Branching laws for representations of real reductive groups of rank one

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PhD Dissertation

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To my mother.

Abstract

Let G be a real reductive group and $G' \subseteq G$ be a reductive subgroup. This thesis is concerned with branching laws for the restriction of G-representations to the subgroup G'. We consider the space $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ of *symmetry breaking operators* where π is a principal series representation of G and τ is a principal series representation of G'. We restrict ourselves to the cases where $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ and where G and G' are of real rank one. We extend previous work by Kobayashi–Speh to the groups

$$(G, G') = \begin{cases} (\mathrm{U}(1, n+1), \mathrm{U}(1, m+1) \times F), & F < \mathrm{U}(n-m), \\ (\mathrm{Sp}(1, n+1), \mathrm{Sp}(1, m+1) \times F), & F < \mathrm{Sp}(n-m), \\ (F_{4(-20)}, \mathrm{Spin}_0(1, 8) \times F), & F < \mathrm{Spin}(8), \end{cases}$$

classifying all symmetry breaking operators between spherical principals series representations in terms of their distribution kernels. The symmetry breaking operators classified belong in most cases to one meromorphic family or are given by its residues. In some special cases additional sporadic symmetry breaking operators occur which we classify explicitly.

For the pair $(G, G') = (\mathrm{U}(1, n+1), \mathrm{U}(1, n))$ we construct a meromorphic family of symmetry breaking operators between scalar principle series representations from the meromorphic family of operators in the spherical case and we determine the residues. We obtain a Plancherel formula for the unitary scalar principal series which is given in terms of symmetry breaking operators, by applying Mackey theory to the open G'-orbit in the real flag variety G/P, where P is a minimal parabolic subgroup of G. We use an analytic continuation procedure to extend the Plancherel formula for the unitary principal series towards the complementary series and towards points of reducibility, obtaining direct integral decompositions and explicit Plancherel formulas for all unitary representations inside the scalar principle series. This includes complementary series, relative discrete series and unitary representations with highest or lowest weight vector. Additional discrete spectra are constructed as residues of symmetry breaking operators.

For the pair $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$ we obtain direct integral decompositions and explicite Plancherel formulas for all unitary representations inside principal series representations of G restricted to G', which are induced from the $\mathrm{O}(n)$-representation $\bigwedge^p(\mathbb{C}^n)$ and characters. We use a similar strategy as before, making use of the results on symmetry breaking operators in this case by Kobayashi–Speh. The representations include complementary series and all unitary G-representations with non-trivial $(\mathfrak{g}, K)$-cohomology and discrete spectrum occurs again as residues of symmetry breaking operators.

Resumé

Lad G være en reel reduktiv gruppe og $G' \subseteq G$ en reduktiv undergruppe. Denne afhandling omhandler branching laws for restriktionen af G-repræsentationer til undergruppen G'. Vi betragter rummet $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ af *symmetry breaking operatorer* hvor π er en principalrække-repræsentation af G og τ er en principalrække-repræsentation af G'. Vi nøjes med at betragte tilfældet hvor $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ og hvor G samt G' har reel rang en. Vi udvider arbejdet af Kobayashi–Speh til grupperne

$$(G, G') = \begin{cases} (\mathrm{U}(1, n+1), \mathrm{U}(1, m+1) \times F), & F < \mathrm{U}(n-m), \\ (\mathrm{Sp}(1, n+1), \mathrm{Sp}(1, m+1) \times F), & F < \mathrm{Sp}(n-m), \\ (F_{4(-20)}, \mathrm{Spin}_0(1, 8) \times F), & F < \mathrm{Spin}(8), \end{cases}$$

hvor vi klassificerer alle symmetry breaking operatorer mellem sfæriske principalrække-repræsentationer ved deres distributionskerner. De symmetry breaking operatorer der bliver klassificeret tilhører, i de fleste tilfælde, til en meromorf familie eller er givet ved deres residier. I nogle specialtilfælde forekommer der yderligere sporadiske symmetry breaking operatorer som vi klassificerer eksplicit.

For parret $(G, G') = (\mathrm{U}(1, n+1), \mathrm{U}(1, n))$ konstruerer vi en meromorf familie af symmetry breaking operatorer mellem skalar principalrække-repræsentationen fra den meromorfe familie af operatorer i det sfæriske tilfælde ,og vi bestemmer dets residier. Vi opnår en Plancherel-formel for den unitære skalar principalrække, som er udtrykt ved symmetry breaking operatorer ved at bruge Mackey teori på den åbne G'-bane i den reelle flag-varietet G/P, hvor P er den minimale parabolske undergruppe i G. Vi bruger analytisk fortsættelse til at udvide Plancherel-formlen for den unitære principalrække mod komplementærrækken og mod reducibilitets punkter, hvorved at opnå direkte integral dekompositioner og eksplicitte Plancherel-formler for alle unitære repræsentationer i skalar principalrækken. Dette inkluderer komplementærerækker, relative diskrete rækker og unitære repræsentationer med højest og lavest vægt vektor. Yderligere diskrete spektre er konstrueret som residier af symmetry breaking operatorer.

For parret $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$ opnår vi en direkte integral dekomposition og eksplicitte Plancherel-formler for alle unitære repræsentationer i principalrække-repræsentationen af G restringeret til G', som er induceret fra $\mathrm{O}(n)$-repræsentationen $\bigwedge^p(\mathbb{C}^n)$ og karakterer. Vi bruger en lignende strategi som tidligere, ved at gøre brug af resultater om symmetry breaking operatorer, i dette tilfælde fra Kobayashi–Speh. Repræsentationerne inkluderer komplementærrækken og alle unitærer G-repræsentationer med ikke-trivielle $(\mathfrak{g}, K)$-cohomologi. Diskret spektrum forekommer igen som residier af symmetry breaking operatorer.

Contents

Preface

This thesis concludes my studies as a PhD student at the Departments of Mathematics at Friedrich-Alexander-University Erlangen-Nürnberg in the first two years and at Aarhus University in the final year. My studies have been supervised by Jan Frahm and funded by the DFG (project 325558309) throughout.

The thesis consists of an introduction and three papers, all concerned with branching laws for representations of real reductive groups of rank one:

- Paper A: Symmetry breaking operators for real reductive groups of rank one.
- Paper B: Branching laws for unitary representations of $\mathrm{U}(1, n+1)$ in the scalar principal series.
- Paper C: Branching of unitary $\mathrm{O}(1, n+1)$-representations with non-trivial $(\mathfrak{g}, K)$-cohomology.

Paper A has been published in the Journal of Functional Analysis and is co-authored with Jan Frahm (doi:10.1016/j.jfa.2020.108568). Up to typesetting the version contained here is unchanged from the published version. The paper contains the results of the first half of my PhD studies. Paper B is a preprint and co-authored with Jan Frahm, as well. Both of these papers are mostly written by myself and most of the results and calculations have been obtained by me with help and guidance of my supervisor. Paper C is a preprint too. The latter two papers contain the main results of the second half of my PhD studies and are in preparation for submission. All papers are self contained and include their own introductions and references.

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Clemens Weiske
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Introduction

Let G be a group and $G' \subseteq G$ be a subgroup. If π is a representation of G, it naturally defines a representation of the subgroup by restriction. The study of branching laws is concerned with understanding representations of the ambient group G as representations of a subgroup. While some properties like for example unitarity are preserved by restriction, other properties are lost. If π is an irreducible representation of G, the restriction $\pi|_{G'}$ is in general not irreducible as a G'-representation anymore. For example, if π is an irreducible unitary representation of a compact Lie group G, π is finite dimensional and the restriction

$$\pi|_{G'} \cong \bigoplus_{\tau \in \widehat{G'}} m(\pi, \tau)\tau$$

decomposes into a direct sum of irreducible G'-representations. Here $\widehat{G}'$ denotes the *unitary dual* of G', i.e. the set of equivalence classes of irreducible unitary (i.e. here finite dimensional) representations of G' and $m(\pi, \tau) \in \mathbb{Z}_{\geq 0}$ is the multiplicity of τ in π which one needs to determine explicitly to solve a branching problem. For classical compact Lie groups such branching laws are known due to Weyl, Littlewood, Murnaghan, Kostant and others going back up to the 1930s. If we consider real reductive non-compact Lie groups, irreducible representations are typically infinite dimensional and restrictions do not admit a direct sum decomposition in general. Considering an irreducible unitary representation π of a reductive Lie group, while not admitting a direct sum decomposition, the restriction admits a decomposition into a direct integral

$$\pi|_{G'} \cong \int_{\widehat{G'}}^{\oplus} m(\pi, \tau)\tau \, d\mu_\pi(\tau)$$

with possibly infinite multiplicities $m(\pi, \tau)$ and a certain measure $d\mu_\pi$ on the unitary dual $\widehat{G}'$ of the subgroup. Proving a branching law includes determining the measure, its support and the multiplicities concretely. In general the support of the measure might contain both a continuous and a discrete part. Especially in the cases with mixed spectrum there has been no systematical treatment of branching laws yet, while many special cases have been studied using various analytical methods (see [Kob19], [ØS19], [MO15], [ØV04], [Rep78] for some examples). Considering non-unitary representations, even a direct integral decomposition does not exist anymore in general but restricted representations can still be studied systematically. In [Kob15] Kobayashi proposed the study of branching laws for smooth admissible representations of real reductive groups by *symmetry breaking operators*. For this let G be real reductive and G' be a reductive subgroup, π be an irreducible smooth admissible G-representation and τ be an irreducible smooth admissible

G'-representation. A symmetry breaking operator is an element of the space

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$$

of continuous linear G'-intertwining operators from the restriction of π into τ and can be thought of as realizing τ inside $\pi|_{G'}$ as a quotient. Then defining the multiplicity

$$m(\pi, \tau) := \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$$

to be the dimension of the space of symmetry breaking operators, this definition is consistent with the multiplicities in the usual sense for compact Lie groups. In this setting branching problems have received a lot of attention recently (see for example [KS15], [KS18], [Cle16], [Cle17], [Möl17], [FØ19b], [FW19], [SZ12]), concerning both special cases of groups and representations as well as systematic results.

This thesis concerns branching laws in both categories of irreducible unitary representations via direct integral decompositions (Paper B and Paper C) and of irreducible smooth admissible representations via symmetry breaking operators (Paper A). We will lay out in this introduction how we use results in the latter setting to prove decompositions in the unitary setting after a brief overview and motivation of our choice of groups and representations. Moreover we give a brief summary of the main results of the papers. For a detailed summary we want to refer the reader to the respective introductions of the papers contained in this thesis.

Real reductive groups of rank one

Let G be a real reductive Lie group and G' be a reductive subgroup. Contrary to the case of compact Lie groups the multiplicites of $m(\pi, \tau)$ as dimensions of spaces of symmetry breaking operators are not necessarily finite. In [KO13] Kobayashi–Oshima showed that assuming G and G' to be defined algebraically over $\mathbb{R}$, we have

$$m(\pi, \tau) = \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$$

for all irreducible smooth admissible representations π of G and τ of G' if and only if the homogeneous space $(G \times G')/\operatorname{diag}(G')$ is a real spherical space. Such pairs of real reductive groups are called *strongly spherical* or *finite multiplicity pairs* and due to their nice property they offer a natural setting for studying symmetry breaking. Strongly spherical real reductive pairs are classified by Kobayashi–Matsuki [KM14] for cases of symmetric pairs and in all generality by Knop–Krötz–Pecher–Schlichtkrull [KKPS19]. By this classification, up to central extension and connected components, the irreducible strongly spherical real reductive pairs with G of real rank one and G' non-compact are given by

$$(G, G') = \begin{cases} (\mathrm{O}(1, n+1), \mathrm{O}(1, m+1) \times F), \\ (\mathrm{U}(1, n+1), \mathrm{U}(1, m+1) \times F), \\ (\mathrm{Sp}(1, n+1), \mathrm{Sp}(1, m+1) \times F), \\ (F_{4(-20)}, \mathrm{Spin}_0(1, 8) \times F), \end{cases} \tag{1}$$

with $0 \leq m \leq n$ and F a certain (possibly trivial) compact factor (see for example Fact II in Paper A for the precise property). Especially the pair $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$

has been in the focus of recent developments in branching problems both in the smooth and in the unitary setting. In [KS15] Kobayashi–Speh classified symmetry breaking operators between the spherical principal series of $(\mathrm{O}(1,n+1),\mathrm{O}(1,n))$. In [KS18] they studied the vector valued cases and gave a full classification of symmetry breaking operators between principal series representations induced from the $\mathrm{O}(n)$-representation $\bigwedge^p(\mathbb{C}^n)$. In [FØ19a] Frahm–Ørsted classified symmetry breaking operators between the spherical principal series of $(\mathrm{O}(1,n+1),\mathrm{O}(1,n))$ and $(\mathrm{U}(1,n+1),\mathrm{U}(1,n))$ via intertwining operators between underlying $(\mathfrak{g},K)$-modules. In [MO15] Möllers–Oshima proved direct integral decompositions for all unitary $\mathrm{O}(1,n+1)$-representations contained in the spherical principal series restricted to $\mathrm{O}(1,m+1)\times\mathrm{O}(n-m)$.

Principal series representations

Let again G be a real reductive group and $P\subseteq G$ be a minimal parabolic subgroup of G with Langlands decomposition $P=MAN$. Let (ξ,V) be a finite dimensional representation of M and $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ be an element of the complexified dual of the Lie algebra of A. We extend (ξ,V) to a representation of P given by $(\xi\otimes e^{\lambda}\otimes\mathbf{1},V_{\xi,\lambda})$, by letting A act by e^{λ} and N trivially. Then the principal series representation $\pi_{\xi,\lambda}$ of G is given by left regular action on the smooth sections of the homogeneous bundle $\mathcal{V}_{\xi,\lambda}$

$$G\times_P V_{\xi,\lambda+\rho}\to G/P$$

over the real flag variety G/P. Here ρ denotes the half sum of positive roots as usual, such that we obtain a unitary representation for λ purely imaginary. Submodules of principal series representations exhaust the equivalence classes of irreducible smooth admissible representations. By the Casselman Embedding Theorem every irreducible smooth admissible representation of a reductive group occurs in a principal series representation as a submodule, which implies that taking the closures of all unitarizable representations occurring in principal series representations as submodules exhaust the unitary dual. Principal series representations have a further advantage, which is the holomorphic dependence on the induction parameter λ in the compact picture.

Symmetry breaking operators

Let G' be a reductive subgroup of G and $P'=M'A'N'$ be a minimal parabolic subgroup of G'. Similarly as before, for a finite dimensional M'-module (η,W) and $\nu\in(\mathfrak{a}'_{\mathbb{C}})^*$ in the complexified dual of the Lie algebra of A' we define the principal series representation $\tau_{\eta,\nu}$ on the smooth sections of the homogeneous bundle $\mathcal{W}_{\eta,\nu}$

$$G'\times_{P'} W_{\eta,\nu+\rho'}\to G/P,$$

associated to the P'-representation $(\eta\otimes e^{\nu+\rho'}\otimes\mathbf{1},W_{\eta,\nu+\rho'})$ on W and where ρ' is again the half sum of positive A'-roots. Consider the space of symmetry breaking operators between principal series representations

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'},\tau_{\eta,\nu}).$$

These operators can be considered as distribution sections of the dual bundle $\mathcal{V}_{\xi^*,-\lambda}$ over G/P with a certain P'-invariance (see part I of Paper A for more details). In this way symmetry breaking operators can be classified in terms of distribution kernels, which satisfy certain differential equations, which encode the M', A' and N' invariance. They are naturally supported on a union of P' orbits in G/P. By the nature of these equations symmetry breaking operators often come in families which depend meromorphically, or holomorphically if suitably normalized, on the induction parameters λ and ν. For example the classification of symmetry breaking operators between spherical principal series of $(\mathrm{O}(1,n+1),\mathrm{O}(1,n))$ in [KS15] consists only of one meromorphic family and its residues. For spherical principal series representations of irreducible strongly spherical real reductive pairs, the existence of a holomorphic family of symmetry breaking operators for every open P'-orbit in G/P is proven in [Möl17] and these families span the space of symmetry breaking operators for generic induction parameters λ and ν.

Knapp–Stein intertwining operators

Let $T_{\xi,\lambda}$ denote the standard Knapp–Stein intertwining operators for $\pi_{\xi,\lambda}$. These are meromorphic families of continuous linear G-intertwining operators

$$T_{\xi,\lambda} : \pi_{\xi,\lambda} \to \pi_{\tilde{w}_0\xi,\tilde{w}_0\lambda},$$

$$T_{\xi,\lambda}f(x) = \int_{\overline{N}} f(x\tilde{w}_0\bar{n})d\bar{n},$$

which can be normalized to depend holomorphically on λ. Here $\overline{N}$ is the nilradical of the parabolic opposite to P and $\tilde{w}_0$ represents the longest Weyl group element. Let τ be an irreducible smooth admissible representation of G'. Clearly a symmetry breaking operator

$$A \in \mathrm{Hom}_{G'}(\pi_{\tilde{w}_0\xi,\tilde{w}_0\lambda}|_{G'},\tau)$$

defines a symmetry breaking operator

$$A \circ T_{\xi,\lambda} \in \mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'},\tau)$$

by composition and if A belongs to a meromorphic family we again obtain a meromorphic family. In the case where there exists a unique open P' orbit in G/P, there has to be a functional equation. Determining a functional equation explicitly is more difficult. Knapp–Stein intertwiners play an important role when dealing with unitarizable representations inside principals series representations. Unitarizable principal series representations outside the unitary principal series (complementary series) and unitarizable composition factors of principle series representations can be unitarized by an inner product which is given by an L^2-pairing and by composition with a standard Knapp–Stein intertwining operator in one argument. Allowing induction parameters λ also for non-unitarizable values, one obtains holomorphic families of bilinear pairings, which unitarize the unitarizable representations inside the principal series for the corresponding induction parameters.

Harmonic analysis and symmetry breaking

Let (G, G') be a strongly spherical real reductive pair. Then since $(G \times G')/\operatorname{diag}(G')$ is real spherical, a minimal parabolic subgroup P' of G' acts with an open orbit on the real flag variety G/P. It follows that the full group G' also acts with an open orbit on G/P. Let $\mathcal{O}$ be such an open G'-orbit. Fix a base point $x_0 \in \mathcal{O}$ and let H be the stabilizer subgroup of x_0 in G'. Then we obtain a natural continuous linear G'-map

$$\Phi : \pi_{\xi,\lambda}|_{G'} \to C^\infty(G'/H, \mathcal{V}^{x_0}_{\xi,\lambda}),$$

by restricting elements of $\pi_{\xi,\lambda}$ to the open orbit $\mathcal{O}$ and letting G' act on the base point. Here $\mathcal{V}^{x_0}_{\xi,\lambda}$ is the homogeneous bundle

$$G' \times_H V^{x_0}_{\xi,\lambda+\rho}|_H \to G'/H$$

over G'/H associated to the representation $V_{\xi,\lambda+\rho}$ twisted by x_0 and restricted to H. In the same way as the principal series representations itself, this map can be understood to depend holomorphically on λ. Let λ be purely imaginary such that $\pi_{\xi,\lambda}$ is unitarizable and denote its unitary closure by $\hat{\pi}_{\xi,\lambda}$. Assuming $C^\infty(G'/H, \mathcal{V}^{x_0}_{\xi,\lambda})$ to be unitarizable with closure $L^2(G'/H, \mathcal{V}^{x_0}_{\xi,\lambda})$ and Φ extending to a unitary map

$$\hat{\pi}_{\xi,\lambda}|_{G'} \to L^2(G'/H, \mathcal{V}^{x_0}_{\xi,\lambda}),$$

by Mackey theory the decomposition of the unitary principal series is reduced to finding a Plancherel formula for the L^2-sections of a homogeneous vector bundle over a homogeneous G'-space for each open G' orbit in G/P. Such Plancherel formulas are known in many cases, for example for symmetric spaces. Assume we have a direct integral decomposition

$$L^2(G'/H, \mathcal{V}^{x_0}_{\xi,\lambda}) \cong \int_{\widehat{G'}}^{\oplus} m(\tau)\tau \, d\mu(\tau)$$

into unitary G'-representations. Then the smooth vectors of this representations being nuclear locally convex topological vector spaces assures the existence of continuous linear intertwining operators

$$A_\tau : C^\infty(G'/H, \mathcal{V}^{x_0}_{\xi,\lambda}) \to \tau^\infty$$

for almost all τ in the support of $d\mu$ by the Gelfand–Kostyuchenko method (see for example [Li18, Theorem 3.3.4]). In the case of Riemannian symmetric spaces for example, these are given by the well known Helgason Fourier transforms. By composition with the map Φ we obtain symmetry breaking operators

$$A_\tau \circ \Phi \in \operatorname{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau^\infty).$$

Embedding τ^∞ into a principal series by Casselman's Theorem we obtain symmetry breaking operators between principal series representations in particular. If A_τ depends meromorphically (or not at all) on λ we obtain meromorphic families in particular. Under these assumptions the knowledge of

- explicit Plancherel formulas for the sections of bundles over the open G'-orbits in terms of the transforms A_τ,

- the meromorphic structure and residues of the symmetry breaking operators $A_\tau \circ \Phi$,
- functional equations for the composition of the symmetry breaking operators and the standard Knapp–Stein intertwining operators occurring in the scalar products of unitarizable representations,

gives a starting point to prove Plancherel formulas for unitary representations inside principal series representations which do not belong to the unitary principal series by analytic continuation. We remark that for the pairs of groups in (1) H is compact and the assumptions made in this section hold (this can be easily seen by generalizing the corresponding results of Paper B and Paper C. See for example Lemma 1.3 and Lemma 6.1 in Paper B). We apply this procedure in Paper B and Paper C in special cases and obtain additional discrete spectra as residues of symmetry breaking operators. In this sense classifying symmetry breaking operators in the smooth admissible case (as in Paper A) directly contributes to decomposing restrictions of unitary representations into direct integrals.

Summary of the main results

In Paper A we obtain a full classification and explicit description of symmetry breaking operators between spherical principal series representations of the strongly spherical real reductive pairs given in (1). This can be seen as a generalization of [KS15], where the case $(\mathrm{O}(1,n+1),\mathrm{O}(1,n))$ is treated and which easily generalizes to all strongly spherical real reductive pairs (G,G') with $G=\mathrm{O}(1,n+1)$ and G' non-compact. We therefore exclude these cases in the article but they can easily be treated with the methods presented in Paper A and [KS15]. In almost all cases the symmetry breaking operators are given by a meromorphic family and its residues and we obtain the residue formulas. Moreover we obtain functional equations for the composition with the standard Knapp–Stein intertwining operators. In contrast to the orthogonal case, for G unitary or symplectic *sporadic* symmetry breaking operators occur, which do not come from a meromorphic family of symmetry breaking operators. We classify the sporadic operators explicitly.

Paper B is concerned with the pair $(G,G')=(\mathrm{U}(1,n+1),\mathrm{U}(1,n))$. We obtain direct integral decompositions for the restriction of all unitary representations inside the scalar principal series representations of G. This includes complementary series, unitary representations with highest or lowest weight vector as well as other relative discrete series representations. The proof uses the ideas outlined in this introduction. We therefore study holomorphic families of symmetry breaking operators between the scalar principal series of G and G'. We obtain these families from the ones between the spherical principal series constructed in Paper A by the translation principle in the sense of [FØ19b]. We study the meromorphic structure of the families and their residues and obtain functional equations for the composition with the standard Knapp–Stein intertwining operators. In this case there is a unique open G'-orbit isomorphic to $G'/\mathrm{U}(n)$ which is a circle bundle over the Riemannian symmetric space G'/K'. This allows us to prove Plancherel formulas for all unitary representations contained in the scalar principal series by an analytic continuation procedure applied to the Plancherel formula for the unitary principal series which we deduce from the Plancherel formula for line bundles over the Riemannian symmetric space G'/K' in [Shi94]. Moreover we obtain a full description of the composition series and unitarizability of composition factors of the scalar principal series representations of G.

Paper C is concerned with the pair $(G,G') = (\mathrm{O}(1,n+1), \mathrm{O}(1,n))$ and we obtain direct integral decompositions for all unitary representations inside the principal series representations of G which are induced from the $\mathrm{O}(n)$-representation $\bigwedge^p(\mathbb{C}^n)$ and characters. This includes complementary series as well as all irreducible unitary $\mathrm{O}(1,n+1)$-representations with non-trivial $(\mathfrak{g},K)$-cohomology, which occur as quotients at the end of the complementary series. In the cases $p=0,n$ this includes additional relative discrete series representations. The outline of the proof is similar to Paper B and we make use of the classification and explicit meromorphic structure of the symmetry breaking operators classified and studied in [KS18].

Outlook

The methods of proof in Paper B and Paper C can easily be generalized to other types of principal series representations of $(G,G') = (\mathrm{O}(1,n+1), \mathrm{O}(1,n)), (\mathrm{U}(1,n+1), \mathrm{U}(1,n))$ as soon as there is enough information about the needed symmetry breaking operators and a Plancherel formula for the bundle over the unique open orbit. In the case of orthogonal groups, the explicit Plancherel theorems are essentially in [Cam18] and in [KS18] holomorphic families of symmetry breaking operators for all principal series representations of $(G,G') = (\mathrm{O}(1,n+1), \mathrm{O}(1,n))$ are constructed. Explicit functional equations and residue formulas for these operators allow us to prove direct integral decompositions for the restriction of all unitary representations contained in principals series representations, i.e. all irreducible unitary G-representations by the same analytic continuation procedure as in Paper C. Obtaining these results is quite hard since the corresponding results for the symmetry breaking operators used in Paper C already require challenging analysis. But with the results of [KS18] and knowledge about the location and nature of unitarizable composition factors and complementary series inside principal series representations of G, we expect to be able to generalize the results to some extent to all unitary G-representations. We hope to work on this problem in the future.

Another case which can be treated in a similar way is the case of symplectic groups. For $(G,G') = (\mathrm{Sp}(1,n+1), \mathrm{Sp}(1,n))$, G' acts on G/P with a unique open orbit with stabilizer $\mathrm{Sp}(n)$ (this can easily be proven using the same proof as of Lemma 1.3 in Paper B, using the notation of Paper A), such that we obtain an $\widehat{\mathrm{Sp}}(1)$-fibration over the Riemannian symmetric space G'/K'. Hence for the spherical principals series, Plancherel formulas for the corresponding bundles are known (see for example [VDP99]) and Paper A yields the starting point for constructing families of symmetry breaking operators via the translation principle.

In view of generalizing the methods of Paper B and Paper C to higher rank groups, an interesting problem would also be to study the structure of the G'-orbits in G/P for all strongly spherical real reductive pairs and to single out the settings where enough harmonic analysis is known to decompose unitary representations via symmetry breaking operators.

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Paper A

Symmetry breaking operators for real reductive groups of rank one

Jan Frahm and Clemens Weiske

Abstract

For a pair of real reductive groups $G' \subset G$ we consider the space $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ of intertwining operators between spherical principal series representations π of G and τ of G', also called *symmetry breaking operators*. Restricting to those pairs (G, G') where $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ and G and G' are of real rank one, we classify all symmetry breaking operators explicitly in terms of their distribution kernels. This generalizes previous work by Kobayashi–Speh for $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$ to the reductive pairs

$$(G, G') = (\mathrm{U}(1, n+1; \mathbb{F}), \mathrm{U}(1, m+1; \mathbb{F}) \times F), \qquad \mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O} \text{ and } F < \mathrm{U}(n-m; \mathbb{F}).$$

In most cases, all symmetry breaking operators can be constructed using one meromorphic family of distributions whose poles and residues we describe in detail. In addition to this family, there may occur some sporadic symmetry breaking operators which we determine explicitly.

Introduction

Group representations occur in various branches of mathematics and mathematical physics. One particular problem in group representation theory is the decomposition of restricted representations. Given an irreducible representation π of a group G and a subgroup $G' \subseteq G$, the restricted representation $\pi|_{G'}$ is in general not irreducible anymore, so one is interested in a decomposition of $\pi|_{G'}$ into irreducible representations of G'.

In the context of finite or compact groups, an irreducible representation π is finite-dimensional and its restriction $\pi|_{G'}$ decomposes into an algebraic direct sum of irreducible representations τ of G' which occur with finite multiplicities $m(\pi, \tau) \in \mathbb{Z}_{\geq 0}$.

In the context of real reductive groups, irreducible representations are typically infinite-dimensional and their restrictions do no longer decompose into direct sums of irreducibles. However, there still exists a notion of multiplicity which describes the occurrence of a given irreducible representation τ of G' inside a restricted representation $\pi|_{G'}$ as a quotient, namely

$$m(\pi, \tau) = \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau).$$

The study of these multiplicities turns out to be very fruitful and has recently attracted a lot of attention.

In contrast to the case of finite or compact groups, the multiplicities for real reductive groups are not necessarily finite. Therefore, Kobayashi [Kob15] has proposed to single out settings where only finite multiplicities occur.

Fact I (see [KO13, Theorem C])**.** *Assume that G and G' are defined algebraically over $\mathbb{R}$. Then*

$$\dim \operatorname{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$$

for all smooth admissible representations π of G and τ of G' if and only if the pair (G, G') is strongly spherical, i.e. the homogeneous space $(G \times G')/\operatorname{diag}(G')$ is real spherical.

For a fixed strongly spherical reductive pair (G, G') one is now interested in determining the multiplicities $m(\pi, \tau)$. It turns out that for some applications such as unitary branching laws (see e.g. [MO15]) or applications to partial differential equations or analytic number theory (see e.g. [FS18, MØZ16b, MØ17]) the mere knowledge of the multiplicities is not sufficient and one needs an explicit description of the operators in

$$\operatorname{Hom}_{G'}(\pi|_{G'}, \tau),$$

so-called *symmetry breaking operators* as advocated by Kobayashi in his ABC program (see [Kob15]). This is the main objective of this paper.

Strongly spherical real reductive pairs have recently been classified in the case of symmetric pairs by Kobayashi–Matsuki [KM14] and in general by Knop–Krötz–Pecher–Schlichtkrull [KKPS19] (The complete list can also be found in [Möl17].) In what follows we restrict to the case where G is of real rank one. Up to central extensions, every simple real reductive group G of rank one is of the form $\mathrm{U}(1, n+1; \mathbb{F})$ with $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $n \geq 0$ or $\mathbb{F} = \mathbb{O}$ and $n = 1$, where we use the interpretation $\mathrm{U}(1, 2; \mathbb{O}) = F_{4(-20)}$.

Fact II (see [KM14] and [KKPS19])**.** *Up to central extensions, the irreducible strongly spherical real reductive pairs (G, G') with G of real rank one and G' non-compact are given by*

$$(G, G') = (\mathrm{U}(1, n+1; \mathbb{F}), \mathrm{U}(1, m+1; \mathbb{F}) \times F)$$

with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $0 \leq m \leq n$ or $\mathbb{F} = \mathbb{O}$ and $0 \leq m \leq n = 1$ and $F < \mathrm{U}(n-m; \mathbb{F})$ such that the action of $\mathrm{U}(1; \mathbb{F}) \times F$ on $\mathbb{F}^{n-m}$ by $(a, c) \cdot z = azc^{-1}$ has an open orbit on the unit sphere in $\mathbb{F}^{n-m}$.

Having fixed the class of groups, we now have to specify a class of representations for which to study symmetry breaking operators. By the Casselman Embedding Theorem every irreducible admissible representation of a reductive group occurs as a quotient inside a principal series representation. Principal series are families of representations induced from a parabolic subgroup $P = MAN$. Here we focus on the case of spherical principal series, i.e. those induced from representations of P which are trivial on M and N. Since $A \cong \mathbb{R}_+$, its representations are parametrized by a complex number $\lambda \in \mathbb{C}$, and we denote the corresponding induced representation of G by π_λ. In the same way the spherical principal series τ_ν ($\nu \in \mathbb{C}$) of G' is constructed. The topic of this paper is a full classification of $\operatorname{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$ for all strongly spherical real reductive pairs (G, G') with G of rank one and all $(\lambda, \nu) \in \mathbb{C}$.

For the pair $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$ of rank one orthogonal groups, symmetry breaking operators between spherical principal series were classified by Kobayashi–Speh [KS15] (see also [KS18] for the vector-valued case). We therefore restrict ourselves to the strongly spherical pairs $(G, G') = (\mathrm{U}(1, n+1; \mathbb{F}), \mathrm{U}(1, m+1; \mathbb{F}) \times F)$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ in this paper.

Multiplicities

For $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ we consider the real reductive pairs

$$(G, G') = (\mathrm{U}(1, n+1; \mathbb{F}), \mathrm{U}(1, m+1; \mathbb{F}) \times F)$$

with $0 \leq m < n$ and $F < \mathrm{U}(n-m; \mathbb{F})$ a closed subgroup. We assume that the pair (G, G') is strongly spherical.

Let π_λ ($\lambda \in \mathbb{C}$) denote the spherical principal series of G. Our normalization is chosen, such that π_λ is unitary for $\lambda \in i\mathbb{R}$ and has a finite-dimensional quotient if and only if $\lambda \in \rho + 2\mathbb{Z}_{\geq 0}$, where $\rho = \frac{1}{2}(p + 2q)$ with $p = n \cdot \dim_{\mathbb{R}} \mathbb{F}$ and $q = \dim_{\mathbb{R}} \mathbb{F} - 1 \in \{1, 3, 7\}$. In the same way we parametrize the spherical principal series τ_ν ($\nu \in \mathbb{C}$) of G' and let $\rho' = \frac{1}{2}(p' + 2q)$ with $p' = m \cdot \dim_{\mathbb{R}} \mathbb{F}$ (see Section 1.1 for the precise definitions). Note that for $m = 0$ we have $\mathfrak{su}(1, 1; \mathbb{F}) = \mathfrak{so}(1, q+1)$, so that G' is a product of an indefinite orthogonal group and a compact group.

To state our results for the multiplicities $\dim \mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$ we define for $m > 0$

$$L = \{(-\rho + q - 1 - 2i, -\rho' + q - 1 - 2j) : i, j \in \mathbb{Z}, 0 \leq j \leq i\}$$

and for $m = 0$

$$L = \{(-\rho + q - 1 - 2i, \pm(1 + 2j)) : i, j \in \mathbb{Z}, 0 \leq j \leq i\},$$

as well as

$$\begin{aligned} S_1 &= \{(-\rho + q - 1 - 2i, \rho' + 2j) : i, j \in \mathbb{Z}, 0 \leq j \leq i\}, \\ S_2 &= \{(-\rho + q - 1 - 2i, \rho' + 2i) : i \in \mathbb{Z}_{\geq 0}\}, \\ S_3 &= \{(-\rho + q - 1, \rho')\}. \end{aligned}$$

Note that for $m = 0$ we have $S_1 \subseteq L$ for $\mathbb{F} = \mathbb{C}$ and $S_2 \cap L = S_3 \cap L = \emptyset$ for $\mathbb{F} = \mathbb{H}$.

Theorem A (Multiplicities)**.** *The multiplicities of symmetry breaking operators between spherical principal series of G and G' are given by*

$$\dim \mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu) = \begin{cases} 1 & \text{for } (\lambda, \nu) \in \mathbb{C}^2 - L, \\ 2 & \text{for } (\lambda, \nu) \in L, \end{cases}$$

except in the following three cases:

(i) For $(G, G') = (\mathrm{U}(1, n+1), \mathrm{U}(1, 1) \times F)$ we have

$$\dim \mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu) = \begin{cases} 1 & \text{for } (\lambda, \nu) \in \mathbb{C}^2 - L, \\ 2 & \text{for } (\lambda, \nu) \in L - S_1, \\ 3 & \text{for } (\lambda, \nu) \in S_1. \end{cases}$$

(ii) For $(G, G') = (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1))$ *we have*

$$\dim \mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu) = \begin{cases} 1 & \text{for } (\lambda,\nu) \in \mathbb{C}^2 - (L \cup S_2), \\ 2 & \text{for } (\lambda,\nu) \in L, \\ 2i+4 & \text{for } (\lambda,\nu) = (-\rho + q - 1 - 2i, \rho' + 2i) \in S_2, \end{cases}$$

(iii) For $(G, G') = (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1) \times \mathrm{U}(1))$ *we have*

$$\dim \mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu) = \begin{cases} 1 & \text{for } (\lambda,\nu) \in \mathbb{C}^2 - (L \cup S_3), \\ 2 & \text{for } (\lambda,\nu) \in L \cup S_3. \end{cases}$$

Regular symmetry breaking operators

We now describe the explicit construction of all symmetry breaking operators in the space $\mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$. For this we realize the representations π_λ as smooth sections of homogeneous line bundles over the flag variety G/P. If τ_ν is realized in the same way as smooth sections over G'/P', $P' = P \cap G'$ a parabolic subgroup of G', then symmetry breaking operators in $\mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$ are given by convolution with a P'-equivariant distribution section over G/P followed by restriction to $G'/P' \subseteq G/P$. For our pairs of groups (G, G') the distribution kernels can most easily be described on the opposite unipotent radical $\overline{N}$ which can be identified with an open dense subset of G/P. In this way, symmetry breaking operators can be classified in terms of certain invariant distributions on $\overline{N}$ (see Section 1.3 for details).

For $G = \mathrm{U}(1, n+1; \mathbb{F})$ the group $\overline{N}$ is a 2-step nilpotent group diffeomorphic to its Lie algebra $\bar{\mathfrak{n}} \cong \mathbb{F}^n \oplus \mathrm{Im}\,\mathbb{F}$, where $\mathrm{Im}\,\mathbb{F} = \{Z \in \mathbb{F} : \overline{Z} = -Z\}$. We write $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ for those distributions on $\bar{\mathfrak{n}}$ which correspond to symmetry breaking operators in $\mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$ and now describe the classification of $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ in detail. For this we may without loss of generality assume that G' is embedded in G such that $\overline{N} \cap G'$ corresponds to the subalgebra $\bar{\mathfrak{n}}' = \mathbb{F}^m \oplus \mathrm{Im}\,\mathbb{F} \subseteq \bar{\mathfrak{n}}$ with $\mathbb{F}^m \subseteq \mathbb{F}^n$ in the first m coordinates.

Since the distribution kernels on G/P corresponding to symmetry breaking operators are P'-equivariant, the support of every such kernel is a union of P'-orbits in G/P. There are three such orbits, an open dense orbit $\mathcal{O}_A$, an intermediate orbit $\mathcal{O}_B$ and a closed orbit $\mathcal{O}_C$. Their intersections with the open dense Bruhat cell $\overline{N}P/P \simeq \bar{\mathfrak{n}}$ are given by

$$\mathcal{O}_A \cap \bar{\mathfrak{n}} = \bar{\mathfrak{n}} - \bar{\mathfrak{n}}', \qquad \mathcal{O}_B \cap \bar{\mathfrak{n}} = \bar{\mathfrak{n}}' - \{0\}, \qquad \mathcal{O}_C \cap \bar{\mathfrak{n}} = \{0\}.$$

For each of the orbits we construct a holomorphic family of distributions which is generically supported on that orbit and contains all distributions in $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ which are supported on that orbit, except for $(\lambda,\nu) \in S_i$ $(i = 1,2,3)$ in the cases (i)-(iii) in Theorem A. The remaining distribution kernels for $(\lambda,\nu) \in S_i$ are all supported on $\{0\}$ and will be described at the end of the introduction. Symmetry breaking operators whose distribution kernels are supported on the whole space $\bar{\mathfrak{n}}$ are called *regular* while operators whose kernels have smaller support are called *singular*. Note that a symmetry breaking operator is a differential operator if and only if its distribution kernel is supported on $\{0\} \subseteq \bar{\mathfrak{n}}$.

For $\mathrm{Re}(\lambda \pm \nu) > -\frac{p''}{2}$ with $p'' = (n-m) \cdot \dim_{\mathbb{R}} \mathbb{F} = p - p' = 2(\rho - \rho')$ we define the following locally integrable function on $\bar{\mathfrak{n}} = \mathbb{F}^n \oplus \mathrm{Im}\,\mathbb{F}$:

$$u^A_{\lambda,\nu}(X, Z) := \frac{1}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})} N(X,Z)^{-2(\nu+\rho')} |X''|^{\lambda-\rho+\nu+\rho'},$$

where $N(X,Z) = (|X|^4 + |Z|^2)^{\frac{1}{4}}$ is the norm function on $\bar{\mathfrak{n}}$ and we have used the notation $X = (X', X'') \in \mathbb{F}^m \oplus \mathbb{F}^{n-m} = \mathbb{F}^n$.

Theorem B (Regular symmetry breaking operators)**.** *$u^A_{\lambda,\nu}$ extends to a family of distributions which depends holomorphically on $(\lambda,\nu) \in \mathbb{C}^2$ and $u^A_{\lambda,\nu} \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ for all $(\lambda,\nu) \in \mathbb{C}^2$. Further, $u^A_{\lambda,\nu} = 0$ if and only if $(\lambda,\nu) \in L$.*

In Corollary 12.4 we show that $\operatorname{Supp} u^A_{\lambda,\nu} = \bar{\mathfrak{n}}$ if and only if $(\lambda,\nu) \in \mathbb{C}^2 - (/\!/ \cup \backslash\!\backslash)$, where

$$/\!/ := \{(\lambda,\nu) \in \mathbb{C}^2, \lambda + \rho - \nu - \rho' \in -2\mathbb{Z}_{\geq 0}\},$$
$$\backslash\!\backslash := \{(\lambda,\nu) \in \mathbb{C}^2, \lambda + \rho + \nu - \rho' \in -2\mathbb{Z}_{\geq 0}\}.$$

Note that $L \subseteq \mathbb{X} := /\!/ \cap \backslash\!\backslash$ since q is odd and p' is even for $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$. For $(\lambda,\nu) \in /\!/ \cup \backslash\!\backslash$ the support of $u^A_{\lambda,\nu}$ is smaller than $\bar{\mathfrak{n}}$ and we now describe these distributions in more detail.

Singular symmetry breaking operators

Write $\bar{\mathfrak{n}} = \mathfrak{v} \oplus \mathfrak{z}$ with $\mathfrak{v} = \mathbb{F}^n$ and $\mathfrak{z} = \operatorname{Im}\mathbb{F}$. Decompose $\mathfrak{v} = \mathfrak{v}' \oplus \mathfrak{v}''$ with $\mathfrak{v}' = \mathbb{F}^m$ and $\mathfrak{v}'' = \mathbb{F}^{n-m}$, so that $\bar{\mathfrak{n}}' = \mathfrak{v}' \oplus \mathfrak{z}$. We write

$$\begin{aligned} p &= \dim \mathfrak{v} = n \cdot \dim_{\mathbb{R}} \mathbb{F}, \\ p' &= \dim \mathfrak{v}' = m \cdot \dim_{\mathbb{R}} \mathbb{F}, \qquad\qquad q = \dim \mathfrak{z} = \dim_{\mathbb{R}} \mathbb{F} - 1. \\ p'' &= \dim \mathfrak{v}'' = (n-m) \cdot \dim_{\mathbb{R}} \mathbb{F}, \end{aligned}$$

Further, let $\Delta_{\mathfrak{v}'}$, $\Delta_{\mathfrak{v}''}$ and $\Box$ denote the Euclidean Laplacians on $\mathfrak{v}'$, $\mathfrak{v}''$ and $\mathfrak{z}$, respectively.

For $(\lambda,\nu) \in \backslash\!\backslash$ with $\lambda + \rho + \nu - \rho' = -2l \in -2\mathbb{Z}_{\geq 0}$ and $\operatorname{Re}\nu \ll 0$ let

$$u^B_{\lambda,\nu}(X,Z) := c^B(\lambda,\nu) N(X,Z)^{-2(\nu+\rho')} \Delta^l_{\mathfrak{v}''} \delta(X'),$$

where $c^B(\lambda,\nu)$ is a meromorphic renormalization parameter defined in (11.1). Further, for $(\lambda,\nu) \in /\!/$ with $\lambda + \rho - \nu - \rho' = -2k \in -2\mathbb{Z}_{\geq 0}$ let

$$u^C_{\lambda,\nu}(X,Z) := \sum_{h+i+2j=k} c_{h,i,j}(\lambda,\nu) \Delta^h_{\mathfrak{v}'} \Delta^i_{\mathfrak{v}''} \Box^j \delta(X,Z),$$

where $c_{h,i,j}(\lambda,\nu)$ are holomorphic functions of $(\lambda,\nu) \in /\!/$ defined in (6.2) and (6.3).

Theorem C (Singular symmetry breaking operators)**.** *(i) $u^B_{\lambda,\nu}$ extends to a family of distributions which depends holomorphically on $(\lambda,\nu) \in \backslash\!\backslash$ and $u^B_{\lambda,\nu} \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ for all $(\lambda,\nu) \in \backslash\!\backslash$. The support of $u^B_{\lambda,\nu}$ is given by*

$$\operatorname{Supp} u^B_{\lambda,\nu} = \begin{cases} \bar{\mathfrak{n}}' & \text{for } (\lambda,\nu) \in \backslash\!\backslash - (\mathbb{X} - L), \\ \{0\} & \text{for } (\lambda,\nu) \in \mathbb{X} - L \end{cases}$$

and for $\lambda + \rho + \nu - \rho' = -2l$ the following residue formula holds:

$$u^A_{\lambda,\nu} = \frac{(-1)^l \pi^{\frac{p''}{2}}}{2^l \Gamma(\frac{p''}{2} + l)} \times \begin{cases} u^B_{\lambda,\nu} & \text{for } m > 0 \text{ and } l \leq \frac{p'}{2}, \\ \frac{\Gamma(\frac{2\nu+p'+2}{4} + \lfloor \frac{2l-p'+2}{4} \rfloor)}{\Gamma(\frac{2\nu+p'+2}{4})} u^B_{\lambda,\nu} & \text{for } m > 0 \text{ and } l > \frac{p'}{2}, \\ \frac{\Gamma(-\frac{\nu}{2} - \lfloor \frac{l}{2} \rfloor)}{\Gamma(-\nu - l)} u^B_{\lambda,\nu} & \text{for } m = 0. \end{cases}$$

(ii) $u^C_{\lambda,\nu}$ *extends to a family of distributions which depends holomorphically on* $(\lambda,\nu) \in //$ *and* $u^C_{\lambda,\nu} \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ *for all* $(\lambda,\nu) \in //$. *The support of* $u^C_{\lambda,\nu}$ *is given by*

$$\operatorname{Supp} u^C_{\lambda,\nu} = \{0\} \qquad \text{for all } (\lambda,\nu) \in //$$

and for $\lambda + \rho - \nu - \rho' = -2k$ *the following residue formulas hold:*

$$u^A_{\lambda,\nu} = \frac{(-1)^k k! \pi^{\frac{p+q}{2}}}{\Gamma(\frac{\nu+\rho'}{2})} \times \begin{cases} \frac{\Gamma(\frac{2\nu+p'}{4})}{\Gamma(\frac{2\nu+p'}{2})} u^C_{\lambda,\nu} & \text{for } m > 0, \\ \frac{\Gamma(\frac{\nu}{2} - \lfloor \frac{k}{2} \rfloor)}{2^k \Gamma(\nu - k)} u^C_{\lambda,\nu} & \text{for } m = 0. \end{cases}$$

Using the three families $u^A_{\lambda,\nu}$, $u^B_{\lambda,\nu}$ and $u^C_{\lambda,\nu}$ of invariant distributions associated to the orbits $\mathcal{O}_A$, $\mathcal{O}_B$ and $\mathcal{O}_C$ we have the following explicit description of $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} \cong \operatorname{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$:

Theorem D (Classification of symmetry breaking operators)**.** *The space* $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ *is given by*

$$\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} = \begin{cases} \mathbb{C} u^A_{\lambda,\nu} & \text{for } (\lambda,\nu) \in \mathbb{C}^2 - L, \\ \mathbb{C} u^B_{\lambda,\nu} \oplus \mathbb{C} u^C_{\lambda,\nu} & \text{for } (\lambda,\nu) \in L. \end{cases}$$

except in the cases (i), (ii) and (iii) of Theorem A with $(\lambda,\nu) \in S_i$.

Theorems C and D show that in almost all cases the holomorphic family $u^A_{\lambda,\nu}$ and its two different renormalizations $u^B_{\lambda,\nu}$ and $u^C_{\lambda,\nu}$ when $(\lambda,\nu) \in L$ spans the space of all symmetry breaking operators. Note that $u^C_{\lambda,\nu}$ is supported at the origin, so the associated symmetry breaking operator is a differential operator. In the cases (i), (ii) and (iii) of Theorem A with $(\lambda,\nu) \in S_i$ there appear additional differential symmetry breaking operators that cannot be obtained as renormalizations of the holomorphic family $A_{\lambda,\nu}$ of operators with distribution kernels $u^A_{\lambda,\nu}$. Following [KS18] we call such operators *sporadic symmetry breaking operators* and now describe them in detail.

Sporadic symmetry breaking operators

To define all sporadic differential symmetry breaking operators we treat the three cases in Theorem A separately. In all three cases the differential operators $\Delta_{\mathfrak{v}'}$, $\Delta_{\mathfrak{v}''}$ and $\Box$ do not generate the algebra of M'-invariant differential operators on $\bar{\mathfrak{n}}$ with constant coefficients. The additional generators give rise to sporadic differential symmetry breaking operators.

(i) $(G, G') = (\mathrm{U}(1, n+1), \mathrm{U}(1,1) \times F)$. For $(\lambda,\nu) \in //$ with $\lambda + \rho - \nu - \rho' = -2k$ we have

$$u^C_{\lambda,\nu} = \sum_{i+2j=k} \frac{2^{-i}\Gamma(\frac{\nu}{2} - j)}{i! j! \Gamma(\frac{p}{2} + i)\Gamma(\frac{\nu}{2} - \lfloor \frac{k}{2} \rfloor)} \Delta^i_{\mathfrak{v}} \left(\frac{\partial}{\partial Z}\right)^{2j} \delta(X, Z).$$

The algebra of M'-invariant differential operators on $\bar{\mathfrak{n}}$ is generated by $\Delta_{\mathfrak{v}}$ and $\frac{\partial}{\partial Z}$ and we obtain another family of distributions by formally replacing $2j$ by $2j+1$:

$$v^C_{\lambda,\nu} := \sum_{i+2j+1=k} \frac{2^{-i}\Gamma(\frac{\nu-1}{2} - j)}{i! \Gamma(j + \frac{3}{2}) \Gamma(\frac{p}{2} + i) \Gamma(\frac{\nu-1}{2} - \lfloor \frac{k-1}{2} \rfloor)} \Delta^i_{\mathfrak{v}} \left(\frac{\partial}{\partial Z}\right)^{2j+1} \delta(X, Z).$$

(ii) $(G,G') = (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1))$. Note that $\bar{\mathfrak{n}} = \mathbb{H} \oplus \operatorname{Im}\mathbb{H}$ and $\bar{\mathfrak{n}}' = \operatorname{Im}\mathbb{H}$. On $\mathbb{H}$ and $\operatorname{Im}\mathbb{H} \subseteq \mathbb{H}$ we consider the usual inner product $\langle X, Y\rangle = \operatorname{Re}(X\overline{Y})$. With respect to this inner product we define three differential operators P_1, P_2, P_3 on $\bar{\mathfrak{n}}$ in terms of their symbols

$$p_1(X,Z) = \langle \mathbf{i}, \overline{X}ZX\rangle, \qquad p_2(X,Z) = \langle \mathbf{j}, \overline{X}ZX\rangle, \qquad p_3(X,Z) = \langle \mathbf{k}, \overline{X}ZX\rangle.$$

Then the algebra of M'-invariant differential operators on $\bar{\mathfrak{n}}$ is generated by $\Delta_{\mathfrak{v}}$, $\square$ and P_1, P_2, P_3. For $k \in \mathbb{Z}_{\geq 0}$ let

$$\mathcal{H}^k(P_1,P_2,P_3)\delta = \{q(P_1,P_2,P_3)\delta(X,Z) : q \in \mathcal{H}^k(\mathbb{R}^3)\},$$

where $\mathcal{H}^k(\mathbb{R}^3)$ denotes the space of homogeneous harmonic polynomials of three variables of degree k.

(iii) $(G,G') = (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1)\times \mathrm{U}(1))$. The subalgebra $\mathfrak{u}(1) \subseteq \mathfrak{sp}(1)$ is spanned by a single element U of the Lie algebra $\mathfrak{sp}(1)$ which is identified with $\operatorname{Im}\mathbb{H} \subseteq \mathbb{H}$. We write $U = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k}$ and note that the only differential operators in $\mathcal{H}^k(P_1,P_2,P_3)$ which are $\mathrm{U}(1)$-invariant occur for $k=0$ (then 1 is $\mathrm{U}(1)$-invariant) and for $k=1$ (then $U_1P_1 + U_2P_2 + U_3P_3$ is $\mathrm{U}(1)$-invariant).

Theorem E (Sporadic symmetry breaking operators)**.** *(i) For $(G,G') = (\mathrm{U}(1,n+1), \mathrm{U}(1,1)\times F)$ and $(\lambda,\nu)\in S_1$ we have*

$$\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} = \mathbb{C}u^B_{\lambda,\nu} \oplus \mathbb{C}u^C_{\lambda,\nu} \oplus \mathbb{C}v^C_{\lambda,\nu}.$$

(ii) For $(G,G') = (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1))$ and $(\lambda,\nu) = (-\rho+q-1-2i, \rho'+2i) \in S_2$ we have

$$\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} = \mathbb{C}u^C_{\lambda,\nu} \oplus \mathcal{H}^{i+1}(P_1,P_2,P_3)\delta.$$

(iii) For $(G,G') = (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1)\times\mathrm{U}(1))$ with $\mathfrak{u}(1) = \mathbb{R}U$, $U = U_1\mathbf{i}+U_2\mathbf{j}+U_3\mathbf{k} \in \operatorname{Im}\mathbb{H} = \mathfrak{sp}(1)$, and $(\lambda,\nu) = (-\rho+q-1, \rho') \in S_3$ we have

$$\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} = \mathbb{C}u^C_{\lambda,\nu} \oplus \mathbb{C}(U_1P_1+U_2P_2+U_3P_3)\delta.$$

Functional equations

For $\lambda \in \mathbb{R}$ the representations π_λ are irreducible and unitarizable if and only if $\lambda \in (-\rho+q-1, \rho-q+1)$ and the corresponding unitary represention form the *complementary series*. The invariant inner product can be expressed using the normalized Knapp–Stein intertwining operators $T_\lambda : \pi_\lambda \to \pi_{-\lambda}$ which are in the non-compact picture on $\bar{\mathfrak{n}} = \mathbb{F}^n \oplus \operatorname{Im}\mathbb{F}$ given by convolution with the holomorphic family of distributions

$$\frac{1}{\Gamma(\lambda)} N(X,Z)^{2(\lambda-\rho)}.$$

Note that the operator T_λ is a differential operator if and only if $\lambda \in -\mathbb{Z}_{\geq 0}$ (see Section 13 for details).

When a complementary series representation π_λ is restricted to G', it decomposes into a direct integral of irreducible unitary representations of G'. We expect the holomorphic family $A_{\lambda,\nu} \in \mathrm{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$ belonging to the distribution kernels $u^A_{\lambda,\nu}$ and its renormalizations to play a key role in this decomposition (see [MO15] for the case $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, m+1) \times \mathrm{O}(n-m))$). To explicitly constuct the direct integral decomposition, compositions of $A_{\lambda,\nu}$ with Knapp–Stein intertwining operators T_λ for G and T'_ν for G' will be important. This motivates the following functional equations:

Theorem F (Functional equations)**.** *For $(\lambda, \nu) \in \mathbb{C}^2$ we have*

$$A_{\lambda,\nu} \circ T_{-\lambda} = \pi^{\frac{p+q}{2}} \frac{\Gamma(\frac{2\lambda+p}{4})}{\Gamma(\frac{2\lambda+p}{2})\Gamma(\frac{\lambda+\rho}{2})} A_{-\lambda,\nu}$$

and

$$T'_\nu \circ A_{\lambda,\nu} = \frac{\pi^{\frac{p'+q}{2}}}{\Gamma(\frac{\nu+\rho'}{2})} \times \begin{cases} \frac{\Gamma(\frac{2\nu+p'}{4})}{\Gamma(\frac{2\nu+p'}{2})} A_{\lambda,-\nu} & \text{if } m > 0, \\ A_{\lambda,-\nu} & \text{if } m = 0. \end{cases}$$

Relation to previous work

The systematic construction and classification of symmetry breaking operators was initiated by T. Kobayashi [Kob15]. Together with B. Speh [KS15] he classified symmetry breaking operators between spherical principal series for $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$. Our work can be seen as an extension of this classification to all strongly spherical reductive pairs (G, G') with G and G' of real rank one. The only other strongly spherical pairs where a full classification of symmetry breaking operators between spherical principal series is known are the pairs $(G, G') = (\mathrm{O}(1, n) \times \mathrm{O}(1, n), \mathrm{O}(1, n))$ treated by Clerc [Cle16, Cle17].

The existence of the meromorphic family of distributions $u^A_{\lambda,\nu}$ and a generic multiplicity one statement were previously obtained by Möllers–Ørsted–Oshima [MØO16] (see also [Möl17]), but the precise multiplicities also for singular parameters as well as the detailed study of the meromorphic nature of $u^A_{\lambda,\nu}$ were missing so far. Only the differential symmetry breaking operators corresponding to the distributions $u^C_{\lambda,\nu}$ were previously constructed by Möllers–Ørsted–Zhang [MØZ16a], but in particular the sporadic differential operators in Theorem E were not known before.

