 0)$ consisting of a finite set Λ with an involution $^\dagger : \Lambda \to \Lambda$ and a preferred element $0 \in \Lambda$ with $0^\dagger = 0$.

Recall that for a closed, oriented and connected surface H_1 is equipped with the intersection pairing. For any real vector space W, let $P(W) := (W \setminus \{0\})/\mathbb{R}_+$.

Definition 4.1.2. A Λ-*marked surface* is given by the following data: $(\Sigma, P, V, \lambda, L)$. Here Σ is a smooth oriented closed surface. P is a finite subset of Σ. We call the elements of P distinguished points of Σ. V assigns to any p in P an element $v(p) \in P(T_p\Sigma)$. We say that $v(p)$ is the direction at p. λ is an assignment of labels from Λ to the points in P, e.g. it is a map $P \to \Lambda$. We say that $\lambda(p)$ is the label of p. Assume Σ splits into connected components $\{\Sigma_\alpha\}$. The L is a Lagrangian subspace of $H_1(\Sigma)$ such that the natural splitting $H_1(\Sigma) \simeq \oplus_\alpha H_1(\Sigma_\alpha)$ induces a splitting $L \simeq \oplus_\alpha L_\alpha$ where $L_\alpha \subset H_1(\Sigma_\alpha)$ is a Lagrangian subspace for each α. By convention the empty set $\emptyset$ is regarded as a Λ-labeled marked surface.

For the sake of brevity, we will refer to a Λ-labeled marked surface as a labeled marked surface. To simplify notation, we shall simply write $\Sigma = (\Sigma, P, V, \lambda, L)$.

Definition 4.1.3. The category $\mathcal{C}(\Lambda)$ of Λ-labeled marked surfaces has Λ-labeled marked surfaces as objects and with morphisms and composition defined as follows. For two (non-empty) Λ-labeled marked surfaces $\Sigma_i, i = 1, 2$ we let $\mathcal{C}(\Lambda)(\Sigma_1, \Sigma_2)$ be the set of pairs $\mathbf{f} = (f, s)$ where s is an integer and f is an isotopy class of an orientation preserving diffeomorphisms $\phi : \Sigma_1 \xrightarrow{\sim} \Sigma_2$ that restricts to a bijection of distinguished points $P_1 \xrightarrow{\sim} P_2$ that preserves directions and labels. For two composable morphisms $\mathbf{f}_1 = (f_1, s_1) : \Sigma_1 \to \Sigma_2$ and $\mathbf{f}_2 = (f_2, s_2) : \Sigma_2 \to \Sigma_3$, their composition $\mathbf{f}_2 \circ \mathbf{f}_1$ is defined to be $= (f_2 \circ f_1, s_2 + s_1 - \sigma((f_2 \circ f_1)_\#(L_1), (f_2)_\#(L_2), L_3))$ where σ is Wall's signature cocycle for triples of Lagrangian subspaces.

Using properties of Wall's signature cocycle we obtain that the composition operation is associative and therefore we obtain the category of Λ-labelled marked surfaces. Observe that Wall's signature cocycle σ is the same as the Maslow index considered in [Tur10].

There is an easy way to make this category into a symmetric monoidal category.

Proposition 4.1.1. *The category $\mathcal{C}(\Lambda)$ is a symmetric monoidal category with $\otimes = \sqcup$ defined as follows. For two labeled marked surfaces $\mathbf{\Sigma}_i = (\Sigma_i, P_i, V_i, \lambda_i, L_i), i = 1, 2$ define their disjoint union $\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2$ to be*

$$(\Sigma_1 \sqcup \Sigma_2, P_1 \sqcup P_2, V_1 \sqcup V_2, \lambda_1 \sqcup \lambda_2, L_1 \oplus L_2).$$

For morphisms $\mathbf{f}_i : \mathbf{\Sigma}_i \to \mathbf{\Sigma}_3, i = 1, 2$ we define $\mathbf{f_1} \sqcup \mathbf{f_2} = (f_1 \sqcup f_2, s_1 + s_2)$. We have an obvious natural transformation Perm : $\mathbf{\Sigma}_1 \sqcup \mathbf{\Sigma}_2 \to \mathbf{\Sigma}_2 \sqcup \mathbf{\Sigma}_1$. The empty surface $\emptyset$ is the unique neutral element with respect to $\sqcup$.

We now describe the operation of orientation reversal. For an oriented surface Σ we let $-\Sigma$ be the oriented surface where we reverse the orientation on each component. For a map g with values in Λ we let $g^\dagger$ be the map, with the same domain and codomain, given by $g^\dagger(x) = g(x)^\dagger$.

Definition 4.1.4. Let $\mathbf{\Sigma} = (\Sigma, P, V, \lambda, L)$ be a Λ-labeled marked surface. Then we define $-\mathbf{\Sigma} := (-\Sigma, P, V, \lambda^\dagger, L)$. We say that $-\mathbf{\Sigma}$ is obtained form $\mathbf{\Sigma}$ by reversal of orientation. For a morphism $\mathbf{f} = (f, s)$ we let $-\mathbf{f} := (f, -s)$.

Factorization

Definition 4.1.5. *Factorization data* for a labeled marked surface $\mathbf{\Sigma}$ consists of a pair (γ, λ) where $\gamma : [0, 1] \hookrightarrow \Sigma$ is an oriented simple closed curve in Σ with base point $\gamma(0)$ whose homology class belong to L, and $\lambda \in \Lambda$. The based loop γ is called *pre-factorization data*. The result of *cutting along* (γ, λ) is the labeled marked surface $\mathbf{\Sigma}_\gamma^\lambda$ which is obtained as follows. The underlying smooth surface is obtained as follows. Choose a tubular neighbour $\gamma \times [-\epsilon, \epsilon], \epsilon << 1$ such that the product orientation agrees with the orientation of Σ. Cut out γ and then collapse the resulting boundaries γ_-, γ_+ (where γ_+ is the component having a collared neighbourhood corresponding to $\gamma \times [0, \epsilon]$) to two points p_- and p_+. The directions at $p_\pm$ are induced by the basepoint $\gamma(0)$, and p_+ is equipped with the label λ, whereas p_- is equipped with $\lambda^\dagger$. The rest of the data making $\mathbf{\Sigma}_\gamma^\lambda$ a labelled marked surface is naturally induced from $\mathbf{\Sigma}$.

We note that factorization is functorial in the following sense

Proposition 4.1.2. *If (γ, λ) is factorization data for $\mathbf{\Sigma}$ and $h = ([f], s) \in \mathcal{C}(\Lambda)(\mathbf{\Sigma}, \mathbf{\Sigma}')$, then $(f(\gamma), \lambda)$ is factoriation data for $\mathbf{\Sigma}'$, and there exists a naturally induced morphism $h_\gamma^\lambda = ([f_\gamma], s) : \mathbf{\Sigma}_\gamma^\lambda \to (\mathbf{\Sigma}')_{f(\gamma)}^\lambda$.*

Axioms

We now turn to the axioms. Let K be a commutative ring (with unit). Let $\mathcal{P}(K)$ be the category of finitely generated projective K-modules. We recall that this is a symmetric monoidal category with the tensor product over K as product and K as unit. For the definition of a strong monoidal functor between symmetric monoidal categories, we refer to [ML98].

The Walker axioms presented below are the ones used in the works of Andersen-Ueno [AU07a, AU07b, AU12, AU15]. These works show that the TUY construction of conformal field theory [TUY89] can be refined to a modular functor and that this modular functor is isomorphic to the Reshetikhin-Turaev Walker modular functor. To show this, they consider the Skein-theoretic construction [BHMV92, BHMV95, Bla00]. Combined with the works of Laszlo [Las98] one gets that the quantum representations defined independently by Hitchin [Hit90] and Axelrod-Della Pietra-Witten [ADPW91] are equivalent to the projective representations from the Reshetikhin-Turaev modular functor. For Turaev's axioms of a modular functor, we refer to chapter 5 in [Tur10].

Definition 4.1.6. For a commutative ring K, let $\mathcal{P}(K)$ be the category of finitely generated projective K-modules. Let Λ be a label set. A *modular functor* V based on (Λ, K) is a strong monoidal functor V

$$(\mathrm{V}, \mathrm{V}_2) : (\mathcal{C}(\Lambda), \sqcup, \emptyset) \to (\mathcal{P}(K), \otimes_K, K)\,,$$

which satisfies the axioms **(a)**, **(b)**, **(c)** and **(d)** specified below.

(a) Assume γ is pre-factorization data. We demand that there is a specified isomorphism

$$g : \bigoplus_{\lambda\in\Lambda} \mathrm{V}(\Sigma_\gamma^\lambda) \xrightarrow{\sim} \mathrm{V}(\Sigma).$$

We let $P_\lambda = P_\lambda(\gamma)$ be projection $\mathrm{V}(\Sigma) \to \mathrm{V}(\Sigma_\gamma^\lambda)$. We refer to g as the factorization (or glueing) isomorphism. It is subject to the axioms **(a1)**, **(a2)** and **(a3)** specified below.

(a1). The isomorphism should be associative in the following sense. Assume that γ_2 is another set of pre-factorization data, disjoint from γ. For any pair $(\lambda, \mu) \in \Lambda^2$ the following diagram is commutative

$$\begin{tikzcd}
\mathrm{V}(\Sigma) \arrow[r, "P_\lambda(\gamma)"] \arrow[d, "P_\mu(\gamma_2)"'] & \mathrm{V}(\Sigma_\gamma^\lambda) \arrow[d, "P_\mu(\gamma_2)"] \\
\mathrm{V}(\Sigma_{\gamma_2}^\mu) \arrow[r, "s'\circ P_\lambda(\gamma)"] & \mathrm{V}\left(\left(\Sigma_\gamma^\lambda\right)_{\gamma_2}^\mu\right)
\end{tikzcd}$$

Here $s' = \mathrm{V}(s)$ where $s = ([f], 0) : \left(\Sigma_{\gamma_2}^\mu\right)_\gamma^\lambda \to \left(\Sigma_\gamma^\lambda\right)_{\gamma_2}^\mu$, f being the obvious identification.

(a2). The isomorphism should be compatible with factorization of morphisms in the following sense. Assume that $h = ([f], s) : \Sigma_1 \to \Sigma_2$ is a morphism, and that γ is pre-factorization data for Σ. For any $\lambda \in \Lambda$ the following diagram is commutative

$$\begin{tikzcd} V(\Sigma_1) \arrow[r, "P_\lambda"] \arrow[d, "V(h)"'] & V((\Sigma_1)^\lambda_\gamma) \arrow[d, "V(h^\lambda_\gamma)"] \\ V(\Sigma_2) \arrow[r, "P_\lambda"] & V((\Sigma_2)^\lambda_{f(\gamma)}) \end{tikzcd}$$

(a3). The isomorphism should be compatible with disjoint union in the following way. Assume that (γ, λ) is a factorization data for Σ_1. For any Σ_2, we see that (γ, λ) is also a choice of glueing data for $\Sigma_1 \sqcup \Sigma_2$ and that there is a canonical morphism $\iota = ([\iota], 0) : (\Sigma_1)^\lambda_\gamma \sqcup \Sigma_2 \longrightarrow (\Sigma_1 \sqcup \Sigma_2)^\lambda_\gamma$. This should induce a commutative diagram

$$\begin{tikzcd} V(\Sigma_1 \sqcup \Sigma_2) \arrow[r, "P_\gamma"] & V((\Sigma_1 \sqcup \Sigma_2)^\lambda_\gamma) \\ V(\Sigma_1) \otimes V(\Sigma_2) \arrow[u, "V_2"] \arrow[r, "P_\gamma \otimes 1"] & V((\Sigma_1)^\lambda_\gamma \otimes V(\Sigma_2). \arrow[u, "V(\iota)\circ V_2"'] \end{tikzcd}$$

(b) For any $\lambda \in \Lambda$ consider a sphere Σ_λ with one distinguished point p labelled with λ, we demand that

$$V(\Sigma_0) \simeq \begin{cases} K \text{ if } \lambda = 0 \\ 0 \text{ if } \lambda \neq 0. \end{cases}$$

(c) For any ordered pair (λ, μ) in Λ^2, consider a sphere $\Sigma_{\lambda,\mu}$ with two distinguished points, labelled by λ and μ respectively. We demand that

$$V(\Sigma_{\lambda,\mu}) \simeq \begin{cases} K \text{ if } \mu = \lambda^\dagger \\ 0 \text{ if } \mu \neq \lambda^\dagger. \end{cases}$$

Modular functors from modular tensor categories

In Turaev's axioms, the surfaces are equipped with points marked both with an object of $\mathcal{V}$ and a sign ϵ. When doing factorization according to Turaev's axioms, one endows the new pair of points with $(V_i, 1)$ and $(V_i, -1)$ respectively, whereas one should endow them with i and $i^\dagger$ respectively according to the Walker axioms if I is the label set. It is to make up for this difference that we must introduce the choice (1.28) in Theorem 10. This choice is only used for the factorization axiom.

4.2 Duality and unitarity

We now move on to consider axioms for a duality and a unitarity on a modular functor V based on a label set Λ. Before we formulate the axioms, consider an arbitrary Λ-labeled marked surface Σ'. Observe that if $p, q \in \Sigma'$ are subject to glueing then so are $p, q \in -\Sigma'$. Oberve that if Σ is the result of glueing Σ along p, q, then $-\Sigma$ is the result of glueing $-\Sigma'$ along the same ordered pair of points.

Definition 4.2.1. Let (V, g) be a modular functor based on Λ and K. A *duality* on V is a perfect pairing

$$\langle \cdot , \cdot \rangle_\Sigma : \mathrm{V}(\Sigma) \otimes \mathrm{V}(-\Sigma) \to K,$$

subject to the axioms **(a)**, **(b)**, **(c)** and **(d)** specified below.

(a) Let $\mathbf{f} = (f, s) : \Sigma_1 \to \Sigma_2$ be a morphism between Λ-labeled marked surfaces. Then

$$\langle \mathrm{V}(\mathbf{f}), \mathrm{V}(-\mathbf{f}) \rangle_{\Sigma_2} = \langle \cdot , \cdot \rangle_{\Sigma_1}.$$

(b) Consider a disjoint union of Λ-labeled marked surfaces $\Sigma = \Sigma_1 \sqcup \Sigma_2$. The modular functor V provides an isomorphism

$$\eta : \mathrm{V}(\Sigma) \otimes \mathrm{V}(-\Sigma) \xrightarrow{\sim} \mathrm{V}(\Sigma_1) \otimes \mathrm{V}(-\Sigma_1) \otimes \mathrm{V}(\Sigma_2) \otimes \mathrm{V}(-\Sigma_2).$$

We demand that with respect to the natural isomorphism $K \otimes K \simeq K$ we have that

$$\langle \cdot , \cdot \rangle_\Sigma = \big(\langle \cdot , \cdot \rangle_{\Sigma_1} \otimes \langle \cdot , \cdot \rangle_{\Sigma_2} \big) \circ \eta.$$

(c) Let γ be pre-factorization data for Σ. The factorization isomorphism induce an isomorphism

$$\bigoplus_{\lambda, \lambda' \in \Lambda} \mathrm{V}(\Sigma_\gamma^\lambda) \otimes \mathrm{V}((-\Sigma)_\gamma^{\lambda'}) \xrightarrow{g \otimes g} \mathrm{V}(\Sigma) \otimes \mathrm{V}(-\Sigma).$$

We have that

$$\langle \cdot , \cdot \rangle_\Sigma = \sum_{\lambda \in \Lambda} \mu_\lambda \langle P_\lambda, P_{\lambda^\dagger} \rangle_{\Sigma_\gamma^\lambda}, \tag{4.1}$$

where $\mu_\lambda \in K$ is invertible.

(d) For a Λ-labeled marked surface Σ we demand that there is an invertible element $\mu \in K^*$ that only depends on the isomorphism class of Σ such that for all $(v, w) \in \mathrm{V}(\Sigma) \times \mathrm{V}(-\Sigma)$ the following equation holds

$$\mu \langle w, v \rangle_{-\Sigma} = \langle v, w \rangle_\Sigma. \tag{4.2}$$

We now move on to present axioms for unitarity. For a complex vector space W with scalar multiplication $(\lambda, w) \mapsto \lambda.w$, let $\overline{W}$ be the complex vector space with the same underlying Abelian group and scalar multiplication given by $(\lambda, w) \mapsto \overline{\lambda}.w$. Here $\overline{\lambda}$ is the complex conjugate of λ. If $P : W \to W'$ is a complex linear map between complex vector spaces, we let $\overline{P} : \overline{W} \to \overline{W'}$ be the induced complex linear map which is set-theoretically identical to P.

Definition 4.2.2 (Unitarity)**.** Let (V, g) be a modular functor based on Λ and $\mathbb{C}$. A *unitary* on V is a positive definite hermitian form for each labeled marked surface Σ

$$(\cdot , \cdot)_\Sigma : \mathrm{V}(\Sigma) \otimes \overline{\mathrm{V}(\Sigma)} \to \mathbb{C},$$

subject to the axioms **(a)**, **(b)**, **(c)** and **(d)** specified below..

(a) Let $\mathbf{f} = (f, s) : \Sigma_1 \to \Sigma_2$ be a morphism between Λ-labeled marked surfaces. Then

$$(\mathrm{V}(f), \mathrm{V}(f))_{\Sigma_2} = (\cdot , \cdot)_{\Sigma_1}.$$

(b) Consider a disjoint union of Λ-labeled marked surface $\Sigma = \Sigma_1 \sqcup \Sigma_2$. Composing with the permutation of the factors, the modular functor V provides an isomorphism

$$\eta : \mathrm{V}(\Sigma) \otimes \overline{\mathrm{V}(\Sigma)} \xrightarrow{\sim} \mathrm{V}(\Sigma_1) \otimes \overline{\mathrm{V}(\Sigma_1)} \otimes \mathrm{V}(\Sigma_2) \otimes \overline{\mathrm{V}(\Sigma_2)}.$$

We demand that with respect to the natural isomorphism $\mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C}$ we have that

$$(\,\cdot\,,\,\cdot\,)_\Sigma = \big((\,\cdot\,,\,\cdot\,)_{\Sigma_1} \otimes (\,\cdot\,,\,\cdot\,)_{\Sigma_2}\big) \circ \eta.$$

(c) Let γ be pre-factorization data for Σ. We have

$$(\,\cdot\,,\,\cdot\,)_\Sigma = \sum_{\lambda \in \Lambda} \kappa_\lambda (P_\lambda, \overline{P_\lambda})_{\Sigma^\lambda_\lambda}, \tag{4.3}$$

where $\kappa_\lambda \in \mathbb{R}_{>0}$ for all λ.

If the modular functor (V, g) also has a duality we demand the unitary and the duality is compatible in the following sense.

(d) For all labeled marked surfaces Σ, we demand that the following diagram is commutative up to a scalar ρ depending only on the isomorphism class of Σ

$$\begin{array}{ccc} \mathrm{V}(\Sigma) & \xrightarrow{\simeq} & \mathrm{V}(-\Sigma)^* \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \overline{\mathrm{V}(\Sigma)^*} & \xrightarrow{\simeq} & \overline{\mathrm{V}(-\Sigma)}. \end{array} \tag{4.4}$$

Here, the horizontal isomorphisms are induced by the duality whereas the vertical isomorphisms are induced by the unitary.

The composition $\omega : \mathrm{V}(\Sigma) \xrightarrow{\simeq} \mathrm{V}(-\Sigma)^* \xrightarrow{\simeq} \overline{\mathrm{V}(-\Sigma)}$ is defined by

$$\langle x, f\rangle_{-\Sigma} = (x, \omega(f))_{-\Sigma},$$

for all (x, f) in $\mathrm{V}(-\Sigma) \times \mathrm{V}(\Sigma)$. The composition $\phi : \mathrm{V}(\Sigma) \xrightarrow{\simeq} \overline{\mathrm{V}(\Sigma)^*} \xrightarrow{\simeq} \overline{\mathrm{V}(-\Sigma)}$ is defined by

$$\langle y, \phi(f)\rangle_\Sigma = (y, f)_\Sigma,$$

for all (y, f) in $\mathrm{V}(\Sigma) \times \mathrm{V}(\Sigma)$. Projective commutativity of (4.4) can be reformulated as the existence of $\rho(\Sigma)$ in $\mathbb{C}$ with

$$\phi = \rho(\Sigma)\omega. \tag{4.5}$$

Definition 4.2.3. A duality $\langle\,\cdot\,,\,\cdot\,\rangle$ and unitarity $(\,\cdot\,,\,\cdot\,)$ satisfy *strict compatibility* if the following holds. For each labelled marked surface Σ we have

$$\mu = \rho = 1,$$

where μ is defined by (4.2) and ρ is factor making the diagram (4.4) commute, as specified by (4.5). If (γ, N) is pre-factorization data, we have for each label $\lambda \in \Lambda$

$$\mu_\lambda = \kappa_\lambda = 1,$$

where μ_λ is defined by (4.1) and κ_λ is defined by (4.3).

4.3 Symplectic characters

As already mentioned above, the construction of $Z_{\mathcal{V}}(q)$ is a slight modification of the construction of Reshetikhin and Turaev. The most interesting part concerns the duality and unitarity. To formulate our result, we must introduce some algebraic notions.

Recall the notation F for the Reshetikhin-Turaev functor.

Definition 4.3.1. A label i is *self-dual* if $i^* = i$, and *non-selfdual* otherwise. If it is selfdual, it is said to be *symplectic* if for some (and hence any) isomorphism $q : V_i \to (V_i)^*$ we have the following identity

$$-q = \left(\mathrm{Id}_{(V_i)^*} \otimes F(\cap_{V_i*})\right) \circ \left(\mathrm{Id}_{(V_i)^*} \otimes q \otimes \mathrm{Id}_{V_i}\right) \circ \left(F(U^-_{V_i}) \otimes \mathrm{Id}_{V_i}\right).$$

It is *non-symplectic* otherwise.

We now introduce the dual of the fundamental group of a modular tensor category.

Definition 4.3.2. Let $(\mathcal{V}, \{V_i\}_{i\in I})$ be a modular tensor category. Let $\Pi(\mathcal{V}, I)^*$ consist of the set of functions $\tilde{\mu} : I \to K^*$ that satisfies $\tilde{\mu}(i)\tilde{\mu}(i^*) = 1$, and such that $\tilde{\mu}(i)\tilde{\mu}(j)\tilde{\mu}(k) \neq 1$, implies $\mathrm{Hom}(\mathbf{1}, V_i \otimes V_j \otimes V_k) = \mathbf{0}$. The set $\Pi(\mathcal{V}, I)^*$ is called the *dual of the fundamental group of a modular tensor category*. An element $\tilde{\mu} \in \Pi(\mathcal{V}, I)^*$ with the property that $\tilde{\mu}$ takes on the values ± 1 on the self-dual simple objects, in such way that $\tilde{\mu}$ is -1 on the symplectic simple objects and 1 on the rest of the self-dual simple objects, is called a *fundamental symplectic character*.

The duality and unitarity are naturally induced from the Reshetikhin-Turaev TQFT. However: orientation reversal is defined a bit differently in Turaev's axioms, and again the identification between a label $(V_i, -1)$ and $(V_{i^*}, 1)$ turn out to be important. To induce the duality, we have the freedom to make an additional choice of (1.28), and the symplectic characters are used in this auxillary choice. Our proof of Theorem 10 is based on the observation that by factorization it essentially reduces to the cases of spheres with with one, two or three labelled marked points. The only case posing a difficulty is the case of a sphere with three labelled marked points, and the definition of symplectic characters is made to deal with exactly this case.

CHAPTER 5

Laplace integrals and Picard-Lefschetz theory

5.1 Classical stationary phase approximation

We now present the notion of a Poincaré asymptotic expansion. For more on this topic, see the treatises [Olv74] and [Won01].

Definition 5.1.1. For any interval $J \subset [0, 2\pi]$ let $S(J) = \{z \in \mathbb{C} : \arg(z) \in J + 2\pi i\mathbb{Z}\}$. Let D be an unbounded subset of $S(J)$ and let $A, B \in \mathbb{C}^{D\times Y}$ where Y is a parameter space. We shall write $A = O(B)$ if for every compact subset $K \subset Y$ there exists a constant $C > 0$, such that for all λ with $|\lambda|$ sufficiently large we have that

$$\mathrm{Sup}_{y\in K}|A(\lambda, y)| \leq C\mathrm{Sup}_{y\in K}|B(\lambda, y)|.$$

We shall write $A = \mathrm{o}(B)$ if for every compact subset $K \subset Y$ we have that

$$\lim_{|\lambda|\to\infty} \frac{\mathrm{Sup}_{y\in K}|A(\lambda, y)|}{\mathrm{Sup}_{y\in K}|B(\lambda, y)|} = 0.$$

We shall write $A \approx_{\lambda\to\infty} B$, if $A - B = O\left(\lambda^{-N}\right)$ for every $N \in \mathbb{N}$.

Definition 5.1.2. Let J be an interval, let $D \subset S(J)$ be unbounded and let $B \in \mathbb{C}^{D\times Y}$, where Y is a parameter space. An *asymptotic expansion of* B is a pair $(b_j)_{j=0}^{\infty}, (c_j)_{j=0}^{\infty} \subset \mathbb{C}^{D\times Y}$ such that for all $j, M \in \mathbb{N}$ we have that $c_{j+1} = \mathrm{o}(c_j)$ and $B = \sum_{j=0}^{M} b_j + o\left(c_M\right)$. This is a *Poincaré asymptotic expansion* if there exists $(d_r)_{r=1}^{m} \subset \mathbb{C}^{D\times Y}$, $(\tilde{b}_{r,j})_{j=0}^{\infty} \subset \mathbb{C}^{Y}$ and $(\beta_j)_{j=0}^{\infty}, (\alpha_j)_{j=0}^{\infty} \subset \mathbb{R}$ such that for all $j \in N$

$$b_j(\lambda, y) = \sum_{r=1}^{m} d_r(\lambda, y)\tilde{b}_{r,j}(y) \log(\lambda)^{\beta_j}\lambda^{\alpha_j}, \qquad c_j = \max(|d_r(\lambda, y)|)_{r=1}^{m} \log(\lambda)^{\beta_j}\lambda^{\alpha_j}.$$

In this case, we write

$$B(\lambda, y) \sim_{\lambda\to\infty} \sum_{r=1}^{m} d_r(\lambda, y) \sum_{j=0}^{\infty} \tilde{b}_{r,j}(y) \log(\lambda)^{\beta_j}\lambda^{\alpha_j}.$$

We shall sometimes write $\sim_{\lambda\to\infty} = \sim$ when the context is clear. Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The following localization result will be used below.

Theorem 5.1.1 (Theorems 7.7.1 [Hö03]). *Let $M \in \mathcal{M}an_d(\mathbb{R})$ be oriented and let $f \in C^\infty(M, \mathbb{H})$. Let $\eta \in \Omega_c^d(M)$. Denote the set of stationary points by $S(f)$. Define $S(f,\eta) = S(f) \cap supp(\eta) \cap (\mathrm{Im}(f))^{-1}(0)$. Let $\mu \in C_c^\infty(V, [0,1])$, where V is a small neighbourhood of $S(f,\eta)$, and $\mu_{|S(f,\eta)} = 1$. Let $\lambda \in (0, \infty)$. Then*

$$\int_M e^{\lambda i f}\eta \approx_{\lambda\to\infty} \int_M e^{\lambda i f}\mu\eta.$$

The following theorem is known as *stationary phase approximation with parameters*.

Theorem 5.1.2 (Theorem 7.7.23 [Hö03]). *Let $U \subset \mathbb{R}_x^n \times \mathbb{R}_y^m$ be an open neighbourhood of $(0,0)$. Let $F \in C^\infty(U, \mathbb{H})$. Assume that F satisfies the following conditions*

$$\frac{\partial F(0,0)}{\partial x} = 0, \quad \mathrm{Im}(F)(0,0) = 0, \quad \det\left(\frac{\partial^2 F(0,0)}{\partial x^2}\right) \neq 0. \tag{5.1}$$

Assume that $u \in C^\infty(U, \mathbb{C})$ is of compact support concentrated in a sufficiently small neighbourhood of $(0,0)$. Then there exists $\{L_{F,j} \in \mathcal{D}_n'(\mathbb{R}^n, 2j)\}_{j\in\mathbb{N}}$ giving an asymptotic expansion

$$e^{-i\lambda F^0(y)}\sqrt{\det\left(\frac{\lambda}{2\pi i}\frac{\partial^2 F}{\partial x^2}\right)^0(y)}\int u(x,y)e^{i\lambda F(x,y)}\,\mathrm{d}\,x \sim_{\lambda\to\infty} \sum_{j=0}^\infty L_{F,j}(u)^0(y)\lambda^{-j}, \tag{5.2}$$

where for functions $G(x,y)$ the notation $G^0(y)$ stands for a function of y only which is in the same residue class modulo the ideal generated by $\frac{\partial F}{\partial x_i}, i = 1, ..., n$, i.e. there exists an open neighbourhood $V \subset U$ of $(0,0)$ and $H_i \in C^\infty(V, \mathbb{C}), i = 1, ..., n$ such that for all $(x,y) \in V$ we have that

$$G(x,y) = G^0(y) + \sum_{i=1}^n H_i(x,y)\frac{\partial F}{\partial x_i}(x,y).$$

5.2 Complex analytic phase and saddle point analysis

5.2.1 The Milnor fibration and the Pham-Picard-Lefschetz-theorem

For $X \in \mathcal{M}an_d(\mathbb{C})$ and $f \in \mathcal{O}(X)$, we let $S(f)$ denote the set of saddles. If $S(f) = \{p\}$, we say (f,p) is an *isolated singularity* of dimension d. For $0 < t << 1$, let $\mathrm{D}(t) = \{z \in \mathbb{C} : |z| < t\}$ and $\mathrm{D}'(t) = \mathrm{D}(t) \setminus \{0\}$.

Theorem 5.2.1 ([Mil68]). *Let $U \subset \mathbb{C}^d$ be an open neigbourhood of 0 and let $f \in \mathcal{O}(U)$ with $S(f) \cap U = \{0\}$ and $f(0) = 0$. There exists $\mu \in \mathbb{N}$ satisfying the following. For every sufficiently small $\epsilon > 0$, there exists $\epsilon' > 0$, such that for all $t \in (0, \epsilon']$, the restriction*

$$f : \mathrm{B}_\epsilon^{2d} \cap f^{-1}(\mathrm{D}'(t)) \to \mathrm{D}'(t) \tag{5.3}$$

is a C^∞ locally trivial fibration with fibres homotopy equivalent to $\bigvee_{j=1}^\mu \mathrm{S}_j^{d-1}$. For each $z \in \mathrm{D}'(t)$, we have $f^{-1}(z) \pitchfork \mathrm{S}_r^{2d-1}, \forall r \in (0, \epsilon]$.

We recall that the fibration (5.3) is called a Milnor fibrationof (f, p). We shall in fact also refer to the map $f : \mathrm{B}_\epsilon^{2d} \cap f^{-1}(\mathrm{D}(t)) \to \mathrm{D}(t)$ as a Milnor fibration. We recall that the $\mu \in \mathbb{N}$ from Theorem 5.2.1 is called the Milnor number of (f, p). The Milnor number is equal to unity if and only if (f, p) is a Morse singularity.

We now fix a Milnor fibration and simplify notation we assume $p = 0$ and write $\mathrm{B} = \mathrm{B}_\epsilon^{2d}$ and $\mathrm{D}'(t) = \mathrm{D}'$. We consider the homological bundle 1.9 $\mathrm{E}_0 = \mathrm{E} \to \mathrm{D}'$ associated with $f_{|\mathrm{B}\setminus f^{-1}(0)}$ and we refer to this as the *homological Milnor fibration*. Introduce the notation $\delta_0 = \delta : \mathrm{S}^1 \to \mathrm{D}'$ for the choice of a simple loop, which encircles 0, and is oriented counter clockwise. We write $\mathrm{M} = \mathrm{M}_\delta \in \mathrm{Aut}(\mathrm{E}(\epsilon))$ for the *monodromomy operator*, where $0 < \epsilon << 1$. Observe that in Section 1.2.2, we used the slightly different notation M_z, where z denoted the saddle point above. Recall from Section 1.2.2 the notion of a vanishing cycle.

Theorem 5.2.2 ([Pha65, Pha67] , [PS71] [Lef50])**.** *Assume that $(f, 0)$ is a Morse singularity, and consider a coordinate neigbourhood (U, z) of 0, such that $f(z) = \sum_{j=1}^d z_j^2$. For each sufficiently small $t > 0$ let $\mathrm{S}_z(t) = \{\sum_{j=1}^d z_j^2 = t, \mathrm{Im}(z_j) = 0\} \subset f^{-1}(t) \cap \mathrm{B}$. Let $\sigma(t) = [\mathrm{S}_z(t)]$ be the associated section of the homological Milnor fibration $\mathrm{E} \to \mathrm{D}'$. Then σ extends to a vanishing cycle and for any $a \in \mathrm{E}(t) = \mathrm{H}_{d-1}(f^{-1}(t) \cap \mathrm{B})$ we have*

$$\mathrm{M}_\delta(a) = a + (-1)^{d(d+1)/2}\langle a, \sigma\rangle\sigma,$$

where $\langle \cdot, \cdot \rangle$ is the intersection pairing. Moreover $\mathrm{M}_\delta(\sigma) = (-1)^d\sigma$.

Theorem 5.2.3 ([A'C73, A'C75, Bri70])**.** *There exists $N \in \mathbb{N}$ with $\left(\mathrm{M}_\delta^N - id\right)^d = 0$. All eigenvalues of M_δ are roots of unity and no Jordan block is of dimension bigger than d.*

Division of forms and the Gelfard-Leray transform

We now recall elements of *Leray's residue theory* [Ler59]. The presentation given here builds on [Pha11]. For a smooth function f, we let $S(f)$ denote the set of stationary points.

Proposition 5.2.4. *Let $M \in \mathcal{M}an_d(\mathbb{R})$ and let $f \in \mathrm{C}^\infty(M, \mathbb{C})$ with $S(f) = \emptyset$. For any $\omega \in \Omega^r_{\mathbb{C}}(M)$, with $\mathrm{d} f \wedge \omega = 0$, there exists a form $\eta \in \Omega^{r-1}_{\mathbb{C}}(M)$ such that $\omega = \mathrm{d} f \wedge \eta$. The restriction of η to any any level set of f is uniquely determined.*

Definition 5.2.1. With notation as in Proposition, we write

$$\eta_{|f^{-1}(c)} = \frac{\omega}{\mathrm{d} f}(c),$$

and $\frac{\omega}{\mathrm{d} f}$ is called the *Gelfand-Leray transform*.