Methods of proof

Our general strategy of classifying symmetry breaking operators follows closely Kobayashi–Speh [KS15] (see Part I and in particular Section 3.2). This reformulates the problem of classifying symmetry breaking operators into a classification problem for invariant distributions on a nilpotent Lie algebra, the nilradical of a parabolic subgroup. However, whereas in [KS15] the relevant Lie algebra is abelian, in our situation we have to deal with a 2-step nilpotent Lie algebra which adds combinatorial difficulties to the computations.

The classification strategy essentially consists of two steps: the classification of all differential symmetry breaking operators (see Part II) and the global study of the distribution kernels of symmetry breaking operators (see Part III). For the classification of differential symmetry breaking operators we apply the F-method (see e.g. [Kob13]) which uses the Euclidean Fourier transform on the nilpotent Lie algebra. Since our Lie algebra is 2-step nilpotent, this results in a system of polynomial differential equations of order 4 which we solve combinatorially. In contrast to all previous applications of the F-method where equations of order 2 were solved, we cannot use classical orthogonal polynomials, but have to systematically solve the equations by hand. In particular, we find that in three special cases sporadic differential operators occur (see Theorem E).

In the global study of invariant distributions corresponding to symmetry breaking operators we follow a more direct approach to study the meromorphic nature of the family $u^A_{\lambda,\nu}$. For the localization of all poles and to obtain $u^C_{\lambda,\nu}$ as renormalization of $u^A_{\lambda,\nu}$, we use polar type coordinates adapted to the nilpotent Lie algebra $\bar{\mathfrak{n}}$ and evaluate explicitly the resulting integrals (see Theorems 9.1 and 10.1). Combined with some combinatorial computations this gives a direct way to obtain differential symmetry breaking operators from regular symmetry breaking operators. We remark that, in addition to the combinatorial construction in Part II, this gives a second and more analytic construction of the distributions $u^C_{\lambda,\nu}$.

While the (unnormalized) family $u^B_{\lambda,\nu}$ is easily obtained from $u^A_{\lambda,\nu}$, it is much harder to find its optimal renormalization and to determine its support. We resolve this problem by explicitly decomposing the relevant distributions with respect to the decomposition $\bar{\mathfrak{n}} = \bar{\mathfrak{n}}' \oplus \mathfrak{v}''$ (see Theorem 11.1). The resulting renormalization even improves the normalization used in [KS15] for the corresponding distributions in the sense that our $u^B_{\lambda,\nu}$ never vanishes. The remaining arguments for the full classification of symmetry breaking operators then work similarly as in [KS15].

Theorem B is shown in Theorem 9.1 and Theorem C is a combination of Theorems 10.1 and 11.1. Theorem E follows from Theorems 7.1 and 7.4 and Corollary 7.6, and together with Theorems 6.1 and 12.1 it implies Theorems A and D. Finally, Theorem F is Theorem 13.6.

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Notation

For two sets $B \subseteq A$ we use the Notation $A - B = \{a \in A : a \notin B\}$. We denote Lie groups by Roman capitals and their corresponding Lie algebras by the corresponding Fraktur lower cases.

Part I

Preliminaries

In this first part we recall from [KS15] the basic theory of symmetry breaking operators between principal series representations and introduce the necessary notation for rank one real reductive groups.

1 Symmetry breaking operators between principal series representations

We recall the basic facts about symmetry breaking operators between principal series representations from [KS15].

1.1 Principal series representations

Let G be a real reductive Lie group and P a minimal parabolic subgroup of G with Langlands decomposition $P = MAN$. For a finite-dimensional representation (ξ, V) of M, a character $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and the trivial representation $\mathbf{1}$ of N we obtain a finite-dimensional representation $(\xi \otimes e^{\lambda} \otimes \mathbf{1}, V_{\xi,\lambda})$ of $P = MAN$. By smooth normalized parabolic induction this representation gives rise to the principal series representation

$$\pi_{\xi,\lambda} := \operatorname{Ind}_P^G(\xi \otimes e^{\lambda} \otimes \mathbf{1})$$

as the left-regular representation of G on the space

$$\{\varphi \in C^{\infty}(G, V) : \ \varphi(gman) = \xi(m)^{-1} a^{-(\lambda+\rho)} \varphi(g) \ \forall man \in MAN\},$$

where $\rho := \frac{1}{2} \operatorname{tr} \operatorname{ad}|_{\mathfrak{n}} \in \mathfrak{a}_{\mathbb{C}}^*$. Let $\mathcal{V}_{\xi,\lambda} := G \times_P V_{\xi,\lambda+\rho} \to G/P$ be the homogeneous vector bundle associated to $V_{\xi,\lambda+\rho}$, then $\pi_{\xi,\lambda}$ identifies with the left-regular action of G on the space of smooth sections $C^{\infty}(G/P, \mathcal{V}_{\xi,\lambda})$.

Now let $G' < G$ be a reductive subgroup. Similarly we let $P' = M'A'N'$ be a minimal parabolic subgroup of G'. For a finite-dimensional representation (η, W) of M' and $\nu \in (\mathfrak{a}'_{\mathbb{C}})^*$ we obtain a finite-dimensional representation $(\eta \otimes e^{\nu} \otimes \mathbf{1}, W_{\eta,\lambda})$ of P' and the corresponding principal series representation

$$\tau_{\eta,\nu} := \operatorname{Ind}_{P'}^{G'}(\eta \otimes e^{\nu} \otimes \mathbf{1}).$$

Again we identify $\tau_{\eta,\nu}$ with the smooth sections $C^{\infty}(G'/P', \mathcal{W}_{\eta,\nu})$ of the homogeneous vector bundle $\mathcal{W}_{\eta,\nu} := G' \times_{P'} W_{\eta,\nu+\rho'} \to G'/P'$, where $\rho' := \frac{1}{2} \operatorname{tr} \operatorname{ad}|_{\mathfrak{n}'}$.

1.2 Symmetry breaking operators

In these realizations the space of symmetry breaking operators between $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$ is given by the continuous linear G'-maps between the smooth sections of the two homogeneous vector bundles

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu}) = \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{V}_{\xi,\lambda})), C^\infty(G'/P', \mathcal{W}_{\eta,\nu})).$$

The Schwartz Kernel Theorem implies that every such operator is given by a G'-invariant distribution section of the tensor bundle $\mathcal{V}_{\xi^*,-\lambda} \boxtimes \mathcal{W}_{\eta,\nu}$ over $G/P \times G'/P'$, where ξ^* is the representation contradigent to ξ. Since G' acts transitively on G'/P' we can consider these distributions as sections on G/P with a certain P'-invariance:

Theorem 1.1 ([KS15, Proposition 3.2]). *There is a natural bijection*

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu}) \xrightarrow{\sim} (\mathcal{D}'(G/P, \mathcal{V}_{\xi^*,-\lambda}) \otimes W_{\eta,\nu+\rho'})^{P'}, \quad T \mapsto u^T.$$

1.3 Restriction to the open Bruhat cell

From now on assume $M' = M \cap G'$, $A' = A \cap G'$ and $N' = N \cap G'$. Since $\overline{N}$ is unipotent we obtain a parameterization of the open Bruhat cell $\overline{N}P/P \subseteq G/P$ in terms of the Lie algebra $\bar{\mathfrak{n}}$ by the map

$$\bar{\mathfrak{n}} \xrightarrow{\exp} \overline{N} \hookrightarrow G \longrightarrow G/P,$$

so that we can consider $\bar{\mathfrak{n}}$ as an open dense subset of G/P. Then the restriction

$$\mathcal{D}'(G/P, \mathcal{V}_{\xi^*,-\lambda}) \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}}, \mathcal{V}_{\xi^*,-\lambda}|_{\bar{\mathfrak{n}}})$$

can be used to define a $\mathfrak{g}$-action on $\mathcal{D}'(\bar{\mathfrak{n}}, \mathcal{V}_{\xi^*,-\lambda}|_{\bar{\mathfrak{n}}}) \cong \mathcal{D}'(\bar{\mathfrak{n}}) \otimes V_{\xi^*,-\lambda+\rho}$ by vector fields. Moreover, since $\mathrm{Ad}(M'A')$ leaves $\bar{\mathfrak{n}}$ invariant, the restriction is further $M'A'$-equivariant. If we assume $P'\overline{N}P = G$, i.e. every P'-orbit in G/P meets the open Bruhat cell $\overline{N}P$, then symmetry breaking operators can be described in terms of $(M'A', \mathfrak{n}')$-invariant distributions on $\bar{\mathfrak{n}}$:

Theorem 1.2 ([KS15, Theorem 3.16]). *Assume $P'\overline{N}P = G$, then there is a natural bijection*

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu}) \xrightarrow{\sim} (\mathcal{D}'(\bar{\mathfrak{n}}) \otimes V_{\xi^*,-\lambda+\rho} \otimes W_{\eta,\nu+\rho'})^{M'A',\mathfrak{n}'}.$$

Given a distribution kernel u^T, the corresponding operator $T \in \mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu})$ is given by

$$T\varphi(h) = \langle u^T, \varphi(h \exp(\,\cdot\,))\rangle. \tag{1.1}$$

In this paper we will only be concerned with the case of spherical principal series representations. We therefore let $\xi = \mathbf{1}$ and $\eta = \mathbf{1}$ be the trivial representations and put $\pi_\lambda = \pi_{\mathbf{1},\lambda}$ and $\tau_\nu = \tau_{\mathbf{1},\nu}$.

For the classification of symmetry breaking operators it will be convenient to also consider invariant distributions on open subsets of $\bar{\mathfrak{n}}$. For an open $M'A'$-invariant subset $\Omega \subseteq \bar{\mathfrak{n}}$ we therefore use the notation

$$\mathcal{D}'(\Omega)_{\lambda,\nu} := (\mathcal{D}'(\Omega) \otimes V_{\mathbf{1},-\lambda+\rho} \otimes W_{\mathbf{1},\nu+\rho'})^{M'A',\mathfrak{n}'},$$

so that $\mathrm{Hom}(\pi_\lambda|_{G'}, \tau_\nu) \cong \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$.

Remark 1.3. The space $\mathcal{D}'(\Omega)_{\lambda,\nu}$ is given by all distributions $u \in \mathcal{D}'(\Omega)$ satisfying

$$u(\mathrm{Ad}(m'a')X) = a'^{-\lambda+\rho+\nu+\rho'}u(X) \qquad \forall\, m' \in M', a' \in A', \tag{1.2}$$
$$d\pi_{-\lambda}(Y)u(X) = 0 \qquad \forall\, Y \in \mathfrak{n}'. \tag{1.3}$$

2 Principal series representations of rank one reductive groups

For $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ let $G = \mathrm{U}(1, n+1; \mathbb{F})$ denote the group of $(n+2) \times (n+2)$ matrices over $\mathbb{F}$ preserving the quadratic form

$$(z_0, z_1, \ldots, z_{n+1}) \mapsto -|z_0|^2 + |z_1|^2 + \cdots + |z_{n+1}|^2.$$

Here we assume that $n \geq 1$ for $\mathbb{F} = \mathbb{C}, \mathbb{H}$ and $n = 1$ for $\mathbb{F} = \mathbb{O}$, using the interpretation $\mathrm{U}(1, 2; \mathbb{O}) \cong F_{4(-20)}$.

Let P be the minimal parabolic subgroup of G with Langlands decomposition $P = MAN$ given by

$$M = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix} : a \in \mathrm{U}(1; \mathbb{F}), b \in \mathrm{U}(n; \mathbb{F}) \right\},$$

$$A = \exp(\mathfrak{a}) \qquad \text{where } \mathfrak{a} = \mathbb{R}H, \quad H = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathbf{0}_n \end{pmatrix},$$

$$N = \exp(\mathfrak{n}) \qquad \text{where } \mathfrak{n} = \left\{ \begin{pmatrix} Z & -Z & X \\ Z & -Z & X \\ X^* & -X^* & \mathbf{0}_n \end{pmatrix} : X \in \mathbb{F}^n, Z \in \operatorname{Im} \mathbb{F} \right\}.$$

Here, for $X = (X_1, \ldots, X_n) \in \mathbb{F}^n$ we write $X^* := (\overline{X}_1, \ldots \overline{X}_n)^T$. Further, we use the notation $\operatorname{Im} \mathbb{F} = \{Z \in \mathbb{F} : \overline{Z} = -Z\}$. Note that $\mathfrak{n}$ is the direct sum of the eigenspaces of $\mathrm{ad}(H)$ to the eigenvalues $+1$ and $+2$. We abbreviate the real dimensions of these eigenspaces by

$$p = n \cdot \dim_{\mathbb{R}} \mathbb{F} \quad \text{and} \quad q = \dim_{\mathbb{R}} \mathbb{F} - 1.$$

We identify $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ by $\lambda \mapsto \lambda(H)$. Then in particular

$$\rho = \frac{1}{2} \operatorname{tr} \mathrm{ad}\,|_{\mathfrak{n}}(H) = \frac{p + 2q}{2}.$$

For $\lambda \in \mathbb{C}$ we consider the spherical principal series representation π_λ in the notation of Section 1.1. Due to Johnson and Wallach the composition series of these representations are well known.

Theorem 2.1 ([JW77, Joh76])**.** *π_λ is irreducible if and only if $\lambda \notin \pm(\rho - q + 1 + 2\mathbb{Z}_{\geq 0})$. π_λ contains an irreducible finite-dimensional submodule if and only if $\lambda \in -\rho - 2\mathbb{Z}_{\geq 0}$.*

2.1 The non-compact picture

Let $\overline{N}$ be the nilradical of the parabolic subgroup opposite to P. Since $\overline{N}$ is unipotent, we identify it with its Lie algebra $\bar{\mathfrak{n}} \cong \mathbb{F}^n \oplus \operatorname{Im}\mathbb{F}$ in terms of the exponential map:

$$\mathbb{F}^n \oplus \operatorname{Im}\mathbb{F} \to \overline{N}, \quad (X,Z) \mapsto \bar{n}_{(X,Z)} := \exp\begin{pmatrix} \frac{Z}{2} & \frac{Z}{2} & X \\ -\frac{Z}{2} & -\frac{Z}{2} & -X \\ X^* & X^* & \mathbf{0}_n \end{pmatrix}.$$

Since $\overline{N}P$ is open and dense in G, the restriction of π_λ to functions on $\overline{N}$ is one-to-one. The resulting realization $I_\lambda \subseteq C^\infty(\bar{\mathfrak{n}})$ of π_λ is called the *non-compact picture of* π_λ. For $g \in \overline{N}MAN$ we write $g = \bar{n}(g)m(g)a(g)n(g)$ for the obvious decomposition. Then the G-action in the non-compact picture is given by

$$\pi_\lambda(g)f(X,Z) = a(g^{-1}\bar{n}_{(X,Z)})^{-(\lambda+\rho)} f(\log \bar{n}(g^{-1}\bar{n}_{(X,Z)})), \tag{2.1}$$

whenever $g^{-1}\bar{n}_{(X,Z)} \in \overline{N}MAN$.

The Lie bracket of $\bar{\mathfrak{n}} \cong \mathbb{F}^n \oplus \operatorname{Im}\mathbb{F}$ is given by

$$[(X,Z),(Y,W)] = (0, 4\operatorname{Im}(XY^*))$$

and the group multiplication is given by

$$(X,Z)\cdot(Y,W) = (X,Z) + (Y,W) + \frac{1}{2}[(X,Z),(Y,W)] = (X+Y, Z+W+2\operatorname{Im}(XY^*)).$$

We write $\bar{\mathfrak{n}} = \mathfrak{v} \oplus \mathfrak{z}$ with $\mathfrak{v} := \mathbb{F}^n$ and $\mathfrak{z} := \operatorname{Im}\mathbb{F}$, noting that $\mathfrak{z}$ is the center of $\bar{\mathfrak{n}}$. We endow $\mathfrak{v}$ and $\mathfrak{z}$ with the usual inner product given by

$$\langle X, Y\rangle = \operatorname{Re}(XY^*)$$

and extend it to $\bar{\mathfrak{n}}$ so that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. Then for all $Z \in \mathfrak{z}$ we obtain a linear skew-symmetric map $J_Z \in \operatorname{End}(\mathfrak{v})$ characterized by

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle.$$

We note that $J_Z X = -4Z\cdot X$, the scalar multiplication of $\mathbb{F}$ on $\mathbb{F}^n$. In particular we have $J_Z^2 = -16|Z|^2 \cdot \mathrm{id}_{\mathfrak{v}}$.

Remark 2.2. (i) Two-step nilpotent Lie algebras and their corresponding groups which satisfy the condition $J_Z^2 = -16|Z|^2\cdot\mathrm{id}_{\mathfrak{v}}$ are said to be of *H-type* and many constructions in this paper could be carried out in the more general setting of H-type groups.

(ii) Those H-type groups which occur as unipotent radicals of semisimple Lie groups of rank one are exactly the H-type groups satisfying the J^2-condition, namely for all $Z, Z' \in \mathfrak{z}$ with $\langle Z, Z'\rangle = 0$ there exists $Z'' \in \mathfrak{z}$ such that $J_Z J_{Z'} = J_{Z''}$ (see [CDKR98]).

We collect some facts that follow directly from the definitions above:

Lemma 2.3. *Let $X, X' \in \mathfrak{v}$ and $Z, Z' \in \mathfrak{z}$.*

(i) $\langle J_Z X, X\rangle = 0$,

(ii) $\langle J_Z X, X'\rangle = -\langle X, J_Z X'\rangle$,

(iii) $|J_Z X|^2 = 16|Z|^2|X|^2$,

(iv) $J_Z J_{Z'} + J_{Z'} J_Z = -32\langle Z, Z'\rangle \cdot \mathrm{id}_{\mathfrak{v}}$.

We now compute the representation π_λ on $\overline{P} = M A \overline{N}$ and on the representative $\tilde{w}_0 = \mathrm{diag}(-1, 1, \mathbf{1}_n)$ of the longest Weyl group element of G with respect to A. For this we first state some matrix decompositions which are easily verified.

For $(X, Z) \in \bar{\mathfrak{n}}$ let

$$N(X, Z) := (|X|^4 + |Z|^2)^{\frac{1}{4}}$$

denote the *norm function* of $\bar{\mathfrak{n}}$.

Lemma 2.4. *(i) Let $m = \mathrm{diag}(a, a, b^{-1}) \in M$ with $a \in \mathrm{U}(1;\mathbb{F})$ and $b \in \mathrm{U}(n;\mathbb{F})$, then*

$$m\bar{n}_{(X,Z)}m^{-1} = \bar{n}_{(aXb, aZa^{-1})}.$$

(ii) Let $t \in \mathbb{R}$ and $a = \exp(tH)$, then

$$a\bar{n}_{(X,Z)}a^{-1} = \bar{n}_{(e^{-t}X, e^{-2t}Z)}.$$

(iii) Let $(X, Z) \neq (0, 0)$, then $\tilde{w}_0\bar{n}_{(X,Z)} = \bar{n}_{(U,V)}man$ with $m \in M$, $n \in N$ and

$$U = \frac{(\frac{1}{4}J_Z - |X|^2)X}{N(X, Z)^4}, \qquad V = \frac{-Z}{N(X, Z)^4}, \qquad a = \exp(2\log(N(X, Z))H).$$

These decompositions immediately imply the following formulas for the action of P and $\tilde{w}_0$:

Proposition 2.5. *(i) For $m = \mathrm{diag}(a^{-1}, a^{-1}, b) \in M$ with $a \in \mathrm{U}(1;\mathbb{F})$ and $b \in \mathrm{U}(n;\mathbb{F})$:*

$$\pi_\lambda(m)u(X, Z) = u(aXb, aZa^{-1}).$$

(ii) For $t \in \mathbb{R}$ and $a = \exp(tH)$:

$$\pi_\lambda(a)u(X, Z) = e^{(\lambda+\rho)t}u(e^t X, e^{2t}Z).$$

(iii) For $(S, T) \in \bar{\mathfrak{n}}$:

$$\pi_\lambda(\bar{n}_{(S,T)})u(X, Z) = u(X - S, Z - T - \tfrac{1}{2}[S, X]).$$

(iv) For the action of $\tilde{w}_0$ we have

$$\pi_\lambda(\tilde{w}_0)u(X, Z) = N(X, Z)^{-2(\lambda+\rho)}u(\sigma(X, Z)).$$

where $\sigma : \bar{\mathfrak{n}} - \{0\} \to \bar{\mathfrak{n}} - \{0\}$ is the inversion given by

$$\sigma(X, Z) = \left(\frac{(\frac{1}{4}J_Z - |X|^2)X}{N(X, Z)^4}, \frac{-Z}{N(X, Z)^4}\right).$$

Note that $X \in \mathbb{F}^n$ is a row vector, so that matrix multiplication is from the right.

Remark 2.6. The map σ is the inversion defining the Kelvin-transform for H-type groups (see [CDKR91, Chapter 4]).

We now use Proposition 2.5 to compute derived representation $d\pi_\lambda$ of $\mathfrak{g}$ on I_λ. To state the action of the Lie algebra $\mathfrak{g}$ let $E_\mathfrak{v}$ denote the Euler operator on $\mathfrak{v}$ and let $E_\mathfrak{z}$ be the Euler operator on $\mathfrak{z}$. Then

$$E := E_\mathfrak{v} + 2E_\mathfrak{z}$$

defines the *weighted Euler operator* on $\bar{\mathfrak{n}}$ which takes into account homogeneity with respect to the action of A (see Lemma 2.4(ii)). For instance, the function $N(X,Z)$ is homogeneous of degree one with respect to the weighted Euler operator, i.e. $EN(X,Z) = N(X,Z)$.

Proposition 2.7. *(i) The element $H \in \mathfrak{a}$ acts by*

$$d\pi_\lambda(H) = E + \lambda + \rho.$$

(ii) For $S \in \mathfrak{v}$:

$$d\pi_\lambda(\mathrm{Ad}(\tilde{w}_0 S)) = -2\langle S, X\rangle(E+\lambda+\rho) + |X|^2\partial_S + \frac{1}{4}\partial_{J_Z S} - \frac{1}{2}|X|^2\partial_{[S,X]} + \frac{1}{8}\partial_{[S,J_Z X]} - \frac{1}{8}\partial_{J_{[S,X]}X}.$$

(iii) For $T \in \mathfrak{z}$:

$$d\pi_\lambda(\mathrm{Ad}(\tilde{w}_0 T)) = -\langle T, Z\rangle(E+E_\mathfrak{v}+\lambda+\rho) + N(X,Z)^4\partial_T + \frac{1}{4}|X|^2\partial_{J_T X} - \frac{1}{16}\partial_{J_T J_Z X}.$$

Proof. (i) follows immediately from Lemma 2.4(ii) by differentiation.

Ad (ii): Let $S \in \mathfrak{v}$. By Proposition 2.5(iii) the derived representation acts by the right-invariant vector fields $d\pi_\lambda(S) = -\partial_S - \frac{1}{2}\partial_{[S,X]}$. Conjugation with $\pi_\lambda(\tilde{w}_0) = \pi_\lambda(\tilde{w}_0^{-1})$ gives, using Proposition 2.5(iv):

$$\begin{aligned}\pi_\lambda(\tilde{w}_0)d\pi_\lambda(-S)\pi_\lambda(\tilde{w}_0^{-1})u(X,Z) = \Bigg(&\langle S, X\rangle 2(E+E_\mathfrak{v}+\lambda+\rho) - \partial_{(|X|^2+\frac{1}{4}J_Z)S} \\ &+ \frac{1}{2}\partial_{[S,(|X|^2-\frac{1}{4}J_Z)X]} - \frac{2\left\langle (|X|^2-\frac{1}{4}J_Z)X, S\right\rangle}{N(X,Z)^4}\partial_{(|X|^2-\frac{1}{4}J_Z)X} \\ &+ \frac{1}{8N(X,Z)^4}\partial_{J_{[S,(|X|^2-\frac{1}{4}J_Z)X]}(|X|^2-\frac{1}{4}J_Z)X}\Bigg)u(X,Z).\end{aligned} \tag{2.2}$$

Note that for $U, S \in \mathbb{F}^n$ we have

$$\langle S, U\rangle U - \frac{1}{16}J_{[S,U]}U = (\mathrm{Re}(SU^*) + \mathrm{Im}(SU^*))U = SU^*U,$$

which is for $U = (|X|^2 - \frac{1}{4}J_Z X)X$ equal to

$$SX^*\left(|X|^2 + \tfrac{1}{4}J_Z\right)\left(|X|^2 - \frac{1}{4}J_Z\right)X = N(X,Z)^4 SX^*X$$
$$= N(X,Z)^4\left(\langle S,X\rangle - \frac{1}{16}J_{[S,X]}X\right)X,$$

so that (2.2) is

$$\left(2\langle S,X\rangle(\rho+\lambda+E) - |X|^2\partial_S - \frac{1}{4}\partial_{J_Z S} + \frac{1}{2}|X|^2\partial_{[S,X]} - \frac{1}{8}\partial_{[S,J_Z X]} + \frac{1}{8}\partial_{J_{[S,X]}X}\right)$$

applied to $u(X,Z)$.

Ad (iii): As in the case before, for $T \in \mathfrak{z}$ the derived representation acts by the right-invariant vector field $d\pi_\lambda(T) = -\partial_T$. Then

$$\pi(\tilde{w}_0)d\pi(-T)\pi(\tilde{w}_0)u(X,Z)$$
$$= \left(\langle T,Z\rangle(E + E_{\mathfrak{v}} + \lambda + \rho) - \frac{1}{4}\partial_{J_T(|X|^2 - \frac{1}{4}J_Z)X} - N(X,Z)^4\partial_T\right)u(X,Z). \qquad \square$$

2.2 Strongly spherical reductive pairs

In view of Facts I and II we consider strongly spherical real reductive pairs of the form

$$(G,G') = (\mathrm{U}(1,n+1;\mathbb{F}), \mathrm{U}(1,m+1;\mathbb{F}) \times F),$$

where $F < \mathrm{U}(n-m;\mathbb{F})$. To avoid arbitrarily large finite groups, we assume throughout the paper that F is connected. The results for non-connected F can easily be deduced from our classification using classical invariant theory for the component group F/F_0. We further assume $0 \le m < n$, excluding the case $m = n$ which leads to the classical theory of intertwining operators between principal series for the group G. Note that for $\mathbb{F} = \mathbb{O}$ we have $n = 1$ which implies $m = 0$ and $G' = \mathrm{Spin}_0(1,8)$.

The intersection $P' := G' \cap P$ is a minimal parabolic subgroup of G' with Langlands decomposition $M'A'N'$ given by $M' = G' \cap M$, $A' = A$ and $N' = G' \cap N$. The opposite parabolic $\overline{P}' = M'A'\overline{N}'$ has unipotent radical $\overline{N}'$ with Lie algebra $\bar{\mathfrak{n}}' \cong \mathbb{F}^m \oplus \operatorname{Im}\mathbb{F}$. Again we write $\bar{\mathfrak{n}}' = \mathfrak{v}' + \mathfrak{z}$ with $\mathfrak{v}' = \mathbb{F}^m$ and $\mathfrak{z} = \operatorname{Im}\mathbb{F}$ and we denote by $\mathfrak{v}'' = \mathbb{F}^{n-m}$ the orthogonal complement of $\bar{\mathfrak{n}}'$ in $\bar{\mathfrak{n}}$, or equivalently $\mathfrak{v}'$ in $\mathfrak{v}$. Then

$$p' = \dim_{\mathbb{R}}\mathfrak{v}' = m \cdot \dim_{\mathbb{R}}\mathbb{F}, \qquad \text{and} \qquad p'' = \dim_{\mathbb{R}}\mathfrak{v}'' = (n-m)\cdot\dim_{\mathbb{R}}\mathbb{F}.$$

Note that for $m = 0$ the Lie algebra $\bar{\mathfrak{n}}' = \mathfrak{z}$ is abelian and hence $\mathfrak{g}' \cong \mathfrak{so}(1,q+1)$.

Remark 2.8. The classification in [KM14] and [KKPS19] (see also [Möl17, Theorem 1.6] for a complete table) shows that the pair (G,G') is strongly spherical if and only if $M'A' = \mathrm{U}(1;\mathbb{F}) \times \mathrm{U}(m;\mathbb{F}) \times F$ has an open orbit on the unit sphere in $\mathfrak{v}'' = \mathbb{F}^{n-m}$. We remark that in this case M' always acts transitively on the unit sphere in $\mathfrak{v}''$. In fact, $\mathfrak{v}'' = \mathbb{F}^{n-m}$ is of real dimension ≥ 2 and therefore its unit sphere is connected. Hence, an open orbit of the compact group M' on the unit sphere has to be the entire sphere. Moreover, the action of $\mathrm{U}(m;\mathbb{F})$ on $\mathbb{F}^{n-m}$ is trivial, so the group $\mathrm{U}(1;\mathbb{F}) \times F$ acts transitively on the unit sphere in $\mathbb{F}^{n-m}$.

In this paper we classify symmetry breaking operators between the spherical principal series representations π_λ of G and τ_ν of G'.

2.3 Orbit structure of G/P

The P'-orbits in G/P will be important in the following since Theorem 1.1 reduces the classification of symmetry breaking operators to the classification of P'-invariant distributions on G/P. Note that we have $N' = \tilde{w}_0 \overline{N}' \tilde{w}_0^{-1}$, so that N' acts on $\tilde{w}_0 \overline{N}'$ by left multiplication.

Proposition 2.9. *The P'-orbits in G/P and their closure relations are*

$$\mathcal{O}_A \xrightarrow{\quad p'' \quad} \mathcal{O}_B \xrightarrow{\quad p'+q \quad} \mathcal{O}_C,$$

where

$$\begin{aligned}
\mathcal{O}_A &= P' \cdot \tilde{w}_0 \bar{n} P = \tilde{w}_0(\overline{N} - \overline{N}')P, \\
\mathcal{O}_B &= P' \cdot \tilde{w}_0 P = \tilde{w}_0 \overline{N}' P, \\
\mathcal{O}_C &= P' \cdot \mathbf{1}_{n+2} P,
\end{aligned}$$

for some $\bar{n} \in \overline{N} - \overline{N}'$. Here $X \xrightarrow{\quad k \quad} Y$ means that Y is a subvariety of $\bar{X}$ of co-dimension k.

For the proof of Proposition 2.9 we write $X = (X', X'') \in \mathbb{F}^m \oplus \mathbb{F}^{n-m}$ instead of X whenever convenient.

Proof of Proposition 2.9. First note that by the Bruhat decomposition $G = \tilde{w}_0 \overline{N} P \sqcup P$. The closed Bruhat cell P clearly becomes the closed orbit $\mathcal{O}_C$ in the quotient G/P. For the open Bruhat cell $\tilde{w}_0 \overline{N} P$ we note that since $P' \tilde{w}_0 = \tilde{w}_0 \overline{P'}$ it suffices to describe the $\overline{P}'$-orbits in $\overline{N}P/P$. The nilpotent group $\overline{N}$ decomposes as $\overline{N} = \overline{N}' \exp(\mathfrak{v}'')$. If we write $\overline{u} = \overline{u}' \exp(X) \in \overline{N}$ with $\overline{u}' \in \overline{N}'$ and $X \in \mathfrak{v}''$, then

$$\overline{n}' a' m' \cdot [\overline{u}' \exp(X) P] = (\overline{n}' \overline{u}'^{(m'a')}) \exp(\mathrm{Ad}(m'a')X) P,$$

where we use the notation $\overline{u}'^{(m'a')} = (m'a') \overline{u}' (m'a')^{-1} \in \overline{N}'$. This shows that the $\overline{P}'$-orbits in $\overline{N}P/P$ are of the form $\overline{N}' \exp(\mathrm{Ad}(M'A')X)P/P$ with $X \in \mathfrak{v}''$ and therefore in one-to-one correspondence with the $\mathrm{Ad}(M'A')$-orbits in $\mathfrak{v}''$. By Remark 2.8 the group $\mathrm{Ad}(M')$ acts transitively on the unit sphere in $\mathfrak{v}''$ and by Lemma 2.4(ii) the group $\mathrm{Ad}(A')$ acts on $\mathfrak{v}''$ by dilations, so the $\mathrm{Ad}(M'A')$-orbits in $\mathfrak{v}''$ are $\mathfrak{v}'' - \{0\}$ and $\{0\}$. Therefore, the P'-orbits in the open Bruhat cell $\tilde{w}_0 \overline{N} P$ are $\mathcal{O}_A = \tilde{w}_0 \overline{N}' \exp(\mathfrak{v}'' - \{0\}) P = \tilde{w}_0(\overline{N} - \overline{N}')P$ and $\mathcal{O}_B = \tilde{w}_0 \overline{N}' \exp(\{0\}) P = \tilde{w}_0 \overline{N}' P$. Finally, the codimensions are easily determined. $\square$

The orbit structure of G/P implies the following:

Corollary 2.10. $P' \overline{N} P = G$. *More precisely,*

$$\mathcal{O}_A \cap \overline{N} = \overline{N} - \overline{N}', \qquad \mathcal{O}_B \cap \overline{N} = \overline{N}' - \{\mathbf{1}\}, \qquad \mathcal{O}_C \cap \overline{N} = \{\mathbf{1}\}.$$

Proof. It is clear that $\mathcal{O}_C \cap \overline{N} = P \cap \overline{N} = \{\mathbf{1}\}$. The remaining two identities are easily verified if one observes that by Lemma 2.4(iii) the map $xP \mapsto \tilde{w}_0 x P$ maps $(\overline{N} - \{\mathbf{1}\})P$ and $(\overline{N}' - \{\mathbf{1}\})P$ to itself and it further maps $\tilde{w}_0 P$ to $\mathbf{1}P$ and vice versa. $\square$

Theorem 1.2 and Corollary 2.10 allow us to reduce the classification of symmetry breaking operators to the classification of certain invariant distributions on the Lie algebra $\bar{\mathfrak{n}}$.

3 Invariant distribution kernels

In this section we give a characterization of the invariant distribution kernels on $\bar{\mathfrak{n}}$ describing symmetry breaking operators in terms of a set of differential equations and describe the strategy that is used in the remaining part of the paper to find all solutions to these equations.

3.1 Differential equations satisfied by symmetry breaking operators

With Remark 1.3 as well as Proposition 2.5(iv) and Proposition 2.7 we immediately obtain the following description of the space $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ of distribution kernels of symmetry breaking operators:

Lemma 3.1. *The space $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ is given by all $u \in \mathcal{D}'(\bar{\mathfrak{n}})$ satisfying:*

(i) $u(aX'b, aX''c, aZa^{-1}) = u(X', X'', Z)$ *for all* $a \in \mathrm{U}(1;\mathbb{F})$, $b \in \mathrm{U}(m;\mathbb{F})$ *and* $c \in F$*,*

(ii) $(E - \lambda + \rho + \nu + \rho')u(X, Z) = 0$*,*

(iii) $D_{\mathfrak{v}}(S)u(X, Z) = 0$ *for all* $S \in \mathfrak{v}'$*,*

(iv) $D_{\mathfrak{z}}(T)u(X, Z) = 0$ *for all* $T \in \mathfrak{z}$*,*

where for $S \in \mathfrak{v}$ *and* $T \in \mathfrak{z}$ *we put*

$$D_{\mathfrak{v}}(S) = 2\langle S, X\rangle(\nu + \rho') + |X|^2\partial_S - \frac{1}{2}|X|^2\partial_{[S,X]} + \frac{1}{4}\partial_{J_Z S} + \frac{1}{8}\partial_{[S,J_Z X]} - \frac{1}{8}\partial_{J_{[S,X]}X},$$
$$D_{\mathfrak{z}}(T) = \langle T, Z\rangle(\nu + \rho' - E_{\mathfrak{v}}) + N(X, Z)^4\partial_T + \frac{1}{4}|X|^2\partial_{J_T X} - \frac{1}{16}\partial_{J_T J_Z X}.$$

Remark 3.2. In case $m > 0$ the subspace $\mathfrak{v}'$ generates $\mathfrak{n}'$, so that the invariance property in Lemma 3.1(iv) follows from the one in Lemma 3.1(iii). For $m = 0$ the property in Lemma 3.1(iii) is trivial since $\mathfrak{v}' = \{0\}$.

3.2 The classification strategy

Let $\Omega \subseteq \bar{\mathfrak{n}}$ be open and M'-invariant. For a closed subset $\Omega' \subseteq \Omega$ we write $\mathcal{D}'_{\Omega'}(\Omega)_{\lambda,\nu} \subseteq \mathcal{D}'(\Omega)_{\lambda,\nu}$ for the subspace of distributions with support contained in Ω'. In particular if $\Omega' = \bar{\mathfrak{n}} - \Omega$ the following sequence is exact:

$$0 \longrightarrow \mathcal{D}'_{\Omega'}(\bar{\mathfrak{n}})_{\lambda,\nu} \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}} - \Omega')_{\lambda,\nu}$$

For $\Omega' = \{0\}$ the space $\mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu}$ consists of those distribution kernels which define differential symmetry breaking operators. The strategy to determine $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ will be to classify $\mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu}$ and $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$ separately and finally to study the restriction map $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} \to \mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$. This strategy was proposed by Kobayashi–Speh in [KS15] and successfully applied in the case $\mathbb{F} = \mathbb{R}$. Since the classification of differential symmetry breaking operators is a question of invariant theory and combinatorics, while the classification of of symmetry breaking operators outside the origin and the continuation of such operators into the origin is a question of distribution theory, we divide the classification into two major parts: the classification of differential symmetry breaking operators (Part II) and the study of meromorphic families of symmetry breaking operators which eventually leads to the full classification (Part III).

Part II

Differential symmetry breaking operators

The main result of this part is the full classification of differential symmetry breaking operators between principal series representations for strongly spherical pairs of the form $(G, G') = (\mathrm{U}(1, n+1; \mathbb{F}), \mathrm{U}(1, m+1; \mathbb{F}) \times F)$ with $0 \leq m < n$ and $F < \mathrm{U}(n-m; \mathbb{F})$. For this we use a Euclidean Fourier transform on $\bar{\mathfrak{n}}$ which gives an equivalent formulation of the classification problem in terms of polynomial solutions to a certain system of differential equations. This approach is essentially the F-method proposed by Kobayashi [Kob13] which was previously applied in several situations where the nilpotent radical is abelian (see e.g. [KØSS15, KKP16, KP16]).

4 The Fourier transformed picture

We use the following normalization of the Fourier transform:

$$\mathcal{F} : \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}}) \to \mathbb{C}[\bar{\mathfrak{n}}], \quad \mathcal{F}u(X, Z) = \int_{\bar{\mathfrak{n}}} e^{-\langle X, X' \rangle - \langle Z, Z' \rangle} u(X', Z')\, d(X', Z').$$

For a differential operator D on $\bar{\mathfrak{n}}$ with polynomial coefficients there exists a unique differential operator $\mathcal{F}(D)$ with polynomial coefficients on the same space such that

$$\mathcal{F}(Du) = \mathcal{F}(D)\mathcal{F}(u) \qquad \text{for } u \in \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}}).$$

This map has the following properties:

$$\mathcal{F}(\langle X, S \rangle) = -\partial_S, \qquad \mathcal{F}(\partial_S) = \langle X, S \rangle$$

and similar for Z. We define

$$\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu} := \mathcal{F}(\mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu}).$$

To characterize the space $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ we have to transform the differential equations in Lemma 3.1. For this let $S_1, \ldots, S_p$ be an orthonormal basis of $\mathfrak{v}$ such that $S_1, \ldots S_{p'}$ is an orthonormal basis of $\mathfrak{v}'$ and $S_{p'+1}, \ldots, S_p$ an orthonormal basis of $\mathfrak{v}''$ and let $T_1, \ldots T_q$ be an orthonormal basis of $\mathfrak{z}$. Then we define the Euclidean Laplacians on $\mathfrak{v}$, $\mathfrak{v}'$, $\mathfrak{v}''$ by

$$\Delta_{\mathfrak{v}} := \partial_{S_1}^2 + \cdots + \partial_{S_p}^2, \qquad \Delta_{\mathfrak{v}'} := \partial_{S_1}^2 + \cdots + \partial_{S_{p'}}^2, \qquad \Delta_{\mathfrak{v}''} := \Delta_{\mathfrak{v}} - \Delta_{\mathfrak{v}'}$$

and the Euclidean Laplacian on $\mathfrak{z}$:

$$\Box = \partial_{T_1}^2 + \cdots + \partial_{T_q}^2.$$

Proposition 4.1. *For $S \in \mathfrak{v}$ and $T \in \mathfrak{z}$ we have*

$$-\mathcal{F}(D_{\mathfrak{v}}(S)) = 2(\nu + \rho' - 1)\partial_S - \langle X, S\rangle\Delta_{\mathfrak{v}} - \frac{1}{2}\partial_{J_Z S}\Delta_{\mathfrak{v}} + \frac{1}{4}\partial_{[S,X]} + 2P_S - 2Q_S,$$
$$-\mathcal{F}(D_{\mathfrak{z}}(T)) = (E_{\mathfrak{v}} + \nu + \rho' - 2)\partial_T - \langle Z, T\rangle(\Delta_{\mathfrak{v}}^2 + \Box) - \frac{1}{4}\partial_{J_T X}\Delta_{\mathfrak{v}} + R_T$$

with

$$P_S = \frac{1}{16}\sum_{j=1}^{q}\partial_{J_{T_j}J_Z S}\partial_{T_j}, \qquad Q_S = \frac{1}{16}\sum_{i=1}^{p}\partial_{J_{[S,S_i]}X}\partial_{S_i}, \qquad R_T = \frac{1}{16}\sum_{j=1}^{q}\partial_{J_{T_j}J_T X}\partial_{T_j}.$$

To show the proposition, we first prove some technical identities.

Lemma 4.2. *For $S \in \mathfrak{v}$ and $T \in \mathfrak{z}$ we have*

(i) $[\Delta_{\mathfrak{v}}, \partial_{J_T X}] = 0,$

(ii) $[\Delta_{\mathfrak{v}}, \partial_{J_Z S}] = 0,$

(iii) $[\Box, \partial_{[J_Z S, X]}] = -32\langle S, X\rangle\Box.$

Proof. Ad (i): Since J_T is skew-symmetric we have

$$[\Delta_{\mathfrak{v}}, \partial_{J_T X}] = 2\sum_{i=0}^{p}\partial_{J_T S_i}\partial_{S_i} = 0.$$

Ad (ii): This is clear since $J_Z S$ is independent of $X \in \mathfrak{v}$.
Ad (iii): We have

$$\begin{aligned}
[\Box, \partial_{[J_Z S, X]}] &= 2\sum_{j=1}^{q}\partial_{[J_{T_j}S, X]}\partial_{T_j} \\
&= 2\sum_{j,k=1}^{q}\langle T_k, [J_{T_j}S, X]\rangle\partial_{T_j}\partial_{T_k} \\
&= 2\sum_{j,k=1}^{q}\langle J_{T_k}J_{T_j}S, X\rangle\partial_{T_j}\partial_{T_k} \\
&= \sum_{j,k=1}^{q}\langle (J_{T_j}J_{T_k} + J_{T_k}J_{T_j})S, X\rangle\partial_{T_j}\partial_{T_k} \\
&= -32\sum_{j,k=1}^{q}\langle T_j, T_k\rangle\langle S, X\rangle\partial_{T_j}\partial_{T_k} \\
&= -32\langle S, X\rangle\Box.
\end{aligned}$$

□

Lemma 4.3. *For $X \in \mathfrak{v}$:*

$$\sum_{i=1}^{p} J_{[X,S_i]}S_i = -16qX.$$

Proof. Since $\langle X, Y\rangle = \mathrm{Re}(XY^*)$, $[X,Y] = 4\,\mathrm{Im}(XY^*)$ and $J_Z X = -4ZX$ we have

$$\begin{aligned}
\sum_{i=1}^{p} J_{[X,S_i]}S_i &= -16\sum_{i=1}^{p}\mathrm{Im}(XS_i^*)S_i \\
&= -16\sum_{i=1}^{p}(XS_i^*)S_i + 16\sum_{i=1}^{p}\langle X, S_i\rangle S_i \\
&= -16X\sum_{i=1}^{p}S_i^*S_i + 16X \\
&= -16(q+1)X + 16X \\
&= -16qX,
\end{aligned}$$

where we have used $\sum_{i=1}^{p} S_i^* S_i = \dim_{\mathbb{R}}\mathbb{F}\cdot\mathbf{1}_n$. □

Proof of Proposition 4.1. Using Lemma 4.2 it is easily seen that

$$\begin{aligned}
\mathcal{F}(|X|^2\partial_S) &= \langle S, X\rangle\Delta_{\mathfrak{v}} + 2\partial_S, \\
\mathcal{F}(|X|^2\partial_{[S,X]}) &= -\partial_{J_Z S}\Delta_{\mathfrak{v}}, \\
\mathcal{F}(\partial_{J_Z S}) &= -\partial_{[S,X]}.
\end{aligned}$$

We further have

$$\begin{aligned}
\mathcal{F}(\partial_{[S,J_Z X]}) &= \mathcal{F}\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\langle X, S_i\rangle\langle Z, T_j\rangle\partial_{[S,J_{T_j}S_i]}\right) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\partial_{S_i}\partial_{T_j}\langle Z, [S, J_{T_j}S_i]\rangle \\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\Big(\langle Z, [S, J_{T_j}S_i]\rangle\partial_{T_j} + \langle T_j, [S, J_{T_j}S_i]\rangle\Big)\,\partial_{S_i} \\
&= -\sum_{i=1}^{p}\sum_{j=1}^{q}\Big(\langle J_{T_j}J_Z S, S_i\rangle\partial_{T_j} + \langle J_{T_j}^2 S, S_i\rangle\Big)\,\partial_{S_j} \\
&= -16P_S + 16q\partial_S
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{F}(\partial_{J_{[S,X]}X}) &= \mathcal{F}\Big(\sum_{i,j=1}^{p}\langle X,S_i\rangle\langle X,S_j\rangle\partial_{J_{[S,S_i]}S_j}\Big)\\
&= \sum_{i,j=1}^{p}\partial_{S_i}\partial_{S_j}\langle X,J_{[S,S_i]}S_j\rangle\\
&= \sum_{i,j=1}^{p}\partial_{S_i}\langle X,J_{[S,S_i]}S_j\rangle\partial_{S_j}\\
&= -\sum_{i,j=1}^{p}\partial_{S_i}\langle J_{[S,S_i]}X,S_j\rangle\partial_{S_j}\\
&= -\sum_{i,j=1}^{p}\Big(\langle J_{[S,S_i]}X,S_j\rangle\partial_{S_i}+\langle J_{[S,S_i]}S_i,S_j\rangle\Big)\partial_{S_j}\\
&= -16Q_S+16q\partial_S,
\end{aligned}
$$

where Lemma 4.3 was used in the last step. Then the first identity follows.

For the second identity a short computation shows that

$$
\begin{aligned}
\mathcal{F}(E_{\mathfrak{v}}) &= -E_{\mathfrak{v}}-p, & \mathcal{F}(|X|^4\partial_T) &= \langle Z,T\rangle\Delta_{\mathfrak{v}}^2,\\
\mathcal{F}(|X|^2\partial_{J_TX}) &= \partial_{J_TX}\Delta_{\mathfrak{v}}, & \mathcal{F}(|Z|^2\partial_T) &= \langle Z,T\rangle\Box+2\partial_T.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
\mathcal{F}(\partial_{J_TJ_ZX}) &= \mathcal{F}\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\langle X,S_i\rangle\langle Z,T_j\rangle\partial_{J_TJ_{T_j}S_i}\right)\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\partial_{S_i}\partial_{T_j}\langle X,J_TJ_{T_j}S_i\rangle\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\Big(\langle X,J_TJ_{T_j}S_i\rangle\partial_{S_i}+\langle S_i,J_TJ_{T_j}S_i\rangle\Big)\partial_{T_j}\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}\langle J_{T_j}J_TX,S_i\rangle\partial_{S_i}\partial_{T_j}+\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{q}\langle S_i,(J_TJ_{T_j}+J_{T_j}J_T)S_i\rangle\partial_{T_j}\\
&= 16R_T-16p\partial_T,
\end{aligned}
$$

where Lemma 2.3(iv) was used in the last step. Putting these identities together shows the second formula. □

Now applying the Fourier transform to the differential equations of Lemma 3.1 we obtain:

Lemma 4.4. *The space $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ is given by all $u\in\mathbb{C}[\bar{\mathfrak{n}}]$ satisfying:*

(i) $(E+\lambda+\rho-\nu-\rho')u(X,Z)=0$,

(ii) $u(aX'b,aX''c,aZa^{-1})=u(X',X'',Z)$ *for all* $a\in\mathrm{U}(1;\mathbb{F})$, $b\in\mathrm{U}(m;\mathbb{F})$ *and* $c\in F$,

(iii) $\Big(2(\nu+\rho'+q-2)\partial_S - (X,S)\Delta_{\mathfrak{v}} - \frac{1}{2}\partial_{J_Z S}\Delta_{\mathfrak{v}} + \frac{1}{4}\partial_{[S,X]} + 2P_S - 2Q_S\Big)u(X,Z) = 0$, *for all* $S \in \mathfrak{v}'$,

(iv) $\Big((E_{\mathfrak{v}}+\nu+\rho'-2)\partial_T - (Z,T)(\Delta_{\mathfrak{v}}^2+\Box) + \frac{1}{4}\partial_{J_T X}\Delta_{\mathfrak{v}} + R_T\Big)u(X,Z) = 0$, *for all* $T \in \mathfrak{z}$.

5 Polynomial Invariants

Consider the ring $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ of polynomials which are invariant under the M' action given in Lemma 4.4(ii), then $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu} \subseteq \mathbb{C}[\bar{\mathfrak{n}}]^{M'}$. The first step in the classification of $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ is to find generators of $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$.

Lemma 5.1. *(i) For $\mathbb{F}=\mathbb{C}$ the group M' acts transitively on $S^{p'-1}\times S^{p''-1}\times\{1\}$.*

(ii) For $\mathbb{F}=\mathbb{H}$ with either $m<n-1$ or $m=n-1$ and $F=\mathrm{Sp}(1)=\mathrm{U}(1;\mathbb{H})$, the group M' acts transitively on $S^{p'-1}\times S^{p''-1}\times S^{q-1}$.

(iii) For $\mathbb{F}=\mathbb{O}$ the group M' acts transitively on $S^{p'-1}\times S^{p''-1}\times S^{q-1}$.

Here we use the convention $S^{-1}=\{0\}\subseteq\mathbb{F}^0$.

Proof. Recall that by Remark 2.8 the group $M'=\mathrm{U}(1;\mathbb{F})\times\mathrm{U}(m;\mathbb{F})\times F$ acts transitively on the unit sphere in $\mathfrak{v}''=\mathbb{F}^{n-m}$.

(i) For $\mathbb{F}=\mathbb{C}$ we have $M'=\mathrm{U}(1)\times\mathrm{U}(m)\times F$ with the action on $\bar{\mathfrak{n}}=\mathbb{C}^m\oplus\mathbb{C}^{n-m}\oplus i\mathbb{R}$ given by $(a,b,c)\cdot(X',X'',Z)=(aX'b,aX''c,Z)$. Since $\mathrm{U}(m)$ acts transitively on $S^{p'-1}=S^{2m-1}$ and $\mathrm{U}(1)\times F$ acts transitively on $S^{p''-1}$ by Remark 2.8, the claim follows.

(ii) For $\mathbb{F}=\mathbb{H}$ we have $M'=\mathrm{Sp}(1)\times\mathrm{Sp}(m)\times F$ with the action on $\bar{\mathfrak{n}}=\mathbb{H}^m\oplus\mathbb{H}^{n-m}\oplus\mathrm{Im}\,\mathbb{H}$ given by $(a,b,c)\cdot(aX'b,aX''c,aZa^{-1})$. Assume first that $m<n-1$, then $\mathrm{Sp}(1)$ cannot act transitively on $S^{p''-1}$, so F has to act transitively on it. Then $\mathrm{Sp}(m)$ acts transitively on $S^{p'-1}=S^{4m-1}$, F acts transitively on $S^{p''-1}$ and $\mathrm{Sp}(1)$ acts transitively on $S^{q-1}=S^2$ and the claim follows. For $m=n-1$ and $F=\mathrm{Sp}(1)=\mathrm{U}(1;\mathbb{H})$, the argument is similar.

(iii) For $\mathbb{F}=\mathbb{O}$ the group $M'=\mathrm{Spin}(7)$ acts on $\mathfrak{v}=\mathfrak{v}''=\mathbb{R}^8$ by the spin representation and on $\mathfrak{z}=\mathbb{R}^7$ by the covering map $\mathrm{Spin}(7)\to\mathrm{SO}(7)$. Clearly the action of $\mathrm{SO}(7)$ on $S^6\subseteq\mathbb{R}^7$ is transitive. The stabilizer of a point in S^6 is isomorphic to $\mathrm{Spin}(6)\simeq\mathrm{SU}(4)$ which acts on $\mathbb{R}^8\simeq\mathbb{C}^4$ by the standard representation and hence transitively on its unit sphere. $\Box$

We now determine the ring $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ of invariants. The most complicated situation occurs for $\mathbb{F}=\mathbb{H}$ and $m=n-1$. For $(X,Z)\in\mathfrak{v}''\oplus\mathfrak{z}=\mathbb{H}\oplus\mathrm{Im}\,\mathbb{H}$ we write

$$p_1(X,Z) := \langle \mathbf{i}, \overline{X}ZX\rangle, \qquad p_2(X,Z) := \langle \mathbf{j}, \overline{X}ZX\rangle, \qquad p_3(X,Z) := \langle \mathbf{k}, \overline{X}ZX\rangle.$$

Lemma 5.2. *(i) For $\mathbb{F}=\mathbb{C}$, the ring $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is generated by $|X'|^2$, $|X''|^2$ and Z.*

(ii) *For $\mathbb{F} = \mathbb{H}$, $m = n-1$ and $F = \{1\}$, the ring $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is generated by $|X'|^2$, $|X''|^2$, $|Z|^2$ and $p_1(X'', Z)$, $p_2(X'', Z)$, $p_3(X'', Z)$.*

(iii) *For $\mathbb{F} = \mathbb{H}$, $m = n-1$ and $F = \exp(\mathbb{R}U) \simeq \mathrm{U}(1)$, $U \in \operatorname{Im}\mathbb{H} = \mathfrak{sp}(1) = \mathfrak{u}(1;\mathbb{H})$, the ring $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is generated by $|X'|^2$, $|X''|^2$, $|Z|^2$ and $\langle U, \overline{X''}ZX''\rangle$.*

(iv) *In all other cases $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is generated by $|X'|^2$, $|X''|^2$ and $|Z|^2$.*

Proof. Statements (i) and (iv) follow immediately from Lemma 5.1, so we only consider the case $\mathbb{F} = \mathbb{H}$ with $m = n-1$. Here $M' = \mathrm{Sp}(1) \times \mathrm{Sp}(n-1) \times F$ acts on $\mathbb{H}^{n-1} \oplus \mathbb{H} \oplus \operatorname{Im}\mathbb{H}$ by $(a, b, c)\cdot(X', X'', Z) = (aX'b, aX''c, aZa^{-1})$. Assume first that $F = \{1\}$, then clearly $|X'|^2$, $|X''|^2$, $|Z|^2$ and $p_1(X'', Z)$, $p_2(X'', Z)$, $p_3(X'', Z)$ are M'-invariant. Further, the transitivity of $\mathrm{Sp}(n-1)$ on the unit sphere in $\mathbb{H}^{n-1}$ implies that the only invariant polynomial in X' is $|X'|^2$. It remains to show that the $\mathrm{Sp}(1)$-invariants in $\mathbb{H} \oplus \operatorname{Im}\mathbb{H}$ are generated by $|X''|^2$, $|Z|^2$ and $p_1(X'', Z)$, $p_2(X'', Z)$, $p_3(X'', Z)$.

The complexification of $\mathrm{Sp}(1)$ is $\mathrm{SL}(2, \mathbb{C})$. The complexification of $\mathbb{H}$ is the direct sum of two copies of the 2-dimensional complex representation V of $\mathrm{SL}(2, \mathbb{C})$. The complexification of $\operatorname{Im}\mathbb{H}$ is the 3-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$. The latter representation can be viewed as the representation of $\mathrm{SL}(2, \mathbb{C})$ on the space F_2 of homogeneous forms on V of degree 2. By [KP96, Exercise 7 in §4] the ring $\mathbb{C}[V \oplus V \oplus F_2]^{\mathrm{GL}(2,\mathbb{C})}$ of $\mathrm{GL}(2, \mathbb{C})$-invariant polynomials on $V \oplus V \oplus F_2$ is generated by ε_0, ε_1 and ε_2 which are given by $\varepsilon_i(v, w, f) = f_i(v, w)$ with $f(sv + tw) = \sum_{i=0}^{2} s^i t^{2-i} f_i(v, w)$. An easy computation shows that $\varepsilon_0, \varepsilon_1, \varepsilon_2$ correspond to p_1, p_2, p_3. Note that an element $\lambda \cdot I$ of the center $\mathbb{C}^\times \cdot I$ of $\mathrm{GL}(2, \mathbb{C})$ acts on V by λ and on F_2 by λ^{-2}. It follows that p_1, p_2 and p_3 generate the ring $\mathbb{C}[\mathbb{H} \oplus \operatorname{Im}\mathbb{H}]^{\mathrm{Sp}(1)\cdot\mathbb{R}^\times}$, where $\lambda \in \mathbb{R}^\times$ acts on $\mathbb{H}$ by λ and on $\operatorname{Im}\mathbb{H}$ by λ^{-2}.
Now let $f \in \mathbb{C}[\mathbb{H} \oplus \operatorname{Im}\mathbb{H}]^{\mathrm{Sp}(1)}$ be merely $\mathrm{Sp}(1)$-invariant. Since the action of $\mathrm{Sp}(1)$ and $\mathbb{R}^\times$ commute, we may assume that $\lambda \in \mathbb{R}^\times$ acts on f by λ^m, $m \in \mathbb{Z}$. If m is odd then $\lambda = -1 \in \mathbb{R}^\times$ acts on f by -1, but on the other hand f is invariant under $-1 \in \mathrm{Sp}(1)$, so $f = 0$. If m is even we write $m = -2k + 4\ell$ with $k, \ell \in \mathbb{Z}_{\geq 0}$, then $f \cdot |X''|^{2k}|Z|^{2\ell}$ is $\mathbb{R}^\times$-invariant and therefore contained in $\mathbb{C}[\mathbb{H} \oplus \operatorname{Im}\mathbb{H}]^{\mathrm{Sp}(1)\cdot\mathbb{R}^\times}$ which is generated by p_1, p_2 and p_3. This shows (ii).

Finally, (iii) follows by inspecting which polynomial expression in p_1, p_2 and p_3 is invariant under $F = \exp(\mathbb{R}U)$. □

In particular Lemma 5.2 implies that the degree of every generator of $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is even. Hence the degree of every element of $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ is even, so that Lemma 4.4(i) implies:

Corollary 5.3. *If $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu} \neq \{0\}$, then $(\lambda, \nu) \in /\!/$.*

6 Classification of differential symmetry breaking operators: holomorphic families

We first treat the case of polynomials in $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ which only depend on $|X'|^2$, $|X''|^2$ and $|Z|^2$. Note that if $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is generated by $|X'|^2$, $|X''|^2$ and $|Z|^2$, then these are in fact all polynomials in $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$.

Let $(\lambda,\nu)\in /\!/$ and write $\lambda+\rho-\nu-\rho'=-2k\in-2\mathbb{Z}_{\geq0}$. We define the distribution

$$\widehat{u}^C_{\lambda,\nu}:=\sum_{h+i+2j=k}c_{h,i,j}(\lambda,\nu)|X'|^{2h}|X''|^{2i}|Z|^{2j},\tag{6.1}$$

where the scalars $c_{h,i,j}(\lambda,\nu)$ are for $m>0$ given by

$$c_{h,i,j}(\lambda,\nu)=\frac{2^{-2i-2h}\Gamma(\frac{2\nu+p'+2}{4})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2}+i)}{h!i!j!\Gamma(\frac{2\nu+p'+2}{4}-j)\Gamma(\frac{p''}{2}+i)\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})}\tag{6.2}$$

and for $m=0$ by

$$c_{h,i,j}(\lambda,\nu)=\frac{2^{-i}\Gamma(\frac{\nu}{2}-j)}{i!j!\Gamma(\frac{p}{2}+i)\Gamma(\frac{\nu}{2}-\lfloor\frac{k}{2}\rfloor)}.\tag{6.3}$$

A close inspection of the gamma factors shows that $c_{h,i,j}(\lambda,\nu)$ is holomorphic in λ and ν and hence $u^C_{\lambda,\nu}$ depends holomorphically on $(\lambda,\nu)\in /\!/$. In the case $m=0$ we have $\mathfrak{v}'=0$ and hence $|X'|^{2h}=0$ for $h>0$, so the summation is over $i+2j=k$ with $h=0$.

Theorem 6.1. *Let $(\lambda,\nu)\in /\!/$, then $\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}\cap\mathbb{C}[|X'|^2,|X''|^2,|Z|^2]=\mathbb{C}\widehat{u}^C_{\lambda,\nu}$. In particular, if $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is generated by $|X'|^2$, $|X''|^2$ and $|Z|^2$, then*

$$\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}=\mathbb{C}\widehat{u}^C_{\lambda,\nu}.$$

The proof of this statement uses the following combinatorial description of differential symmetry breaking operators:

Proposition 6.2. *Let $\lambda+\rho-\nu-\rho'=-2k\in-2\mathbb{Z}_{\geq0}$ and let $\widehat{u}\in\mathbb{C}[\bar{\mathfrak{n}}]$ be of the form*

$$\widehat{u}=\sum_{h+i+2j=k}c_{h,i,j}|X'|^{2h}|X''|^{2i}|Z|^{2j},$$

with scalars $c_{h,i,j}\in\mathbb{C}$. Then $\widehat{u}\in\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ if and only if the following relations are satisfied:

(i) If $m>0$ and $h+i+2j=k$:

$$h(\lambda+\rho+\nu-\rho'+2i)c_{h,i,j}=(i+1)(2i+p'')c_{h-1,i+1,j},\tag{6.4}$$

for $h>0$ and

$$jc_{h,i,j}=4(h+1)(h+2)(2h+p'+2)c_{h+2,i,j-1}+4(h+1)(i+1)(2i+p'')c_{h+1,i+1,j-1},\tag{6.5}$$

for $j>0$.

(ii) If $m=0$ and $h+i+2j=k$:

$$j(\nu+\rho'-q-2j)c_{h,i,j}=2(i+1)(i+2)(2i+p)(2i+p+2)c_{h,i+2,j-1}\tag{6.6}$$

for $j>0$.

To prove the proposition we need some technical identities.

Lemma 6.3. *Let $h,i,j\geq0$.*

(i) $\Delta_{\mathfrak{v}}|X'|^{2h}|X''|^{2i} = 2h(2h+p'-2)|X'|^{2h-2}|X''|^{2i} + 2i(2i+p''-2)|X'|^{2h}|X''|^{2i-2}$.

For $S \in \mathfrak{v}'$ we have

(ii) $\partial_{[S,X]}|X'|^{2h}|X''|^{2i}|Z|^{2j} = \frac{j}{h+1}\partial_{J_Z S}|X'|^{2h+2}|X''|^{2i}|Z|^{2j-2}$,

(iii) $P_S|X'|^{2h}|X''|^{2i}|Z|^{2j} = -2j\partial_S|X'|^{2h}|X''|^{2i}|Z|^{2j}$,

(iv) $Q_S|X'|^{2h}|X''|^{2i} = q\partial_S|X'|^{2h}|X''|^{2i}$.

For $T \in \mathfrak{z}$ we have

(v) $\partial_{J_T X}|X'|^{2h}|X''|^{2i} = 0$,

(vi) $R_T|X'|^{2h}|X''|^{2i}|Z|^{2j} = -4j(h+i)\langle Z,T\rangle|X'|^{2h}|X''|^{2i}|Z|^{2j-2}$.

Proof. Ad (i): This follow from $\Delta_{\mathfrak{v}} = \Delta_{\mathfrak{v}'} + \Delta_{\mathfrak{v}''}$ and the well known identity

$$\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}(x_1^2+\cdots+x_p^2)^l = l(l+p-2)(x_1^2+\cdots+x_p^2)^{l-2}.$$

Ad (ii): Clearly

$$\begin{aligned}\partial_{[S,X]}|X'|^{2h}|X''|^{2i}|Z|^{2j} &= 2j\langle[S,X],Z\rangle|X'|^{2h}|X''|^{2i}|Z|^{2j-2}\\ &= 2j\langle J_Z S,X\rangle|X'|^{2h}|X''|^{2i}|Z|^{2j-2}\\ &= \frac{j}{h+1}\partial_{J_Z S}|X'|^{2h+2}|X''|^{2i}|Z|^{2j-2}.\end{aligned}$$

Ad (iii): We have

$$P_S|X'|^{2h}|X''|^{2i}|Z|^{2j} = \frac{j}{8}\partial_{J_Z J_Z S}|X'|^{2h}|X''|^{2i}|Z|^{2j-2} = -2j\partial_S|X'|^{2h}|X''|^{2i}|Z|^{2j}.$$

Ad (iv): The statement follows immediately by Lemma 4.3.
Ad (v): Since $\partial_{J_T X} = \partial_{J_T X'} + \partial_{J_T X''}$ this follows from $J_T|_{\mathfrak{v}'} \in \mathfrak{so}(\mathfrak{v}')$ and $J_T|_{\mathfrak{v}''} \in \mathfrak{so}(\mathfrak{v}'')$.
Ad (vi): We have

$$\begin{aligned}R_T|X'|^{2h}|X''|^{2i}|Z|^{2j} &= \frac{j}{8}\partial_{J_Z J_T X}|X'|^{2h}|X''|^{2i}|Z|^{2j-2}\\ &= -\frac{j}{4}\Big(h\langle J_Z X', J_T, X'\rangle|X'|^{2h-2}|X''|^{2i}|Z|^{2j-2}\\ &\qquad + i\langle J_Z X'', J_T, X''\rangle|X'|^{2h}|X''|^{2i-2}|Z|^{2j-2}\Big).\end{aligned}$$

Now

$$\begin{aligned}\langle J_Z X', J_T X'\rangle &= \frac{1}{2}\left(\langle J_{Z+T}X', J_{Z+T}X'\rangle - \langle J_Z X', J_Z X'\rangle - \langle J_T X', J_T X'\rangle\right)\\ &= 16|X'|^2\langle Z,T\rangle\end{aligned}$$

and similarly

$$\langle J_Z X'', J_T X''\rangle = 16|X''|^2\langle Z,T\rangle,$$

which implies the statement. □

Proof of Proposition 6.2. First let $m > 0$ and $S \in \mathfrak{v}'$. Using Lemma 6.3 we obtain that

$$
\begin{aligned}
-\mathcal{F}(D_{\mathfrak{v}}(S))u(X,Z) = \sum_{h+i+2j=k} c_{h,i,j}\Big(& -2i(2i+p''-2)\langle S,X\rangle|X'|^{2h}|X''|^{2i-2}|Z|^{2j}\\
& + 2h(2\nu+2\rho'-4j-2h-p'-2q)\langle S,X\rangle|X'|^{2h-2}|X''|^{2i}|Z|^{2j}\\
& - 2h(h-1)(2h+p'-2)\langle J_Z S,X\rangle|X'|^{2h-4}|X''|^{2i}|Z|^{2j}\\
& - 2ih(2i+p''-2)\langle J_Z S,X\rangle|X'|^{2h-2}|X''|^{2i-2}|Z|^{2j}\\
& + \frac{j}{2}\langle J_Z S,X\rangle|X'|^{2h}|X''|^{2i}|Z|^{2j-2}\Big).
\end{aligned}
$$

After rearrangement the coefficients of $\langle S,X\rangle$ and $\langle J_Z S,X\rangle$ have to vanish separately since $\langle S,J_Z S\rangle = 0$, so (6.4) and (6.5) follow.

For $m=0$, $D_{\mathfrak{v}}(S)u(X,Z)=0$, since $\mathfrak{v}'=\{0\}$. For $T\in\mathfrak{z}$ we have by Lemma 6.3

$$
\begin{aligned}
-\mathcal{F}(D_{\mathfrak{z}}(T))u(X,Z) = \sum_{h+i+2j=k} \langle Z,T\rangle c_{h,i,j}\Big(& 2j(\nu+\rho'-2j-q)|X|^{2i}|Z|^{2j-2}\\
& - 4i(i-1)(2i+p-2)(2i+p-4)|X|^{2i-4}|Z|^{2j}\Big).
\end{aligned}
$$

As in the previous case, (6.6) follows after regrouping the summands and comparing coefficients. ☐

Proof of Theorem 6.1. Clearly (6.4) implies that

$$
c_{h,i,j} = \frac{\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2}+i)}{h!i!\Gamma(\frac{p''}{2}+i)\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})} c_{k-2j,0,j}.
$$

Then (6.5) becomes

$$
jc_{k-2j,0,j} = 16\left(\frac{2\nu+p'-2}{4}-j\right)c_{k-2j+2,0,j-1},
$$

which implies

$$
c_{k-2j,0,j} = 2^{4j}\frac{\Gamma(\frac{2\nu+p'-2}{4})}{j!\Gamma(\frac{2\nu+p'-2}{4}-j)}c_{k,0,0}.
$$

If $m=0$ the proof works analogously. ☐

7 Classification of differential symmetry breaking operators: sporadic operators

We now treat the cases where $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is not generated by $|X'|^2$, $|X''|^2$ and $|Z|^2$.

7.1 The complex case

Let $\mathbb{F}=\mathbb{C}$, then $\mathbb{C}[\bar{\mathfrak{n}}]^{M'}$ is generated by $|X'|^2$, $|X''|^2$ and Z. To simplify the formulas, we identify $Z\in\operatorname{Im}\mathbb{C}=\mathbf{i}\mathbb{R}$ with $\operatorname{Im}Z\in\mathbb{R}$. Let $\widehat{u}^C_{\lambda,\nu}$ be defined as in (6.1). We define an additional polynomial $\widehat{v}^C_{\lambda,\nu}\in\mathbb{C}[\bar{\mathfrak{n}}]$ by

$$
\widehat{v}^C_{\lambda,\nu}(X,Z) := \sum_{i+2j+1=k}\frac{2^{-i}\Gamma(\frac{\nu-1}{2}-j)}{i!\Gamma(j+\frac{3}{2})\Gamma(\frac{p}{2}+i)\Gamma(\frac{\nu-1}{2}-\lfloor\frac{k-1}{2}\rfloor)}|X|^{2i}Z^{2j+1}.
$$

Theorem 7.1. *Let $(G, G') = (\mathrm{U}(1, n + 1), \mathrm{U}(1, m + 1) \times F)$ be strongly spherical and $(\lambda, \nu) \in //$ with $\lambda + \rho - \nu - \rho' = -2k \in -2\mathbb{Z}_{\geq 0}$. Then*

$$\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu} = \begin{cases} \mathbb{C}\widehat{u}^C_{\lambda,\nu} \oplus \mathbb{C}\widehat{v}^C_{\lambda,\nu} & \text{for } m = 0 \text{ and } \nu \in 1 + 2\mathbb{Z}_{\geq 0},\ 0 < \nu \leq k, \\ \mathbb{C}\widehat{u}^C_{\lambda,\nu} & \text{otherwise.} \end{cases}$$

To show the theorem, we first prove an analog of Proposition 6.2 in this case.

Proposition 7.2. *Let $\lambda + \rho - \nu - \rho' = -2k \in -2\mathbb{Z}_{\geq 0}$ and let $\widehat{u} \in \mathbb{C}[\bar{\mathfrak{n}}]$ be of the form*

$$\widehat{u} = \sum_{h+i+j=k} c_{h,i,j} |X'|^{2h} |X''|^{2i} Z^j$$

with scalars $c_{h,i,j} \in \mathbb{C}$. Then $\widehat{u} \in \mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ if and only if the following relations are satisfied:

(i) If $m > 0$ and $h + i + j = k$:

$$2h(\lambda + \rho + \nu - \rho' + 2i)c_{h,i,j} = (i + 1)(2i + p'')c_{h-1,i+1,j} \tag{7.1}$$

for $h > 0$ and

$$jc_{h,i,j} = 8(h+1)(h+2)(2h+p'+2)c_{h+2,i,j-2}+8(h+1)(i+1)(2i+p'')c_{h+1,i+1,j-2} \tag{7.2}$$

for $j > 0$, where we set $c_{h,i,-1} = 0$.

(ii) If $m = 0$ and $h + i + j = k$:

$$j(\nu + \rho' - q - j)c_{h,i,j} = 4(i + 1)(i + 2)(2i + p)(2i + p + 2)c_{h,i+2,j-2} \tag{7.3}$$

for $j > 0$, where we set $c_{h,i,-1} = 0$.

Again we first state some technical identities, which follow in the same way as the ones in Lemma 6.3.

Lemma 7.3. *(i) $\partial_{[S,X]}|X'|^{2h}|X''|^{2i}Z^j = \frac{j}{2(h+1)}\partial_{J_Z S}|X'|^{2h+2}|X''|^{2i}Z^{j-2}$*

(ii) $P_S|X'|^{2h}|X''|^{2i}Z^j = -j\partial_S|X'|^{2h}|X''|^{2i}Z^j$

(iii) $R_T|X'|^{2h}|X''|^{2i}Z^j = -2j(h + i)|X'|^{2h}|X''|^{2i}Z^{j-1}$

Proof of Proposition 7.2. First let $m > 0$. Combining Lemma 6.3 and Lemma 7.3 we obtain

$$\begin{aligned} &- \mathcal{F}(D_{\mathfrak{v}}(S))u(X, Z) \\ &\quad= \sum_{h+i+j=k} c_{h,i,j}\Big(- 2i(2i + p'' - 2)\langle S, X\rangle|X'|^{2h}|X''|^{2i-2}Z^j \\ &\qquad+ 2h(2\nu + 2\rho' - 2j - 2h - p' - 2q)\langle S, X\rangle|X'|^{2h-2}|X''|^{2i}Z^j \\ &\qquad- 2h(h - 1)(2h + p' - 2)\langle J_T S, X\rangle|X'|^{2h-4}|X''|^{2i}Z^{j+1} \\ &\qquad- 2ih(2i + p'' - 2)\langle J_T S, X\rangle|X'|^{2h-2}|X''|^{2i-2}Z^{j+1} \\ &\qquad+ \frac{j}{4}\langle J_T S, X\rangle|X'|^{2h}|X''|^{2i}Z^{j-1}\Big). \end{aligned}$$

Since S and J_TS are orthogonal this implies the statement in this case.

Now let $m=0$, then combining Lemma 6.3 and Lemma 7.3 we obtain

$$-\mathcal{F}(D_{\mathfrak{z}}(T))u(X,Z)=\sum_{h+i+j=k}c_{h,i,j}\Big(j(\nu+\rho'-1-j)|X|^{2i}Z^{j-1}\\ -4i(i+1)(2i+p''-2)(2i+p''-4)|X|^{2i-4}Z^{j+1}\Big),$$

which proves the proposition in this case. □

Now we can prove Theorem 7.1.

Proof of Theorem 7.1. First let $m>0$. Then (7.2) implies that $c_{h,i,1}=0$ for all possible h,i, which implies that all $c_{h,i,j}=0$ for odd j. Now the statement follows from Theorem 6.1.

Let us now consider the case $m=0$ and write $c_{i,j}=c_{0,i,j}$. First it is clear that (7.3) separates odd and even degrees in Z. For the even degrees in Z the theorem follows in the same way as Theorem 6.1. Assume $\nu\neq 2l+1$ for $0\leq 2l\leq k-1$. Then (7.3) implies that $c_{i,j}=0$ first for $j=1$ and then recursively for all odd j. Now if $\nu=2l+1$ for some $0\leq 2l\leq k-1$, then (7.3) implies $c_{i,2j+1}=0$ for all $j<l$ and further

$$\begin{aligned}c_{0,i,2j+1}&=\frac{\Gamma(\lfloor\frac{k-1}{2}\rfloor+\frac{3}{2})\Gamma(\frac{\nu-1}{2}-j)\Gamma(\frac{p}{2}+k-1-2\lfloor\frac{k-1}{2}\rfloor)}{4^{\lfloor\frac{k-1}{2}\rfloor-j}i!\Gamma(j+\frac{3}{2})\Gamma(\frac{\nu-1}{2}-\lfloor\frac{k-1}{2}\rfloor)\Gamma(\frac{p}{2}+i)}c_{0,k-1-2\lfloor\frac{k-1}{2}\rfloor,2\lfloor\frac{k-1}{2}\rfloor+1}\\ &=\text{const}\times\frac{2^{2j}\Gamma(\frac{\nu-1}{2}-j)}{i!\Gamma(j+\frac{3}{2})\Gamma(\frac{p}{2}+i)\Gamma(\frac{\nu-1}{2}-\lfloor\frac{k-1}{2}\rfloor)}.\end{aligned}$$ □

7.2 The quaternionic case

Let $\mathbb{F}=\mathbb{H}$ and $m=n-1$, then $(G,G')=(\mathrm{Sp}(1,n+1),\mathrm{Sp}(1,n)\times F)$ is real spherical for all $F<\mathrm{Sp}(1)$. The only connected subgroups of $\mathrm{Sp}(1)$ are (up to conjugation) $F=\{1\}$, $\mathrm{U}(1)$ and $\mathrm{Sp}(1)$. The case $F=\mathrm{Sp}(1)$ was already treated by Theorem 6.1. We now discuss the case $F=\{1\}$, then the statement for $F=\mathrm{U}(1)$ will follow easily.

We use the orthonormal basis

$$S_1=1,\quad S_2=\mathbf{i},\quad S_3=\mathbf{j},\quad S_4=\mathbf{k}$$

of $\mathbb{H}$ and the orthonormal basis

$$T_1=\mathbf{i},\quad T_2=\mathbf{j},\quad T_3=\mathbf{k}$$

of $\operatorname{Im}\mathbb{H}$, so that an element $(X,Z)\in\bar{\mathfrak{n}}$ is given by

$$(X,Z)=\sum_{i=1}^{4}X_iS_i+\sum_{j=1}^{3}Z_jT_j,$$

for real variables X_i and Z_j.

We recall the $\mathrm{Sp}(1)$-invariants $p_j(X,Z)=\langle T_j,\overline{X}ZX\rangle$ on $\mathbb{H}\oplus\operatorname{Im}\mathbb{H}$ from Section 5 and note the following derivatives for $S\in\mathbb{H}$, $T\in\operatorname{Im}\mathbb{H}$:

$$\partial_Sp_j(X,Z)=2\langle T_j,\overline{X}ZS\rangle,\qquad\partial_Tp_j(X,Z)=\langle T_j,\overline{X}TX\rangle.$$

For a polynomial $q \in \mathbb{C}[p_1, p_2, p_3]$ in three variables we denote by $q(p_1, p_2, p_3)$ the polynomial on $\mathbb{H}^n \oplus \operatorname{Im}\mathbb{H}$ given by

$$q(p_1, p_2, p_3)(X, Z) = q(p_1(X'', Z), p_2(X'', Z), p_3(X'', Z)).$$

For $l \in \mathbb{Z}_{\geq 0}$ define the set

$$\mathcal{H}^l(p_1, p_2, p_3) := \{q(p_1, p_2, p_3),\ q \in \mathcal{H}^l(\mathbb{R}^3)\}.$$

Theorem 7.4. *Let $(G, G') = (\mathrm{Sp}(1, n+1), \mathrm{Sp}(1, n))$ and $(\lambda, \nu) \in //$ with $\lambda + \rho - \nu - \rho' = -2k \in -2\mathbb{Z}_{\geq 0}$. Then*

$$\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu} = \begin{cases} \mathbb{C}\widehat{u}^C_{\lambda,\nu} \oplus \mathcal{H}^{\frac{k}{2}}(p_1, p_2, p_3) & \text{for } (n, m) = (1, 0),\ k > 0 \text{ even and } \nu + \rho' = k + 4, \\ \mathbb{C}\widehat{u}^C_{\lambda,\nu} & \text{otherwise.} \end{cases}$$

The proof is split into Propositions 7.9 and 7.12.

Remark 7.5. Since $\dim \mathcal{H}^\ell(\mathbb{R}^3) = 2\ell + 1$ we have for $n = 1$

$$\dim \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu} = k + 2 \qquad \text{for } (\lambda, \nu) = (-(k+1), k+1).$$

In particular, the dimension can become arbitrarily large while both π_λ and τ_ν have at most 4 composition factors.

We now deduce from Theorem 7.4 (the case $F = \{1\}$) the remaining case $F = \mathrm{U}(1)$. Realizing $\mathrm{Sp}(1)$ as the group of unit quaternions we have $\mathfrak{sp}(1) = \operatorname{Im}\mathbb{H}$. Then the Lie algebra of $F = \mathrm{U}(1)$ is generated by a single element U which we write as $U = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} \in \operatorname{Im}\mathbb{H} = \mathfrak{sp}(1)$.

Corollary 7.6. *Let $(G, G') = (\mathrm{Sp}(1, n+1), \mathrm{Sp}(1, n) \times \mathrm{U}(1))$, where the Lie algebra $\mathfrak{u}(1)$ of the $\mathrm{U}(1)$-factor is generated by $U = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} \in \operatorname{Im}\mathbb{H} = \mathfrak{sp}(1)$. Then for $(\lambda, \nu) \in //$ with $\lambda + \rho - \nu - \rho' = -2k \in -2\mathbb{Z}_{\geq 0}$ we have*

$$\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu} = \begin{cases} \mathbb{C}\widehat{u}^C_{\lambda,\nu} \oplus \mathbb{C}(U_1p_1 + U_2p_2 + U_3p_3) & \text{for } (n, m) = (1, 0) \text{ and } \nu + \rho' = 6, \\ \mathbb{C}\widehat{u}^C_{\lambda,\nu} & \text{otherwise.} \end{cases}$$

Proof. Since $\widehat{u}^C_{\lambda,\nu}$ is $\mathrm{U}(1)$-invariant, by Theorem 7.4 it suffices to show that

$$\mathcal{H}^\ell(p_1, p_2, p_3)^{\mathrm{U}(1)} = \begin{cases} \mathbb{C}1 & \text{for } \ell = 0, \\ \mathbb{C}(U_1p_1 + U_2p_2 + U_3p_3) & \text{for } \ell = 1, \\ \{0\} & \text{else.} \end{cases}$$

Clearly $\mathbb{C}[p_1, p_2, p_3]^{\mathrm{U}(1)}$ is generated by $U_1p_1+U_2p_2+U_3p_3$, so every polynomial in $\mathcal{H}^\ell(p_1, p_2, p_3)$ is of the form $f(U_1p_1 + U_2p_2 + U_3p_3)$ with $f \in \mathbb{C}[t]$ homogeneous of degree ℓ. Since $\Delta_p[f(U_1p_1 + U_2p_2 + U_3p_3)] = (U_1^2 + U_2^2 + U_3^2)f''(U_1p_1 + U_2p_2 + U_3p_3)$ this implies $f'' = 0$, so either $f = 0$ or $\ell \in \{0, 1\}$. For $\ell = 0$ the polynomial f is constant and for $\ell = 1$ it is a constant multiple of t and the statement follows. □

The case $n > 1$

To apply the terms occurring in Proposition 4.1 to a general invariant polynomial, we first prove some basic identities for the invariants p_1, p_2 and p_3.

Lemma 7.7. *Let $q \in \mathbb{C}[p_1, p_2, p_3]$ and write $\Delta_p = \sum_{j=1}^3 \frac{\partial^2}{\partial p_j^2}$, then*

(i) $\Delta_{\mathfrak{v}}(q(p_1, p_2, p_3)) = 4|X|^2|Z|^2(\Delta_p q)(p_1, p_2, p_3)$,

(ii) $\Box(q(p_1, p_2, p_3)) = |X|^4(\Delta_p q)(p_1, p_2, p_3)$.

Proof. Ad (i): By the product rule we have

$$\Delta_{\mathfrak{v}}(q(p_1, p_2, p_3)) = \sum_{i=1}^4 \sum_{j,k=1}^3 \frac{\partial^2 q}{\partial p_j \partial p_k}(p_1, p_2, p_3)\frac{\partial p_j}{\partial X_i}(X, Z)\frac{\partial p_k}{\partial X_i}(X, Z) + \sum_{j=1}^3 \frac{\partial q}{\partial p_j}(p_1, p_2, p_3)\Delta_{\mathfrak{v}}(p_j(X, Z)).$$

For $j, k \in \{1, 2, 3\}$ we have

$$\begin{aligned} \sum_{i=1}^4 \frac{\partial p_j}{\partial X_i}(X, Z)\frac{\partial p_k}{\partial X_i}(X, Z) &= 4\sum_{i=1}^4 \langle T_j, \overline{X}ZS_i\rangle\langle T_k, \overline{X}ZS_i\rangle \\ &= -4\sum_{i=1}^4 \langle T_j, \overline{X}ZS_i\rangle\langle ZXT_k, S_i\rangle \\ &= -4\langle T_j, \overline{X}ZZXT_k\rangle \\ &= 4|X|^2|X|^2\langle T_j, T_k\rangle \\ &= 4|X|^2|Z|^2\delta_{j,k}. \end{aligned}$$

Further, for every $j \in \{1, 2, 3\}$

$$\Delta_{\mathfrak{v}} p_j(X, Z) = 2\sum_{i=1}^4 \langle T_j, \overline{S}_i Z S_i\rangle = 0,$$

where the last step follows from the identity $\mathbf{i}Z\mathbf{i} + \mathbf{j}Z\mathbf{j} + \mathbf{k}Z\mathbf{k} = Z$ for $Z \in \operatorname{Im}\mathbb{H}$ which is easily verified.

Ad (ii): This is similar to (i) using for $j, k \in \{1, 2, 3\}$ the identity

$$\begin{aligned} \sum_{i=1}^3 \frac{\partial p_j}{\partial Z_i}(X, Z)\frac{\partial p_k}{\partial Z_i}(X, Z) &= \sum_{i=1}^3 \langle T_j, \overline{X}T_iX\rangle\langle T_k, \overline{X}T_iX\rangle \\ &= \sum_{i=1}^3 \langle T_j, \overline{X}T_iX\rangle\langle XT_k\overline{X}, T_i\rangle \\ &= \langle T_j, \overline{X}XT_k\overline{X}X\rangle \\ &= |X|^4\langle T_j, T_k\rangle \\ &= |X|^4\delta_{j,k}, \end{aligned}$$

as well as $\Box(p_j(X, Z)) = 0$ which is clear since $p_j(X, Z)$ is linear in Z. $\Box$

Lemma 7.8. *Let $q \in \mathcal{H}^\ell(\mathbb{R}^3)$, then for $S \in \mathfrak{v}'$ we have*

(i) $\partial_S |X'|^{2h} = 2h\langle X', S\rangle |X'|^{2h-2}$,

(ii) $$\Delta_{\mathfrak{v}}(|X'|^{2h}|X''|^{2i}q(p_1,p_2,p_3)) = 2h(2h+p'-2)|X'|^{2h-2}|X''|^{2i}q(p_1,p_2,p_3) + 4i(i+2\ell+1)|X'|^{2h}|X''|^{2i-2}q(p_1,p_2,p_3),$$

(iii) $$\begin{aligned}\partial_{J_Z S}\Delta_{\mathfrak{v}}(|X'|^{2h}|X''|^{2i}q(p_1,p_2,p_3)) &= 4h(h-1)(2h+p'-2)\langle Z,[S,X']\rangle |X'|^{2h-4}|X''|^{2i}q(p_1,p_2,p_3)\\ &\quad + 8ih(i+2\ell+1)|X'|^{2h-2}|X''|^{2i-2}\langle Z,[S,X']\rangle q(p_1,p_2,p_3),\end{aligned}$$

(iv) $$\partial_{[S,X]}(|Z|^{2j}q(p_1,p_2,p_3)) = 2j\langle Z,[S,X']\rangle |Z|^{2j-2}q(p_1,p_2,p_3) + |Z|^{2j}\partial_{[S,X']}q(p_1,p_2,p_3),$$

(v) $$P_S(|X'|^{2h}|Z|^{2j}q(p_1,p_2,p_3)) = -4hj|X'|^{2h-2}|Z|^{2j}\langle X',S\rangle q(p_1,p_2,p_3) + \frac{1}{8}h|X'|^{2h-2}|Z|^{2j}\partial_{[J_Z S,X']}q(p_1,p_2,p_3),$$

(vi) $$Q_S(|X'|^{2h}|X''|^{2i}q(p_1,p_2,p_3)) = 2hq|X'|^{2h-2}|X''|^{2i}\langle X',S\rangle q(p_1,p_2,p_3) + \frac{1}{8}h|X'|^{2h-2}|X''|^{2i}\partial_{J_{[S,X']}X''}q(p_1,p_2,p_3).$$

Proof. For the whole proof we remark that $q(p_1,p_2,p_3)$ is homogeneous in X of degree 2ℓ and homogeneous in Z of degree ℓ.

Ad (i): This is clear.

Ad (ii): This follows by an easy application of the product rule, Lemma 7.7(i) and the identities $\Delta_{\mathfrak{v}}|X'|^{2h} = 2h(2h+p'-2)|X'|^{2h-2}$ and $\Delta_{\mathfrak{v}}|X''|^{2i} = 4i(i+1)|X''|^{2i-2}$.

Ad (iii): Here $\partial_{J_Z S}$ is applied to (ii) using

$$\partial_{J_Z S}|X'|^{2h} = 2h\langle J_Z S, X'\rangle |X'|^{2h-2} = 2h\langle Z,[S,X']\rangle |X'|^{2h-2}.$$

Ad (iv): This is the product rule.

Ad (v): We compute

$$\begin{aligned}&P_S(|X'|^{2h}|Z|^{2j}q(p_1,p_2,p_3))\\ &= \frac{1}{16}\sum_{a=1}^{3}\partial_{J_{T_a}J_Z S}(|X'|^{2h})\partial_{T_a}(|Z|^{2j}q(p_1,p_2,p_3))\\ &= \frac{h}{8}|X'|^{2h-2}\sum_{a=1}^{3}\langle J_{T_a}J_Z S, X'\rangle\Big[2j\langle T_a,Z\rangle |Z|^{2j-2}q(p_1,p_2,p_3) + |Z|^{2j}\partial_{T_a}q(p_1,p_2,p_3)\Big]\\ &= \frac{hj}{4}\langle J_Z^2 S, X'\rangle |X'|^{2h-2}|Z|^{2j-2}q(p_1,p_2,p_3)\\ &\qquad + \frac{h}{8}|X'|^{2h-2}|Z|^{2j}\sum_{a=1}^{3}\langle T_a,[J_Z S,X']\rangle\partial_{T_a}q(p_1,p_2,p_3)\\ &= -4hj\langle S,X'\rangle |X'|^{2h-2}|Z|^{2j}q(p_1,p_2,p_3) + \frac{h}{8}|X'|^{2h-2}|Z|^{2j}\partial_{[J_Z S,X']}q(p_1,p_2,p_3).\end{aligned}$$

Ad (vi): We first note that $[S,S_i]=0$ whenever $S_i\in\mathfrak{v}''$, so that

$$Q_S=\frac{1}{16}\sum_{i=1}^{p'}\partial_{J_{[S,S_i]}X}\partial_{S_i}.$$

Then clearly $\partial_{J_{[S,S_i]}X}|X''|^{2i}=0$. Further, since $J_{[S,S_i]}|_{\mathfrak{v}'}\in\mathfrak{so}(\mathfrak{v}')$ we have $\partial_{J_{[S,S_i]}X}|X'|^{2h}=0$. Hence

$$\begin{aligned}
&Q_S(|X'|^{2h}|X''|^{2i}q(p_1,p_2,p_3))\\
&=\frac{1}{8}h\sum_{a=1}^{p}\partial_{J_{[S,S_a]}X}\big(\langle S_a,X'\rangle|X'|^{2h-2}|X''|^{2i}q(p_1,p_2,p_3)\big)\\
&=\frac{1}{8}h|X'|^{2h-2}|X''|^{2i}\sum_{a=1}^{p'}\Big(\langle S_a,J_{[S,S_a]}X'\rangle q(p_1,p_2,p_3)+\langle S_a,X'\rangle\partial_{J_{[S,S_a]}X''}q(p_1,p_2,p_3)\Big)\\
&=\frac{1}{8}h|X'|^{2h-2}|X''|^{2i}\left(-\left\langle X',\sum_{a=1}^{p'}J_{[S,S_a]}S_a\right\rangle q(p_1,p_2,p_3)+\partial_{J_{[S,X']}X''}q(p_1,p_2,p_3)\right)
\end{aligned}$$

and the claimed formula follows with Lemma 4.3. □

Proposition 7.9. *For* $(G,G')=(\mathrm{Sp}(1,n+1),\mathrm{Sp}(1,n))$ *with* $n>1$ *and* $(\lambda,\nu)\in /\!/$ *we have*

$$\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}=\mathbb{C}\widehat{u}^C_{\lambda,\nu}.$$

Proof. Let $\widehat{u}\in\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$, then $\widehat{u}$ is an $\mathrm{Sp}(n-1)\times\mathrm{Sp}(1)$-invariant polynomial on $\mathbb{H}^{n-1}\oplus(\mathbb{H}\oplus\operatorname{Im}\mathbb{H})$ which is homogeneous of degree $2k$. By Lemma 5.2 it can therefore be written in the form

$$\widehat{u}(X',X'',Z)=\sum_{h+i+2j+2\ell=k}|X'|^{2h}|X''|^{2i}|Z|^{2j}q_\ell^{h,i,j}(p_1,p_2,p_3)$$

with $q_\ell^{h,i,j}\in\mathbb{C}[p_1,p_2,p_3]$ homogeneous of degree ℓ and $\lambda+\rho-\nu-\rho'=-2k$. Since $p_1^2+p_2^2+p_3^2=|X''|^4|Z|^2$ we can, thanks to the Fischer decomposition, without loss of generality assume that $q_\ell^{h,i,j}\in\mathcal{H}^\ell(\mathbb{R}^3)$. This makes the above decomposition unique.

By Lemma 7.8 the differential operator $-\mathcal{F}(D_{\mathfrak{v}}(S))$ in Proposition 4.1 applied to $\widehat{u}$ takes the following form:

$$\begin{aligned}
\sum_{h+i+2j+2\ell=k}\Big[&4h(\nu+\rho'-1)|X'|^{2h-2}|X''|^{2i}|Z|^{2j}\langle X',S\rangle\\
&-2h(2h+p'-2)|X'|^{2h-2}|X''|^{2i}|Z|^{2j}\langle X',S\rangle\\
&-4i(i+2\ell+1)|X'|^{2h}|X''|^{2i-2}|Z|^{2j}\langle X',S\rangle\\
&-2h(h-1)(2h+p'-2)|X'|^{2h-4}|X''|^{2i}|Z|^{2j}\langle Z,[S,X']\rangle\\
&-4hi(i+2\ell+1)|X'|^{2h-2}|X''|^{2i-2}|Z|^{2j}\langle Z,[S,X']\rangle\\
&+\tfrac{1}{2}j|X'|^{2h}|X''|^{2i}|Z|^{2j-2}\langle Z,[S,X']\rangle+\tfrac{1}{4}|X'|^{2h}|X''|^{2i}|Z|^{2j}\partial_{[S,X']}\\
&-8hj|X'|^{2h-2}|X''|^{2i}|Z|^{2j}\langle X',S\rangle+\tfrac{1}{4}h|X'|^{2h-2}|X''|^{2i}|Z|^{2j}\partial_{[J_ZS,X']}\\
&-4hq|X'|^{2h-2}|X''|^{2i}|Z|^{2j}\langle X',S\rangle-\tfrac{1}{4}h|X'|^{2h-2}|X''|^{2i}|Z|^{2j}\partial_{J_{[S,X']}X''}\Big]q_\ell^{h,i,j}(p_1,p_2,p_3).
\end{aligned}$$

After rearrangement we obtain

$$\begin{aligned}
&\sum_{h+i+2j+2\ell=k} \frac{1}{4}|X'|^{2h}|X''|^{2i}|Z|^{2j}\partial_{[S,X']}q_\ell^{h,i,j}(p_1,p_2,p_3)\\
&+\sum_{h+i+2j+2\ell=k-1}|X'|^{2h}|X''|^{2i}|Z|^{2j}\Big[4(h+1)(\nu-h-2j-1)\langle X',S\rangle q_\ell^{h+1,i,j}(p_1,p_2,p_3)\\
&\qquad -4(i+1)(i+2\ell+2)(X',S)q_\ell^{h,i+1,j}(p_1,p_2,p_3)\\
&\qquad +\frac{1}{4}(h+1)(\partial_{[J_Z S,X']}-\partial_{J_{[S,X']}X''})q_\ell^{h+1,i,j}(p_1,p_2,p_3)\Big]\\
&+\sum_{h+i+2j+2\ell=k-2}|X'|^{2h}|X''|^{2i}|Z|^{2j}\Big[\frac{1}{2}(j+1)\langle Z,[S,X']\rangle q_\ell^{h,i,j+1}(p_1,p_2,p_3)\\
&\qquad -2(h+1)(h+2)(2h+p'+2)\langle Z,[S,X']\rangle q_\ell^{h+2,i,j}(p_1,p_2,p_3)\\
&\qquad -4(h+1)(i+1)(i+2\ell+2)\langle Z,[S,X']\rangle q_\ell^{h+1,i+1,j}(p_1,p_2,p_3)\Big]=0.
\end{aligned}\tag{7.4}$$

We claim that if $q\in\mathbb{C}[p_1,p_2,p_3]$ is harmonic, then the polynomials $\partial_{[S,X']}q(p_1,p_2,p_3)$, $\langle X',S\rangle q(p_1,p_2,p_3)$ and $(\partial_{[J_Z S,X']}-\partial_{J_{[S,X']}X''})q_\ell^{h+1,i,j}(p_1,p_2,p_3)$ are all harmonic in X', X'' and Z. In fact, all polynomials are linear in X' and hence harmonic in X'. They are further harmonic in X'' since the operators $\partial_{[S,X']}$, $\langle X',S\rangle$, $\partial_{[J_Z S,X']}$ and $\partial_{J_{[S,X']}X''}$ commute with the Laplacian $\Delta_{\mathfrak{v}''}$ on $\mathfrak{v}''$. This is obvious for the first three operators and follows from Lemma 4.2 for the fourth one. Finally, the operators $\partial_{[S,X']}$, $\langle X',S\rangle$ and $\partial_{J_{[S,X']}X''}$ also commute with the Laplacian $\Box$ on $\mathfrak{z}$ whence the corresponding polynomials are harmonic in Z. For the remaining polynomial we have by Lemma 4.2

$$\Box\partial_{[J_Z S,X']}q(p_1,p_2,p_3)=\partial_{[J_Z S,X']}\Box q(p_1,p_2,p_3)-32\langle S,X'\rangle\Box q(p_1,p_2,p_3)=0.$$

The polynomial $\langle Z,[S,X']\rangle q(p_1,p_2,p_3)$ is also harmonic in X' and X'' for the same reasons as above. It is in general not harmonic in Z, so we decompose it as

$$\begin{aligned}
\langle Z,[S,X']\rangle q(p_1,p_2,p_3)=&\underbrace{\langle Z,[S,X']\rangle q(p_1,p_2,p_3)-\frac{|Z|^2}{2\ell+1}\partial_{[S,X']}q(p_1,p_2,p_3)}_{q(p_1,p_2,p_3)_+:=}\\
&+\frac{|Z|^2}{2\ell+1}\partial_{[S,X']}q(p_1,p_2,p_3)
\end{aligned}$$

with $q(p_1,p_2,p_3)_+$ and $\partial_{[S,X']}q(p_1,p_2,p_3)$ harmonic in X', X'' and Z. Inserting this into

(7.4) gives

$$\begin{aligned}
&\sum_{h+i+2j+2\ell=k}|X'|^{2h}|X''|^{2i}|Z|^{2j}\frac{1}{2\ell+1}\Bigg[\frac{1}{4}(2j+2\ell+1)\partial_{[S,X']}q_\ell^{h,i,j}(p_1,p_2,p_3)\\
&\qquad\qquad -2(h+1)(h+2)(2h+p'+2)\partial_{[S,X']}q_\ell^{h+2,i,j-1}(p_1,p_2,p_3)\\
&\qquad\qquad -4(h+1)(i+1)(i+2\ell+2)\partial_{[S,X']}q_\ell^{h+1,i+1,j-1}(p_1,p_2,p_3)\Bigg]\\
&+\sum_{h+i+2j+2\ell=k-1}|X'|^{2h}|X''|^{2i}|Z|^{2j}\Bigg[4(h+1)(\nu-h-2j-1)\langle X',S\rangle q_\ell^{h+1,i,j}(p_1,p_2,p_3)\\
&\qquad\qquad -4(i+1)(i+2\ell+2)\langle X',S\rangle q_\ell^{h,i+1,j}(p_1,p_2,p_3)\\
&\qquad\qquad +\frac{1}{4}(h+1)(\partial_{[J_ZS,X']}-\partial_{J_{[S,X']}X''})q_\ell^{h+1,i,j}(p_1,p_2,p_3)\Bigg]\\
&+\sum_{h+i+2j+2\ell=k-2}|X'|^{2h}|X''|^{2i}|Z|^{2j}\Bigg[-2(h+1)(h+2)(2h+p'+2)q_\ell^{h+2,i,j}(p_1,p_2,p_3)_+\\
&\qquad\qquad -4(h+1)(i+1)(i+2\ell+2)q_\ell^{h+1,i+1,j}(p_1,p_2,p_3)_+\\
&\qquad\qquad +\frac{1}{2}(j+1)q_\ell^{h,i,j+1}(p_1,p_2,p_3)_+\Bigg]=0, \qquad (7.5)
\end{aligned}$$

where we use the convention $q_\ell^{h,i,-1}=0$. Now all terms in square brackets are harmonic in X', X'' and Z, so uniqueness of the Fischer decomposition implies that for fixed h, i and j the sum of the three square brackets with $2\ell=k-h-i-2j$, $2\ell=k-h-i-2j-1$ and $2\ell=k-h-i-2j-2$, respectively, vanishes. Since $k-h-i-2j$ is either even or odd, this sum contains either only the first and the third square bracket or only the second one. Further, the first square bracket is of degree $2\ell=k-h-i-2j$ in X'' whereas the third square bracket is of degree $2\ell=k-h-i-2j-2$ in X'', so the square brackets have to vanish separately. For the first square bracket this implies

$$\begin{aligned}
\partial_{[S,X']}\Bigg(&\frac{1}{4}(2j+2\ell+1)q_\ell^{h,i,j}(p_1,p_2,p_3)-2(h+1)(h+2)(2h+p'+2)q_\ell^{h+2,i,j-1}(p_1,p_2,p_3)\\
&-4(h+1)(i+1)(i+2\ell+2)q_\ell^{h+1,i+1,j-1}(p_1,p_2,p_3)\Bigg)=0.
\end{aligned}$$

Now $[S,X']$ ranges over $[\mathfrak{v}',\mathfrak{v}']=\mathfrak{z}$, so the term in the brackets is independent of $T\in\mathfrak{z}$. This implies that for $\ell>0$ the term in the brackets vanishes. For $j=0$ this means that $q_\ell^{h,i,0}=0$ whenever $\ell>0$ since $q_\ell^{h,i,-1}=0$. Inductively, the same equation gives $q_\ell^{h,i,j}=0$ whenever $\ell>0$. Since $q_0^{h,i,h}=c_{h,i,j}\in\mathcal{H}^0(\mathbb{R}^3)=\mathbb{C}$ are scalars, $\widehat{u}$ is of the form

$$\widehat{u}=\sum_{h+i+2j=k}c_{h,i,j}|X'|^{2h}|X''|^{2i}|Z|^{2j}$$

and the rest follows from Theorem 6.1. □

The case $n = 1$

For $n = 1$ we have $\mathfrak{v}' = \{0\}$ and therefore we only have the differential equation for $T \in \mathfrak{z}$ in Proposition 4.1. We apply the terms occurring in this equation separately to a general invariant polynomial.

Lemma 7.10. *Let $q \in \mathcal{H}^\ell(\mathbb{R}^3)$, then for $T \in \mathfrak{z}$ we have*

(i) $\partial_T(|Z|^{2j} q(p_1,p_2,p_3)) = 2j|Z|^{2j-2}\langle T, Z\rangle q(p_1,p_2,p_3) + |Z|^{2j}\partial_T q(p_1,p_2,p_3)$,

(ii) $\partial_{J_T X}\Delta_{\mathfrak{v}}(|X|^{2i} q(p_1,p_2,p_3)) = 4i(i+2\ell+1)|X|^{2i-2}\partial_{J_T X} q(p_1,p_2,p_3)$,

(iii) $$R_T(|X|^{2i}|Z|^{2j} q(p_1,p_2,p_3)) = -4ij|X|^{2i}|Z|^{2j-2}\langle Z, T\rangle q(p_1,p_2,p_3) - 2(i+2j+2\ell+1)|X|^{2i}|Z|^{2j}\partial_T q(p_1,p_2,p_3),$$

(iv) $\Delta_{\mathfrak{v}}^2(|X|^{2i} q(p_1,p_2,p_3)) = 16(i-1)i(i+2\ell)(i+2\ell+1)|X|^{2i-4} q(p_1,p_2,p_3)$,

(v) $\Box(|Z|^{2j} q(p_1,p_2,p_3)) = 2j(2j+2\ell+1)|Z|^{2j-2} q(p_1,p_2,p_3)$.

Proof. Ad (i): This is the product rule.

Ad (ii): Since $J_T|_{\mathfrak{v}'} \in \mathfrak{so}(\mathfrak{v}')$ and $J_T|_{\mathfrak{v}''} \in \mathfrak{so}(\mathfrak{v}'')$ we have $\partial_{J_T X}|X'|^{2h} = \partial_{J_T X}|X''|^{2i} = 0$ and the formula follows from Lemma (7.8(ii)).

Ad (iii): We first compute $R_T q(p_1,p_2,p_3)$:

$$\begin{aligned}
R_T q(p_1,p_2,p_3) &= \frac{1}{16}\sum_{a=1}^{3}\partial_{J_{T_a}J_T X}\sum_{b=1}^{3}\partial_{T_a} p_b(X,Z)\frac{\partial q}{\partial p_b}(p_1,p_2,p_3) \\
&= \frac{1}{16}\sum_{a,b=1}^{3}\partial_{J_{T_a}J_T X}\partial_{T_a} p_b(X,Z)\frac{\partial q}{\partial p_b}(p_1,p_2,p_3) \\
&\qquad + \frac{1}{16}\sum_{a,b,c=1}^{3}\partial_{J_{T_a}J_T X} p_c(X,Z)\partial_{T_a} p_b(X,Z)\frac{\partial^2 q}{\partial p_b \partial p_c}(p_1,p_2,p_3) \\
&= 2\sum_{a,b=1}^{3}\langle T_b, \overline{X} T_a T_a T X\rangle\frac{\partial q}{\partial p_b}(p_1,p_2,p_3) \\
&\qquad + 2\sum_{a,b,c=1}^{3}\langle T_c, \overline{X} Z T_a T X\rangle\langle T_b, \overline{X} T_a X\rangle\frac{\partial^2 q}{\partial p_b \partial p_c}(p_1,p_2,p_3) \\
&= -6\sum_{b=1}^{3}\partial_T p_b(X,Z)\frac{\partial q}{\partial p_b}(p_1,p_2,p_3) \\
&\qquad + 2\sum_{b,c=1}^{3}\langle T_c, \overline{X} Z X T_b \overline{X} T X\rangle\frac{\partial^2 q}{\partial p_b \partial p_c}(p_1,p_2,p_3) \\
&= -6\partial_T q(p_1,p_2,p_3) + 2\sum_{b,c=1}^{3}\langle T_b \overline{X} Z X T_c, \overline{X} T X\rangle\frac{\partial^2 q}{\partial p_b \partial p_c}(p_1,p_2,p_3).
\end{aligned}$$

Now note that $Z_1 Z_2 Z_3 = \langle Z_1, Z_3\rangle Z_2 - \langle Z_1, Z_2\rangle Z_3 - \langle Z_3, Z_2\rangle Z_1$ modulo $\mathbb{R}\mathbf{1}$ for $Z_1, Z_2, Z_3 \in \operatorname{Im}\mathbb{H}$, so that

$$\langle T_b \overline{X} Z X T_c, \overline{X} T X\rangle = \delta_{b,c}|X|^4\langle Z, T\rangle - \langle T_b, \overline{X} Z X\rangle\langle T_c, \overline{X} T X\rangle - \langle T_b, \overline{X} T X\rangle\langle T_c, \overline{X} Z X\rangle.$$

This gives

$$\sum_{b,c=1}^{3}\langle T_b\overline{X}ZXT_c,\overline{X}TX\rangle\frac{\partial^2 q}{\partial p_b\partial p_c}(p_1,p_2,p_3)$$
$$=\langle Z,T\rangle|X|^4(\Delta_p q)(p_1,p_2,p_3)-2\partial_Z\partial_T q(p_1,p_2,p_3)=-2(\ell-1)\partial_T q(p_1,p_2,p_3)$$

and hence

$$R_T q(p_1,p_2,p_3)=-2(2\ell+1)\partial_T q(p_1,p_2,p_3).$$

Further, since $J_{T_a}J_T=J_{T'}-16\langle T,T_a\rangle\mathrm{id}_{\mathfrak{v}}$ for some $T'\in\operatorname{Im}\mathbb{F}$ and $J_{T'}\in\mathfrak{so}(\mathfrak{v})$, we have

$$\partial_{J_{T_a}J_TX}|X|^{2i}=-32i\langle T,T_a\rangle|X|^{2i}.$$

Then

$$\begin{aligned}
&R_T(|X|^{2i}|Z|^{2j}q(p_1,p_2,p_3))\\
&=\frac{1}{16}\sum_{a=1}^{3}\partial_{J_{T_a}J_TX}\Big(2j\langle Z,T_a\rangle|X|^{2i}|Z|^{2j-2}q(p_1,p_2,p_3)+|X|^{2i}|Z|^{2j}\partial_{T_a}q(p_1,p_2,p_3)\Big)\\
&=-2i|X|^{2i}\sum_{a=1}^{3}\langle T,T_a\rangle\Big(2j\langle Z,T_a\rangle|Z|^{2j-2}q(p_1,p_2,p_3)+|Z|^{2j}\partial_{T_a}q(p_1,p_2,p_3)\Big)\\
&\quad+4j|X|^{2i}|Z|^{2j-2}\sum_{a,b=1}^{3}\langle Z,T_a\rangle\langle T_b,\overline{X}ZT_aTX\rangle\frac{\partial q}{\partial p_b}(p_1,p_2,p_3)+|X|^{2i}|Z|^{2j}R_Tq(p_1,p_2,p_3)\\
&=-4ij|X|^{2i}|Z|^{2j-2}\langle Z,T\rangle q(p_1,p_2,p_3)-2(i+2j+2\ell+1)|X|^{2i}|Z|^{2j}\partial_Tq(p_1,p_2,p_3),
\end{aligned}$$

where we have used $\sum_{a=1}^3\langle Z,T_a\rangle\langle T_b,\overline{X}ZT_aTX\rangle=\langle T_b,\overline{X}Z^2TX\rangle=-|Z|^2\partial_Tp_b(X,Z)$ in the last step.

Ad (iv): This follows simply by applying Lemma 7.8(ii) twice.

Ad (v): Apply the product rule and Lemma 7.7(ii). □

Lemma 7.11. *Let $0\neq q\in\mathcal{H}^\ell(\mathbb{R}^3)$, then $\partial_{J_TX}q(p_1,p_2,p_3)=0$ for all $T\in\operatorname{Im}\mathbb{H}$ if and only if $\ell=0$.*

Proof. We have

$$\begin{aligned}
J_{T_1}X&=4(X_2S_1-X_1S_2+X_4S_3-X_3S_4)\\
J_{T_2}X&=4(X_3S_1-X_4S_2-X_1S_3+X_2S_4)\\
J_{T_3}X&=4(X_4S_1+X_3S_2-X_2S_3-X_1S_4).
\end{aligned}$$

Since $\Delta_pq=0$ implies $\Delta_{\mathfrak{v}}q(p_1,p_2,p_3)=0$ by Lemma 7.7(i) we have

$$\begin{aligned}
0&=\frac14\left(\frac{\partial}{\partial X_2}\partial_{J_{T_1}X}+\frac{\partial}{\partial X_3}\partial_{J_{T_2}X}+\frac{\partial}{\partial X_4}\partial_{J_{T_3}X}\right)q(p_1,p_2,p_3)\\
&=\Bigg(-X_1\left(\frac{\partial^2}{\partial X_2^2}+\frac{\partial^2}{\partial X_3^2}+\frac{\partial^2}{\partial X_4^2}\right)+\left(X_2\frac{\partial}{\partial X_2}+X_3\frac{\partial}{\partial X_3}+X_4\frac{\partial}{\partial X_4}\right)\frac{\partial}{\partial X_1}\\
&\qquad+3\frac{\partial}{\partial X_1}\Bigg)q(p_1,p_2,p_3)\\
&=(E_{\mathfrak{v}}+3)\frac{\partial}{\partial X_1}q_j(p_1,p_2,p_3)=(2\ell+2)\frac{\partial}{\partial X_1}q_j(p_1,p_2,p_3).
\end{aligned}$$

Hence $\frac{\partial}{\partial X_1}q(p_1,p_2,p_3)$ must vanish, which implies $q=0$ or $\ell=0$. □

Proposition 7.12. *For* $(G, G') = (\mathrm{Sp}(1,2), \mathrm{Sp}(1,1))$ *and* $(\lambda, \nu) \in \backslash\backslash$ *with* $\lambda + \rho - \nu - \rho' = -2k$ *we have*

$$\mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu} = \begin{cases} \mathbb{C}\widehat{u}^C_{\lambda,\nu} \oplus \mathcal{H}^{\frac{k}{2}}(p_1, p_2, p_3) & \text{for } k > 0 \text{ even and } \nu + \rho' = k + 4, \\ \mathbb{C}\widehat{u}^C_{\lambda,\nu} & \text{otherwise.} \end{cases}$$

Note that for $k = 0$ we have $\mathbb{C}\widehat{u}^C_{\lambda,\nu} = \mathcal{H}^0(p_1, p_2, p_3) = \mathbb{C}1$.