Proposition 5.2.5. *If M is oriented, $f \in \mathrm{C}^\infty(M, \mathbb{R})$ and $\omega \in \Omega^d_c(M)$ we have*

$$\int_M \omega = \int_{\mathbb{R}} \left(\int_{f=t} \frac{\omega}{\mathrm{d} f} \right) \mathrm{d} t.$$

If $\eta \in \Omega^{d-1}_{\mathbb{C},c}(M)$, we have

$$\frac{\mathrm{d}}{\mathrm{d} t} \int_{f=t} \eta = \int_{f=t} \frac{\mathrm{d} \eta}{\mathrm{d} f}.$$

We now consider the holomorphic case which is also treated in Chapter 10 [AGZV12].

Proposition 5.2.6. *Let $X \in \mathcal{M}an_d(\mathbb{C})$ and let $f \in \mathcal{O}(X)$ with $S(f) = \emptyset$. Observe $\mathrm{d}\, f \wedge \omega = 0$ for any $\omega \in \Omega_{Hol}(X)$. With notation as in Proposition 5.2.5, if ω is holomorphic, then the form η can locally be choosen to be holomorphic. In particular, for each $z \in \mathbb{C}$, the Gelfand-Leray transform $\frac{\omega}{\mathrm{d}\, f}(z)$ is a holomorphic form on $f^{-1}(z)$. If ω is a holomorphic d-form, η is a holomorphic $d-1$ form, and $\psi(z) \subset f^{-1}(z)$ is a family of $d-1$ cycles depending continuously on z, then the assignments*

$$z \mapsto \int_{\psi(z)} \frac{\omega}{\mathrm{d}\, f}(z), \quad z \mapsto \int_{\psi(z)} \eta$$

are both holomorphic. Furthermore, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\, z} \int_{\psi(z)} \eta = \int_{\psi(z)} \frac{\mathrm{d}\, \eta}{\mathrm{d}\, f}.$$

5.2.2 Malgrange's asymptotic expansions

We now return to the homological Milnor fibration (1.9). Given a holomorphic frame $\sigma = (\sigma_1, ..., \sigma_\mu)$ of the homological Milnor fibration, the associated *Picard-Fuchs equation* is the linear ordinary differential equation

$$\frac{\mathrm{d}\, I}{\mathrm{d}\, t} = A_\sigma I, \tag{5.4}$$

defined by $\nabla = \mathrm{d} - A_\sigma \,\mathrm{d}\, t$, where $\nabla = \nabla_0$ is the Gauss-Manin connection associated with (1.9). Recall the notion of a regular singularity of a linear system (see for instance [MS16]). Let $\mathbb{C}\{t\}$ denote the algebra of germs of holomorphic functions at $t = 0$. The following theorem, due to Malgrange, is of fundamental importance for our purposes.

Theorem 5.2.7 ([Mal74])**.** *The Picard-Fuchs equations (5.4) have a regular singularity at $t = 0$. Let $\mathcal{A} \subset \mathbb{Q} \cap (0, 1]$ be the set of arguments of eigenvalues of the monodromy operator M_δ . Let σ be a vanishing cycle of (1.9) defined on a fixed sector S, and let ω be a holomorphic $(d, 0)$ form defined in a neighbourhood of 0. For $0 < |t|$ sufficently small, and $t \in S$, there exists $\{d_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{N}$ and $\{f_{\alpha,\beta}\}_{\alpha\in\mathcal{A},\, 0\leq\beta\leq d_\alpha} \subset t^{-1}\mathbb{C}\{t\}$ giving a convergent expansion*

$$\int_{\sigma(t)} \frac{\omega}{\mathrm{d}\, f} = \sum_{\alpha\in\mathcal{A}} \sum_{\beta=0}^{d_\alpha} f_{\alpha,\beta}(t) t^\alpha \log(t)^\beta. \tag{5.5}$$

For each $\alpha \in \mathcal{A}$ the number $d_\alpha + 1$ is less than or equal to the maximal dimension of any Jordan block of M_δ associated with the eigenvalue $\exp(2\pi i\alpha)$. If η is a holomorphic $n-1$ form defined in a neighbourhood of 0, we have

$$\lim_{t\to 0_+} \int_{\sigma(t)} \eta = 0. \tag{5.6}$$

The expansion (5.5) follows from the well-known form of solutions to regular singularities, together with Theorem 5.2.3, and an analysis of the growth of solutions. The following results of Malgrange shows that if we assume an analycity condition on the phase f we can relax the non-degeneracy hypothesis in stationary phase approximation. Below λ denotes a positive real parameter.

Theorem 5.2.8 ([Mal74]). *Let (f,p) be an isolated singularity of dimension d. Let X be an open neighbourhood of p and let $(\omega,[\psi]) \in \Omega^{d,0}_{Hol}(X) \times \mathrm{H}_d(X, X \cap \{\mathrm{Re}(f) < \mathrm{Re}(f(p))\})$. If X is sufficiently small, there exists an unbounded set $\mathcal{A} \subset \mathbb{Q}_{<0}$, $\{d_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{N}$ and a sequence $\{c_{\alpha,\beta}\}_{\alpha\in\mathcal{A},0\leq\beta\leq d_\alpha} \subset \mathbb{C}$ giving an asymptotic expansion*

$$\int_\psi e^{-\lambda f}\omega \sim_{\lambda\to\infty} e^{-\lambda f(p)} \sum_{\alpha\in\mathcal{A}} \sum_{\beta=0}^{d_\alpha} c_{\alpha,\beta}\lambda^\alpha \log(\lambda)^\beta. \tag{5.7}$$

The set $\exp(-2\pi i\mathcal{A})$ is a subset of the set of eigenvalues of M and for each $\alpha \in \mathcal{A}$ the number $d_\alpha + 1$ is less than or equal to the maximal dimension of any Jordan block associated with $\exp(-2\pi i\alpha)$.

For any manifold M, and any closed subset F we let $\mathcal{D}'_n(M,F)$ denote the space of currents of dimension n with support in F.

Theorem 5.2.9 ([Mal74]). *Let $M \in \mathcal{M}an_n(\mathbb{R})$ be real analytic and oriented. Let $f \in C^\infty(M,\mathbb{R})$ be real analytic. Let $p \in S(f)$. There exists an unbounded set $\mathcal{A} \subset \mathbb{Q}_{<0}$ which is a union of finitely many arithmetic progressions and which have the following properties. For every sufficiently small open neighbourhood B of p there exsists $\{d_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{N}$ and $\{c_{\alpha,\beta}\}_{\alpha\in\mathcal{A},\, 0\leq\beta\leq d_\alpha} \subset \mathcal{D}'_n(B, \{f = f(p)\} \cap B)$ such that for every $\omega \in \Omega^n_c(B)$ we have that*

$$\int_M e^{\lambda i f}\omega \sim_{\lambda\to\infty} e^{\lambda i f(p)} \sum_{\alpha\in\mathcal{A}} \sum_{\beta=0}^{d_\alpha} c_{\alpha,\beta}(\omega)\lambda^\alpha \log(\lambda)^\beta. \tag{5.8}$$

Moreover, we have $d_\alpha \leq n-1$ for every $\alpha \in \mathcal{A}$. If in addition p is a local maximum of f, there exists $\{b_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{N}$ and$\{u_{\alpha,\beta}\}_{\alpha\in\mathcal{A},\, 0\leq\beta\leq bd_\alpha)\in\mathcal{A}\times\mathbb{N}} \subset \mathcal{D}'_n(B, \{f = f(p)\} \cap B)$ such that

$$\int_M e^{\lambda f}\omega \sim_{\lambda\to\infty} e^{\lambda f(p)} \sum_{\alpha\in\mathcal{A}} \sum_{\beta=0}^{b_\alpha} u_{\alpha,\beta}(\omega)\lambda^\alpha \log(\lambda)^\beta. \tag{5.9}$$

Moreover, we have $b_\alpha \leq n-1$ for every $\alpha \in \mathcal{A}$. In both cases, all of the currents are of finite order.

For an explicit construction of the currents, we refer to [AGZV12]. In our study of quantum invariants, we shall use the following variant, which will be proven below.

Theorem 5.2.10 ([AP18a]). *Let $f \in C^\infty(\mathbb{R}^n,\mathbb{C})$. Assume $p \in S(f)$ and that p is a maximum of $a = \mathrm{Re}(f)$. Assume $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are both real analytic near p. Let ϕ be a smooth function with compact support contained in a small neighborhood D of p.*

1. *If $\mathrm{Hess}(f)_p$ is non-degenerate on a subspace of $T_p\mathbb{R}^n$ of co-dimension one and D is sufficiently small, then there exists $m \in \mathbb{N}$ and $\{c_\alpha(\phi)\}_{\alpha=0}^\infty \subset \mathbb{C}$ such that*

$$\int_{\mathbb{R}^n} e^{\lambda f(x)}\phi(x)\ \mathrm{d}x \sim_{\lambda\to\infty} e^{\lambda f(p)}\lambda^{-\frac{n-1}{2}} \sum_{\alpha=0}^\infty c_\alpha(\phi)\lambda^{-\alpha/m}.$$

2. *Let $\check{f} \in \mathcal{O}(V)$ be a holomorphic extension of f to an open neighbourhood $V \subset \mathbb{C}^n$ of p. If $S(\check{f}) = \{p\}$ and $a - a(p)$ has an isolated zero at p and if D is sufficiently small, then there exists an unbounded set $\mathcal{A} \subset \mathcal{Q}_{<0}$, a finite set $B \subset \mathbb{N}$ and $\{c_{\alpha,\beta}(\phi)\}_{\alpha\in\mathcal{A},\ \beta\in B} \subset \mathbb{C}$ such that*

$$\int_{\mathbb{R}^n} e^{\lambda f(x)}\phi(x)\ \mathrm{d}x \sim_{\lambda\to\infty} e^{\lambda f(p)} \sum_{\alpha\in\mathcal{A},\beta\in B} c_{\alpha,\beta}(\phi)\lambda^\alpha \log(\lambda)^\beta.$$

Remark 5.2.11. We stress that Theorem 5.2.9 impose no condition on the Hessian nor does it impose that the stationary point (resp. maximum) is an isolated stationary point (resp. maximum). Theorem 5.2.9 is essentially a local result and in Malgranges article [Mal74], he deals with the case where M is an open subset of $\mathbb{R}^n$.

Remark 5.2.12. In the second case in Theorem 5.2.10, we have that the set $\exp(2\pi i\mathcal{A})$ is a subset of the set of eigenvalues of the monodromy operator of $(\check{f}, p)$ and for each $\alpha \in \mathcal{A}$ we have $c(\phi)_{\alpha,\beta} = 0$ if $\beta + 1$ is greater than or equal to the maximal dimension of any Jordan block associated with $\exp(2\pi i\alpha)$.

We shall present proofs of Theorem 5.2.8 and Theorem 5.2.9 below.

Corollary 5.2.13. *Assume that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ is a complex valued function which satisfy (5.1). If F is real analytic near $(0,0)$, one can choose the function F^0 in equation (5.2) in Theorem 6.2.2 to be real analytic near 0. If F is real and real analytic, then F^0 can be chosen to be real and real analytic as well.*

Proof. This will be proven in the first step of the proof of Theorem 5.2.10 when F is complex valued, and the last statement follows from the Weierstrass preparation theorem for real analytic functions. □

Corollary 5.2.13 can be used to provide a bound for powers of $\log(k)$ appearing in the asymptotic expansions (5.8) and (5.9).

Corollary 5.2.14. *Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Assume f is real analytic near a stationary point p. Assume that ϕ is a smooth function of compact support contained in a small neighborhood D of p. Assume that $\mathrm{Hess}(f)_p$ is non-degenerate on a subspace of dimension $m < n$ and let $q = n - m - 1$. All the d_α in (5.8) are bounded by q. Similarly, if p is a maximum of f, then all the b_α in (5.9) are bounded by q.*

Proof. This is a straightforward application of Corollary 5.2.13 and the theorems of Malgrange on asymptotic expansions of oscillatory integrals presented above in (5.8) and (5.9). □

Recall the *Gamma function* Γ which for $\mathrm{Re}(z) > 0$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}\, t,$$

and extended holormorphically to $\mathbb{C} \setminus (-\mathbb{N})$ by the functional equation $\Gamma(z+1) = z\Gamma(z)$.

Proposition 5.2.15. *For $l \in \mathbb{N}, \lambda > 0$ and $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > -1$ we have*

$$\int_0^\infty e^{-t\lambda} t^\alpha \log(t)^l \, \mathrm{d}\, t = \frac{\mathrm{d}^l}{\mathrm{d}\,\alpha^l} \left(\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \right).$$

We now prove Theorem 5.2.8. First we recall that given a pair of spaces $A \subset Z$ we get a long exact sequence in homology

$$\cdots \xrightarrow{\partial} H_n(A) \to H_n(Z) \to H_n(Z, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Proof of Theorem 5.2.8. We may assume without loss of generality that $f : X \to f(X)$ is a Milnor fibration. Let $X^- = X \cap \mathrm{Re}(f)^{-1}(\mathrm{Re}(p), \infty))$. Choose a small sector $S \subset \mathrm{D}'$ which contains $(0, \delta]$ for sufficiently small δ. For every $t \in (0, \delta]$ we get by excision an isomorphism $\partial_t : \mathrm{H}_d(X, X^-) \xrightarrow{\sim} \mathrm{H}_{d-1}(V_t)$. As S is contractible, the fibration restricts to a globally trivial fibration over S. It follows that we can choose a continuous family $\psi_t \subset f^{-1}(t), t \in S$ such that $\sigma(t) = \partial\psi_t$ is parallel with respect to the Gauss-Manin connection and $\partial_t([\psi]) = [\sigma(t)]$. We have $\int_\psi e^{-\lambda f}\omega \approx_{\lambda\to\infty} \int_{\psi_t} e^{-\lambda f}\omega$, for evey $t \in (0, \delta]$. As X is contractible, we can for each λ find a holomorphic $n-1$ form $\Psi(\lambda)$ with $\mathrm{d}\,\Psi(\lambda) = e^{-\lambda f}\omega$. By Stokes Theorem we get $\int_{\psi_t} e^{-\lambda f}\omega = \int_{\sigma(t)} \psi(\lambda)$. From (5.6) we get that this quantity tends to 0 as t tends to 0. Thus we get by Proposition 5.2.6

$$\int_{\sigma(t)} \Psi(\lambda) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}\,s} \int_{\sigma(s)} \Psi(\lambda)\ \mathrm{d}\,s = \int_0^t \int_{\sigma(s)} \frac{e^{-\lambda f}\omega}{\mathrm{d}\,f}\ \mathrm{d}\,s = \int_0^t e^{-s\lambda} \int_{\sigma(s)} \frac{\omega}{\mathrm{d}\,f}\ \mathrm{d}\,s. \tag{5.10}$$

As σ is parallel with respect to the Gauss-Manin connection, we can combine the expansion (5.5) from Theorem 5.2.7 and Proposition 5.2.15 to the right hand side of (5.10) to obtain (5.7). This concludes the proof. □

Remark 5.2.16. In the literature it is common to discuss in connection with the method of saddle point analysis the *method of steepest descent*. The idea is to consider Laplace integrals with holomorphic integrand over a cycle Δ which can be continuously deformed into regions where the phase f is real and either increasing or decresing according to whether one considers $\int_\delta \exp(-\lambda f)$ or $\int_\Delta \exp(\lambda f)$. This idea was used in the proof above and our definition of a Picard-Lefschetz thimble introduced in Section 1.2.2 is motivated by it.

We now turn to a sketch proof of Theorem 5.2.9.

Sketch proof of Theorem 5.2.9. We consider the integral $I(\lambda) = \int \exp(\lambda f)\omega$ in the case where f has a maximum at p. Using Hironaka's resolution of singularities [Hir64, AHV18] (more precisely, a version which is stated in [Ati70]) we can reduce to the case where the phase function is a monomial P. Write $I(\lambda) = \int_0^\infty e^{-t} \int_{P=t} \frac{\omega}{f}\ \mathrm{d}\,t$. The proof consists in showing that the Gelfand-Leray transform $J(t) = \int_{P=t} \frac{\omega}{f}$ admits an expansion for small t of the same form as in Theorem 5.2.7 and then integrate it againts the Laplace kernel using Proposition 5.2.15. We recall that there exists a correspondence between terms in a small t expansion of an integrable function $h(t)$ and poles in the left-half plane $\{\mathrm{Re}(s) < 0\}$ of its Mellin transform $h^*(s) = \int_0^\infty f(x)x^{s-1}\ \mathrm{d}\,x$. See Theorem 4 in [FGD95]. We have

$$J^*(s) = \int_0^\infty J(x)x^{s-1}\ \mathrm{d}\,x = \int_{\mathbb{R}^n} P^{s-1}\omega.$$

The expressions $\int_{\mathbb{R}^n} P^{s-1}\omega$ is the well-known homogeneous distribution. It has a well-known meromorphic extension to the left half-plane, see for instance [Hö3] or [AGZV12]. For a detailed computation, which shows that the coefficients of the final expansion are finite order currents evaluated on ω, see also Chapter 7 in [AGZV12]. □

We now give the proof of Theorem 5.2.10.

Proof of Theorem 5.2.10. Let $I(k) = \int_{\mathbb{R}^n} e^{kf(x)}\phi(x)\ \mathrm{d}\,x$. Without loss of generality, we can assume that p is the origin and that $f(0) = 0$. We shall start by reducing the first case to the second case. By using the theorem of stationary phase approximation with parameters we first show that $I(k)$ has an asymptotic expansion in terms of products of negative powers of k and 1-dimensional oscillatory integrals $I_j(k)$, all of which have the the same phase f^0 whose imaginary and real parts are real analytic functions. Assume that U is a coordinate neighborhood centered at the fixed point. We can apply a linear transformation in order to obtain coordinates (x, y) on $U \subset \mathbb{R}^{n-1} \times \mathbb{R}$ such that $\mathrm{Hess}(f)_0$ is non-degenerate on $T_0\mathbb{R}^{n-1} = \mathrm{Span}(\partial x_1, ..., \partial x_{n-1})$. It follows from our assumptions that f satisfies the conditions for stationary phase approximation with parameters. With the notation as above, write $g(y) = f^0(y)$. Then there are smooth functions $q_j(x, y)$ such that

$$f(x,y) = \sum_{j=1}^{n-1} q_j(x,y)\frac{\partial f}{\partial x_j}(x,y) + g(y). \tag{5.11}$$

We wish to argue that we can arrange that the functions $\mathrm{Im}(g)$ and $\mathrm{Re}(g)$ are both real analytic and that $\mathrm{Re}(g) \leq 0$. In Proposition 7.7.13 in [Hor03], Hörmander proves that the expansion (5.2) is valid for any function $g'(y)$ for which there exists functions $q'_1(x,y), ..., q'_{n-1}(x,y)$ such that (5.11) holds near 0, with g replaced by g', and q_j replaced by q'_j. Hence it will suffice to prove the following assertion: Assume that the real and imaginary parts of a function $s(x, y)$ are both real analytic and that $f_1, ..., f_{n-1}$ are functions whose real and imaginary parts are both real analytic and

$$\begin{cases} f_j(0,0) = 0, \text{ for } j = 1, ..., n-1, \\ \det\left(\frac{\partial f_j}{\partial x_r}(0,0)\right) \neq 0. \end{cases}$$

Then there are functions $h(y), q_j(x, y), j = 1, ..., n-1$, with real analytic real and imaginary parts, such that in a neighborhood of 0 we have

$$s(x,y) = \sum_{j=1}^{n-1} q_j(x,y) f_j(x,y) + h(y). \tag{5.12}$$

We see that we can arrange that $\mathrm{Im}(g), \mathrm{Re}(g)$ are real analytic by using this assertion in the case where $s = f$ and $f_j = \partial f / \partial x_j$.

We now prove the assertion. Shrink the domain to assume that the following power series expansions are valid near 0

$$f_j(x,y) = \sum c^j_{a,b} x^a y^b, \quad s(x,y) = \sum c_{a,b} x^a y^b,$$

where a ranges over $\mathbb{N}^{n-1}$ and b range over $\mathbb{N}$. Let $U \subset \mathbb{R}^n$ be a small enough domain such that these series are absolutely convergent. Consider the domain $U_1 + iU_2 \subset \mathbb{C}^n$ with coordinates (z, w) with $z = x_1 + ix_2$ and $w = y_1 + iy_2$. We can extend f_j and s to holomorphic functions $\check{f}_j$ and $\check{s}$ on $U_1 + iU_2$ by

$$\check{f}_j(z,w) = \sum c^j_{a,b} z^a w^b, \quad \check{s}(z,w) = \sum c_{a,b} z^a w^b.$$

Observe that

$$\frac{\partial \check{f}_j}{\partial z_j}(x_1, y_1) = \frac{\partial f_j}{\partial x_j}(x_1, y_1).$$

Therefore it will suffice to prove the holomorphic version of the assertion. This can be proven using the Weierstrass preparation theorem, analogously to how the proof of Theorem 7.5.7 in [Hor03] relies on Malgrange's preparation theorem, which is Theorem 7.5.6 in [Hor03].

Thus we obtain an asymptotic expansion such that for all $M \in \mathbb{N}$ we have

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}} e^{kf(x,y)}\phi(x,y)\ \mathrm{d}\,x\\ &= e^{kg(y)}\sqrt{\frac{2i\pi}{\det\left(k\frac{\partial^2 f}{\partial x^2}\right)^0(y)}}\sum_{j=0}^{M} k^{-j}L_{f,j}(\phi)^0(y) + O\left(k^{-M-1-\frac{n-1}{2}}\right)\end{aligned}$$

where $\mathrm{Im}(g), \mathrm{Re}(g)$ are real analytic. Observe that as $\int_{\mathbb{R}^{n-1}} e^{kf(x,y)}\phi(x,y)\ \mathrm{d}\,x$ is bounded as a function of k, we see that it is also true that $\mathrm{Re}(g) \le 0$. As the estimate is uniform for small y, we have

$$I(k) = \int_{\mathbb{R}} e^{kg(y)}\sqrt{\frac{2i\pi}{\det\left(k\frac{\partial^2 f}{\partial x^2}\right)^0(y)}}\sum_{j=0}^{M} k^{-j}L_{f,j}(\phi)^0(y)\ \mathrm{d}\,y + O\left(k^{-M-1-\frac{n-1}{2}}\right). \tag{5.13}$$

Introduce

$$I_j(k) = \int_{\mathbb{R}} e^{kg(y)}\sqrt{\frac{2i\pi}{\det\left(\frac{\partial^2 f}{\partial x^2}\right)^0(y)}}L_{f,j}(\phi)^0(y)\ \mathrm{d}\,y,$$

and rewrite (5.13) as

$$I(k) = k^{-\frac{n-1}{2}}\sum_{j=0}^{M} k^{-j}I_j(k) + O\left(k^{-M-1-\frac{n-1}{2}}\right).$$

Without loss of generality, we can assume that $\frac{\mathrm{d}\,g}{\mathrm{d}\,y}(0) = 0$. Otherwise the conclusion of the theorem is true with all coefficients being $c_\alpha(\phi) = 0$. From (5.12) and $\mathrm{d}\,f_0 = 0$ we also deduce $g(0) = f(0,0)$, which implies that $\mathrm{Re}(g)(y)$ has a maximum at $y = 0$. Write $\mathrm{Re}(g)(y) = a_1(y)$. As we are in the one-dimensional case, a real analytic function is either constant or has isolated zeroes and isolated stationary points. Thus, by appealing to the expansions (5.8) and (5.9) we can assume without loss of generality that the holomorphic extension $\check{g}(y)$ has an isolated stationary point at $y = 0$ and $a_1(y)$ has an isolated zero at $y = 0$. Hence the existence of an expansion can be reduced to the second case. We shall return to the form of the expansion after having proven its existence.

We now deal with the second case. Write

$$\int_U e^{kf(x)}\phi(x)\ \mathrm{d}\,x = \int_U e^{kf(x)}T(\phi)(x)\ \mathrm{d}\,x + \int_U e^{kf(x)}R(\phi)(x)\ \mathrm{d}\,x$$

where $T(\phi)(x)$ is a Taylor polynomial of ϕ at $x = 0$ of very high degree such that $R(\phi)(x)$ vanishes to very high order at $x = 0$, say to order m. Let $a = \mathrm{Re}(f)$. By the Cauchy-Schwartz inequality we have

$$\left|\int_U e^{kf(x)}R(\phi)(x)\ \mathrm{d}\,x\right| \le \int_U |e^{kf(x)}R(\phi)(x)|\ \mathrm{d}\,x \le \left(\mathrm{Vol}(U)\int_U e^{2ka(x)}(R(\phi)(x))^2\ \mathrm{d}\,x\right)^{\frac{1}{2}}.$$

We can expand $J_1(k) = \int_U e^{2ka(x)}(R(\phi)(x))^2 \, \mathrm{d}\, x$ by using Malgrange's theorem for Laplace integrals. The coefficients of that expansion will be of the form $b_{\alpha,\beta} = B^{2a}_{\alpha,\beta}(R(\phi))$, where $B^{2a}_{\alpha,\beta}$ is a distribution of finite order supported at the level set $\{x : a(x) = 0\}$. By assumption this level set meets $\mathrm{supp}(\phi)$ only at the origin at which $R(\phi)(x)$ vanishes to a high order m. It follows that for any $M \in \mathbb{N}$ we can choose m large enough to ensure that

$$\int_U e^{kf(x)} R(\phi)(x) \, \mathrm{d}\, x = O\left(k^{-M}\right). \tag{5.14}$$

We can expand $J_2(k) = \int_U e^{k\check{f}(x)} T(\phi)(x) \, \mathrm{d}\, x$ by using Malgrange's theorem for oscillatory integrals with holomorphic phase. Here we use that we can think of U as a real n chain inside $\mathbb{C}^n$, and $T(\phi)(x) \, \mathrm{d}\, x_1 \cdots \mathrm{d}\, x_n$ can be seen as a holomorphic n form on $\mathbb{C}^n$. Combining the expansion of $J_2(k)$ with the estimate (5.14) gives a partial expansion of $I(k)$. Continuing this way, by Taylor expandanding ϕ to all orders, we get a full asymptotic expansion of $I(k)$.

We now return to the form of the expansion in the first case. We appeal to Example 11.1.3 in [AGZV12] where the case of an oscillatory line integral with holomorphic phase is treated. Indeed, this case reduces to the case $g(y) = y^m$, and it is proven that the asymptotic expansion is of the form (for holommorphic $q(z)$ and over an appropriate contour through 0)

$$\int e^{ky^m} q(y) \, \mathrm{d}\, y \sim \sum_{j=1}^{\infty} c_j(q) k^{-\frac{j}{m}}.$$

This finishes the proof $\square$

5.3 Resurgence phases

At this point, the reader may want to recall the central definitions from resurgence given in the introduction to this thesis. Écalle's theory of resurgence [É81a, É81b] was originally developed in the context of analytic ODE's. For an introduction to Écalle's theory we refer to [Sau07] and the monograph [MS16]. In this section we are concerned with applying elements of the theory to the study of asymptotic expansions of Laplace integrals. More precisely, we shall prove Theorem 5 which generalizes and summarizes results from the works of Malgrange [Mal74], Pham [Pha83], Berry-Howls [BH91], Howls [How97] and Delabaere-Howls [DH02] mentioned in the introduction.

We introduce the Laplace transform along a curve. For a function $\varphi(\zeta)$ and an oriented curve $\gamma \subset \mathbb{C}$ we introduce the Laplace transform

$$\mathcal{L}_\gamma(\varphi)(\lambda) = \int_\gamma e^{-\lambda\zeta} \varphi(\zeta) \, \mathrm{d}\, \zeta.$$

We note that the Laplace transform is the formal inverse of the Borel transform.

Proposition 5.3.1. *Let κ be a complex number with $\mathrm{Re}(\kappa) > 0$ and let $m \in \mathbb{N}$. We have that*

$$\mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(\lambda^{-\kappa} \log(\lambda)^m) = \lambda^{-\kappa} \log(\lambda)^m, \qquad \mathcal{B} \circ \mathcal{L}_{\mathbb{R}_+}(\zeta^{\kappa-1} \log(\zeta)^m) = \zeta^{\kappa-1} \log(\zeta)^m.$$

Proof. We compute

$$\begin{aligned}\mathcal{L}_{\mathbb{R}_+}\left(\mathcal{B}\left(\lambda^{-\kappa}\log(\lambda)^m\right)\right) &= \mathcal{L}_{\mathbb{R}_+}\left((-1)^m \frac{\partial^m}{\partial\kappa^m}\left(\frac{\zeta^{\kappa-1}}{\Gamma(\kappa)}\right)\right)\\ &= (-1)^m \frac{\partial^m}{\partial\kappa^m}\left(\frac{\mathcal{L}_{\mathbb{R}_+}(\zeta^{\kappa-1})}{\Gamma(\kappa)}\right)\\ &= (-1)^m \frac{\partial^m}{\partial\kappa^m}\left(\lambda^{-\kappa}\right) = \lambda^{-\kappa}\log(\lambda)^m.\end{aligned}$$

For the last equality we used Proposition 5.2.15. Computing $\mathcal{B}\circ\mathcal{L}_{\mathbb{R}_+}\left(\zeta^{\kappa-1}\log(\zeta)^m\right)$ is done similarly. □

5.3.1 Saddle point analysis of resurgence phases

We now recall the setup from Section 1.2.2. Fix a resurgence phase $(Y, f) \in \mathcal{M}\mathrm{an}_d(\mathbb{C})/\mathbb{C}$. Write $f(Y) = Z$ and $C = Z \setminus \Omega$ where $\Omega = f(S)$, and $S \subset Y$ is the set of saddle points. For $\eta \in \Omega$ recall the notation $\mathrm{D}'(\eta) = \mathrm{D}\setminus\{\eta\}$ for the small punctured disc and introduce the notation $\delta_\eta \in \pi_1(\mathrm{D}'(\eta))$ for a small loop homotopic to $\partial\,\mathrm{D}'(\eta)$. We now turn to the Laplace integral

$$\mathrm{I}(\lambda, \Delta(\sigma,\gamma), \omega) = \int_{\Delta(\sigma,\gamma)} e^{-\lambda f}\,\omega$$

and the proof of Theorem 5.

Proof of Theorem 5. Recall that $\Delta = \Delta(\sigma,\gamma)$ denotes a Picard-Lefschetz thimble emanating from a saddle point $z \in S \cap f^{-1}(\eta)$. By properties of the Gelfand-Leray transform, we have that

$$\mathrm{I}(\lambda, \Delta, \omega) = \int_0^\infty e^{-\lambda\gamma(t)}\left(\int_{\sigma(\gamma(t))}\frac{\omega}{\mathrm{d}f}\right)\dot{\gamma}(t)\ \mathrm{d}\,t. \tag{5.15}$$

Recall that $\Delta(\sigma,\gamma)(t) = \sigma(\gamma(t))$ where $\sigma(\gamma(t))$ is obtained by parallel transport of σ along the Gauss-Manin connection ∇. As ω is holomorphic, we can deform γ close to the saddle point η without changing the integral. Thus we may assume $\dot{\gamma}(t) = \lambda^{-1}/|\lambda^{-1}|$ when t is sufficiently close to 0. Write $\alpha = \lambda^{-1}/|\lambda^{-1}|$. It follows that for sufficiently small $\epsilon > 0$ we can write

$$\int_0^\infty e^{-\lambda\gamma(t)}\left(\int_{\sigma(\gamma(t))}\frac{\omega}{\mathrm{d}f}\right)\dot{\gamma}(t)\ \mathrm{d}\,t = \int_\eta^{\eta+\epsilon\alpha} e^{-\lambda t}\left(\int_{\sigma(\hat{\gamma}(t))}\frac{\omega}{\mathrm{d}f}\right)\alpha\ \mathrm{d}\,t + \mathcal{E}(\lambda),$$

where $\hat{\gamma} : [0,1] \to C \cup \{\eta\}$ is the straight line $\hat{\gamma}(t) = (1-t)\eta + t(\eta+\epsilon\alpha)$ and $e^{\lambda\eta}\mathcal{E}(\lambda)$ is exponentially decreasing. Now use Theorem 5.2.7 and Proposition 5.2.15 to establish (1.10). □

Remark 5.3.2. As the expansion (1.10) was obtained by deforming γ near η to the curve $\eta + \exp(-i\theta)\mathbb{R}_{\geq 0}$ where $\theta = \arg(\lambda)$, it is clear that the coefficients $c^{\omega}_{\alpha,\beta}$ are indeed independent of γ.

Theorem 5.3.3. *In case z is a Morse singularity there exists $\{c_\alpha\}_{\alpha=0}^{\infty} \subset \mathbb{C}$ such that the asymptotic expansion takes the form*

$$\mathrm{I}(\lambda, \Delta(\sigma,\gamma), \omega) \sim_{\lambda\to\infty} e^{-\lambda f(z)} \lambda^{-\frac{d}{2}} \sum_{\alpha=0}^{\infty} c_\alpha \lambda^{-\alpha}. \tag{5.16}$$

We shall present two proofs. The first one makes use of the Picard-Lefschetz theorem, whereas the second one is more geometric.

First proof Theorem 5.3.3. As we are considering a Morse singularity the Milnor number is equal to one. Thus the rank of the homological Milnor fibration is equal to unity and The Picard-Lefschetz Theorem (Theorem 5.2.2) implies that the monodromy operator M is equal to $(-1)^d \,\mathrm{Id}$. Therefore it follows from well-known results on the form of solutions to linear systems in terms of their monodromy representations, that the expansion (5.5) is in fact of the form

$$\int_{\sigma(t)} \frac{\omega}{\mathrm{d} f} = g(t) t^{\frac{d}{2}-1},$$

where $g(t)$ is holomorphic. See for instance Lemma 1.35 and Corollary 1.37 in [MS16]. With notation as in Theorem 5 this implies that $\mathcal{A} \subset -\frac{d}{2} - \mathbb{N}$. Moreover there is exactly one Jordan block of size 1 so we have $d_\alpha = 0$ for every $\alpha \in \mathcal{A}$. $\square$

The next proof given here is an adaption from a similar proof given in [How97].