Proof. As in the proof of Proposition 7.9 we can write $\widehat{u} \in \mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$ as

$$\widehat{u} = \sum_{i+2j+2\ell=k} |X|^{2i}|Z|^{2j} q^{i,j}_\ell(p_1, p_2, p_3)$$

with $q^{i,j}_\ell \in \mathcal{H}^\ell(\mathbb{R}^3)$. By Lemma 7.10 the differential operator $-\mathcal{F}(D_{\mathfrak{z}}(T))$ in Proposition 4.1 applied to $\widehat{u}$ takes the following form:

$$\begin{aligned} \sum_{i+2j+2\ell=k} \Bigg[& 2j(\nu + \rho' + 2i + 2\ell - 2)|X|^{2i}|Z|^{2j-2}\langle Z, T\rangle \\ & + (\nu + \rho' + 2i + 2\ell - 2)|X|^{2i}|Z|^{2j}\partial_T \\ & - i(i + 2\ell + 1)|X|^{2i-2}|Z|^{2j}\partial_{J_T X} - 4ij|X|^{2i}|Z|^{2j-2}\langle Z, T\rangle \\ & - 2(i + 2j + 2\ell + 1)|X|^{2i}|Z|^{2j}\partial_T \\ & - 16i(i-1)(i+2\ell)(i+2\ell+1)|X|^{2i-4}|Z|^{2j}\langle Z, T\rangle \\ & - 2j(2j + 2\ell + 1)|X|^{2i}|Z|^{2j-2}\langle Z, T\rangle \Bigg] q^{i,j}_\ell(p_1, p_2, p_3). \end{aligned}$$

After rearrangement we obtain

$$\begin{aligned} & \sum_{i+2j+2\ell=k} (\nu + \rho' - 4j - 2\ell - 4)|X|^{2i}|Z|^{2j}\partial_T q^{i,j}_\ell(p_1, p_2, p_3) \\ & - \sum_{i+2j+2\ell=k-1} (i+1)(i+2\ell+2)|X|^{2i}|Z|^{2j}\partial_{J_T X} q^{i+1,j}_\ell(p_1, p_2, p_3) \\ & + \sum_{i+2j+2\ell=k-2} |X|^{2i}|Z|^{2j} \Bigg[2(j+1)(\nu + \rho' - 2j - 5)\langle Z, T\rangle q^{i,j+1}_\ell(p_1, p_2, p_3) \\ & \qquad - 16(i+1)(i+2)(i+2\ell+2)(i+2\ell+3)\langle Z, T\rangle q^{i+2,j}_\ell(p_1, p_2, p_3) \Bigg]. \end{aligned} \tag{7.6}$$

As in the proof of Proposition 7.9 we note that for $q \in \mathcal{H}^\ell(\mathbb{R}^3)$ the two polynomials $\partial_T q(p_1, p_2, p_3)$ and $\partial_{J_T X} q(p_1, p_2, p_3)$ are harmonic in X and Z, the latter one due to Lemma 4.2. We further decompose

$$\langle Z, T\rangle q(p_1, p_2, p_3) = \underbrace{\langle Z, T\rangle q(p_1, p_2, p_3) - \frac{|Z|^2}{2\ell + 1}\partial_T q(p_1, p_2, p_3)}_{q(p_1,p_2,p_3)_+ :=} + \frac{|Z|^2}{2\ell + 1}\partial_T q(p_1, p_2, p_3),$$

then (7.6) rewrites as

$$\sum_{i+2j+2\ell=k} |X|^{2i}|Z|^{2j}\frac{1}{2\ell+1}\Bigg[(2\ell+1)(\nu+\rho'-4j-2\ell-4)\partial_T q_\ell^{i,j}(p_1,p_2,p_3)$$
$$+2j(\nu+\rho'-2j-3)\partial_T q_\ell^{i,j}(p_1,p_2,p_3)$$
$$-16(i+1)(i+2)(i+2\ell+2)(i+2\ell+3)\partial_T q_\ell^{i+2,j-1}(p_1,p_2,p_3)\Bigg]$$
$$-\sum_{i+2j+2\ell=k-1}(i+1)(i+2\ell+2)|X|^{2i}|Z|^{2j}\partial_{J_TX} q_\ell^{i+1,j}(p_1,p_2,p_3)$$
$$+\sum_{i+2j+2\ell=k-2}|X|^{2i}|Z|^{2j}\Bigg[2(j+1)(\nu+\rho'-2j-5)q_\ell^{i,j+1}(p_1,p_2,p_3)_+$$
$$-16(i+1)(i+2)(i+2\ell+2)(i+2\ell+3)q_\ell^{i+2,j}(p_1,p_2,p_3)_+\Bigg]. \tag{7.7}$$

Now all terms inside square brackets are both harmonic in X and Z and we can apply the uniqueness of the Fischer decomposition. Further, a similar argument as in the proof of Proposition 7.9 comparing degrees then shows that for fixed i and j each square bracket has to vanish separately for the respective value of ℓ. The second square brackets vanish if and only if for all i and j

$$i(i+2\ell+1)\partial_{J_TX} q_\ell^{i,j}(p_1,p_2,p_3)=0.$$

By Lemma 7.11 this implies that $q_\ell^{i,j}=0$ whenever $i>0$ and $\ell>0$. If we assume that $\ell=0$, then $q_\ell^{i,j}=c_{i,j}\in\mathcal{H}^0(\mathbb{R}^3)=\mathbb{C}$ is scalar and we obtain the distributions $\widehat{u}^C_{\lambda,\nu}$. We therefore assume $\ell>0$ for the rest of the proof. Since $q(p_1,p_2,p_3)_+=0$ if and only if $q=0$, the last square bracket vanishes if and only if for all i and j:

$$2(j+1)(\nu+\rho'-2j-5)q_\ell^{i,j+1}=16(i+1)(i+2)(i+2\ell+2)(i+2\ell+3)q_\ell^{i+2,j}. \tag{7.8}$$

Plugging this into the first square bracket further gives

$$(\nu+\rho'-4j-2\ell-4)\partial_T q_\ell^{i,j}(p_1,p_2,p_3)=0. \tag{7.9}$$

Now for $i=2$ we have $q_\ell^{2,j}=0$ by our previous considerations and the right hand side of (7.8) vanishes. Thus, either $q_\ell^{0,j+1}=0$ or $\nu+\rho'-2j-5=0$. In the latter case, (7.9) implies that $(2j+2\ell+3)\partial_T q_\ell^{0,j+1}(p_1,p_2,p_3)=0$ and hence $q_\ell^{0,j+1}=0$ as $\ell>0$.

Summarizing, we have shown that for $\ell>0$ we have $q_\ell^{i,j}=0$ whenever $i>0$ or $j>0$. The only possible solution in addition to $\widehat{u}^C_{\lambda,\nu}$ is therefore

$$\widehat{u}=q_\ell^{0,0}(p_1,p_2,p_3)$$

with $2\ell=k>0$. This polynomial indeed satisfies (7.8), but it only satisfies (7.9) if $\nu+\rho'=2\ell+4=k+4$. $\square$

Part III

Classification of symmetry breaking operators

Using the classification of differential symmetry breaking operators from the previous part, we now complete the classification of all symmetry breaking operators between spherical principal series representations for strongly spherical pairs of the form $(G, G') = (\mathrm{U}(1, n+1; \mathbb{F}), \mathrm{U}(1, m+1; \mathbb{F}) \times F)$ with $0 \leq m < n$ and $F < \mathrm{U}(n-m; \mathbb{F})$.

8 Solutions outside the origin

We first determine the space $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$ of invariant distributions defined outside the origin.

Proposition 8.1. *For any $(\lambda, \nu) \in \mathbb{C}^2$, the space $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$ is spanned by the non-zero distribution*

$$\frac{1}{\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})} N(X,Z)^{-2(\nu+\rho')} |X''|^{\lambda-\rho+\nu+\rho'}.$$

Remark 8.2. By Lemma A.2, we have for $(\lambda, \nu) \in \backslash\backslash$ and $l \in \mathbb{Z}_{\geq 0}$ given by $\lambda+\rho+\nu-\rho' = -2l$:

$$\frac{1}{\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})} |X''|^{\lambda-\rho+\nu+\rho'} = (-1)^l \frac{\pi^{\frac{p''}{2}}}{2^{2l}\Gamma(\frac{p''+2l}{2})} (\Delta^l_{\mathfrak{v}''}\delta)(X'').$$

Otherwise

$$\operatorname{Supp} \frac{1}{\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})} |X''|^{\lambda-\rho+\nu+\rho'} = \mathfrak{v}''.$$

The strategy to prove this proposition is to use the action $\pi_\lambda(\tilde{w}_0)$ of the longest Weyl group element $\tilde{w}_0$. On the group level this corresponds to switching between the open dense subsets $\overline{N}MAN$ and $\tilde{w}_0\overline{N}MAN$ of G. The advantage of working on $\tilde{w}_0\overline{N}MAN$ is that the left-action of N' on $\tilde{w}_0\overline{N}$ is by translations on $\overline{N}$.

Proof of Proposition 8.1. First note that since $N(X,Z) \neq 0$ for all $(X,Z) \in \bar{\mathfrak{n}} - \{0\}$, the involution $\pi_{-\lambda}(\tilde{w}_0)$ from Proposition 2.5(iv) defines a bijection from $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})$ onto itself. We first determine the image of $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$ under $\pi_{-\lambda}(\tilde{w}_0)$. It is easy to see that every $v \in \pi_{-\lambda}(\tilde{w}_0)(\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu})$ is M'-invariant and homogeneous of degree $\lambda - \rho + \nu + \rho'$.

Further, on homogeneous distributions the differential operators $D_{\mathfrak{v}}(S)$ and $D_{\mathfrak{z}}(T)$ are given by

$$D_{\mathfrak{v}}(S) = \pi_{-\lambda}(\tilde{w}_0)d\pi_{-\lambda}(S)\pi_{-\lambda}(\tilde{w}_0) \qquad \text{and} \qquad D_{\mathfrak{z}}(T) = \pi_{-\lambda}(\tilde{w}_0)d\pi_{-\lambda}(T)\pi_{-\lambda}(\tilde{w}_0)$$

with $d\pi_{-\lambda}(S) = -\partial_S - \frac{1}{2}\partial_{[S,X]}$ and $d\pi_{-\lambda}(T) = -\partial_T$ (see the proof of Proposition 2.7). Hence, every $v \in \pi_{-\lambda}(\tilde{w}_0)(\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu})$ satisfies $\partial_T v = 0$ for all $T \in \mathfrak{z}$ and $(\partial_S + \frac{1}{2}\partial_{[S,X]})v = 0$ for all $S \in \mathfrak{v}'$. This implies that $v(X,Z) = v(X'')$ is independent of X' and Z. By Remark 2.8 the group M' acts transitively on the unit sphere in $\mathfrak{v}''$, so the M'-invariance and the homogeneity condition imply by Lemma A.2 that

$$\pi_{-\lambda}(\tilde{w}_0)(\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}) = \mathbb{C}\frac{|X''|^{\lambda-\rho+\nu+\rho'}}{\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})}.$$

Applying $\pi_{-\lambda}(\tilde{w}_0)$ to this distribution shows the claim. $\square$

9 Analytic continuation of invariant distributions

For $\mathrm{Re}(\nu) \ll 0$ and $\mathrm{Re}(\lambda+\nu) \gg 0$ we define the following locally integrable function on $\bar{\mathfrak{n}}$:

$$u^A_{\lambda,\nu}(X,Z) := \frac{1}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})}N(X,Z)^{-2(\nu+\rho')}|X''|^{\lambda-\rho+\nu+\rho'}.$$

In this section we will prove:

Theorem 9.1. *$u^A_{\lambda,\nu}$ extends to a family of distributions that depends holomorphically on $(\lambda,\nu) \in \mathbb{C}^2$ and $u^A_{\lambda,\nu} \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ for all $(\lambda,\nu) \in \mathbb{C}^2$.*

To prove this statement we use polar coordinates on the H-type Lie algebra $\bar{\mathfrak{n}}$. For this let

$$\mathbb{S} := \{(\omega,\eta) \in \mathfrak{v} \oplus \mathfrak{z} = \bar{\mathfrak{n}} : N(\omega,\eta) = 1\}.$$

By Lemma B.2 there exists a unique smooth measure $d\mathbb{S}$ on $\mathbb{S}$ so that for all $\varphi \in C_c(\bar{\mathfrak{n}})$ we have

$$\int_{\bar{\mathfrak{n}}} \varphi(X,Z)\,d(X,Z) = \int_{\mathbb{R}_+}\int_{\mathbb{S}} \varphi(r\omega, r^2\eta)r^{2\rho-1}\,d\mathbb{S}(\omega,\eta)\,dr,$$

where $d(X,Z)$ denotes the Lebesgue measure on $\bar{\mathfrak{n}}$ normalized by the inner product.

For $(\omega,\eta) \in \mathbb{S}$ we write $\omega'' = (\omega_{p'+1},\dots,\omega_p) \in \mathfrak{v}''$.

Lemma 9.2. *$|\omega''|^\lambda$ defines a meromorphic family of generalized functions on $\mathbb{S}$ with simple poles for $\lambda \in -p'' - 2\mathbb{Z}_{\geq 0}$.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{S})$ be a test function. Further, choose $\chi \in C_c^\infty(\mathbb{R}_+)$ with

$$\int_{\mathbb{R}_+} \chi(r)r^{p+2q-1}\,dr = 1.$$

Then we can write

$$\int_{\mathbb{S}} |\omega''|^\lambda\varphi(\omega,\eta)\,d\mathbb{S}(\omega,\eta) = \int_{\mathbb{R}_+} r^{p+2q-1}\int_{\mathbb{S}} |r\omega''|^\lambda \cdot r^{-\lambda}\chi(r)\varphi(\omega,\eta)\,d\mathbb{S}(\omega,\eta)\,dr.$$

The formula $\tilde{\varphi}_\lambda(r\omega, r^2\eta) := r^{-\lambda}\chi(r)\varphi(\omega,\eta)$ defines a smooth compactly supported function on $\mathbb{R}_+ \times \mathbb{S} \cong \mathbb{R}^{p+q} - \{0\}$, i.e. $\tilde{\varphi}_\lambda \in C_c^\infty(\mathbb{R}^{p+q} - \{0\}) \subseteq C_c^\infty(\bar{\mathfrak{n}})$, that depends holomorphically on $\lambda \in \mathbb{C}$. We then have

$$\int_{\mathbb{S}} |\omega''|^\lambda \varphi(\omega,\eta)\, d\mathbb{S}(\omega,\eta) = \int_{\mathbb{R}^{p+q}} |X''|^\lambda \tilde{\varphi}_\lambda(X,Z)\, d(X,Z),$$

where $X = (X', X'') \in \mathbb{R}^{p'} \times \mathbb{R}^{p''} \cong \mathbb{R}^p$. Now the claim follows from Lemma A.1. $\square$

Proof of Theorem 9.1. Let $\varphi \in C_c^\infty(\bar{\mathfrak{n}})$ be a test function. By Lemma B.2

$$\langle u^A_{\lambda,\nu}, \varphi\rangle = \frac{1}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})} \int_{\mathbb{R}_+} r^{\lambda+\rho-\nu-\rho'-1} \times \int_{\mathbb{S}} |\omega''|^{\lambda-\rho+\nu+\rho'} \varphi((r\omega, r^2\eta))\, d\mathbb{S}(\omega,\eta)\, dr. \quad (9.1)$$

which extends holomorphically to an entire function in λ and ν by Lemma A.1 and Lemma 9.2 since the inner integral is an even function of r. By Proposition 8.1, the restriction of $u^A_{\lambda,\nu}$ to $\bar{\mathfrak{n}} - \{0\}$ is contained in $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$. But for $\operatorname{Re}\nu \ll 0$ and $\operatorname{Re}(\lambda+\nu) \gg 0$ the distribution is sufficiently regular, so that the differential equations hold on all of $\bar{\mathfrak{n}}$. Then $u^A_{\lambda,\nu} \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ for $\operatorname{Re}\nu \ll 0$ and $\operatorname{Re}(\lambda+\nu) \gg 0$ and hence for all $(\lambda,\nu) \in \mathbb{C}^2$ by analytic continuation. $\square$

Remark 9.3. The normalizing factor $\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})$ is chosen such that it has a simple pole at $(\lambda,\nu) \in (/\!/ \cup \backslash\!\backslash) - \mathbb{X}$ and a pole of second order at $(\lambda,\nu) \in \mathbb{X}$.

Remark 9.4. That the distributions $N(X,Z)^{-2(\nu+\rho')}|X''|^{\lambda-\rho+\nu+\rho'}$ extend meromorphically in $(\lambda,\nu) \in \mathbb{C}^2$ also follows from [MØO16], but Theorem 9.1 gives the precise normalization factor which makes the family of distributions depend holomorphically on $(\lambda,\nu) \in \mathbb{C}^2$.

In addition, from (9.1) as well as Remark 8.2 and Lemma A.1 we immediately obtain:

Corollary 9.5. *For $(\lambda,\nu) \in \mathbb{C}^2 - /\!/$ we have*

$$\operatorname{Supp} u^A_{\lambda,\nu} = \begin{cases} \bar{\mathfrak{n}} & \text{for } (\lambda,\nu) \in \mathbb{C}^2 - (/\!/ \cup \backslash\!\backslash), \\ \bar{\mathfrak{n}}' & \text{for } (\lambda,\nu) \in \backslash\!\backslash - \mathbb{X}. \end{cases}$$

For $(\lambda,\nu) \in /\!/$ we have $\operatorname{Supp} u^A_{\lambda,\nu} \subseteq \{0\}$.

In the following we write $A_{\lambda,\nu} \in \operatorname{Hom}_{G'}(\pi_\lambda|_{G'}, \tau_\nu)$ for the symmetry breaking operator corresponding to $u^A_{\lambda,\nu}$ via Theorem 1.2.

10 Residues at the origin

Let $(\lambda,\nu) \in /\!/$ with $\lambda+\rho-\nu-\rho' = -2k \in -2\mathbb{Z}_{\geq 0}$. In Section 6 we found a holomorphic family of polynomials $\widehat{u}^C_{\lambda,\nu} \in \mathbb{C}[\bar{\mathfrak{n}}]_{\lambda,\nu}$. The corresponding distributions $u^C_{\lambda,\nu}$ are for $m > 0$ given by

$$u^C_{\lambda,\nu} = \sum_{h+i+2j=k} \frac{2^{-2i-2h}\Gamma(\frac{2\nu+p'+2}{4})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2}+i)}{h!i!j!\Gamma(\frac{2\nu+p'+2}{4}-j)\Gamma(\frac{p''}{2}+i)\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})} \Delta^h_{\mathfrak{v}'}\Delta^i_{\mathfrak{v}''}\square^j\delta, \quad (10.1)$$

and for $m=0$ by

$$u^C_{\lambda,\nu}=\sum_{i+2j=k}\frac{2^{-i}\Gamma(\frac{\nu}{2}-j)}{i!j!\Gamma(\frac{\nu}{2}-\lfloor\frac{k}{2}\rfloor)\Gamma(\frac{p}{2}+i)}\Delta^i_{\mathfrak{v}}\Box^j\delta. \tag{10.2}$$

In this section we obtain $u^C_{\lambda,\nu}$ as a renormalization of the holomorphic family $u^A_{\lambda,\nu}$ and show that it is the only differential symmetry breaking operator that occurs as a renormalization of the family $A_{\lambda,\nu}$. Note that in Part II, $u^C_{\lambda,\nu}$ has been obtained combinatorially as solution of the differential equations of Lemma 3.1.

Theorem 10.1. *Let $(\lambda,\nu)\in /\!/$ and $k\in\mathbb{Z}_{\geq 0}$ given by $\lambda+\rho-\nu-\rho'=-2k$.*

(i) $u^C_{\lambda,\nu}\in\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ and the following residue formula holds:

$$u^A_{\lambda,\nu}=\frac{(-1)^k k!\pi^{\frac{p+q}{2}}}{\Gamma(\frac{\nu+\rho'}{2})}\times\begin{cases}\frac{\Gamma(\frac{2\nu+p'}{4})}{\Gamma(\frac{2\nu+p'}{2})}u^C_{\lambda,\nu} & \text{for } m>0,\\ \frac{\Gamma(\frac{\nu}{2}-\lfloor\frac{k}{2}\rfloor)}{2^k\Gamma(\nu-k)}u^C_{\lambda,\nu} & \text{for } m=0.\end{cases}$$

(ii) The distribution $u^C_{\lambda,\nu}$ is non-zero for all $(\lambda,\nu)\in /\!/$ and $\operatorname{Supp} u^C_{\lambda,\nu}=\{0\}$.

Proof. First, $\operatorname{Supp} u^C_{\lambda,\nu}\subseteq\{0\}$ by definition. To prove $u^C_{\lambda,\nu}\neq 0$ we consider the case $m>0$ first. Here the summand of $u^C_{\lambda,\nu}$ for $i=j=0$ is given by

$$\frac{1}{2^{2k}k!\Gamma(\frac{p''}{2})}\Delta^k_{\mathfrak{v}'}\delta,$$

which is always non-zero. Since the distributions $\Delta^h_{\mathfrak{v}'}\Delta^i_{\mathfrak{v}''}\Box^j\delta$, $h,i,j\in\mathbb{Z}_{\geq 0}$, are linearly independent, it follows that $u^C_{\lambda,\nu}\neq 0$. For $m=0$ one shows that the summand of $u^C_{\lambda,\nu}$ for $j=\lfloor\frac{k}{2}\rfloor$ does not vanish, so we have proven (ii).

To show (i) we abbreviate $\gamma:=\lambda-\rho+\nu+\rho'$ and $\gamma':=-2(\nu+\rho')$. We prove the claimed residue formula for $\operatorname{Re}(\lambda+\nu)\gg 0$, the general case then follows by analytic continuation.

By Lemma B.3 and Lemma A.2 we have for $2\rho+\gamma+\gamma'=-2k\in-2\mathbb{Z}_{\geq 0}$:

$$\begin{aligned}(-1)^k&\frac{2(2k)!}{k!\Gamma(\frac{2\rho+\gamma+\gamma'}{2})}\int_{\bar{\mathfrak{n}}}N(X,Z)^{\gamma'}|X''|^{\gamma}\varphi(X,Z)\,d\bar{\mathfrak{n}}\\ &=(-1)^k\frac{4(2k)!}{k!\Gamma(\frac{2\rho+\gamma+\gamma'}{2})}\int_{\mathbb{R}_+}r^{2\rho+\gamma+\gamma'-1}\int_0^1 x^{p+\gamma-1}(1-x^4)^{\frac{q}{2}-1}\\ &\qquad\times\int_{S^{p-1}}|\omega''|^{\gamma}\int_{S^{q-1}}\varphi(rx\omega,r^2\sqrt{1-x^4}\eta)\,d\eta\,d\omega\,dx\,dr.\end{aligned}$$

Since the integral over x,ω and η is an even function in r, by Lemma A.1:

$$=\frac{d^{2k}}{dr^{2k}}\bigg|_{r=0}2\int_0^1 x^{p+\gamma-1}(1-x^4)^{\frac{q}{2}-1}\int_{S^{p-1}}|\omega''|^{\gamma}\int_{S^{q-1}}\varphi(rx\omega,r^2\sqrt{1-x^4}\eta)\,d\eta\,d\omega\,dx.$$

By Lemma C.1 and by substituting $x^4=y$:

$$\begin{aligned}=\sum_{i+2j=2k}&\frac{(2k)!}{2}\sum_{|\alpha|=i}\frac{1}{\alpha!}\sum_{|\beta|=j}\frac{1}{\beta!}\int_0^1 y^{\frac{\gamma+p+i}{4}-1}(1-y)^{\frac{q+j}{2}-1}\,dy\\ &\times\int_{S^{p-1}}\omega^{\alpha}|\omega''|^{\gamma}\,d\omega\int_{S^{q-1}}\eta^{\beta}\,d\eta\frac{\partial^{i+j}}{\partial X^{\alpha}\partial Z^{\beta}}\varphi(0,0).\end{aligned}$$

Since the integrals over the spheres vanish for odd length multi-indices α and β we obtain by evaluation of the x integral:

$$= \sum_{i+2j=k} \frac{(2k)!}{2} \sum_{|\alpha|=i} \frac{1}{(2\alpha)!} \sum_{|\beta|=j} \frac{1}{(2\beta)!} B\left(\frac{\gamma+p+2i}{4}, \frac{q}{2}+j\right) \\ \times \int_{S^{p-1}} \omega^{2\alpha} |\omega''|^{\gamma}\, d\omega \int_{S^{q-1}} \eta^{2\beta}\, d\eta \frac{\partial^{2i+2j}}{\partial X^{2\alpha} \partial Z^{2\beta}} \varphi(0,0).$$

Evaluating the integrals over the spheres with Lemma B.1 we obtain:

$$= \sum_{i+2j=k} 2^{-2j-2i+1}(2k)! \sum_{|\alpha|=i} \sum_{|\beta|=j} B\left(\frac{\gamma+p+2i}{4}, \frac{q}{2}+j\right) \frac{\pi^{\frac{p}{2}} \Gamma(\frac{\gamma+p''}{2}+|\alpha''|)}{\alpha! \Gamma(\frac{\gamma+p}{2}+i)\Gamma(\frac{p''}{2}+|\alpha''|)} \\ \times \frac{\pi^{\frac{q}{2}}}{\beta! \Gamma(\frac{q}{2}+j)} \frac{\partial^{2i+2j}}{\partial X^{2\alpha}\partial Z^{2\beta}} \varphi(0,0)$$

$$= \pi^{\frac{p+q}{2}} \sum_{i+2j=k} \sum_{|\alpha|=i} \sum_{|\beta|=j} \frac{2^{-2j-2i+1}(2k)! \Gamma(\frac{\gamma+p+2i}{4}) \Gamma(\frac{\gamma+p''}{2}+|\alpha''|)}{\alpha!\beta! \Gamma(\frac{\gamma+p}{2}+i)\Gamma(\frac{p''}{2}+|\alpha''|)\Gamma(\frac{\gamma+p+2q+2k}{4})} \frac{\partial^{2i+2j}}{\partial X^{2\alpha}\partial Z^{2\beta}} \varphi(0,0)$$

$$= \pi^{\frac{p+q}{2}} \sum_{h+i+2j=k} \sum_{|\alpha'|=h} \sum_{|\alpha''|=i} \sum_{|\beta|=j} \frac{2^{-2j-2i-2h+1}(2k)! \Gamma(\frac{\gamma+p+2i+2h}{4}) \Gamma(\frac{\gamma+p''}{2}+i)}{\alpha'!\alpha''!\beta! \Gamma(\frac{\gamma+p}{2}+i+h)\Gamma(\frac{p''}{2}+i)\Gamma(\frac{\gamma+p+2q+2k}{4})} \\ \times \frac{\partial^{2h+2i+2j}}{\partial X'^{2\alpha'}\partial X''^{2\alpha''}\partial Z^{2\beta}} \varphi(0,0)$$

$$= \left\langle \pi^{\frac{p+q}{2}} \sum_{h+i+2j=k} \frac{2^{-2j-2i-2h+1}(2k)! \Gamma(\frac{\gamma+p+2i+2h}{4}) \Gamma(\frac{\gamma+p''}{2}+i)}{h!i!j! \Gamma(\frac{\gamma+p}{2}+i+h)\Gamma(\frac{p''}{2}+i)\Gamma(\frac{\gamma+p+2q+2k}{4})} \Delta_{\mathfrak{v}'}^{h} \Delta_{\mathfrak{v}''}^{i} \Box^{j} \delta, \varphi \right\rangle,$$

where we have used the Multinomial Theorem C.2 in the last step. Using the duplication identity $\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi} 2^{1-2z}\Gamma(2z)$ the final expression can be brought into the claimed form. □

Remark 10.2. Although we exclude the case $\mathbb{F} = \mathbb{R}$ (i.e. $q = 0$) in our considerations, the computations in Theorem 10.1(i) also remain valid in this case. The resulting operators $C_{\lambda,\nu} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1})$ are the conformally covariant differential operators first dicovered by A. Juhl [Juh09] which have previously been obtained as residues of $A_{\lambda,\nu}$ in [KS15]. However, even in the case $q = 0$ the proof of Theorem 10.1(i) gives a new and more direct approach to obtaining Juhl's operators as residues of a holomorphic family of distributions than that of [KS15].

Corollary 9.5 together with Theorem 10.1(i) implies the following:

Corollary 10.3. *$u^A_{\lambda,\nu} = 0$ if and only if $(\lambda,\nu) \in L$.*

Proof. By Corollary 9.5 we know that $u^A_{\lambda,\nu} = 0$ implies $(\lambda,\nu) \in /\!/$. We first consider the case $m > 0$. Then by Theorem 10.1(i) we have for $(\lambda,\nu) \in /\!/$ with $\lambda+\rho-\nu-\rho' = -2k \in -2\mathbb{Z}_{\geq 0}$:

$$u^A_{\lambda,\nu} = (-1)^k \frac{\pi^{\frac{p+q}{2}} k! \Gamma(\frac{2\nu+p'}{4})}{\Gamma(\frac{\nu+\rho'}{2})\Gamma(\frac{2\nu+p'}{2})} u^C_{\lambda,\nu}.$$

Since $u^C_{\lambda,\nu} \neq 0$ by Theorem 10.1(ii), we have $u^A_{\lambda,\nu} = 0$ if and only if the gamma factor vanishes which is by the duplication formula for the gamma function equivalent to $\nu \in -\rho' + q - 1 - 2\mathbb{Z}_{\geq 0}$. Since $(\lambda, \nu) \in /\!/$ this holds if and only if $(\lambda, \nu) \in L$.
Now let $m = 0$, then by the same arguments $u^A_{\lambda,\nu} = 0$ if and only if the gamma factor

$$\frac{\Gamma(\frac{\nu}{2} - \lfloor\frac{k}{2}\rfloor)}{\Gamma(\nu - k)\Gamma(\frac{\nu+\rho'}{2})}$$

vanishes. Using the duplication formula this is a constant multiple of

$$\frac{\Gamma(\frac{\nu}{2} - \lfloor\frac{k}{2}\rfloor)}{\Gamma(\frac{\nu}{2} - \frac{k}{2})\Gamma(\frac{\nu}{2} - \frac{k-1}{2})\Gamma(\frac{\nu+\rho'}{2})}.$$

Now for k even we have $\Gamma(\frac{\nu}{2} - \lfloor\frac{k}{2}\rfloor) = \Gamma(\frac{\nu}{2} - \frac{k}{2})$, so the gamma factor vanishes if and only if $\nu \in (-\rho' - 2\mathbb{Z}_{\geq 0}) \cup ((k-1) - 2\mathbb{Z}_{\geq 0}) = (k-1) - 2\mathbb{Z}_{\geq 0}$. This is equivalent to $(\lambda, \nu) = (-\rho + q - 1 - (k + 2\ell), k - 2\ell - 1) = (-\rho + q - 1 - 2i, \pm(\rho' - q + 1 + 2j))$ with $2i = k + 2\ell \in 2\mathbb{Z}_{\geq 0}$ and $0 \leq j \leq i$. For k odd the arguments are similar. □

11 Singular symmetry breaking operators

Let $(\lambda, \nu) \in \backslash\backslash$ and $l \in \mathbb{Z}_{\geq 0}$ given by $\lambda + \rho + \nu - \rho' = -2l$. We define the following renormalization factors:

$$c^B(\lambda, \nu) := \begin{cases} \frac{1}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})} & \text{for } m > 0 \text{ and } l \leq \frac{p'}{2}, \\ \frac{\Gamma(\frac{2\nu+p'+2}{4})}{\Gamma(\frac{2\nu+p'+2}{4} + \lfloor\frac{2l-p'+2}{4}\rfloor)\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})} & \text{for } m > 0 \text{ and } l > \frac{p'}{2}, \\ \frac{1}{\Gamma(-\frac{\nu}{2} - \lfloor\frac{l}{2}\rfloor)} & \text{for } m = 0. \end{cases} \tag{11.1}$$

For $\operatorname{Re}\nu \ll 0$ we define a family of distributions on $\bar{\mathfrak{n}}$ by

$$u^B_{\lambda,\nu} := c^B(\lambda, \nu) N(X, Z)^{-2(\nu+\rho')} \Delta^l_{\mathfrak{v}''} \delta(X'').$$

Theorem 11.1. *Let $(\lambda, \nu) \in \backslash\backslash$ and $l \in \mathbb{Z}_{\geq 0}$ given by $\lambda - \rho + \nu + \rho' = -2l$.*

(i) $u^B_{\lambda,\nu}$ extends to a family of distributions that depends holomorphically on $\nu \in \mathbb{C}$ and $u^B_{\lambda,\nu} \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}$ for all $(\lambda, \nu) \in \backslash\backslash$.

(ii) The following residue formula holds:

$$u^A_{\lambda,\nu} = \frac{(-1)^l \pi^{\frac{p''}{2}}}{2^l \Gamma(\frac{p''}{2} + l)} \times \begin{cases} u^B_{\lambda,\nu} & \text{for } m > 0 \text{ and } l \leq \frac{p'}{2}, \\ \frac{\Gamma(\frac{2\nu+p'+2}{4} + \lfloor\frac{2l-p'+2}{4}\rfloor)}{\Gamma(\frac{2\nu+p'+2}{4})} u^B_{\lambda,\nu} & \text{for } m > 0 \text{ and } l > \frac{p'}{2}, \\ \frac{\Gamma(-\frac{\nu}{2} - \lfloor\frac{l}{2}\rfloor)}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})} u^B_{\lambda,\nu} & \text{for } m = 0. \end{cases}$$

(iii) The following identity holds:

$$u^B_{\lambda,\nu} = c^B(\lambda, \nu) \sum_{k=0}^{l} \frac{2^{2l-2k} l! (k + \frac{p''}{2})_{l-k}}{k!} \Bigg(\sum_{i+2j=l-k} \frac{(-1)^{i+j} 2^i \Gamma(\frac{\nu+\rho'}{2} + i + j)}{i! j! \Gamma(\frac{\nu+\rho'}{2})} \\ \times |X'|^{2i} N(X', Z)^{-2(\nu+\rho')-4i-4j} \Bigg) \Delta^k_{\mathfrak{v}''} \delta(X'').$$

(iv) $u^B_{\lambda,\nu}$ *is non-zero for all* $(\lambda,\nu)\in \backslash\backslash$*, more precisely:*

$$\operatorname{Supp} u^B_{\lambda,\nu} = \begin{cases} \{0\} & \textit{for } (\lambda,\nu)\in \mathbb{X}-L,\\ \bar{\mathfrak{n}}' & \textit{otherwise.}\end{cases}$$

Proof. By Remark 8.2 it is clear that the residue formula (ii) holds for $\operatorname{Re}\nu \ll 0$, and then Theorem 9.1 implies that $u^B_{\lambda,\nu}$ extends meromorphically in ν. We now show the identity (iii) and afterwards deduce (i) and (iv) from it.

Let $\operatorname{Re}\nu \ll 0$ and write $N(X,Z)^{-2(\nu+\rho')} = f(|X''|^2)$ with $f(x) = (x^2+2ax+b)^{-\frac{\nu+\rho'}{2}}$ and $a = |X'|^2$, $b = N(X',Z)^4$. By the Multinomial Theorem (C.2) we have for $\varphi\in C_c^\infty(\bar{\mathfrak{n}})$:

$$\Delta^l_{\mathfrak{v}''}\left(f(|X''|^2)\varphi(X,Z)\right)\Big|_{X''=0} = \sum_{\substack{\alpha\in\mathbb{Z}^{p''}_{\geq 0},\\ |\alpha|=l}} \frac{l!}{\alpha!}\left(\frac{\partial}{\partial X''}\right)^{2\alpha}\left(f(|X''|^2)\varphi(X,Z)\right)\Big|_{X''=0}.$$

Since odd powers of partial derivatives applied to $f(|X''|^2)$ vanish at $X''=0$, we see by successively applying the product rule

$$= \sum_{\substack{\alpha,\beta\in\mathbb{Z}^{p''}_{\geq 0},\\ |\alpha|=l,\beta\leq\alpha}} \frac{l!(2\alpha)!}{\alpha!(2\beta)!(2\alpha-2\beta)!}\left(\frac{\partial}{\partial X''}\right)^{2\beta} f(|X''|^2)\cdot\left(\frac{\partial}{\partial X''}\right)^{2\alpha-2\beta}\varphi(X,Z)\Big|_{X''=0}.$$

Then successively applying Faà di Bruno's Formula (C.1) yields

$$= \sum_{\substack{\alpha,\beta\in\mathbb{Z}^{p''}_{\geq 0},\\ |\alpha|=l,\beta\leq\alpha}} \frac{l!(2\alpha)!}{\alpha!\beta!(2\alpha-2\beta)!} f^{(|\beta|)}(0)\left(\frac{\partial}{\partial X''}\right)^{2\alpha-2\beta}\varphi(X,Z)\Big|_{X''=0}.$$

Again by Faà di Bruno's Formula (C.1) we have

$$f^{(|\beta|)}(x)\Big|_{x=0} = \sum_{i+2j=|\beta|}(-1)^{i+j}2^i\frac{|\beta|!}{i!j!}\frac{\Gamma(\frac{\nu+\rho'}{2}+i+j)}{\Gamma(\frac{\nu+\rho'}{2})}b^{-\frac{\nu+\rho'}{2}-i-j}a^i,$$

so we obtain

$$u^B_{\lambda,\nu} = c^B(\lambda,\nu)\sum_{\substack{\alpha,\beta\in\mathbb{Z}^{p''}_{\geq 0},\\ |\alpha|=l,\beta\leq\alpha}} \frac{l!(2\alpha)!|\beta|!}{\alpha!\beta!(2\alpha-2\beta)!}$$
$$\times\sum_{i+2j=|\beta|}\frac{(-1)^{i+j}2^i\Gamma(\frac{\nu+\rho'}{2}+i+j)}{i!j!\Gamma(\frac{\nu+\rho'}{2})}|X'|^{2i}N(X',Z)^{-2(\nu+\rho')-4i-4j}\delta^{(2\alpha-2\beta)}(X'').$$

Writing $\alpha=\beta+\gamma$ and regrouping gives

$$u^B_{\lambda,\nu} = c^B(\lambda,\nu)\sum_{k=0}^{l}\left(\sum_{i+2j=l-k}\frac{(-1)^{i+j}2^i\Gamma(\frac{\nu+\rho'}{2}+i+j)}{i!j!\Gamma(\frac{\nu+\rho'}{2})}|X'|^{2i}N(X',Z)^{-2(\nu+\rho')-4i-4j}\right)$$
$$\times\sum_{|\gamma|=k}\frac{l!(l-k)!}{(2\gamma)!}\left(\sum_{|\beta|=l-k}\frac{(2\beta+2\gamma)!}{(\beta+\gamma)!\beta!}\right)\delta^{(2\gamma)}(X'').$$

Note that

$$\frac{(2\beta+2\gamma)!}{(\beta+\gamma)!}=\frac{(2\gamma)!}{\gamma!}\prod_{h=1}^{p''}\frac{(2\gamma_h+1)_{2\beta_h}}{(\gamma_h+1)_{\beta_h}}=\frac{(2\gamma)!}{\gamma!}\prod_{h=1}^{p''}2^{2\beta_h}(\gamma_h+\tfrac{1}{2})_{\beta_h}=\frac{2^{2|\beta|}(2\gamma)!(\gamma+\frac{1}{2})_\beta}{\gamma!},$$

where we have used the Pochhammer symbol $(a)_n=a(a+1)\cdots(a+n-1)$ and its multi-index analog $(\alpha)_\beta=(\alpha_1)_{\beta_1}\cdots(\alpha_{p''})_{\beta_{p''}}$. Then the sum over β can be computed using Lemma C.2 and we obtain

$$\begin{aligned}u^B_{\lambda,\nu}&=c^B(\lambda,\nu)\sum_{k=0}^{l}\left(\sum_{i+2j=l-k}\frac{(-1)^{i+j}2^i\Gamma(\frac{\nu+\rho'}{2}+i+j)}{i!j!\Gamma(\frac{\nu+\rho'}{2})}|X'|^{2i}N(X',Z)^{-2(\nu+\rho')-4i-4j}\right)\\&\qquad\qquad\times\sum_{|\gamma|=k}\frac{2^{2l-2k}l!(k+\frac{p''}{2})_{l-k}}{\gamma!}\delta^{(2\gamma)}(X'')\\&=c^B(\lambda,\nu)\sum_{k=0}^{l}\left(\sum_{i+2j=l-k}\frac{(-1)^{i+j}2^i\Gamma(\frac{\nu+\rho'}{2}+i+j)}{i!j!\Gamma(\frac{\nu+\rho'}{2})}|X'|^{2i}N(X',Z)^{-2(\nu+\rho')-4i-4j}\right)\\&\qquad\qquad\times\frac{2^{2l-2k}l!(k+\frac{p''}{2})_{l-k}}{k!}\Delta^k_{\mathfrak{v}''}\delta(X''),\end{aligned}$$

where we have again used the Multinomial Theorem (C.2) in the second step. This shows (iii) for $\operatorname{Re}\nu\ll 0$ and the general case follows by analytic continuation. Note that the distributions $\Delta^k_{\mathfrak{v}''}\delta(X'')$ $(0\le k\le l)$ are linearly independent.

We now show (i) and (iv) for $m>0$. First note that $(\lambda,\nu)\in L$ if and only if $\nu=-\rho'+q-1-2j$ with $2j\le l-\frac{p'}{2}-1$ which can only happen if $l>\frac{p'}{2}$. Since both $u^A_{\lambda,\nu}$ and the gamma factor in the residue formula (ii) vanish for precisely those values of ν, it follows from Theorem 9.1 that $u^B_{\lambda,\nu}$ is holomorphic in ν, so we have shown (i). Now for $(\lambda,\nu)\notin L$ the gamma factor in the residue formula (ii) is non-zero, so $u^B_{\lambda,\nu}$ is a non-zero multiple of $u^A_{\lambda,\nu}$ and therefore $\operatorname{Supp}u^B_{\lambda,\nu}=\operatorname{Supp}u^A_{\lambda,\nu}$, so that (iv) follows from Corollaries 9.5 and 10.3 in this case. For $(\lambda,\nu)\in L$ with $\nu=-\rho'+q-1-2j$, $2j\le l-\frac{p'}{2}-1$, the normalization factor $c^B(\lambda,\nu)$ is regular. Now, the term for $k=l$ in the identity (iii) equals

$$c^B(\lambda,\nu)N(X',Z)^{-2(\nu+\rho')}\Delta^l_{\mathfrak{v}''}\delta(X'').$$

Since also $N(X',Z)^{-2(\nu+\rho')}$ is regular with $\operatorname{Supp}(N(X',Z)^{-2(\nu+\rho')})=\bar{\mathfrak{n}}'$ by Corollary B.4 and all other terms for $0\le k<l$ have support contained in $\bar{\mathfrak{n}}'$, this implies (iv) for $(\lambda,\nu)\in L$.

Finally we show (i) and (iv) for $m=0$. Here $(\lambda,\nu)\in L$ if and only if ν is an odd integer $\ge -l$. These are precisely the poles of $\Gamma(-\frac{\nu-1}{2}-\lfloor\frac{l+1}{2}\rfloor)$. The gamma factor in the residue formula (ii) can with the duplication formula be rewritten as

$$\frac{\Gamma(-\frac{\nu}{2}-\lfloor\frac{l}{2}\rfloor)}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})}=\frac{\Gamma(-\frac{\nu}{2}-\lfloor\frac{l}{2}\rfloor)}{\Gamma(-\nu-l)}=\frac{2^{\nu+l+1}\sqrt{\pi}\Gamma(-\frac{\nu}{2}-\lfloor\frac{l}{2}\rfloor)}{\Gamma(-\frac{\nu}{2}-\frac{l}{2})\Gamma(-\frac{\nu}{2}-\frac{l-1}{2})}.$$

Since $\{\frac{l}{2},\frac{l-1}{2}\}=\{\lfloor\frac{l+1}{2}\rfloor-\frac{1}{2},\lfloor\frac{l}{2}\rfloor\}$ this equals

$$=\frac{2^{\nu+l+1}\sqrt{\pi}}{\Gamma(-\frac{\nu-1}{2}-\lfloor\frac{l+1}{2}\rfloor)}$$

and therefore the gamma factor vanishes precisely where $u^A_{\lambda,\nu}$ vanishes. It follows that $u^B_{\lambda,\nu}$ is holomorphic on $\backslash\backslash$, so we have shown (i). For $(\lambda,\nu)\notin L$ the same argument as for $m>0$ shows (iv). For $(\lambda,\nu)\in L$ the term for $k=l$ in the identity (iii) equals

$$c^B(\lambda,\nu)|Z|^{-(\nu+\rho')}\Delta^l_{\mathfrak{v}}\delta(X).$$

Since ν is an odd integer, the normalization factor $c^B(\lambda,\nu)$ is regular, and since $\rho'=q=\dim\mathfrak{z}$, we have $\operatorname{Supp}(|Z|^{-(\nu+\rho')})=\bar{\mathfrak{n}}'$ by Lemma A.1. This implies (iv) for $(\lambda,\nu)\in L$ and the proof is complete. □

In the following we write $B_{\lambda,\nu}$ for the symmetry breaking operator corresponding to $u^B_{\lambda,\nu}$.

12 Classification of symmetry breaking operators

In this section we give a full classification of symmetry breaking operators between spherical principal series of G and G' in terms of the operators $A_{\lambda,\nu}$ and $B_{\lambda,\nu}$ and the previously classified differential symmetry breaking operators (see Part II).

Theorem 12.1. *For all strongly spherical pairs of the form $(G,G')=(\mathrm{U}(1,n+1;\mathbb{F}),\mathrm{U}(1,m+1;\mathbb{F})\times F)$ with $\mathbb{F}=\mathbb{C},\mathbb{H},\mathbb{O}$, $0\le m<n$ and $F<\mathrm{U}(n-m;\mathbb{F})$ we have*

$$\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}=\begin{cases}\mathbb{C}u^A_{\lambda,\nu} & \text{if } (\lambda,\nu)\in\mathbb{C}^2-//,\\ \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu} & \text{if } (\lambda,\nu)\in //-L,\\ \mathbb{C}u^B_{\lambda,\nu}\oplus\mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu} & \text{if } (\lambda,\nu)\in L.\end{cases}$$

The rest of this section is devoted to the proof of this statement.

12.1 Continuation of solutions outside of the origin

We first study for which parameters the distribution solutions outside the origin in $\bar{\mathfrak{n}}$ can be extended to the whole $\bar{\mathfrak{n}}$, i.e. for which $(\lambda,\nu)\in\mathbb{C}^2$ the restriction map $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}\to\mathcal{D}'(\bar{\mathfrak{n}}-\{0\})_{\lambda,\nu}$ is onto. Since the latter space is one-dimensional, we can equivalently determine the cases where the restriction map is trivial.

Proposition 12.2. *The following are equivalent:*

(i) $(\lambda,\nu)\in //-L$,

(ii) The restriction $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}\to\mathcal{D}'(\bar{\mathfrak{n}}-\{0\})_{\lambda,\nu}$ *is identical zero,*

(iii) $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu}=\mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu}$.

The proof of Proposition 12.2 works analogously as the proof of [KS15, Proposition 11.7]. Therefore we need to use the following facts (see [KS15, Lemma 11.10, Lemma 11.11]):

Lemma 12.3. *(i) Let d_μ be a differential operator on $\mathbb{R}^n$ which is holomorphic in $\mu \in \mathbb{C}$ and let $v_\mu \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution which is meromorphic in μ, such that for differential operators d_i on $\mathbb{R}^n$ and distributions $v_j \in \mathcal{D}'(\mathbb{R}^n)$ we have the expansions*

$$d_\mu = d_0 + \mu d_1 + \mu^2 d_2 + \cdots \qquad \text{and} \qquad v_\mu = \mu^{-1} v_{-1} + v_0 + \mu v_1 + \cdots .$$

If there exists an $\varepsilon > 0$ such that that $d_\mu v_\mu = 0$ for all $\mu \in \mathbb{C}$ with $0 < |\mu| < \varepsilon$, then

$$d_0 v_{-1} = 0 \qquad \text{and} \qquad d_0 v_0 + d_1 v_{-1} = 0.$$

(ii) Let E be the weighted Euler-operator on $\mathbb{R}^{p+q}$ which is in coordinates $(x_1, \ldots, x_p, z_1, \ldots, z_q)$ given by

$$\sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} + 2 \sum_{j=1}^{q} z_j \frac{\partial}{\partial z_j}.$$

Let $v \in \mathcal{D}'(\mathbb{R}^{p+q})$ be a distribution with $\operatorname{Supp} v = \{0\}$*, then for every integer $k \in \mathbb{Z}$, $(E+k)v = 0$ if and only if $(E+k)^2 = 0$.*

Proof. (i) follows immediatiely from the Laurent expansion of $d_\mu v_\mu$.

Ad (ii): Since v_μ is supported in the origin, we have

$$v = \sum_{\alpha \in \mathbb{Z}^p_{\geq 0}, \beta \in \mathbb{Z}^q_{\geq 0}} w_{\alpha,\beta} \delta^{((\alpha,0)+(0,\beta))}$$

for scalars $w_{\alpha,\beta} \in \mathbb{C}$ which are almost all zero. Then by homogeneity and since different derivatives of the Dirac delta are linearly independent it follows that

$$(E+k)v = \sum_{\alpha \in \mathbb{Z}^p_{\geq 0}, \beta \in \mathbb{Z}^q_{\geq 0}} (k - p - 2q - |\alpha| - 2|\beta|) w_{\alpha,\beta} \delta^{((\alpha,0)+(0,\beta))} = 0,$$

if and only if

$$(E+k)^2 v = \sum_{\alpha \in \mathbb{Z}^p_{\geq 0}, \beta \in \mathbb{Z}^q_{\geq 0}} (k - p - 2q - |\alpha| - 2|\beta|)^2 w_{\alpha,\beta} \delta^{((\alpha,0)+(0,\beta))} = 0. \qquad \square$$

Proof of Proposition 12.2. The equivalence of (ii) and (iii) follows directly from the exactness of the sequence

$$0 \longrightarrow \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu} \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu},$$

so it remains to show the equivalence of (i) and (ii). By Proposition 8.1, $u^A_{\lambda,\nu}|_{\bar{\mathfrak{n}}-\{0\}}$ vanishes if and only if $\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})$ has a pole, i.e. if and only if $(\lambda,\nu) \in /\!/$. Hence $u^A_{\lambda,\nu}|_{\bar{\mathfrak{n}}-\{0\}}$ is a non-zero element of $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$ for all $(\lambda,\nu) \notin /\!/$. For $(\lambda,\nu) \in L$, the distribution $u^B_{\lambda,\nu}|_{\bar{\mathfrak{n}}-\{0\}}$ is a non-zero element of $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu}$ by Theorem 11.1(iv). So we are left to show that for $(\lambda_0,\nu_0) \in /\!/ - L$ the restriction map $\mathcal{D}'(\bar{\mathfrak{n}})_{\lambda_0,\nu_0} \to \mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda_0,\nu_0}$ is trivial, i.e. every $w \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda_0,\nu_0}$ has $\operatorname{Supp} w \subseteq \{0\}$. Write $\lambda_0 + \rho - \nu_0 - \rho' = -2k$ with $k \in \mathbb{Z}_{\geq 0}$. For all $(\lambda,\nu) \in \mathbb{C}$ with $\lambda + \nu = \lambda_0 + \nu_0$, consider the parameter $\mu = \lambda + \rho - \nu - \rho' + 2k$.

Then $v_{\lambda,\nu} := \Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})u^A_{\lambda,\nu}$ depends meromorphically on μ and has a simple pole at $\mu = 0$, such that near (λ_0, ν_0):

$$v_{\lambda,\nu} = \mu^{-1}v_{-1} + v_0 + \mu v_1 + \ldots,$$

for distributions $v_i \in \mathcal{D}'(\bar{\mathfrak{n}})$. By Corollary 10.3 the distribution v_{-1} is non-trivial for $(\lambda_0, \nu_0) \notin L$. Further,

$$v_{\lambda,\nu}|_{\bar{\mathfrak{n}}-\{0\}} = \frac{1}{\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})}N(X,Z)^{-2(\nu+\rho')}|X''|^{\lambda-\rho+\nu+\rho'},$$

which depends holomorphically on $(\lambda, \nu) \in \mathbb{C}^2$, so $v_{-1}|_{\bar{\mathfrak{n}}-\{0\}} = 0$ and $v_0|_{\bar{\mathfrak{n}}-\{0\}} \neq 0$ spans $\mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda_0,\nu_0}$ by Proposition 8.1. If now $w \in \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda_0,\nu_0}$, then there exists a constant $c \in \mathbb{C}$ such that

$$w|_{\bar{\mathfrak{n}}-\{0\}} = cv_0|_{\bar{\mathfrak{n}}-\{0\}}.$$

In particular the distribution $w' := w - cv_0 \in \mathcal{D}'(\bar{\mathfrak{n}})$ has support contained in $\{0\}$. Now, by Lemma 3.1(ii) we have

$$((E + 2\rho + 2k) - \mu \cdot \mathbf{1})v_{\lambda,\nu} = (E - (\lambda - \rho - \nu - \rho'))v_{\lambda,\nu} = 0,$$

so Lemma 12.3(i) implies

$$(E + 2\rho + 2k)v_{-1} = 0 \qquad \text{and} \qquad (E + 2\rho + 2k)v_0 - v_{-1} = 0.$$

Hence

$$(E + 2\rho + 2k)^2 w' = (E + 2\rho + 2k)^2 w - (E + 2\rho + 2k)^2 cv_0 = 0,$$

because also $(E + 2\rho + 2k)w = 0$. Lemma 12.3(ii) then implies $(E + 2\rho + 2k)w' = 0$, so $c(E + 2\rho + 2k)v_0 = 0$ and therefore $cv_{-1} = 0$. Since $v_{-1} \neq 0$ we must have $c = 0$ and $\operatorname{Supp} w = \operatorname{Supp} w' \subseteq \{0\}$, i.e. $w|_{\bar{\mathfrak{n}}-\{0\}} = 0$. ☐

12.2 Proof of the main theorem

We can finally prove Theorem 12.1. Using the exact sequence

$$0 \longrightarrow \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu} \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}} - \{0\})_{\lambda,\nu},$$

Corollary 5.3 and Proposition 12.2 imply the statement for $(\lambda, \nu) \notin L$. For $(\lambda, \nu) \in L$, Proposition 8.1 implies that $\dim \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu} \leq \dim \mathcal{D}'(\bar{\mathfrak{n}})_{\lambda,\nu} \leq \dim \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu} + 1$. Since $u^B_{\lambda,\nu} \notin \mathcal{D}'_{\{0\}}(\bar{\mathfrak{n}})_{\lambda,\nu}$ for $(\lambda, \nu) \in L$ due to its support, Theorem 11.1 implies the statement for $(\lambda, \nu) \in L$. ☐

Combining Theorem 6.1, Corollary 9.5, Corollary 10.3 and Theorem 11.1:

Corollary 12.4. *(i) For* $(\lambda, \nu) \in \mathbb{C}^2$*:*

$$\operatorname{Supp} u^A_{\lambda,\nu} = \begin{cases} \emptyset & \text{for } (\lambda,\nu) \in L, \\ \{0\} & \text{for } (\lambda,\nu) \in /\!/ - L, \\ \bar{\mathfrak{n}}' & \text{for } (\lambda,\nu) \in \backslash\!\backslash - \mathbb{X}, \\ \bar{\mathfrak{n}} & \text{otherwise.} \end{cases}$$

(ii) For $(\lambda,\nu) \in \backslash\backslash$*:*

$$\operatorname{Supp} u^B_{\lambda,\nu} = \begin{cases} \{0\} & \textit{for } (\lambda,\nu) \in \mathbb{X} - L, \\ \bar{\mathfrak{n}}' & \textit{otherwise.} \end{cases}$$

(iii) For $(\lambda,\nu) \in //$*:*

$$\operatorname{Supp} u^C_{\lambda,\nu} = \{0\}.$$

13 Functional equations

Let $T_\lambda : \pi_\lambda \to \pi_{-\lambda}$ be the normalized Knapp–Stein intertwining operator for G given in the non-compact picture on $\bar{\mathfrak{n}}$ by convolution with

$$\frac{1}{\Gamma(\lambda)} N(X,Z)^{2(\lambda-\rho)}.$$

Similarly, we denote by $T'_\nu : \tau_\nu \to \tau_{-\nu}$ the normalized Knapp–Stein intertwining operator for G' which is defined for $m > 0$ by convolution with

$$\frac{1}{\Gamma(\nu)} N(X,Z)^{2(\nu-\rho')},$$

and for $m = 0$ by convolution with

$$\frac{1}{\Gamma(\frac{\nu}{2})} |Z|^{\nu-\rho'}.$$

By Lemma A.1 and Corollary B.4 both T_λ and T'_ν are holomorphic in the parameters λ and ν and non-trivial for all $\lambda,\nu \in \mathbb{C}$. Since the normalized Knapp–Stein operators are intertwining, they define maps between spaces of symmetry breaking operators by composition:

$$\begin{aligned} &\operatorname{Hom}_{G'}(\pi_\lambda|_{G'},\tau_\nu) \to \operatorname{Hom}_{G'}(\pi_{-\lambda}|_{G'},\tau_\nu), && A \mapsto A \circ T_{-\lambda}, \\ &\operatorname{Hom}_{G'}(\pi_\lambda|_{G'},\tau_\nu) \to \operatorname{Hom}_{G'}(\pi_\lambda|_{G'},\tau_{-\nu}), && A \mapsto T'_\nu \circ A. \end{aligned}$$

Formulas expressing the image of a symmetry breaking operator under such a composition map in terms of another symmetry breaking operator are called functional equations. These equations are particularly interesting if the Knapp–Stein operator or the symmetry breaking operator is a differential operator. For T_λ this happens precisely for $\lambda = -k \in -\mathbb{Z}_{\geq 0}$, where T_λ is given by convolution with (see Corollary B.4)

$$\frac{(-1)^k 2^{-k} k! \pi^{\frac{p+q}{2}}}{\Gamma(\frac{\rho+k}{2})} \sum_{i+2j=k} \frac{2^{-i}\Gamma(\frac{p+2i}{4})}{i!j!\Gamma(\frac{p+2i}{2})} \Delta^i_{\mathfrak{v}} \square^j \delta(X,Z).$$

Remark 13.1. Similarly for $m > 0$ and $\nu = -k \in -\mathbb{Z}_{\geq 0}$, T'_ν is given by convolution with

$$\frac{(-1)^k 2^{-k} k! \pi^{\frac{p'+q}{2}}}{\Gamma(\frac{\rho'+k}{2})} \sum_{i+2j=k} \frac{2^{-i}\Gamma(\frac{p'+2i}{4})}{i!j!\Gamma(\frac{p'+2i}{2})} \Delta^i_{\mathfrak{v}'} \square^j \delta(X',Z).$$

For $m = 0$ and $\nu = -2k \in -2\mathbb{Z}_{\geq 0}$, Lemma A.1 implies that the operator T'_ν is given by convolution with

$$(-1)^k \frac{\pi^{\frac{q}{2}}}{2^{2k}\Gamma(\frac{\rho'+2k}{2})} \square^k \delta(Z).$$

We choose the maximal compact subgroup $K = \mathrm{U}(1;\mathbb{F}) \times \mathrm{U}(n+1;\mathbb{F})$ of G. The spherical principal series representations π_λ contain a unique (up to scalar multiples) K-spherical vector $\mathbf{1}_\lambda \in I_\lambda$ which we normalize by $\mathbf{1}_\lambda(0) = 1$. We first find an explicit formula for $\mathbf{1}_\lambda$:

Lemma 13.2. *For $(X,Z) \in \bar{\mathfrak{n}}$ we have*

$$\mathbf{1}_\lambda(X,Z) = ((1+|X|^2)^2 + |Z|^2)^{-\frac{\lambda+\rho}{2}}.$$

Proof. By the Iwasawa decomposition $G = KAN$ we can decompose $\bar{n}_{(X,Z)} = kan$ with $k \in K$, $a = \exp(tH) \in A$ and $n \in N$. An easy computation shows that

$$e^{2t} = (1+|X|^2)^2 + |Z|^2,$$

then the claim follows. □

The intersection $K' = K \cap G' = \mathrm{U}(1;\mathbb{F}) \times \mathrm{U}(m+1;\mathbb{F}) \times F$ is a maximal compact subgroup of G' and we denote by

$$\mathbf{1}'_\nu(X',Z) = ((1+|X'|^2)^2 + |Z|^2)^{-\frac{\nu+\rho'}{2}}$$

the unique K'-spherical vector of τ_ν, normalized by $\mathbf{1}'_\nu(0) = 1$. Note that for $m = 0$ this equals $(1+|Z|^2)^{-\frac{\nu+\rho'}{2}}$.