Second proof of Theorem 5.3.3. We shall assume for notational convenience that $f(z) = 0, \theta = 0$ and that $f(z_1, ..., z_d) = \sum_{j=1}^{d} z_j^2$. As the Milnor-number is 1, we may assume that $\Delta(s) = \{f = s, \mathrm{Im}(z_j) = 0, j = 1, ..., d\}$. Write $\omega = g(z_1, ..., z_d) \,\mathrm{d}\, z_1 \cdots \mathrm{d}\, z_2$. Let $B^d(t)$ be the standard ball of radius t and real dimension d, i.e. $\Delta(t) = \partial B^d(t)$ where we think of $\mathbb{R}^d$ as a subspace of $\mathbb{C}^d$ where the coordinates u_j on $\mathbb{R}^d$ are defined by $u_j = \mathrm{Re}(z_j)$. We have the standard spherical coordinates $u_1 = r\cos(\theta_1), u_d = r\prod_{l=1}^{d-1} \sin(\theta_l)$, and for $j = 2, ..., d-1$ we have $u_j = r\sin(\theta_j)\prod_{l=1}^{j-1}\cos(\theta_l)$. Here $0 \le r \le \sqrt{t}, 0 \le \theta_l \le \pi, l = 1, ..., d-2$ and $0 \le \theta_{d-1} \le 2\pi$. This is a real analytic change of variables. We recall the formula

$$\mathrm{d}\, u_1 \cdots \mathrm{d}\, u_d = r^{d-1} \sin^{d-2}(\theta_1) \sin^{d-3}(\theta_2) \cdots \sin(\theta_{d-2}) \,\mathrm{d}\, r \,\mathrm{d}\, \theta_1 \cdots \mathrm{d}\, \theta_{d-1}. \tag{5.17}$$

Up to an exponentially suppressed term $\mathcal{E}(\lambda)$ we have

$$\mathrm{I}(\lambda) = \int_0^{\epsilon} e^{-\lambda t} \int_{\partial B^d(t)} \frac{\omega}{\mathrm{d}\, t} \,\mathrm{d}\, t. \tag{5.18}$$

Now we have

$$\int_{\partial B^d(t)} \frac{\omega}{\mathrm{d}\, t} = \int_{B^d(t)} \mathrm{d} \frac{\omega}{\mathrm{d}\, t} = \frac{\mathrm{d}}{\mathrm{d}\, t} \int_{B^d(t)} g \,\mathrm{Vol} = \sum_{|\alpha|=0}^{\infty} \frac{\partial^\alpha g}{\partial u^\alpha}(0) \frac{\mathrm{d}}{\mathrm{d}\, t} \int_{B^d(t)} u^\alpha \,\mathrm{d}\, u_1 \cdots \mathrm{d}\, u_d.$$

Here we used Proposition 5.2.6. We can use (5.17) to rewrite the terms of the right hand

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\, t} \int_{B^d(t)} u^\alpha \,\mathrm{d}\, u_1 \cdots \mathrm{d}\, u_d \\ &= \frac{\mathrm{d}}{\mathrm{d}\, t} \int_0^{t^{1/2}} r^{|\alpha|+d-1} \,\mathrm{d}\, r \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} Q_\alpha(\theta_1, ..., \theta_{d-1}) \,\mathrm{d}\, \theta_1 \cdots \mathrm{d}\, \theta_{d-1} = t^{\frac{|\alpha|+d}{2}-1} C(\alpha), \end{aligned}$$

where $Q_\alpha(\theta_1, ..., \theta_{d-1}) = r^{-|\alpha|-d+1} \prod_{j=1}^{d} (u_j(r, \theta))^{\alpha_j}$. Write $\alpha = (\alpha_1, ..., \alpha_d)$. As we have that $\int_0^\pi \cos^{\alpha_j}(\theta) \sin^m(\theta) \, \mathrm{d}\,\theta = 0$ unless α_j is even, we conclude that $C(\alpha) = 0$, unless every $\alpha_1, ..., \alpha_{d-1}$ are even. When α_{d-1} is even, we also have $\int_0^{2\pi} \sin^{\alpha_d}(\theta_{d-1}) \cos^{\alpha_{d-1}}(\theta_{d-1}) \, \mathrm{d}\,\theta_{d-1} = 0$, unless α_d is even. We conclude that $C(\alpha) = 0$, unless $\alpha_1, ..., \alpha_d$ are all even. It follows that we have a convergent expansion $\int_{\partial B^d(t)} \frac{\omega}{\mathrm{d}\,t} = t^{\frac{d}{2}-1} \sum_{\alpha=0}^{\infty} c_\alpha t^\alpha$. Inserting this into (5.18), and applying Proposition 5.2.6, we obtain (5.16). □

5.3.2 Resurgence properties of the Borel transform

We now define the notion of localized monodromy, relevant for the resurgence relations (1.13).

Definition 5.3.1. A resurgence phase is said to have *localized monodromy* if the following holds. Let $\mathrm{B} = \cup_{z \in S} \mathrm{B}(z)$ where each $\mathrm{B}(z)$ is a small ball centered at $z \in S$. For every $\eta \in \Omega$ the operator $\mathrm{var}_{\delta_\eta} := \mathrm{Mon}_\delta - \mathrm{Id}$ factors

$$\begin{tikzcd} \mathrm{H}_{d-1}(f^{-1}(x)) \arrow[r] \arrow[dr, "\mathrm{var}_{\delta_\eta}"'] & \mathrm{H}_{d-1}(f^{-1}(x), f^{-1}(x) \setminus \mathrm{B}) \arrow[d, "\widetilde{\mathrm{var}}_{\delta_\eta}"] \\ & \mathrm{H}_{d-1}(f^{-1}(x)) \end{tikzcd}$$

where $x = \delta_\eta(1)$ and $\widetilde{\mathrm{var}}_{\delta_\eta}([y]) = \mathrm{var}_\delta(y)$. Moreover, we have the inclusion

$$\mathrm{var}_{\delta_\eta}\left(\mathrm{H}_{d-1}(f^{-1}(x))\right) \subset i_*\left(\mathrm{H}_{d-1}(f^{-1}(x) \cap \mathrm{B})\right)$$

where i_* is induced from the inclusion $i : f^{-1}(x) \cap \mathrm{B} \hookrightarrow f^{-1}(x)$.

We believe it to be a theorem that this localization principle holds for every resurgence phase. Similar localization principles concerning monodromy of singularities are common in the literature. There are many results due to Pham in this direction. See [Pha11] and [Vas02].

Remark 5.3.4. For Theorem 6 to hold, we only need lozalized monodromy with respect to vanishing cycles.

We now turn to the proof of Theorem 6.

Proof of Theorem 6. Observe that (1.11) follows from Proposition 5.3.1, and thus the Borel transform clearly defines a resurgent function as stated. We observe that (1.12) follows from equations (1.11), (5.15) and Proposition 5.3.1. Finally, we turn to (1.13). It follows from the definition that (up to a change of parameter)

$$\mathrm{Var}_\delta\left(\mathcal{B}_{\sigma,\omega}\right) = \int_{\mathrm{var}_\delta(\sigma)} \frac{\omega}{\mathrm{d}\,f}.$$

However, the assumption that f have localized monodromy in the sense of Definition 5.3.1 ensures that $\mathrm{var}(\sigma)$ is itself a vanishing cycle, and thus we can conclude by appealing again to (1.11). □

5.3.3 Rapid decay homology

The following generalizes some of the results from [Pha83] to the context of resurgence phases.

Definition 5.3.2. A *family of supports* on a topological space T is a family Φ of closed subsets of T which satisfy the following two conditions. If $A, B \in \Phi$, then $A \cup B \in \Phi$. If $A \in \Phi$ and $B \subset A$ is a closed subset, then $B \in \Phi$.

For $n \in \mathbb{N}$, let $\Delta^{(n)}$ be the standard n-simplex, defined as the convex hull of the standard basis vectors of $\mathbb{R}^{n+1}$. Let $\Delta^{(n)}(T)$ be the set of continuous maps $\sigma : \Delta^{(n)} \to T$, also called n-simplices in T. Let R be a ring, and let $C_n(T, R) := \bigoplus_{\sigma \in \Delta^{(n)}(T)} R_\sigma$ be the space of n-chains in T (with coefficients in R). Let Φ be a family of supports on T.

Definition 5.3.3. For any $\chi \in R^{\Delta^{(n)}(T)}$ define

$$\operatorname{supp}(\chi) = \bigcup_{\sigma \in \chi^{-1}(R \setminus \{0\})} \sigma(\Delta^{(n)}).$$

Define $C_n(T, \Phi, R) \subset R^{\Delta^{(n)}(T)}$ to be the R-module of χ's such that $\operatorname{supp}(\chi) \in \Phi$ and each $p \in T$ admits a neighbourhood U which meets only finitely many $\sigma(\Delta^{(n)})$ with $\chi(\sigma) \neq 0$.

The ordinary boundary operator δ extends to $C_n(T, \Phi, R)$, and $(C_*(T, \Phi, R), \delta)$ form a chain complex, i.e. $\delta^2 = 0$. We denote the associated homology R-module by $\mathrm{H}_*(T, \Phi, R)$.

We now return to our resurgence phase $f : Y \to \mathbb{C}$, and the reader might want to recall the notation from the beginning of Section 5.3.1. Let $\arg(\lambda) = \theta$.

Definition 5.3.4. For $\theta \in \mathbb{R}$ and $c > 0$ let $S_c^-(\theta) = \{z \in \mathbb{C} \mid \operatorname{Re}(ze^{i\theta}) \leq c\}$. Let $\Phi(f, \theta)$ be the family of supports which consists of closed subsets $A \subset Y$ such that $A \cap f^{-1}(S_c^-(\theta))$ is compact for every $c > 0$.

Definition 5.3.5. For $\eta \in \Omega$ let $\mathcal{Y}_\eta(\lambda)$ be the set of homotopy-classes of paths $\gamma : [0, \infty) \to C \cup \{\eta\}$ which start at η and satisfy that $t \mapsto \operatorname{Re}(\lambda(\eta - \gamma(t))$ is strictly increasing and unbounded. For $z \in S$ let $\mathcal{P}(z, \lambda)$ be the vector space of vanishing cycles near z defined on a small contractible sector $S(\lambda) \subset \mathrm{D}'_{f(z)}$ with $S(\lambda) \cap (\mathbb{R}_+ e^{-i\arg(\lambda)} + f(z)) \neq \emptyset$.

Next we introduce the notion of a system of cuts. Here we adopt the convention that a deformation retraction of a space A onto a subspace $B \subset A$ is a homotopy $h_t : A \to A$ with the properties that $h_0 = \mathrm{Id}$, $h_1(A) = B$ and that $h_{t|B} = \mathrm{Id}_B$. This is sometimes referred to as a strong deformation retraction in the literature as some authors relax the last condition and only require $h_t(B) = B$.

Definition 5.3.6. A *system of cuts* is a choice of disjoint paths $(\gamma_\eta \in \mathcal{Y}_\eta(\lambda))$ which satisfies the following. The space $\cup_{\eta \in \Omega} \gamma_\eta \cup \mathrm{D}'(\eta)$ is a deformation retract of $f(X)$ along h with

$$\operatorname{Re}(\lambda h(q, t)) \leq \operatorname{Re}(\lambda q), \ \forall (q, t) \in Z \times [0, 1]. \tag{5.19}$$

We have the following theorem where we work with coefficient ring $R = \mathbb{C}$.

Theorem 5.3.5. *We have for each $z \in S$ and $\gamma \in \mathcal{Y}_{f(z)}(\lambda)$ a linear map*

$$\Delta(\,\cdot\,,\gamma) : \mathcal{P}(z,\lambda) \to \mathrm{H}_d(Y,\Phi)$$

defined by associating to a vanishing cycle σ the Picard-Lefschetz thimble $\Delta(\sigma,\gamma)$. Given a system of cuts $(\gamma_\eta)_{\eta\in\Omega}$, we get an induced isomorphism

$$\oplus_{z\in S}\Delta(\,\cdot\,,\gamma_{f(z)}) : \bigoplus_{z\in S(f)} \mathcal{P}(z,\lambda) \xrightarrow{\sim} \mathrm{H}_d(Y,\Phi).$$

To prove this theorem we recall some notions from algebraic topology. A *cofibration* is a continous map betweem topological space $i : Y \to W$ which satisfy the following condition. Given a continous map $\tilde{g}_0 : W \to V$ and homotopy $g_t : Y \to V$ with $\tilde{g}_0 \circ i = g_0$, there exists a homotopy $\tilde{g}_t : W \to V$ with $\tilde{g}_t \circ i = g_t$.

Proposition 5.3.6. *Assume $\pi : X \to Z$ is a fibre bundle, and $h : Z \times [0,1] \to Z$ is a deformation retraction onto a closed subspace $A \subset Z$ satisfying that $\iota : A \hookrightarrow Z$ is a cofibration. Then h lifts to a deformation retraction of X onto $\pi^{-1}(A)$.*

For Proposition 5.3.6 to hold, it is important that we consider strong a deformation retraction, i.e. $h_{t|Z} = \mathrm{Id}_Z$ for all t.

Let us also recall the principle of excision. If $A_0 \supset A_1 \supset A_2$ are topological spaces, such that the closure of A_2 is contained in the interior of A_1, then we have an isomorphism

$$\mathrm{H}_*(A_0, A_1) \simeq \mathrm{H}_*(A_0 \setminus A_2, A_1 \setminus A_2).$$

Finally, we a recall the long exact sequence associated to a triple of spaces $R \subset S \subset T$

$$\cdots \xrightarrow{\delta} \mathrm{H}_n(S,R) \to \mathrm{H}_n(T,R) \to \mathrm{H}_n(T,S) \xrightarrow{\delta} \mathrm{H}_{n-1}(S,R) \to \cdots \tag{5.20}$$

Proof of Theorem 5.3.5. We shall start by showing that $\mathrm{H}_*(Y,\Phi)$ is equivalent to a projective limit of ordinary (relative) homology groups. For $c > 0$ let $S_c^+(\theta) = \{z \in \mathbb{C} \mid \mathrm{Re}(ze^{i\theta}) \geq c\}$. We shall argue that we have a an isomorphism of chain complexes

$$C_*(Y,\Phi) \simeq \lim_{c\to+\infty} C_*(Y, f^{-1}(S_c^+(\theta)), \tag{5.21}$$

where $C_*(Y, f^{-1}(S_c^+(\theta)) = C_*(Y,R)/C_*(f^{-1}(S_c^+(\theta))$, and the right hand side of (5.21) is the projective limit of the directed system $(C_*(Y, f^{-1}(S_c^+(\theta)))_{c>0}$. It follows immediately from Definition 5.3.3 and Definition 5.3.4 that for any $\chi \in C_n(Y,\Phi)$ and any $c > 0$ there exists at most finitely many $\sigma \in \Delta^{(r)}(Y)$ such that $\chi(\sigma) \neq 0$ and $\sigma(\Delta^{(r)}) \cap f^{-1}(S_c^-(\theta)) \neq \emptyset$. It follows that we can define a linear map $F_c : C_r(Y,\Phi) \to C_r(Y, f^{-1}(S_c^+(\theta)))$ by

$$F_c(\chi) = \sum_{\substack{\sigma\in\Delta^{(r)}(Y):\chi(\sigma)\neq 0,\\ \sigma(\Delta^{(r)})\cap f^{-1}(S_c^-(\theta))\neq\emptyset}} \chi(\sigma)\cdot\sigma.$$

We observe that F_c is clearly surjective. Moreover, as $\Delta^{(r)}$ is compact, it holds that for any $\sigma \in \Delta^{(r)}(Y)$ there must exists some $c' > 0$ such that $\sigma(\Delta^{(d)}) \cap f^{-1}(S_{c'}^-(\theta)) \neq \emptyset$. From this we deduce that $\chi = 0$ if and only if $F_c(\chi) = 0$ for every $c > 0$. This bi-implication together with the surjective of F_c for each $c > 0$, imply that the F_c's induce an isomorphism (5.21). As this is a chain map, we get an induced isomorphism

$$\mathrm{H}_*(Y, \Phi) \simeq \lim_{c \to +\infty} \mathrm{H}_*(Y, f^{-1}(S_c^+(\theta)). \tag{5.22}$$

We now show the summands $\mathrm{H}_*(Y, f^{-1}(S_c^+(\theta)), c > 0$ localizes to the preimage of a system of cuts in a way which is compatible with the $c \to \infty$ limit. Consider a system of cuts $(\gamma_\eta)_{\eta \in \Omega}$. The deformation retraction of $f(Y)$ onto $\cup_{\eta \in \Omega}(\gamma_\eta \cup \mathrm{D}'(\eta))$ can be lifted to a deformation retraction of $Y \setminus f^{-1}(\Omega)$ onto $\bigcup_{\eta \in \Omega} f^{-1}(\mathrm{D}'(\eta) \cup \gamma_\eta)$. By gluing it with the identity on $f^{-1}(\Omega)$, this extends to a deformation retraction of Y onto $\bigcup_{\eta \in \Omega} f^{-1}(\mathrm{D}'(\eta) \cup \gamma_\eta)$. Here we use proposition 5.3.6. Thus we get an induced isomorphism

$$G_c : \mathrm{H}_*(Y, f^{-1}(S_c^+(\theta)) \xrightarrow{\sim} \mathrm{H}_* \left(\bigcup_{\eta \in \Omega} f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta)), \bigcup_{\eta \in \Omega} f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta)) \right)$$

and because of (5.19), these combine with (5.22) to given an isomorphism

$$\begin{aligned} G : \mathrm{H}_*(Y, \Phi) &\xrightarrow{\sim} \lim_{c \to +\infty} \mathrm{H}_* \left(\bigcup_{\eta \in \Omega} f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta)), \bigcup_{\eta \in \Omega} f^{-1}((\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta)) \right) \\ &\simeq \lim_{c \to +\infty} \mathrm{H}_* \left(\bigcup_{\eta \in \Omega} f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta)), \bigcup_{\eta \in \Omega} f^{-1}((\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta)) \right) \\ &\simeq \lim_{c \to +\infty} \bigoplus_{\eta \in \Omega} \mathrm{H}_* \left(f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta)), f^{-1}((\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta)) \right) \end{aligned}$$

We now fix $z \in \Omega, c > 0$ and consider a summand

$$\mathrm{H}_\eta = \mathrm{H}_* \left(f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta)), f^{-1}((\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta)) \right).$$

We now show that in dimension d each summand H_η localize to the set of saddle points of f. For the sake of notational convenience we will write $\gamma_\eta \cup \mathrm{D}'(\eta) = \mathrm{D}' \cup [0, \infty)$ and $(\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta) = [c, \infty)$. This notation is justified by a natural homotopy equivalence of pairs. A small homological argument will show that for every $0 < \epsilon << c$ we have a natural isomorphism

$$\mathrm{H}_d(f^{-1}(\mathrm{D}' \cup [0, \infty)), f^{-1}[c, \infty)) \simeq \mathrm{H}_d(f^{-1}(\mathrm{D}' \cup [0, \infty)), f^{-1}[\epsilon, \infty)).$$

This follows from the long exact sequence (5.20) associated with the triple $f^{-1}(\mathrm{D}' \cup [0, \infty)) \supset f^{-1}[\epsilon, \infty) \supset f^{-1}[c, \infty)$, the triviality of the fibration above $[\epsilon, \infty)$ and the fact that $f^{-1}[c, \infty)$ is a deformation retract of $f^{-1}[\epsilon, \infty)$.

Choose pairwise disjoint open subets $(Y_x)_{x \in f^{-1}(\eta) \cap S}$ such that $(Y_x, f_{|Y_x})$ is a Milnor fibration of (f, x) for every $x \in S \cap f^{-1}(\eta)$. Let $B_\eta = \bigcup_{x \in f^{-1}(\eta)} Y_x$ Let $A_0 = f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta))$. Let $A_1 = f^{-1}((\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta))$ and let $A_2 = f^{-1}((\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta)) \cap (Y \setminus B_\eta)$. We can choose ϵ small enough so that $A_2 = f^{-1}[\epsilon', \infty)$ with $\epsilon' > \epsilon$. In particular, the closure of A_2 is contained in the interior of A_1. Therefore we can apply exicision to obtain that

$$\mathrm{H}_\eta \simeq \mathrm{H}_* \left(B_\eta \cap f^{-1}(\gamma_\eta \cup \mathrm{D}'(\eta)), B_\eta \cap f^{-1}((\gamma_\eta \cup \mathrm{D}'(\eta)) \cap S_c^+(\theta)) \right)$$

Now one can argue as in the proof of Theorem 5.7 to show that each summand H_η has a basis consisting of vanishing cycles of (f, x). It is clear that their lifts by parallel transport (along the Gauss-Manin connection) define elements in the projective limit (5.22). □

CHAPTER 6

Quantization of moduli spaces and quantum representations

6.1 Quantization

We shall discuss two forms of quantization schemes known as geometric Kähler quantization and Toeplitz quantization. In both cases, a crucial ingredient is the notion of a prequantum bundle, as defined below in Definition 6.1.1. The use of prequantum bundles in quantization, was invented independently by Kostant [Kos70] and Souriau [Sou70]. It is is also thoroughly discussed in the monograph of Woodhouse [Woo92]. The presentation below is inspired by the survey article of Schlichenmaier [Sch12] and the articles [KS01] and [And12] of Karabegov-Schlichenmaer and Andersen respectively.

Definition 6.1.1. Let (M, ω) be a symplectic manifold. A *prequantum bundle* is a Hermitian line bundle with unitary connection $(L, h, \nabla) \to M$, with curvature $F_\nabla = -i\omega$.

A prequantum line bundle exists if and only if $\left[\frac{\omega}{2\pi}\right] \in H^2(M, \mathbb{Z})$. This is known as the *integrality condition*. Inequivalent choices are parametrized by $H^1(M, U(1))$. A proof of these assertions can be found in [Woo92].

We fix from now on and until Section 6.3 a compact symplectific manifold (M, ω) of real dimension $2n_0$, together with a prequantum line bundle (L, h, ∇). For each $k \in \mathbb{Z}$, let $L^k := L^{\otimes k}$ and equip L^k with the induced connection $\nabla^{(k)}$ and the induced Hermitian form $h^{(k)}$. We write $h' = h^{(-1)}$ for the induced metric on $\tau : L^* \to M$. We think of k as $1/\hbar$ and it is often referred to as the *quantum level*. We get a volume form $\Omega = \omega^{n_0}/(n_0!(2\pi)^{n_0})$ and a Hermitian inner product

$$(s, t)^{(k)}_{\mathrm{L}^2} = \int_M h^{(k)}(s, t)\, \Omega,$$

defined on the the Hilbert space $\mathrm{L}^2(L^k)$ of square integrable sections of L^k.

6.1.1 Geometric Kähler quantization

If J is a complex structure on M, L^k aquires the structure of a holomorphic line bundle with $\overline{\partial}^{(k)}$ operator defined by $\pi^{(0,1)} \circ \nabla^{(k)}$ and Chern connection given by $\nabla^{(k)}$. We fix a Kähler structure J from now on and until section 6.1.4.

Definition 6.1.2. The *level k quantum Hilbert space* is by definition $\mathrm{H}_k = \mathrm{H}^0((M,J), L^k)$. The *level k geometric Kähler quantization* $Q^{(k)} : C^\infty(M) \to \mathrm{End}(\mathrm{H}_k)$ is defined by

$$Q^{(k)}(f)(s) = \pi^{(k)}(\nabla^{(k)}_{X_f^{(k)}} s + if \cdot s), \quad f \in C^\infty(M),\ s \in \mathrm{H}_k$$

where $X_f^{(k)}$ is the Hamiltonian vector field with respect to $k\omega$ and $\pi^{(k)} : \mathrm{L}^2(L^k) \to \mathrm{H}_k$ is the orthogonal projection. Let $K^{(k)} \in \Gamma^\infty(M \times M, L^k \boxtimes (L^*)^k)$ be the integral kernel of the orthogonal projector, i.e. such that for all $s \in C^\infty(M, L^k)$ and all $x \in M$ we have

$$\pi^{(k)}(s)(x) = \int_M K^{(k)}(x,y)s(y)\,\Omega(y).$$

We let $\|\cdot\|$ be the operator norm on $\mathrm{End}(\mathrm{H}_k)$ defined using the L^2 inner product.

Although we shall not need it, we shall also define the notion of metaplectic quantization. Let δ_J be a squareroot of the canonoical line bundle of (M,J). The *level k Hilbert space* arising from the half-form corrected geometric quantization is $H^0((M,J), L^k \otimes \delta_J)$.

6.1.2 Toeplitz quantization

The use of Toeplitz operators in quantization was introduced by Berezin in [Ber74] and Boutet de Monvel-Guillemin in [BdMG81].

Definition 6.1.3. The *level k Toeplitz quantization* $T^{(k)} : C^\infty(M) \to \mathrm{End}(\mathrm{H}_k)$ is defined by

$$T_f^{(k)}(s) = \pi^{(k)}(f \cdot s), \quad f \in C^\infty(M),\ s \in \mathrm{H}_k\,.$$

The $T_f^{(k)}$ is called the *level k Toeplitz operator associated with* f. The *Toeplitz quantization scheme* consists in the assignment $T : C^\infty(M) \to \prod_{k=0}^\infty \mathrm{End}(\mathrm{H}_k)$ given by $T(f) = (T_f^{(m)})_{m=0}^\infty$.

The following theorem shows that Toeplitz operators exhibits the correct asymptotic semi-classical behaviour.

Theorem 6.1.1 ([BMS94])**.** *For every $(f,g) \in C^\infty(M)^2$ we have*

$$\|T_f^{(k)} \circ T_g^{(k)} - T_{fg}^{(k)}\| = O\left(k^{-1}\right), \quad \|ik[T_f^{(k)}, T_g^{(k)}] - T_{\{f,g\}}^{(k)}\| = O\left(k^{-1}\right).$$

For every $f \in C^\infty(M)$ there exists $C > 0$ such that

$$|f|_\infty - \frac{C}{k} \le \|T_f^{(k)}\| \le |f|_\infty.$$

In particular $\lim_{k\to\infty}\|T_f^{(k)}\| = |f|_\infty$. Morever, the map $T^{(k)} : C^\infty(M) \to \mathrm{End}(\mathrm{H}_k)$ is surjective at each level k.

The following formula, known as the *Tuynman lemma* [Tuy87], relates geometric and Toeplitz quantization. For every $f \in C^\infty(M)$ we have $Q^{(k)}(f) = iT^{(k)}_{f - \frac{1}{2k}\Delta(f)}$.

6.1.3 Expansion of the Bergman kernel

The aim of this subsection is present an asymptotic expansion of the Bergman kernel due Schlichenmaier and Karabegov [KS01]. Following [KS01] it is convenient to work on the associated $U(1)$ bundle.

Definition 6.1.4. Let $\tau : X \to M$ be the $U(1)$ bundle defined by $X = \{\alpha \in L^* \mid h'(\alpha) = 1\}$. Let $\tilde{\Omega}$ be the unique $U(1)$ invariant volume form with $\int_X \tau^*(f)\tilde{\Omega} = \int_M f\Omega$ for every $f \in C^\infty(M)$ and let $\mathrm{L}^2(X)$ be complex vector space of square-integrable $\mathbb{C}$-valued functions, with respect to the inner product $(f,g) = \int_X f\overline{g}\,\tilde{\Omega}$. Let $\tilde{\mathrm{H}}_k \subset C^\infty(X)$ be the subspace of functions which are the restriction of holomorphic functions on $L^* \setminus \{0\}$ that are k-homogenous with respect to the $U(1)$ action. Let $\hat{\pi}^{(k)} : \mathrm{L}^2(X) \to \tilde{\mathrm{H}}_k$ be the orthogonal projection. The *level k Bergman kernel on* $\mathrm{L}^2(X)$ is the $\hat{K}^{(k)} \in C^\infty(X \times X)$ that for all $f \in \mathrm{L}^2(X)$ and all $x \in X$ we have

$$\hat{\pi}^{(k)}(f)(x) = \int_X \hat{K}^{(k)}(x,y)f(y)\,\tilde{\Omega}(y).$$

Proposition 6.1.2 ([KS01]). *Let $e : L^* \setminus \mathbf{0} \to \prod_k \mathrm{H}_k$ be the map such that for each $s \in \mathrm{H}_k$ one has $(s, e^{(k)}_\alpha)_{\mathrm{L}^2} = \alpha^{\otimes k}(s)$. We have*

$$\hat{K}^{(k)}(\alpha,\beta) = (e^{(k)}_\beta, e^{(k)}_\alpha)_{\mathrm{L}^2}.$$

Thus the Bergman Kernel extends to a holomorphic function on $L^* \setminus \mathbf{0} \times \overline{L^* \setminus \mathbf{0}}$ by

$$\hat{K}^{(k)}(c\alpha, d\beta) = c\overline{d}\hat{K}^{(k)}(\alpha,\beta). \tag{6.1}$$

By extension of (6.1) we can define a smooth function on M by $u_k(x) = \hat{K}^{(k)}(\alpha(x),\alpha(x))$, where α is a smooth section of X defined near x.

Theorem 6.1.3 ([Zel98]). *There exists $(b_v)_{v=0}^\infty \subset C^\infty(M)$ giving a Poincaré asymptotic expansion $u_k \sim_{k\to\infty} \sum_{v=0}^\infty k^{n_0 - v} b_v$ and $b_0 = 1$.*

We now commence a local asymptotic expansion of the Bergman Kernel near the diagonal. Consider an arbitrary point $x_0 \in M$. Let $s : U \to L$ be a a non-vanishing local holomorphic section on a contractible complex coordinate neighbourhood U centered at x_0. We write $s^* : U \to L^*$ for the dual section. This induces a local smooth section $\alpha : U \to X$. Now define Φ by

$$\Phi = \log(h'(s^*)) = -\log(h(s)).$$

Now let $\tilde{\Phi} \in C^\infty(U \times \overline{U})$ be an *almost analytical extension of Φ along the diagonal*. This means the following. Write $y = (z_1, z_2)$ for the holomorphic coordinates on $U \times U$ naturally induced by the coordinates z, and let $\Delta : U \to U \times U$ be the map given by $\Delta(z) = (z,z)$. That $\tilde{\Phi}$ extends Φ on the diagonal means that the following equation holds

$$\tilde{\Phi} \circ \Delta = \Phi. \tag{6.2}$$

That $\tilde{\Phi} \in C^\infty(U \times \overline{U})$ is almost analytical along the diagonal means that for any (possibly empty) string $(v_1, ..., v_m)$ with $v_i \in \{z_1, \bar{z}_1, z_2, \bar{z}_2\}$ we have

$$0 = \frac{\partial^{m+1}\tilde{\Phi}}{\partial v_1 \cdots \partial v_m \partial \bar{z}_1} \circ \Delta = \frac{\partial^{m+1}\tilde{\Phi}}{\partial v_1 \cdots \partial v_m \partial z_2} \circ \Delta. \tag{6.3}$$

For the existence of almost analytical extensions please see [KS01]. For $i = 1, 2$ let $\pi_i : U \times U \to U$ be the projection. We define $\chi \in C^\infty(U \times U)$ as follows

$$\chi = \tilde{\Phi} - 1/2(\Phi \circ \pi_1 + \Phi \circ \pi_2).$$

Shrinking U further if necessary, we can assume that for $yU \times U \setminus \Delta(U)$ we have

$$\mathrm{Re}(\chi)(y) < 0 \tag{6.4}$$

See [KS01] for an explanation of this.

Theorem 6.1.4 ([Zel98], [KS01]). *We have a Poincaré asymptotic expansion*

$$\hat{K}^{(k)}(\alpha(x_1), \alpha(x_2)) \sim_{k\to\infty} e^{k\chi(x_1,x_2)} \sum_{v=0}^{\infty} \tilde{b}_v(x_1, x_2) k^{n_0 - v}, \tag{6.5}$$

where, for each $v \in \mathbb{N}$, the function $\tilde{b}_v$ is an almost analytic extension of the b_v from Theorem 6.1.3.

We have the following important theorem due to Bordemann-Meinrenken-Schlichenmaier and Schlichenmaier.

Theorem 6.1.5 ([BMS94], [Sch00, Sch01]). *For every $(f, g) \in C^\infty(M)^2$ there is a unique sequence $(C_j^(f, g))_{j=0}^\infty \subset C^\infty(M)$ such that for every $M \in \mathbb{N}$ we have*

$$\|T_f^{(k)} \circ T_g^{(k)} - \sum_{j=0}^{M} T^{(k)}_{C_j(f,g)} k^{-j}\| = O(k^{-M-1}),$$

This induces a so-called $\star$ product, giving a deformation quantization. See [KS01] for more details.

6.1.4 The Hitchin connection

In the last few sections, we considered a fixed Kähler structure. From a physical point of view, this is auxillary data. A natural way to remedy this is to consider a family of Kähler structures and provide natural isomorphisms between the quantum Hilbert spaces parametrized by this family. From now on we will write $\mathrm{H}_k(J)$ to stress the dependence of a specific Kähler structure J. For any smooth bundle $\pi : W \to M$ and any manifold $\mathcal{T}$ a smooth map $\mathcal{T} \to \Gamma^\infty(M, W)$ is defined to be a smooth section of $\pi_M^*(W) \to \mathcal{T} \times M$.

Definition 6.1.5. A smooth family of Kähler structures parametrized by a manifold $\mathcal{T}$ is a smooth map

$$J : \mathcal{T} \to \Gamma^\infty(M, \mathrm{End}(TM))$$

such that $(M, \omega, J(\sigma))$ is a Kähler manifold for each $\sigma \in \mathcal{T}$.

We will only consider families of Kähler structures for which we can define a smooth vector bundle $\mathrm{H}_k \to \mathcal{T}$ whose fiber at a point $\sigma \in \mathcal{T}$ is $\mathrm{H}_k(J(\sigma))$. We shall asssume they form a smooth subbundle of the trivial bundle $\Gamma^\infty(M, L^k) \to \mathcal{T}$. To identify the fibers of the bundle of quantization we introduce the notion of a Hitchin connection.

Definition 6.1.6. A *Hitchin connection* is a connection on $\mathrm{H}_k \to \mathcal{T}$ which is the restriction of a connection on $\Gamma^\infty(M, L^k) \to \mathcal{T}$ of the form

$$\nabla^t + u$$

where ∇^t is the trivial connection and u is a differential form on $\mathcal{T}$ with values in differential operators on L^k.

There is natural way to differentiate J along a vector field V on $\mathcal{T}$ which we denote by $V[J]$. By differentiating the equation $J^2 = -\mathrm{Id}$, one can show that $V[J]$ decompose $V[J]_\sigma = V[J]''_\sigma + V[J]'_\sigma \in \Gamma^\infty(M, (T^{(1,0)}M_\sigma \otimes_{\mathbb{C}} (T^{(0,1)}M^*_\sigma \oplus T^{(0,1)}M_\sigma \otimes_{\mathbb{C}} (T^{(1,0)}M_\sigma)^*)$. Here $M_\sigma = (M, J(\sigma))$. We will asume that J is *holomorphic* in the sense that for any complex vector field V on $\mathcal{T}$ with $(1,0)$-part V' and $(0,1)$-part V'' we have

$$V'[J]_\sigma = V[J]'_\sigma, \quad V''[J]_\sigma = V[J]''_\sigma.$$

As ω is non-degenerate we can define $\tilde{G} : \Gamma^\infty(\mathcal{T}, T_{\mathbb{C}}\mathcal{T}) \to \Gamma^\infty(\pi^*_M(T_{\mathbb{C}}M \otimes_{\mathbb{C}} T_{\mathbb{C}}M))$ such that $\tilde{G}(V) \cdot \omega = V[J]$. Hence there is a map G defined on the real vector fields on $\mathcal{T}$ with $\tilde{G}(V) = G(V) + \overline{G}(V)$ for all real vector fields V on $\mathcal{T}$. We observe that for each $\sigma \in \mathcal{T}$ we have $G(V)_\sigma \in C^\infty(S^2(T_\sigma))$. Now we are ready to define the notion of rigidity.

Definition 6.1.7. A holomorphic family J of Kähler structures is *rigid* if $\overline{\partial}_\sigma(G(V)_\sigma) = 0$ for all vector fields on $\mathcal{T}$ and all points $\sigma \in \mathcal{T}$.

For any $B \in C^\infty(M, S^2(T_{\mathbb{C}}M))$ we define a second-order differential operator by $\Delta_B := \nabla^2_B + \nabla_{\delta B}$ where δB is the divergence of B. For each $\sigma \in \mathcal{T}$ we get a Ricci potential F_σ satisfying $\int_M F_\sigma \omega^m = 0$. This gives a smooth family of Ricci potentials $F : \mathcal{T} \to C^\infty(M)$. We have the following existence result.

Theorem 6.1.6 ([And12],[AL14])**.** *Suppose J is a rigid holomorphic family of Kähler structures, and that there exists $n_1 \in \mathbb{Z}$ such that the first Chern class of ω is $n_1[\omega] \in H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{R}) = 0$. Then the following expression defines a Hitchin connection*

$$\nabla^{H,k}_V = \nabla^t_V + \frac{1}{4k + 2n_1}\left(\Delta_{G(V)} + 2\,\mathrm{d}\,F \cdot G(V) + 4kV'[F]\right).$$

If $(M, J(\sigma))$ have no non-zero global holomorphic vector field for any $\sigma \in \mathcal{T}$, then $\nabla^{H,k}$ is projectively flat.

We shall often refer to $\nabla^{H,k}$ as the level k Hitchin-connection. We remark that there is also a notion of a Hitchin connection suitable for metaplectic quantization and an analogous existence theorem. See [AGL12, AL14]. There are also stronger existence theorems concerning Hitchin connections due to Andersen-Rasmussen [AR17].

Asymptotic expansion of parallel transport of the Hitchin connection

Let $\gamma : [0,1] \to \mathcal{T}$ be a path. Write $\gamma(t) = \sigma_t$. We let

$$\mathrm{P}^{(k)}_\gamma \in \mathrm{Hom}(\mathrm{H}_k(\sigma_0), \mathrm{H}_k(\sigma_1)) \tag{6.6}$$

denote parallel transport with respect to the Hitchin connection along γ. Below we will introduce a sequence of functions f_u to be used for an asymptotic expansion of (6.6).

Definition 6.1.8. Let $n_1 \in \mathbb{Z}$ be as in Theorem 6.1.6. We define the shifted Berezin-Toeplitz star product $\tilde{\star}$ by

$$f \tilde{\star} g = \sum_{j=0}^{\infty} \tilde{C}_j(f,g)(k+n_1/2)^{-j}$$

where

$$\sum_{j=0}^{\infty} C_j(f,g)h^{-j} = \sum_{j=0}^{\infty} \tilde{C}_j(f,g)(k+n_1/2)^{-j}.$$

For a vector field V on $\mathcal{T}$ and a family of smooth functions f on M let $A(V)$ be the formal operator defined by

$$A(V)(f) = V[F]\tilde{\star} f - V[F]f + (c_0(V)\tilde{\star} f - C(V)(f))\, k^{-1}.$$

Here

$$C(V) = 1/4\left(\Delta_{\tilde{G}}(V) - 2\nabla_{\tilde{G}(V)\,\mathrm{d}\,F} - \Delta_{\tilde{G}}(V)(F) + 2n_1 V[F]\right),$$

and

$$c_0(V) = 1/2\left(-\Delta_{\tilde{G}}(V)(F) + n_1 V[F]\right).$$

Let $g : [0,1] \to C^{\infty}(\mathcal{M})$ be the curve with

$$\begin{cases} \dot{g} = -(\pi^{(1,0)}(\dot{\gamma}(t))[F]g(t), \\ g(0) = 1. \end{cases}$$

Let $g_\gamma = g(1)$, and let $A_j(V)$ be the part of degree $-j$ in k.Let $f_0 = g_\gamma$. Define f_u recursively by

$$\sum_{u=0}^{\infty} \left(\pi^{(1,0)}(\dot{\gamma})\right)(f_u)k^{-u} = \sum_{u=0}^{\infty} A_u\left(\pi^{(1,0)}(\dot{\gamma})\right)(f_u)k^{-u}.$$

Finally, for a function $f \in C^{\infty}(M)$, we let $T^{(k)}_{f,(\sigma_1,\sigma_0)} = (T^{(k)}_{f,\sigma_1})_{|\mathrm{H}_k(\sigma_0)}$, where $T^{(k)}_{f,\sigma}$ denote the Toeplitz operator associated to f with respect to $J(\sigma)$. We have the following asymptotic expansion due to Andersen.

Theorem 6.1.7 ([And06]). *With respect to the operator norm, we have*

$$\|\mathrm{P}^{(k)}_{\gamma} - \sum_{u=0}^{m} T^{(k)}_{f_u,(\sigma_1,\sigma_0)}\,(k+n_1/2)^{-u}\| = O(k^{-(m+1)}). \tag{6.7}$$

Stricly speaking, this result is not stated explicitly in [And06] but follows from results obtained in that paper.

6.2 Quantum representations

Fix a holomorphic rigid family of Kähler structures $J : \mathcal{T} \to \Gamma^{\infty}(M, \mathrm{End}(TM))$. satisfying the condition of Theorem 6.1.6, and let $\nabla^{H,k}$ be the associated projectively flat Hitchin connection defined in Theorem 6.1.6. From now on, we shall assume in addition that M is simply connected. As we have fixed the family, we will often suppress J from the notation. For all $\sigma \in \mathcal{T}$ we write $\mathrm{H}_k(J(\sigma)) = \mathrm{H}_k(\sigma)$.

6.2.1 prequantum actions and quantum actions

Definition 6.2.1. Let Γ be a group. A prequantum action of Γ consists of an action of Γ on M by symplectomorphisms and an action of Γ on $\mathcal{T}$ by diffeomorphisms which are compatible with each other in the sense that for all $\varphi \in \Gamma$ and for all $\sigma \in \mathcal{T}$ the associated symplectomorphism $\varphi : (M, J(\sigma)) \to (M, J(\varphi.\sigma))$ is holomorphic.

Fix a prequantum action of Γ on $(\mathcal{T}, M)$. We note the following proposition.

Proposition 6.2.1. *For each $\varphi \in \Gamma$ and each $k \in \mathbb{N}$, there is a unique (up to a $U(1)$-factor) smooth section $\tilde{\varphi}_k \in \Gamma^\infty(M, \varphi^*(L^k) \otimes L^{-k})$ which is parallel with respect to the induced connection $\widetilde{\nabla}_k = \varphi^*(\nabla^{(k)}) \otimes I + I \otimes \nabla^{(-k)}$ and unitary with respect to $\varphi^*(h^{(k)}) \otimes h^{(-k)}$. The section induces an isomorphism of bundles $\varphi_k^* : \mathrm{H}_k \to \varphi^*(\mathrm{H}_k)$ which covers φ and preserves the Hitchin connection. One can choose $\tilde{\varphi}_k = \tilde{\varphi}_1^{\otimes k}$.*

Remark 6.2.2. We can and will always assume that $\tilde{\varphi}_k = \tilde{\varphi}_1^{\otimes k}$.

Proof. As L is a prequantum line bundle and as φ is a symplectomorphism, it is easily seen that $\widetilde{\nabla}_k$ is a flat connection. As M is assumed to be simply connected, existence and uniqueness of the parallel section follow, and the two properties follow from the fact that $\nabla^{(k)}$ and $h^{(k)}$ are compatible, and from the construction of the Hitchin connection. □

We shall consider prequantum actions compatible with a real analytic structure.

Definition 6.2.2. A real analytic structure on $(\mathcal{T}, M, L)$ is given by a real analytic structure on L and M such that the projection $L \to M$ is a real analytic map and such that the Hermitian metric h is real analytic and such that for any $\sigma \in \mathcal{T}$ it is true that any local holomorphic section of $L \to (M, J(\sigma))$ is real analytic. A prequantum action of Γ on a triple $(\mathcal{T}, M, L)$ with a real analytic structure is said to be a real analytic prequantum action if for any $\varphi \in \Gamma$, the associated symplectomorphism $\varphi : M \to M$ is real analytic.

We now define the projective action associated to the prequantum action.

Definition 6.2.3. For any $k \in \mathbb{N}$ and any $\sigma \in \mathcal{T}$ we have a homomorphism

$$\mathrm{Z}_k : \Gamma \to \mathrm{PGL}(\mathrm{H}_k(\sigma)) \tag{6.8}$$

which for each element $\varphi \in \Gamma$ admits a representative $\mathrm{Z}_k(\varphi)$ constructed as follows. Choose a smooth curve $\gamma : I \to \mathcal{T}$ starting at $\varphi.\sigma$ and ending at σ. By definition $\mathrm{Z}_k(\varphi)$ acts via the composition

$$\mathrm{Z}_k(\varphi) : \mathrm{H}_k(\sigma) \xrightarrow{\varphi_k^*} \mathrm{H}_k(\varphi.\sigma) \xrightarrow{\mathrm{P}_\gamma^{(k)}} \mathrm{H}_k(\sigma).$$

The homomorphism (6.8) is referred to as the *quantum representation*.

Quantum representations are related to the problem of quantizing a symplectomorphism. This has been considered by several authors in other contexts, see for instance the works of Zeldith [Zel97, Zel03], Bolte [BK98] or the works of Galasso and Paoletti [GP18]. We note the following proposition.

Proposition 6.2.3. *Assume that M^φ is a smooth submanifold. Observe that $\tilde{\varphi}$ restricts to a unitary endomorphism of the line bundle $L_{|M^\varphi} \to M^\varphi$. This endomorphism can be written as multiplication by $e^{i\theta}$ where $\theta \bmod 2\pi\mathbb{Z}$ is constant along connected components of M^φ.*

Proof. Let $\gamma : I \to M^\varphi$ be a smooth curve contained in an open subset of M on which L admits a smooth local frame α. Write $\nabla\alpha = \psi \otimes \alpha$ where ψ is a 1-form. Write $\tilde{\varphi} = \vartheta\,(\alpha^* \otimes \varphi^*(\alpha))$ where ϑ is a smooth function. We must show that $\frac{\mathrm{d}}{\mathrm{d}\,t}(\vartheta \circ \gamma) = 0$. As $\tilde{\varphi}_\nu$ is parallel we have $0 = \widetilde{\nabla}(\tilde{\varphi}) = (\mathrm{d}\,\vartheta - \psi + \varphi^*(\psi))\,(\alpha^* \otimes \varphi^*(\alpha))$. Therefore $\mathrm{d}\,\vartheta = \psi - \varphi^*(\psi)$, and from this we conclude

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(\vartheta \circ \gamma) = \mathrm{d}\,\vartheta(\dot{\gamma}) = \psi(\dot{\gamma}) - \varphi^*(\psi)(\dot{\gamma}) = \psi(\dot{\gamma} - \mathrm{d}\,\varphi(\dot{\gamma})) = \psi(\dot{\gamma} - \dot{\gamma}) = 0.$$

This concludes the proof. □

6.2.2 Asymptotic expansions of quantum characters

Fix $\sigma \in \mathcal{T}$. Let $\varphi \in \Gamma$ and let $\gamma : I \to \mathcal{T}$ be a smooth curve starting at $\varphi.\sigma$ and ending at σ. We are interested in calculating the asymptotic of $\mathrm{tr}(Z_k(\varphi))$ as $k \to +\infty$. We introduce the parameter $\tilde{k} = k + \frac{n_1}{2}$.

We shall use Theorem 6.1.7 and Theorem 6.1.4 to give an expansion of $\mathrm{tr}(Z_k(\varphi))$ in terms of oscillatory integrals. To do this, we must use the fact that the two Bergman kernels are naturally related.

Lemma 6.2.4. *For two unit norm sections $\psi_1 : U_1, \to L, \psi_2 : U_2 \to L$ we have for all $p_1 \in U_1$ and $p_2 \in U_2$ that*

$$K_\sigma^{(k)}(p_1, p_2) = B_\sigma^{(k)}(\psi_1^*(p_1), \psi_2^*(p_2)) \left(\psi_1^k(p_1)\right) \otimes \left(\psi_2^k(p_2)\right)^*$$

where for a frame η of L, we let η^* be the dual co-frame.

Choose an open covering $\{U_w\}_{w\in W}$ of $M^\varphi = \{x \in M : \varphi(x) = x\}$ such that U_w is an open complex coordinate ball with centre $z_w \in M^\varphi$ and which admits a holomorphic non-vanishing section $s_w : U_w \to L$. By shrinking U_w if necessary, we can and will assume that s_w is defined on $\varphi(U_w)$. We may assume that s_w is defined on $\varphi(U_w)$. Define $\{\chi_w \in C^\infty(U_w \times U_w)\}_{w\in W}$ as in Theorem 6.1.4. Let $\alpha_w = s_w/(|s_w|)$ and define $\theta \in C^\infty(U_w, \mathbb{R}/2\pi\mathbb{Z})$ by

$$\tilde{\varphi}_1(\alpha_w) = e^{i\theta_w}\varphi^*(\alpha_w).$$

Definition 6.2.4. Let $R = (\mathrm{Id}, \varphi) : M \to M \times M$. Define $\widehat{\varphi}_w \in C^\infty(U_w, \mathbb{C})$ by

$$\widehat{\varphi}_w = i\theta_w + \chi_w \circ R.$$

We are now ready to expand $\mathrm{tr}\,(Z_k(\varphi))$ as a sum of products of powers of $\tilde{k}$ and oscillatory integrals.

Theorem 6.2.5 ([AP18b]). *There exists a sequence of smooth compactly supported top forms $\Omega_n^w(\varphi) \in \Omega^{2n_0}(U_w)$ giving an asymptotic expansion*

$$\operatorname{tr}(Z_k(\varphi)) = \tilde{k}^{n_0} \sum_{n=0}^{N} \sum_{w} \left(\int_{U_w} e^{\tilde{k}\widehat{\varphi}}\, \Omega_n^w(\varphi) \right) \tilde{k}^{-n} + O(k^{n_0-(N+1)}) \tag{6.9}$$

for each $N \in \mathbb{N}$. If the prequantum action is real analytic, there is a function $\widehat{\varphi} : V \to \mathbb{C}/2\pi i\mathbb{Z}$ where V is an open neighbourhood of M^φ whose imaginary part and real part are both real analytic and which satisfies

$$\widehat{\varphi}_{|U_w} = \widehat{\varphi}_w \mod 2\pi i\mathbb{Z}.$$

Proof. By (6.7) we have for any $m \in \mathbb{N}$

$$\operatorname{tr}(Z_k(\varphi)) = \sum_{u=0}^{m} \operatorname{tr}\left(T^{(k)}_{f_u,(\sigma,\varphi.\sigma)} \circ \varphi_k^* \right) \tilde{k}^{-u} + O(k^{-(m+1)}).$$

As φ is a symplectomorphism, the following formula is valid for any holomorphic section s of L^k and any $x \in M$.

$$\begin{aligned} &(\pi_{\sigma_1}^{(k)} \circ (f\varphi_k^*)(s))(x) \\ &= \int_M K_{\sigma_1}^{(k)}(x,y) f(y) \tilde{\varphi}^{\otimes k}(s)(\varphi^{-1}(y))\, \Omega(y) \\ &= \int_M K_{\sigma_1}^{(k)}(x,\varphi(y)) f(\varphi(y)) \tilde{\varphi}^{\otimes k}(s)(y)\, \Omega(y). \end{aligned}$$

An operator $P = \int B(x,y)\ \mathrm{d}\,y$ given by a smooth integral kernel is trace class and its trace is given by

$$\operatorname{tr}(P) = \int B(y,y)\ \mathrm{d}\,y.$$

From equation 5.3 in [KS01] it follows that away from the diagonal the Bergman kernel is locally uniformly $O(k^{-N})$ for every $N \in \mathbb{N}$. By choosing a partition of unity (μ_w) subordinate to the cover $\{U_w\}$ of M^φ and combining the above considerations, we arrive at

$$\operatorname{tr}\left(T^{(k)}_{f,(\sigma_0,\sigma_1)} \circ \varphi_k^* \right) \approx \sum_w \int \mu_w(y) f(\varphi(y)) B_{\sigma_1}^{(k)}(\alpha_w^*(y), \alpha_w^*(\varphi(y))) e^{ki\theta_w(y)}\, \Omega(y), \tag{6.10}$$

where $\approx$ means equality up to addition of a function of k which is $O(k^{-N})$ for every $N > 0$. Recall $\{\tilde{b}_v\}$ from Theorem 6.1.4 and define for $u, v \geq 0$ the function $f_{u,v}^w \in C^\infty(U_w)$ as follows

$$f_{u,v}^w = \mu_w \cdot f_u \circ \varphi \cdot \tilde{b}_v \circ R.$$

Applying (6.5) to the integrand of (6.10) we obtain the following expansion

$$\begin{aligned} \operatorname{tr}(Z_k(\varphi)) &= \sum_{u=0}^{m} \operatorname{tr}\left(T^{(k)}_{f_u,(\sigma_0,\sigma_1)} \circ \varphi_k^* \right) \tilde{k}^{-u} + O\left(k^{-(m+1)} \right) \\ &= k^{n_0} \sum_{u,v=0}^{N} \sum_w \left(\int_{U_w} f_{u,v}^w e^{k\widehat{\varphi}_w}\, \Omega \right) k^{-v} \tilde{k}^{-u} \\ &+ O(k^{n_0-(N+1)}). \end{aligned}$$

Here $\widehat{\varphi}_w$ is as defined in Definition 6.2.4. This proves the first part of the theorem by a simple power series substitution relating k^{-1} to $\tilde{k}^{-1}$.

In order to prove the second half, we first make the following observation. Suppose $V \subset \mathbb{C}^n$ is an open neighborhood of the origin and g a real analytic function on V, then there is a preferred almost analytical extension $\hat{g}$ to $V \times V$ which satisfies $\hat{g}(v_1, v_2) = \overline{\hat{g}(v_2, v_1)}$, and the map $g \mapsto \hat{g}$ is $\mathbb{R}$-linear. Writing $g(v) = \sum c_{a,b} v^a \overline{v}^b$, near 0 for $a, b \in \mathbb{N}^n$, we take $\hat{g}(v_1, v_2) = \sum c_{a,b} v_1^a \overline{v_2}^b$. As g is real-valed we have $c_{b,a} = \overline{c_{a,b}}$. Hence $\hat{g}$ satisfies all of the desired properties.

Assume that the prequantum action is real analytic. Note that this entails that the real and imaginary parts of $\widehat{\varphi}_w$ are both real analytic. The potential ambiguity in defining $\widehat{\varphi}$ lies in the choice of the section $s = s_w$ the choice of a real analytic extension $\tilde{\Phi}_s(y_1, y_2)$ of Φ_s, and the choice of θ_w gives a $2\pi i\mathbb{Z}$ ambiguity which we shall ignore for now for notational convenience. Write $\widehat{\varphi}_w = \widehat{\varphi}_s$, where s is the chosen section. We claim that if we choose $\tilde{\Phi}_s = \hat{\Phi}_s$ then $\widehat{\varphi}_s$ will in fact be independent of s. Any other choice of s will be of the form $s' = e^g s$ for some holomorphic function g. With the obvious notation we have that

$$\Phi_{s'} = -g - \overline{g} + \Phi_s, \quad i\theta_{s'} = i\theta_s + 2^{-1}(g - g \circ \varphi + \overline{g} \circ \varphi - \overline{g}). \tag{6.11}$$

From (6.11), holomorphicity of g and linearity of $r \mapsto \hat{r}$ we observe $\hat{\Phi}_{s'}(y_1, y_2) = -g(y_1) - \overline{g}(y_2) + \hat{\Phi}_s(y_1, y_2)$. Combining these observations, we can make the following computation

$$\begin{aligned}
\widehat{\varphi}_{s'}(z) &= i\theta_{s'}(z) + \Big(\hat{\Phi}_{s'}(z, \varphi(z)) - 2^{-1}\left(\Phi_{s'}(\varphi(z)) + \Phi_{s'}(z)\right)\Big) \\
&= i\theta_s(z) + \Big(\hat{\Phi}_s(z, \varphi(z)) - 2^{-1}\left(\Phi_s(\varphi(z)) + \Phi_s(z)\right)\Big) \\
&\quad + 2^{-1}(g(z) - g(\varphi(z)) + \overline{g}(\varphi(z)) - \overline{g}(z)) \\
&\quad - g(z) - \overline{g}(\varphi(z)) \\
&\quad + 2^{-1}\left(g(z) + \overline{g}(z) + g(\varphi(z)) + \overline{g}(\varphi(z))\right) \\
&= i\theta_s(z) + \Big(\hat{\Phi}_s(z, \varphi(z)) - 2^{-1}\left(\Phi_s(\varphi(z)) + \Phi_s(z)\right)\Big) \\
&= \widehat{\varphi}_s(z).
\end{aligned}$$

This concludes the proof. ☐

In order to analyze further the expansion given in (6.9) we turn to the integrals

$$\mathrm{I}(w, n, k) = \int_{U_w} e^{\tilde{k}\widehat{\varphi}}\, \Omega_n^w(\varphi).$$

The large k behavior of $\mathrm{I}(w, n, k)$ localize to $U_w \cap M^\varphi$, as the real part of $\widehat{\varphi}_w$ is strictly negative away from the fixed point set cf. (6.4). This follows from Proposition 5.1.1. We prove that the set of fixed points of φ correspond to stationary points and we proceed to examine the Hessian of $\widehat{\varphi}_w$ at a fixed point.

As the analysis of the phases are purely local, we may omit the index w. We focus on a holomorphic coordinate chart U centered at a fixed point p with Kähler coordinates z satisfying

$$-i\omega(p) = 2^{-1} \sum_l \mathrm{d}\, z_l \wedge \mathrm{d}\, \overline{z}_l. \tag{6.12}$$

Let $y = (y_1, y_2)$ be the holomorphic coordinates on $U \times U$ which are naturally induced by z. For the rest of Subsection 6.2.5 we use the coordinates z on U, and the coordinates y on $U \times U$.

Recall that the construction of the phase $\widehat{\varphi}$ function involves a choice of holomorphic frame

$$s : U \to L \setminus \{0\}. \tag{6.13}$$

Define $\zeta = \zeta_s$ by writing the lift $\tilde{\varphi}$ with respect to the frames s and $\varphi^*(s)$ as follows

$$\tilde{\varphi}(s)(z) = \zeta(z)s(\varphi(z)).$$

As ζ is non-vanishing, we can write (shrinking U if necessary) $\zeta = \exp(\lambda + i\theta)$ where θ and λ are smooth real valued functions. In the notation above, $\theta_w = \theta$. As $\tilde{\varphi}$ is unitary we can write $\tilde{\varphi}(\exp(2^{-1}\Phi)s) = \exp(i\theta)\left(\exp(2^{-1}\Phi \circ \varphi)\varphi^*(s)\right)$. From this we conclude that

$$\lambda = 2^{-1}(\Phi \circ \varphi - \Phi). \tag{6.14}$$

We shall now prove that all fixed points of φ are indeed stationary points of $\widehat{\varphi}$. Moreover, we can arrange that χ has a stationary point at (p, p) by choosing the holomorphic section s in (6.13) suitably. Write

$$\mathrm{I} = \mathrm{I}_{n_0 \times n_0}$$

for the identity matrix of dimension n_0.

Lemma 6.2.6 ([AP18b]). *We have*

$$U \cap M^{\varphi} \subset \{q \in U \mid \mathrm{d}\,\widehat{\varphi}_q = 0\}.$$

Moreover, we can choose the holomorphic section s in (6.13) *such that*

$$\mathrm{d}\,\Phi_p = 0, \quad \mathrm{d}\,\chi_{(p,p)} = 0, \quad \frac{\partial^2 \Phi}{\partial z^2}(p) = 0, \quad \frac{\partial^2 \Phi}{\partial z \partial \bar{z}}(p) = 2^{-1}\,\mathrm{I}. \tag{6.15}$$

Proof. Let q be a fixed point of φ and let us show that q is a stationary point by comparing the derivatives of $\chi \circ R$ with the derivatives of $i\theta$. By differentiating (6.2) and using (6.3) we get

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \tilde{\Phi}}{\partial y_1} \circ \Delta, \quad \frac{\partial \Phi}{\partial \bar{z}} = \frac{\partial \tilde{\Phi}}{\partial \bar{y}_2} \circ \Delta. \tag{6.16}$$

By using (6.3) and (6.16) one obtains

$$\begin{aligned} \frac{\partial \chi \circ R}{\partial z}(q) &= 2^{-1}\left(\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial \bar{z}}\frac{\partial \overline{\varphi}}{\partial z} - \frac{\partial \Phi}{\partial z}\frac{\partial \varphi}{\partial z}\right)(q), \\ \frac{\partial \chi \circ R}{\partial \bar{z}}(q) &= 2^{-1}\left(\frac{\partial \Phi}{\partial \bar{z}}\frac{\partial \overline{\varphi}}{\partial \bar{z}} - \frac{\partial \Phi}{\partial \bar{z}} - \frac{\partial \Phi}{\partial z}\frac{\partial \varphi}{\partial \bar{z}}\right)(q). \end{aligned} \tag{6.17}$$

In order to calculate the derivatives of $i\theta$ at q we shall first relate the derivatives of $\log(\zeta)$ to Φ using that $\tilde{\varphi}$ is parallel with respect to $\widetilde{\nabla}$, and then calculate the derivatives of $i\theta$ using the identity (6.14). Here $\widetilde{\nabla}$ is the connection on $\varphi^*(L) \otimes L^*$ naturally induced from ∇. Write

$\vartheta = \log(\zeta)$. Define the $(1,0)$-form ψ by $\nabla s = \psi \otimes s$. Write $\widetilde{\nabla}(\varphi^*(s) \otimes s^*) = \kappa \otimes \varphi^*(s) \otimes s^*$ where κ is a 1-form. We have $\kappa = \varphi^*(\psi) - \psi$. Thinking of $\tilde{\varphi}$ as a section of $\varphi^*(L) \otimes L^*$ we can write $\tilde{\varphi} = \exp(\vartheta)\varphi^*(s) \otimes s^*$. By parallelity of $\tilde{\varphi}$ we get

$$0 = \widetilde{\nabla}\tilde{\varphi} = (\mathrm{d}\,\vartheta + \kappa)\left(\exp(\vartheta)\varphi^*(s) \otimes s^*\right)$$

which is equivalent to $\mathrm{d}\,\vartheta = \psi - \varphi^*(\psi)$. In particular

$$\frac{\partial \vartheta}{\partial z_l} = \psi^l - \sum_j \psi^j \circ \varphi \frac{\partial \varphi^j}{\partial z_l}, \quad \frac{\partial \vartheta}{\partial \bar{z}_l} = -\sum_j \psi^j \circ \varphi \frac{\partial \varphi^j}{\partial \bar{z}_l}.$$

As h, ∇ are compatible, we have

$$\mathrm{d}\,\Phi = -\,\mathrm{d}\left(\log(h(s,s)\right) v = -\psi - \overline{\psi}$$

from which we get

$$\frac{\partial \Phi}{\partial z_l} = -\psi^l, \quad \frac{\partial \Phi}{\partial \bar{z}_l} = -\overline{\psi^l}. \tag{6.18}$$

Combining this with the expressions for the derivatives of ϑ given above, we get

$$\frac{\partial \vartheta}{\partial z} = \left(\frac{\partial \Phi}{\partial z} \circ \varphi\right) \frac{\partial \varphi}{\partial z} - \frac{\partial \Phi}{\partial z}, \quad \frac{\partial \vartheta}{\partial \bar{z}} = \left(\frac{\partial \Phi}{\partial z} \circ \varphi\right) \frac{\partial \varphi}{\partial \bar{z}}.$$

Combining with (6.14) this gives

$$\begin{aligned}
\frac{\partial i\theta}{\partial z} &= \frac{\partial\left(\vartheta - 2^{-1}\left(\Phi \circ \varphi - \Phi\right)\right)}{\partial z} \\
&= 2^{-1}\left(\left(\frac{\partial \Phi}{\partial z} \circ \varphi\right) \frac{\partial \varphi}{\partial z} - \left(\frac{\partial \Phi}{\partial \bar{z}} \circ \varphi\right) \frac{\partial \overline{\varphi}}{\partial z} - \frac{\partial \Phi}{\partial z}\right).
\end{aligned}$$

As θ, Φ are real functions, we obtain

$$\begin{aligned}
\frac{\partial i\theta}{\partial z} &= 2^{-1}\left(\left(\frac{\partial \Phi}{\partial z} \circ \varphi\right) \frac{\partial \varphi}{\partial z} - \left(\frac{\partial \Phi}{\partial \bar{z}} \circ \varphi\right) \frac{\partial \overline{\varphi}}{\partial z} - \frac{\partial \Phi}{\partial z}\right), \\
\frac{\partial i\theta}{\partial \bar{z}} &= -2^{-1}\left(\left(\frac{\partial \Phi}{\partial \bar{z}} \circ \varphi\right) \frac{\partial \overline{\varphi}}{\partial \bar{z}} - \left(\frac{\partial \Phi}{\partial z} \circ \varphi\right) \frac{\partial \varphi}{\partial \bar{z}} - \frac{\partial \Phi}{\partial \bar{z}}\right).
\end{aligned} \tag{6.19}$$

Comparing (6.19) with (6.17) we see that q is a stationary point of $\widehat{\varphi}$.

We now proceed to prove (6.15). We see that if we perform the transformation $s \mapsto \exp(r)s$ then the $(1,0)$-form ψ transform as follows $\psi \mapsto \psi + \partial r$. Consider the holomorphic function r given by $r = \sum_l \frac{\partial \Phi}{\partial z_l}(p) z_l + \sum_{l \leq j} \frac{\partial^2 \Phi}{\partial z_l \partial z_j}(p) z_l z_j$. Replace s by $\acute{s} = \exp(r)s$, and observe that by (6.18) we have

$$\mathrm{d}\,\acute{\Phi}_p = 0, \quad \frac{\partial^2 \acute{\Phi}}{\partial z_l \partial z_j}(p) = 0.$$

We now turn to the fourth equation. We recall that as ∇ is the Chern connection of h, and as L is a prequantum line bundle, we have

$$\bar{\partial}\partial \log(h(\acute{s}))_p = F_{\nabla}(p) = -i\omega(p) = 2^{-1} \sum_l \mathrm{d}\, z_l \wedge \mathrm{d}\, \bar{z}_l.$$

Thus we obtain the desired equation which is seen to hold independently of the choice of section

$$\frac{\partial^2 \acute{\Phi}}{\partial z \partial \bar{z}}(p) = 2^{-1}\,\mathrm{I}.$$

We now turn to the remaining equation. By (6.16) we see that

$$\frac{\partial \acute{\chi}}{\partial y} \circ \Delta = \left(2^{-1}\tfrac{\partial \acute{\Phi}}{\partial z}, \quad -2^{-1}\tfrac{\partial \acute{\Phi}}{\partial z}\right), \quad \frac{\partial \acute{\chi}}{\partial \bar{y}} \circ \Delta = \left(-2^{-1}\tfrac{\partial \acute{\Phi}}{\partial \bar{z}}, \quad 2^{-1}\tfrac{\partial \acute{\Phi}}{\partial \bar{z}}\right).$$

Therefore $\acute{\chi}$ has a stationary point at (p, p) as $\acute{\Phi}$ has. □

From now on we assume $s : U \to L \setminus \{0\}$ from (6.13) has been chosen as in Lemma 6.2.6.

Lemma 6.2.7. *Assume $Q : U \to \mathbb{C}$ is a smooth function defined on an open subset U of $\mathbb{C}^m$. Assume $R : V \to U$ is a smooth map defined on an open subset V of $\mathbb{C}^l$. If $R(z)$ is a stationary point of Q then the Hessian of G at z is given by $\mathrm{Hess}(Q \circ R) = \mathrm{d}\, R^t \mathrm{Hess}(Q)\, \mathrm{d}\, R$.*

Assume that M^φ is a smooth submanifold or that the action is real analytic such that M^φ is a real algebraic subvariety. For each component $Y \subset M^\varphi$ there exists $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that for any $q \in Y$ the linear endomorphism $\tilde{\varphi}_q^{\otimes k} : L_q^k \to L_q^k$ is given by multiplication by $e^{ik\theta}$. Let $\{\theta_j \in \mathbb{R}/2\pi\mathbb{Z}\}$ be the set of arguments defined in this way. Recall (from Section 1.2.1) that the fixed point set M^φ is said to be non-degenerate if every component $Y \subset M^\varphi$ is a smooth submanifold of M and $TY = \mathrm{Ker}\,(\mathrm{d}\,\varphi - \mathrm{Id})_{|Y}$. We have the following theorem.

Theorem 6.2.8 ([AP18a]). *Assume M^φ is non-degenerate. For each θ_j, define $2m_j$ as the maximal integer occuring as the dimension of a component of M^φ on which $\tilde{\varphi}$ is given by multiplication by $\exp(i\theta_j)$. There exists a sequence of smooth densities Ω_α^j on M^φ giving an asymptotic expansion*

$$\mathrm{tr}\,(Z_k(\varphi)) \sim \sum_j e^{i\tilde{k}\theta_j} \tilde{k}^{m_j} \sum_{\alpha=0}^{\infty} \tilde{k}^{-\frac{\alpha}{2}} \int_{M^\varphi} \Omega_\alpha^j. \tag{6.20}$$

Proof. The proof consists in three steps. First we compute the Hessian at a stationary point, second we adress the question of non-degeneracy, and third we apply stationary phase approximation. Only the last step uses the assumption that M^φ is non-degenerate.