We now evaluate the Knapp–Stein operators T_λ and T'_ν and the symmetry breaking operators $A_{\lambda,\nu}$ on these spherical vectors. The relevant integral in this context was computed by Frahm–Su [FS18]. To keep this paper self-contained, we include a full proof.

Proposition 13.3 ([FS18, Section 4.2])**.** *For $(\lambda,\nu) \in \mathbb{C}^2$ with $|\operatorname{Re}\nu| < \operatorname{Re}\lambda + \frac{p''}{2}$:*

$$\int_{\bar{\mathfrak{n}}} N(X,Z)^{-2(\nu+\rho')} |X''|^{\lambda-\rho+\nu+\rho'} ((1+|X|^2)^2 + |Z|^2)^{-\frac{\lambda+\rho}{2}}\, d(X,Z) \\ = \frac{\pi^{\frac{p+q}{2}} \Gamma(\frac{2\lambda+p}{4}) \Gamma(\frac{\lambda+\rho-\nu-\rho'}{2}) \Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})}{\Gamma(\frac{p''}{2}) \Gamma(\frac{2\lambda+p}{2}) \Gamma(\frac{\lambda+\rho}{2})}.$$

Proof. We abbreviate $a := -2(\nu+\rho')$, $b := \lambda-\rho+\nu+\rho'$ and $c := -2(\lambda+\rho)$. Using polar coordinates we find

$$\int_{\bar{\mathfrak{n}}} N(X,Z)^a |X''|^b ((1+|X|^2)^2 + |Z|^2)^{\frac{c}{4}}\, d(X,Z) \\ = \frac{8\pi^{\frac{p+q}{2}}}{\Gamma(\frac{p'}{2})\Gamma(\frac{p''}{2})\Gamma(\frac{q}{2})} \int_{\mathbb{R}_+^3} ((r^2+s^2)^2+t^2)^{\frac{a}{4}} ((1+r^2+s^2)^2+t^2)^{\frac{c}{4}} r^{p'-1} s^{p'+b-1} t^{q-1}\, dr\, ds\, dt.$$

We integrate over t first using [GR65, 3.259 (3)]

$$= \frac{4\pi^{\frac{p+q}{2}} \Gamma(-\frac{a+4b+2q}{4})}{\Gamma(\frac{p'}{2})\Gamma(\frac{p''}{2})\Gamma(-\frac{a+4b}{4})} \int_{\mathbb{R}_+^2} (1+r^2+s^2)^{\frac{c+2q}{2}} (r^2+s^2)^{\frac{a}{2}} r^{p'-1} s^{p''+b-1} \\ \times {}_2F_1\left(-\frac{a}{4}, \frac{q}{2}; -\frac{a+c}{4}; 1 - \frac{(1+r^2+s^2)^2}{(r^2+s^2)^2}\right) dr\, ds.$$

Using polar coordinates on $\mathbb{R}^2$ we find

$$= \frac{4\pi^{\frac{p+q}{2}}\Gamma(-\frac{a+4b+2q}{4})}{\Gamma(\frac{p'}{2})\Gamma(\frac{p''}{2})\Gamma(-\frac{a+4b}{4})} \int_0^{\frac{\pi}{2}} \cos^{p'-1}\phi \sin^{p''+b-1}\phi \, d\phi \int_0^\infty x^{p+b+a-1}(1+x^2)^{\frac{c+2q}{2}} \times {}_2F_1\left(-\frac{a}{4}, \frac{q}{2}; -\frac{a+c}{4}; 1 - \frac{(1+x^2)^2}{x^4}\right) dx.$$

Calculating the first integral and using the Euler identity [AAR99, 1, (2.2.7)]

$$= \frac{2\pi^{\frac{p+q}{2}}\Gamma(\frac{p''+b}{2})\Gamma(-\frac{a+4b+2q}{4})}{\Gamma(\frac{p''}{2})\Gamma(\frac{p+b}{2})\Gamma(-\frac{a+4b}{4})} \int_0^\infty x^{p+2q+a+b+c-1} \times {}_2F_1\left(-\frac{c}{4}, -\frac{a+c+2q}{4}; -\frac{a+c}{4}, 1 - \frac{(1+x^2)^2}{x^4}\right) dx.$$

Substituting $y = \frac{1+x^2}{x^2}$

$$= \frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{p''+b}{2})\Gamma(-\frac{a+4b+2q}{4})}{\Gamma(\frac{p''}{2})\Gamma(\frac{p+b}{2})\Gamma(-\frac{a+4b}{4})} \int_1^\infty (y-1)^{-\frac{p+2q+a+b+c}{2}-1} \times {}_2F_1\left(-\frac{c}{4}, -\frac{a+c+2q}{4}; -\frac{a+c}{4}, 1 - y^2\right) dy.$$

Expanding the hypergeometric function, using the Euler integral representation [AAR99, 1, (2.3.17)] yields

$$= \frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{p''+b}{2})}{\Gamma(\frac{p''}{2})\Gamma(\frac{q}{2})\Gamma(\frac{p+b}{2})} \int_0^\infty t^{-\frac{a+c+2q}{4}-1}(1+t)^{\frac{a}{4}} \int_1^\infty (y-1)^{-\frac{p+2q+a+b+c}{2}-1}(1+y^2t)^{\frac{c}{4}} \, dy \, dt.$$

Evaluating the inner integral using [GR65, 3.254 (2)] we find

$$= \frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{p''+b}{2})\Gamma(-\frac{p+2q+a+b+c}{2})\Gamma(\frac{p+2q+a+b}{2})}{\Gamma(\frac{p''}{2})\Gamma(\frac{q}{2})\Gamma(\frac{p+b}{2})\Gamma(-\frac{c}{2})} \int_0^\infty t^{-\frac{a+2q}{4}-1}(1+t)^{\frac{a}{4}} \times {}_2F_1\left(\frac{p+2q+a+b}{4}, \frac{p+2q+a+b+2}{4}; \frac{2-c}{4}; -t^{-1}\right) dt.$$

Substituting $x = t^{-1}$ we have

$$= \frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{p''+b}{2})\Gamma(-\frac{p+2q+a+b+c}{2})\Gamma(\frac{p+2q+a+b}{2})}{\Gamma(\frac{p''}{2})\Gamma(\frac{q}{2})\Gamma(\frac{p+b}{2})\Gamma(-\frac{c}{2})} \int_0^\infty x^{\frac{q}{2}-1}(1+x)^{\frac{a}{4}} \times {}_2F_1\left(\frac{p+2q+a+b}{4}, \frac{p+2q+a+b+2}{4}; \frac{2-c}{4}; -x\right) dx.$$

Then [GR65, 7.512 (5)] yields

$$= \frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{p''+b}{2})\Gamma(\frac{p+b}{4})\Gamma(-\frac{p+2q+a+b+c}{2})\Gamma(\frac{p+2q+a+b}{2})}{\Gamma(\frac{p''}{2})\Gamma(\frac{p+b}{2})\Gamma(-\frac{c}{2})\Gamma(\frac{p+2q+b}{4})} \times {}_3F_2\left(\frac{p+2q+a+b}{4}, -\frac{p+2q+a+b+c}{4}, \frac{q}{2}; \frac{2-c}{4}, \frac{p+2q+b}{4}; 1\right).$$

Since $-p+2q-a-b-c=\lambda+\rho+\nu+\rho'=p+2q+b$, this is

$$=\frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{p''+b}{2})\Gamma(\frac{p+b}{4})\Gamma(-\frac{p+2q+a+b+c}{2})\Gamma(\frac{p+2q+a+b}{2})}{\Gamma(\frac{p''}{2})\Gamma(\frac{p+b}{2})\Gamma(-\frac{c}{2})\Gamma(\frac{p+2q+b}{4})}{}_2F_1\left(\frac{p+2q+a+b}{4},\frac{q}{2};\frac{2-c}{4};1\right).$$

Then evaluating the hypergeometric function with [AAR99, Theorem 2.2.2] and applying the duplication formula of the gamma function yields the desired formula. □

For the case $m=0$ we further need the following integral:

Proposition 13.4. *For $\nu\in\mathbb{C}$ with $\operatorname{Re}\nu>0$ we have*

$$\int_{\mathfrak{z}}|Z|^{\nu-\rho'}(1+|Z|^2)^{-\frac{\nu+\rho'}{2}}\,dZ=\frac{\pi^{\frac{q}{2}}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+\rho'}{2})}.$$

Proof. Using polar coordinates on $\mathbb{R}^q$ the integral is equal to

$$\frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}\int_0^\infty r^{\nu-1}(1+r^2)^{-\frac{\nu+\rho'}{2}}\,dr.$$

Then substituting $r^2=t$ this is equal to

$$\frac{\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}B\left(\frac{\nu}{2},\frac{\rho'}{2}\right)=\frac{\pi^{\frac{q}{2}}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+\rho'}{2})}.$$ □

Corollary 13.5. *For the spherical vectors $\mathbf{1}_\lambda$ and $\mathbf{1}'_\nu$ we have*

$$T_\lambda\mathbf{1}_\lambda=\frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{2\lambda+p}{4})}{\Gamma(\frac{2\lambda+p}{2})\Gamma(\frac{\lambda+\rho}{2})}\mathbf{1}_{-\lambda},\qquad T'_\nu\mathbf{1}'_\nu=\frac{\pi^{\frac{p'+q}{2}}}{\Gamma(\frac{\nu+\rho'}{2})}\times\begin{cases}\frac{\Gamma(\frac{2\nu+p'}{4})}{\Gamma(\frac{2\nu+p'}{2})}\mathbf{1}'_{-\nu} & \text{for } m>0,\\ \mathbf{1}'_{-\nu} & \text{for } m=0\end{cases}$$

and

$$A_{\lambda,\nu}\mathbf{1}_\lambda=\frac{\pi^{\frac{p+q}{2}}\Gamma(\frac{2\lambda+p}{4})}{\Gamma(\frac{p''}{2})\Gamma(\frac{2\lambda+p}{2})\Gamma(\frac{\lambda+\rho}{2})}\mathbf{1}'_\nu.$$

Proof. Since $A_{\lambda,\nu}:\pi_\lambda\to\tau_\nu$ is intertwining and since $K'\subseteq K$, the image $A_{\lambda,\nu}\mathbf{1}_\lambda$ of the spherical vector $\mathbf{1}_\lambda$ must be K'-invariant, i.e. K'-spherical. Hence it is enough to compute $A_{\lambda,\nu}\mathbf{1}_\lambda(0)=\langle u^A_{\lambda,\nu},\mathbf{1}_\lambda\rangle$. The last identity follows from Proposition 13.3. The remaining two identities follow similarly by setting $\lambda-\rho+\nu+\rho'=0$ in Proposition 13.3 for $m>0$, and for $m=0$ by Proposition 13.4. □

Since the compositions of symmetry breaking operators and Knapp–Stein operators are again symmetry breaking operators and these are generically unique, we can read off the following functional equations by evaluating at the spherical vector and employing Corollary 13.5.

Theorem 13.6. *For $(\lambda,\nu)\in\mathbb{C}^2$ we have*

$$T'_\nu\circ A_{\lambda,\nu}=\frac{\pi^{\frac{p'+q}{2}}}{\Gamma(\frac{\nu+\rho'}{2})}\times\begin{cases}\frac{\Gamma(\frac{2\nu+p'}{4})}{\Gamma(\frac{2\nu+p'}{2})}A_{\lambda,-\nu} & \text{for } m>0,\\ A_{\lambda,-\nu} & \text{for } m=0,\end{cases}$$

$$A_{\lambda,\nu}\circ T_{-\lambda}=\pi^{\frac{p+q}{2}}\frac{\Gamma(\frac{2\lambda+p}{4})}{\Gamma(\frac{2\lambda+p}{2})\Gamma(\frac{\lambda+\rho}{2})}A_{-\lambda,\nu}.$$

Remark 13.7. Theorem 13.6 can be combined with the residue formulas Theorem 10.1(i) and Theorem 11.1(ii) to obtain formulas for the composition of $B_{\lambda,\nu}$ and $C_{\lambda,\nu}$ with the Knapp–Stein operators T_λ and T'_ν. For example for $m > 0$, $(\lambda,\nu) \in \mathbb{X}$ and $l, k \in \mathbb{Z}_{\geq 0}$ given by $\lambda + \rho + \nu - \rho' = -2l$, $\lambda + \rho - \nu - \rho' = -2k$ with $l \geq k$ we obtain the following identity involving only differential operators:

$$T'_{k-l} \circ C_{-k-l-\frac{p''}{2},k-l} = (-1)^{l+k}\pi^{\frac{p'+q}{2}} \frac{l!\Gamma(\frac{2l-2k+p'}{4})}{k!\Gamma(\frac{2l-2k+p'}{2})\Gamma(\frac{l-k+\rho'}{2})} C_{-k-l-\frac{p''}{2},l-k},$$

and for $(\lambda,\nu) \in L$ with $\nu \in -\rho' - 2\mathbb{Z}_{\geq 0}$ we obtain $T'_\nu \circ B_{\lambda,\nu} = 0$.

Appendix

A Homogeneous generalized functions

Let $n \geq 1$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -n$ the function

$$u_\lambda(x) = \frac{|x|^\lambda}{\Gamma(\frac{\lambda+n}{2})}$$

is locally integrable and hence defines a distribution $u_\lambda \in \mathcal{D}'(\mathbb{R}^n)$ which is homogeneous of degree λ.

Lemma A.1 ([GS68] Chapter I, 3.5 and 3.9)**.** *The family of distributions $u_\lambda \in \mathcal{D}'(\mathbb{R}^n)$ extends holomorphically to an entire function in $\lambda \in \mathbb{C}$. For $\lambda \notin -n - 2\mathbb{Z}_{\geq 0}$ we have* $\operatorname{Supp} u_\lambda = \mathbb{R}^n$ *and for $\lambda = -n-2N \in -n-2\mathbb{Z}_{\geq 0}$ we have*

$$u_{-n-2N}(x) = (-1)^N \frac{\pi^{\frac{n}{2}}}{2^{2N}\Gamma(\frac{n+2N}{2})}(\Delta^N\delta)(x),$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^n$.

The following result follows immediately from the classification of homogeneous distributions on $\mathbb{R}$ in [GS68, Chapter I, 3.11]:

Lemma A.2. *Let $F < \mathrm{O}(n)$ be a compact subgroup which acts transitively on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. If $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree λ and invariant under the action of F, then u is a scalar multiple of u_λ.*

B Integral formulas

For $\alpha \in \mathbb{Z}^p_{\geq 0}$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_p$ and $\alpha! = \alpha_1! \cdots \alpha_p!$.

Lemma B.1. *(i) Let $S^{p-1} \subseteq \mathbb{R}^p$ be the unit sphere and $\alpha \in \mathbb{Z}^p_{\geq 0}$, then*

$$\int_{S^{p-1}} \omega_1^{2\alpha_1} \ldots \omega_p^{2\alpha_p}\, d\omega = 2^{-2|\alpha|+1} \frac{(2\alpha)!\pi^{\frac{p}{2}}}{\alpha!\Gamma(\frac{p}{2}+|\alpha|)}.$$

(ii) For $p = p' + p''$ and $\omega \in S^{p-1}$, $\alpha \in \mathbb{Z}^p_{\geq 0}$ we write $\omega'' = (\omega_{p'+1}, \ldots \omega_p)$ and $\alpha'' = (\alpha_{p'+1}, \ldots, \alpha_p)$. Then for $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > -p'' - 2|\alpha''|$ we have

$$\int_{S^{p-1}} \omega_1^{2\alpha_1} \ldots \omega_p^{2\alpha_p} |\omega''|^\gamma\, d\omega = 2^{-2|\alpha|+1} \frac{\pi^{\frac{p}{2}}(2\alpha)!\Gamma(\frac{\gamma+p''}{2}+|\alpha''|)}{\alpha!\Gamma(\frac{\gamma+p}{2}+|\alpha|)\Gamma(\frac{p''}{2}+|\alpha''|)}.$$

Proof. Ad (i): By the repeated use of the coordinates $(-1,1)\times S^{m-1}\to S^m$, $(r,\eta)\mapsto\omega=(r,\sqrt{1-r^2}\eta)$ with $d\omega=(1-r^2)^{\frac{m-2}{2}}\,dr\,d\eta$ and the beta integral we find

$$\int_{S^{p-1}}\omega_1^{2\alpha_1}\dots\omega_p^{2\alpha_p}\,d\omega=2\prod_{l=1}^{p-1}\frac{\Gamma(\alpha_l+\frac{1}{2})\Gamma(\frac{p-l}{2}+\alpha_{l+1}+\dots+\alpha_p)}{\Gamma(\frac{p+1-l}{2}+\alpha_l+\dots+\alpha_p)}$$
$$=\frac{2}{\Gamma(\frac{p}{2}+|\alpha|)}\prod_{l=1}^{p}\Gamma(\alpha_l+\tfrac{1}{2})=2^{-2|\alpha|+1}\frac{(2\alpha)!\pi^{\frac{p}{2}}}{\alpha!\Gamma(\frac{p}{2}+|\alpha|)}.$$

Ad (ii): Similar as in (i) we find

$$\begin{aligned}&\int_{S^{p-1}}\omega_1^{2\alpha_1}\dots\omega_p^{2\alpha_p}|\omega''|^\gamma\,d\omega\\&=\int_{-1}^{1}r^{2\alpha_1}(1-r^2)^{\frac{\gamma+p-3}{2}+\alpha_2+\dots+\alpha_p}dr\int_{S^{p-2}}\omega_2^{2\alpha_2}\dots\omega_p^{2\alpha_p}|\omega''|^\gamma\,d\omega\\&=B\left(\alpha_1+\tfrac{1}{2},\tfrac{\gamma+p-1}{2}+\alpha_2+\dots+\alpha_p\right)\int_{S^{p-2}}\omega_2^{2\alpha_2}\dots\omega_p^{2\alpha_p}|\omega''|^\gamma\,d\omega\\&=\dots=\prod_{l=1}^{p'}B\left(\alpha_l+\tfrac{1}{2},\tfrac{\gamma+p-l}{2}+\alpha_{l+1}+\dots\alpha_p\right)\int_{S^{p''-1}}\omega_{p'+1}^{2\alpha_{p'+1}}\dots\omega_p^{2\alpha_p}\,d\omega.\end{aligned}$$

The remaining integral can be evaluated with (i). □

Fix $p,q\geq 1$ and let $\mathbb{S}:=\{(\omega,\eta)\in\mathbb{R}^p\times\mathbb{R}^q,|\omega|^4+|\eta|^2=1\}\subseteq\mathbb{R}^{p+q}$. By $d(X,Z)=dX\,dZ$ we denote the normalized Lebesgue measure on $\mathbb{R}^p\times\mathbb{R}^q\cong\mathbb{R}^{p+q}$.

Lemma B.2 ([Cow82, Chapter I])**.** *There exists a unique smooth measure $d\mathbb{S}$ on $\mathbb{S}$ such that for every $\varphi\in C_c^\infty(\mathbb{R}^{p+q})$:*

$$\int_{\mathbb{R}^{p+q}}\varphi(X,Z)\,d(X,Z)=\int_{\mathbb{R}_+}r^{p+2q-1}\int_{\mathbb{S}}\varphi((r\omega,r^2\eta))\,d\mathbb{S}(\omega,\eta)\,dr.$$

Lemma B.3. *For the measure $d\mathbb{S}$ of Lemma B.2 the following integral formula holds for all $\varphi\in C_c^\infty(\mathbb{S})$:*

$$\int_{\mathbb{S}}\varphi(\omega,\eta)\,d\mathbb{S}(\omega,\eta)=2\int_0^1x^{p-1}(1-x^4)^{\frac{q}{2}-1}\int_{S^{p-1}}\int_{S^{q-1}}\varphi(x\omega,\sqrt{1-x^4}\eta)\,d\eta\,d\omega\,dx.$$

Proof. Let $\psi\in C_c^\infty(\mathbb{R}^{p+q})$. Using polar coordinates we find

$$\int_{\mathbb{R}^{p+q}}\psi(X,Z)\,d(X,Z)=\int_{\mathbb{R}_+}s^{p-1}\int_{S^{p-1}}\int_{\mathbb{R}_+}t^{q-1}\int_{S^{q-1}}\psi(s\omega,t\eta)\,d\eta\,dt\,d\omega\,ds.$$

Now choosing coordinates $(s,t)=(rx,r^2\sqrt{1-x^4})$ we obtain

$$=2\int_{\mathbb{R}_+}r^{p+2q-1}\int_0^1x^{p-1}(1-x^4)^{\frac{q}{2}-1}\int_{S^{p-1}}\int_{S^{q-1}}\psi(rx\omega,r^2\sqrt{1-x^4}\eta)\,d\eta\,d\omega\,dx\,dr.$$

Then Lemma B.2 implies the statement. □

For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -(p+2q)$ the function

$$v_\lambda(X,Z) = \frac{1}{\Gamma(\frac{\lambda+p+2q}{2})}(|X|^4+|Z|^2)^{\frac{\lambda}{4}}$$

is locally integrable and hence defines a distribution $v_\lambda \in \mathcal{D}'(\mathbb{R}^{p+q})$.

Corollary B.4. *The family of distributions $v_\lambda \in \mathcal{D}'(\mathbb{R}^{p+q})$ extends holomorphically to an entire function in $\lambda \in \mathbb{C}$. For $\lambda \notin -(p+2q)-2\mathbb{Z}_{\geq 0}$ we have $\operatorname{Supp} v_\lambda = \mathbb{R}^{p+q}$ and for $\lambda = -(p+2q)-2N \in -(p+2q)-2\mathbb{Z}_{\geq 0}$ we have*

$$v_{-(p+2q)-2N}(X,Z) = \frac{(-1)^N 2^{-N} N! \pi^{\frac{p+q}{2}}}{\Gamma(\frac{2N+p+2q}{4})} \sum_{i+2j=N} \frac{2^{-i}\Gamma(\frac{p+2i}{4})}{i!j!\Gamma(\frac{p+2i}{2})} \Delta^i \Box^j \delta(X,Z),$$

where $\Delta = \sum_{i=1}^p \frac{\partial^2}{\partial X_i^2}$ and $\Box = \sum_{j=1}^q \frac{\partial^2}{\partial Z_j^2}$ are the Laplacians on $\mathbb{R}^p$ and $\mathbb{R}^q$.

Proof. Using the polar coordinates $(r\omega, r^2\eta)$, $(\omega,\eta) \in \mathbb{S}$, of Lemma B.2, the distribution v_λ is given by

$$\frac{1}{\Gamma(\frac{\lambda+p+2q}{2})} r^{\lambda+p+2q-1},$$

which is holomorphic in $\lambda \in \mathbb{C}$ by Lemma A.1. For $\lambda \notin -(p+2q)-2\mathbb{Z}_{\geq 0}$ we have $\operatorname{Supp} r^{\lambda+p+2q-1} = \mathbb{R}_+$ and hence $\operatorname{Supp} v_\lambda = \mathbb{R}^{p+q}$. For $\lambda = -(p+2q)-2N$, $N \in \mathbb{Z}_{\geq 0}$, Lemma A.1 shows that

$$\frac{1}{\Gamma(\frac{\lambda+p+2q}{2})} r^{\lambda+p+2q-1} = (-1)^N \frac{N!}{2(2N)!} \delta^{(2N)}(r).$$

Let $\varphi \in C_c^\infty(\bar{\mathfrak{n}})$. Then by Lemma B.3 and Lemma C.1 we have

$$\frac{d^{2N}}{dt^{2N}}\bigg|_{t=0} \int_{\mathbb{S}} \varphi(r\omega, r^2\eta)\, d\mathbb{S} = 2 \sum_{i+2j=2N} \sum_{|\alpha|=i} \sum_{|\beta|=j} \frac{(2N)!}{\alpha!\beta!} \int_0^1 x^{p+i-1}(1-x^4)^{\frac{q+j}{2}-1}\, dx$$
$$\times \int_{S^{p-1}} \omega^\alpha \, d\omega \int_{S^{q-1}} \eta^\beta \, d\eta \left(\frac{\partial}{\partial X}\right)^\alpha \left(\frac{\partial}{\partial Z}\right)^\beta \varphi(0,0).$$

Now the integrals over the spheres vanish for odd-length multi-indices and the remaining integrals can be computed using Lemma B.1. We obtain

$$= \sum_{i+2j=N} \sum_{|\alpha|=i} \sum_{|\beta|=j} 2^{-2i-2j+1} \pi^{\frac{p+q}{2}} \frac{(2N)! B(\frac{p+2i}{4}, \frac{q}{2}+j)}{\alpha!\beta!\Gamma(\frac{p}{2}+i)\Gamma(\frac{q}{2}+j)} \left(\frac{\partial}{\partial X}\right)^{2\alpha} \left(\frac{\partial}{\partial Z}\right)^{2\beta} \varphi(0,0),$$

which is by the Multinomial Theorem (C.2) equal to

$$= \left\langle \sum_{i+2j=N} 2^{-2i-2j+1} \pi^{\frac{p+q}{2}} \frac{(2N)!\Gamma(\frac{p+2i}{4})}{\Gamma(\frac{2N+p+2q}{4}) i! j! \Gamma(\frac{p}{2}+i)} \Delta_{\mathfrak{v}}^i \Box^j \delta, \varphi \right\rangle. \qquad \Box$$

C Combinatorial identities

We recall Faà di Bruno's Formula for the higher derivatives of the composition of two real-valued functions f and g on $\mathbb{R}$:

$$\frac{d^n}{dx^n}\Big[(f\circ g)(x)\Big] = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1!\cdots k_n!}\left[\frac{d^{k_1+\cdots+k_n}f}{dx}\circ g\right](x)\prod_{m=1}^{n}\left(\frac{1}{m!}\frac{d^m g}{dx^m}\right)^{k_m}. \quad \text{(C.1)}$$

Lemma C.1. *Let $f \in C^\infty(\mathbb{R}^2)$, then*

$$\frac{d^k}{dt^k}\bigg|_{t=0} f(tx, t^2y) = \sum_{i+2j=k} \frac{k!}{i!j!} x^i y^j \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0).$$

Proof. It is clear that the left hand side is an expression of the form

$$\sum_{i+2j=k} c_{ijk} x^i y^j \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0)$$

with certain non-negative integer coefficients c_{ijk}. By riffle shuffle permutation the coefficients c_{ijk} are divisible by $\binom{k}{i}$ and the normalized coefficients $\tilde{c}_{ijk} := \binom{k}{i}^{-1} c_{ijk}$ satisfy

$$\frac{d^{2j}}{dt^{2j}}\bigg|_{t=0} \varphi(t^2y) = \tilde{c}_{ijk} \frac{\partial^j \varphi}{\partial y^j}(0) \qquad \forall\, \varphi \in C^\infty(\mathbb{R}).$$

By Faà di Bruno's Formula (C.1) we have $\tilde{c}_{ijk} = \frac{(2j)!}{j!}$. Hence $c_{ijk} = \frac{(i+2j)!}{i!j!}$. □

For commuting variables $x_1, \ldots, x_n$ and $m \in \mathbb{Z}_{\geq 0}$ the Multinomial Theorem holds:

$$(x_1 + \cdots + x_n)^m = \sum_{\substack{\alpha \in \mathbb{Z}^n_{\geq 0} \\ |\alpha| = m}} \frac{m!}{\alpha_1! \cdots \alpha_n!} x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \quad \text{(C.2)}$$

For $a \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$ let $(a)_n = a(a+1)\cdots(a+n-1)$ denote the Pochhammer symbol. For multi-indices $\alpha \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$ we further write

$$(x)_\alpha = (x_1)_{\alpha_1} \cdots (x_n)_{\alpha_n}.$$

Lemma C.2. *For any $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$ we have*

$$\sum_{|\alpha|=m} \frac{(x)_\alpha}{\alpha!} = \frac{(x_1 + \cdots + x_n)_m}{m!}.$$

Proof. The proof is by induction on n. For $n = 1$ the statement is obvious. For the induction step write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ and $\alpha = (\beta, k)$ with $\beta \in \mathbb{Z}^{n-1}_{\geq 0}$ and $0 \leq k \leq m$, then

$$\begin{aligned}
\sum_{|\alpha|=m} \frac{(x)_\alpha}{\alpha!} &= \sum_{k=0}^{m} \frac{(x_n)_k}{k!} \sum_{|\beta|=m-k} \frac{(x')_\beta}{\beta!} \\
&= \sum_{k=0}^{m} \frac{(x_n)_k (x_1 + \cdots + x_{n-1})_{m-k}}{k!(m-k)!} \\
&= \frac{(x_1 + \cdots + x_{n-1} + x_n)_m}{m!},
\end{aligned}$$

where, in the last step, we have used the identity

$$\sum_{k=0}^{m} \binom{m}{k} (a)_k (b)_{m-k} = (a+b)_m,$$

which is equivalent to the Vandermonde identity

$$\sum_{k=0}^{m} \binom{a}{k} \binom{b}{m-k} = \binom{a+b}{m},$$

since $\frac{1}{k!}(a)_k = (-1)^k \binom{-a}{k}$. $\square$

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Paper B

Branching laws for unitary representations of $\mathrm{U}(1, n+1)$ in the scalar principal series

Jan Frahm and Clemens Weiske

Abstract

We find complete branching laws and explicit Plancherel formulas for the restriction of irreducible unitary representations of $\mathrm{U}(1, n+1)$ to $\mathrm{U}(1, n)$ which are contained in the scalar principal series. These unitary representations belong either to the unitary principal series or the complementary series or two types of unitarizable quotients of the scalar principal series. For quotients which are unitary highest or lowest weight representations we obtain a discrete decomposition into unitary highest or lowest weight representations. In most other cases the decomposition has a continuous and a discrete part. The Plancherel formulas are given in terms of holomorphic families of $\mathrm{U}(1, n)$-intertwining operators and the discrete spectrum is constructed as residues of these families.

Introduction

Branching laws describe the decomposition of a representation π of a group G into irreducible representations for a subgroup $G' \subseteq G$. They arise for example in physics when the symmetries of a system are reduced, but they also appear in several branches of mathematics. Of particular relevance are branching problems in the category of unitary Lie group representations since these representations are important in harmonic analysis and its applications, for instance to symmetries in quantum mechanics.

For a compact Lie group G, an irreducible unitary representation π is automatically finite-dimensional and its restriction $\pi|_{G'}$ decomposes into a direct sum

$$\pi|_{G'} \cong \bigoplus_{\tau \in \widehat{G'}} m(\pi, \tau)\, \tau$$

of irreducible unitary representations τ of G' with multiplicities $m(\pi, \tau) \in \mathbb{Z}_{\geq 0}$. Using the Lie correspondence, the branching problem can to a large extent be translated into an algebraic problem for Lie algebra representations and there exist formulas for the multiplicites $m(\pi, \tau)$ which are of an algebraic or combinatorial nature.

When G is non-compact, irreducible unitary representations are typically infinite-dimensional and their restrictions decompose into direct integrals instead of direct sums:

$$\pi|_{G'} \cong \int_{\widehat{G'}}^{\oplus} m(\pi, \tau)\, \tau \, d\mu_\pi(\tau),$$

where $d\mu_\pi$ is a certain measure on the unitary dual $\widehat{G}'$ of G and $m(\pi,\tau) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ a multiplicity function. Solving the branching problem for $\pi|_{G'}$ consists of determining $d\mu_\pi$ and $m(\pi,\tau)$ explicitly. We remark that the measure $d\mu_\pi$ may contain both a discrete and a continuous part in general.

While the orbit method for nilpotent Lie groups provides a precise answer to the branching problem for all unitary representations, no general theory is available for the class of real reductive groups. In the 1990s Kobayashi [Kob94] initiated the study of discretely decomposable branching problems for real reductive groups, i.e. the measure $d\mu_\pi$ is discrete and the direct integral decomposition is in fact a direct sum. In this case algebraic and geometric methods are available to study and obtain explicit branching laws (see e.g. [Osh15] and references therein). But especially the case where $\pi|_{G'}$ contains both continuous and discrete spectrum has so far not been treated systematically since here more delicate analytical problems arise (see e.g. [Kob19, MO15, ØS19, ØV04, Var11, Xie94] for some special cases).

In this paper we propose a general strategy to obtain branching laws for unitary representations of real reductive groups and illustrate this technique for the groups $(G,G') = (\mathrm{U}(1,n+1),\mathrm{U}(1,n))$ and all representations π which occur inside the principal series induced from a character. The idea consists essentially of two steps. We first decompose the unitary principal series and representations which are sufficiently close to it by applying the Mackey machinery to the open G'-orbits in the generalized flag variety for G and using harmonic analysis on the corresponding homogeneous spaces. This allows us to state the relevant branching laws in terms of families of intertwining operators between principal series representations of G and G', also called *symmetry breaking operators* by Kobayashi [Kob15]. The second step is an analytic continuation procedure which allows to decompose also unitary representations of G which occur as quotients of principal series representations for singular parameters. In our special case $(G,H) = (\mathrm{U}(1,n+1),\mathrm{U}(1,n))$ the analytic continuation produces additional discrete components when moving the induction parameters away from the unitary principal series. In particular, we are able to decompose both unitary principal series representations, complementary series representations, unitary highest/lowest weight representations and relative discrete series representations.

We now describe our results in detail.

Unitary representations in the scalar principal series

Let $G = \mathrm{U}(1,n+1)$ and $\pi_{s,\lambda}$ ($\lambda \in \mathbb{C}$, $s \in \mathbb{Z}$) be the scalar principal series of G which contains the one-dimensional K-type $e^{-is\theta} \boxtimes \mathbb{C}$ of $K \cong \mathrm{U}(1) \times \mathrm{U}(n+1)$. We choose our normalization such that $\pi_{s,\lambda}$ is unitary for $\lambda \in i\mathbb{R}$ and such that $\pi_{0,\lambda}$ has a finite dimensional quotient if and only if $\lambda \in \rho + 2\mathbb{Z}_{\geq 0}$, where $\rho = n+1$. Then besides the unitary principal series there is a complementary series in many cases. More precisely $\pi_{s,\lambda}$ is a complementary series representation if and only if

$$\lambda \in \begin{cases} (-\rho+|s|, \rho-|s|) & \text{if } |s| < \rho, \\ (-1,1) & \text{if } |s| \geq \rho \text{ and } s+\rho \equiv 1 \mod 2 \end{cases}$$

(see Corollary 3.3). For $\lambda \in -\rho \pm s - 2\mathbb{Z}_{\geq 0}$, the representation $\pi_{s,\lambda}$ is reducible and contains an infinite dimensional unitarizable quotient $\pi^{\llcorner}_{s,\lambda}$ without highest or lowest weight vector and for $\lambda = \rho + 2i - |s| \leq 0$, $i \in \mathbb{Z}_{\geq 0}$ the representation $\pi_{s,\lambda}$ contains an infinite dimensional unitarizable quotient $\pi^{\sqsubset}_{s,\lambda}$ with highest or lowest weight vector, depending on the sign of s. The superscipts $\llcorner$ and $\sqsubset$ are used to represent the K-type picture of the composition factors (see Figure 3.1 and Figure 3.2). Moreover we remark that both types of quotients can occur in the same principal series representation. A full description of the composition series of $\pi_{s,\lambda}$ and unitarizability of composition factors is obtained in Theorem 3.4.

Let $G' = \mathrm{U}(1,n)$ embedded in G in the usual way. Then we denote the scalar principal series for G' by $\tau_{t,\nu}$ ($\nu \in \mathbb{C}$, $t \in \mathbb{Z}$) similarly normalized, replacing ρ by $\rho' = n$. We denote the corresponding unitarizable composition factors in the same way with the superscripts $\llcorner$ and $\sqsubset$. For the formulation of the main results we define the following discrete set:

$$D'_t = \{\nu \in \mathbb{R} : \nu = \rho' + 2j - |t| < 0, j \in \mathbb{Z}_{\geq 0}\}$$

Note that D'_t is precisely the set of parameters $\nu \neq 0$ such that $\tau_{t,\nu}$ contains a unitarizable highest/lowest weight representation.

Main results

The scalar principal series representations can be realized as sections of a homogeneous line bundle over the real flag variety G/P (see Section 1.2 for details). Then G' acts with an open orbit on G/P and the orbit is as a G'-space isomorphic to $G'/\mathrm{U}(n)$ (see Section 1.1). If $\pi_{s,\lambda}$ is a unitary principal series representation we obtain a unitary map

$$\Phi : \pi_{s,\lambda} \to L^2(G'/\mathrm{U}(n))$$

into the space $L^2(G'/\mathrm{U}(n))$ whose Plancherel formula and direct integral decomposition is well-known for example by [Shi94] and [FJ77]. We lift the Plancherel formula of $L^2(G'/\mathrm{U}(n))$ to the representation $\pi_{s,\lambda}$ using the map Φ. More precisely the Fourier transforms for $G'/\mathrm{U}(n)$ become symmetry breaking operators under composition with Φ (see Lemma 7.4). In this way we deduce the branching law for the unitary principal series by Mackey theory.

Theorem A (see Theorem 8.4(i))**.** *For the unitary principal series, $\lambda \in i\mathbb{R}$, we have*

$$\pi_{s,\lambda}|_{G'} \cong \bigoplus_{t \in \mathbb{Z}} \left(\int_{i\mathbb{R}}^{\oplus} \tau_{t,\nu}\, d\nu \oplus \bigoplus_{-\nu \in D'_t} \tau^{\sqsubset}_{t,\nu} \right).$$

Moreover we obtain the explicit Plancherel formula (see Lemma 7.6) which is given in terms of symmetry breaking operators with meromorphic dependence on $\lambda \in \mathbb{C}$. For the complementary series and the unitarizable quotients the scalar product of the unitary closures is given in terms of standard Knapp–Stein intertwining operators. In this setting the map Φ is also well defined for λ in an open interval $(-1,1)$ around the unitary axis (see Section 6 and Section 7 for the relevant statements). This way we immediately obtain branching laws and Plancherel formulas for the small complementary series $\lambda \in (-1,1)$ in the same way as in the case of the unitary principal series. Again the Plancherel formula is of meromorphic dependence of λ by the symmetry breaking operators and additionally by the Knapp–Stein intertwining operators occurring in the scalar products of the complementary series. This allows us to carefully analytically continue this formula on the real axis. We collect discrete components as residues of symmetry breaking operators and prove the following branching laws for the whole complementary series.

Theorem B (see Theorem 8.4(ii)). *For $\pi_{s,\lambda}$ in the complementary series we have*

$$\pi_{s,\lambda}|_{G'} \cong \bigoplus_{t\in\mathbb{Z}} \left(\int_{i\mathbb{R}}^{\oplus} \tau_{t,\nu}\, d\nu \oplus \bigoplus_{-\nu\in D'_t} \tau^{\sqsubset}_{t,\nu} \right) \oplus \bigoplus_{t=-\rho'+1}^{\rho'-1} \left(\bigoplus_{k\in[0,\frac{-\lambda-1-|s-t|}{2})\cap\mathbb{Z}} \tau_{t,-\lambda-1-|s-t|-2k} \right).$$

Note that all additional discrete spectrum is in the complementary series of G'. By formally continuing the Plancherel formula (see Lemma 7.6) along the real axis beyond the complementary series (see Section 8) we can further obtain branching laws for all unitarizable quotients. Here we collect additional discrete summands of unitarizable quotients $\tau^{\llcorner}_{t,\nu}$ besides the complementary series. Since the quotients $\pi^{\llcorner}_{s,\lambda}$ and $\pi^{\sqsubset}_{s,\lambda}$ occur in the same representation $\pi_{s,\lambda}$, a careful analysis of the Plancherel formula for the whole representation is needed to deduce the branching laws for the quotients and in particular the discrete decomposition of $\pi^{\sqsubset}_{s,\lambda}$.

Theorem C (see Theorem 8.4(iii) and Theorem 8.4(iv)). *(i) For $i\in\mathbb{Z}_{\geq 0}$ and $\lambda=-\rho-2i\pm s\leq 0$, we have for the unitary quotient*

$$\pi^{\llcorner}_{s,\lambda}|_{G'} \cong \bigoplus_{t\in\mathbb{Z}} \left(\int_{i\mathbb{R}}^{\oplus} \tau_{t,\nu}\, d\nu \oplus \bigoplus_{-\nu\in D'_t} \tau^{\sqsubset}_{t,\nu} \right) \oplus \bigoplus_{t\in\mathbb{Z},\, |t|\geq\rho'} \bigoplus_{k\in[0,\frac{-\lambda-1-|s-t|}{2})\cap\mathbb{Z}} \tau^{\llcorner}_{t,-\lambda-1-|s-t|-2k}$$
$$\oplus \bigoplus_{t=-\rho'+1}^{\rho'-1} \left(\bigoplus_{k\in[0,\frac{-\lambda-\rho-|s-t|+|t|}{2}]\cap\mathbb{Z}} \tau^{\llcorner}_{t,-\lambda-1-|s-t|-2k} \right.$$
$$\left. \oplus \bigoplus_{k\in(\frac{-\lambda-\rho-|s-t|+|t|}{2},\frac{-\lambda-1-|s-t|}{2})\cap\mathbb{Z}} \tau_{t,-\lambda-1-|s-t|-2k} \right).$$

(ii) For $i\in\mathbb{Z}_{\geq 0}$ and $\lambda=\rho+2i\mp s\leq 0$, we have for the unitary highest resp. lowest weight representation $\pi^{\sqsubset}_{s,\lambda}$ the discrete branching law

$$\pi^{\sqsubset}_{s,\lambda}|_{G'} \cong \bigoplus_{a=\pm s-i}^{\infty} \bigoplus_{j=0}^{i} \tau^{\sqsubset}_{j+a,\rho'+j-a}.$$

The second statement of the Theorem above is contained in the main result of [Kob08] where it is obtained algebraically while our approach uses analytic methods to deduce the decomposition and also gives an explicit Plancherel formula (see Corollary 8.3).

Symmetry breaking operators

For the proof of the main theorem by analytic continuation the crucial ingredient is the meromorphic dependence of the symmetry breaking operators occurring in the Plancherel formulas. These operators are obtained by composition of the map Φ with the Fourier transform on $G'/\mathrm{U}(n)$ and are elements of the space

$$\mathrm{Hom}_{G'}(\pi_{s,\lambda}|_{G'}, \tau_{t,\nu}).$$

In [FW20] we classified symmetry breaking operators between the spherical principal series representations of G and G'. In the spherical case the space $\mathrm{Hom}_{G'}(\pi_{0,\lambda}|_{G'}, \tau_{0,\nu})$ is generically one-dimensional and symmetry breaking operators are given by a holomorphic family of integral kernel operators and their residues. We use the translation principle in the sense of [FØ19b] to construct families of symmetry breaking operators $A^{s,t}_{\lambda,\nu}$ for the scalar case (see Section 2). In contrast to [FW20] we find the optimal normalization of these families and their residues by studying the explicit K'-sprectrum of the operators as proposed in [FØ19a]. We define the set

$$L := \{(-\rho - 2i, -\rho' - 2j) \in \mathbb{C}^2, i, j \in \mathbb{Z}_{\geq 0},\, i \geq j\}.$$

Then consistent with the spherical case we find the holomorphic family $A^{s,t}_{\lambda,\nu}$ to vanish if and only if $(\lambda + |s-t| - |t|, \nu - |t|) \in L$. Further we obtain two normalizations of $A^{s,t}_{\lambda,\nu}$ along two families of complex lines. For $\nu \in -\rho' + |t| - 2\mathbb{Z}_{\geq 0}$ we have

$$\tilde{A}^{s,t}_{\lambda,\nu} = \Gamma\left(\frac{\lambda + \rho - \nu - \rho' + |s-t|}{2}\right) A^{s,t}_{\lambda,\nu},$$

and for $\lambda + \rho - \nu - \rho' + |s-t| \in -2\mathbb{Z}_{\geq 0}$ we have

$$C^{s,t}_{\lambda,\nu} = \Gamma\left(\frac{\nu + \rho' - t}{2}\right)\Gamma\left(\frac{\nu + \rho' + t}{2}\right) A^{s,t}_{\lambda,\nu}.$$

Theorem D (see Lemma 5.7 and Corollary 5.8)**.** *The operators $\tilde{A}^{s,t}_{\lambda,\nu}$ and $C^{s,t}_{\lambda,\nu}$ are non-vanishing holomorphic families of symmetry breaking operators on the complex lines where they are defined. They are linearly independent if and only if $(\lambda + |s-t| - |t|, \nu - |t|) \in L$.*

These two families of symmetry breaking operators play a central role in the proof of the main theorems since they give the contribution of the two different types of unitarizable quotients in the decomposition of the unitary representations.

Functional equations

The second crucial ingredient of the proof in the main theorem are the Knapp–Stein intertwining operators which occur in the scalar products of the unitary representations outside the unitary principal series. Therefore we prove functional equations for the composition with symmetry breaking operators. More precisely, for $f \in \pi_{s,\lambda}$ the holomorphic family of G-intertwining operators

$$T_{s,\lambda} : \pi_{s,\lambda} \to \pi_{s,-\lambda}$$

is given by

$$T_{s,\lambda}f(x) = \frac{1}{\Gamma(\lambda)} \int_{\bar{N}} f(x\tilde{w}_0\bar{n})d\bar{n},$$

where $\tilde{w}_0$ is a representative of the longest Weyl group element of G. Similarly we denote by $T'_{t,\nu}$ the corresponding Knapp–Stein intertwiner for G'.

Theorem E (see Theorem 4.6)**.** *For* $(\lambda,\nu) \in \mathbb{C}^2$ *and* $s,t \in \mathbb{Z}$ *we have*

$$T'_{t,\nu} \circ A^{s,t}_{\lambda,\nu} = \frac{2^{1-\nu-\rho'}\pi^{\rho'}}{\Gamma(\frac{\nu+\rho'+t}{2})\Gamma(\frac{\nu+\rho'-t}{2})} A^{s,t}_{\lambda,-\nu},$$

$$A^{s,t}_{\lambda,\nu} \circ T_{s,-\lambda} = \frac{2^{1-\lambda-\rho}\pi^{\rho}}{\Gamma(\frac{\lambda+\rho-s}{2})\Gamma(\frac{\lambda+\rho+s}{2})} A^{s,t}_{-\lambda,\nu}.$$

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Notation

For two sets $B \subseteq A$ we use the Notation $A - B = \{a \in A : a \notin B\}$. We denote Lie groups by Roman capitals and their corresponding Lie algebras by the corresponding Fraktur lower cases.

1 Preliminaries

Let $G = \mathrm{U}(1, n+1)$, $n > 1$ and let P be a minimal parabolic subgroup of G with Langlands decomposition $P = MAN$. Then $M \cong \mathrm{U}(1)\times\mathrm{U}(n)$, $A = \exp(\mathfrak{a}) \cong \exp(\mathbb{R})$ and $N = \exp(\mathfrak{n})$ where $\mathfrak{n} \cong \mathbb{C}^n \oplus i\mathbb{R}$ is the $(2n+1)$-dimensional Heisenberg algebra. Explicitly we realize G as complex $(n+2)\times(n+2)$-matrices preserving the quadratic form

$$(z_0, z_1, \ldots, z_{n+1}) \mapsto -|z_0|^2 + |z_1|^2 + \cdots + |z_{n+1}|^2$$

and for the minimal parabolic we choose

$$M = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix} : a \in \mathrm{U}(1), b \in \mathrm{U}(n) \right\},$$

$$A = \exp(\mathfrak{a}) \qquad \text{where } \mathfrak{a} = \mathbb{R}H, \quad H = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathbf{0}_n \end{pmatrix},$$

$$N = \exp(\mathfrak{n}) \qquad \text{where } \mathfrak{n} = \left\{ \begin{pmatrix} Z & -Z & X \\ Z & -Z & X \\ X^* & -X^* & \mathbf{0}_n \end{pmatrix} : X \in \mathbb{C}^n, Z \in i\mathbb{R} \right\}.$$

Here, for $X = (X_1, \ldots, X_n) \in \mathbb{C}^n$ we write $X^* := (\overline{X}_1, \ldots \overline{X}_n)^T$. Clearly $\mathfrak{n}$ is the direct sum of the eigenspaces of $\mathrm{ad}(H)$ to the eigenvalues $+1$ and $+2$. We identify $\mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}$ by $\lambda \mapsto \lambda(H)$ such that

$$\rho = \frac{1}{2} \operatorname{tr} \operatorname{ad}|_{\mathfrak{n}}(H) = n + 1.$$

Moreover let $\overline{N}$ be the nilradical of the parabolic opposite to P and $\bar{\mathfrak{n}}$ its Lie-algebra. Then $\bar{\mathfrak{n}} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z} \cong i\mathbb{R}$ is the center of $\bar{\mathfrak{n}}$ and $\mathfrak{v} \cong \mathbb{C}^n$ the orthogonal compliment satisfying $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$.

Since $\overline{N}$ is unipotent, we identify it with its Lie algebra $\bar{\mathfrak{n}} \cong \mathbb{C}^n \oplus i\mathbb{R}$ in terms of the exponential and choose coordinates:

$$\mathbb{C}^n \oplus i\mathbb{R} \to \overline{N}, \quad (X, Z) \mapsto \bar{n}_{(X,Z)} := \exp \begin{pmatrix} \frac{Z}{2} & \frac{Z}{2} & X \\ -\frac{Z}{2} & -\frac{Z}{2} & -X \\ X^* & X^* & \mathbf{0}_n \end{pmatrix}.$$

Similarly let $G' = \mathrm{U}(1, n)$ embedded into G in the upper left corner. Then $P' := G' \cap P$ is a minimal parabolic of G' with Langlands decomposition $P' = M'AN'$. Let $\bar{\mathfrak{n}}'$ be the Lie-algebra of the nilradical of the opposite parabolic of P' which is the $(2n-1)$-dimensional Heisenberg algebra and decomposes into $\bar{\mathfrak{n}}' = \mathfrak{v}' \oplus \mathfrak{z}$ with $[\mathfrak{v}', \mathfrak{v}'] = \mathfrak{z}$ as above. Then similarly

$$\rho' = \frac{1}{2} \operatorname{tr} \operatorname{ad}|_{\mathfrak{n}'}(H) = n.$$

Let $\tilde{w}_0 := \mathrm{diag}(-1, -1, \mathbf{1}_n)$ be a representative of the longest Weyl-group element. The following matrix decompositions are easily computed.

Lemma 1.1. *(i) Let $m = \mathrm{diag}(a, a, b^{-1}) \in M$ with $a \in \mathrm{U}(1; \mathbb{F})$ and $b \in \mathrm{U}(n; \mathbb{F})$, then*

$$m\bar{n}_{(X,Z)}m^{-1} = \bar{n}_{(aXb, aZa^{-1})}.$$

(ii) *Let* $t \in \mathbb{R}$ *and* $a = \exp(tH)$*, then*

$$a\bar{n}_{(X,Z)}a^{-1} = \bar{n}_{(e^{-t}X, e^{-2t}Z)}.$$

(iii) *Let* $(X, Z) \neq (0, 0)$*, then* $\tilde{w}_0\bar{n}_{(X,Z)} = \bar{n}_{(U,V)}man$ *with* $m \in M$*,* $n \in N$ *and*

$$U = \frac{-(|X|^2 + Z)X}{N(X,Z)^4}, \qquad V = \frac{-Z}{N(X,Z)^4}, \qquad a = \exp(2\log(N(X,Z))H).$$

1.1 P'-orbit structure of G/P

Following [FW20, I.2.3] the P'-orbit structure of G/P is given in the following way:

Lemma 1.2. *The* P'*-orbits in* G/P *and their closure relations are*

$$\mathcal{O}_A \overset{2}{\text{———}} \mathcal{O}_B \overset{2n-1}{\text{———}} \mathcal{O}_C,$$

where

$$\begin{aligned}
\mathcal{O}_A &= P' \cdot \tilde{w}_0\bar{n}P = \tilde{w}_0(\overline{N} - \overline{N}')P, \\
\mathcal{O}_B &= P' \cdot \tilde{w}_0 P = \tilde{w}_0\overline{N}'P, \\
\mathcal{O}_C &= P' \cdot \mathbf{1}_{n+2}P,
\end{aligned}$$

for some $\bar{n} \in \overline{N} - \overline{N}'$*. Here* $X \overset{k}{\text{——}} Y$ *means that* Y *is a subvariety of* $\bar{X}$ *of co-dimension* k*.*

In particular the orbit $\mathcal{O}_A$ is open and we will show in the following that $\mathcal{O}_A$ is in fact a G'-orbit.