We now give the computation of the Hessian. The equations (6.15) makes it significantly easier to analyze the Hessian of P at a fixed point, together with Lemma 6.2.7. Write

$$\frac{\partial \varphi}{\partial z}(p) = H, \quad \frac{\partial \varphi}{\partial \bar{z}}(p) = K.$$

The Hessian of $\widehat{\varphi}$ at p with respect to $(\partial z, \partial \bar{z})$ is given as follows

$$\mathrm{Hess}(\widehat{\varphi})_p = 4^{-1} \begin{pmatrix} \mathrm{I} & 0 \\ H & K \\ 0 & \mathrm{I} \\ \overline{K} & \overline{H} \end{pmatrix}^t \begin{pmatrix} 0 & 0 & -\mathrm{I} & 2\mathrm{I} \\ 0 & 0 & 0 & -\mathrm{I} \\ -\mathrm{I} & 0 & 0 & 0 \\ 2\mathrm{I} & -\mathrm{I} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ H & K \\ 0 & \mathrm{I} \\ \overline{K} & \overline{H} \end{pmatrix}. \tag{6.21}$$

In fact this can be can be rephrased with the following formula valid at our fixed point p

$$\mathrm{Hess}(\widehat{\varphi}) = \mathrm{d}\, R^t \mathrm{Hess}(\chi)\, \mathrm{d}\, R$$

where $R = (\mathrm{Id}, \varphi)$. We first show that the Hessian of $i\theta$ vanish at p from which we conclude that $\mathrm{Hess}(\widehat{\varphi})_p = \mathrm{Hess}(\chi \circ R)_p$, and we then derive a coordinate expression for $\mathrm{Hess}(\chi \circ R)_p$ using Lemma 6.2.6 and Lemma 6.2.7.

From (6.12) we see that the fact that $\mathrm{d}\,\varphi_p$ is a symplectomorphism with respect to ω_p is equivalent to the following two equations

$$H^t\overline{H} - \overline{K}^t K = \mathrm{I}, \quad H^t\overline{K} - \overline{K}^t H = 0. \tag{6.22}$$

Combining equation (6.19) with equation (6.15) and equation (6.22)

$$\begin{aligned}
\frac{\partial^2 i\theta}{\partial z^2}(p) &= 2^{-1}\left(\frac{\partial \varphi}{\partial z}^t \left(\frac{\partial^2 \Phi}{\partial z^2}\circ\varphi\right)\frac{\partial \varphi}{\partial z}\right)(p) \\
&\quad + 2^{-1}\left(\frac{\partial \overline{\varphi}}{\partial z}^t \left(\frac{\partial^2 \Phi}{\partial \overline{z}\partial z}\circ\varphi\right)\frac{\partial \varphi}{\partial z}\right)(p) \\
&\quad - 2^{-1}\left(\frac{\partial \varphi}{\partial z}^t \left(\frac{\partial^2 \Phi}{\partial z\partial \overline{z}}\circ\varphi\right)\frac{\partial \overline{\varphi}}{\partial z}\right)(p) \\
&\quad - 2^{-1}\left(\frac{\partial \overline{\varphi}}{\partial z}^t \left(\frac{\partial^2 \Phi}{\partial \overline{z}^2}\circ\varphi\right)\frac{\partial \overline{\varphi}}{\partial z}\right)(p) \\
&\quad - 2^{-1}\frac{\partial^2 \Phi}{\partial z^2}(p) \\
&\quad = 4^{-1}\left(\overline{K}^t H - H^t\overline{K}\right) = 0.
\end{aligned}$$

Similarly one can use the equations (6.19), (6.15) together with (6.22) to show that $\frac{\partial^2 i\theta}{\partial \overline{z}\partial z}(p) = \frac{\partial^2 i\theta}{\partial z\partial \overline{z}}^t(p) = 0$. This shows that $\mathrm{Hess}(\widehat{\varphi})_p = \mathrm{Hess}(\chi \circ R)_p$.

We claim that the Hessian of χ at (p,p) with respect to the ordered basis $(\partial y_1, \partial y_2, \partial \overline{y}_1, \partial \overline{y}_2)$ is equal to the following matrix

$$\mathrm{Hess}(\chi)_{(p,p)} = 4^{-1}\begin{pmatrix} 0 & 0 & -\mathrm{I} & 2\mathrm{I} \\ 0 & 0 & 0 & -\mathrm{I} \\ -\mathrm{I} & 0 & 0 & 0 \\ 2\mathrm{I} & -\mathrm{I} & 0 & 0 \end{pmatrix}.$$

We shall calculate the Hessian of χ as four block matrices and we start with $\frac{\partial^2 \chi}{\partial y^2}\circ\Delta(p)$. Differentiating equation (6.16) and using (6.3) we get

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{\partial^2 \tilde{\Phi}}{\partial y_1^2}\circ\Delta. \tag{6.23}$$

Combining equation (6.15) and equation (6.23) we obtain

$$\frac{\partial^2 \chi}{\partial y_1^2}\circ\Delta(p) = \frac{\partial^2 \tilde{\Phi}}{\partial y_1^2}\circ\Delta(p) - 2^{-1}\frac{\partial^2 \Phi}{\partial z^2}(p) = 0.$$

Another two applications of (6.3) gives

$$\frac{\partial^2 \chi}{\partial y_1\partial y_2}\circ\Delta(p) = \frac{\partial^2 \chi}{\partial y_2\partial y_1}^t\circ\Delta(p) = 0$$

and

$$\frac{\partial^2 \chi}{\partial y_2^2} \circ \Delta(p) = -2^{-1} \frac{\partial^2 \Phi}{\partial z^2}(p) = 0.$$

Thus we see

$$\frac{\partial^2 \chi}{\partial y^2} \circ \Delta(p) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The computations of $\frac{\partial^2 \chi}{\partial y \partial \overline{y}} \circ \Delta(p)$ and $\frac{\partial^2 \chi}{\partial \overline{y}^2} \circ \Delta(p)$ are similar. We can now appeal to Lemma 6.2.7 as

$$\mathrm{d}\, R = \begin{pmatrix} \mathrm{I} & 0 \\ H & K \\ 0 & \mathrm{I} \\ \overline{K} & \overline{H} \end{pmatrix}.$$

This concludes the computation of the Hessian.

We now address the question of non-degeneracy of the Hessian of $\widehat{\varphi}$. Denote the symmetric bilinear form associated to $\mathrm{Hess}(\widehat{\varphi})_p$ by Ψ. We shall prove that

$$\{\xi \in T_p M_{\mathbb{C}} : \Psi(\xi, \eta) = 0,\ \forall \eta \in T_p M_{\mathbb{C}}\} = \mathrm{Ker}(\mathrm{d}\, \varphi_p - \mathrm{Id}). \tag{6.24}$$

Moreover, we shall prove that Ψ is non-degenerate on any real complementary subspace to $\mathrm{Ker}(\mathrm{d}\, \varphi_p - \mathrm{Id})$.

Let $\eta \in T_p M_{\mathbb{C}}$ be any vector not fixed by $\mathrm{d}\, \varphi$. We shall argue that

$$\Psi(\eta, \overline{\eta}) \neq 0 \tag{6.25}$$

which will imply that the symmetric bilinear form Ψ is non-degenerate on any subspace which is complimentary to $\mathrm{Ker}(\mathrm{d}\, \varphi_p - \mathrm{Id})$ and closed under conjugation.

Write $\mathrm{d}\, R_p(\eta) = (v', w', v'', w'')$. We have $\mathrm{d}\, R_p(\overline{\eta}) = (\overline{v''}, \overline{w''}, \overline{v'}, \overline{w'})$, as $\mathrm{d}\, \varphi$ is the complex-linear extension of a real linear endomorphism of $T_p M$. Let $(\,\cdot\,,\,\cdot\,)$ be the standard Hermitian metric on $\mathbb{C}^{n_0}$, given by $(u, y) = u^t \overline{y}$ where we recall that $n_0 = \dim_{\mathbb{C}}(M)$. According to (6.21) we have

$$-4\Psi(\eta, \overline{\eta}) = (v', v') + (w', w') - 2(v', w') + (v'', v'') + (w'', w'') - 2(w'', v'').$$

For any two vectors κ, μ the real part of $(\kappa, \kappa) + (\mu, \mu) - 2(\kappa, \mu)$ is equal to $|\kappa - \mu|^2$. Thus (6.25) holds, as

$$-4\,\mathrm{Re}\,(\Psi(\eta, \overline{\eta})) = |v' - w'|^2 + |v'' - w''|^2 \neq 0.$$

We finish by proving the inclusion

$$\{\xi \in T_p M_{\mathbb{C}} : \Psi(\xi, \eta) = 0,\ \forall \eta \in T_p M_{\mathbb{C}}\} \supset \mathrm{Ker}\left(\mathrm{d}\, \varphi_p - \mathrm{Id}\right).$$

To that end, assume ξ is fixed by $\mathrm{d}\, \varphi_p$. Write ξ' for the $(1,0)$ part of ξ and write ξ'' for the $(0,1)$ part of ξ. We compute

$$\begin{aligned} -4\Psi(\,\cdot\,, \xi) &= \begin{pmatrix} {\pi^{(1,0)}}^t & {\mathrm{d}'\, \varphi_p}^t & {\pi^{(0,1)}}^t & {\mathrm{d}''\, \varphi_p}^t \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{I} & -2\,\mathrm{I} \\ 0 & 0 & 0 & \mathrm{I} \\ \mathrm{I} & 0 & 0 & 0 \\ -2\,\mathrm{I} & \mathrm{I} & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi' \\ \xi' \\ \xi'' \\ \xi'' \end{pmatrix} \\ &= \left(\pi^{(0,1)} - \mathrm{d}''\, \varphi_p\right)^t \xi' + \left(\mathrm{d}'\, \varphi_p - \pi^{(1,0)}\right)^t \xi''. \end{aligned}$$

That ξ is fixed by $\mathrm{d}\,\varphi_p$ is equivalent to

$$\xi' = H\xi' + K\xi'', \quad \xi'' = \overline{K}\xi' + \overline{H}\xi''.$$

We have

$$\mathrm{d}'\,\varphi_p = \begin{pmatrix} H & K \end{pmatrix}, \quad \mathrm{d}''\,\varphi_p = \begin{pmatrix} \overline{KH} \end{pmatrix}, \quad \pi^{(1,0)} = \begin{pmatrix} \mathrm{I} & 0 \end{pmatrix}, \quad \pi^{(0,1)} = \begin{pmatrix} 0 & \mathrm{I} \end{pmatrix}.$$

Thus we can write

$$\begin{aligned}\left(\pi^{(0,1)} - \mathrm{d}''\,\varphi_p\right)^t \xi' + \left(\mathrm{d}'\,\varphi_p - \pi^{(1,0)}\right)^t \xi'' &= \begin{pmatrix} -\overline{K}^t\xi' \\ \xi' - \overline{H}^t\xi' \end{pmatrix} + \begin{pmatrix} H^t\xi'' - \xi'' \\ K^t\xi'' \end{pmatrix}. \\ &= \begin{pmatrix} \left(H^t\overline{K} - \overline{K}^tH\right)\xi' + \left(H^t\overline{H} - \overline{K}^tK - I\right)\xi'' \\ \left(I - \overline{H}^tH + K^t\overline{K}\right)\xi' + \left(K^t\overline{H} - \overline{H}^tK\right)\xi'' \end{pmatrix} = 0.\end{aligned}$$

The last equation follows from (6.22).

We are now ready to apply Theorem to $\mathrm{tr}(Z_k(\varphi))$. For notational convenience we shall assume that all the components of M^φ are of the same dimension $2m_0$. Let e_0 be the codimension of M^φ, i.e. $2n_0 - 2m_0 = e_0$. Choose each holomorphic trivialization $s_w : U_w \to L \setminus \{0\}$ appearing in the covering of M^φ, such that each U_w admits smooth real coordinates $(u_w, v_w) : U_w \to \mathbb{R}^{2n_0}$ that satisfies $\{u_w = 0\} = U_w \cap M^\varphi$. With notation as in Theorem 6.2.5, define $g_n^w \in C^\infty(U_w)$ by $\Omega_n^w(\varphi) = g_n^w \,\mathrm{d}\, u_w \wedge \mathrm{d}\, v_w$. We can now rewrite the expansion in Theorem 6.2.5 as

$$\mathrm{tr}(Z_k(\varphi)) = \tilde{k}^{n_0} \sum_{n=0}^{M} \sum_{w} \tilde{k}^{-n} \int \left(\int g_n^w(u_w, v_w) e^{\tilde{k}\widehat{\varphi}_w(u_w,v_w)} \,\mathrm{d}\, u_w \right) \mathrm{d}\, v_w + O(k^{n_0-(M+1)}). \tag{6.26}$$

As the real part of χ is negative away from the diagonal, we see that (6.24) that the phase function $\widehat{\varphi}_w$ satisfy condition (5.1). We now wish to apply stationary phase approximation to the double integrals

$$I(n, w, \tilde{k}) = \int \left(\int g_n^w(u_w, v_w) e^{\tilde{k}\widehat{\varphi}_w(u_w,v_w)} \,\mathrm{d}\, u_w \right) \mathrm{d}\, v_w.$$

We shall use freely that a function of k or $\tilde{k}$ is $O(k^{-M})$ if and only if it is $O(\tilde{k}^{-M})$. Let $\mathcal{I}$ be the ideal generated by $\frac{\partial\widehat{\varphi}_w}{\partial u_w^1}, \ldots, \frac{\partial\widehat{\varphi}_w}{\partial u_w^{e_0}}$. With notation as above, we observe that for any smooth function $G(u_w, v_w)$, we have $G^0(v_w) = G(0, v_w)$ as $\frac{\partial\widehat{\varphi}_w}{\partial u_w}(0, v_w) = 0$ by Lemma 6.2.6 and $G(u_w, v_w) - G^0(v_w) \in \mathcal{I}$. Thus by Hörmander's theorem of stationary phase approximation with parameters (Theorem 5.1.2 above) we get the following expansion

$$\begin{aligned}&\int g_n^w(u_w, v_w) e^{\tilde{k}\widehat{\varphi}_w(u_w,v_w)} \,\mathrm{d}\, u_w + O\left(\tilde{k}^{-(N+1+e_0/2)}\right) \\ &= e^{\tilde{k}\widehat{\varphi}(0,v_w)} \tilde{k}^{-2^{-1}e_0} \sqrt{\frac{2i\pi}{\det\left(\frac{\partial^2\widehat{\varphi}_w}{\partial u_w^2}(0, v_w)\right)}} \sum_{r=0}^{N} \tilde{k}^{-r} L_{\widehat{\varphi}_w,r}(g_n^w)(0, v_w).\end{aligned} \tag{6.27}$$

From Lemma 6.2.6 we easily deduce that $\widehat{\varphi}_w(0, v_w) = i\theta_j$, for some j. Define $\Omega_{n,r}^w$ by

$$\Omega_{n,r}^w(v_w) = \sqrt{\frac{2i\pi}{\det\left(\frac{\partial^2\widehat{\varphi}_w}{\partial u_w^2}(0, v_w)\right)}} L_{\widehat{\varphi}_w,r}(g_n^w)(0, v_w) \,\mathrm{d}\, v_w.$$

This form has compact support inside $U_l \cap M^\varphi$. Using (6.27) we have the following estimate

$$I(n,w,\tilde{k}) = e^{i\tilde{k}\theta_j}\tilde{k}^{-2^{-1}e_0}\sum_{r=0}^{N}\tilde{k}^{-r}\int_{M^\varphi}\Omega^w_{n,r} + O(k^{-(N+1+e_0/2)}). \tag{6.28}$$

For each θ_j let W_j be the set of w' such that $U_{w'} \cap M^\varphi$ lies in a component of M^φ on which $\tilde{\varphi}$ is given by multiplication by $\exp(i\theta_j)$. Define $\Omega^j_{n,r} = \sum_{w\in W_j}\Omega^w_{n,r}$. Observe that $n_0 - \frac{1}{2}e_0 = m_0$. Using (6.28), we can rewrite (6.26) as follows

$$\operatorname{tr}(Z_k(\varphi)) = \tilde{k}^{m_0}\sum_j e^{ik\theta_j}\sum_{n,r=0}^{N}\tilde{k}^{-n-r}\int_{M^\varphi}\Omega^j_{n,r} + O(k^{m_0-(N+1)}).$$

Defining $\Omega^j_\alpha = \sum_{n+r=\alpha}\Omega^j_{n,r}$, we see that the theorem holds. □

By appealing to the results of Chapter 5, we can prove a stronger result if we impose the real analycity condition.

Theorem 6.2.9 ([AP18a]). *Assume the prequantum action is real analytic. Assume that all $z \in M^\varphi$ satisfy one of the following three conditions:*

- *z is a smooth point with $T_zM^\varphi = \operatorname{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})$,*
- *$\dim(\operatorname{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})) \leq 1$, or*
- *z is an isolated saddle point of the germ of $\widehat{\varphi}_{\mathbb{C}}$ at z.*

Then there exists for each θ_j an unbounded set $A_j \subset \mathbb{Q}_{\leq 0}$, $n_j \in \mathbb{Q}_{\geq 0}$, $d_j \in \mathbb{N}$, and a sequence $\{c^j_{\alpha,\beta}\}_{\alpha\in A_j,\ 0\leq\beta\leq d_j} \subset \mathbb{C}$ giving an asymptotic expansion

$$\operatorname{tr}(Z_k(\varphi)) \sim_{k\to\infty} \sum_j e^{\tilde{k}\theta_j}\tilde{k}^{n_j}\sum_{\alpha\in A_j}\sum_{\beta=0}^{d_j} c^j_{\alpha,\beta}\tilde{k}^\alpha\log(\tilde{k})^\beta.$$

If the first or second condition holds for all $z \in M^\varphi \cap \widehat{\varphi}^{-1}(\theta_j)$ then $d_j = 0$ and $n_j = m_j$.

Proof. By Theorem 6.2.5 we have

$$\operatorname{tr}(Z_k(\varphi)) = \tilde{k}^{n_0}\sum_{n=0}^{N}\sum_{w}\left(\int_{U_w} e^{\tilde{k}\widehat{\varphi}}\,\Omega^w_n(\varphi)\right)\tilde{k}^{-n} + O(k^{n_0-(N+1)}).$$

Hence it will be enough to show the existence of an asymptotic expansion of integrals of the following form

$$I(\tilde{k}) = \int_U e^{\tilde{k}\widehat{\varphi}(x)}\phi(x)\ \mathrm{d}\,x$$

where U is an open neighborhood of a fixed point z and ϕ is a smooth function of compact support concentrated in a small coordinate ball centered at z. The case where z is a smooth point of M^φ with $T_zM^\varphi = \operatorname{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})$ is dealt with in the proof of Theorem 6.2.8. The case with $\dim\operatorname{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id}) \leq 1$ can be dealt with by appealing to the first part of Theorem 5.2.10 and the second step in the proof of Theorem 6.2.8. The case where $\widehat{\varphi}_{\mathbb{C}}$ has an isolated stationary case is dealt with by appealing to the second part of Thereom 5.2.10. Here we use that any maximum of $\operatorname{Re}(\widehat{\varphi})$ is a fixed point by (6.4) and therefore a stationary point of $\widehat{\varphi}$ by Lemma 6.2.6 and therefore a stationary point of $\widehat{\varphi}_{\mathbb{C}}$. □

6.3 The moduli space of flat connections on a surface

This section is influenced by Hitchin's article [Hit90]. We turn our attention to the situation where M is the moduli space of flat connections over surface. Recall the notation from the beginning of Section 1.2.1. We let $\mathcal{M}^*(G,\Sigma,P,C) \subset \mathcal{M}(G,\Sigma,P,C)$ denote the subset of gauge equivalence classes of irreducible connections. Now let Σ be a closed oriented surface of genus ≥ 2. Choose $p \in \Sigma$ together with a projective tangent p. As in Section 1.2.1 we choose $m \in \mathbb{N}$ with $(m,n) = 1$ and consider the generator $C_m = \exp(\frac{2\pi i m}{n}) \in Z(G)$. We have $\mathcal{M}^*(G,\Sigma,p,C_m) = \mathcal{M}(G,\Sigma,p,C_m)$, i.e., in the coprime case the moduli space introduced in Section 1.2.1 consists of irreducible connections.

Theorem 6.3.1 ([Fre95] Propositions 3.17 and 5.19)**.** *Let $\mathcal{M} \in \{\mathcal{M}(G,\Sigma,P,C_m), \mathcal{M}^*(G,\Sigma)\}$. Then $\mathcal{M}$ is smooth manifold which supports a symplectic form ω and there exists a prequantum bundle*

$$\mathcal{L}_{CS} \to \mathcal{M}.$$

The symplectic form ω is called the Atiyah-Bott-Goldman form and was introduced in [AB83, Gol84]. The prequantum bundle $\mathcal{L}_{\text{CS}}$ is induced from the Chern-Simons lines introduced in Chapter 2.

Theorem 6.3.2. *The spaces $\mathcal{M}^*(G,\Sigma)$ and $\mathcal{M}(G,\Sigma,p,C_m)$ are both simply connected and for both of these spaces the Atiyah-Bott-Goldman form satisfy the integrality condition. The space $\mathcal{M}(G,\Sigma,p,C_m)$ is compact whereas $\mathcal{M}^*(G,\Sigma)$ is non-compact.*

We note that this implies that the two moduli spaces in question support a prequantum line bundle as in Theorem 6.3.1.The statement concerning compactness is easily verified using the character variety description. The simple connectedness is proven in for instance [DW97]. Let $[A] \in \mathcal{M}^*(G,\Sigma)$. There exists an isomorphism $T_{[A]}\mathcal{M}^*(G,\Sigma) \simeq H^1(\Sigma, \mathrm{Ad}_A)$. For tangents represented by Ad_A valued 1-forms η, ν the symplectic form is defined by

$$\omega(\eta,\nu) = \int_\Sigma \langle \eta \wedge \nu \rangle$$

where $\langle \cdot, \cdot \rangle$ is the invariant, non-degenerate symmetric bilinear form on $\mathfrak{g}$ choosen in Chapter 2.

6.3.1 The Narasimhan-Seshadri correspondence

Let $\mathcal{T}_\Sigma$ be Teichmüller space. Fix for now a complex structure $\sigma \in \mathcal{T}_\Sigma$ on Σ. Let $X = \Sigma_\sigma$ be the resulting Riemann surface.

Definition 6.3.1. Define the *slope* $\mu(E)$ of a holomorphic vector bundle $E \to X$ to be $\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}$. A holomorphic vector bundle is said to be *semistable* if for every holomorphic subbundle $F \subset E$ we have $\mu(E) \geq \mu(F)$. It is said to be *stable*, if for every proper subbundle $F \subset E$ we have $\mu(E) > \mu(F)$.

Semi-stable bundles admits what is called a Harder-Narasimhan filtration.

Theorem 6.3.3. *Any semi-stable vector bundle E admits a unique filtration of destabilizing subbundles of maximal rank $E = E_0 \supset E_1 \supset \cdots \supset E_m = 0$, i.e. E_i/E_{i+1} is stable and the isomorphism type of the bundle $Gr(E) = \bigoplus_{i=1}^m E_i/E_{i+1}$ is an invariant.*

We make the following definition.

Definition 6.3.2. Two semistable bundles E, E' are *S-equivalent* if $\mathrm{Gr}(E)$ is isomorphic to $\mathrm{Gr}(E')$ as a holomorphic vector bundle.

The following is due to Mumford, and is an example of a moduli space construced through *geometric invariant theory*. For the theorem to hold, the assumption that we are considering a Riemann surface of genus $g \geq 2$ is important.

Theorem 6.3.4. *There exists a moduli space $\mathcal{N}(X, r, \mathcal{O}_X)$ of S-equivalence classes of semistable holomorphic vector bundles $E \to X$ of rank r and trivial determinant. The space $\mathcal{N}(X, r, \mathcal{O}_X)$ is a projective variety and the smooth locus $\mathcal{N}^*(X, r, \mathcal{O}_X)$ consists of the S-equivalence classes of stable holomorphic vector bundles. Moreover, there exists a moduli space $\mathcal{N}(X, r, [p])$ of stable holomorphic vector bundles $E \to X$ of rank r and with fixed determinant given by the divisor $[p]$. This is a smooth projective variety.*

For a precise definition of what it means that the projective variety $\mathcal{N}(X, r, \mathcal{O}_X)$ is a moduli space, see for instance [New78]. Morally, it means that a holomorphic map $Y \to \mathcal{N}(X, r, \mathcal{O}_X)$ is equivalent to a family of semistable bundles (of rank r and trivial determinant) parametrized by Y. Let $\mathcal{N}$ be either of these moduli spaces introduced above. This variety supports a holomorphic line bundle $\mathcal{L} \to \mathcal{N}$ constructed by Quillen [Qui85].

Remark 6.3.5. Two stable holomorphic vector bundles are S-equivalent if and only if they are isomorphic as holomorphic vector bundles.

We have the following important theorem due to Narasimhan and Seshadri.

Theorem 6.3.6 ([NS64])**.** *Let $P \to \Sigma$ be a flat G bundle, and let $E \to \Sigma$ be the induced hermitian vector bundle with a unitary connection ∇ and a parallel unit norm section of* $\det(E)$*. There exists a holomorphic structure on $E \to X$ such that ∇ is the Chern connection. This induces a homeomorphism $\mathcal{M}(G, \Sigma) \simeq \mathcal{N}(X, n, \mathcal{O}_X)$ which maps the irreducible flat connections diffeomorphically to the smooth locus of stable holomorphic bundles, and the induced complex structure $J(\sigma)$ on $\mathcal{M}(G, \Sigma)$ is compatible with Atiyah-Bott-Goldman form ω. Similarly, there exists a diffemorphism $\mathcal{M}(G, \Sigma, p, C_m) \simeq \mathcal{N}(X, n, [p])$, and the induced complex structure $J(\sigma)$ is compatible with the symplectic form on $\mathcal{M}(G, \Sigma, p, C_m)$. Through these identifications, the Chern-Simons line bundles are identified with Quillen's line bundle $\mathcal{L}$.*

These correspondences are both known under the name of the *Narasimhan-Seshadri correspondence*. Let us remark that the complex structures induced by the Narasimhan-Seshadri correspondence can also be described purely in terms of gauge theory in the following way. The complex structure σ induces a unique Riemaniann metric of scalar curvature -1 and hence a Hodge $*$ operater $\Omega^1(\Sigma, \mathrm{Ad}_A) \to \Omega^1(\Sigma, \mathrm{Ad}_A)$ (here we use that Ad_A is a Hermitian vector bundle). Thus we can define an almost complex structure $J(\sigma)$ through $J(\sigma)[\alpha] = [-*(\alpha)]$, where α is the unique harmonic representative of its cohomology class.

6.3.2 The Verlinde bundle

We have the following theorem.

Theorem 6.3.7 ([Hit90])**.** *Let $M \in \{\mathcal{M}^*(G,\Sigma), \mathcal{M}(G,\Sigma,p,C_m)\}$, and equip M with the Atiyah-Bott-Goldman form. The family of Kähler structures $J : \mathcal{T}_\Sigma \to \Gamma^\infty(M, TM)$ form a rigid family.*

We stress that this result is a reformulation of results due to Hitchin [Hit90] and Axelrod-Della Pietra-Witten [ADPW91]. We make the following definition.

Definition 6.3.3. The *level k Verlinde bundle* $\mathrm{H}_k \to \mathcal{T}_\Sigma$ is the complex vector bundle whose fiber at a point $\sigma \in \mathcal{T}_\Sigma$ is $\mathrm{H}_k(\sigma) = H^0\left((M, J(\sigma)), \mathcal{L}^{\otimes k}\right)$.

Remark 6.3.8. As Teichmüller space is contractible the Verlinde bundle is topologically trivial and one may wonder: what is the importance of the Hitchin connection in this setup? The answer is that the Hitchin connection is compatible with the action of the mapping class group, and therefore it can be used to define the so-called quantum representations whereas a mapping class group invariant trivialization does not exists.

6.4 The asymptotic expansion conjecture for mapping tori

6.4.1 The coprime case

We now focus on the co-prime case and write $\mathcal{M} = \mathcal{M}(G,\Sigma,p,C_m)$. Let $\Gamma = \Gamma(\Sigma, p, v_p, C_m)$.

Remark 6.4.1. We remark that we could of course neglect C_m from the notation and write $\Gamma = \Gamma(\Sigma, p, v_p)$, as there is only one puncture. Our choice of notation is meant to emphasize the connection to the modular functor $\mathrm{V}_{G,k}$.

Proposition 6.4.2. *The mapping class group Γ acts on $\mathcal{T}_\Sigma$ and $\mathcal{M}$ and these combine to form a prequantum action which is real analytic.*

Proof. It is shown in [AHJ+17] that the setup of the co-prime case fit together to form a prequantum action. This is seen to be a real analytic action as follows. Consider the corresponding character variety $\tilde{X}$ in this co-prime case for the complexified group $\mathrm{SL}(n,\mathbb{C})$, which carries a natural complex structure coming from $\mathrm{SL}(n,\mathbb{C})$. Once we choose a point $\sigma \in \mathcal{T}_\Sigma$ we get a holomorphic family of rank n semi-stable complex vector bundles with trivial determinant parametrized by an open neighborhood V of $\mathcal{M}$ in $\tilde{X}$. This gives a holomorphic map from V to $(\mathcal{M}, J(\sigma))$ which therefore restricts to a real analytic isomorphism from $\mathcal{M}$ to $(\mathcal{M}, J(\sigma))$, showing that the Narasimhan-Seshadri identification is real-analytic. Concerning the prequantum Chern-Simons line bundle, we see that it is the restriction of a holomorphic line bundle to $\mathcal{M}$, thus also real analytic. Further, the Takhtajan-Zograf formula [TZ87, TZ88, TZ89, TZ91] for its Hermitian structure proves its real analyticity. □

Definition 6.4.1. The induced level k quantum representation is denoted by $\mathrm{Z}_{G,k}$.

These projective representations are the ones constructed by Hitchin [Hit90] and Axelrod, Della Pietra and Witten [ADPW91] and introduced above in (1.4). We can now give a proof of the main theorems from Section 1.2.1.

Proofs of Theorems 1, 2 and 3. Theorem 1 is simply an application of Theorem 6.2.8 together with the following fact. It is shown in [AHJ+17] that there is a unique choice of $\tilde{\varphi}$ such that on a component Y of the fixed point set $\mathcal{M}^{\varphi}$ the lift $\tilde{\varphi}$ is given by multiplication by $\exp(2\pi i \mathrm{S}(x))$ for any $x \in \iota^{*-1}(Y)$, where $\iota^* : \mathcal{M}(G, T_\varphi, L, C_m) \to \mathcal{M}^{\varphi}$ is the pullback of the inclusion $\iota\Sigma \hookrightarrow T_\varphi$. We implicitly assume that $\tilde{\varphi}$ has been choosen accordingly. Theorem 2 follows from Theorem 6.2.5 and Theorem 3 follows from Theorem 6.2.9. □

Let us also adress the projective ambiguity. In [AP16] Andersen and Poulsen are able to calculate the curvature of what is now known to be the Hitchin-connection.

Theorem 6.4.3 ([AP16, AP17b, AP18c]). *We have*

$$F_{\nabla^H} = \frac{ik(n^2-1)}{12(k+n)\pi}\omega_{\mathcal{T}}.$$

Here $\omega_{\mathcal{T}}$ is the Weil-Petersson form.

Using this, it is easy to control the k-dependence of the framing corretion. See [Ioo18].

Remark 6.4.4. In the case of Theorem 2, we remark that one can in fact say more about the contributions to the sets $\mathcal{A}_\theta$, which come from non-smooth points of $\mathcal{M}^{\varphi}$. A contribution from an isolated saddle z can be characterized in terms of the monodromy of $(\widetilde{\varphi}_{\mathbb{C}}, z)$. For the details, we appeal to Remark 5.2.12. A similar remark holds for the d_θ.

6.4.2 The punctured torus case

We now discuss the punctured torus case. Recall that $G = \mathrm{SU}(2)$ in this case. We consider the surface $(S^1 \times S^1, p, v_p)$ where $p \in S^1 \times S^1$ is a puncture and $v_p \in T_p S^1 \times S^1 \setminus \{0\}/\mathbb{R}_{>0}$. For $l \in (-2, 2)$ let $\mathcal{M}_l$ be the moduli space introduced in Section 1.2.1. Let a, b be the standard generators of $\pi_1(S^1 \times S^1)$. The trace coordinates on the character variety construction of the full moduli space gives an identification

$$\mathcal{M}_l \xrightarrow{\sim} \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2 - xyz = l\}. \tag{6.29}$$

For more details, see Goldman's article [Gol97]. For $l \in (-2, 2)$ we remark that $\mathcal{M}_l$ is smooth and diffeomorphic to S^2. Through the identification (6.29) the spaces $\mathcal{M}_l$ form concentric spheres, so we have a fixed isomorphism

$$H^2(\mathcal{M}_l, \mathbb{R}) \simeq H^2(S^2, \mathbb{R}) \simeq \mathbb{R}. \tag{6.30}$$

The space $\mathcal{M}_{-2}$ consists of a single point, while $\mathcal{M}_2$ is isomorphic to $\mathbb{T}^2/\{\pm 1\}$, and contains four singularities. This moduli space is known as the pillow case.

For $l \in (-2, 2)$, let us discuss in more detail the integrality condition. The moduli space $\mathcal{M}_l$ is quantizable at level k exactly when ω_l satisfy the integrality condition $[\frac{\omega_l}{k2\pi}] \in H^2(\mathcal{M}_l, \mathbb{Z})$. Recall that l determines a choice of a conjugacy class $C_l \in C_G$, such that for any $c_l \in C_l$ we have $\mathrm{tr}(c_l) = l$. Recall that $\mathrm{U}(1) \subset G$ is a maximal torus. Let $\mathfrak{h} \simeq i\mathbb{R}$ be its Lie algebra. We can write $l = 2\cos(\lambda_l)$ where $i\lambda_l \in \mathfrak{h}$ satisfy that $\exp(i\lambda_l) \in C_l$. Here exp denotes the exponential $\exp : \mathfrak{g} \to G$. By Proposition 6.7 in [Jef94] the symplectic form ω_l depend linearly on λ_l. Because of the fixed isomorphisms (6.30), this means that there exists a non-empty lattice $\tilde{\mathcal{Q}}(k) \subset i\mathbb{R}$, such that for $l \in 2\cos(\tilde{\mathcal{Q}}) \cap (-2,2)$ the (level k) integrality condition is fulfilled. With notation as in 1.2.1 we have that $\mathcal{Q} = \cup_{k\in\mathbb{N}} \mathcal{Q}(k)$, where $\mathcal{Q}(k) = \cos(\tilde{\mathcal{Q}}(k)) \cap (-2,2)$. This is a dense subset.

The general construction of a Hitchin connection in [And12] applies to the case of $\mathcal{M}_l$, for all $l \in \mathcal{Q}$, with its Chern-Simons line bundle constructed in [AHJ+17] and its family of complex structures parametrized by Teichmüller space $\mathcal{T}$ of $S^1 \times S^1$ constructed in the works of Daskalopoulos and Wentworth and Mehta and Seshadri [DW97, MS80]. The mapping class group $\Gamma = \Gamma(S^1 \times S^1, p, v_p, C_l)$ acts on this setup and the same argument as in the co-prime case shows that this is in fact a real analytic action (the analogue of Remark 6.4.1 applies here as well). Further, we get in this case, since the complex dimension of $\mathcal{T}$ is one, by Theorem 4.8 of [AL14] that any of the Hitchin connections provided above will be projectively flat. Thus we get a projective representation of $\Gamma(S^1 \times S^1, p, v_p)$. For any mapping class $\varphi \in \Gamma$ and any $l \in \mathcal{Q}$, we consider a lift

$$Z_{l,k}(\varphi) \in GL(\mathrm{H}_k(\sigma)).$$

In this case, parallel transport also has an asymptotic expansion in terms of Toeplitz operators as in (6.7). We stress that if k_0 is minimal with $l \in \mathcal{Q}(k_0)$, then $Z_{l,k}$ defined for levels $k \in k_0\mathbb{N}$.

Proof of Theorem 4. This is an application of Theorem 6.2.8 and Theorem 6.2.9 to this situation together with the proof of Proposition 5.1 in [Bro98] which shows that for l inside an open dense subset $A \subset (-2,2)$ the fixed point set $\mathcal{M}_l^\varphi$ is finite and non-degenerate. □

Example with degenerate fixed point set

In this subsection, we consider in more detail the action of Γ on $\mathcal{M}_l$. We compute the fixed point set $\mathcal{M}_l^\varphi$ in two examples. Our findings illustrate that 1) for Anasov mapping classes the fixed point set is generically non-degenerate as proven by Brown [Bro98] and 2) one can find examples where $\mathcal{M}_l^\varphi$ is degenerate but satisfies the condition of Theorem 2.

Let $t_a \in \Gamma$ be the Dehn-twist about a, and let $t_b \in \Gamma$ be the Dehn-twist about b. We have a explicit expression of the action of t_a, t_b on the moduli space M_l in terms of the coordinates (x, y, z) introduced above. As explained in [Gol97] we have

$$t_a(x,y,z) = (x, z, xz - y), \quad t_b(x,y,z) = (xy - z, y, x). \tag{6.31}$$

Consider the composition

$$\varphi = t_b^{-1} \circ t_a. \tag{6.32}$$

It follows by Penner's work [Pen88] that φ is a Anasov homeomorphism.

We shall consider the action of φ, φ^2 on $\mathcal{M}_{-1/4}$. Let $p = \left(-1, \frac{1}{2}, \frac{1}{2}\right)$ and $q = \left(\frac{1}{2}, -1, -1\right)$. We shall prove that $\mathcal{M}_{-1/4}^\varphi = \{p, q\}$, and we have

$$\begin{aligned} \dim(\mathrm{Ker}(\mathrm{d}\,\varphi_p - I)) &= \dim(\mathrm{Ker}(\mathrm{d}\,\varphi_q - I)) = 0, \\ \dim(\mathrm{Ker}(\mathrm{d}\,\varphi_p^2 - I)) &= \dim(\mathrm{Ker}(\mathrm{d}\,\varphi_q^2 - I)) = 1. \end{aligned} \tag{6.33}$$

These assertions are proven by a direct calculation presented below.

We analyze the fixed point set of the mapping class (6.32) in detail. Introduce the function $L(x,y,z) = x^2 + y^2 + z^2 - 2 - xyz$ such that $\mathcal{M}\mathrm{an}_l$ is the level-set $L(x,y,z) = l$. We start by giving a coordinate expression for the action of φ. We observe that $t_b^{-1}(x,y,z) = (z,y,zy-x)$ as is easily deduced from (6.31). From this we get the expression

$$\varphi(x,y,z) = (xz - y, z, z(xz-y) - x). \tag{6.34}$$

From the coordinate expression (6.34) one verifies that $\varphi(p) = p$, and $\varphi(q) = q$. As any fixed point of φ is a fixed point of φ^2, we can prove the set-theoretical identities by arguing that $\mathcal{M}^{\varphi^2}_{-1/4} \subset \{p,q\}$. Observe that q' is a fixed point of φ^2 if and only if $\varphi(q') = \varphi^{-1}(q')$. From the coordinate expression (6.34) one sees that

$$\varphi^{-1}(x,y,z) = (yx - z, y(yx-z) - x, y).$$

Thus we see by comparing that (x,y,z) is a fixed point of φ^2 if and only if

$$xz - y = yx - z,\ y(yx-z) - x = z,\ z(zx-y) - x = y. \tag{6.35}$$

The first of the equations appearing in (6.35) is equivalent to

$$x(z-y) = y - z. \tag{6.36}$$

Now, let (x,y,z) be a fixed point. From (6.36) we see that we must either have that $x = -1$, or that $z = y$.

We shall start by assuming that $x = -1$. Hence the two lower equations appearing in (6.35) reads

$$-y^2 - yz + 1 = z,\ -z^2 - yz + 1 = y \tag{6.37}$$

from which we learn that

$$y - z = -z^2 - yz + 1 - (-y^2 - yz + 1) = y^2 - z^2 = (y-z)(y+z).$$

Hence we can assume that either $y = z$ or $y + z = 1$. If $y = z$ we have

$$0 = L(-1,y,y) + \frac{1}{4} = 3y^2 - \frac{3}{4}.$$

This equation has exactly two solutions given by $y = -1/2$ and $y = 1/2$. We see however that the identities $y = z = -1/2$ would violate (6.37). It follows that we must have $y = z = \frac{1}{2}$, which corresponds to the solution p. Assume now that $z + y = 1$. Then we have

$$0 = L(-1,y,1-y) + \frac{1}{4} = y^2 - y + 1/4.$$

This equation has only one solution $y = 1/2$ which again corresponds to the point p.

We now assume that $z = y$. The second equation of (6.35) now reads $z^2x - z^2 - x = z$. This is equivalent to $(z^2-1)x = z + z^2$ which can be rewritten as

$$(z+1)(z-1)x = z(z+1). \tag{6.38}$$

Observe that this implies that $z \neq 1$. If $z = -1$, we get

$$0 = L(x, -1, -1) + \frac{1}{4} = x^2 - x + 1/4.$$

This has precisely one solution namely $x = 1/2$, and we recover the solution p. Assume now that $z \neq -1$. Then (6.38) implies that $x = \frac{z}{z-1}$. Thus we get

$$0 = L(z/(z-1), z, z) + \frac{1}{4} = \frac{(z+1)(2z-1)(2z^2 - 7z + 7)}{(z-1)^2}.$$

This equation has exactly two real solutions $z = -1$ and $z = 1/2$. As we have discarded -1, this implies that $z = 1/2$, and as we also have $x = \frac{z}{z-1}$, we conclude $x = -1$. This corresponds to the point q.

We have now proven the claims about the fixed point sets (6.33), and it remains to compute the differentials. We start with some general considerations. From (6.34) we compute that with respect to the $\partial x, \partial y, \partial z$ basis of $T\mathbb{R}^3$ we have

$$\mathrm{d}\,\varphi = \begin{pmatrix} z & -1 & x \\ 0 & 0 & 1 \\ z^2 - 1 & -z & 2xz - y \end{pmatrix}.$$

We have

$$T_p\mathcal{M}_{-1/4} = \operatorname{Ker}(\mathrm{d}\,L), \quad \mathrm{d}\,L = (2x - yz, 2y - xz, 2z - xy).$$

We now turn to the point p. As $\mathrm{d}\,L_p = \left(0, \frac{-3}{2}, \frac{-3}{2}\right)$ we conclude that $T_p\mathcal{M}_{-1/4} = \mathbb{R}\partial x \oplus \mathbb{R}(\partial y - \partial z)$. With respect to the basis $\partial x, \partial y, \partial z$ of $T_p\mathbb{R}^3$ we have

$$\mathrm{d}\,\varphi_p = \begin{pmatrix} -1 & -1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Introduce the basis $v_1 = \partial x, v_2 = \partial y - \partial z$. With respect to the (v_1, v_2) basis we have

$$\mathrm{d}\,\varphi_p = \begin{pmatrix} -1 & -3/2 \\ 0 & -1 \end{pmatrix}.$$

It follows that 1 is not an eigenvalue of $\mathrm{d}\,\varphi_p$. We now consider $\mathrm{d}\,\varphi_p^2$. With respect to the basis (v_1, v_2) it has matrix given by

$$\mathrm{d}\,\varphi_p^2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

It follows that $\dim(\operatorname{Ker}(\mathrm{d}\,\varphi_p^2 - I)) = 1$.

We now turn to the point q. We have

$$\mathrm{d}\,\varphi_q = \begin{pmatrix} 1/2 & -1 & -1 \\ 0 & 0 & 1 \\ -3/4 & -1/2 & -3/2 \end{pmatrix}.$$

We have $\mathrm{d}\, L_q = \left(\frac{-9}{4}, \frac{3}{2}, \frac{3}{2}\right)$. Thus we have $T_q\mathcal{M}_{-1/4} = \mathbb{R}(\partial y - \partial z) \oplus \mathbb{R}(\partial x + 3/2\partial y)$. Introduce the basis $w_1 = \partial y - \partial z, w_2 = \partial x + 3/2\partial y$. We have $\mathrm{d}\, \varphi_q(w_1) = -w_1$ and $\mathrm{d}\, \varphi_q(w_2) = 3w_1 - w_2$. Thus we see that with respect to w_1, w_2, we have

$$\mathrm{d}\, \varphi_q = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{d}\, \varphi_q^2 = \begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix}.$$

From these expressions, it is easy to conclude that $\dim(\mathrm{Ker}(\mathrm{d}\, \varphi_q - I)) = 0$ and $\dim(\mathrm{Ker}(\mathrm{d}\, \varphi_q^2 - I)) = 1$. Thus (6.33) holds.

CHAPTER 7

Resurgence in quantum topology

The use of concepts from resurgence in quantum topology was pioneered by Witten [Wit11], Garoufalidis [Gar08] and more recently Gukov-Marino-Putrov [GMP16]. For this chapter, the reader may wish to recall the notations and definitions from the introduction and Sections 3.2.1, 3.2.2 and 3.2.2.

7.1 Seifert fibered homology spheres

Let $X = \Sigma((p_1/q_1), ..., (p_n/q_n))$ be as in the Section 1.2.3. Without loss of generality we can assume that $p_2, ..., p_n$ are odd. The homeomorphism type of X is unaltered under a transformation $q_j \mapsto q_j + y_j$ for any choice of integers $y_1, ..., y_n$ such that $(pj, q_j + y_j) = 1$ and

$$\sum_{j=1}^{n} \frac{q_j}{p_j} = \sum_{j=1} \frac{q_j + y_j}{p_j}. \tag{7.1}$$

If q_j is odd for $j > 1$, we perform the transformation $q_j \mapsto q_j + p_j$ and $q_1 \mapsto q_1 - p_1$ which does not change the sum (7.1). Hence we can assume without loss of generality that $q_2, ..., q_n$ are all even. Finally, changing the sign of p_j and q_j simultaneously if needed, we can assume that $p_j > 0$ for each j. In fact, we shall only need to assume $P > 0$. Recall that $\pm 1 = H = P \sum_{j=1} \frac{q_j}{pj}$. Note that this implies that q_1 is odd.

7.1.1 Computation of the quantum invariant

We shall present the computation of $\tau_k(X)$ from [LR99]. It shows the usefullness of rewriting quantum invariants as contour integrals, an idea we pursue in Section **??**.

Proposition 7.1.1 ([LR99],[Hik05b]). *We have*

$$\tau_k(X) = B_k \sum_{\substack{r=0, \\ k \nmid r}}^{2kP} \exp\left(\frac{-r^2 \pi i}{2Pk}\right) \frac{\prod_{j=1}^{n} \sin\left(\frac{r\pi}{kp_j}\right)}{\sin\left(\frac{r\pi}{k}\right)^{n-2}}, \tag{7.2}$$

where

$$B_k = \frac{\exp\left(\frac{H3\pi i}{4} - \frac{i\pi}{2k}\left(3 - \frac{H}{P} + 12\sum_{j=1}^{n} S(q_j, p_j)\right)\right)}{\sin\left(\frac{\pi}{k}\right)\sqrt{P2k}}.$$

Sketch of proof. The proof is an application of the formula for rational surgeries (3.15). Let L be the surgery link, as depicted in Figure~1.2. We order the components as follows: the unlink with framing 0 (corresponding to the $\mathrm{SL}(2,\mathbb{Z})$ matrix S) is first, and the rest of the components are ordered from left to right in the diagram. Choose

$$\left\{ U_j = \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix} \right\}_{j=1}^{n} \subset \mathrm{SL}(2,\mathbb{Z}).$$

We denote a coloring of L by $(\alpha,\beta) = (\alpha,\beta_1,\beta_2,...,\beta_n) \in \{1,...,k-1\}^{n+1}$. We have the identity $J_{(\alpha,\beta)}(L) = \prod_{j=1}^{n} \sin\left(\frac{\alpha\beta_j\pi}{k}\right)\left(\sin\left(\frac{\pi}{k}\right)\sin\left(\frac{\alpha\pi}{k}\right)^{n-1}\right)^{-1}$. This is easily proven using induction on n, the definition of colored Jones polynomial in terms of the graphical calculus and the formula for the $S_{i,j}$-matrix given in Theorem 3.2.10. We compute

$$\begin{aligned}
\sum_{\lambda\in \mathrm{Col}(L)} J_\lambda(L) \prod_{c\in\pi_0(L)} \rho_k(\Lambda(c))_{\lambda(c),1} &= \sum_{\alpha,\beta_1,...,\beta_n=1}^{k-1} \frac{\prod_{j=1}^{n}\sin\left(\frac{\alpha\beta_j\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)\sin\left(\frac{\alpha\pi}{k}\right)^{n-1}} \rho_k(S)_{\alpha,1}\prod_{j=1}^{n}\rho_k(U_j)_{\beta_j,1} \\
&= \sum_{\alpha=1}^{k-1} \frac{\rho_k(S)_{\alpha,1}}{\sin\left(\frac{\pi}{k}\right)\sin\left(\frac{\alpha\pi}{k}\right)^{n-1}} \sum_{\beta_1,...,\beta_n=1}^{k-1}\prod_{j=1}^{n}\sin\left(\frac{\alpha\beta_j\pi}{k}\right)\rho_k(U_j)_{\beta_j,1} \\
&= \frac{\sqrt{\frac{2}{k}}}{\sin\left(\frac{\pi}{k}\right)} \sum_{\alpha=1}^{k-1}\frac{1}{\sin\left(\frac{\alpha\pi}{k}\right)^{n-2}} \\
&\times \sum_{\beta_1,...,\beta_n=1}^{k-1}\left(\frac{k}{2}\right)^{\frac{n}{2}}\prod_{j=1}^{n}\left(\rho_k(S)_{\alpha,\beta_j}\rho_k(U_j)_{\beta_j,1}\right) \\
&= \frac{\sqrt{\frac{2}{k}}}{\sin\left(\frac{\pi}{k}\right)}\left(\sum_{\alpha=1}^{k-1}\frac{\left(\frac{k}{2}\right)^{\frac{n}{2}}\prod_{j=1}^{n}\rho_k(SU_j)_{\alpha,1}}{\sin\left(\frac{\alpha\pi}{k}\right)^{n-2}}\right).
\end{aligned} \tag{7.3}$$

For the third equality we used (3.11). We now focus on the term $\left(\sum_{\alpha=1}^{k-1}\frac{\left(\frac{k}{2}\right)^{\frac{n}{2}}\prod_{j=1}^{n}\rho_k(SU_j)_{\alpha,1}}{\sin\left(\frac{\alpha\pi}{k}\right)^{n-2}}\right)$. By (3.12) we get

$$\begin{aligned}
\left(\frac{k}{2}\right)^{\frac{n}{2}}\prod_{j=1}^{n}\rho_k(SU_j)_{\alpha,1} &= \left(\frac{k}{2}\right)^{\frac{n}{2}}\prod_{j=1}^{n}\frac{\mathrm{sign}(p_j)}{\sqrt{2k|p_j|}}\exp\left(\frac{-\pi i}{4}\Phi(SU_j)+\frac{\pi i}{2kp_j}r_j\right) \\
&\times \sum_{\substack{t_j \mod 2kp_j,\\ t_j\equiv\alpha \mod 2k}} \exp\left(\frac{-\pi i q_j t_j^2}{kp_j}\right)2\sin\left(\frac{\pi t_j}{kp_j}\right) \\
&= \left(\frac{\mathrm{sign}(P)}{\sqrt{|P|}}\exp\left(\sum_{j=1}^{n}\frac{-\pi i}{4}\Phi(SU_j)+\frac{\pi i}{2kp_j}r_j\right)\right) \\
&\times \left(\sum_{l=0}^{P-1}\times\exp\left(\frac{-\pi i H(\alpha+2kl)^2}{kP}\right)\prod_{j=1}^{n}\sin\left(\frac{\pi(\alpha+2kl)}{kp_j}\right)\right).
\end{aligned} \tag{7.4}$$

For the third equality we used that the j'th factor only depends on $l_j \mod p_j$ and that the p_j's are pairwise coprime. For the fourth equality we used $\sum_{j=1}^{n}\frac{q_j}{p_j} = \frac{H}{P}$. Observe that our assumption that $p_j > 0$ implies that $\frac{\operatorname{sign}(P)}{\sqrt{|P|}} = \frac{1}{\sqrt{P}}$. We now assemble the label independent factors in (7.3), (7.4) and (3.15). We have $\sigma(L) = \sum_{j=1}^{n} \operatorname{sign}(p_j q_j) - \operatorname{sign}\left(\frac{H}{P}\right)$, so by using (3.14), we obtain

$$\sum_{c \in \pi_0(L)} \Phi(U_c) - 3\sigma(L) = 3\operatorname{sign}\left(\frac{H}{P}\right) + \sum_{j=1}^{n} \Phi(SU_j) = 3H + \sum_{j=1}^{n} \Phi(SU_j).$$

Observe that $\operatorname{sign}\left(\frac{H}{P}\right) = H$, as $P > 0$ and $H = \pm 1$. We compute

$$\begin{aligned} C(k) &= \frac{\sqrt{\frac{2}{k}}}{\sin\left(\frac{\pi}{k}\right)\sqrt{P}} \exp\left(\sum_{j=1}^{n} \frac{-\pi i}{4}\Phi(SU_j) + \frac{\pi i}{2kp_j} r_j + \frac{\pi i(k-2)}{4k}\left(3H + \sum_{j=1}^{n} \Phi(SU_j)\right)\right) \\ &= \frac{\sqrt{\frac{2}{k}}}{\sin\left(\frac{\pi}{k}\right)\sqrt{P}} \exp\left(\frac{H3\pi i}{4} - \frac{i\pi}{2k}\left(3 - \frac{H}{P} + 12\sum_{j=1}^{n} S(q_j, p_j)\right)\right) = 2B_k. \end{aligned}$$

The second equality follows from a straightforward computation using the definition of Φ equation (3.13) and that $\sum_j q_j/p_j = H/P$. Combining (3.15) with (7.3) and (7.4) we get

$$\begin{aligned} \tau_k(X) &= C(k) \sum_{\alpha=1}^{k-1} \sum_{l=0}^{P-1} \frac{\exp\left(\frac{-\pi i H(\alpha+2kl)^2}{kP}\right) \prod_{j=1}^{n} \sin\left(\frac{\pi(\alpha+2kl)}{kp_j}\right)}{\sin\left(\frac{\alpha\pi}{k}\right)^{n-2}} \\ &= C(k) \sum_{\substack{r=\alpha+2kn \\ 1 \le \alpha \le k-1 \\ 0 \le n \le P-1}} \frac{\exp\left(\frac{-\pi i H r^2}{kP}\right) \prod_{j=1}^{n} \sin\left(\frac{\pi r}{kp_j}\right)}{\sin\left(\frac{r\pi}{k}\right)^{n-2}}. \end{aligned}$$

Define $\eta(r) = \exp\left(\frac{-\pi i H r^2}{kP}\right) \prod_{j=1}^{n} \sin\left(\frac{\pi r}{kp_j}\right) \left(\sin\left(\frac{r\pi}{k}\right)\right)^{2-n}$. We have $\eta(r) = \eta(-r)$ and $\eta(r) = \eta(r + 2kPm)$ for all $m \in \mathbb{Z}$. Using this and $(P-j)2k + a \equiv -j2k + a \mod 2kP$, we see

$$\sum_{\substack{r=\alpha+2kn \\ 1 \le \alpha \le k-1 \\ 0 \le n \le P-1}} \eta(r) = \frac{1}{2}\left(\sum_{\substack{r=\alpha+2kn \\ 1 \le \alpha \le k-1 \\ 0 \le n \le P-1}} \eta(r) + \sum_{\substack{r=2kn-a \\ 1 \le \alpha \le k-1 \\ 1 \le n \le P}} \eta(r)\right) = \frac{1}{2}\sum_{\substack{r=0 \\ k \nmid r}}^{2kP} \eta(r).$$

This finishes the proof. □

Define the meromorphic function $F \in \mathcal{M}(\mathbb{C})$ and $g \in \mathbb{C}[y]$ by

$$F(y) = -\left(\sinh\left(\frac{y}{2}\right)\right)^{2-n} \prod_{j=1}^{n} \sinh\left(\frac{y}{2p_j}\right), \quad g(y) = \frac{Hiy^2}{8\pi P}.$$

Define $\gamma = \gamma(H)$ to be the contour from $(-1-i)\infty$ to $(1+i)\infty$ when $H > 0$, and from $(-1+i)\infty$ to $(-i+1)\infty$ when $H < 0$. Define

$$Z_\infty(x) = \frac{1}{2\pi i} \frac{\sqrt{P\pi i 8}}{\sqrt{H}} \sum_{n=0}^{\infty} \frac{F^{(2n)}(0)\left(\frac{i8P\pi}{H}\right)^n}{(2n)!} \frac{\Gamma\left(n+\frac{1}{2}\right)}{x^{n+\frac{1}{2}}}.$$

Observe that $\gamma(H)$ is a steepest descent path for g. Recall the definition of the normalized quantum invariant $\widetilde{Z}_k$ given in (1.14). Lawrence and Rozansky shows the following result.

Theorem 7.1.2 ([LR99]). *We have*

$$\widetilde{Z}_k(X) = \frac{1}{2\pi i}\int_\gamma F(y)e^{kg(y)}\ \mathrm{d}\,y - \sum_{m=1}^{2P-1} Res\left(\frac{F(y)e^{kg(y)}}{1-e^{-ky}}, y = 2\pi i m\right). \tag{7.5}$$

There exists a set $R(X)$ of finitely many non-zero rational numbers modulo the integers and non-vanishing polynomials $Z_\theta(z) \in \mathbb{C}[z]$, $\theta \in R(X)$ of degree at most $n-3$ such that

$$\sum_{\theta\in R(X)} e^{2\pi i k\theta}\, Z_\theta(k) = -\sum_{m=1}^{2P-1} Res\left(\frac{F(y)e^{kg(y)}}{1-e^{-ky}}, y = 2\pi i m\right). \tag{7.6}$$

We have an asymptotic expansion

$$\widetilde{Z}_k(X) \sim \sum_{\theta\in R(X)} e^{2\pi i k\theta}\, Z_\theta(k) + Z_\infty(k). \tag{7.7}$$

Recall the following elementary result.

Proposition 7.1.3. *Let $z_0 \in \mathbb{C}$ and let $g \in \mathcal{M}_{z_0}(\mathbb{C})$ be the germ of a meromorphic function with a pole at z_0. Assume $w_0 \in \mathbb{C}$ and that $z \in \mathcal{O}_{w_0}(\mathbb{C})$ satisfies $z(w_0) = z_0, \dot{z}(w_0) \neq 0$. If either z_0 is a simple pole, or $z(w)$ is linear in w, then we have*

$$Res(g(z(w)), w = w_0) = \frac{Res(g(z), z = z(w_0))}{\dot{z}(w_0)}.$$

We now turn to Theorem 7.1.2.

Proof of Theorem 7.1.2. We start with (7.5). Define the functions $h_n(\beta, x)$ and $f_n(\beta, x)$ given by

$$h_n(\beta, x) = \frac{-ie^{-\frac{H\pi i}{2kP}\beta^2} e^{\frac{2\pi i\beta x}{k}}}{2\left(\sin\left(\frac{\beta\pi}{k}\right)\right)^{n-2}(1-e^{-2\pi i\beta})} = \frac{f_n(\beta,x)}{1-e^{-2\pi i\beta}}.$$

Let C be a contour in $\mathbb{C}$ which follows a line through the origin from $(-1+i)\infty$ to $(1-i)\infty$, except for a deviation close to the origin around a clockwise semi-circle below the line. If $\frac{H}{P}$ is negative, this line is rotated $\pi/2$ in the clockwise direction. We claim that for every $y \in \mathbb{R}$ there exists a positive constant $a > 0$ with the property that $h_n(\beta, x) = \mathcal{O}(e^{-a\beta})$ for any sufficiently large $|\beta|$ on $C + y$. Consider first the case $H/P > 0$ and parametrize $C + y$ by $t \mapsto \beta(t) = t(-1+i) + y$. Observe that the numerator of $h_n(\beta)$ becomes dominated by the factor $e^{-\frac{t^2 H}{P}}$ whereas the denominator becomes unbounded. The case $H/P < 0$ is similar. It follows that whenever $C + y, y \in \mathbb{R}$ does not meet any poles of $h_n(\beta, x)$, we can define $\Theta_n^y(x)$ by

$$\Theta_n^y(x) = \int_{C+y} h_n(\beta, x)\ \mathrm{d}\,\beta.$$

We shall write $\Theta_n^0 = \Theta_n$.

Clearly the denominator $\left(2\sin\left(\frac{\beta\pi}{k}\right)\right)^{n-2}(1-e^{-2\pi i\beta})$ is invariant under $\beta \mapsto \beta + 2Pk$. Thus we get

$$h_n(\beta + 2Pk, x) = h_n(\beta, x - Hk)e^{4\pi P i x} \tag{7.8}$$

and similarly

$$h_n(\beta - 2Pk, x) = h_n(\beta, x + Hk)e^{-4\pi Pix}.$$

Recall that $H \in \{\pm 1\}$. Let us assume for now that $H = 1$ so that $H/P > 0$. We will keep H in the notation, as a convenience to the reader who wishes to consider the case $H = -1$ also. Observe that $h_N(\beta, x)$ have poles at $\mathbb{Z} \subset \mathbb{R}$. It follows that when we push the contour $C \mapsto C + 2PK$, we move it across the poles $0, 1, 2, ..., 2PK - 1$. By proposition 7.1.3, we get for $m \in \mathbb{Z} \setminus k\mathbb{Z}$ that $\mathrm{Res}(h_m(\beta, x), \beta = m) = \frac{f_n(m,x)}{2\pi i}$. By using the residue Theorem and the transformation property (7.8) we obtain

$$\begin{aligned}
\Theta_n(x) &= \int_C h_n(\beta, x) \ \mathrm{d}\,\beta \\
&= \int_{C+2Pk} h_n(\beta, x) \ \mathrm{d}\,\beta + 2\pi i \sum_{m=0}^{2Pk-1} \mathrm{Res}(h_m(\beta, x), \beta = m) \\
&= \theta_n(x - Hk)e^{4\pi i Px} + 2\pi i \sum_{l=0}^{2P-1} \mathrm{Res}(h_m(\beta, x), \beta = lk) + \sum_{\substack{m=0,\\ k\nmid m}}^{2Pk-1} f_n(m, x).
\end{aligned} \tag{7.9}$$

When $2Px \in \mathbb{Z}$, the computation (7.9) yields

$$\sum_{\substack{m=0,\\ k\nmid m}}^{2Pk-1} f_n(m, x) = \Theta_n(x) - \Theta_n(x - Hk) + 2\pi i \sum_{l=0}^{2P-1} \mathrm{Res}(h_m(\beta, x), \beta = lk).$$

As we are assuming $H = 1$ we get

$$\Theta_n(x) - \Theta_n(x - Hk) = \int_C h_n(\beta, x) - h_n(\beta, x - Hk) \ \mathrm{d}\,\beta = \int_C f_n(\beta, x) \ \mathrm{d}\,\beta.$$

Thus we can write

$$\sum_{\substack{m=0,\\ k\nmid m}}^{2Pk-1} f_n(m, x) = \int_C f_n(\beta, x) \ \mathrm{d}\,\beta + 2\pi i \sum_{l=0}^{2P-1} \mathrm{Res}(h_m(\beta, x), \beta = lk). \tag{7.10}$$

Observe that (7.2) can be rewritten as follows

$$\tau_k = B_k \sum_{\substack{m=0,\\ k\nmid m}}^{2Pk-1} \sum_{\epsilon \in \{\pm 1\}} \left(\prod_{j=1}^{n} \epsilon(j) \right) f_n(m, x_\epsilon)$$

where $x_\epsilon = \frac{1}{2} \sum_{j=1}^{n} \frac{\epsilon(j)}{p_j}$. As (7.10) is linear in f_n, we obtain the following equation

$$\frac{\tau_k}{B_k} = \int_C f(\beta)\, \mathrm{d}\,\beta - 2\pi i \sum_{m=1}^{2P-1} \mathrm{Res}\left(\frac{f(\beta)}{1 - e^{-2\pi i\beta}}, \beta = mk \right)$$

where $f(\beta) = q^{-\frac{H\beta^2}{4P}} \left(\sin\left(\frac{\beta\pi}{k}\right)\right)^{2-n} \prod_{j=1}^{n} \sin\left(\frac{\beta\pi}{2p_j}\right)$. Now (7.5) follows upon making the change of variable $y = \frac{2\pi i\beta}{k}$, recalling $\tau_k(S^2 \times S^1) = \frac{\sqrt{\frac{k}{2}}}{\sin\left(\frac{\pi}{k}\right)}$ and observing that $iC = \gamma$ and that $f(\beta) = e^{kg(y)}F(y)$.

We now turn to (7.6). We claim that

$$\operatorname{Res}\left(\frac{F(y)e^{kg(y)}}{1-e^{-ky}}, y = 2\pi im\right) \in k^{-1}e^{kg(2\pi im)}\mathbb{C}[k].$$

To see this, we start by noting that as $\sinh(z/2)$ have a simple zero at $z = 2\pi ikm$, we have an expression of the form

$$\frac{1}{1-e^{-ky}} = \frac{2e^{k\frac{y}{2}}}{\sinh\left(\frac{ky}{2}\right)} = \frac{c_{-1}}{k(y-2\pi im)} + \sum_{j=0}^{\infty} c_j k^j (y-2\pi im)^j$$

which is valid for y near $2\pi ikm$. The sequence $\{c_j\}_{j=-1}^{\infty} \subset \mathbb{C}$ is independent of k, m because of periodicity. We can write

$$F(y) = \sum_{j=1}^{n-2} \frac{a_{-j}(m)}{(y-2\pi im)^j} + \sum_{j=0}^{\infty} a_j(m)(y-2\pi im)^j.$$

Thus it is clear that

$$\operatorname{Res}\left(\frac{F(y)e^{kg(y)}}{1-e^{-ky}}, y = 2\pi im\right) = e^{kg(2\pi im)}\left(k^{-1}c_{-1}a_0(m) + \sum_{j=1}^{n-2} a_{-j}(m)c_{j-1}k^{j-1}\right)$$

Now it is a matter of defining $R(X)$ to be the set of $\theta \in \{g(2\pi im)\}$ with non-vanishing contribution, and then checking that the k^{-1} terms cancel.

Finally, we turn to (7.7). We must compute the asymptotic expansion of the Laplace integral $I(k) = \frac{1}{2\pi i}\int_\gamma F(y)e^{kg(y)}\,\mathrm{d}\,y$. To that end, we indtroduce the variable t defined by $-t = g(y) = \frac{Hiy^2}{8\pi P}$. Thus we have

$$\begin{aligned} \mathrm{I}(k) &= \frac{1}{2\pi i}\int_\gamma F(y)e^{kg(y)}\,\mathrm{d}\,y = \frac{1}{2\pi i}\int_0^\infty e^{-kt}\int_{g=t}\frac{F}{\mathrm{d}\,g}\,\mathrm{d}\,t \\ &= \frac{1}{2\pi i}\int_0^\infty e^{-kt}\frac{\sqrt{P\pi i8}}{\sqrt{t}\sqrt{H}}F\left(\sqrt{\frac{i8P\pi t}{H}}\right)\,\mathrm{d}\,t \\ &= \int_0^\infty e^{-kt}\frac{\sqrt{P2}}{\sqrt{t}\sqrt{\pi Hi}}F\left(\sqrt{\frac{i8P\pi t}{H}}\right)\,\mathrm{d}\,t. \end{aligned} \tag{7.11}$$

For the third equation we used that $\mathrm{d}\,y = \frac{\sqrt{\pi iP8}}{2\sqrt{t}\sqrt{H}}\,\mathrm{d}\,t$, and that $F(y) = F(-y)$. From this fact we also deduce that for y near 0 we have $F(y) = \sum_{n=0}^{\infty}\frac{F^{(2n)}(0)}{(2n)!}y^{2n}$. By Proposition 5.2.15, we get the following asymptotic expansion

$$\begin{aligned} \mathrm{I}(k) &\sim_{k\to\infty} \frac{1}{2\pi i}\frac{\sqrt{P\pi i8}}{\sqrt{H}}\sum_{n=0}^{\infty}\frac{F^{(2n)}(0)\left(\frac{i8P\pi}{H}\right)^n}{(2n)!}\int_0^\infty e^{-kt}t^{n-\frac{1}{2}}\,\mathrm{d}\,t \\ &= \frac{1}{2\pi i}\frac{\sqrt{P\pi i8}}{\sqrt{H}}\sum_{n=0}^{\infty}\frac{F^{(2n)}(0)\left(\frac{i8P\pi}{H}\right)^n}{(2n)!}\frac{\Gamma\left(n+\frac{1}{2}\right)}{k^{n+\frac{1}{2}}} = \mathrm{Z}_\infty(k). \end{aligned}$$

This completes the proof. $\square$

7.1.2 The moduli space and complex Chern-Simons values

We now begin our investion of $\mathcal{M}(\mathrm{SL}(2,\mathbb{C}),X)$. We build on [FS90] and to align with the choices of orientation made therein we can and will assume that $H=1$, as only $\frac{H}{P}$ is fixed. We have the following presentation of $\pi_1(X)$ (which can be obtained by the procedure described in Section 3.1.2)

$$\pi_1(X) \simeq \left\langle h, x_1, ..., x_n \mid x_1x_2\cdots x_n, x_j^{p_j}h^{-q_j}, [x_j,h], j=1,...,n \right\rangle .$$

Due to work of Fintushel and Stern [FS90] much is known about $\mathcal{M}(\mathrm{SU}(2),X)$, and we shall now recall a few of their results. As X is an integral homology sphere, the only reducible representation into $SU(2)$ is the trivial one. For an irreducible representation $\rho : \pi_1(X) \to \mathrm{SU}(2)$ at most $n-3$ of the $\rho(x_j)$ are $\pm I$, and if exactly $n-m$ of the $\rho(x_j)$ are equal to $\pm I$, then the component of ρ in $\mathcal{M}(\mathrm{SU}(2),X)$ is of dimension $2(n-m)-6$.