Lemma 1.3. *(i)* *Let* $K_0' := \mathrm{Stab}_{G'}(\overline{n}_{e_n}P)$*. Then* $K_0' = \mathbf{1} \times \mathrm{U}(n) \subseteq K' = \mathrm{U}(1) \times \mathrm{U}(n)$*.*

(ii) *We have* $G' \cdot \overline{n}_{e_n}P = P' \cdot \overline{n}_{e_n}P = (\overline{N} - \overline{N}')P$ *is the open* P' *orbit in* G/P*.*

For the proof of this lemma we make use of the explicit action of G' on $G/P \cong K/M$ and of the Iwasawa decomposition of elements of $\overline{N}$. Therefor consider the map

$$K/M \to S^{2\rho-1} \subseteq \mathbb{C}^{n+1}$$

given by

$$k = \begin{pmatrix} a & & \\ & b & * \\ & c & * \end{pmatrix} \mapsto \overline{\left(\frac{b}{a}, \frac{c}{a}\right)}.$$

Lemma 1.4. *(i)* *The map*

$$K/M \to S^{2\rho-1} \subseteq \mathbb{C}^{n+1}$$

$$k = \begin{pmatrix} a & & \\ & b & * \\ & c & * \end{pmatrix} \to \overline{\left(\frac{b}{a}, \frac{c}{a}\right)}$$

is a G *equiavriant isomorphism with the action of* G *on* $S^{2\rho-1}$ *given by*

$$g \cdot \omega = \frac{(\overline{g}(1,\omega)^t)'}{(\overline{g}(1,\omega)^t)^1},$$

where $(\,\cdot\,)^1$ *is the first and* $(\,\cdot\,)'$ *the remainign coordinates of the vector.*

(ii) We have $\overline{n}_{(X,Z)} = \kappa(\overline{n}_{(X,Z)})e^{H(\overline{n}_{(X,Z)})}n \in KAN$, *with*

$$\kappa(\overline{n}_{(X,Z)}) = \begin{pmatrix} a & & \\ & b & * \\ & c & * \end{pmatrix},$$

with

$$a = \frac{1+|X|^2+Z}{\sqrt{(1+|X|^2)^2+|Z|^2}}, \qquad b = \frac{1-|X|^2-Z}{\sqrt{(1+|X|^2)^2+|Z|^2}}$$

and

$$c = \frac{2X^*}{\sqrt{(1+|X|^2)^2+|Z|^2}},$$

and

$$H(\overline{n}_{(X,Z)}) = \frac{1}{2}\log((1+|X|^2)^2+|Z|^2)H.$$

Proof. (i) follows from (ii) which is an easy computation. □

Using the KAN decomposition of the Lemma above, we obtain a map

$$\bar{\mathfrak{n}} \to K/M \cong S^{2n+1},$$

by multiplying with $\operatorname{diag}(a^{-1}, a^{-1}, \mathbf{1}_n) \in M$ from the right:

$$\overline{n}_{(X,Z)} \mapsto \left(\frac{1-|X|^2+Z}{1+|X|^2-Z}, \frac{2X}{1+|X|^2-Z}\right). \tag{1.1}$$

Proof of Lemma 1.3. Ad (i): From Lemma 1.4(i) and (1.1) it follows immediately that $K_0' = \mathrm{U}(n)$, embedded in K' in the bottom right corner.

Ad(ii): By Lemma 1.1(i) we have $\overline{n}_{e_n}P = \tilde{w}_0\overline{n}_{-e_n}P$ such that by Lemma 1.2 we have $P' \cdot \overline{n}_{e_n}P = (\overline{N} - \overline{N}')P$, since w_0 fixes $(\overline{N} - \overline{N}')P$ again by Lemma 1.1(i). By the Bruhat-decompositon we have $G' = P' \sqcup N'\tilde{w}_0P'$ and $N'\tilde{w}_0 = \tilde{w}_0\overline{N}'$ obviously fixes $(\overline{N} - \overline{N}')P$. □

Hence in particular as a G'-space $\mathcal{O}_A \cong G'/K_0'$.

1.2 Principal series representations

Let (ξ, V) be a finite-dimensional representation of M, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ a character and $\mathbf{1}$ the trivial representation of N. We obtain a finite-dimensional representation $(\xi \otimes e^\lambda \otimes \mathbf{1}, V_{\xi,\lambda})$ of $P = MAN$. By smooth normalized induction this representation gives rise to the principal series representation

$$\pi_{\xi,\lambda} := \operatorname{Ind}_P^G(\xi \otimes e^\lambda \otimes \mathbf{1})$$

as the left-regular representation of G on the space $C^\infty(G/P, \mathcal{V}_{\xi,\lambda})$ of smooth sections of the homogeneous vector bundle $\mathcal{V}_{\xi,\lambda} := G \times_P V_{\xi,\lambda+\rho} \to G/P$ associated to $V_{\xi,\lambda+\rho}$.

Similarly for the subgroup G', let (η, W) be a finite-dimensional representation of M' and $\nu \in (\mathfrak{a}_{\mathbb{C}}')^*$. We obtain a finite-dimensional representation $(\eta \otimes e^\nu \otimes \mathbf{1}, W_{\eta,\lambda})$ of P' and the corresponding principal series representation

$$\tau_{\eta,\nu} := \operatorname{Ind}_{P'}^{G'}(\eta \otimes e^\nu \otimes \mathbf{1}),$$

by left-regular G'-action on the smooth sections $C^\infty(G'/P', \mathcal{W}_{\eta,\nu})$ of the homogeneous vector bundle $\mathcal{W}_{\eta,\nu} := G' \times_{P'} W_{\eta,\nu+\rho'} \to G'/P'$.

1.3 Symmetry breaking operators

Symmetry breaking operators between $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$ are given by continuous linear G'-maps between the smooth sections of the two homogeneous vector bundles

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'},\tau_{\eta,\nu}) = \mathrm{Hom}_{G'}(C^\infty(G/P,\mathcal{V}_{\xi,\lambda})), C^\infty(G'/P',\mathcal{W}_{\eta,\nu})).$$

By the Schwartz Kernel Theorem every operator is given by a G'-invariant distribution section of the tensor bundle $\mathcal{V}_{\xi^*,-\lambda}\boxtimes\mathcal{W}_{\eta,\nu}$ over $G/P\times G'/P'$, where ξ^* is the representation contradigent to ξ. Since G' acts transitively on G'/P' we can consider these distributions as sections on G/P with a certain P'-invariance:

Theorem 1.5 ([KS15, Proposition 3.2]). *There is a natural bijection*

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'},\tau_{\eta,\nu}) \xrightarrow{\sim} (\mathcal{D}'(G/P,\mathcal{V}_{\xi^*,-\lambda})\otimes W_{\eta,\nu+\rho'})^{P'}.$$

We obtain a parameterization of the open Bruhat cell $\overline{N}P/P\subseteq G/P$ in terms of $\bar{\mathfrak{n}}$ by

$$\bar{\mathfrak{n}} \xrightarrow{\exp} \overline{N} \hookrightarrow G \longrightarrow G/P,$$

so that we can consider $\bar{\mathfrak{n}}$ as an open dense subset of G/P. Then the restriction

$$\mathcal{D}'(G/P,\mathcal{V}_{\xi^*,-\lambda}) \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}},\mathcal{V}_{\xi^*,-\lambda}|_{\bar{\mathfrak{n}}})$$

defines a $\mathfrak{g}$-action on $\mathcal{D}'(\bar{\mathfrak{n}},\mathcal{V}_{\xi^*,-\lambda}|_{\bar{\mathfrak{n}}})\cong\mathcal{D}'(\bar{\mathfrak{n}})\otimes V_{\xi^*,-\lambda+\rho}$ by vector fields. Moreover, since $\mathrm{Ad}(M'A')$ leaves $\bar{\mathfrak{n}}$ invariant, the restriction is further $M'A'$-equivariant. Since $P'\overline{N}P=G$ by Lemma 1.2, every P'-orbit in G/P meets the open Bruhat cell $\overline{N}P$, then symmetry breaking operators can be described in terms of $(M'A',\mathfrak{n}')$-invariant distributions on $\bar{\mathfrak{n}}$:

Theorem 1.6 ([KS15, Theorem 3.16]). *There is a natural bijection*

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'},\tau_{\eta,\nu}) \xrightarrow{\sim} (\mathcal{D}'(\bar{\mathfrak{n}})\otimes V_{\xi^*,-\lambda+\rho}\otimes W_{\eta,\nu+\rho'})^{M'A',\mathfrak{n}'}.$$

In the following we denote the space $(\mathcal{D}'(\bar{\mathfrak{n}})\otimes V_{\xi^*,-\lambda+\rho}\otimes W_{\eta,\nu+\rho'})^{M'A',\mathfrak{n}'}$ by $\mathcal{D}'(\bar{\mathfrak{n}})^{\xi,\eta}_{\lambda,\nu}$.

1.4 Symmetry breaking operators in the spherical case

For the spherical principal series, i.e. ξ resp. η being the trivial M resp. M'-representation $\mathbf{1}$, symmetry breaking operators have been classified in [FW20] in a more general setting which includes our pair of groups of interest (G,G') and we recall parts of the main results which are needed for the purpose of this article. Therefore we define the following subsets of $\mathbb{C}^2$.

$$\begin{aligned}
/\!/ &:= \{(\lambda,\nu)\in\mathbb{C}^2, \lambda+\rho-\nu-\rho'\in-2\mathbb{Z}_{\geq 0}\},\\
\backslash\!\backslash &:= \{(\lambda,\nu)\in\mathbb{C}^2, \lambda+\rho+\nu-\rho'\in-2\mathbb{Z}_{\geq 0}\},\\
\mathbb{X} &:= /\!/ \cap \backslash\!\backslash,\\
L &:= \{(-\rho-2i,-\rho'-2j)\in\mathbb{C}^2, i,j\in\mathbb{Z}_{\geq 0},\ i\geq j\}.
\end{aligned}$$

Let $u^A_{\lambda,\nu}$ be the distribution on $\bar{\mathfrak{n}}$ given for $\mathrm{Re}(\lambda\pm\nu)>-1$ by

$$u^A_{\lambda,\nu}(X,Z) = \frac{1}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})}N(X,Z)^{-2(\nu+\rho')}|X_n|^{\lambda-\rho+\nu+\rho'},$$

where $N(X,Z) = (|X|^4 + |Z|^2)^{\frac{1}{4}}$ is the norm function on $\bar{\mathfrak{n}}$ where we use the notation $X = (X', X_n) \in \mathbb{C}^{n-1} \oplus \mathbb{C}$.

Theorem 1.7 ([FW20] Theorem B). *$u^A_{\lambda,\nu}$ extends to a family of distributions which depends holomorphically on $(\lambda,\nu) \in \mathbb{C}^2$ and $u^A_{\lambda,\nu} \in \mathcal{D}'(\bar{\mathfrak{n}})^{\mathbf{1},\mathbf{1}}_{\lambda,\nu}$ for all $(\lambda,\nu) \in \mathbb{C}^2$. Further, $u^A_{\lambda,\nu} = 0$ if and only if $(\lambda,\nu) \in L$.*

For the holomorphic family $u^A_{\lambda,\nu}$ the following holds.

Theorem 1.8 ([FW20] Corollary 12.4). *For $(\lambda,\nu) \in \mathbb{C}^2$:*

$$\operatorname{Supp} u^A_{\lambda,\nu} = \begin{cases} \emptyset & \text{for } (\lambda,\nu) \in L, \\ \{0\} & \text{for } (\lambda,\nu) \in /\!/ - L, \\ \bar{\mathfrak{n}}' & \text{for } (\lambda,\nu) \in \backslash\!\backslash - \mathbb{X}, \\ \bar{\mathfrak{n}} & \text{otherwise.} \end{cases}$$

Following [FW20], the full classification of $\mathcal{D}'(\bar{\mathfrak{n}})^{\mathbf{1},\mathbf{1}}_{\lambda,\nu}$ is in our case given by $u^A_{\lambda,\nu}$ for $(\lambda,\nu) \notin L$ and for $(\lambda,\nu) \in L$ the space $\mathcal{D}'(\bar{\mathfrak{n}})^{\mathbf{1},\mathbf{1}}_{\lambda,\nu}$ is two-dimensional and a basis can be given by two different renormalizations of $u^A_{\lambda,\nu}$ along the lines in $/\!/$ resp. $\backslash\!\backslash$. We will obtain a basis for $(\lambda,\nu) \in L$ by a different approach in Section 5.1.

2 Symmetry breaking for the scalar principle series

In this section we use the translation principle to classify symmetry breaking operators between principle series representations of (G, G'), where we induce from a character of the subgroup $\mathrm{U}(1) \subseteq M, M'$ instead of the trivial M resp. M' representation as in the spherical case.

2.1 The translation principle

Let $(\chi, \mathbf{E})$ be an irreducible finite-dimensional representation of G and $i : E \hookrightarrow \mathbf{E}$ be the unique irreducible subrepresentation of $\mathbf{E}|_P$ which is of the form $(\sigma \otimes e^{\mu} \otimes \mathbf{1})$ with respect to the Langlands decomposition $P = MAN$, where σ is an irreducible M representation and $\mu \in \mathfrak{a}^*_{\mathbb{C}}$. Let further $E^{\vee}$ denote the dual of E. In [FØ19b] Möllers–Ørsted showed the following, which is called *translation principle* and allows us to construct new symmetry breaking operators form the ones we classified before.

Theorem 2.1 (See [FØ19b] Theorem 2.3 and Proposition 2.4). *For every P'-invariant quotient $p : \mathbf{E}|_{P'} \twoheadrightarrow E'$ with $E' \cong \sigma' \otimes e^{\mu'} \otimes \mathbf{1}$ as $P' = M'A'N'$ representation, there is a unique linear map*

$$\begin{aligned}\Phi : \operatorname{Hom}_{G'}(&\operatorname{Ind}_P^G(\xi \otimes e^{\lambda} \otimes \mathbf{1}), \operatorname{Ind}_{P'}^{G'}(\xi' \otimes e^{\nu} \otimes \mathbf{1})) \\ &\to \operatorname{Hom}_{G'}(\operatorname{Ind}_P^G((\xi \otimes \sigma) \otimes e^{\lambda+\mu} \otimes \mathbf{1}), \operatorname{Ind}_{P'}^{G'}((\xi' \otimes \sigma') \otimes e^{\nu+\mu'} \otimes \mathbf{1}))\end{aligned}$$

with the property that for every symmetry breaking operator A with corresponding distribution kernel u^A, the distribution kernel $u^{\Phi(A)}$ of the image $\Phi(A)$ is given by $u^{\Phi(A)} = \varphi \otimes u^A$, the multiplication with the smooth section $C^{\infty}(G/P, G \times_P E^{\vee}) \otimes E'$ defined by

$$\Phi(g) = p \circ \chi(g) \circ i \in \operatorname{Hom}_{\mathbb{C}}(E, E') \simeq E^{\vee} \otimes E', \qquad g \in G.$$

In particular Φ maps holomorphic families to holomoprhic families (see [FØ19b, Remark 2.5]).

2.2 Scalar principal series representations

The irreducible representations of $\mathrm{U}(1)$ are one-dimensional and parametrized by integers. We extend these representations to M and denote them for $s \in \mathbb{Z}$ by ξ_s, acting on $m = \mathrm{diag}(a, a, b) \in M$, $a \in \mathrm{U}(1), b \in \mathrm{U}(n)$ by

$$\xi_s(m) = a^s.$$

Similarly we denote the equivalent M' representations which we obtain by restriction to M' by ξ'_t for $t \in \mathbb{Z}$. Let

$$\pi_{s,\lambda} := \mathrm{Ind}_P^G(\xi_s \otimes e^\lambda \otimes \mathbf{1}) \qquad \text{and} \qquad \tau_{t,\nu} := \mathrm{Ind}_{P'}^{G'}(\xi'_t \otimes e^\nu \otimes \mathbf{1}).$$

Then in the non-compact picture the action of $\pi_{s,\lambda}$ is given by

$$\pi_{s,\lambda}(g)\varphi(\bar{n}) = e^{-(\lambda+\rho)\log a(g^{-1}\bar{n})}\xi_s(m(g^{-1}\bar{n}))^{-1}\varphi(\bar{n}(g^{-1}\bar{n})).$$

Following Theorem 1.6 we consider the space

$$\left(\mathcal{D}'(\bar{\mathfrak{n}}) \otimes V_{\xi_{-s},-\lambda+\rho} \otimes W_{\xi'_t,\nu+\rho'}\right)^{M'A',\mathfrak{n}'}$$

which we denote in the following by $\mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu}$. Then for the distribution kernels of symmetry breaking operators by Lemma 1.1 the following holds:

Lemma 2.2. *Let $u \in \mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu}$. Then*

(i) $(E - \lambda + \rho + \nu + \rho')u = 0$,

(ii) $u(aX'b^{-1}, aX_n, Z) = a^{s-t}u(X, Z)$ *for* $a \in \mathrm{U}(1), b \in \mathrm{U}(n-1)$,

(iii) $d\pi_{-s,-\lambda}(X)u = 0$, *for all* $X \in \bar{\mathfrak{n}}'$.

Let $(\chi_+, \mathbf{E}_+)$, $\mathbf{E}_+ = \mathbb{C}^{n+2}$ be the defining representation of G. Then

$$i_+ : E_+ = \mathbb{C} \hookrightarrow \mathbf{E}_+, \qquad z \mapsto (z, z, 0, \dots, 0)^T$$

is the maximal subrepresentation on which N acts trivially and $E_+|_P \cong \xi_1 \otimes e^1 \otimes \mathbf{1}$. Now we choose the P'-invariant quotient to be

$$p_+ : \mathbf{E}_+ \twoheadrightarrow \mathbb{C}, \qquad (z_1, \dots z_{n+2})^T \mapsto z_{n+2},$$

which is the trivial representation $E_{p_+} := \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$ of P'. Then the $\mathrm{Hom}_\mathbb{C}(E_+, E_{p_+})$ valued function φ_{p_+} is on $\overline{N}$ given for $(X, Z) \in \mathbb{C}^n \oplus i\mathbb{R}$ by

$$p_+ \circ \chi_+\left(\bar{n}_{(X,Z)}\right) \circ i_+(1) = p_+(1 + |X|^2 + Z, 1 - |X|^2 - Z, 2X^*) = 2\overline{X}_n.$$

On the other hand we can choose the P' invariant quotient

$$q_+ : \mathbf{E}_+ \twoheadrightarrow \mathbb{C},$$

$$(z_1, \dots z_{n+2})^T \mapsto z_1 - z_2,$$

which is the representation $e_{q_+} := \xi_1' \otimes e^{-1} \otimes \mathbf{1}$ of P', such that we obtain the $\mathrm{Hom}_{\mathbb{C}}(E_+, E_{q_+})$ valued function φ_{q_+} is on $\overline{N}$ given for $(X, Z) \in \mathbb{C}^n \oplus i\mathbb{R}$ by

$$q_+ \circ \chi_+ \left(\bar{n}_{(X,Z)}\right) \circ i_+(1) = 2(|X|^2 + Z).$$

By Theorem 2.1 we obtain maps

$$\overline{X}_n : \mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu} \to \mathcal{D}'(\bar{\mathfrak{n}})^{s+1,t}_{\lambda+1,\nu}, \qquad u \mapsto \overline{X}_n u,$$

$$(|X|^2 + Z) : \mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu} \to \mathcal{D}'(\bar{\mathfrak{n}})^{s+1,t+1}_{\lambda+1,\nu-1}, \qquad u \mapsto (|X|^2 + Z)u.$$

Now consider the dual representation $(\mathbf{E}_-, \chi_-) := (\mathbf{E}_+^\vee, \chi_+^\vee)$, with $\mathbf{E}_- = \mathbb{C}^{n+2}$ which we can also consider as G acting on row vectors on the right with inverse elements. Then we have a maximal subrepresentation

$$i_- : E_- = \mathbb{C} \hookrightarrow \mathbf{E}_-, \qquad z \mapsto (z, -z, 0, \dots, 0)$$

which acts trivial on N and $E_-|_P = \xi_{-1} \otimes e^1 \otimes \mathbf{1}$. Similarly we choose the P'-invariant quotients to be

$$p_- : \mathbf{E}_- \twoheadrightarrow \mathbb{C}, \qquad (z_1, \dots, z_{n+2}) \mapsto z_{n+2},$$

which is again trivial on P', and

$$q_- : \mathbf{E}_- \twoheadrightarrow \mathbb{C}, \qquad (z_1, \dots, z_{n+2}) \mapsto z_1 + z_2,$$

which is given by $\xi_{-1}' \otimes e^{-1} \otimes \mathbf{1}$ on P'. In this case we obtain the functions φ_{p_-} and φ_{q_-} to be given for $(X, Z) \in \mathbb{C}^n \oplus i\mathbb{R}$ by

$$p_- \circ \chi_- \left(\bar{n}_{(X,Z)}\right) \circ i_-(1) = p_-(1 + |X|^2 - Z, -1 + |X|^2 - Z, -2X) = -2X_n,$$

and by

$$q_- \circ \chi_- \left(\bar{n}_{(X,Z)}\right) \circ i_-(1) = 2(|X|^2 - Z).$$

such that we obtain by Theorem 2.1 the maps

$$X_n : \mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu} \to \mathcal{D}'(\bar{\mathfrak{n}})^{s-1,t}_{\lambda+1,\nu}, \qquad u \mapsto X_n u,$$

$$(|X|^2 - Z) : \mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu} \to \mathcal{D}'(\bar{\mathfrak{n}})^{s-1,t-1}_{\lambda+1,\nu-1}, \qquad u \mapsto (|X|^2 - Z)u.$$

In the following we write $X_n^k : \mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu} \to \mathcal{D}'(\bar{\mathfrak{n}})^{s-k,t}_{\lambda+k,\nu}$ for the map given by applying X_n successively k-times and similarly for the other maps. Moreover we abuse notation and write for $k \in \mathbb{Z}$ and Y one of the maps $X_n, \overline{X}_n, (|X|^2 + Z), (|X|^2 - Z)$:

$$Y^k := \begin{cases} Y^k & \text{if } k \geq 0, \\ \overline{Y}^{-k} & \text{if } k < 0. \end{cases}$$

2.3 A meromorphic family of symmetry breaking operators

By the translation principle we define the following meromorphic family of distributions in $\mathcal{D}'(\bar{\mathfrak{n}})^{s,t}_{\lambda,\nu}$.

$$u^A_{(\lambda,s),(\nu,t)}(X,Z) := \frac{\Gamma(\frac{\lambda+\rho+\nu-\rho'-|s-t|}{2})\Gamma(\frac{\lambda+\rho-\nu-\rho'-|s-t|}{2}-|t|)}{\Gamma(\frac{\lambda+\rho+\nu-\rho'+|s-t|}{2})\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2})}\overline{X}_n^{s-t}(|X|^2+Z)^t u^A_{\lambda-|s-t|-|t|,\nu+|t|},$$

which is for $\mathrm{Re}(\lambda\pm\nu) > -1+|s-t|+2|t|$ given by

$$u^A_{(\lambda,s),(\nu,t)}(X,Z) = \frac{\overline{X}_n^{s-t}(|X|^2+Z)^t N(X,Z)^{-2(\nu+\rho'+|t|)}|X_n|^{\lambda-\rho+\nu+\rho'-|s-t|}}{\Gamma(\frac{\lambda+\rho+\nu-\rho'+|s-t|}{2})\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2})}.$$

We denote the corresponding integral operators by $A^{s,t}_{\lambda,\nu}$. We show in Section 5 that $A^{s,t}_{\lambda,\nu}$ is indeed holomorphic in (λ,ν) and that it is an optimal renormalization (see Proposition 5.4 and Lemma 5.6) by explicitly computing the eigenvalues on the K'-types of $\pi_{s,\lambda}$.

3 Composition series of the scalar principle series

We study the composition series of the scalar principle series representations $\pi_{s,\lambda}$. This is done in the compact picture of the representations using the spectrum generating operator as established by Branson–Ólafsson–Ørsted in [BÓØ96]. This idea has already been applied to spherical principal series representations of G by Frahm–Ørsted in [FØ19a]. Let $(\pi_{s,\lambda})_{\mathrm{HC}}$ be the underlying Harish-Chandra module of $\pi_{s,\lambda}$. Then as K-representation $(\pi_{s,\lambda})_{\mathrm{HC}} =_K L^2(K/M,\mathcal{L}_s)$, where $\mathcal{L}_s$ is the homogeneous vector bundle $K\times_M \xi_s \to K/M$, such that

$$(\pi_{s,\lambda})_{\mathrm{HC}} \cong_K \bigoplus_{\alpha_1,\alpha_2=0}^{\infty} e^{i(\alpha_1-\alpha_2-s)\theta}\otimes\mathcal{H}^{(\alpha_1,\alpha_2)}(\mathbb{C}^{n+1})$$

as $\mathrm{U}(1)\times\mathrm{U}(n+1)$ representation, where $\mathcal{H}^{(\alpha_1,\alpha_2)}(\mathbb{C}^{n+1})$ is the space of harmonic polynomials on $\mathbb{C}^n$ which are holomorphic of degree α_1 and antiholomorphic of degree α_2, which is an irreducible $\mathrm{U}(n+1)$-representation. We denote the K-types $e^{i(\alpha_1-\alpha_2-s)\theta}\otimes\mathcal{H}^{(\alpha_1,\alpha_2)}(\mathbb{C}^{n+1})$ in the following by $\mathcal{E}_s(\alpha_1,\alpha_2)$. Then following [FØ19a, Section 5.2] we can reach at most the K-types $\mathcal{E}_s(\alpha_1\pm1,\alpha_2)$ and $\mathcal{E}_s(\alpha_1,\alpha_2\pm1)$ with a single action of $\mathfrak{g}_\mathbb{C}$, in the sense that for every $\lambda'\in\mathbb{C}$, the set of K-types one reaches with a single action of $\mathfrak{g}_\mathbb{C}$ via $d\pi_{s,\lambda'}$ is a subset of $\{(\alpha_1\pm1,\alpha_2),(\alpha_1,\alpha_2\pm1)\}$ and it is equal for some $\lambda'\in\mathbb{C}$. Reaching resp. not reaching a K-type (β_1,β_2) in such a way from a K-type (α_1,α_2) will be denoted by

$$(\alpha_1,\alpha_2)\leftrightarrow(\beta_1,\beta_2) \quad \text{resp.} \quad (\alpha_1,\alpha_2)\not\leftrightarrow(\beta_1,\beta_2).$$

Reaching resp. not reaching a K-type (β_1,β_2) from a K-type (α_1,α_2) by a single action of $\mathfrak{g}_\mathbb{C}$ via $d\pi_{s,\lambda}$ will be denoted by

$$(\alpha_1,\alpha_2)\to(\beta_1,\beta_2) \quad \text{resp.} \quad (\alpha_1,\alpha_2)\not\to(\beta_1,\beta_2).$$

We fix the bilinear for $B(X,Y) = \frac{1}{2}\operatorname{tr}(XY)$ on $\mathfrak{g}$, which satisfies $B(H,H) = 1$ and choose a $(-B)$-orthonormal basis $\{X_{\mathfrak{v},j},\ i = 1,\dots,n\}$ of $\mathfrak{k} \cap (\mathfrak{v} + \overline{\mathfrak{v}})$, with

$$X_{\mathfrak{v},j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_j \\ 0 & -e_j & \mathbf{0}_n \end{pmatrix},$$

and a $(-B)$-orthonormal basis $\{X_{\mathfrak{z}}\}$ of $\mathfrak{k} \cap (\mathfrak{z} + \overline{\mathfrak{z}})$, with

$$X_{\mathfrak{z}} = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & \mathbf{0}_n \end{pmatrix}.$$

Following [BÓØ96], the spectrum generating operator $\mathcal{P}$ is defined as

$$\mathcal{P} = C_{\mathfrak{v}} + C_{\mathfrak{z}},$$

with

$$C_{\mathfrak{v}} := -\sum_{j=1}^{n} X_{\mathfrak{v},j}^2, \quad \text{and} \quad C_{\mathfrak{z}} := -\frac{1}{2}X_{\mathfrak{z}}^2,$$

as an element of the universal enveloping algebra $U(\mathfrak{k})$ of $\mathfrak{k}$. Since the left action of K and the right action of $\mathcal{P}$ commute on $(\pi_{s,\lambda})_{\mathrm{HC}}$, the right action of $\mathcal{P}$ acts on each K-type $\mathcal{E}_s(\alpha_1,\alpha_2)$ by a scalar $\sigma_s(\alpha_1,\alpha_2)$.

Proposition 3.1 (See [BÓØ96] Lemma 4.4 and Theorem 2.7)**.** *(i) If $(\alpha_1,\alpha_2) \leftrightarrow (\beta_1,\beta_2)$, then $(\alpha_1,\alpha_2) \not\rightarrow (\beta_1,\beta_2)$ if and only if*

$$\sigma_s(\beta_1,\beta_2) - \sigma_s(\alpha_1,\alpha_2) + 2\lambda = 0.$$

(ii) If $T : \pi_{s,\lambda} \to \pi_{s,\lambda'}$ a continuous intertwining operator which acts on the K-types (α_1,α_2) by scalars t_{α_1,α_2}, then for all K-types $(\alpha_1,\alpha_2) \leftrightarrow (\beta_1,\beta_2)$:

$$(\sigma_s(\beta_1,\beta_2) - \sigma_s(\alpha_1,\alpha_2) + 2\lambda)t_{\alpha_1,\alpha_2} = (\sigma_s(\beta_1,\beta_2) - \sigma_s(\alpha_1,\alpha_2) + 2\lambda')t_{\beta_1,\beta_2}.$$

(iii) $\pi_{s,\lambda}$ is a complementary series representation for $\lambda \in (-R_s,R_s)$, where

$$2R_s := \min_{(\beta_1,\beta_2)\leftrightarrow(\alpha_1,\alpha_2)} \{|\sigma_s(\beta_1,\beta_2) - \sigma_s(\alpha_1,\alpha_2)|\}.$$

The first part of the proposition lets us find submodules of $\pi_{s,\lambda}$ and the second part lets us decide which submodules outside the unitarizable principal series are unitary, since it lets us decide for which λ the standard Knapp–Stein operators

$$T_{s,\lambda} : \pi_{s,\lambda} \to \pi_{s,-\lambda}$$

are positive and the third part immediately yields if $\pi_{s,\lambda}$ is a complementary series representation.

Lemma 3.2. *The eigenvalues of the spectrum generating operator are given by*

$$\sigma_s(\alpha_1,\alpha_2) = 2\alpha_1(\alpha_1 + \rho - 1 - s) - 2\alpha_2(\alpha_2 + \rho - 1 + s) - \frac{s^2}{2}.$$

Proof. Following [FØ19a, Section 5.2], the statement holds for $s = 0$. Since $X_{\mathfrak{v},j}$ does not contain a $\mathfrak{u}(1)$ part of $\mathfrak{k}$ and $\mathfrak{m}$, the action of $C_{\mathfrak{v}}$ is independent of s and $X_{\mathfrak{z}} = \operatorname{diag}(-i,-i,\mathbf{0}_n) + \operatorname{diag}(2i,0,\mathbf{0}_n) \in \mathfrak{k}$. Now $\operatorname{diag}(-i,-i,\mathbf{0}_n) \in \mathfrak{m}$ acts on $\mathcal{E}_s(\alpha_1,\alpha_2)$ by is and $\operatorname{diag}(2i,0,\mathbf{0}_n)$ by $2i(\alpha_1-\alpha_2-s)$. Hence $C_{\mathfrak{z}}$ acts by $\frac{1}{2}(2\alpha_1-2\alpha_2-s)^2$, such that

$$\sigma_s(\alpha_1,\alpha_2) = \sigma_0(\alpha_1,\alpha_2) - \frac{1}{2}(2\alpha_1-2\alpha_2)^2 + \frac{1}{2}(2\alpha_1-2\alpha_2-s)^2,$$

which is the stated identity. □

Proposition 3.1(iii) and Lemma 3.2 implies the following Corollary about the complementary series of $\pi_{s,\lambda}$.

Corollary 3.3. *$\pi_{s,\lambda}$ is a complementary series representation if and only if*

$$\lambda \in \begin{cases} (-\rho+|s|,\rho-|s|) & \text{if } |s|<\rho, \\ (-1,1) & \text{if } |s|\geq\rho \text{ and } s+\rho\equiv 1 \mod 2. \end{cases}$$

Lemma 3.2 and Proposition 3.1 now allow us to concretely describe the composition series of $\pi_{s,\lambda}$. Therefor we define the following K-modules:

$$H^+_{j,k} := \bigoplus_{\alpha_1\geq j,\alpha_2\geq k} \mathcal{E}(\alpha_1,\alpha_2), \qquad F^+_{j,k} := \bigoplus_{\alpha_1\leq j,\alpha_2\leq k} \mathcal{E}(\alpha_1,\alpha_2),$$

$$L^-_{j,k,l} := \bigoplus_{j\leq\alpha_1\leq k,\alpha_2\geq l} \mathcal{E}(\alpha_1,\alpha_2), \qquad L^+_{j,k,l} := \bigoplus_{\alpha_1\geq l,j\leq\alpha_2\leq k} \mathcal{E}(\alpha_1,\alpha_2),$$

and let $H^-_{j,k} := H^+_{k,j}$ and $F^-_{j,k} := F^+_{k,j}$.

Theorem 3.4. *$\pi_{s,\lambda}$ is irreducible if and only if $\lambda \notin \pm(\rho+2\mathbb{Z}_{\geq 0}-s) \cup \pm(\rho+2\mathbb{Z}_{\geq 0}+s)$. Let $j \in \mathbb{Z}_{\geq 0}$.*

(i) Let $\lambda = \rho+2j\pm s$, $\pm s \geq -j$.

- *a) $\pi_{s,\lambda}$ has two reducible submodules realized in $H^\pm_{j\pm s+1,0}$ and $H^\pm_{0,j+1}$ and an irreducible submodule realized on $H^\pm_{j\pm s+1,j+1}$.*
- *b) The unitarizable composition factors are the submodule $H^\pm_{j\pm s+1,j+1}$ and for $j = 0$ $L^\pm_{0,1,s+1}$, for $s+j=0$ $L^\mp_{0,1,j+1}$ and for $s=j=0$ the one dimensional subquotient $F^+_{1,1}$.*
- *c)* $\ker T_{s,\lambda} = H^\pm_{j\pm s+1,0} + H^\pm_{0,j+1}$.

(ii) Let $\lambda = \rho+2j\pm s$, $-\rho-j < \pm s < -j$.

- *a) $\pi_{s,\lambda}$ has an irreducible submodule realized in $H^\pm_{0,j+1}$.*
- *b) The unitarizable composition factor is the submodule $H^\pm_{0,j+1}$ and for $j=0$ the quotient $\pi_{s,\lambda}/H^\pm_{0,j+1}$.*
- *c)* $\ker T_{s,\lambda} = H^\pm_{0,j+1}$.

(iii) Let $\lambda = \rho+2j\pm s$, $-\rho-2j \leq \pm s \leq -\rho-j$.

a) $\pi_{s,\lambda}$ *has two irreducible submodules realized in* $H^{\pm}_{0,j+1}$ *and* $L^{\pm}_{0,-\rho-j\mp s,0}$.

b) *The unitarizable composition factors are* $H^{\pm}_{0,j+1}$ *and* $L^{\pm}_{0,-\rho-j\mp s,0}$ *and for* $\lambda = 1$, *the quotient* $\pi_{s,\lambda}/(H^{\pm}_{0,j+1} + L^{\pm}_{0,-\rho-j\mp s,0})$.

c) $\ker T_{s,\lambda} = H^{\pm}_{0,j+1} + L^{\pm}_{0,-\rho-j\mp s,0}$.

(iv) Let $\lambda = \rho + 2j \pm s$, $\pm s < -\rho - 2j$.

a) $\pi_{s,\lambda}$ *has two reducible submodules realized in* $H^{\pm}_{0,j+1}$ *and* $L^{\pm}_{0,-\rho-j\mp s,0}$ *and an irreducible submodule realized in* $L^{\pm}_{j+1,-\rho-s\mp s,0}$.

b) *The unitarizable composition factors are the quotients* $\pi_{s,\lambda}/H^{\pm}_{0,j+1}$ *and* $\pi_{s,\lambda}/L^{\pm}_{0,-\rho\mp s-j,0}$ *and for* $\lambda = -1$ *the submodule realized in* $L^{\pm}_{j+1,-\rho-s\mp s,0}$.

c) $\ker \Gamma(\lambda) T_{s,\lambda} = L^{\pm}_{j+1,-\rho-s\mp s,0}$.

(v) Let $\lambda = -\rho - 2j - (\pm s)$, $\pm s \geq -j$.

a) $\pi_{s,\lambda}$ *has two reducible submodules realized in* $L^{\pm}_{0,j,0}$ *and* $L^{\mp}_{0,j+s,0}$ *and a irreducible finite dimensional submodule* $F^{\pm}_{j+s,j}$.

b) *The unitarizable composition factors are the quotient* $\pi_{s,\lambda}/(L^{\pm}_{0,j,0} + L^{\mp}_{0,j\pm s,0})$ *and for* $s = j = 0$ *the one dimensional submodule* $F^{+}_{1,1}$.

c) $\ker T_{s,\lambda} = L^{\pm}_{0,j,0} + L^{\pm}_{0,j\pm s,0}$.

(vi) Let $\lambda = -\rho - 2j - (\pm s)$, $-\rho - j < \pm s < -j$.

a) $\pi_{s,\lambda}$ *has an irreducible submodules realized in* $L^{\pm}_{0,j,0}$.

b) *The unitary composition factor is the quotient* $\pi_{s,\lambda}/L^{\pm}_{0,j,0}$ *and for* $j = 0$ *the submodule* $L^{\pm}_{0,j,0}$.

c) $\ker T_{s,\lambda} = L^{\pm}_{0,j,0}$.

(vii) Let $\lambda = -\rho - 2j - (\pm s)$, $-\rho - 2j \leq \pm s \leq -\rho - j$, *this case is given in (iv) by writing* $\lambda = \rho + 2(-\rho - j \mp s) \pm s$.

(viii) Let $\lambda = -\rho - 2j - (\pm s)$, $s < -\rho - 2j$. *This case is given in (iii) by writing* $\lambda = \rho + 2(-\rho - j \mp s) \pm s$.

Proof. In the following we indentify the K-types of $(\pi_{s,\lambda})_{\mathrm{HC}}$ with the lattice $\mathbb{Z}^2_{\geq 0}$ in $\mathbb{R}^2$. If $(\alpha_1, \alpha_2) \not\to (\alpha_1 \pm 1, \alpha_2)$ and $(\alpha_1 \pm 1, \alpha_2) \to (\alpha_1, \alpha_2)$, we indicate this by a horizontal line between the rays of K-types $(\alpha_1, *)$ and $(\alpha_1 \pm 1, *)$ and an arrow labeling the line, pointing towards $(\alpha_1, *)$. By Proposition 3.1 we can read off the submodules from a picture like this. Note that $\sigma_s(\alpha_1, \alpha_2) = \sigma_{-s}(\alpha_2, \alpha_1)$. Therefor we will only carry out the proof for $\lambda = \rho + 2j + s$ and $\lambda = -\rho - 2j + s$, since the other two cases can be obtained simply by exchanging α_1 and α_2. Moreover we have

$$\sigma_s(\beta_1, \beta_2) - \sigma_s(\alpha_1, \alpha_2) + 2\lambda = -(\sigma_s(\alpha_1, \alpha_2) - \sigma_s(\beta_1, \beta_2) - 2\lambda),$$

such that replacing λ by $-\lambda$ fixes the lines of the K-type picture, but flips the direction of the arrows which indicate submodules. Hence it is enough to study the composition series for case $\lambda = \rho + 2j + s$ concretely.

Following Proposition 3.1, the scalars t_{α_1,α_2} with which the Knapp–Stein intertwiner $T_{s,\lambda}$ acts on the K-type (α_1,α_2) are up to a scalar given by

$$t_{\alpha_1,\alpha_2} = c_\lambda \frac{(\frac{\rho-s-\lambda}{2})_{\alpha_1}(\frac{\rho+s-\lambda}{2})_{\alpha_2}}{(\frac{\rho-s+\lambda}{2})_{\alpha_1}(\frac{\rho+s+\lambda}{2})_{\alpha_2}},$$

where c_λ is a renormalization factor which makes this expression non-zero and regular. Then a submodule u of $\pi_{s,\lambda}$ is unitarizable, if and only if $T_{s,\lambda}|u$ is definite, which is equivalent to

$$\tilde{c}_\lambda \frac{(\frac{\rho-s-\lambda}{2})_{\alpha_1}(\frac{\rho+s-\lambda}{2})_{\alpha_2}}{(\frac{\rho-s+\lambda}{2})_{\alpha_1}(\frac{\rho+s+\lambda}{2})_{\alpha_2}}$$

being definite for all K-types (α_1,α_2) which occur in u and $\tilde{c}_\lambda$ is chosen to make the expression non-zero and regular only for the K-types in u. A (sub-)quotient of $\pi_{s,\lambda}$ is unitarizable if and only if the corresponding submodule of $\pi_{s,-\lambda}$ with the same K-types is unitarizable. Here again, setting s to $-s$ is equivalent to replacing α_1 by α_2. Replacing λ by $-\lambda$ is up to renormalisation just the inversion of t_{α_1,α_2}, such that positivity is preserved. Hence the study of unitarity can also be restricted to one case. Only for the kernels of the Knapp–Stein intertwiners we need to consider the cases λ and $-\lambda$. We mark the unitarizable composition factors with red background color in the K-type pictures. Let $\lambda = \rho + 2j + s$. Then we have:

1. $(\alpha_1,\alpha_2) \not\to (\alpha_1+1,\alpha_2)$ if and only if $\alpha_1 = -\rho - j$.
2. $(\alpha_1+1,\alpha_2) \not\to (\alpha_1,\alpha_2)$ if and only if $\alpha_1 = s + j$.
3. $(\alpha_1,\alpha_2) \not\to (\alpha_1,\alpha_2+1)$ if and only if $\alpha_2 = -\rho - j - s$.
4. $(\alpha_1,\alpha_2+1) \not\to (\alpha_1,\alpha_2)$ if and only if $\alpha_2 = j$.

The first condition never holds, since $\alpha_1 \geq 0$ and the second condition always holds for $\alpha_1 = j + s$ if $j + s \geq 0$. The third condition holds for $\alpha_2 = -\rho - j - s$ if and only if $j + s \leq -\rho$, and the last condition always holds for $\alpha_2 = j$. By Proposition 3.1 the corresponding K-type picture is given by Figure 3.1 and Figure 3.2. This concludes the proof of part (a). For the scalars t_{α_1,α_2} with which the Knapp–Stein intertwiner $T_{s,\lambda}$ acts on the K-type (α_1,α_2) we have up to normalization

$$t_{\alpha_1,\alpha_2} = \frac{(-j-s)_{\alpha_1}(-j)_{\alpha_2}}{(\rho+j)_{\alpha_1}\Gamma(\rho+j+s+\alpha_2)}.$$

Let $j + s \geq 0$. Then $t_{\alpha_1,\alpha_2} = 0$ for all $\alpha_1 > j + s$ and all $\alpha_2 > j$ such that $\ker T_{s,\lambda} = H^+_{s+j+1,0} + H^+_{0,j+1}$. Moreover, due to the signs of the factors in the numerator, the scalars $\Gamma(-j-s)\Gamma(-j)t_{\alpha_1,\alpha_2}$ are positive on $H^+_{j+s+1,j+1}$ and if $j = 0$ on $L^\pm_{0,1,s+1}$ and if $s + j = 0$ on $L^\mp_{0,1,j+1}$ and if $s = j = 0$ on $F^+_{1,1}$.

Let $-\rho < j + s < 0$. Then $t_{\alpha_1,\alpha_2} = 0$ if and only if $\alpha_2 > j$ such that $\ker T_{s,\lambda} = H^+_{0,j+1}$ and $\Gamma(-j)t_{\alpha_1,\alpha_2}$ is strictly positive on $H^+_{0,j+1}$ and for $j = 0$, t_{α_1,α_2} is positive for all $\alpha_2 = 0$.

Let $j+s < -\rho$ and $\lambda \geq 0$. then in particular $j \geq -\rho-j-s \geq 0$. Then again $t_{\alpha_1,\alpha_2} = 0$ if and only if $\alpha_2 > j$ or $\alpha_2 \leq -\rho - j - s$, such that $\ker T_{s,\lambda} = H^+_{0,j+1} + L^+_{0,-\rho-j-s}$. Again $\Gamma(-j)t_{\alpha,\alpha'}$ is positive on $H^+_{0,j+1}$, and $\Gamma(\rho+j+s)t_{\alpha_1,\alpha_2}$ is positive on $L^+_{0,-\rho-j-s}$. Moreover

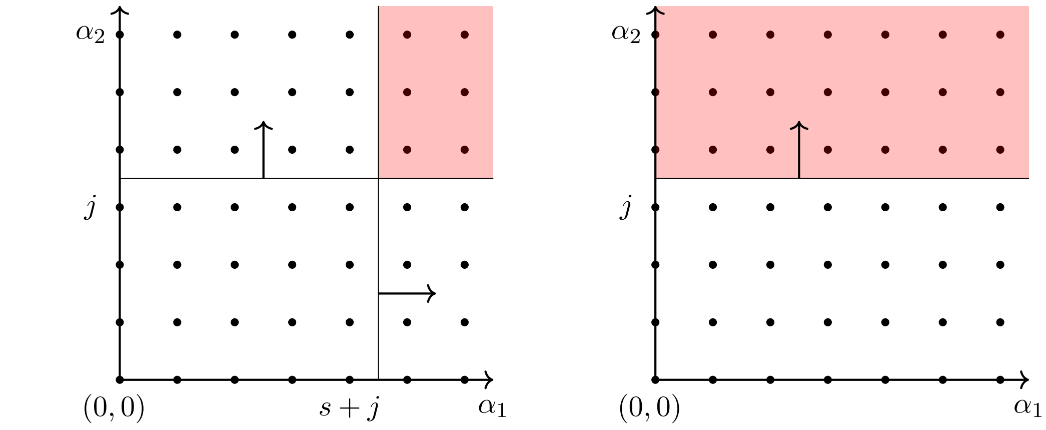

Figure 3.1: K-type pictures for $\lambda = \rho + 2j + s$ with $s \geq -j$ (left) and $-\rho - j < s < -j$ (right) with unitarizable composition factors (red).

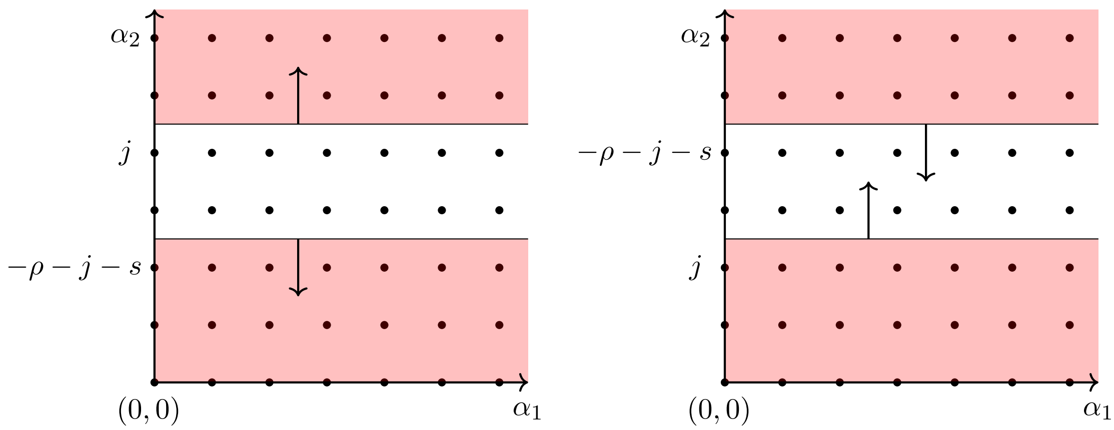

Figure 3.2: K-type picture for $\lambda = \rho + 2j + s$ with $-\rho - 2j \leq s \leq -\rho - j$ (left) and $s < -\rho - 2j$ (right) with unitarizable composition factors (red).

for $j = -\rho - j - s$, i.e. $\lambda = 0$, $\Gamma(-j)t_{\alpha_1,\alpha_2}$ is positive for $\alpha_2 = j$. And for $j = -\rho - j - s + 1$, i.e. $\lambda = 1$, we have $(-1)^j t_{\alpha_1,j} > 0$ for all α_1.

Finally let $\lambda < 0$. Then $j < -\rho - j - s$ and t_{α_1,α_2} vanishes of and only if $\alpha_2 > j$ and all $\alpha_2 \leq -\rho - j - s$, hence everywhere. But the normalization $\Gamma(-\rho - j - s)t_{\alpha_1,\alpha_2}$ vanishes if and only if $j < \alpha_2 \leq -\rho - j - s$, such that $\ker T_{s,\lambda} = L^+_{j+1,-\rho-j-s}$. Moreover $\Gamma(-\rho - j - s)t_{\alpha_1,\alpha_2}$ is strictily positive if and only if $\alpha_2 \leq j$ or $\alpha_2 > -\rho - j - s$. $\Gamma(-j)\Gamma(\rho + j + s)t_{\alpha_1,\alpha_2}$ is regular but indefinite on $L^+_{j+1,-\rho-j-s,0}$ for $-\rho - j - s \neq j + 1$. For $-\rho - j - s = j + 1$, i.e. $\lambda = -1$ we have $\Gamma(-j)\Gamma(\rho + j + s)t_{\alpha_1,j+1} > 0$ for all α_1.

Let now $\lambda = -\rho - 2j - s$. Then we have up to scalar

$$t_{\alpha_1,\alpha_2} = \frac{(\rho + j)_{\alpha_1}(\rho + j + s)_{\alpha_2}}{\Gamma(-j - s + \alpha_1)\Gamma(-j + \alpha_2)}.$$

Then we can easily read off the kernels of $T_{s,\lambda}$. □

Remark that for $j \in \mathbb{Z}_{\geq 0}$ and $\lambda = \rho + 2j \pm s \leq 0$, The sub-quotient realized on $L^{\pm}_{0,j,0}$ contains a highest or lowest K-weight. We denote the corresponding unitary completion by

$$\pi^{\sqsubset}_{s,\lambda}.$$

For $\lambda = \rho+2j\pm s$, we denote the unitary completions of the unitarizable quotients realized on $H^{\pm}_{\max\{0,j\pm s+1\},j+1}$ by

$$\pi^{\llcorner}_{s,\lambda}.$$

We abuse notation and denote the unitary completions of $\pi_{s,\lambda}$ for $\lambda \in i\mathbb{R}$ also by $\pi_{s,\lambda}$. Then in particular for $\lambda = 0$, $|s| \geq \rho$ and $s+\rho \equiv 0 \mod 2$,

$$\pi_{s,\lambda} = \pi^{\sqsubset}_{s,\lambda} \oplus \pi^{\llcorner}_{s,\lambda}.$$

We use the notation $\tau^{\sqsubset}_{t,\nu}$, $\tau^{\llcorner}_{t,\nu}$ and $\tau_{t,\nu}$ similarly for the unitary completions for G'.

4 Functional equations

Composition with Knapp–Stein intertwiners maps symmetry breaking operators to symmetry breaking operators. In the non-compact picture these are given by convolution with the holomorphic family

$$\frac{1}{\Gamma(\lambda)}(|X|^2+Z)^s N(X,Z)^{2(\lambda-\rho-s)},$$

and for general $f \in \pi_{s,\lambda}$ by

$$T_{s,\lambda}f(x) = \frac{1}{\Gamma(\lambda)}\int_{\bar{\mathfrak{n}}} f(x\tilde{w}_0\overline{n}_{(X,Z)})d(X,Z).$$

Then by Lemma 1.4(ii),

$$\mathbf{1}_{s,\lambda}(\overline{n}_{(X,Z)}) := (1+|X|^2+Z)^{-s}((1+|X|^2)^2+Z^2)^{-\frac{\lambda+\rho-s}{2}} \in \pi_{s,\lambda}$$

is an element of the one-dimensional K-type $\mathcal{E}_s(0,0)$.

Proposition 4.1. *For* $\mathrm{Re}(a) > -2$ *and* $\mathrm{Re}(b)+\mathrm{Re}(c) < -2n$ *we have*

$$\int_{\bar{\mathfrak{n}}}|X_n|^a(1+|X|^2+Z)^b((1+|X|^2)^2+|Z|^2)^{\frac{c}{2}}d(X,Z) = \frac{\pi^\rho 2^{1+b+c}\Gamma(\frac{2+a}{2})\Gamma(-\frac{a+2b+2c+2n+2}{2})}{\Gamma(-\frac{c}{2})\Gamma(-\frac{c+2b}{2})}.$$

Proof. Using polar coordinates on $\mathbb{R}^{2n-2}$ and $\mathbb{R}^2$ and on the radial directions $\mathbb{R}^2_+$ aftewards, we find the integral to be equal to

$$\mathrm{Vol}(S^{2n-3})\,\mathrm{Vol}(S^1)\int_0^{\frac{\pi}{2}}\cos^{2n-3}\phi\sin^{a+1}\phi\,d\phi \\ \times\int_{\mathbb{R}_+} r^{p+a-1}\int_{\mathbb{R}}(1+r^2-is)^b((1+r^2)^2+s^2)^{\frac{c}{2}}\,ds\,dr.$$

Evaluating the first integral and substituting $s = (1 + r^2)t$ we find

$$= \frac{\operatorname{Vol}(S^{2n-3})\operatorname{Vol}(S^1)B(n-1, \frac{a+2}{2})}{2} \int_{\mathbb{R}_+} r^{2n+a-1}(1+r^2)^{b+c+1}\,dr \\ \times \int_{\mathbb{R}} (1-is)^b(1+s^2)^{\frac{c}{2}}\,ds.$$

Evaluating the first integral with [GR65, 3.251 (11)] and substituting $e^{-i\varphi} = \frac{1-is}{\sqrt{1+s^2}}$ we obtain

$$= \frac{\operatorname{Vol}(S^{2n-3})\operatorname{Vol}(S^1)B(n-1, \frac{a+2}{2})B(\frac{p+a}{2}, -\frac{a+2b+2c+2n+2}{2})}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ib\varphi}\cos^{-b-c-2}\varphi\,d\varphi,$$

which we can evaluate with [GR65, 3.892 (2)] to obtain the result. □

Lemma 4.2. *$T_{s,\lambda}|_{\mathcal{E}_s(\alpha_1,\alpha_2)} = t_{\alpha_1,\alpha_2}\cdot \operatorname{id}_{\mathcal{E}_s(\alpha_1,\alpha_2)}$ for scalars t_{α_1,α_2} given by*

$$t_{\alpha_1,\alpha_2} = \frac{2^{1-\lambda-\rho}\pi^{\rho}(\frac{\rho-s-\lambda}{2})_{\alpha_1}(\frac{\rho+s-\lambda}{2})_{\alpha_2}}{\Gamma(\frac{\lambda+\rho-s}{2}+\alpha_1)\Gamma(\frac{\lambda+\rho+s}{2}+\alpha_2)}.$$

Proof. For $(\alpha_1, \alpha_2) = (0, 0)$, it is enough to evaluate $T_{s,\lambda}\mathbf{1}_{s,\lambda}$ at a specific value, which we choose to be $\tilde{w}_0$. Then for suitable λ and s

$$\begin{aligned} T_{s,\lambda}\mathbf{1}_{s,\lambda}(\tilde{w}_0) &= \frac{1}{\Gamma(\lambda)}\int_{\bar{\mathfrak{n}}} \mathbf{1}_{s,\lambda}(\overline{n}_{(X,Z)})d(X,Z) \\ &= \frac{1}{\Gamma(\lambda)}\int_{\mathbb{R}^{2n+1}} (1+|X|^2-iZ)^{-s}((1+|X|^2)^2+Z^2)^{-\frac{\lambda+\rho-s}{2}}d(X,Z) \end{aligned}$$

Then the statement follows from Proposition 4.1 and Proposition 3.1(ii) by analytic continuation. □

Corollary 4.3. *The operator*

$$\tilde{T}_{s,\lambda} := \begin{cases} \frac{\Gamma(\frac{-\lambda+\rho-|s|}{2})}{\Gamma(-\frac{\lambda}{2}+1)}T_{s,\lambda} & \text{for } |s|\geq\rho \text{ and } |s|+\rho\equiv 0 \mod 2, \\ \frac{\Gamma(\frac{-\lambda+\rho-|s|}{2})}{\Gamma(-\frac{\lambda-1}{2})}T_{s,\lambda} & \text{for } |s|\geq\rho \text{ and } |s|+\rho\equiv 1 \mod 2, \\ T_{s,\lambda} & \text{otherwise.} \end{cases}$$

defines a family of non-vanishing operators which is holomorphic in λ for fixed $s\in\mathbb{Z}_{\geq 0}$.