Let $L(p_1,...,p_n) \subset \mathbb{N}^n$ be the the set of n-tuples $l=(l_1,...,l_n)$ which satisfies the following condition. We have $0 \le l_1 \le p_1 - 1$ and $0 \le l_j \le \frac{p_j-1}{2}$, for $j=2,...,n$ and there exists at least three distinct $l_{j_1} < l_{j_2} < l_{j_3}$ with $l_{j_t} \neq 0$ for $t=1,2,3$. The following proposition is an adaption of Lemma 2 in [BC06] and Lemma 2.1 and Lemma 2.2 in [FS90].

Proposition 7.1.4. *Let $l=(l_1,l_2,...,l_n) \in L(p_1,...,p_n)$. Then there exists matrices $Q_j \in \mathrm{SL}(2,\mathbb{C})$ and a representation $\rho_l : \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ with*

$$\rho_l(x_1) = Q_1 \begin{pmatrix} e^{\frac{\pi i l_1}{p_1}} & 0 \\ 0 & e^{\frac{-l\pi i}{p_1}} \end{pmatrix} Q_1^{-1}, \quad \rho_l(x_j) = Q_j \begin{pmatrix} e^{\frac{2\pi i l_j}{p_j}} & 0 \\ 0 & e^{\frac{-2\pi i l_j}{p_j}} \end{pmatrix} Q_j^{-1}$$

for $j=2,\dots n$. For any non-trivial representation $\rho : \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ there exists $l' \in \mathbb{N}^n$ with at most $n-3$ of the elements of the set $\{l'_j\}$ being divisible by p_j such that ρ is of the form

$$\rho(x_1) = S_1 \begin{pmatrix} e^{\frac{\pi i l'_1}{p_1}} & 0 \\ 0 & e^{\frac{-\pi i l'_1}{p_1}} \end{pmatrix} S_1^{-1}, \quad \rho(x_j) = S_j \begin{pmatrix} e^{\frac{2\pi i l'_j}{p_j}} & 0 \\ 0 & e^{\frac{-2\pi i l'_j}{p_j}} \end{pmatrix} S_j^{-1}. \tag{7.12}$$

for some $S_1,...,S_n \in \mathrm{SL}(2,\mathbb{C})$.

For the representation ρ_l we can in fact choose $Q_j = I$ for $j \neq j_2, j_3$. Before commencing the proof let us introduce the following notation

$$\exp(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$$

which should not cause any ambiguities as long as the context shows that we are dealing with a matrix.

Proof. We start with the construction of ρ_l. Introduce the matrices

$$X_1 = \exp(\pi i l_1/p_1), \quad X_j = \exp(2\pi i l_j/p_j)$$

for $j \in \{2, ..., n\} \setminus \{j_2, j_3\}$. Rewrite the relation $\prod_{j=1}^{n} x_j = 1$ as the equivalent relation $x_{j_3+1} \cdots x_n x_1 \cdots x_{j_1} \cdots x_{j_2} \cdots x_{j_3-1} = x_{j_3}^{-1}$. Assume we have chosen $Q_{j_2}, Q_{j_3} \in \mathrm{SL}(2, \mathbb{C})$ such that

$$X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q_{j_2} X_{j_2} Q_{j_2}^{-1} \cdots X_{j_3-1} = Q_{j_3}^{-1} X_{j_3}^{-1} Q_{j_3}. \tag{7.13}$$

Taking $Q_j = I$ for $j \notin \{j_2, j_3\}$, we can define $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ by

$$\rho(x_j) = Q_j X_j Q_j^{-1},\ \rho(h) = X_1^{p_1}.$$

To see this, observe that $B := X_1^{p_1} = (-I)^{l_1}$ is central and as q_1 is odd whereas q_j is even for $j \geq 2$, we also have $X_j^{p_j} = B^{q_j}, \forall j$. The last relation in $\pi_1(X)$ is ensured by (7.13). Observe that it will suffice to choose $Q \in \mathrm{SL}(2, \mathbb{C})$ with

$$\mathrm{tr}\left(X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q X_{j_2} Q^{-1} \cdots X_{j_3-1}\right) = 2\cos\left(\frac{2\pi l_{j_3}}{p_{j_3}}\right) < 2 \tag{7.14}$$

because this will ensure that there exists some $T \in \mathrm{SL}(2, \mathbb{C})$ with

$$T X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q X_{j_2} Q^{-1} \cdots X_{j_3-1} T^{-1} = X_{j_3}.$$

For (7.14) we used our assumption on j_3. Write

$$\begin{aligned}
&X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots X_{j_2-1} = \exp(ia),\\
&X_{j_2} = \exp(ib),\\
&X_{j_2+1} \cdots X_{j_3-1} = \exp(ic),\\
&X_{j_3} = \exp(id).
\end{aligned}$$

Define $Q = \begin{pmatrix} u & -v \\ 1 & 1 \end{pmatrix}$ for u, v to be chosen below. Assume $u + v = 1$ so that $Q \in \mathrm{SL}(2, \mathbb{C})$. We compute

$$\begin{aligned}
&X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q X Q^{-1} \cdots X_{j_3-1}\\
&\qquad = \begin{pmatrix} ue^{i(a+b+c)} + ve^{i(a-b+c)} & uve^{i(a+b-c)} - uve^{i(a-b-c)} \\ e^{i(b+c-a)} - e^{i(c-a-b)} & ue^{-i(a+b+c)} + ve^{i(b-a-c)} \end{pmatrix}.
\end{aligned}$$

We have

$$\begin{aligned}
&\mathrm{tr}\begin{pmatrix} ue^{i(a+b+c)} + ve^{i(a-b+c)} & uve^{i(a+b-c)} - uve^{i(a-b-c)} \\ e^{i(b+c-a)} - e^{i(c-a-b)} & ue^{-i(a+b+c)} + ve^{i(b-a-c)} \end{pmatrix}\\
&\qquad = 2u\cos(a+b+c) + 2v\cos(a+c-b).
\end{aligned}$$

It follows that we must solve

$$\begin{pmatrix} \cos(a+b+c) & \cos(a+c-b) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2\cos(d) \\ 1 \end{pmatrix}. \tag{7.15}$$

Using the trigonometric identity $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ we get

$$\begin{aligned}\det\begin{pmatrix}\cos(a+b+c) & \cos(a+c-b)\\ 1 & 1\end{pmatrix} &= \cos((a+c)+b) - \cos((a+c)-b)\\ &= -2\sin(a+c)\sin(b).\end{aligned}$$

Thus it remains to argue $a+c \notin \pi\mathbb{Z}$ and $b \notin \pi\mathbb{Z}$. Assume towards a contradiction that $a+c = \pi m$ for some $m \in \mathbb{Z}$. Hence we would have $P(a+c) = Pm\pi$ which would imply

$$l_{j_1} 2^{\epsilon} \prod_{t \neq j_1} p_t = 0 \mod p_{j_1}$$

for $\epsilon \in \{0,1\}$, with $\epsilon = 0$ for $j_1 = 1$. This is a contradiction, as $2^{\epsilon}\prod_{t\neq j_1} p_t$ is invertible in $\mathbb{Z}/p_{j_1}\mathbb{Z}$ and $1 \leq l_{j_1} \leq (p_{j_1}-1)/2^{\epsilon}$. Here we use co-primality. Similarly, one sees that $b \notin \pi\mathbb{Z}$. Thus we can solve (7.15), and this concludes the first part of the proposition.

Now let $\rho : \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ be an arbitrary non-trivial representation. As remarked before any non-trivial representation is irreducible since X is integral homology three sphere. Since $\rho(h)$ computes with the image of ρ, we see that $\rho(h) = \pm I$. Hence the relation $x_j^{p_j} = h^{q_j}$ implies that $\rho(x_j)^{p_j} = \pm I$, and for $j = 2,\dots,n$ we must have $\rho(x_j)^{p_j} = I$, since q_j is even. Hence ρ must be of the form (7.12) for some $l' \in \mathbb{N}^n$. It only remains to argue that at most $n-3$ of the $\rho(x_j)$ are $\pm I$. If not, the relation $x_1x_2\cdots x_n = 1$ implies that there is $j_1 < j_2$ with $\rho(x_{j_1})\rho(x_{j_2}) = \pm I$. As p_{j_1} and p_{j_2} are relatively coprime, this is only possible if $\rho(x_{j_1}) = \pm 1$ and $\rho(x_{j_2}) = \pm 1$. This would imply that $\rho(\pi_1(X)) \subset \{\pm 1\} = Z(\mathrm{SU}(2))$ which contradicts the fact that ρ is irreducible since it was assumed non-trivial. □

We now compute the Chern-Simons values of the representations constructed above.

Proposition 7.1.5 ([AP18b])**.** *For $l = (l_1,\dots,l_n) \in L(p_1,\dots,p_n)$, let $\rho_l : \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ be the associated representation defined in Proposition 7.1.4. We have that*

$$\mathrm{S}(\rho_l) = \frac{-\left(P\left(\frac{l_1}{p_1} + \sum_{j=2}^{n}\frac{2l_j}{p_j}\right)\right)^2}{4P} \mod \mathbb{Z}. \tag{7.16}$$

The formula (7.16) was proven for SU(2) connections by Kirk and Klassen and it is stated in Theorem 5.2 in [KK90].

Proof of Proposition 7.1.5. Let $K \subset X$ be the n'th exceptional fiber. Let Y be the complement of a tubular neighborhood of K in X. Removing K has the effect on π_1 of removing the relation $x_n^{p_n} = h^{-q_n}$, i.e. we have a presentation

$$\pi_1(Y) \simeq \left\langle h, x_1,\dots,x_n \mid x_1x_2\cdots x_n, \forall j, [x_j,h], \text{and } x_1^{p_1}h^{-q_1},\dots,x_{n-1}^{p_{n-1}}h^{-q_{n-1}}\right\rangle. \tag{7.17}$$

As the meridian and longitude of ∂Y we can take $\mu = x_n^{p_n}h^{q_n}$ and $\lambda = x_n^{-p_1\cdots p_{n-1}}h^c$ respectively, where $c = \sum_{j=1}^{n-1}\frac{p_1\cdots p_{n-1}q_j}{p_j}$.

Let $\rho : \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ be any irreducible representation. Let $l \in \mathbb{N}^n$ be the n-tuple such that (7.12) holds. To prove formula (7.16) it will suffice to show

$$\mathrm{S}(\rho) = \frac{-\left(P\left(\frac{l_1}{p_1} + \sum_{j=2}^{n}\frac{2l_j}{p_j}\right)\right)^2}{4P} \mod \mathbb{Z}.$$

Introduce the two quantitites $\epsilon = P\left(\frac{l_1}{p_1} + \sum_{j=2}^{n} \frac{2l_j}{p_j}\right)$ and $\eta = \frac{\epsilon}{P}$.

The proof of (7.16) presented here consists analogously with the proof of Theorem 5.2 in [KK90] of two parts. In the first part, we find a path of $\mathrm{SL}(2,\mathbb{C})$ connections on X connecting ρ_l to an abelian representation ρ_0. In fact ρ_0 will be an $\mathrm{SU}(2)$ connection on X. In the second part, we then find a path from ρ_0 to the trivial representation ρ^{triv} and we then apply Kirk and Klassens formula (2.2.2). The only difference from the proof in [KK90] is that we need to explicitly ensure that our paths stay away from parabolic representations. The relevant paths are chosen such that λ, μ are mapped to the maximal $\mathbb{C}^*$ torus of diagonal matrices.

After conjugating by S_n^{-1} we have $\rho(x_n) = \exp_M\left(\frac{2\pi i l_n}{p_n}\right)$. Consider the subset $S \subset \mathrm{Hom}\left(\pi_1(Y), \mathrm{SL}(2,\mathbb{C})\right)$ of representations $\tilde{\rho}$ satisfying

$$\tilde{\rho}(h) = \rho(h),\ \mathrm{tr}(\tilde{\rho}(x_1)) = 2\cos\left(\frac{\pi i l_1}{p_1}\right),\ \mathrm{tr}(\tilde{\rho}(x_j)) = 2\cos\left(\frac{2\pi i l_j}{p_j}\right),\ \text{for } j \geq 2.$$

By considering the presentation (7.17), we see that S is naturally homeomorphic to the product of $n-1$ conjugacy classes

$$S \simeq \left[\exp_M\left(\frac{\pi i l_1}{p_1}\right)\right] \times \bigtimes_{j=2}^{n-1} \left[\exp_M\left(\frac{2\pi i l_j}{p_j}\right)\right].$$

Here $[Q]$ denotes the $\mathrm{SL}(2,\mathbb{C})$ conjugacy class of $Q \in \mathrm{SL}(2,\mathbb{C})$. Therefore the connectedness of $\mathrm{SL}(2,\mathbb{C})$ implies that S is connected. Write $\rho = \rho_1$. Choose a smooth path ρ_t in S connecting ρ_1 to $\rho_0 \in S$ given by

$$\rho_0(x_1) = \exp_M\left(-\frac{\pi i l_1}{p_1}\right),\ \rho_0(x_j) = \exp_M\left(-\frac{2\pi i l_j}{p_j}\right),\ j = 2, ..., n-1$$

and $\rho_0(x_n) = \exp_M\left(\frac{\pi i l_1}{p_1} + \sum_{j=2}^{n-1} \frac{2\pi i l_j}{p_j}\right)$. We can choose the arc ρ_t such that $\rho_t(x_n) = \exp_M(2\pi i f(t))$ for a smooth function $f(t)$. In particular, we must have $f(0) = \frac{l_1}{2p_1} + \sum_{j=1}^{n-1} \frac{l_j}{p_j}$ and $f(1) = \frac{l_n}{p_n}$. Notice that $f(0) = \frac{\eta}{2} - f(1)$. As q_n is even and c is odd we have the following two equalities

$$\begin{aligned} \rho_t(\mu) = \rho_t(x_n)^{p_n} \rho_t(h)^{q_n} &= \exp_M(2\pi i p_n f(t)), \\ \rho_t(\lambda) = \rho_t(x_n)^{-p_1 \cdots p_{n-1}} \rho_t(h)^c &= \exp_M\left(-2\pi i p_1 \cdots p_{n-1} f(t) + \pi i\right). \end{aligned}$$

Define $\alpha_1(t) = p_n f(t)$ and $\beta_1(t) = -\frac{P}{p_n} f(t) + \frac{1}{2}$. We have that

$$-2\int_0^1 \alpha_1'(t)\beta_1(t)\ \mathrm{d}t = -\frac{\epsilon^2}{4P} + \frac{p_n \epsilon}{2P} + \frac{\epsilon l_n}{p_n} \mod \mathbb{Z}. \tag{7.18}$$

For the second part we use the fact that $H_1(Y) \simeq \mathbb{Z}$ with generator μ to conclude that the abelian $\mathrm{SU}(2)$ connection ρ_0 can be connected to the trivial representation ρ^{triv} by a path of $\mathrm{SU}(2)$ representations σ_t with $\sigma_t(\mu) = \exp_M(2\pi i t \alpha_1(0))$ and $\sigma_t(\lambda) = \exp_M(2\pi i \beta(0))$. Let $\alpha_0(t) = t\alpha_1(0)$ and $\beta_0(t) = \beta(0)$. As $\mathrm{S}(\rho^{\mathrm{triv}}) = 0$, we can apply Kirk and Klassen's formula (2.2.2) to obtain

$$-\mathrm{S}(\rho) = \mathrm{S}(\rho^{\mathrm{triv}}) - \mathrm{S}(\rho) = -2\int_0^1 \alpha_0'(t)\beta_0(t)\ \mathrm{d}t - 2\int_0^1 \alpha_1'(t)\beta_1(t)\ \mathrm{d}t.$$

We have

$$-2\int_0^1 \alpha_1'(t)\beta_1(t)\;\mathrm{d}\,t = 2\frac{\epsilon^2}{4P} + 2P(f(1))^2 - \frac{p_n\epsilon}{2P} - 2\epsilon f(1) + l_n.$$

Comparing this with (7.18) we get that

$$\begin{aligned}
-\,\mathrm{S}(\rho) &= -\frac{\epsilon^2}{4P} + \frac{p_n\epsilon}{2P} + \frac{\epsilon l_n}{p_n} \\
&\quad + 2\frac{\epsilon^2}{4P} + 2P(f(1))^2 - \frac{p_n\epsilon}{2P} - 2\epsilon f(1) \mod \mathbb{Z} \\
&= \frac{\epsilon^2}{4P} + 2f(1)(Pf(1) - \epsilon/2) \mod \mathbb{Z} \\
&= \frac{\epsilon^2}{4P} + 2f(1)P\left(-\frac{l_1}{2p_1} - \sum_{j=2}^{n-1}\frac{l_j}{p_j}\right) \mod \mathbb{Z} \\
&= \frac{\epsilon^2}{4P} \mod \mathbb{Z}.
\end{aligned}$$

This is what we wanted. □

Brieskorn integral homology spheres

The family of Brieskorn integral homology spheres ($n = 3$) is very special due to the fact that the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections $\mathcal{M}(\mathrm{SL}(2,\mathbb{C}),\Sigma(p_1,p_2,p_3))$ is finite with cardinality given by the $\mathrm{SL}(2,\mathbb{C})$ Casson invariant introduced by Curtis [Cur01, Cur03]

$$\lambda_{\mathrm{SL}(2,\mathbb{C})}(\Sigma(p_1,p_2,p_3)) = (p_1-1)(p_2-1)(p_3-1)/4.$$

This is shown by Boden and Curtis [BC06]. Prior to this and in relation to Floer homology, Fintuschel and Stern [FS90] analyzed the SU(2) moduli space $\mathcal{M}(\mathrm{SU}(2),X)$ of the Seifert fibered three-manifold X considered in this thesis and their work shows that the components are even dimensional manifolds with top dimension $2n-6$. This is in stark contrast to the finiteness of the moduli space $\mathcal{M}(\mathrm{SL}(2,\mathbb{C}),\Sigma(p_1,p_2,p_3))$. Moreover, the situation at hand is also special because all the Chern-Simons values of flat $\mathrm{SL}(2,\mathbb{C})$-connections we encounter are in $\mathbb{R}/\mathbb{Z}$. In the three fibered case, this is naturally explained by work of Kitano and Yamaguchi [KY16] which gives a decomposition

$$\mathcal{M}(\mathrm{SL}(2,\mathbb{C}),\Sigma)) = \mathcal{M}(\mathrm{SL}(2,\mathbb{R}),\Sigma)) \bigcup_{\mathcal{M}(\mathrm{U}(1),\Sigma)} \mathcal{M}(\mathrm{SU}(2),\Sigma)).$$

Where $\Sigma = \Sigma(p_1,p_2,p_3)$. Thus we pay special attention to the class of Brieskorn integral homology spheres and we obtain the following corollary.

Corollary 7.1.6. *If p_1,p_2,p_3 are odd primes and $X = \Sigma(p_1,p_2,p_3)$, the Chern-Simons action $\mathrm{S}_{\mathbb{C}} : \mathcal{M}(\mathrm{SL}(2,\mathbb{C})(X)) \to \mathbb{R}/\mathbb{Z}$ is injective.*

Proof. The number of representations found in Proposition 7.1.4 is equal to $(p_1-1)(p_2-1)(p_3-1)/4$. This is equal to the number of squares in the ring $\mathbb{Z}/4P\mathbb{Z} \simeq \mathbb{Z}/4\mathbb{Z} \oplus_{j=1}^3 \mathbb{Z}/p_j\mathbb{Z}$ whose components in the last three factors are invertible. We can now appeal to (1.17) which will be proven below. □

7.1.3 The Borel transform and complex Chern-Simons

We now present the proof of Theorem 7.

Proof of Theorem 7. We start by giving a characterization of which of the phases in (7.5) give a non-zero contribution. Recall the definition of $\mathcal{T}(\mu)$ given in (1.18).

The set of phases $R(X)2\pi i$ in (7.7) consists of the values $g(2\pi i m) = \frac{-m^2 2\pi i}{4P}$, $m = 1, ..., 2P-1$ for which

$$\sum_{x \in \mathcal{T}(-m^2/4P)} \operatorname{Res}\left(\frac{F(y)e^{kg(y)}}{1-e^{-ky}}, y = 2\pi i x\right) \neq 0. \tag{7.19}$$

Thus we must prove that if (7.19) holds, then there exists $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ with $\mathrm{S}(\rho) = \frac{-m^2}{4P} \mod \mathbb{Z}$. We start by noting that the set of poles of F is given by

$$\mathcal{P}_F = \{2\pi i m \mid m \in \mathbb{Z} \text{ and } m \text{ is divisible by at most } n-3 \text{ of the } p_j\text{'s}\}. \tag{7.20}$$

It follows that if $\tilde{m}$ is divisible by at least $n-2$ of the p_j, then $F(y)$ does not have a pole at $y = 2\pi i \tilde{m}$ and we get for integral k

$$\begin{aligned}\operatorname{Res}\left(\frac{F(y)e^{kg(y)}}{1-e^{-Ky}}, y = 2\pi i \tilde{m}\right) &= F(2\pi i \tilde{m})e^{kg(2\pi i \tilde{m})}\operatorname{Res}\left(\frac{1}{1-e^{-ky}}, y = 2\pi i \tilde{m}\right) \\ &= F(2\pi i \tilde{m})e^{kg(2\pi i \tilde{m})}\frac{1}{k}.\end{aligned}$$

As we already noted above, Lawrence and Rozansky checked that all the k^{-1} terms cancels, so it follows that

$$\sum_{\substack{\tilde{m} \in \mathcal{T}(-m^2/4P), \\ \text{and } \tilde{m} \text{ is divisible by at least } n-2 \text{ of the } p_j}} \operatorname{Res}\left(\frac{F(y)e^{Kg(y)}}{1-e^{-Ky}}, y = 2\pi i \tilde{m}\right) = 0.$$

Therefore we see that if (7.19) holds, then there is some $\tilde{m} \in \mathcal{T}(-m^2/4P)$ which is divisible by at most $n-3$ of the p_j. By Proposition 7.1.5 this implies that there exists some $l \in L(p_1, ..., p_n)$ with

$$\frac{-m^2}{4P} = \frac{-\tilde{m}^2}{4P} = \mathrm{S}(\rho_l),$$

where the first equality is by definition of $\mathcal{T}(-m^2/4P)$. This establishes $R(X) \subset \mathrm{CS}^*_{\mathbb{C}}$ and we get (1.15).

We now compute $\mathrm{CS}^*_{\mathbb{C}}$. We want to show it is equal to the set

$$\mathcal{W}(p_1, ..., p_n) := \left\{\frac{-m^2}{4P} \mod \mathbb{Z} : m \in \mathbb{Z} \text{ and } m \text{ is divisible by at most } n-3 \text{ of the } p_j\text{'s}\right\}.$$

It is clear that $\mathrm{CS}^*_{\mathbb{C}} \subset \mathcal{W}(p_1, ..., p_n)$. We must show that for any $y \in \mathbb{Z}$ which is not divisible by more than at most three of the p_j we can find $l = (l_1, ..., l_n) \in L(p_1, ..., p_n)$ which solves the congruence equation

$$y^2 = \frac{-\left(P\left(\frac{\pi l_1}{p_1} + \sum_{j=2}^{n} \frac{2\pi l_j}{p_j}\right)\right)^2}{4P} \mod \mathbb{Z} \tag{7.21}$$

For $x \in \mathbb{Z}$ and $d \in \mathbb{N}$ let $[x]_d$ denote the congruence class of x in the quotient ring $\mathbb{Z}/d\mathbb{Z}$. Since p_j is odd for $j \geq 2$, it follows that $4p_1, p_2, ..., p_n$ are also pairwise co-prime. Hence the Chinese remainder theorem applies and the natural ring homomorphism $q : \mathbb{Z} \to \mathbb{Z}/4p_1\mathbb{Z} \oplus_{j=2}^{n} \mathbb{Z}/p_j\mathbb{Z}$, given by $x \mapsto ([x]_{4p_1}, ..., [x]_{p_n})$, descends to an isomorphism of rings

$$\overline{q} : \mathbb{Z}/4P\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/4p_1\mathbb{Z} \bigoplus_{j=2}^{n} \mathbb{Z}/p_j\mathbb{Z}.$$

It follows that (7.21) is in fact equivalent to the following n congruence equations

$$[y]^2{}_{4p_1} = \left[l_1 \prod_{j=2}^{n} p_j + 2p \left(\sum_{j=2}^{n} l_j \prod_{t \neq j} p_t \right) \right]^2_{4p_1}, \tag{7.22}$$

$$[y]^2_{p_j} = \left[2l_j \prod_{t \neq j} p_t \right]^2_{p_j}, \ \forall j \geq 2.$$

The coprimality conditions ensures that $2 \prod_{t \neq j} p_t$ is an invertible element in $\mathbb{Z}/p_j\mathbb{Z}$ and therefore solving the last $n-1$ of the equations in (7.22) can indeed be done with $0 \leq l_j \leq (p_j - 1)/2$. It remains only to consider the first of the equations in (7.22). We note that

$$(cp_1 + j)^2 - (dp_1 - j)^2 = (2(c+d) + (c^2 - d^2)p_1)p_1$$

and if c and d have the same parity this is divisible by $4p_1$. Here we use that $[2(c+d)]_4 = [c^2 - d^2] = 0$, if c and d have the same parity. It follows that the squares $[x]^2_{4p_1}$ occur in a repeating pattern which is symmetric around multiples of p_1

$$\begin{array}{cccccccc} x: & p_1 - j & \ldots & p_1 - 1 & p_1 & p_1 + 1 & \ldots & p_1 + j \\ [x]^2_{4p_1}: & [p_1 - j]^2_{4p_1} & \ldots & [p_1 - 1]^2_{4p_1} & [p_1]^2_{4p_1} & [p_1 - 1]^2_{4p_1} & \ldots & [p_1 - j]^2_{4p_1} \end{array}$$

In particular $[y]^2_{4p_1} \in \{[0]_{4p_1}, [1]^2_{4p_1}, [2]^2_{4p_1}, ..., [p_1 - 2]^2_{4p_1} [p_1 - 1]^2_{4p_1}\}$, and equation (7.22) is reduced to

$$[y]^2_{4p_1} = \left[l_1 \prod_{t=2}^{n} p_t \right]^2_{4p},$$

which is independent of $l_2, ..., l_n$. As $\prod_{t=2}^{n} p_t$ is invertible modulo $4p_1$, we can for every $m = 1, .., p_1 - 1$ find a unique $x_m \in \{1, ..., p-1\}$ and $d_m \in \mathbb{N}$ with $x_m \prod_{t=2}^{n} p_t = m + d_m p_1$. As multiplication is linear, we must have $x_{p_1 - m} = p_1 - x_m$ hence $(p_1 - x_m) \prod_{t=2}^{n} p_t = p_1 - m + d_{p-m} p_1$. Thus

$$\begin{aligned} p_1(d_m + d_{m-p} + 1) &= \left(x_m \prod_{t=2}^{n} p_t - m \right) + \left((p_1 - x_m) \prod_{t=2}^{n} p_t - (p_1 - m) \right) + p_1 \\ &= p_1 \prod_{t \geq 2}^{n} p_t = P. \end{aligned}$$

As $p_2, ..., p_n$ are all odd, this implies that d_m and d_{p-m} have the same parity which imply

$$\left\{ \left[x_m \prod_{t=2}^{n} p_t \right]^2_{4p_1}, \left[x_{p-m} \prod_{t=2}^{n} p_t \right]^2_{4p_1} \right\} = \{[m]^2_{4p_1}, [p_1 - m]^2_{4p_1}\}.$$

It follows that we can in fact solve (7.22) with $l_1 \in \{0, ..., p-1\}$. Thus we have shown that $\mathrm{CS}^*_{\mathbb{C}} = \mathcal{W}(p_1, ..., p_n)$.

We now turn to $\mathcal{B}(\mathrm{Z}_\infty)$. The expression for the Borel transform (1.16) is a direct consequence of (7.11) and the Proposition 5.3.1. As $F(-y) = F(y)$ we note that the factor $F(\sqrt{8\pi i P\zeta})$ gives a well-defined meromorphic function. Thus $\mathcal{B}(\mathrm{Z}_\infty)(\zeta)$ is a multivalued meromorphic function with a square root singularity at 0 and with singularities for $\sqrt{8\pi i P\zeta} \in \mathcal{P}_F$ where $\mathcal{P}_F$ is the set of poles of $F(y)$. This set was computed above (see equation (7.20)) and we conclude that the poles of $\mathcal{B}(\mathrm{Z}_\infty)(\zeta)$ occur at $\zeta_m = \frac{-\pi m^2}{2iP} = \frac{-m^2}{4P}\frac{2\pi}{i}$ with $m \in \mathbb{Z}$ being divisible by less than or equal to $n-3$ of the p_j's. This concludes the proof of (1.17). $\square$

7.1.4 Resummation of the quantum invariant

We now prove Theorem 8.

Proof of Theorem 8. It easily follows from (1.16) that

$$F(\zeta) = \mathcal{B}(\mathrm{Z}_\infty)\left(\frac{\zeta^2}{i8P\pi}\right)\frac{\zeta H}{P4}. \tag{7.23}$$

Now (1.20) and (1.21) follows from the residue Theorem, equation (7.11) and Theorem 7.1.2. Here it is understood that in definition (1.19) the integrals $\oint_{y=2\pi i x} \mathrm{d}\, y$ are understood to be over sufficiently small loops encircling $2\pi i x$. $\square$

7.1.5 The q-series, modularity and $\hat{Z}$ invariants

By first considering the rational function $(z^P - z^{-P})^{-2-n}\prod_{j=1}^n (z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}})$ it is easy to see that $F(y) = \sum_{m=0}^\infty \chi(m) e^{\frac{ym}{2P}} \in \mathbb{Z}[[e^{\frac{y}{2P}}]]$.

Definition 7.1.1. Introduce for $|q| < 1$ the q-series $\Psi_{p_1,...,p_n} = \Psi$ defined by

$$\Psi(q) = \sum_{m=0}^\infty \chi(m) q^{\frac{m^2}{2p}}.$$

Observe that we can rewrite (1.16) as

$$\mathcal{B}(\mathrm{Z}_\infty)(\zeta) = \frac{\sqrt{P\pi 2}}{\sqrt{\zeta}\sqrt{Hi}} F(\sqrt{i8P\pi\zeta}).$$

Thus one can clearly relate the q series to $\widetilde{Z}_k$ through the Borel transform as observed by Gukov-Putrov-Marino in [GMP16]. This is also clear from the equation (7.23). It is believed that Ψ is (a normalization of) the new topological invariants $\widehat{Z}_a(q)(M)$.

For $n = 3$, Lawrence and Zagier have shown in [LZ99] that the quantum invariant can be recovered as a certain radial limit of Ψ. This was extended (in some cases) to $n = 4$ by Hikami in [Hik05d]. The series Ψ have interesting aritmetic properties, for $n = 3$ the coefficients $\chi(m)$ are periodic functions of period $2P$. For $n = 3$ it is the so-called

Eichler integral of a mock modular form with weight 3/2. The connection between quantum invariants and number theory was further pursued by Hikami in a number of articles, including [Hik04, Hik05b, Hik05a, Hik06, Hik11], and by Hikami et al. in [BHL11, HL14]. Let us also mention the work [DG15] by Dimofte-Garoufalidis which connects modularity in quantum topology with complex Chern-Simons theory.

It is interesting to observe that Ψ is obtained [GMP16] from the Borel transform through a resummation process reminiscent of the median resummation of [CG11]. Moreover as explained in [CCF+18] there is a Mock/false modular form duality related to $\widehat{Z}_a(q)(M)$, i.e. there exists an associated pair of a so-called Mock modular form and a so-called false modular form, and these are related by a $q \mapsto q^{-1}$ transformation and have the same transseries expression near $q \to 1$. This is quite possibly connected to the conjecture 2 in [Gar08] (called the symmetry conjecture).

7.2 Surgeries on the figure eight knot

We now turn to the hyperbolic three-manifolds $M_{r/s}$. We have the following result, due to Andersen-Hansen

Theorem 7.2.1 ([AH06]). *Choose $c, d \in \mathbb{Z}$ with $rd - cs = 1$. Define*

$$\chi_{n,K}(x,y) = \sin\left(\frac{\pi}{s}(x - nd)\right) e^{2\pi i K\left(\frac{dn^2}{s} + \frac{r}{4s}x^2 - \frac{n}{s}x - xy\right)} \frac{S_\kappa\left(-\pi + 2\pi(x-y)\right)}{S_\kappa\left(-\pi + 2\pi(x+y)\right)}.$$

Then we have that

$$\tau_k(M_{r/s}) = \nu k q^\mu \sum_{n \in \mathbb{Z}/|s|\mathbb{Z}} \int_{C_1(k)\times C_2(k)} \cot(\pi k x)\tan(\pi k y)\chi_{n,k}(x,y)\ \mathrm{d}\,y\,\mathrm{d}\,x$$

where $\nu, \mu \in \mathbb{C}^*$ and $C_1(k)$ is a simple closed contour which encircles the set $\{m/k : m = 1, 2, ..., k-1\}$, and $C_2(k)$ is a simple closed contour encircling $\{(m+1/2)/k : m = 0, 1, ..., k-1\}$. Both contours are oriented anti-clockwise.

We now introduce Faddeev's quantum dilogarithim [AK15, Fad95, FK94, Kas97].

Definition 7.2.1. The *quantum dilogarithm* with parameter $\kappa = \pi/k \in (0,1)$ is given as follows

$$S_\kappa(z) = \exp\left(\frac{1}{4}\int_{\widetilde{C}} \frac{e^{zy}}{\sinh(\pi y)\sinh(\kappa y) y}\ \mathrm{d}\,y\right),$$

for $|\mathrm{Re}(z)| < \kappa + \pi$, and $\widetilde{C}$ is the contour $(-\infty, -1/2) \cup \Delta \cup (1/2, \infty)$ where Δ is the half-circle from $-1/2$ to $1/2$ in the upper halfplane.

The quantum dilogarithm satisfies the following functional equation.

Proposition 7.2.2. *For $|Re(\zeta)| < \pi$ we have*

$$(1 + e^{i\zeta})S_\kappa(\zeta + \kappa) = S_\kappa(\zeta - \kappa). \tag{7.24}$$

The function $x \mapsto S_\kappa(-\pi + 2\pi x)$ admits an analytic extension to

$$A_k := \mathbb{C} \setminus \left\{\frac{m}{k} + \frac{1}{2k} : m = k, k+1, ...,\right\}.$$

If $m \in \mathbb{N}$ then the set $\{\frac{n}{k} + \frac{1}{2k} : n = mk, mk+1, ..., (m+1)k-1\}$ consists of poles of order m whereas the set $\{\frac{n}{k} + \frac{1}{2k} : n = -mk, -mk+1, ..., -mk+k-1\}$ consists of zeroes of order m.