Proof. The Corollary follows immediately from Lemma 4.2. □

Remark 4.4. We renormalize the Knapp–Stein intertwiners for G' in the same way to obtain non-vanishing holomorphic families $\tilde{T}'_{t,\nu}$

Let $s\in\mathbb{Z}$. Then (see [FØ19a, Appendix B.2] for details)

$$\mathcal{E}_s(\alpha_1,\alpha_2)|_{K'} = \bigoplus_{\substack{0\leq\alpha_1'\leq\alpha_1 \\ 0\leq\alpha_2'\leq\alpha_2}} e^{i(\alpha_1-\alpha_2-s)\theta}\otimes\mathcal{H}^{(\alpha_1',\alpha_2')}(\mathbb{C}^n). \tag{4.1}$$

The concrete embedding

$$I_{(\alpha_1',\alpha_2')\to(\alpha_1,\alpha_2)}:\mathcal{H}^{(\alpha_1',\alpha_2')}(\mathbb{C}^n)\to\mathcal{H}^{(\alpha_1,\alpha_2)}(\mathbb{C}^{n+1})$$

is given by (see [FØ19a, (B.6)])

$$\phi(z')\mapsto \begin{cases} \phi(z')z_{n+1}^{\alpha_1-\alpha_2-\alpha_1'+\alpha_2'}P_{\alpha_2-\alpha_2'}^{\alpha_1-\alpha_2-\alpha_1'+\alpha_2',\alpha_1'+\alpha_2'+n-1}(1-2|z_{n+1}|^2) \\ \qquad\qquad\qquad\qquad \text{for } \alpha_1-\alpha_2\geq\alpha_1'-\alpha_2', \\ \phi(z')\overline{z}_{n+1}^{-(\alpha_1-\alpha_2-\alpha_1'+\alpha_2')}P_{\alpha_1-\alpha_1'}^{-(\alpha_1-\alpha_2-\alpha_1'+\alpha_2'),\alpha_1'+\alpha_2'+n-1}(1-2|z_{n+1}|^2) \\ \qquad\qquad\qquad\qquad \text{for } \alpha_1-\alpha_2\leq\alpha_1'-\alpha_2'. \end{cases} \tag{4.2}$$

Here $P_n^{\alpha,\beta}$ are the classical Jacobi polynomials. Let $\mathcal{E}_t'(\alpha_1',\alpha_2'):=e^{i(\alpha_1'-\alpha_2'-t)\theta}\otimes\mathcal{H}^{(\alpha_1',\alpha_2')}(\mathbb{C}^n)$ be the K'-types of $\tau_{t,\nu}$. Then by comparing the U(1)-factor we see that

$$\operatorname{Hom}_{K'}(\mathcal{E}_s(\alpha_1,\alpha_2)|_{K'},\mathcal{E}_t'(\alpha_1',\alpha_2'))\neq\{0\}$$

if and only if

$$\alpha_1'-\alpha_2'-t=\alpha_1-\alpha_2-s.$$

Let w.l.o.g. $s\geq t$. Then for the vector $\mathbf{1}_{t,\nu}'\in\mathcal{E}_t'(0,0)$ which is in the non-compact picture given by

$$\mathbf{1}_{t,\nu}'(\overline{n}_{(X',0,Z)}):=(1+|X'|^2+Z)^{-t}((1+|X'|^2)^2+Z^2)^{-\frac{\nu+\rho'-t}{2}},$$

we have

$$I_{(0,0)\to(s-t,0)}\mathbf{1}_{t,\nu}'(z',z_{n+1})=z_{n+1}^{s-t}\mathbf{1}_{t,\nu}'(z').$$

Consider the operator

$$A_{\lambda,\nu}^{s,t}:\pi_{s,\lambda}\to\tau_{t,\nu}.$$

Then by Schur's Lemma, $A_{\lambda,\nu}^{s,t}$ maps $I_{(0,0)\to(s-t,0)}\mathbf{1}_{t,\nu}'$ to a scalar multiple of $\mathbf{1}_{t,\nu}'$.

Lemma 4.5. *For $(\lambda,\nu)\in\mathbb{C}^2$ and $(s,t)\in\mathbb{Z}^2$ we have*

$$A_{\lambda,\nu}^{s,t}I_{(0,0)\to(|s-t|,0)}\mathbf{1}_{t,\nu}'=(-1)^t\frac{2^{1-\lambda-\rho}\pi^\rho}{\Gamma(\frac{\lambda+\rho+|s-t|+t}{2})\Gamma(\frac{\lambda+\rho+|s-t|-t}{2})}\mathbf{1}_{t,\nu}'.$$

Proof. Let $s\geq t$. By Schur's Lemma it is enough to show the identity evaluated at one element which we choose to be $\tilde{w}_0\in K'$. By Lemma 1.4(ii) we have in the non-compact picture

$$I_{(0,0)\to(s-t,0)}\mathbf{1}_{t,\nu}'(\overline{n}_{(X,Z)})=\left(\frac{2X_n}{1+|X|^2-Z}\right)^{s-t}\mathbf{1}_{s,\lambda}(\overline{n}_{(X,Z)})=(2X_n)^{s-t}\mathbf{1}_{t,\lambda+s-t}(\overline{n}_{(X,Z)}).$$

Then following Section 2, we have

$$A_{\lambda,\nu}^{s,t}I_{(0,0)\to(s-t,0)}\mathbf{1}_{t,\nu}'(\tilde{w}_0)=\langle u_{(\lambda,s),(\nu,t)}^A,I_{(0,0)\to(s-t,0)}\mathbf{1}_{t,\nu}'(\tilde{w}_0\exp(\,\cdot\,))\rangle.$$

As an element of $C^\infty(\mathcal{V}_{\xi_{-s},-\lambda+\rho})$ we have following Lemma 1.1(iii),

$$u_{(s,\lambda),(t,\nu)}^A(\tilde{w}_0\overline{n}_{(X,Z)})=\frac{1}{\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2})\Gamma(\frac{\lambda-\rho+\nu+\rho'+|s-t|}{2})}(-1)^t|X_n|^{\lambda-\rho+\nu+\rho'-s+t}\overline{X}_n^{s-t},$$

such that

$$A^{s,t}_{\lambda,\nu} I_{(0,0)\to(s-t,0)} \mathbf{1}'_{t,\nu}(\tilde{w}_0) = 2^{s-t}(-1)^t \frac{2^{s-t}(-1)^t}{\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2})\Gamma(\frac{\lambda-\rho+\nu+\rho'+|s-t|}{2})}$$
$$\times \int_{\bar{\mathfrak{n}}} |X_n|^{\lambda-\rho+\nu+\rho'+s-t}(1+|X|^2+Z)^{-t}((1+|X|^2)^2+|Z|^2)^{-\frac{\lambda+\rho+s-2t}{2}} d(X,Z)$$

can be evaluated using Proposition 4.1.

For $s \leq t$ the argument works analogously. $\square$

Theorem 4.6. *For $(\lambda, \nu) \in \mathbb{C}^2$ and $s, t \in \mathbb{Z}$ we have*

$$T'_{t,\nu} \circ A^{s,t}_{\lambda,\nu} = \frac{2^{1-\nu-\rho'}\pi^{\rho'}}{\Gamma(\frac{\nu+\rho'+t}{2})\Gamma(\frac{\nu+\rho'-t}{2})} A^{s,t}_{\lambda,-\nu},$$

$$A^{s,t}_{\lambda,\nu} \circ T_{s,-\lambda} = \frac{2^{1-\lambda-\rho}\pi^{\rho}}{\Gamma(\frac{\lambda+\rho-s}{2})\Gamma(\frac{\lambda+\rho+s}{2})} A^{s,t}_{-\lambda,\nu}.$$

Proof. The Theorem follows from Lemma 4.2 and Lemma 4.5. $\square$

5 K'-type analysis

Let

$$\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) := I_{(\alpha'_1,\alpha'_2)\to(\alpha_1,\alpha_2))}\left(\mathcal{E}'_t(\alpha'_1,\alpha'_2)\right) \in \mathcal{E}_s(\alpha_1,\alpha_2)$$

be the (α'_1,α'_2) K'-type of $\tau_{t,\nu}$ in the (α_1,α_2) K-type of $\pi_{s,\lambda}$. Then by (4.2) the map

$$R_{(\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)} : \mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) \to \mathcal{E}'_t(\alpha'_1,\alpha'_2)$$

which is given by

$$R_{(\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)}\phi(z',z_{n+1}) = \begin{cases} \left(\frac{\partial^{s-t}}{\partial z_{n+1}^{s-t}}\phi\right)(z',0), & \text{if } s \geq t, \\ \left(\frac{\partial^{t-s}}{\partial \overline{z_{n+1}}^{t-s}}\phi\right)(z',0) & \text{if } s < t, \end{cases}$$

defines K'-isomorphisms. Then since by (4.1), $\mathcal{E}_s(\alpha_1,\alpha_2)$ decmposes into K' with multiplicities lesser or equal to one, $A^{s,t}_{\lambda,\nu}$ must act by scalar multiples of $R_{(\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)}$ on $\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2))$. We define these scalars by

$$A^{s,t}_{\lambda,\nu}|_{\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2))} = a^{s,t}_{\lambda,\nu}\left((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)\right) R_{(\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)}.$$

Moreover every $(\mathfrak{g}', K')$-intertwining operator $(\pi_{s,\lambda})_{\mathrm{HC}} \to (\tau_{t,\nu})_{\mathrm{HC}}$ can be described in such a way. In [FØ19a] introduced a method using the spectrum generating operator to describe such scalars defining $(\mathfrak{g}', K')$-intertwiners as the solution of an infinite set of equations depending linearly on the induction parameters. This way they found the scalars $a^{0,0}_{\lambda,\nu}$ and the proofs can easily be generalized to find the scalars $a^{s,t}_{\lambda,\nu}$ for all $s, t \in \mathbb{Z}$.

Therefore let

$$\omega : \mathfrak{g}_{\mathbb{C}} \to C^\infty(K/M),$$
$$\omega(X)(k) := B(\mathrm{Ad}(k^{-1})X, H)$$

and let $m(\omega(X))$ denote the multiplication operator. The map ω is called *cocycle*. We define for $X \in \mathfrak{g}'$

$$\omega^{(\beta_1,\beta_2),(\beta_1',\beta_2')}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}(X) := \mathrm{proj}_{\mathcal{E}_s((\beta_1,\beta_2),(\beta_1',\beta_2'))} \circ m(\omega(X))|_{\mathcal{E}_s((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))}$$

which is a non-zero K'-ismorophism if and only if

$$((\beta_1,\beta_2),(\beta_1',\beta_2')) \in \{((\alpha_1 \pm 1,\alpha_2),(\alpha_1' \pm 1,\alpha_2')),((\alpha_1 \pm 1,\alpha_2),(\alpha_1',\alpha_2' \mp 1)),$$
$$((\alpha_1,\alpha_2 \pm 1),(\alpha_1',\alpha_2' \pm 1)),((\alpha_1,\alpha_2 \pm 1),(\alpha_1' \mp 1,\alpha_2'))\}.$$

$((\beta_1,\beta_2),(\beta_1',\beta_2'))$ being contained in this set we will denote in the following by

$$((\beta_1,\beta_2),(\beta_1',\beta_2')) \leftrightarrow ((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')).$$

Similarly let ω' be the cocycle of $\mathfrak{g}'$ and we define

$$\omega^{(\beta_1',\beta_2')}_{(\alpha_1',\alpha_2')}(X) := \mathrm{proj}_{\mathcal{E}_t'((\beta_1',\beta_2'))} \circ m(\omega'(X))|_{\mathcal{E}_t'((\alpha_1',\alpha_2'))}$$

which defines a non-zero K'-isomorphism if and only if $(\beta_1',\beta_2') \leftrightarrow (\alpha_1',\alpha_2')$, i.e. if and only if

$$(\beta_1',\beta_2') \in \{(\alpha_1' \pm 1,\alpha_2'),(\alpha_1',\alpha_2' \pm 1)\}.$$

Then clearly

$$R_{(\beta_1,\beta_2),(\beta_1',\beta_2')} \circ \omega^{(\beta_1,\beta_2),(\beta_1',\beta_2')}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')} = \lambda^{(\beta_1,\beta_2),(\beta_1',\beta_2')}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')} \omega^{(\beta_1',\beta_2')}_{(\alpha_1',\alpha_2')} \circ R_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}. \tag{5.1}$$

for non-zero scalars $\lambda^{(\beta_1,\beta_2),(\beta_1',\beta_2')}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}$. Recall the eigenvalues $\sigma_s(\alpha_1,\alpha_2)$ of the spectrum generating operator given in Lemma 3.2. We denote the eigenvalues of the spectrum generating operator for G' similarly by $\sigma_t'(\alpha_1',\alpha_2')$.

Theorem 5.1 ([FØ19a] Theorem 3.4)**.** *Let $a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ be scalars defining a $(\mathfrak{g}',K')$-intertwiner $(\pi_{s,\lambda})_{\mathrm{HC}} \to (\tau_{t,\nu})_{\mathrm{HC}}$. Then for all $\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \neq 0$ and $\mathcal{E}_t'(\beta_1',\beta_2') \neq 0$,*

$$\sum_{\substack{(\beta_1,\beta_2),\\ ((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \leftrightarrow ((\beta_1,\beta_2),(\beta_1,\beta_1'))}} (\sigma_s(\alpha_1,\alpha_2) - \sigma_s(\beta_1,\beta_2) + 2\lambda) \lambda^{(\beta_1,\beta_2),(\beta_1',\beta_2')}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')} a((\beta_1,\beta_2),(\beta_1',\beta_2'))$$
$$= (\sigma_t'(\alpha_1',\alpha_2') - \sigma_t'(\beta_1',\beta_2') + 2\nu) a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')).$$

Then finding the proportionality constants works in the same way as in the case $s = t = 0$ (see [FØ19a, Lemma 3.7]).

Lemma 5.2. *For all $\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \neq 0$ and $\mathcal{E}_t'(\beta_1',\beta_2') \neq 0$*

$$\sum_{\substack{(\beta_1,\beta_2),\\ ((\alpha_1,\alpha_2),\alpha_1',\alpha_2') \leftrightarrow ((\beta_1,\beta_2),(\beta_1,\beta_1'))}} \lambda^{(\beta_1,\beta_2),(\beta_1',\beta_2')}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')} = 1,$$

$$\sum_{\substack{(\beta_1,\beta_2),\\ ((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \leftrightarrow ((\beta_1,\beta_2),(\beta_1,\beta_1'))}} (\sigma_s(\alpha_1,\alpha_2) - \sigma_s(\beta_1,\beta_2)) \lambda^{(\beta_1,\beta_2),(\beta_1',\beta_2')}_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}$$
$$= \sigma_t'(\alpha_1',\alpha_2') - \sigma_t'(\beta_1',\beta_2') + 2 + 2|s-t|.$$

Proof. By [FØ19a, Chapter 5.3], for $X \in \mathfrak{g}'$, the cocycles $\omega(X) = \omega'(X)$ are given by a homogeneous polynomial in the coordinates $z_1, \dots z_n$ and $\overline{z_1}, \dots \overline{z_n}$ of degree one, such that the multiplication with the cocylce commutes with the maps $R_{(\alpha_1,\alpha_2),(\alpha_1',\alpha_2')}$ which implies with (5.1) the first identity.

Let $\lambda = -1$ and $\nu = |s-t|$. Then by [FW20, Theorem 10.1], $u^C_{\lambda-|s-t|-|t|,\nu+|t|}$ is a non-zero multiple of

$$\Delta_{\mathfrak{v}''}^{|s-t|+|t|}\delta(X,Z).$$

let w.l.o.g. $s \geq t$ such that by the translation principle $u^{s,t}_{\lambda,\nu}$ is a non-zero multiple of

$$\frac{1}{|s-t|!}D^{s-t}_{\overline{X_n}}\delta(X,Z),$$

where $D_{\overline{X_n}}$ is the differential operator

$$D_{\overline{X}_n} := \frac{\partial}{\partial \operatorname{Re} X_n} - i\frac{\partial}{\partial \operatorname{Im} X_n}.$$

Then the corresponding intertwining operator $(\pi_{s,-1})_{\mathrm{HC}} \to (\tau_{t,|s-t|})_{\mathrm{HC}}$ is given by the scalars $a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \equiv 1$ for all $\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \neq 0$ and $\mathcal{E}_t'(\beta_1',\beta_2') \neq 0$. Then Theorem 5.1 implies the second identity. For $s<t$ the argument is the same by replacing $D_{\overline{X}_n}$ by $\overline{D_{\overline{X}_n}}$. □

We define for $s,t \in \mathbb{Z}$,

$$(s,t)_- := -\frac{|s-t|-s+t}{2}, \qquad (s,t)_+ := \frac{|s-t|+s-t}{2},$$

such that

$$(s,t)_- = \begin{cases} 0 & \text{for } s \geq t, \\ s-t & \text{for } s<t \end{cases}$$

and

$$(s,t)_+ = \begin{cases} s-t & \text{for } s \geq t, \\ 0 & \text{for } s<t. \end{cases}$$

Calculating the proportionality constants with the identities of Lemma 5.2, Theorem 5.1 yields the following.

Lemma 5.3. *Every $(\mathfrak{g}',K')$-intertwining operator $(\pi_{s,\lambda})_{\mathrm{HC}} \to (\tau_{t,\nu})_{\mathrm{HC}}$ is for all K'-types $\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \neq 0$ and $\mathcal{E}_t'(\beta_1',\beta_2') \neq 0$ given by scalars $a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ satisfying*

$$\begin{aligned}(\alpha_1+\alpha_2+\rho')&(\nu-t+\rho'+2\alpha_1')a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))\\ &= (\alpha_1'+\alpha_2+\rho'+(s,t)_+)(\lambda-s+\rho+2\alpha_1)a((\alpha_1+1,\alpha_2),(\alpha_1'+1,\alpha_2'))\\ &\quad+(\alpha_1-\alpha_1'-(s,t)_+)(\lambda-s-\rho-2\alpha_2+2)a((\alpha_1,\alpha_2-1),(\alpha_1'+1,\alpha_2')),\end{aligned} \tag{5.2}$$

$$\begin{aligned}(\alpha_1+\alpha_2+\rho')&(\nu+t-\rho'-2\alpha_1'+2)a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))\\ &= (\alpha_1'+\alpha_2+\rho'+(s,t)_--1)(\lambda+s-\rho-2\alpha_1+2)a((\alpha_1-1,\alpha_2),(\alpha_1'-1,\alpha_2'))\\ &\quad+(\alpha_1-\alpha_1'-(s,t)_-+1)(\lambda+s+\rho+2\alpha_2)a((\alpha_1,\alpha_2+1),(\alpha_1'-1,\alpha_2')),\end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&(\alpha_1+\alpha_2+\rho')(\nu+t+\rho'+2\alpha_2'+2)a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))\\
&\qquad=(\alpha_1'+\alpha_2+\rho'+(s,t)_+)(\lambda+s+\rho+2\alpha_2)a((\alpha_1,\alpha_2+1),(\alpha_1',\alpha_2'+1))\\
&\qquad\quad+(\alpha_1-\alpha_1'-(s,t)_+)(\lambda+s-\rho-2\alpha_1+2)a((\alpha_1-1,\alpha_2),(\alpha_1',\alpha_2'+1)), \qquad (5.4)
\end{aligned}
$$

$$
\begin{aligned}
&(\alpha_1+\alpha_2+\rho')(\nu-t-\rho'-2\alpha_2')a((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))\\
&\qquad=(\alpha_1'+\alpha_2+\rho'+(s,t)_--1)(\lambda-s-\rho-2\alpha_2+2)a((\alpha_1,\alpha_2-1),(\alpha_1',\alpha_2'-1))\\
&\qquad\quad+(\alpha_1-\alpha_1'-(s,t)_-+1)(\lambda-s+\rho+2\alpha_1)a((\alpha_1+1,\alpha_2),(\alpha_1',\alpha_2'-1)). \qquad (5.5)
\end{aligned}
$$

Proposition 5.4. *The scalars $t^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ given by*

$$
\begin{aligned}
&t^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))\\
&\quad=\sum_{k=0}^{\infty}\frac{(\alpha_1'-\alpha_1+(s,t)_+)_k(\alpha_1'+\alpha_2+\rho'+(s,t)_+)_k(\frac{\nu-t+\rho'}{2})_{\alpha_1'}(\frac{\nu+t+\rho'}{2})_{\alpha_2'}}{k!(k+|s-t|)!\Gamma(\frac{\lambda-t+|s-t|+\rho}{2}+\alpha_1'+k)\Gamma(\frac{\lambda+t+|s-t|+\rho}{2}+\alpha_2'+k)}\\
&\qquad\qquad\times\left(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)_k\left(\frac{\lambda+\rho+\nu-\rho'+|s-t|}{2}\right)_k
\end{aligned}
$$

define $(\mathfrak{g}',K')$-intertwining operators $(\pi_{s,\lambda})_{\mathrm{HC}}\to(\tau_{t,\nu})_{\mathrm{HC}}$ which are holomorphic in λ and ν.

Proof. The proof is the same as the proof of [FØ19a, Proposition 5.7] using the reparametrisation $p=\alpha_1+\alpha_2$, $q_1=\alpha_1'+(s,t)_+$ and $q_2=\alpha_2'-(s,t)_-$. Holomorphicity is clear by the definition of the scalars. ☐

Since the dimension of $\mathrm{Hom}_{G'}(\pi_{s,\lambda}|_{G'},\tau_{t,\nu})$ is generically one (see [Möl17]) we immediately obtain by Lemma 4.5 the following corollary.

Corollary 5.5.

$$A^{s,t}_{\lambda,\nu}|_{\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))}=(-1)^t2^{1-\lambda-\rho}\pi^{\rho}t^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')).$$

Lemma 5.6. $A^{s,t}_{\lambda,\nu}=0$ *if and only if* $(\lambda+|s-t|-|t|,\nu-|t|)\in L$.

Proof. By the translation principle we have that

$$X_n^{s-t}(|X|^2-Z)^t u^A_{(\lambda,s),(\nu,t)}=\frac{\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2}+|t|)}{\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2})}u^A_{\lambda+|s-t|+|t|,\nu-|t|},$$

such that $A^{s,t}_{\lambda,\nu}=0$ implies $(\lambda+|s-t|+|t|,\nu-|t|)\in L$ or $(\lambda+|s-t|-|t|,\nu-|t|)\in /\!/$ and $(\lambda+|s-t|+|t|,\nu-|t|)\notin /\!/$. Hence we can assume $(\lambda+|s-t|-|t|,\nu-|t|)\in /\!/$. Let therefore $l\in\mathbb{Z}_{\geq 0}$ such that

$$\lambda+\rho-\nu-\rho'+|s-t|=-2l.$$

Consider $t^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')$. Then we can in particular assume the summation index k to be lesser or equal to l such that

$$\frac{\lambda-t+|s-t|}{2}+\alpha_1'+k\leq\frac{\nu-t+\rho'}{2}+\alpha_1',$$

$$\frac{\lambda+t+|s-t|}{2}+\alpha_2'+k\leq\frac{\nu+t+\rho'}{2}+\alpha_2'.$$

Then $A^{s,t}_{\lambda,\nu}=0$ if and only if

$$\frac{(\frac{\nu-t+\rho'}{2})_{\alpha_1'}(\frac{\nu+t+\rho'}{2})_{\alpha_2'}}{\Gamma(\frac{\nu-t+\rho'}{2}+\alpha_1')\Gamma(\frac{\nu+t+\rho'}{2}+\alpha_2')}=0$$

for all $\alpha_1',\alpha_2'\geq 0$, which holds if and only if $\nu-|t|\in-\rho'-2\mathbb{Z}_{\geq 0}$, i.e. $(\lambda+|s-t|-|t|,\nu-|t|)\in L$. □

5.1 Two normalizations of $A^{s,t}_{\lambda,\nu}$

We define the following meromorphic families of symmetry breaking operators as renormalizations of $A^{s,t}_{\lambda,\nu}$ along different complex lines in $\mathbb{C}^2$.

For $\nu\in-\rho'+|t|-2\mathbb{Z}_{\geq 0}$ we define

$$\tilde{A}^{s,t}_{\lambda,\nu}:=\Gamma\left(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)A^{s,t}_{\lambda,\nu},$$

and for $\lambda+\rho-\nu-\rho'+|s-t|\in-2\mathbb{Z}_{\geq 0}$ we define

$$C^{s,t}_{\lambda,\nu}:=\Gamma\left(\frac{\nu+\rho'-t}{2}\right)\Gamma\left(\frac{\nu+\rho'+t}{2}\right)A^{s,t}_{\lambda,\nu}.$$

We denote the eigenvalues of these operators on the K'-types $\mathcal{E}_{s,t}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ by $\tilde{a}^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ resp. $c^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$. Then we have by definition for $\nu=-\rho'-2j+|t|$,

$$\begin{aligned}&\tilde{a}^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))\\&\quad=\sum_{k=0}^{\infty}\frac{(\alpha_1'-\alpha_1+(s,t)_+)_k(\alpha_1'+\alpha_2+\rho'+(s,t)_+)_k\Gamma\left(\frac{\lambda+\rho-|t|+|s-t|}{2}+j+k\right)}{k!(k+|s-t|)!\Gamma(\frac{\lambda-t+|s-t|+\rho}{2}+\alpha_1'+k)\Gamma(\frac{\lambda+t+|s-t|+\rho}{2}+\alpha_2'+k)}\\&\qquad\times\left(\frac{|t|-t}{2}-j\right)_{\alpha_1'}\left(\frac{|t|+t}{2}-j\right)_{\alpha_2'}\left(\frac{\lambda+\rho+|t|+|s-t|}{2}-\rho'-j\right)_k\end{aligned}\tag{5.6}$$

and for $\lambda+\rho-\nu-\rho'=-2l$

$$\begin{aligned}c^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))&=\sum_{k=0}^{l}\frac{(\alpha_1'-\alpha_1+(s,t)_+)_k(\alpha_1'+\alpha_2+\rho'+(s,t)_+)_k}{k!(k+|s-t|)!}\\&\times\left(\frac{\nu+\rho'-t}{2}+\alpha_1'+k-l\right)_{l-k}\left(\frac{\nu+\rho'+t}{2}+\alpha_2'+k-l\right)_{l-k}(-l)_k(\nu-l)_k\end{aligned}\tag{5.7}$$

We remark that for $s=t=0$, $C^{0,0}_{\lambda,\nu}$ defined this way is non-zero scalar multiple of the differential operator $C_{\lambda,\nu}$ and for $(\lambda,\nu)\in L$, $\tilde{A}^{0,0}_{\lambda,\nu}$ is a non-zero scalar multiple of the singular operator $B_{\lambda,\nu}$ in the notation of [FW20].

Lemma 5.7. *(i) For $\nu\in-\rho'+|t|-2\mathbb{Z}_{\geq 0}$, the operators $\tilde{A}^{s,t}_{\lambda,\nu}$ define a family of symmetry breaking operators which is holomorphic in λ and non-vanishing for all $\lambda\in\mathbb{C}$.*

(ii) For $\lambda+\rho-\nu-\rho'+|s-t|\in -2\mathbb{Z}_{\geq 0}$, the operators $C^{s,t}_{\lambda,\nu}$ define a family of symmetry breaking operators which is holomorphic in ν and non-vanishing for all $\nu\in\mathbb{C}$.

Proof. Holomorphicity can easily be read off the eigenvalues of Proposition 5.4. Moreover by definition $\tilde{A}^{s,t}_{\lambda,\nu}$ and $C^{s,t}_{\lambda,\nu}$ can only vanish if $A^{s,t}_{\lambda,\nu}$ vanishes, i.e. if $\nu=-\rho'-2j+|t|$ for some $j\in\mathbb{Z}_{\geq 0}$ and $\lambda+\rho-\nu-\rho'+|s-t|=-2l$ for some $l\in\mathbb{Z}_{\geq 0}$. Then choosing $\alpha_1'=\alpha_2'=0$ and $\alpha_1,=(s,t)_+$ it is easy to see that the corresponding eigenvalues of $\tilde{A}^{s,t}_{\lambda,\nu}$ and $C^{s,t}_{\lambda,\nu}$ on the K'-type $((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ is non-zero. □

Corollary 5.8. *The operators $\tilde{A}^{s,t}_{\lambda,\nu}$ and $C^{s,t}_{\lambda,\nu}$ are linearly independent if and only if $(\lambda-|t|+|s-t|,\nu-|t|)\in L$.*

Proof. First we remark that $\tilde{A}^{s,t}_{\lambda,\nu}$ and $C^{s,t}_{\lambda,\nu}$ are defined at the same time if and only if $(\lambda-|t|+|s-t|,\nu-|t|)\in L$. Hence we only proof linear independence in this case.

Let $t\geq 0$. Then clearly the eigenvalues $\tilde{a}^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))=0$ for all $\alpha_1'>j$. But clearly $c^{s,t}_{\lambda,\nu}((\alpha_1'+(s,t)_+,\alpha_2),(\alpha_1',\alpha_2'))\neq 0$ for all α_1' such that the linear independence follows. For $t<0$ the argument works analogously. □

Corollary 5.9. *Let $(\lambda+|s-t|-|t|,\nu-|t|)=(-\rho-2i,-\rho'-2j)\in L$ with $\nu\geq 0$. Then*

$$\tilde{a}^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))=(-1)^i\frac{j!}{(i-j)!\Gamma(|t|-j)}c^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$$

for all $t\geq 0$ and $\alpha_1'\leq j$ and $\alpha_1\leq i+(s,t)_+$ as well as for all $t<0$ and $\alpha_2'\leq j$ and $\alpha_2\leq i+(s,t)_-$, as well as for all $t\geq 0$ and all $\alpha_1'\leq j$ and for all $t<0$ and $\alpha_2'\leq j$ if $|t|-j-\rho'\leq i$.

Proof. First we have that the scalar factor in the identity is the quotient of the renormalizing factors of $\tilde{A}^{s,t}_{\lambda,\nu}$ and $C^{s,t}_{\lambda,\nu}$,

$$(-1)^i\frac{j!}{(i-j)!\Gamma(|t|-j)}=\frac{\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2})}{\Gamma(\frac{\nu+\rho'+t}{2})\Gamma(\frac{\nu+\rho'-t}{2})}.$$

Let $t\geq 0$. Then for all $\alpha_1'\leq j$, both $\tilde{a}^{s,t}_{\lambda,\nu}(\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ and $c^{s,t}_{\lambda,\nu}(\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ do not vanish everywhere and are scalar multiples of both sums defining the eigenvalues have the same number of summands. Then if $|t|-j-\rho'\leq i$, $\lambda+\rho+\nu-\rho'+|s-t|$ is an even non-positive integer which is bigger than $2j-2i$ such that for all $((\alpha_1,\alpha_2),(\alpha_1',\alpha_2'))$ with $\alpha_1'\leq j$, both eigenvalues have $\min(\alpha_1-\alpha_1'-(s,t)_+,i+j-|t|-\rho')$ summands.

If $|t|-j-\rho'\leq i$, the eigenvalues have the same number of summands if and only if $\alpha_1-\alpha_1'-(s,t)_+\leq i-j$ for all $\alpha_1'\leq j$, i.e. if $i-j\leq\alpha_1-\alpha_1'-(s,t)_+$.

For $t<0$ the proof works analogously. □

We define a new symmetry breaking operator $\tilde{\tilde{A}}^{s,t}_{\lambda,\nu}$ for $(\lambda-|t|+|s-t|,\nu-|t|)\in L$:

$$\tilde{\tilde{A}}^{s,t}_{\lambda,\nu}:=\tilde{A}^{s,t}_{\lambda,\nu}-(-1)^i\frac{j!}{(i-j)!\Gamma(|t|-j)}C^{s,t}_{\lambda,\nu}.$$

Lemma 5.10. *Let $(\lambda+|s-t|-|t|,\nu-|t|)=(-\rho-2i,-\rho'-2j)\in L$ with $\nu\geq 0$ and $|t|-j-\rho'\leq i$. Then for $t\geq 0$ ($t<0$), $\tilde{\tilde{a}}^{s,t}_{\lambda,\nu}=0$ for all $\alpha_1\leq i$ ($\alpha_2\leq i$).*

Proof. Let $s \geq 0$. The scalars $\tilde{\tilde{a}}^{s,t}_{\lambda,\nu}$ with $\alpha_1 \leq i$ define a G'-intertwining operator from the submodule of $\pi_{s,\lambda}$ which is realized on the K-types $\alpha_1 \leq i$ to $\tau_{t,\nu}$ by restriction. Hence its image must be a submodule of $\tau_{t,\nu}$, which contains for $t \geq 0$ ($t < 0$) the two submodules $\tau^{\sqsubset}_{t,\nu}$ realized on the K'-types $\alpha'_1 \leq j$ ($\alpha'_2 \leq j$) and $\tau^{\llcorner}_{t,\nu}$ realized on the K'-types $\alpha'_1 > |t| - \rho' - j$ ($\alpha'_2 > |t| - \rho' - j$). The K-types with $\alpha_1 \leq i$ contain only K' types $\alpha'_1 \leq \alpha_1 \leq i$ such that the only non-trivial candidate for the image is $\tau^{\sqsubset}_{t,\nu}$ for $t \geq 0$. But by Corollary 5.9, $\tilde{\tilde{a}}^{s,t}_{\lambda,\nu} = 0$ for all $\alpha'_1 \leq j$ in this case.

For $s < 0$ the proof works analogously. □

The following Corollary is easily read off the eigenvalues $a^{s,t}_{\lambda,\nu}$.

Corollary 5.11. *Let $i, j \in \mathbb{Z}_{\geq} 0$.*

(i) If $\lambda \in -\rho - 2i + s$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha_1 \leq i$ and all $\alpha_2 \leq i - s$.

(ii) If $\lambda \in -\rho - 2i - s$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha_1 \leq i + s$ and all $\alpha_2 \leq i$.

(iii) If $\lambda \in \rho + 2i - s$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha_1 \leq s - \rho - i$.

(iv) If $\lambda \in \rho + 2i + s$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha_2 \leq -s - \rho - i$.

(v) If $\nu = -\rho' - 2j + t$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha'_1 > j$ and all $\alpha'_2 > j - t$ if $t < j$.

(vi) If $\nu = -\rho' - 2j - t$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha'_1 > j + t$ if $t > -j$ and all $\alpha'_2 > j$.

(vii) If $\nu \in \rho' + 2j - t$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha'_1 > t - \rho' - j$.

(viii) If $\nu \in \rho' + 2j + t$, $a^{s,t}_{\lambda,\nu}((\alpha_1,\alpha_2),(\alpha'_1,\alpha'_2)) = 0$ for all $\alpha'_2 > -t - \rho' - j$.

6 Structure of the open Orbit as homogeneous G'-space

By Lemma 1.3 we can define a continous linear G'-equivariant map

$$\Phi : \pi_{s,\lambda} \to C^\infty(G'/K'_0)$$

given by

$$f \mapsto f|_{\mathcal{O}_A}(\,\cdot\,\overline{n}_{e_n}).$$

By Mackey theory the restriction to an open subset carries enough information for our purposes of decomposing unitary representations.

Let for $g \in G$, $g = \overline{n} e^{\overline{H}(g)} \overline{\kappa}(g) \in \overline{N}AK$ be the $\overline{N}AK$ Iwasawa-decomposition. Then by Lemma 1.1 and Lemma 1.3 we have for $X_n \in \mathbb{C}^\times$

$$\overline{n}_{(X',X_n,Z)} = \overline{n}_{(X',0,Z)} e^{-\log|X_n| H} \operatorname{diag}\left(\frac{X_n}{|X_n|}, \frac{X_n}{|X_n|}, \mathbf{1}_n\right) \overline{n}_{e_n} \times \operatorname{diag}\left(\frac{\overline{X}_n}{|X_n|}, \frac{\overline{X}_n}{|X_n|}, \mathbf{1}_n\right) e^{\log|X_n| H}, \quad (6.1)$$

as an element of $\overline{N}' A (K'/K'_0) \overline{n}_{e_n} P$.

Lemma 6.1. *For* $\operatorname{Re}(\lambda) > -1$*, the map* Φ *extends to an* G'*-equivariant map*

$$\pi_{s,\lambda} \to L^2(G'/K_0').$$

The map is unitary for $\lambda \in i\mathbb{R}$.

Recall that by the Iwasawa-decomposition the following integral formula holds

$$\int_{G'/K'} f(g)dg = \int_{\overline{N}'\times\mathfrak{a}} f(\overline{n}e^X)e^{2\rho'(X)}d\overline{N}'dX, \tag{6.2}$$

Moreover, let $\omega \in \mathbb{C}$ with $|\omega| = 1$ and $\Xi = \operatorname{diag}(\omega, \omega, \mathbf{1}_n) \in M'$. Then by Lemma 1.1(i)

$$\Xi\overline{n}_{e_n} = \overline{n}_{\omega e_n}\Xi. \tag{6.3}$$

Proof of Lemma 6.1. Let $f \in \pi_{s,\lambda}$. We choose representatives $\Xi = \operatorname{diag}(\omega, \ldots, \omega, 1)$ of $K'/K_0' \cong U(1)$. By (6.2)

$$\begin{aligned}
\|\Phi f\|^2_{L^2(G'/K_0')} &= \int_{G'} |f(g\overline{n}_{e_n})|^2 dg \\
&= \int_{\overline{N}'\times\mathfrak{a}\times S^1} |f(\overline{n}_{(X',0,Z)}e^{rH}\Xi\overline{n}_{e_n})|^2 e^{2\rho' r} d(X',Z)\; dr\; d\omega \\
&= \int_{\overline{N}'\times\mathfrak{a}\times S^1} |f(\overline{n}_{(X',e^{-r}\omega,Z)})|^2 e^{-2r(\operatorname{Re}(\lambda)+\rho-\rho')} d(X',Z)\; dr\; d\omega \\
&= \int_{\overline{N}'\times\mathbb{R}_+\times S^1} |f(\overline{n}_{(X',t\omega,Z)})|^2 t^{2(\operatorname{Re}(\lambda)+\rho-\rho')-1} d(X',Z)\; dt\; d\omega, \\
&= \int_{\mathcal{O}_A} |f(\overline{n}_{(X',X_n,Z)})|^2 |X_n|^{2\operatorname{Re}(\lambda)} d(X,Z).
\end{aligned}$$

In particular this equation implies unitarity of the map for $\operatorname{Re}(\lambda) = 0$. Now by Lemma 1.4(ii), we have that

$$\overline{n}_{(X',X_n,Z)} = ke^{\frac{1}{2}\log((1+|X'|^2+|X_n|^2)^2+|Z|^2)}n \in KAN,$$

such that there exists a non-negative constant c_f such that

$$|f(\overline{n}_{(X',X_n,Z)})|^2 \leq c_f((1+|X'|^2+|X_n|^2)^2+|Z|^2)^{-(\operatorname{Re}(\lambda)+\rho)}.$$

Hence

$$\begin{aligned}
\|\Phi f\|_{L^2(G'/K_0')} \leq& c_f \int_{\overline{N}} ((1+|X'|^2+|X_n|^2)^2+|Z|^2)^{-(\operatorname{Re}(\lambda)+\rho)}|X_n|^{2\operatorname{Re}(\lambda)} d(X,Z) \\
=& \tilde{c}_f \int_{(\mathbb{R}_+)^3} ((1+r^2+s^2)^2+t^2)^{-(\operatorname{Re}(\lambda)+\rho)} r^{2n-3} s^{2\operatorname{Re}(\lambda)+1}\; dr\; ds\; dt,
\end{aligned}$$

where $\tilde{c}_f = 2\operatorname{Vol}(S^{2n-3})\operatorname{Vol}(S^1)c_f$. Then substituting t we have

$$\begin{aligned}
\|\Phi f\|_{L^2(G'/K_0')} \leq \tilde{c}_f \int_{\mathbb{R}_+} (1+t^2)^{-\operatorname{Re}(\lambda)+\rho}\; dt \\
\times \int_{(\mathbb{R}_+)^2} (1+r^2+s^2)^{-2(\operatorname{Re}(\lambda)+\rho)+1} r^{2n-3} s^{2\operatorname{Re}(\lambda)+1} dr\; ds.
\end{aligned}$$

Here the first integral converges for all $\mathrm{Re}(\lambda) > -\rho + \frac{1}{2} = -n - \frac{1}{2}$. Using polar coordinates on $(\mathbb{R}_+)^2$ we find the second integral to be equal to

$$\int_0^{\frac{\pi}{2}} \cos^{2n-3}\phi \sin^{2\,\mathrm{Re}(\lambda)+1}\phi \, d\phi \int_0^\infty x^{(\mathrm{Re}(\lambda)+n)-1}(1+x)^{-2(\mathrm{Re}(\lambda)+\rho)+1}\, dx,$$

which converges for all $\mathrm{Re}(\lambda) > -1$. □

Corollary 6.2. *(i) We have*

$$H(e^{-rH}\overline{n}_{(-X,0,-Z)}\overline{n}_{(Y,0,W)}) = rH + \log(N((X', e^{-r}, Z)\cdot(-Y,0,-W))^2)H.$$

(ii) We have for $X_n \in \mathbb{C}^\times$ and $g \in G'$ with $\overline{n}_{(X',X_n,Z)} \in g\overline{n}_{e_n}P$,

$$\mathrm{pr}_{\mathrm{U}(1)}(\kappa(g^{-1})) = \frac{\overline{X}_n(|X|^2 - Z)}{|X_n|N(X,Z)^2}.$$

Here $\mathrm{pr}_{\mathrm{U}(1)}$ denotes the projection to the $\mathrm{U}(1)$-factor of $K = \mathrm{U}(1) \times \mathrm{U}(n+1)$.

Proof. Ad (i): By Lemma 1.1(ii) we have

$$H(e^{-rH}\overline{n}_{(-X,0,-Z)}\overline{n}_{(Y,0,W)}) = H(\overline{n}_{(-e^rX,0,-e^{2r}Z)}\overline{n}_{(e^rY,0,e^{2r}W)}) - rH,$$

which is by Lemma 1.4(ii) equal to the stated formula.

Ad (ii): We have

$$\mathrm{diag}\left(\frac{X_n}{|X_n|}, \frac{X_n}{|X_n|}, \mathbf{1}_n\right)\kappa(g^{-1}) = \kappa\left(e^{\log|X_n|H}\overline{n}_{(-X',0,-Z)}\right) = \kappa\left(\overline{n}_{(-|X_n|^{-1}X',0,-|X_n|^{-2}Z)}\right).$$

Then the statement follows from Lemma 1.4(ii). □

Combining the results of this section we obtain the following Lemma.

Lemma 6.3. *Let $f \in \pi_{0,\lambda}$.*

$$A_{(0,\lambda),(0,\nu)}f(h) = \frac{1}{\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})}\int_{G'/K'_0} e^{-(\nu+\rho')H(g^{-1}h)}\Phi f(g)\, dg.$$

In particular the integral in the formula of the Lemma above is K'-invariant such that the spherical case it can be described on the Riemannian symmetric space G'/K' by the kernel $e^{-(\nu+\rho')H(g^{-1}*)}$, which is just the standard Fourier transform on G'/K'.

7 Harmonic analysis on the open orbit

We define $\tau_s : K' \to \mathbb{C}^\times$ to be the character given by $K' = \mathrm{U}(1) \times \mathrm{U}(n) \ni (k_0, k) \mapsto k_0^s$. We follow [Shi94] to obtain a spherical inversion formula on $L^2(G'/K'_0)$. Then we define the left and right K'-equivariant functions

$$\phi_{s,\nu}(g) := \int_{K'} e^{(\nu-\rho')H(g^{-1}k)}\tau_s(k^{-1}\kappa(g^{-1}k))\, dk.$$

Then clearly $\phi_{s,\lambda} \in C^\infty(G'/K', \mathcal{L}_s)$, where $\mathcal{L}_s = G' \times_{K'} \tau_s$ is the homogeneous bundle induced by τ_s.

We define the s-spherical transform for $f \in C_c^\infty(G'/K', \mathcal{L}_s)$

$$\hat{f}(s,\nu) := \int_{G'/K'} \phi_{-s,-\nu}(g) f(g)\, dg,$$

and the c-function

$$c(s,\nu) := \frac{2^{\rho'-\nu}(\rho'-1)!\Gamma(\nu)}{\Gamma(\frac{\nu+\rho'+s}{2})\Gamma(\frac{\nu+\rho'-s}{2})}.$$

Shimeno obtained the following s-spherical inversion formula for $\mathrm{U}(1,n)$.

Theorem 7.1 ([Shi94] Theorem 8.3)**.** *Let $D_s' = \{\nu \in \mathbb{R} : \nu = \rho' + 2j - |s| + \in \mathbb{Z}_{<0},\, j \in \mathbb{Z}_{\geq 0}\}$. For a function $f \in^{\tau_{-s}} (C_c^\infty(G'/K', \mathcal{L}_s))$*

$$f(\mathbf{1}) = \frac{1}{4\pi}\int_{i\mathbb{R}} \hat{f}(s,\nu)\frac{d\nu}{|c(s,\nu)|^2} + \sum_{\nu \in D_s} \hat{f}(s,\nu)\,\mathrm{Res}_{\mu=\nu}\left((c(s,\mu)c(s,-\mu))^{-1}\right).$$

We remark that in the formula of the theorem above in [Shi94] the discrete summands occur with a factor of $-i$ which is due to the different interpretation of the residues at $\nu \in D_s$ contrary to $i\nu \in iD_s$.

Lemma 7.2.

$$\phi_{s,\nu}(g^{-1}h) = \int_{K'} \tau_s(\kappa(h^{-1}k))e^{(\nu-\rho')H(h^{-1}k)}\tau_{-s}(\kappa(g^{-1}k))e^{-(\nu+\rho')H(g^{-1}k)}\, dk.$$

Proof. First note that $h^{-1}gk = h^{-1}\kappa(gk)e^{H(gk)}n$, and since A normalizes N' we have

$$\kappa(h^{-1}gk) = \kappa(h^{-1}\kappa(gk)), \qquad H(h^{-1}gk) = H(gk) + H(h^{-1}\kappa(gk)), \tag{7.1}$$

such that

$$\phi_{s,\nu}(g^{-1}h) = \int_{K'} \tau_{-s}(\kappa(g^{-1}\kappa(gk)))\tau_s(\kappa(h^{-1}\kappa(gk))e^{(-\nu+\rho')H(g^{-1}\kappa(gk))}e^{(\nu-\rho')H(h^{-1}\kappa(gk))}\, dk.$$

By the formula

$$\int_{K'} F(\kappa(gk))\, dk = \int_{K'} F(k)e^{-2\rho H(g^{-1}k)}\, dk$$

we obtain the Lemma. ☐

We define the following integral transform whenever it exists. For $f \in C_c^\infty(G'/K', \mathcal{L}_s)$:

$$\tilde{f}(s,\nu,k) := \int_{G'/K'} e^{-(\nu+\rho')H(g^{-1}k)}\tau_{-s}(\kappa(g^{-1}k))f(g)\, dg.$$

Then since M' normalizes N' and commutes with A we have for $m \in M'$ and $n \in N'$:

$$\tilde{f}(s,\nu,kmn) = \tau_{-s}(m)\tilde{f}(s,\nu,k).$$

By (7.1) we have for $r \in \mathbb{R}$:

$$\tilde{f}(s,\nu,ke^{rH}) = e^{-(\nu+\rho')r}\tilde{f}(s,\nu,k),$$

such that $\tilde{f}(s,\nu,\cdot\,) \in \tau_{s,\nu}$.

Corollary 7.3 (Inversion formula)**.** *For $f \in C_c^\infty(G'/K',\mathcal{L}_s)$ we have*

$$f(g) = \frac{1}{4\pi}\int_{i\mathbb{R}}\int_{K'} e^{(\nu-\rho')H(g^{-1}k)}\tau_s(\kappa(g^{-1}k))\tilde{f}(s,\nu,k)\,dk\frac{d\nu}{|c(s,\nu)|^2}$$
$$+\sum_{\nu\in D_s}\int_{K'} e^{(\nu-\rho')H(g^{-1}k)}\tau_s(\kappa(g^{-1}k))\tilde{f}(s,\nu,k)\,dk\,\mathrm{Res}_{\mu=\nu}\left((c(s,\mu)c(s,-\mu))^{-1}\right)$$

and

$$\|f\|^2_{L^2(G')} = \frac{1}{4\pi}\int_{i\mathbb{R}}\int_{K'}\|\tilde{f}(s,\nu,k)\|^2_{L^2(K')}\frac{d\nu}{|c(s,\nu)|^2}$$
$$+\sum_{\nu\in D_s}(\tilde{f}(s,\nu,k),\tilde{f}(s,-\nu,k))_{L^2(K')}\,dk\,\mathrm{Res}_{\mu=\nu}\left((c(s,\mu)c(s,-\mu))^{-1}\right).$$

Proof. Let $f \in C_c^\infty(G'/K_0',\mathcal{L}_s)$ and define for $g\in G'$

$$f'(g') := \int_{K'}\tau_s(k)f(gkg')\,dk.$$

Then clearly $f'(\mathbf{1}) = f(g)$ and $f' \in^{\tau_{-s}} (C_c^\infty(G'/K',\mathcal{L}_s))$, such that by Theorem 7.1

$$f(g) = \frac{1}{4\pi}\int_{i\mathbb{R}}\hat{f}'(s,\nu)\frac{d\nu}{|c(s,\nu)|^2}+\sum_{\nu\in D_s}\hat{f}'(s,\nu)\,\mathrm{Res}_{\mu=\nu}\left((c(s,\mu)c(s,-\mu))^{-1}\right).$$

Now

$$\hat{f}'(s,\nu) = \int_{G'}\int_{K'}\phi_{-s,-\nu}(g')\tau_s(k)f(gkg')\,dk\,dg'$$
$$= \int_{G'} f(gg')\phi_{-s,-\nu}(g')\,dg' = \int_{G'} f(gg')\phi_{s,\nu}(g'^{-1})\,dg' = \int_{G'} f(g')\phi_{s,\nu}(g'^{-1}g)\,dg'.$$

By Lemma 7.2 this is

$$\int_{K'} e^{(\nu-\rho')H(g^{-1}k)}\tau_s(\kappa(g^{-1}k))\tilde{f}(s,\nu,k)\,dk. \qquad \square$$

Note that the integrand of the integral in the inversion formula in Corollary 7.3 is $K\cap M'$ invariant such that using the formula (see [Kna86, Chapter V.6])

$$\int_{K'} f(k)\,dk = \int_{\overline{N}'} f(\kappa(\overline{n}))e^{-2\rho' H(\bar{n})}\,d\overline{n},$$

we obtain the inversion formula in the non-compact picture to be

$$f(g) = \frac{1}{4\pi}\int_{i\mathbb{R}}\int_{\overline{N}'} e^{(\nu-\rho')H(g^{-1}\overline{n})}\tau_s(\kappa(g^{-1}\overline{n}))\tilde{f}(s,\nu,\overline{n})\,d\overline{n}\frac{d\nu}{|c(s,\nu)|^2}$$
$$+\sum_{\nu\in D_s}\int_{\overline{N}'} e^{(\nu-\rho')H(g^{-1}\overline{n})}\tau_s(\kappa(g^{-1}\overline{n}))\tilde{f}(s,\nu,\overline{n})\,d\overline{n}\,\mathrm{Res}_{\mu=\nu}\left((c(s,\mu)c(s,-\mu))^{-1}\right). \tag{7.2}$$

Considering the Fourier summands of f we obtain the maps

$$L^2(G'/K_0') \to L^2(G'/K',\mathcal{L}_t),$$

$$f \mapsto \hat{f}_t,$$

with

$$\hat{f}_t(g) := \int_{\mathrm{U}(1)} f(g\,\mathrm{diag}(\omega,\omega,\mathbf{1}_{n-1}))\omega^t\,d\omega$$

Then for $f \in C_c^\infty(\mathcal{O}_A)$ we define

$$\Phi_t f(g) := \widehat{(\Phi f)}_t(g) = \int_{\mathrm{U}(1)} f(g\,\mathrm{diag}(\omega,\omega,\mathbf{1}_{n-1})\overline{n}_{e_n})\omega^t\,d\omega.$$

Now similar to Lemma 6.3 we obtain the following by Corollary 6.2.

Lemma 7.4. *Let $f \in \pi_{s,\lambda}$ with $\mathrm{Re}(\lambda) > -1$. Then $A^{s,t}_{\lambda,\nu} f(g) = \widetilde{\Phi_t f}(t,\nu,g)$.*

In this sense the symmetry breaking operator $A^{s,t}_{\lambda,\nu}$ can be defined in terms of Fourier transforms on G'/K'.

For functions f, f' on G/P we define the pairing

$$(f,f') = \int_K f(g)f'(g)dg.$$

Then assuming $f, f' \in C_c^\infty(G/P, \mathcal{V}_{\xi,\lambda})$,

$$\langle f,f'\rangle_{s,\lambda} := \begin{cases} (f,\overline{f'}) & \text{if } \lambda \in i\mathbb{R}, \\ (f,\tilde{T}_{s,\lambda}\overline{f'}) & \text{if } \lambda \in \mathbb{R}-\{0\}, \end{cases}$$

defines an inner-product $\|\cdot\|_{s,\lambda}$ such that $\pi_{s,\lambda}$ is a unitary representation for all $\lambda \in i\mathbb{R}$ and all $\lambda \in \mathbb{R}$ if $\pi_{s,\lambda}$ belongs to the complementary series and on all unitarizable quotients $\pi^{\sqsubset}_{s,\lambda}$ and $\pi^{\llcorner}_{s,\lambda}$ for $\lambda \in \mathbb{R}$.

For G' it will be more convenient to work with $\tau^{\llcorner}_{t,\nu}$ and $\tau^{\sqsubset}_{t,\nu}$ realized as submodules of $\tau_{t,\nu}$ We therefore introduce a renormalization

$$\tilde{\tilde{T}}'_{t,\nu} := \frac{\Gamma(\frac{\nu+\rho'+t}{2})\Gamma(\frac{\nu+\rho'-t}{2})}{2^{1-\nu-\rho'}\pi^{\rho'}} T'_{t,\nu}.$$

We abuse notation to introduce a pairing $(\cdot,\cdot)$ for functions on G'/P' and define the pairing $\langle\cdot,\cdot\rangle_{t,\nu}$ for $f, f' \in C_c^\infty(G'/P', \mathcal{W}_{\eta,\nu})$, with $\eta \in \widehat{\mathrm{U}}(1)$ by

$$\langle f,f'\rangle_{t,\nu} = \begin{cases} (f,\overline{f'}) & \text{if } \nu \in i\mathbb{R}, \\ (f,\tilde{\tilde{T}}'_{t,\nu}\overline{f'}) & \text{if } \nu \in \mathbb{R}-\{0\}. \end{cases}$$

Then this pairing is a scalar product on $\tau_{t,\nu}$ for $\nu \in i\mathbb{R}$ and whenever $\tau_{t,\nu}$ are complementary series representations and on all unitary submodules. We define the following holomorphic functions in $\lambda, \in \mathbb{C}$ for $s \in \mathbb{Z}$:

$$t(s,\lambda) := \begin{cases} \frac{2^{1+\lambda-\rho}\pi^\rho}{\Gamma(\frac{-\lambda+\rho+|s|}{2})\Gamma(-\frac{\lambda}{2}+1)} & \text{if } |s| \geq \rho \text{ and } |s|+\rho \equiv 0 \mod 2, \\ \frac{2^{1+\lambda-\rho}\pi^\rho}{\Gamma(\frac{-\lambda+\rho+|s|}{2})\Gamma(-\frac{\lambda-1}{2})} & \text{if } |s| \geq \rho \text{ and } |s|+\rho \equiv 1 \mod 2, \\ \frac{2^{1+\lambda-\rho}\pi^\rho}{\Gamma(\frac{-\lambda+\rho+s}{2})\Gamma(\frac{-\lambda+\rho-s}{2})} & \text{otherwise,} \end{cases}$$

Then in particular by Theorem 4.6, for all $s,t \in \mathbb{Z}$, $(\lambda,\nu) \in \mathbb{C}^2$,

$$A^{s,t}_{-\lambda,\nu} \circ \tilde{T}_{s,\lambda} = t(s,\lambda) A^{s,t}_{\lambda,\nu}, \qquad \tilde{\tilde{T}}'_{t,\nu} \circ A^{s,t}_{\lambda,\nu} = A^{s,t}_{\lambda,-\nu}. \tag{7.3}$$

Moreover for $\nu = -\rho' - 2j + |t| \geq 0$, the composition

$$\tilde{\tilde{T}}'_{t,\nu} \circ \tilde{A}_{\lambda,\nu}$$

is holomorphic in λ, since by Lemma 4.2 ,$\tilde{\tilde{T}}'_{t,\nu}$ has poles only on K'-types on which by (5.6), $\tilde{A}^{s,t}_{\lambda,\nu}$ vanishes globally anyway.

We further define the meromorphic functions

$$c(s,\lambda,t,\nu) := \frac{c(t,\nu)c(t,-\nu)}{\Gamma\left(\frac{\lambda+\rho+\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{-\lambda+\rho+\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{-\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)},$$

$$c_{\sqsubset}(s,\lambda,t,\nu) := \frac{\Gamma\left(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)}{\operatorname{Res}_{\mu=\nu}(c(t,\mu)^{-1}c(t,-\mu)^{-1})\Gamma\left(\frac{\lambda+\rho+\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{-\lambda+\rho+\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{-\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)},$$

$$c_{\llcorner}(s,\lambda,t,\nu) := \frac{c(t,\nu)c(t,-\nu)\Gamma\left(\frac{\nu+\rho'-t}{2}\right)^2\Gamma\left(\frac{\nu+\rho'+t}{2}\right)^2}{\Gamma\left(\frac{\lambda+\rho+\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{-\lambda+\rho+\nu-\rho'+|s-t|}{2}\right)\Gamma\left(\frac{-\lambda+\rho-\nu-\rho'+|s-t|}{2}\right)}.$$

By comparing the Gamma factors in the formulas of $c(s,\lambda,t,\nu)$ and $c_{\sqsubset}(s,\lambda,t,\nu)$ we immediately obtain the following result about the regularity of the inverse of these functions.

Corollary 7.5. *(i) For* $\mathrm{Re}(\nu) = 0$,

$$\frac{1}{c(s,\lambda,t,\nu)}$$

is a regular function for all $\mathrm{Re}(\lambda) \in (-1-|s-t|, 1+|s-t|)$.

(ii) For $-\nu \in D'_t$,

$$c_{\sqsubset}(s,\lambda,t,\nu)^{-1}$$

is a regular function for all $\mathrm{Re}(\lambda) < 0$.

Now applying Corollary 7.3 to the norms $\|\hat{f}_s\|_\lambda$ we obtain the following Plancherel formulas.