These properties of the quantum dilogarithm are due to Faddeev. A proof can be found in an appendix in [AH06]. It is well-known that the semiclassical asymptotics of Faddeev's quantum dilogarithm S is given by Euler's dilogarithm.

Theorem 7.2.3. *For $\zeta \in (\{Re(z) = \pm\pi\} \cap \{Im(z) \geq 0\}) \cup \{Re(z) < \pi\}$ we have*

$$S_\kappa(\zeta) = \exp\left(\frac{k}{2\pi i}\operatorname{Li}_2\left(-e^{i\zeta}\right) + I_\kappa(\zeta)\right)$$

where $I_\kappa(\zeta) = \frac{1}{4}\int_{C(R)} \frac{e^{z\zeta}}{z\sin(\pi z)}\left(\frac{1}{\sinh(\kappa z)} - \frac{1}{\kappa z}\right) \mathrm{d}\, z$.

This is the motivation behind Conjecture 1. We now recall the parametrization of the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections on $M_{r/s}$. As mentioned in the introduction, this builds on work by Riley [Ril85], Klassen [Kla91] and Kirk-Klassen [KK90, KK91]. We have the following presentation of the fundamental group of $M_{r/s}$

$$\pi_1(M_{r/s}) \simeq \langle x, y \mid [x^{-1}, y]x = y[x^{-1}, y], x^r(yx^{-1}y^{-1}x^2y^{-1}x^{-1}y)^s = 1\rangle$$

where $\mu = x$ and $\lambda = yx^{-1}y^{-1}x^2y^{-1}x^{-1}y$ correspond to the preferred meridian and longitude. Consider the following two equations

$$\begin{cases} v^{-r} &= \left(\frac{w-v^2}{1-v^2w}\right)^s, \\ v^2w &= (1-v^2w)(w-v^2). \end{cases} \tag{7.25}$$

Given a solution (v, w) to (7.25) with $v^2 \neq 1$, one can define for $(s, u+1) = (v, w)$ a representation $\rho_{(s,u+1)} : \pi_1(M_{r/s}) \to \mathrm{SL}(2,\mathbb{C})$ where $\rho_{(t,l)}$ is given by

$$\rho_{(t,l)}(x) = \begin{pmatrix} t & t^{-1} \\ 0 & t^{-1} \end{pmatrix}, \rho_{(t,l)}(y) = \begin{pmatrix} t & 0 \\ -tl & t^{-1} \end{pmatrix}.$$

We have

Theorem 7.2.4 (Andersen-Hansen, [AH06] Theorem 2)**.** *The map $(x, y) \mapsto [\rho_{(e^{\pi i x}, e^{2\pi i y}-1)}]$ gives a surjection from the set of critical points (x, y) of the phase functions $\Phi_n^{\alpha,\beta}$ with $x \notin \mathbb{Z}$ onto $\mathcal{M}^*(\mathrm{SL}(2,\mathbb{C}), M_{r/s})$, and $[\rho_{(e^{\pi i x}, e^{2\pi i y}-1)}]$ is conjugate to an $\mathrm{SU}(2)$ representation if and only if $(x, y) \in \{(x, y) \in \mathbb{R} \times \mathbb{C} : e^{2\pi i y} \in]-\infty, 0[\}$. Moreover, we have that*

$$\Phi_n^{\alpha,\beta}(x, y) = \mathrm{S}_{\mathbb{C}}([\rho_{(e^{\pi i x}, e^{2\pi i y}-1)}) \mod \mathbb{Z}.$$

Before we give the proof of Theorem 9, we make the following remark.

Remark 7.2.5. As we are only interested in the large k asymptotic, we may consider the restriction of the phase functions $\Phi_n^{\alpha,\beta}$ to bounded subsets of $\mathbb{C}^2$, containing finitely many saddle points, that are mapped surjectively to CS. Thus the phase restricted phase functions are proper smooth submersions over the complement of their critical values. By the Ehresmann Lemma, this implies that they are resurgence phases, and thus it is meaningfull to speak of Picard-Lefschetz thimbles.

Proof of Theorem 9. The proof is an application of Theorem 5 and of Theorem 6.

The existence of the expansion (1.23) follows from (1.22) and (5.16). The fact that the Borel transforms $\mathcal{B}(Z_\theta)$ have singularities corresponding to (shifted) Chern-Simons values as given in (1.24) follows from the very first part of Theorem (6). The existence of a decomposition (1.25) giving the resurgence relations (1.26) follows from (1.13). $\square$

CHAPTER 8

Analytic extensions of quantum invariants

In this section, we present details on work in progress, which is joint with Andersen.

8.1 Analytic expressions for R matrices

At this point, the reader may want to recall the definitions and notations from Section 3.2.1. For the convenience of the reader we here recall some of the notation. Let

$$\kappa(k) = \pi/k,\ t(k) = \exp(i\kappa/2),\ s(k) = (t(k))^2,\ q(k) = (s(k))^2,$$

Below we shall simply write κ, t, s and q but we make the obvious remark that these can be extended to analytic functions.

Recall that the colored Jones polynomial is constructed using the graphical calculus and the modular tensor category assoicated with the $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$-modules $V^{(1)}, V^{(2)}, ..., V^{(k-1)}$ introduced in Section 3.2.1. Recall that for each $V^{(l)}$ we introduced a distinguished basis $e^{(m)}_{-l_m}, ..., e^{(m)}_{l_m}, l = 2l_m + 1$. We have the following expression for the inverse $C^{-1}(n,m)$ braiding.

Proposition 8.1.1. *Let $0 < n, m < k$, and consider the inverse braiding isomorphism $C^{-1}(n,m) : V^{(n)} \otimes V^{(m)} \to V^{(m)} \otimes V^{(n)}$. Write*

$$C^{-1}(n,m)\left(e^{(n)}_i \otimes e^{(m)}_j\right) = \sum C^{-1}(n,m)^{v,w}_{i,j} e^{(m)}_v \otimes e^{(n)}_w.$$

With this notation we have

$$C^{-1}(n,m)^{v,w}_{i,j} = \frac{(\bar{s}-s)^{v-j}}{[v-j]!}\frac{[l_m+v]!}{[l_m+j]!}\frac{[l_n-w]!}{[l_n-i]!}\bar{t}^{-(v-j)(v-j+1)+4ij-2(v-j)(j-i)}\delta^{i+j}_{v+w}, \tag{8.1}$$

for (i,j,v,w) satisfying

$$\begin{aligned} l_m &\geq v \geq j \geq -l_m, \\ l_n &\geq i \geq w \geq -l_n. \end{aligned} \tag{8.2}$$

The proof is a modification of the proof of Corollary 2.32 in [KM91].

Proof. Recall that the braiding is given by $C(z) = P(R.z)$, where $P(x \otimes y) = y \otimes x$ and R is the universal R matrix given by Theorem 3.2.7

$$R = \frac{1}{4k} \sum_{n,a,b} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{ab+(b-a)n+n} X^n K^a \otimes Y^n K^b$$

where the sum is over all $0 \leq n < k$ and $0 \leq a, b < 4k$. Observe that $C^{-1}(z) = R^{-1}.P(z)$ and as explained in [RT91] and [KM91], the inverse of R is given by $R^{-1} = (S \otimes \mathrm{Id})(R)$, where S is the antipode. By Definition 3.2.8 we have $S(X) = -sX$ and $S(K) = \overline{K}$. As S is anti-multiplicative we get $S(X^n K^a) = (-s)^n \overline{K}^a X^n$. As $s^l = t^{2l} = \bar{t}^{(-2l)}$ this implies

$$R^{-1} = \frac{1}{4k} \sum_{n,a,b} \frac{(\bar{s}-s)^n}{[n]!} \bar{t}^{ab+(b-a)n-n} \cdot \overline{K}^a X^n \otimes Y^n K^b.$$

Recall that by Definition 3.2.9 the left action of $\tilde{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ on $V^{(r)}$ is given by

$$Xe_j^{(r)} = [l_r + j + 1] e_{j+1}^{(r)}, \quad Ye_j^{(r)} = [l_r - j + 1] e_{j-1}^{(r)}, \quad Ke_j^{(r)} = s^j e_j^{(r)}.$$

Thus we readily compute that

$$\begin{aligned} C^{-1}(n,m)\left(e_i^{(n)} \otimes e_j^{(m)}\right) =& R^{-1}.(e_j^{(m)} \otimes e_i^{(n)}) \\ =& \frac{1}{4k} \sum_{u,a,b} \frac{(\bar{s}-s)^u}{[u]!} \bar{t}^{ab+(b-a)u-u} \cdot \overline{K}^a X^u \left(e_j^{(m)}\right) \otimes Y^n K^b \left(e_i^{(n)}\right) \\ =& \frac{1}{4k} \sum_{u,a,b} \frac{(\bar{s}-s)^u}{[u]!} \bar{t}^{ab+(b-a)u-u+2(j+u)a-2ib} \\ & \times \frac{[l_m+j+u]!}{[l_m+j]!} \frac{[l_n-i+u]!}{[l_n-i]!} \left(e_{j+u}^{(m)} \otimes e_{i-u}^{(n)}\right). \end{aligned} \tag{8.3}$$

Introduce now $v = i - u$ and $w = j + u$. We now focus on the exponent of the $\bar{t}$ factor in $C^{-1}(n,m)_{i,j}^{w,v}$. For every $u \in \mathbb{Z}$ and $x, y \in \frac{1}{2}\mathbb{Z}$ we have

$$4k = \sum_{0 \leq a,b < 4k} \bar{t}^{ab} = \sum_{0 \leq a,b < 4r} \bar{t}^{(a+u-2y)(b+u-2x)}.$$

We can rewrite the exponent of $\bar{t}$ in the coefficient of $e_{j+u}^{(m)} \otimes e_{i-u}^{(n)}$ in (8.3) as follows

$$ab + (b-a)u - u + 2a(j+u) - 2ib = (a+u-2i)(b+u+2j) - u(u+1) - 2u(j-i) + 4ij.$$

So for fixed $0 < u < k$ we have

$$\frac{1}{4k} \sum_{0 \leq a,b < 4r} \bar{t}^{ab+(b-a)u-u+2a(j+u)-2ib} = \bar{t}^{-u(u+1)-2u(j-i)+4ij}. \tag{8.4}$$

Introduce now $v = i - u$ and $w = j + u$. Thus $u = w - j$. Comparing (8.4) and (8.3) we get

$$C^{-1}(n,m)_{i,j}^{w,v} = \frac{(\bar{s}-s)^{w-j}}{[w-j]!} \frac{[l_m+w]!}{[l_m+j]!} \frac{[l_n-v]!}{[l_n-i]!} \bar{t}^{-(w-j)(w-j+1)+4ij-2(w-j)(j-i)} \delta_{v+w}^{i+j},$$

valid for $w \in \{j, ..., j+k-1\} \cap \{-l_m, ..., l_m\}, v \in \{i-k+1, ..., i\} \cap \{-l_n, ..., l_n\}$. However, by definition of l_m, l_n we have $\{j, ..., j+k-1\} \cap \{-l_m, ..., l_m\} = \{j, ..., l_m\}$ and $\{i-k+1, ..., i\} \cap \{-l_n, ..., l_n\} = \{-l_n, ..., i\}$. This finishes the proof $\square$

We next introduce a meromorphic function needed to extend the R-matrixes analytically.

Definition 8.1.1. Define $\Xi = \Xi_k \in \mathcal{M}(\mathbb{C})$ by

$$\Xi(z) = \frac{S_\kappa\left(\pi - (2z+1)\kappa\right)}{S_\kappa\left(\pi - \kappa\right)}.$$

We have the following result.

Lemma 8.1.2. *The function Ξ has poles in the negative integers and zeros in $\{k, k+1, \dots\}$, and for a positive integer m we have*

$$[m]! = \frac{t^{m(m+1)}}{(i2\sin(\kappa))^m}\Xi(m).$$

Proof. We have

$$[m]! = \prod_{j=1}^{m} \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{q^{\sum_{j=1}^{m}\frac{j}{2}}}{(2i\sin(\frac{\pi}{k}))^m}\prod_{j=1}^{m}(1-q^{-j}) = \frac{q^{\frac{m(m+1)}{4}}}{(2i\sin(\frac{\pi}{k}))^m}\prod_{j=1}^{m}(1-q^{-j}). \tag{8.5}$$

The functional equation (7.24) gives

$$1 + e^{i(\zeta+\pi)} = \frac{S_\kappa(\zeta+\pi-\kappa)}{S_\kappa(\zeta+\pi+\kappa)}.$$

Taking $\zeta = -2\pi j/k$ gives

$$(1-q^{-j}) = \frac{S_\kappa\left(\frac{-\pi(2j+1)}{k} + \pi\right)}{S_\kappa\left(\frac{-\pi(2j-1)}{k} + \pi\right)}.$$

It follows that

$$\prod_{j=1}^{m}(1-q^{-j}) = \frac{S_\kappa\left(\frac{-\pi(2m+1)}{k} + \pi\right)}{S_\kappa\left(\frac{-\pi}{k} + \pi\right)} = \frac{S_\kappa\left(-(2m+1)\kappa + \pi\right)}{S_\kappa\left(-\kappa + \pi\right)}.$$

Inserting this expression into (8.5) gives the desired result. ◻

We now work our way towards an analytic expression for $C^{\pm}(n,m)_{i,j}^{v,w}$ in terms of the indices n, m, i, j, v, w which we want to think of as complex variables.

Definition 8.1.2. Define $R_k^+ = R^+ \in \mathcal{M}(\mathbb{C}^2 \times \mathbb{C}^4)$ by

$$\begin{aligned} &R^+(y_1, y_2, z_1, z_2, z_3, z_4) = \\ &\frac{\Xi((y_1-1)/2 + z_4)\Xi_x((y_2-1)/2 - z_3)}{\Xi_x(z_4 - z_1)\Xi((y_1-1)/2 + z_1)\Xi((y_2-1)/2 - z_2)} \\ &\times t^{Q_+((y_1-1)/2, (y_2-1)/2, z_1, z_2, z_3, z_4)} \end{aligned} \tag{8.6}$$

where $Q_+ \in \mathbb{Z}[y_1, y_2, z_1, z_2, z_3, z_4]$ is given by

$$\begin{aligned} Q_+(y_1, y_2, z_1, z_2, z_3, z_4) =& 4z_1z_2 - 2(z_4 - z_1)(z_1 - z_2) - (z_4 - z_1)(z_4 - z_1 + 1) \\ &+ (y_1 + z_4)(y_1 + z_4 + 1) + (y_2 - z_3)(y_2 - z_3 + 1) \\ &- (z_4 - z_1)(z_4 - z_1 + 1) - (y_1 + z_1)(y_1 + z_1 + 1) \\ &- (y_2 - z_2)(y_2 - z_2 + 1). \end{aligned}$$

Define $R_k^- = R^- \in \mathcal{M}(\mathbb{C}^2 \times \mathbb{C}^4)$ by

$$\begin{aligned}&R^-(y_1,y_2,z_1,z_2,z_3,z_4) = \\ &(-1)^{z_3-z_2}\frac{\Xi((y_2-1)/2+z_3)\Xi((y_1-1)/2-z_4)}{\Xi(z_3-z_2)\Xi((y_2-1)/2+z_2)\Xi((y_1-1)/2-z_1)} \\ &\times t^{Q_-((y_1-1)/2,(y_2-1)/2,z_1,z_2,z_3,z_4)}\end{aligned} \tag{8.7}$$

where $Q_- \in \mathbb{Z}[y_1,y_2,z_1,z_2,z_3,z_4]$ is defined by

$$\begin{aligned}Q_-(y_1,y_2,z_1,z_2,z_3,z_4) =&(z_3-z_2)(z_3-z_2+1) - 4z_1z_2 + 2(z_3-z_2)(z_2-z_1) \\ &(y_1-z_4)(y_1-z_4+1) + (y_2+z_3)(y_2+z_3+1) \\ &-(z_3-z_2)(z_3-z_2+1) - (y_1-z_1)(y_1-z_1+1) \\ &-(y_2+z_2)(y_2+z_2+1).\end{aligned}$$

The functions R^+, R^- are analytic extensions of the coefficients of the R matrices.

Theorem 8.1.3. *For (i,j,v,w) satisfying (3.6) we have*

$$C(n,m)_{i,j}^{v,w} = R^+(n,m,i,j,v,w)\delta_{v+w}^{i+j} \tag{8.8}$$

For (i,j,v,w) satisfying (8.2) we have

$$C^{-1}(n,m)_{i,j}^{v,w} = R^-(n,m,i,j,v,w)\delta_{v+w}^{i+j}. \tag{8.9}$$

Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The pole divisor D_+ of R^+ is equal to

$$\begin{aligned}&\{z_4 - z_1 \in k+\mathbb{N}\} \cup \{(y_1-1)/2 + z_2 \in k+\mathbb{N}\} \cup \{(y_2-1)/2 - z_2 \in k+\mathbb{N}\} \\ &\cup \{(y_2-1)/2 + z_4 \in -\mathbb{N}^*\} \cup \{(y_2-1)/2 - z_3 \in -\mathbb{N}^*\} \subset \mathbb{C}^2 \times \mathbb{C}^4.\end{aligned} \tag{8.10}$$

The pole divisor D_- of R^- is equal to

$$\begin{aligned}&\{(y_2-1)/2 + z_3 \in -\mathbb{N}^*\} \cup \{(y_1-1)/2 - z_4 \in -\mathbb{N}^*\} \cup \{z_3 - z_2 \in k+\mathbb{N}\} \\ &\cup \{(y_2-1)/2 + z_1 \in k+\mathbb{N}\} \cup \{(y_1-1)/2 - z_1 \in k+\mathbb{N}\} \subset \mathbb{C}^2 \times \mathbb{C}^4.\end{aligned} \tag{8.11}$$

Proof. The equations (8.10) and (8.11) follow from Lemma 8.1.2. The proofs of (8.8) and (8.9) are straightforward computations using Lemma 8.1.2.

We begin with (8.8). It follows from the equation (3.5) that if we define the polynomial $P(X_1,X_2,X_3,X_4) \in \mathbb{Z}[X_1,X_2,X_3,X_4]$ by

$$P(X_1,X_2,X_3,X_4) = 4X_1X_2 - 2(X_4-X_1)(X_1-X_2) - (X_4-X_1)(X_4-X_1+1),$$

then

$$C(n,m)_{i,j}^{v,w} = \delta_{v+w}^{i+j}\frac{(s-\bar{s})^{w-i}}{[w-i]!}\frac{[l_n+w]!}{[l_n+i]!}\frac{[l_m-v]!}{[l_m-j]!}t^{P(i,j,v,w)}.$$

We now rewrite the quantum factorials using Lemma 8.1.2

$$
\begin{aligned}
&C(n,m)_{i,j}^{v,w} t^{-P(i,j,v,w)} \\
&= \delta_{v+w}^{i+j} \frac{(s-\bar{s})^{w-i}}{\frac{t^{(w-i)(w-i+1)}}{(2i\sin(\kappa))^{(w-i)}}\Xi(w-i)} \frac{\frac{t^{(l_n+w)(l_n+w+1)}}{(2i\sin(\kappa))^{(l_n+w)}}\Xi(l_n+w)}{\frac{t^{(l_n+i)(l_n+i+1)}}{(2i\sin(\kappa))^{(l_n+i)}}\Xi(l_n+i)} \frac{\frac{t^{(l_m-v)((l_m-v+1)}}{(2i\sin(\kappa))^{(l_m-v)}}\Xi(l_m-v)}{\frac{t^{(l_m-j)(l_m-j+1)}}{(2\sin(\kappa))^{(l_m-j)}}\Xi(l_m-j)} \\
&\delta_{v+w}^{i+j} = \frac{\Xi(l_n+w)\Xi(l_m-v)}{\Xi(w-i)\Xi(l_n+i)\Xi(l_m-j)} \\
&\times (2i\sin(\kappa))^{2(w-i)-(l_n+w)+(l_n+i)-(l_m-v)+(l_m-j)} \\
&\times t^{(l_n+w)(l_n+w+1)+(l_m-v)(l_m-v+1)} \\
&\times t^{-(w-i)(w-i+1)-(l_n+i)(l_n+i+1)-(l_m-j)(l_m-j+1)} \\
&= \delta_{v+w}^{i+j} \frac{\Xi(l_n+w)\Xi(l_m-v)}{\Xi(w-i)\Xi(l_n+i)\Xi(l_m-j)} (2i\sin(\kappa))^{-i+v-j+w} \\
&\times t^{(l_n+w)(l_n+w+1)+(l_m-v)(l_m-v+1)} \\
&\times t^{-(w-i)(w-i+1)-(l_n+i)(l_n+i+1)-(l_m-j)(l_m-j+1)}.
\end{aligned}
\tag{8.12}
$$

Put $(y_1, y_2, z_1, z_2, z_3, z_4) = (n, m, i, j, v, w)$ and compare (8.12) with (8.6). This gives (8.8). We now turn to (8.9). It follows from (8.1) that if we define $N(X_1, X_2, X_3, X_4) \in \mathbb{Z}[X_1, X_2, X_3, X_4]$ by

$$
N(X_1, X_2, X_3, X_4) = -(X_3 - X_2)(X_3 - X_2 + 1) + 4X_1X_2 - 2(X_3 - X_2)(X_2 - X_1),
$$

then we have

$$
C^{-1}(m,n)_{i,j}^{w,v} = \delta_{v+w}^{i+j} \frac{(\bar{s}-s)^{w-j}}{[w-j]!} \frac{[l_m+w]!}{[l_m+j]!} \frac{[l_n-v]!}{[l_n-i]!} \bar{t}^{N(i,j,w,v)}.
$$

Just as above, we can use Lemma 8.1.2 to get

$$
\begin{aligned}
&C^{-1}(n,m)_{i,j}^{w,v} t^{N(i,j,w,v)} \\
&= \delta_{v+w}^{i+j} \frac{(\bar{s}-s)^{w-j}}{\frac{t^{(w-j)(w-j+1)}}{(2i\sin(\kappa))^{(w-j)}}\Xi(w-j)} \frac{\frac{t^{(l_m+w)(l_m+w+1)}}{(2i\sin(\kappa))^{(l_m+w)}}\Xi(l_m+w)}{\frac{t^{(l_m+j)(l_m+j+1)}}{(2i\sin(\kappa))^{(l_m+j)}}\Xi(l_m+j)} \frac{\frac{t^{(l_n-v)((l_n-v+1)}}{(2i\sin(\kappa))^{(l_n-v)}}\Xi(l_n-v)}{\frac{t^{(l_n-i)(l_n-i+1)}}{(2i\sin(\kappa))^{(l_n-i)}}\Xi(l_n-i)} \\
&= \delta_{v+w}^{i+j} \frac{(-1)^{w-j}\Xi(l_m+w)\Xi(l_n-v)}{\Xi(w-j)\Xi(l_m+j)\Xi(l_n-i)} \\
&\times (2i\sin(\kappa))^{2(w-j)-(l_m+w)+(l_n-i)-(l_n-v)+(l_m+j)} \\
&\times t^{(l_n-v)(l_n-v+1)+(l_m+w)(l_m+w+1)} \\
&\times t^{-(w-j)(w-j+1)-(l_n-i)(l_n-i+1)-(l_m+j)(l_m+j+1)} \\
&= \delta_{v+w}^{i+j} (-1)^{w-j} \frac{\Xi(l_m+w)\Xi(l_n-v)}{\Xi(w-j)\Xi(l_m+j)\Xi(l_m-i)} (2i\sin(\kappa))^{w-i+v-j} \\
&\times t^{(l_n-v)(l_n-v+1)+(l_m+w)(l_m+w+1)} \\
&\times t^{-(w-j)(w-j+1)-(l_n-i)(l_n-i+1)-(l_m+j)(l_m+j+1)}.
\end{aligned}
\tag{8.13}
$$

Now it is a matter of putting $(y_1, y_2, z_1, z_2, z_3, z_4) = (n, m, i, j, w, v)$ and comparing (8.13) with (8.7). This finishes the proof. $\square$

The R matrix approach to the Jones polynomial

We now take a closer look at how to compute the colored Jones polynomial using the graphical calculus or, in other words, using the Reshetikhin-Turaev functor F. Recall the colored tangles $X^{\pm}_{V,W}, \cup_V, \cup^-_V, \cap_V, \cap^-_V, \phi_V, \phi^-_V$ depicted in Section 8.1.

Proposition 8.1.4. *Let δ^l_j be the Kronecker delta. We have*

$$F(\cup^-_n)(1) = \sum_j \bar{s}^{2j} e_j^{-(n)} \otimes e_j^{(n)}, \quad F(\cap^-_n)\left(e_j^{(n)} \otimes e_l^{-(n)}\right) = \delta^l_j s^{2j},$$

where $e^{-(n)}_{-l_n}, ..., e^{-(n)}_{l_n} \in V^{(n)*}$ is the dual basis of the distinguished basis of $V^{(n)}$.

Proof. Let (A, R, v) be a Ribbon Hopf algrebra, let V be a finitely generated left A module, and let μ in A be any invertible element satisfying (3.4). According to Theorem 3.6 in [KM91] we have $F(\cap^-_V)(x \otimes f) = f(\mu.x)$ for any $(x \otimes f) \in V \otimes V^*$ and $F(\cup^-_V)(1) = \sum_j e^j \otimes \mu^{-1}.e_j$ for any basis e_j of V (and dual basis e^j of V^*). Now turn to the $\tilde{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ case. We know from Theorem 3.2.10 that $\mu = K^2$. We recall from Theorem 3.2.9 that $K.e^{(n)}_j = s^j e^{(n)}_j$. □

Motivated by this, we make the following definitions.

Definition 8.1.3. A *factorized* diagram of a framed oriented link L is a blackboard framed diagram D such that every component is involved in at least one crossing, and every crossing c is of type $X^{\pm}$ with notation as in Definition 3.1.12. The *framed graph* Γ have its set of vertices V given by the crossings C, and its set of directed edges E is given by the connected components of $D \setminus C$. In accordance with the framing, an edge may contain directed kinks.

We fix from now on a factorized diagram D. We introduce some helpful notation. From now on, we write $\pi_0(L) = \pi_0$ and $m = |\pi_0|$. With notations as in Definition 3.1.12 we denote the crossings by $c^{\pm} \in C^{\pm}$, the kinks by $\varphi^{\pm} \in \Phi^{\pm}$, the caps by $\cap^{\pm} \in \bigcap^{\pm}$ and the the cups are denoted by $\cup^{\pm} \in \bigcup^{\pm}$. We introduce maps the maps $i, j, v, w : C \to E$, according to the Figure 8.1 We also have obvious inclusion maps $\cup^- \sqcup \cap^- \to E$, and $E \sqcup \Phi^+ \sqcup \Phi^- \to \pi_0(L)$,

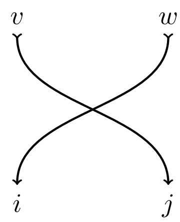


Figure 8.1: The edge maps

for which we shall not introduce any notation. Given $\lambda \in \mathrm{Col}(L, \Lambda(k))$, define $T(\lambda)$ to be the subset of $x \in \left(\frac{1}{2}\mathbb{Z}\right)^E$ with the following property. For every crossing c of type X^+ or X^- respectively, the tuple $(\lambda(i(c)), \lambda(j(c)), x(i(c)), x(j(c)), x(v(c)), x(w(c))$ satisfy (3.6) or (8.2) respectively (with $n = \lambda(i(c))$ and $m = \lambda(j(c))$) and further we have that

$$x(i(c)) + x(j(c)) - x(v(c)) - x(w(c)) = 0.$$

The set $T(\lambda)$ is independent of k.

Definition 8.1.4. Define $\mathcal{J}_\lambda(L) \in \mathcal{M}(\mathbb{C})$ by

$$\begin{aligned}
\mathcal{J}_\lambda(L)(z) = \sum_{x \in T(\lambda)} & \prod_{\cup \in \bigcup^-} (s(z))^{-2x(\cup)} \prod_{\cap \in \bigcap^-} (s(z))^{2x(\cap^-)} \\
\times & \prod_{\varphi \in \Phi^+} (t(z))^{-\lambda(\varphi)(\lambda(\varphi)+1)} \prod_{\varphi \in \Phi^-} (t(z))^{\lambda(\varphi)(\lambda(\varphi)+1)} \\
\times & \prod_{c \in C^+} R_z^+(\lambda(i(c)), \lambda(j(c)), x(i(c)), x(j(c)), x(v(c)), x(w(c))) \\
\times & \prod_{c \in C^-} R_z^-(\lambda(i(c)), \lambda(j(c)), x(i(c)), x(j(c)), x(v(c)), x(w(c))).
\end{aligned}$$

Proof of Theorem 11 . This follows from Theorem 8.1.3 and Proposition 8.1.4. It can be seen by using the graphical calculus introduced in Section and the write out the morphism in matrix notation. The fact that every component is involved in a least one crossing ensures that we can think of the basis vectors as labelled by $T(\lambda)$. □

8.2 Semi-classical analysis and complex Chern-Simons theory

We define the semi-classical approximation Φ from Conjecture 2, and compare it with Yoon's potential to support the conjecture. The idea is that (1.29) should follow from this comparison and Theorem 2.2.6.

We define for each crossing c the map $\pi_c : \mathbb{C}^{\pi_0} \times \mathbb{C}^E \to \mathbb{C}^6$ given by

$$\pi_c = (y_{i(c)}, y_{j(c)}, z_{i(c)}, _{j(c)}, z_{v(c)}, z_{w(c)}).$$

Motivated by Theorem 7.2.3 and Proposition and Theorem 11 we introduce the following function.

Definition 8.2.1. Define $\Phi_+ \in \mathcal{M}(\mathbb{C}^2 \times \mathbb{C}^4)$ by

$$\begin{aligned}
\Phi_+(y_1, y_2, z_1, z_2, z_3, z_4) = & -\operatorname{Li}_2\left(e^{-\pi i(y_1+2z_4)}\right) - \operatorname{Li}_2\left(e^{-\pi i(y_2-2z_3)}\right) \\
& + \operatorname{Li}_2\left(e^{-\pi i(2z_4-2z_1)}\right) + \operatorname{Li}_2\left(e^{-\pi i(y_1+2z_1)}\right) \\
& + \operatorname{Li}_2\left(e^{-\pi i(y_2-2z_2)}\right) - (i\pi)^2 \widetilde{Q}_+(y_1, y_2, z_1, z_2, z_3, z_4)
\end{aligned}$$

Define $\Phi_- \in \mathcal{M}(\mathbb{C}^2 \times \mathbb{C}^4)$ by

$$\begin{aligned}
\Phi_-(y_1, y_2, z_1, z_2, z_3, z_4) = & -\operatorname{Li}_2\left(e^{-\pi i(y_2+2z_3)}\right) - \operatorname{Li}_2\left(e^{-\pi i(y_1-2z_4)}\right) \\
& + \operatorname{Li}_2\left(e^{-\pi i(2z_3-2z_2)}\right) + \operatorname{Li}_2\left(e^{-\pi i(y_2+2z_2)}\right) \\
& + \operatorname{Li}_2\left(e^{-\pi i(y_1-2z_1)}\right) + (i\pi)^2 \widetilde{Q}_-(y_1, y_2, z_1, z_2, z_3, z_4).
\end{aligned}$$

Define the *formal semiclassical approximation* of $\mathcal{J}$ to be the meromorphic multivalued function $\Phi \in \mathcal{M}(\mathbb{C}^{\pi_0} \times \mathbb{C}^E)$ given by

$$\Phi = \sum_{c \in C^+} \Phi_+ \circ \pi_c + \sum_{c \in C^-} \Phi_- \circ \pi_c.$$

Let R denote the set of connected components of $\mathbb{R}^2 \setminus D$. Recall that these are the regions in Yoon's terminology [Yoo18]. We introduce for each crossing c a complex variable u_c, and for each region r we introduce a complex variable w_r. This gives coordinates on $\mathbb{C}^C$ and $\mathbb{C}^R$. Let b denote the number of regions.

Proposition 8.2.1. *Let $K : \mathbb{C}^{\pi_0} \times \mathbb{C}^E \to \mathbb{C}^C$ be the map defined by*

$$u_c = z_{v(c)} + z_{w(c)} - z_{i(c)} - z_{j(c)}.$$

We have

$$K\left(\mathbb{C}^{\pi_0} \times \mathbb{C}^E\right) \subset \{\sum_{c \in C} u_c = 0\}.$$

Proof. This follows easily from the fact that every edge is associated with exactly two crossings. □

Definition 8.2.2. Define $I : \mathbb{C}^{\pi_0} \times \mathbb{C}^R \to \mathbb{C}^{\pi_0} \times \mathbb{C}^E$ by

$$z_e = w_{r(e)} - w_{l(e)}, \quad y_e = 2m_e,$$

where $w_{r(e)}$ is the region to the right of z_e and $w_{l(e)}$ is the region to the left of z_e. Define $I_0 : C^2 \times \mathbb{C}^4 \to \mathbb{C}^2 \times \mathbb{C}^4$ by

$$I_0(m_1, m_2, w_l, w_r, w_d, w_0) = \left(\frac{m_1}{2}, \frac{m_2}{2}, w_l - w_d, w_d - w_r, w_l - w_o, w_o - w_r\right).$$

Recall the linear map K introduced above.

Theorem 8.2.2. *Let $\Delta : \mathbb{C} \to \mathbb{C}^R$ be the linear map given by $\Delta(z) = (z, ..., z)$. Let $C_0 = \mathbb{C}^{\pi_0} \times \Delta(\mathbb{C}) \subset \mathbb{C}^{\pi_0} \times \mathbb{C}^R$. The map I induces a linear isomorphism*

$$I : \mathbb{C}^{\pi_0} \times \mathbb{C}^R / C_0 \xrightarrow{\sim} \mathrm{Ker}(K).$$

Let us now compare the $I^*(\Phi)$ with $\mathbb{W}_0$. Define $I_0 : C^2 \times \mathbb{C}^4 \to \mathbb{C}^2 \times \mathbb{C}^4$ by

$$I_0(m_1, m_2, w_l, w_r, w_d, w_0) = \left(\frac{m_1}{2}, \frac{m_2}{2}, w_l - w_d, w_d - w_r, w_l - w_o, w_o - w_r\right).$$

For every crossing c we have

$$I_0 \circ \tilde{\pi}_c = \pi_c \circ I.$$

Using the functional equations for the dilogarithm stated in Theorem 2.2.4, one can show

$$\mathbb{W}_+ - \Phi_+ \circ I_0 = 4\pi^2(m_l + w_l - w_d)(m_r + w_r - w_d) + (i\pi)^2 Y_+$$
$$\mathbb{W}_- - \Phi_- \circ I_0 = 4\pi^2(w_l - w_d - m_l)(w_r - w_d - m_r) - (i\pi)^2 Y_-$$

where $Y_\pm$ is a polynomial of degree at most 2.

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