Lemma 7.6. *For $\lambda \in i\mathbb{R}$ and $f \in \pi_{s,\lambda}$*

$$\|\hat{f}_t\|_{s,\lambda} = \frac{1}{4\pi}\int_{i\mathbb{R}} \|A^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} \frac{d\nu}{c(s,\lambda,t,\nu)} + \sum_{-\nu \in D'_t} \|\tilde{A}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} \frac{1}{c_{\sqsubset}(s,\lambda,t,\nu)},$$

And for $\lambda \in (-1,1) - \{0\}$

$$\|\hat{f}_t\|_{s,\lambda} = \frac{1}{4\pi}\int_{i\mathbb{R}} \|A^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu + \sum_{-\nu \in D'_t} \|\tilde{A}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} \frac{t(s,\lambda)}{c_{\sqsubset}(s,\lambda,t,\nu)}.$$

Proof. The first part is clear by Corollary 7.3, Lemma 6.1 and Lemma 7.4 by the functional equations for $\tilde{\tilde{T}}'_{t,\nu}$. For the second part, first clearly $\langle \hat{f}_t, f\rangle_{s,\lambda} = \langle \hat{f}_t, \hat{f}_t\rangle_{s,\lambda}$. If $\lambda \in (-1,1) - \{0\}$, then both $\hat{f}_t$ and $\tilde{T}_{s,\lambda} f$ can be considered as $L^2(G'/K'_0)$-functions such that in this case

$$\|\hat{f}_t\|^2_{s,\lambda} = \frac{1}{4\pi}\int_{i\mathbb{R}} (A^{s,t}_{\lambda,\nu} f, A^{s,t}_{-\lambda,\nu} \circ \tilde{T}_{s,\lambda} f)_{L^2(K')} \frac{d\nu}{c(s,\lambda,t,\nu)} + \sum_{-\nu \in D'_t} (A^{s,t}_{\lambda,\nu} f, \tilde{\tilde{T}}'_{t,\nu} \circ \tilde{A}^{s,t}_{-\lambda,\nu} \circ \tilde{T}_{s,\lambda} f)_{L^2(K')} \frac{1}{c_{\sqsubset}(s,\lambda,t,\nu)}.$$

Then using the functional equations (7.3) implies the statement. □

8 Analytic continuation

Consider the Plancherel formula of Lemma 7.6. On $G/P \cong K/M$, the left hand side written as $(\hat{f}_t, \tilde{T}_{s,\lambda}\overline{\hat{f}_t})$ depends holomorphically on λ for fixed s, t. Writing the norms of the right hand side similarly in terms of the K'-pairing $(\cdot,\cdot)$, the right hand side depends at least meromorphically on λ. Hence the goal of this section is to analytically continue the right hand side to obtain a Plancherel formula on $i\mathbb{R} \cup (-\infty, 0)$ which is valid on the complementary series and on unitary composition factors, including the unitary highest and lowest K'-weight representations $\pi^{\sqsubset}_{s,\lambda}$.

Proposition 8.1. *For $\lambda \in (-\infty, 0)$ and $f \in \pi_{s,\lambda}$*

$$\begin{aligned}\|\hat{f}_t\|_{s,\lambda} = {} & \frac{1}{4\pi}\int_{i\mathbb{R}} \|A^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu + \sum_{-\nu \in D'_t} \|\tilde{A}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu}\, , \frac{t(s,\lambda)}{c_{\sqsubset}(s,\lambda,t,\nu)} \\ & + \sum_{k \in [0, \frac{-\lambda-1-|s-t|}{2}) \cap \mathbb{Z}} \|C^{s,t}_{\lambda,-\lambda-1-|s-t|-2k} f\|^2_{t,\nu} \operatorname{Res}_{\mu=-\lambda-1-|s-t|-2k} \left(\frac{t(s,\lambda)}{c_{\llcorner}(s,\lambda,t,\mu)}\right).\end{aligned}$$

Proof. Let $\lambda \in (-1, 0)$. We rewrite the formula of Lemma 7.6 as

$$(\hat{f}_t, \tilde{T}_{s,\lambda}\overline{\hat{f}_t}) = \frac{1}{4\pi}\int_{i\mathbb{R}} (A^{s,t}_{\lambda,\nu} f, A^{-s,-t}_{\lambda,-\nu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu + \sum_{-\nu \in D'_t} (\tilde{A}^{s,t}_{\lambda,\nu} f, \tilde{\tilde{T}}'_{t,\nu} \tilde{A}^{-s,-t}_{\lambda,\nu}\overline{f}) \frac{t(s,\lambda)}{c_{\sqsubset}(s,\lambda,t,\nu)}. \tag{8.1}$$

By Lemma 5.7 and Corollary 7.5, the sum on the right hand side of (8.1) is already holomorphic in $\lambda \in (-\infty, 0)$ such that we only need to continue the integral part of the formula. By Corollary 7.5, the integral is already holomorphic for $\lambda \in (-1-|s-t|, 0)$. We prove the statement by induction. Consider the Theorem holding for $\lambda \in (-|s-t|-2k-1, -|s-t|-2k)$. Then

$$\frac{1}{c(s,\lambda,t,\nu)}$$

has exactly one simple pole for $\mathrm{Re}(\nu) \in (0,1)$ at $\nu = \lambda+1+|s-t|+2k$. Hence moving the contour of integration from $i\mathbb{R}$ to $i\mathbb{R}+1$, we have

$$\begin{aligned} &\int_{i\mathbb{R}} (A^{s,t}_{\lambda,\nu} f, A^{-s,-t}_{\lambda,-\nu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu \\ &\qquad = \int_{i\mathbb{R}+1} (A^{s,t}_{\lambda,\nu} f, A^{-s,-t}_{\lambda,-\nu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu \\ &\qquad - 2\pi \operatorname{Res}_{\mu=\lambda+1+|s-t|+2k} \left((A^{s,t}_{\lambda,\mu} f, \tilde{\tilde{T}}'_{t,\mu} A^{-s,-t}_{\lambda,\mu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\mu)} \right), \end{aligned} \tag{8.2}$$

where along the lines $\mu = \lambda+1+|s-t|+2k$,

$$\begin{aligned} &\operatorname{Res}_{\mu=\lambda+1+|s-t|+2k} \left((A^{s,t}_{\lambda,\mu} f, \tilde{T}'_{t,\mu} A^{-s,-t}_{\lambda,\mu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\mu)} \right) \\ &\qquad = -\operatorname{Res}_{\mu=-\lambda-1-|s-t|-2k} \left((A^{s,t}_{\lambda,\mu} f, \tilde{T}'_{t,\mu} A^{-s,-t}_{\lambda,\mu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\mu)} \right) \\ &\qquad = -\|C^{s,t}_{\lambda,-\lambda-1-|s-t|-2k} f\|^2_{t,\nu} \operatorname{Res}_{\mu=-\lambda-1-|s-t|-2k} \left(\frac{t(s,\lambda)}{c_{\llcorner}(s,\lambda,t,\mu)} \right). \end{aligned}$$

Now for $\mathrm{Re}(\nu) = 1$, $c(s,\lambda,t,\nu)^{-1}$ is a regular function in λ for $\lambda \in (-|s-t|-2k-2, -|s-t|-2k)$ such that (8.2) defines an analytic continuation of (8.1) onto $(-|s-t|-2k-2, 0)$. Let $\lambda \in (-|s-t|-2k-2, -|s-t|-2k-1]$. Then for $\mathrm{Re}(\nu) \in (0,1)$, $c(s,\lambda,t,\nu)^{-1}$ has exactly one simple pole at $-\nu = \lambda+1+|s-t|+2k$. Hence moving the contour of integration of the integral of (8.2) we have

$$\begin{aligned} &\int_{i\mathbb{R}+1} (A^{s,t}_{\lambda,\nu} f, A^{-s,-t}_{\lambda,-\nu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu \\ &\qquad = \int_{i\mathbb{R}} (A^{s,t}_{\lambda,\nu} f, A^{-s,-t}_{\lambda,-\nu}\overline{f}) \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu \\ &\qquad + 2\pi \operatorname{Res}_{\mu=-\lambda-1-|s-t|-2k} \left((A^{s,t}_{\lambda,\mu} f, \tilde{T}'_{t,\mu} A^{-s,-t}_{\lambda,\mu}\overline{f}) \frac{t(s,\lambda)}{t'(t,\mu) c(s,\lambda,t,\mu)} \right). \end{aligned}$$

Again for $\mathrm{Re}(\nu) = 0$, $c(s,\lambda,t,\nu)^{-1}$ is a regular function in λ for $\lambda \in (-|s-t|-2k-3, -|s-t|-2k-1)$ such that (8.2) defines an analytic continuation of (8.1) onto $(-|s-t|-2k-3, 0)$. $\square$

By Lemma 5.10, we can rewrite the statement of Proposition 8.1 for $\lambda - |t| + |s-t| \in -\rho - 2\mathbb{Z}_{\geq 0}$, using the operator $\tilde{\tilde{A}}^{s,t}_{\lambda,\nu}$.

Corollary 8.2. *Let* $(\lambda + |s-t| - |t|, \nu - |t|) = (-\rho - 2i, -\rho' - 2j) \in L$ *with* $\nu \geq 0$ *and* $|t| - j - \rho' \leq i$. *Then*

$$\left(\frac{\Gamma(\frac{\lambda+\rho-\nu-\rho'+|s-t|}{2})}{\Gamma(\frac{\nu+\rho'+t}{2})\Gamma(\frac{\nu+\rho'-t}{2})}\right)^2 \|C^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} = \|\tilde{A}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} + \|\tilde{\tilde{A}}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu}$$

Proof. The Corollary follows immediately from Corollary 5.9. □

Corollary 8.3. *For* $\lambda - |t| + |s-t| \in -\rho - 2\mathbb{Z}_{\geq 0}$ *and* $f \in \pi_{s,\lambda}$

$$\begin{aligned}\|\hat{f}_t\|_{s,\lambda} = \frac{1}{4\pi}\int_{i\mathbb{R}} \|A^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} \frac{t(s,\lambda)}{c(s,\lambda,t,\nu)}\, d\nu + \sum_{\substack{-\nu\in D'_t,\\ \nu>-\lambda-1-|s-t|}} \|\tilde{A}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu}\, , \frac{t(s,\lambda)}{c_{\sqsubset}(s,\lambda,t,\nu)} \\ + \sum_{\substack{-\nu\in D'_t,\\ \nu\leq-\lambda-1-|s-t|}} \left(2\|\tilde{A}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu} + \|\tilde{\tilde{A}}^{s,t}_{\lambda,\nu} f\|^2_{t,\nu}\right) \frac{t(s,\lambda)}{c_{\sqsubset}(s,\lambda,t,\mu)} \\ + \sum_{\substack{k\in[0,\frac{-\lambda-1-|s-t|}{2})\cap\mathbb{Z},\\ \lambda+1+|s-t|+2k\notin D'_t}} \|C^{s,t}_{\lambda,-\lambda-1-|s-t|-2k} f\|^2_{t,\nu} \operatorname{Res}_{\mu=-\lambda-1-|s-t|-2k}\left(\frac{t(s,\lambda)}{c_{\llcorner}(s,\lambda,t,\mu)}\right).\end{aligned}$$

8.1 Proof of the main theorem

Form the Plancherel formulas of Proposition 8.1 we obtain the following unitary branching laws.

Theorem 8.4. *(i) For the unitary principal series,* $\lambda \in i\mathbb{R}$, *we have*

$$\pi_{s,\lambda}|_{G'} \cong \bigoplus_{t\in\mathbb{Z}} \left(\int^{\oplus}_{i\mathbb{R}} \tau_{t,\nu}\, d\nu \oplus \bigoplus_{-\nu\in D'_t} \tau^{\sqsubset}_{t,\nu}\right).$$

(ii) For the complementary series (see Corollary 3.3) we have

$$\pi_{s,\lambda}|_{G'} \cong \bigoplus_{t\in\mathbb{Z}} \left(\int^{\oplus}_{i\mathbb{R}} \tau_{t,\nu}\, d\nu \oplus \bigoplus_{-\nu\in D'_t} \tau^{\sqsubset}_{t,\nu}\right) \oplus \bigoplus_{t=-\rho'+1}^{\rho'-1} \left(\bigoplus_{k\in[0,\frac{-\lambda-1-|s-t|}{2})\cap\mathbb{Z}} \tau_{t,-\lambda-1-|s-t|-2k}\right).$$

(iii) For $i \in \mathbb{Z}_{\geq 0}$ *and* $\lambda = -\rho - 2i \pm s \leq 0$, *we have for the unitary quotients*

$$\begin{aligned}\pi^{\llcorner}_{s,\lambda}|_{G'} \cong \bigoplus_{t\in\mathbb{Z}} \left(\int^{\oplus}_{i\mathbb{R}} \tau_{t,\nu}\, d\nu \oplus \bigoplus_{-\nu\in D'_t} \tau^{\sqsubset}_{t,\nu}\right) \oplus \bigoplus_{t\in\mathbb{Z},\,|t|\geq\rho'} \bigoplus_{k\in[0,\frac{-\lambda-1-|s-t|}{2})\cap\mathbb{Z}} \tau^{\llcorner}_{t,-\lambda-1-|s-t|-2k} \\ \oplus \bigoplus_{t=-\rho'+1}^{\rho'-1} \left(\bigoplus_{k\in[0,\frac{-\lambda-\rho-|s-t|+|t|}{2}]\cap\mathbb{Z}} \tau^{\llcorner}_{t,-\lambda-1-|s-t|-2k} \right. \\ \left. \oplus \bigoplus_{k\in(\frac{-\lambda-\rho-|s-t|+|t|}{2},\frac{-\lambda-1-|s-t|}{2})\cap\mathbb{Z}} \tau_{t,-\lambda-1-|s-t|-2k} \right).\end{aligned}$$

(iv) For $i \in \mathbb{Z}_{\geq 0}$ and $\lambda = \rho + 2i \mp s \leq 0$, we have for the unitary highest resp. lowest weight representations $\pi^{\sqsubset}_{s,\lambda}$ the discrete branching law

$$\pi^{\sqsubset}_{s,\lambda}|_{G'} \cong \bigoplus_{a=\pm s-i}^{\infty} \bigoplus_{j=0}^{i} \tau^{\sqsubset}_{j+a,\rho'+j-a}.$$

Proof. Ad(i). The statment follows already from Lemma 7.6.

Ad(ii). For $\lambda > -1$, the statement also follows already from Lemma 7.6, hence we can assume $|s| \leq \rho$. Then in particular

$$-\lambda - 1 - |s-t| < \rho' - |t|$$

for all $s, t \in \mathbb{Z}$, such that for $k \in [0, \frac{-\lambda-1-|s,t|}{2})$, the operator $C^{s,t}_{\lambda,-\lambda-1-|s-t|-2k}$ always maps into the complementary series which implies the statement.

Ad(iii). Let $\lambda = -\rho - 2i \pm s \leq 0$. Then assuming $\nu = -\lambda - 1 - |s-t| - 2k$ with $k \in [0, \frac{-\lambda-1-|s-t|}{2})$ and $(\lambda + |s-t| - |t|, \nu - |t|) \in L$ implies that $\nu = -\rho' - 2j + |t|$ with

$$|t| - j - \rho' \leq i - \frac{|s-t| + s - |t|}{2}.$$

Since all of the operators in the formula of Corollary 8.3 do not vanish for arbitrarily high K-types (α_1, α_2), and since the image of $\tilde{A}^{s,t}_{\lambda,\nu}$ is $\tau^{\sqsubset}_{t,\nu}$ while the image of $\tilde{\tilde{A}}^{s,t}_{\lambda,\nu}$ is $\tau^{\llcorner}_{t,\nu}$, we obtain the stated identity by sorting representations into complementary series and submodules.

Ad(iv). Let $\lambda = \rho + 2i - s = -\rho - 2\tilde{i} + s \leq 0$ with $\tilde{i} = s - \rho - i \geq i$. The quotient $\pi^{\sqsubset}_{s,\lambda}$ is realized on K-types with $\alpha_1 \leq i$. Then by Corollary 5.11, $A^{s,t}_{\lambda,\nu}$ vanishes for all $\alpha_1 \leq \tilde{i}$, such that the integral does not contribute to the restriction $\pi^{\sqsubset}_{s,\lambda}|_{G'}$. Moreover the images of the operators $\tilde{\tilde{A}}^{s,t}_{\lambda,\nu}$ for $-\nu \in D'_t$ and $C^{s,t}_{\lambda,\nu}$ for $-\nu \notin D'_t$ is $\tau^{\llcorner}_{t,\nu}$ which contains arbitrary high K' types in both directions, such that both cannot contribute to the decomposition of $\pi^{\sqsubset}_{s,\lambda}$ since its K'-types are bounded in one direction. Now $\tilde{A}^{s,t}_{\lambda,\nu}$ vanishes for all $\alpha_1 \leq \tilde{i}$ if $(\lambda+|s-t|-|t|, \nu-|t|) \notin L$. Let $t \geq 0$ and $\nu = -\rho'-2j+t$ such that $(\lambda+|s-t|-t, \nu-t) \in L$, which is equivalent to

$$j \leq i, \quad \text{if } s \geq t, \qquad j - t \leq i - s, \quad \text{if } s < t.$$

Moreover for $\tilde{A}^{s,t}_{\lambda,\nu}$ to define an non-vanishing operator from the quotient $\pi^{\sqsubset}_{s,\lambda}|_{G'}$ to the submodule $\tau^{\sqsubset}_{t,\nu}$, for every K'-type (α'_1, α'_2) contained in $\tau^{\sqsubset}_{t,\nu}$ (i.e. $\alpha'_1 \leq j$) there must exist a K-type (α_1, α_2) contained in $\pi^{\sqsubset}_{s,\lambda}$ (i.e. $\alpha_1 \leq i$) such that $\alpha_1 - \alpha_2 - \alpha'_1 + \alpha'_2 = s - t$ and $\alpha'_1 \leq \alpha_1$, $\alpha'_2 \leq \alpha_2$ which holds if and only if $j \leq i - (s,t)_+$, i.e.

$$j \leq i, \quad \text{if } s < t, \qquad j - t \leq i - s, \quad \text{if } s \geq t,$$

such that in particular both conditions

$$j \leq i, \quad j - t \leq i - s \tag{8.3}$$

must be satisfied for all $t \geq 0$. Moreover by (5.6), the image of $\tilde{A}^{s,t}_{\lambda,\nu}$ is $\tau^{\sqsubset}_{t,\nu}$. In all of these cases the eigenvalue $\tilde{a}^{s,t}_{\lambda,\nu}((j + (s,t)+, \alpha_2), (j, \alpha'_2)) \neq 0$ such that all remaining operators do in fact not vanish for a K-type with $\alpha_1 \leq i$.

Let $t < 0$. Then $\tau^{\sqsubset}_{t,\nu}$ containes K'-types (α'_1, α'_2) with $\alpha'_1 \in \mathbb{Z}_{\geq 0}$ arbitrary while $\pi^{\sqsubset}_{s,\lambda}$ only containes K-types (α_1, α_2) with $\alpha_1 \leq i$. Hence $\tilde{A}^{s,t}_{\lambda,\nu}$ never defines an non-trivial intertwining operator between the quotient $\pi^{\sqsubset}_{s,\lambda}|_{G'}$ and $\tau^{\sqsubset}_{t,\nu}$. Moreover assuming the conditions (8.3) to hold for $\lambda = \rho + 2i - s \leq 0$ implies

$$j - t \leq i - s \leq -\rho - i < 0$$

which is a contradiction for $t < 0$. Further for $t \geq 0$ and $\lambda = \rho + 2i - s$ and j, t satisfying (8.3), we obtain

$$j - t \leq i - s \leq -\rho - i \leq -\rho - j,$$

such that $t \geq \rho$ and every $-\rho' - 2j + t \in -D'_t$. Then introducing the notation $a = t - j$ implies the formula for $s \geq 0$. For $s < 0$ the argument works similarly. $\square$

Remark 8.5. The result for unitary highest weight modules Theorem 8.4(iv) was obtained before in a more general setting by Kobayashi in [Kob08, Theorem 8.11] under the isomorphism

$$\pi^{\sqsubset}_{s,\rho+2i-s} \cong \pi^{\mathrm{U}(n+1,1)}_{\mu},$$

with $\mu = (i, 0, \ldots, 0, i - s) \in \mathbb{Z}^{n+2}$ in the notation of the article.

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Paper C

Branching of unitary $\mathrm{O}(1, n+1)$-representations with non-trivial $(\mathfrak{g}, K)$-cohomology

Clemens Weiske

Abstract

Let $G = \mathrm{O}(1, n+1)$ with maximal compact subgroup K and let Π be an irreducible unitary representation of G with non-trivial $(\mathfrak{g}, K)$-cohomology. Then Π occurs inside a principal series representation of G induced from the $\mathrm{O}(n)$-representation $\bigwedge^p(\mathbb{C}^n)$ and characters of a minimal parabolic subgroup of G at the limit of the complementary series. Considering the subgroup $G' = \mathrm{O}(1, n)$ of G with maximal compact subgroup K', we prove branching laws and explicit Plancherel formulas for the restrictions to G' of all unitary representations occurring in such principal series, including the complementary series, all unitary G-representations with non-trivial $(\mathfrak{g}, K)$-cohomology and further relative discrete series representations in the cases $p = 0, n$. Discrete spectra are constructed explicitly as residues of G'-intertwining operators which resemble the Fourier transforms on vector bundles over the Riemannian symmetric space G'/K'.

Introduction

Unitary representations of reductive Lie groups with non-trivial $(\mathfrak{g}, K)$-cohomology appear in several branches of mathematics, for example in the theory of locally symmetric spaces where for a Lie-group G with finite center, maximal compact subgroup K and discrete cocompact subgroup Γ, by the Matsushima–Murakami formula (see [BW80, VII, Theorem 3.2])

$$H^*(\Gamma \backslash G/K, \mathbb{C}) = \bigoplus_{\pi \in \widehat{G}} m(\Gamma, \pi) H^*(\mathfrak{g}, K; \pi_K)$$

the cohomology of $\Gamma\backslash G/K$ is given by $(\mathfrak{g}, K)$-cohomologies of unitary representations of G with multiplicities, which are essentially the dimensions of spaces of automorphic forms on $\Gamma\backslash G/K$. All representations with non-trivial $(\mathfrak{g}, K)$-cohomology are constructed and their cohomologies calculated in [VZ84]. The unitary ones are well known for the indefinite orthogonal group $\mathrm{O}(1, n+1)$ and classified for example for $\mathrm{GL}(n, \mathbb{R})$ (see [Spe83]). In our case of interest $\mathrm{O}(1, n+1)$ the unitary cohomological representations occur as limits of complementary series representations in the Fell topology on the unitary dual and on the automorphic dual in the sense of Burger–Sarnak [BS91]. Restrictions of these representations are of particular importance in the latter setting, since in [BS91] it is proven that the restriction of automorphic representations to certain subgroups is again automorphic for the subgroup. In [SV11] it is proven that the restrictions of the cohomological representations of $\mathrm{O}(1,n+1)$ to $\mathrm{O}(1,n)$ contain a certain cohomological representation of the subgroup discretely. The main result of this article is the full branching law for the cohomological representations of $\mathrm{O}(1,n+1)$ restricted to $\mathrm{O}(1,n)$, extending the result of [SV11].

For an irreducible unitary representation π of a reductive Lie group G which is typically infinite dimensional the restriction to a subgroup G' decomposes into a direct integral

$$\pi|_{G'} \cong \int_{\widehat{G'}}^{\oplus} m(\pi,\tau)\tau \, d\mu_\pi(\tau)$$

with a certain measure $d\mu_\pi$ on the unitary dual $\widehat{G}'$ of G' and possibly infinite multiplicities $m(\pi,\tau)$. Only in special cases the support of the measure is discrete and in general it might contain a continuous and a discrete part. Many special cases have been studied recently using analytic methods (e.g. [Kob19], [ØS19], [MO15]). In our case the cohomological representations of $\mathrm{O}(1,n+1)$ can be realized as quotients of principal series representations induced from the $\mathrm{O}(n)$-representation $\bigwedge^p(\mathbb{C}^n)$ at the limit of the complementary series. To obtain the direct integral decomposition of the cohomological representations we prove branching laws for the unitary principal series and use an analytic continuation procedure to extend the result onto the complementary series and towards the cohomological representations. In particular we obtain branching laws for the complementary series and also for all other unitarizable quotients which occur within these principal series. More precisely we collect discrete components in the decomposition as residues of G'-intertwining operators, so called *symmetry breaking operators* by Kobayashi [Kob15] and we make use of the detailed classification and study of these operators in the relevant case by Kobayashi–Speh [KS18].

Main results

Let $G = \mathrm{O}(1,n+1)$, $n > 1$ and let $P = MAN \subseteq G$ be a minimal parabolic subgroup. Then $M \cong \mathrm{O}(1) \times \mathrm{O}(n)$. Consider the representation

$$\left(\alpha \otimes \bigwedge^p(\mathbb{C}^n)\right) \otimes e^\lambda \otimes \mathbf{1}$$

of MAN on the vector space $V^{\pm}_{p,\lambda}$ where we use the superscript $+$ if α is the trivial irreducible $\mathrm{O}(1)$ representation and $-$ if it is the non-trivial one and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ which we identify by $\mathbb{C}$ by mapping the half sum of all positive roots ρ to $\frac{n}{2}$. Let $\pi^{\pm}_{p,\lambda}$ be the principal series representation of G on the smooth sections of the homogeneous bundle

$$G \times_P V^{\pm}_{p,\lambda+\rho} \to G/P$$

over the real flag variety G/P. Our normalization is chosen such that $\pi^{\pm}_{p,\lambda}$ is unitary for $\lambda \in i\mathbb{R}$ and such that $\pi^{\pm}_{p,\lambda}$ contains a unique submodule $\Pi_{p,\pm}$ whose underlying $(\mathfrak{g},K)$-module has non-trivial $(\mathfrak{g},K)$-cohomology for $\lambda = p - \rho$.

Let $G' = \mathrm{O}(1,n)$ embedded in G such that $P' = G' \cap P$ is a minimal parabolic subgroup of G'. Similarly we consider the $P' = M'AN'$ representation

$$\left(\alpha \otimes \bigwedge^q(\mathbb{C}^{n-1})\right) \otimes e^\nu \otimes \mathbf{1}$$

on the vector space $W^{\pm}_{q,\nu}$ and denote by $\tau^{\pm}_{q,\nu}$ the principal series representation which is given by the smooth sections of the bundle

$$G' \times_{P'} W^{\pm}_{q,\nu+\rho'} \to G'/P',$$

where ρ' is the obvious and under the identification above equal to $\frac{n-1}{2}$. Our normalization is again such that the unitary principal series is given on the imaginary axis and such that $\tau^{\pm}_{q,\nu}$ contains a cohomological representation $\Pi'_{q,\pm}$ as a submodule for $\nu = q - \rho'$.

For G and G' we denote the unitary closure of unitarizable representations π in the following by $\hat{\pi}$. For the unitary principal series we prove the following branching laws. For the uniform formulation for all $p = 0, \dots, n$ we set $\hat{\tau}^{\pm}_{q,\nu} = \{0\}$ for $q = -1, n$.

Theorem A (Branching laws for the unitary principal series (see Lemma 12.2))**.** *For $\lambda \in i\mathbb{R}$ and $p \neq \frac{n}{2}$ we have*

$$\hat{\pi}^{\pm}_{p,\lambda}|_{G'} \cong \bigoplus_{\alpha=+,-} \bigoplus_{q=p-1,p} \int^{\oplus}_{i\mathbb{R}} \hat{\tau}^{\alpha}_{q,\nu}\, d\nu.$$

For $\lambda \in i\mathbb{R}$ and $p = \frac{n}{2}$ we have

$$\hat{\pi}^{\pm}_{p,\lambda}|_{G'} \cong \widehat{\Pi}'_{\frac{n}{2},+} \oplus \widehat{\Pi}'_{\frac{n}{2},-} \oplus \bigoplus_{\alpha=+,-} \bigoplus_{q=p-1,p} \int^{\oplus}_{i\mathbb{R}} \hat{\tau}^{\alpha}_{q,\nu}\, d\nu.$$

If $p \neq \rho = \frac{n}{2}$, there is a complementary series. More precisely $\pi^{\pm}_{p,\lambda}$ is a complementary series representation if and only if $\lambda \in (-|\rho - p|, |\rho - p|)$, for which we also prove unitary branching laws and where complementary series of G' occur discretely. We formulate the result only for the negative half of the complementary series. The result for the positive parameters follows by duality.

Theorem B (see Theorem 13.6)**.** *For $\lambda \in (-|\rho - p|, 0)$ we have*

$$\hat{\pi}^{\pm}_{p,\lambda}|_{G'} \cong \bigoplus_{\alpha=+,-} \bigoplus_{q=p-1,p} \left(\int^{\oplus}_{i\mathbb{R}} \hat{\tau}^{\alpha}_{q,\nu}\, d\nu \oplus \bigoplus_{k \in [0, \frac{-\lambda-1+(\alpha\frac{1}{2})}{2}) \cap \mathbb{Z}} \hat{\tau}^{\alpha}_{q,\lambda+\frac{1}{2}+2k} \right).$$

Moreover we prove unitary branching laws for the unitarizable quotients $\Pi_{p,\pm}$ whose underlying $(\mathfrak{g}, K)$-modules have non-trivial $(\mathfrak{g}, K)$-cohomology, sitting as quotients at the limit of the complementary series. Here complementary series as well as cohomological representations occur in the discrete spectrum.

Theorem C (see Theorem 13.7)**.** *(i) For the one dimensional quotients we have*

$$\widehat{\Pi}_{0,\pm}|_{G'} \cong \widehat{\Pi}'_{0,\pm}, \qquad \widehat{\Pi}_{n+1,\pm}|_{G'} \cong \widehat{\Pi}'_{n,\pm}.$$

(ii) For $0 < p \leq \frac{n}{2}$ we have

$$\widehat{\Pi}_{p,\pm}|_{G'} \cong \widehat{\Pi}'_{p,\pm} \oplus \bigoplus_{k \in (0, \rho'-p+1) \cap \mathbb{Z}} \tau^{\pm(-1)^k}_{p-1,p-1-\rho'+k} \oplus \bigoplus_{\alpha=+,-} \int^{\oplus}_{i\mathbb{R}} \hat{\tau}^{\alpha}_{p-1,\nu}\, d\nu.$$

(iii) For n odd and $p = \frac{n+1}{2}$ we have

$$\widehat{\Pi}_{\frac{n+1}{2},\pm}|_{G'} \cong \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{\frac{n-1}{2},\nu}\, d\nu.$$

(iv) For $\frac{n+1}{2} < p \leq n$ we have

$$\widehat{\Pi}_{p,\pm}|_{G'} \cong \widehat{\Pi}'_{p-1,\pm} \oplus \bigoplus_{k\in(0,p-1-\rho')\cap\mathbb{Z}} \tau^{\pm(-1)^k}_{p-1,\rho'-p+1+k} \oplus \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{p-1,\nu}\, d\nu.$$

Speh–Venkataramana proved the inclusions

$$\begin{aligned} &\widehat{\Pi}'_{p,\pm} \subseteq \widehat{\Pi}_{p,\pm}|_{G'} && p < \frac{n+1}{2}, \\ &\widehat{\Pi}'_{p-1,\pm} \subseteq \widehat{\Pi}_{p,\pm}|_{G'} && p > \frac{n+1}{2}, \end{aligned}$$

(see [SV11, Theorem 1.4]) and the theorem above gives the full decomposition of the cohomological representations $\widehat{\Pi}_{p,\pm}|_{G'}$.

For $p \neq 0, n$, the representations $\Pi_{p,\pm}$ are the only proper unitarizable composition factors of $\pi^{\pm}_{p,\lambda}$. For $p = 0, n$ there are additional unitarizable composition factors $I_{p,j,\pm}$ for each positive integer j, occuring as quotients in $\pi^{\pm}_{p,\lambda}$ for $\lambda = -\rho - j$. We prove branching laws for the closures of these representations as well. Here complementary series, a cohomological representation, as well as the corresponding quotients for the subgroup $I'_{q,k,\pm}$, with $q = 0, n-1$ and k positive integers occur discretely.

Theorem D (see Theorem 13.8). *(i) For $p = 0$ we have*

$$\hat{I}_{0,j,\pm}|_{G'} \cong \widehat{\Pi}'_{1,\mp} \oplus \bigoplus_{k=1}^{j} \hat{I}'_{0,k,\pm(-1)^{k+j}} \oplus \bigoplus_{k\in(0,\rho')\cap\mathbb{Z}} \hat{\tau}^{\pm(-1)^{k+j}}_{0,-\rho'+k} \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{0,\nu}\, d\nu.$$

(ii) For $p = n$ we have

$$\hat{I}_{n,j,\pm}|_{G'} \cong \widehat{\Pi}'_{n-1,\pm} \oplus \bigoplus_{k=1}^{j} \hat{I}'_{n-1,k,\pm(-1)^{k+j}} \oplus \bigoplus_{k\in(0,\rho')\cap\mathbb{Z}} \hat{\tau}^{\pm(-1)^{k+j}}_{n-1,-\rho'+k} \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{n-1,\nu}\, d\nu.$$

For all branching laws we obtain an explicit Plancherel theorem (see Corollary 13.5). We remark that the decompositions in the spherical case, i.e. $n = 0$ have been proven before by Möllers–Oshima in [MO15] by a different approach which is likely to generalize to arbitrary p. Our method of proof offers a more systematically approach which does not rely on the nilradical to be abelian.

Method of proof

The subgroup G' acts on the real flag variety G/P with an open orbit which is as a G'-space given by a $\mathbb{Z}/2\mathbb{Z}$-fibration over the Riemannian symmetric space G'/K', where K'

is the maximal compact subgroup of G' (see Proposition 2.3 and Lemma 9.1). Restriction to the open orbit naturally induces a G'-map

$$\Phi : \pi^{\pm}_{p,\lambda} \to L^2\left(G'/K', \bigwedge^p(\mathbb{C}^n)\right)$$

if $\operatorname{Re}\lambda > -\frac{1}{2}$ (see Lemma 9.4). The Plancherel and inversion formula for the space $L^2(G'/K', \bigwedge^p(\mathbb{C}^n))$ is essentially due to [Cam18]. It is given in terms of Fourier transforms on $L^2(G'/K', \bigwedge^p(\mathbb{C}^n))$, which are G'-intertwining maps into principal series $\tau^{\pm}_{q,\nu}$. By composition of the map Φ and the Fourier transforms we obtain elements of the space $\operatorname{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\pm}_{q,\nu})$ of symmetry breaking operators, which are in this special case classified by Kobayashi–Speh in [KS18]. The symmetry breaking operators we obtain in this procedure are given by families of integral kernel operators with meromorphic dependence on λ and ν and the meromorphic structure of the operators is studied in [KS18] in great detail. This allows us to carefully analytically continue the Plancherel formula of $L^2(G'/K', \bigwedge^p(\mathbb{C}^n))$ in λ over the critical point $\lambda = -\frac{1}{2}$ on the real axis, towards the complementary series and unitarizable quotients $\Pi_{p,\pm}$.

Structure of this article

In Section 1 we recall some facts for symmetry breaking operators between principal series representations and establish the necessary notation for principal series representations of G in Section 2. In Section 3 we discuss the restriction to the identity component of the representations in question which will be used for arguments later in the article. In Section 4 and Section 5 we study the composition series of the principal series representations and give criteria for reducibility and unitarizability. In Section 6 we recall the classification of symmetry breaking operators between $\pi^{\pm}_{p,\lambda}|_{G'}$ and $\tau^{\pm}_{q,\nu}$ from [KS18] as well as their meromorphic structure. We recall functional equations of symmetry breaking operators and the standard Knapp–Stein intertwining operators in Section 7 and extend them to operators into quotients of principal series representations in Section 8. We establish the structure of the open G' orbit in G/P as a homogeneous G'-space in Section 9 and prove a Plancherel formula for the corresponding space in Section 11 using the Plancherel formula for the restriction to the connected component of [Cam18] (Section 10). We lift these results to the representation $\pi^{\pm}_{p,\lambda}$ around the unitary axis in Section 12. The main result here is Theorem 12.1, by which the Fourier transform on the homogeneous G'-space is essentially given by symmetry breaking operators classified by Kobayashi–Speh on the principal series. Finally Section 13 is dedicated to the proof of the main theorems where we analytically continue the Plancherel formula around the unitary axis towards the complementary series and the unitary composition factors.

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Notation

For two sets $B \subseteq A$ we use the Notation $A - B = \{a \in A : a \notin B\}$. We denote Lie groups by Roman capitals and their corresponding Lie algebras by the corresponding Fraktur lower cases.

1 Symmetry breaking operators between principal series representations

We recall the basic facts about symmetry breaking operators between principal series representations from [KS18].

1.1 Principal series representations

Let G be a real reductive Lie group and P a minimal parabolic subgroup of G with Langlands decomposition $P = MAN$. For a finite-dimensional representation (ξ, V) of M, a character $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and the trivial representation $\mathbf{1}$ of N we obtain a finite-dimensional representation $(\xi \otimes e^{\lambda} \otimes \mathbf{1}, V_{\xi,\lambda})$ of $P = MAN$. By smooth normalized parabolic induction this representation gives rise to the principal series representation

$$\pi_{\xi,\lambda} := \operatorname{Ind}_P^G(\xi \otimes e^{\lambda} \otimes \mathbf{1})$$

as the left-regular representation of G on the space

$$\{\varphi \in C^\infty(G, V) : \ \varphi(gman) = \xi(m)^{-1} a^{-(\lambda+\rho)} \varphi(g) \ \forall man \in MAN\},$$

where $\rho := \frac{1}{2} \operatorname{tr} \operatorname{ad}|_{\mathfrak{n}} \in \mathfrak{a}_{\mathbb{C}}^*$. Let $\mathcal{V}_{\xi,\lambda} := G \times_P V_{\xi,\lambda+\rho} \to G/P$ be the homogeneous vector bundle associated to $V_{\xi,\lambda+\rho}$. Then $\pi_{\xi,\lambda}$ identifies with the left-regular action of G on the space of smooth sections $C^\infty(G/P, \mathcal{V}_{\xi,\lambda})$.

Now let $G' < G$ be a reductive subgroup. Similarly we let $P' = M'A'N'$ be a minimal parabolic subgroup of G'. For a finite-dimensional representation (η, W) of M' and $\nu \in (\mathfrak{a}'_{\mathbb{C}})^*$ we obtain a finite-dimensional representation $(\eta \otimes e^{\nu} \otimes \mathbf{1}, W_{\eta,\lambda})$ of P' and the corresponding principal series representation

$$\tau_{\eta,\nu} := \operatorname{Ind}_{P'}^{G'}(\eta \otimes e^{\nu} \otimes \mathbf{1}).$$

Again we identify $\tau_{\eta,\nu}$ with the smooth sections $C^\infty(G'/P', \mathcal{W}_{\eta,\nu})$ of the homogeneous vector bundle $\mathcal{W}_{\eta,\nu} := G' \times_{P'} W_{\eta,\nu+\rho'} \to G'/P'$, where $\rho' := \frac{1}{2} \operatorname{tr} \operatorname{ad}|_{\mathfrak{n}'}$.

1.2 Symmetry breaking operators

In these realizations the space of symmetry breaking operators between $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$ is given by the continuous linear G'-maps between the smooth sections of the two homogeneous vector bundles

$$\operatorname{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu}) = \operatorname{Hom}_{G'}(C^\infty(G/P, \mathcal{V}_{\xi,\lambda})), C^\infty(G'/P', \mathcal{W}_{\eta,\nu})).$$

The Schwartz Kernel Theorem implies that every such operator is given by a G'-invariant distribution section of the tensor bundle $\mathcal{V}_{\xi^*,-\lambda} \boxtimes \mathcal{W}_{\eta,\nu}$ over $G/P \times G'/P'$, where ξ^* is the representation contradigent to ξ. Since G' acts transitively on G'/P' we can consider these distributions as sections on G/P with a certain P'-invariance:

Theorem 1.1 ([KS15, Proposition 3.2])**.** *There is a natural bijection*

$$\operatorname{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu}) \xrightarrow{\sim} (\mathcal{D}'(G/P, \mathcal{V}_{\xi^*,-\lambda}) \otimes W_{\eta,\nu+\rho'})^{P'}, \quad T \mapsto u^T.$$

In our case of interest the dimension of $\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu})$ is in particular bounded by 1.

Theorem 1.2 ([SZ12] Theorem B)**.** *For $(G, G') = (\mathrm{O}(1, n+1), \mathrm{O}(1, n))$ we have*

$$\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \leq 1$$

for all irreducible Casselman–Wallach representations π of G and τ of G'.

1.3 Restriction to the open Bruhat cell

From now on assume $M' = M \cap G'$, $A' = A \cap G'$ and $N' = N \cap G'$. Let $\overline{N}$ be the nilradical of the parabolic opposite to P. Since $\overline{N}$ is unipotent we obtain a parameterization of the open Bruhat cell $\overline{N}P/P \subseteq G/P$ in terms of the Lie algebra $\bar{\mathfrak{n}}$ by the map

$$\bar{\mathfrak{n}} \xrightarrow{\exp} \overline{N} \hookrightarrow G \longrightarrow G/P,$$

so that we can consider $\bar{\mathfrak{n}}$ as an open dense subset of G/P. Then the restriction

$$\mathcal{D}'(G/P, \mathcal{V}_{\xi^*,-\lambda}) \longrightarrow \mathcal{D}'(\bar{\mathfrak{n}}, \mathcal{V}_{\xi^*,-\lambda}|_{\bar{\mathfrak{n}}})$$

can be used to define a $\mathfrak{g}$-action on $\mathcal{D}'(\bar{\mathfrak{n}}, \mathcal{V}_{\xi^*,-\lambda}|_{\bar{\mathfrak{n}}}) \cong \mathcal{D}'(\bar{\mathfrak{n}}) \otimes V_{\xi^*,-\lambda+\rho}$ by vector fields. Moreover, since $\mathrm{Ad}(M'A')$ leaves $\bar{\mathfrak{n}}$ invariant, the restriction is further $M'A'$-equivariant. If we assume $P'\overline{N}P = G$, i.e. every P'-orbit in G/P meets the open Bruhat cell $\overline{N}P$, then symmetry breaking operators can be described in terms of $(M'A', \mathfrak{n}')$-invariant distributions on $\bar{\mathfrak{n}}$:

Theorem 1.3 ([KS15, Theorem 3.16])**.** *Assume $P'\overline{N}P = G$, then there is a natural bijection*

$$\mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu}) \xrightarrow{\sim} (\mathcal{D}'(\bar{\mathfrak{n}}) \otimes V_{\xi^*,-\lambda+\rho} \otimes W_{\eta,\nu+\rho'})^{M'A',\mathfrak{n}'}.$$

Given a distribution kernel u^T, the corresponding operator $T \in \mathrm{Hom}_{G'}(\pi_{\xi,\lambda}|_{G'}, \tau_{\eta,\nu})$ is given by

$$T\varphi(h) = \langle u^T, \varphi(h\exp(\,\cdot\,))\rangle. \tag{1.1}$$

2 Principal series representations of rank one orthogonal groups

Let $G = \mathrm{O}(1, n+1)$ denote the group of $(n+2) \times (n+2)$ matrices over $\mathbb{R}$ preserving the quadratic form

$$(z_0, z_1, \ldots, z_{n+1}) \mapsto -|z_0|^2 + |z_1|^2 + \cdots + |z_{n+1}|^2.$$

Let P be the minimal parabolic subgroup of G with Langlands decomposition $P = MAN$ given by

$$M = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix} : a \in \mathrm{O}(1), b \in \mathrm{O}(n) \right\},$$

$$A = \exp(\mathfrak{a}) \qquad \text{where } \mathfrak{a} = \mathbb{R}H, \quad H = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathbf{0}_n \end{pmatrix},$$

$$N = \exp(\mathfrak{n}) \qquad \text{where } \mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & X \\ 0 & 0 & X \\ X^T & -X^T & \mathbf{0}_n \end{pmatrix} : X \in \mathbb{R}^n \right\}.$$

We identify $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ by $\lambda \mapsto \lambda(H)$. Then in particular

$$\rho = \frac{1}{2} \operatorname{tr} \operatorname{ad}|_{\mathfrak{n}}(H) = \frac{n}{2}.$$

Consider the finite dimensional representations

$$\xi = \alpha \otimes \sigma_p,$$

of $M = \mathrm{O}(1) \times \mathrm{O}(n)$, with $\alpha \in \{\mathbf{1}, \mathrm{sgn}\} \cong \widehat{\mathrm{O}}(1)$ and $\sigma_p = \bigwedge^p (\mathbb{C}^n)$ with $p \in \{0, \dots, n\}$. We define the principal series representations

$$\pi^{\pm}_{p,\lambda} := \mathrm{Ind}(\xi \otimes e^{\lambda} \otimes \mathbf{1})$$

where we use the index $+$ if $\alpha = \mathbf{1}$ and $-$ if $\alpha = \mathrm{sgn}$.

Similarly we consider the finite dimensional M' representations

$$\eta = \alpha \otimes \delta_q$$

with α as above and $\delta_q = \bigwedge^q (\mathbb{C}^{n-1})$ with $q \in \{0, \dots, n-1\}$ and denote the corresponding prinicpal series representations by $\tau^{\pm}_{q,\nu}$.

2.1 The non-compact picture

Let $\overline{N}$ be the nilradical of the parabolic subgroup opposite to P. Since $\overline{N}$ is unipotent, we identify it with its Lie algebra $\bar{\mathfrak{n}} \cong \mathbb{R}^n$ in terms of the exponential map:

$$\mathbb{R}^n \to \overline{N}, \quad X \mapsto \bar{n}_X := \exp \begin{pmatrix} 0 & 0 & X \\ 0 & 0 & -X \\ X^* & X^* & \mathbf{0}_n \end{pmatrix}.$$

Since $\overline{N}P$ is open and dense in G, the restriction of $\pi^{\pm}_{p,\lambda}$ to functions on $\overline{N}$ is one-to-one. The resulting realization in $C^\infty(\bar{\mathfrak{n}})$ of $\pi^{\pm}_{p,\lambda}$ is called the *non-compact picture of* $\pi^{\pm}_{p,\lambda}$. For $g \in \overline{N}MAN$ we write $g = \bar{n}(g)m(g)a(g)n(g)$ for the obvious decomposition. Then the G-action in the non-compact picture is given by

$$\pi^{\pm}_{p,\lambda}(g)f(X, Z) = \xi^{-1}(m(g^{-1}\bar{n}_X))a(g^{-1}\bar{n}_X)^{-(\lambda+\rho)}f(\log \bar{n}(g^{-1}\bar{n}_X)), \tag{2.1}$$

whenever $g^{-1}\overline{n}_X \in \overline{N}MAN$.

Let $\tilde{w}_0 = \mathrm{diag}(-1, 1, \mathbf{1}_n)$, then $\tilde{w}_0$ represents the longest Weyl group element of G with respect to A. The following Lemma is easily verified by standard calculations.

Lemma 2.1. *(i) Let $m = \mathrm{diag}(a, a, b^{-1}) \in M$ with $a \in \mathrm{O}(1)$ and $b \in \mathrm{O}(n)$, then*

$$m\bar{n}_X m^{-1} = \bar{n}_{aXb}.$$

(ii) Let $t \in \mathbb{R}$ and $a = \exp(tH)$, then

$$a\bar{n}_X a^{-1} = \bar{n}_{e^{-t}X}.$$

(iii) Let $X \neq 0$, then $\tilde{w}_0\bar{n}_X = \bar{n}_U man$ with $n \in N$ and

$$U = \frac{-X}{|X|^2}, \qquad a = \exp(2\log(|X|)H).$$

$$m = \mathrm{diag}\left(-1, -1, \psi_n(X)\right),$$

with

$$\psi_n(X) = \mathbf{1}_n - \frac{2X^T X}{|X|^2} \in \mathrm{O}(n)$$

.

These decompositions immediately imply the following formulas for the action of P and $\tilde{w}_0$:

Proposition 2.2. *(i) For $m = \mathrm{diag}(a^{-1}, a^{-1}, b) \in M$ with $a \in \mathrm{O}(1)$ and $b \in \mathrm{O}(n)$:*

$$\pi^{\pm}_{p,\lambda}(m)u(X) = \xi^{-1}(m)u(aXb).$$

(ii) For $t \in \mathbb{R}$ and $a = \exp(tH)$:

$$\pi^{\pm}_{p,\lambda}(a)u(X) = e^{(\lambda+\rho)t}u(e^t X).$$

(iii) For $Y \in \mathbb{R}^n$:

$$\pi^{\pm}_{p,\lambda}(\bar{n}_Y)u(X) = u(X - Y).$$

(iv) For the action of $\tilde{w}_0$ we have

$$\pi_\lambda(\tilde{w}_0)u(X) = \xi^{-1}(\mathrm{diag}(-1, -1, \psi_n(X)))|X|^{-2(\lambda+\rho)}u(\sigma(X)).$$

where $\sigma : \bar{\mathfrak{n}} - \{0\} \to \bar{\mathfrak{n}} - \{0\}$ is the inversion given by

$$\sigma(X) = \frac{-X}{|X|^2}.$$

Note that $X \in \mathbb{R}^n$ is a row vector, so that matrix multiplication is from the right.

2.2 Orbit structure of G/P

By [FW20, Proposition 2.9], the P'-orbits in G/P are given by the following.

Proposition 2.3. *The P'-orbits in G/P and their closure relations are*

$$\mathcal{O}_A \overset{1}{\text{——}} \mathcal{O}_B \overset{n-1}{\text{——}} \mathcal{O}_C,$$

where

$$\begin{aligned}\mathcal{O}_A &= P' \cdot \tilde{w}_0 \bar{n} P = \tilde{w}_0(\overline{N} - \overline{N}')P,\\ \mathcal{O}_B &= P' \cdot \tilde{w}_0 P = \tilde{w}_0 \overline{N}' P,\\ \mathcal{O}_C &= P' \cdot \mathbf{1}_{n+2} P,\end{aligned}$$

for some $\bar{n} \in \overline{N} - \overline{N}'$. Here $X \overset{k}{\text{——}} Y$ means that Y is a subvariety of $\bar{X}$ of co-dimension k.

In particular the orbit $\mathcal{O}_A$ is open in G/P.

3 The component group G/G_0

We study the restriction of representations of G to the identity component G_0.

3.1 Global characters of $\mathrm{O}(1, n + 1)$

$G = \mathrm{O}(1, n+1)$ is a disconnected group with four connected components and the identity component G_0 is isomorphic to $\mathrm{SO}_0(1, n + 1)$ and the component group $G/G_0 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence there are four global characters $\chi_{\pm,\pm}$ of G which are restricted to the subgroup $M = \mathrm{O}(1) \times \mathrm{O}(n)$ given by

$$\chi_{+,+}|_M = \mathbf{1} \otimes \mathbf{1}, \qquad \chi_{+,-}|_M = \mathbf{1} \otimes \det,$$

$$\chi_{-,+}|_M = \mathrm{sgn} \otimes \mathbf{1}, \qquad \chi_{-,-}|_M = \mathrm{sgn} \otimes \det.$$

We remark that $\chi_{+,-}$ is the determinant on G.

Note that as $\mathrm{O}(n)$-representations we have

$$\bigwedge\nolimits^p (\mathbb{C}^n) \cong \bigwedge\nolimits^{n-p} (\mathbb{C}^n) \otimes \det$$

such that for the principal series representation $\pi^{\pm}_{p,\lambda}$ we have

$$\chi_{+,-} \otimes \pi^{\pm}_{p,\lambda} \cong \pi^{\pm}_{n-p,\lambda}, \qquad \chi_{-,+} \otimes \pi^{\pm}_{p,\lambda} \cong \pi^{\mp}_{p,\lambda}, \qquad \chi_{-,-} \otimes \pi^{\pm}_{p,\lambda} \cong \pi^{\mp}_{n-p,\lambda}.$$

3.2 Restriction to the identity component

The following lemma is similar to [KS18, Lemma 15.2].

Lemma 3.1. *Let π be an irreducible admissible representation of G.*

(i) If $\chi \otimes \pi \not\cong \pi$ for all $\chi \in \{\chi_{+,-}, \chi_{-,+}, \chi_{-,-}\}$ then $\pi|_{G_0}$ is irreducible.

(ii) If $\chi_0 \otimes \pi \cong \pi$ *for* $\chi_0 \in \{\chi_{+,-}, \chi_{-,+}, \chi_{-,-}\}$ *and* $\chi \otimes \pi \ncong \pi$ *for all* $\chi_0 \neq \chi \in \{\chi_{+,-}, \chi_{-,+}, \chi_{-,-}\}$, *then* $\pi|_{G_0} = \pi^{(+)} \oplus \pi^{(-)}$ *decomposes into two non-isomorphic irreducible* G_0 *representations* $\pi^{(+)}$ *and* $\pi^{(-)}$

In the following we denote by

$$\bar{\pi}^{\pm}_{p,\lambda} := \pi^{\pm}_{p,\lambda}|_{G_0}$$

the restriction to the identity component. We immediately obtain the following

Lemma 3.2. *(i) If* $p \neq \frac{n}{2}$, *the restriction* $\bar{\pi}^{\pm}_{p,\lambda}$ *is irreducible as* G_0*-representation if and only if* $\pi^{\pm}_{p,\lambda}$ *is irreducible as a* G*-representation. If* $\pi^{\pm}_{p,\lambda}$ *is reducible, the composition series of* $\bar{\pi}^{\pm}_{p,\lambda}$ *is given by the composition factors of* $\pi^{\pm}_{p,\lambda}$ *restricted to* G_0.

(ii) If $p = \frac{n}{2}$, *the restriction* $\bar{\pi}^{\pm}_{p,\lambda} = \bar{\pi}^{\pm(+)}_{p,\lambda} \oplus \bar{\pi}^{\pm(-)}_{p,\lambda}$ *is always reducible and decomposes into two non-isomorphic* G_0*-representations* $\bar{\pi}^{\pm(+)}_{p,\lambda}$ *and* $\bar{\pi}^{\pm(-)}_{p,\lambda}$. *The representations* $\bar{\pi}^{\pm(\pm)}_{p,\lambda}$ *are irreducible if and only if* $\pi^{\pm}_{p,\lambda}$ *is irreducible. If* $\pi^{\pm}_{p,\lambda}$ *is reducible, the composition series of* $\bar{\pi}^{\pm(\pm)}_{p,\lambda}$ *is given by the composition factors of* $\pi^{\pm}_{p,\lambda}$ *restricted to* G_0, *which are contained in* $\bar{\pi}^{\pm(\pm)}_{p,\lambda}$.

We use the corresponding notation $\bar{\tau}^{\pm}_{q,\nu}$ and $\bar{\tau}^{\pm(\pm)}_{q,\nu}$ for the components of the restriction to $G'_0 \cong \mathrm{SO}_0(1,n)$.

4 Composition series of $\pi^{\pm}_{p,\lambda}$

We recall results about the composition series of $\pi^{\pm}_{p,\lambda}$ and give explicit realizations as kernels of the standard Knapp–Stein intertwining operators.

4.1 Irreducibility of principal series representations

Theorem 4.1 ([KS18] Theorem 2.18)**.** $\pi^{\pm}_{p,\lambda}$ *is reducible if and only if*

$$\lambda \in (-\rho - 1 - \mathbb{Z}_{\geq 0}) \cup (\rho + 1 + \mathbb{Z}_{\geq 0}) \cup \{\rho - p, -\rho + p\}.$$

4.2 The Knapp–Stein intertwining operator

The composition series is closely connected to the Knapp–Stein intertwining operators which we introduce in this section. Following [KS18, Chapter 8] we define the (normalized) Knapp–Stein intertwining operator $T_{p,\lambda}$ as an element of $\mathcal{D}'(\mathbb{R}^n) \otimes \mathrm{End}_{\mathbb{C}}(\xi)$ by

$$T_{p,\lambda} = \begin{cases} \frac{1}{\Gamma(\lambda)} |X|^{2(\lambda-\rho)} \sigma_p(\psi_n(X)) & \text{if } p \neq \frac{n}{2}, \\ \frac{1}{\Gamma(\lambda+1)} |X|^{2(\lambda-\rho)} \sigma_p(\psi_n(X)) & \text{if } p = \frac{n}{2}. \end{cases}$$

This defines a non-vanishing holomorphic family of intertwining operators

$$T_{p,\lambda} : \pi^{\pm}_{p,\lambda} \to \pi^{\pm}_{p,-\lambda}.$$

The composition series of $\pi^{\pm}_{p,\lambda}$ is given in the following proposition which is [KS18, Proposition 2.18 and Proposition8.17]

Proposition 4.2. *If $\pi^{\pm}_{p,\lambda}$ is reducible it has composition series of length two and if additionally $\lambda \neq 0$ it has a unique irreducible submodule given by $\ker T_{p,\lambda}$. If $\lambda = 0$ and $\pi^{\pm}_{p,\lambda}$ is reducible (hence $p = \frac{n}{2}$) it decomposes into the direct sum of two irreducible representations which are given by*

$$\ker\left(T_{p,0} \pm \frac{\pi^p}{p!}\,\mathrm{id}\right).$$

Similarly we denote by $T'_{q,\nu}$ the Knapp–Stein intertwining operator for the subgroup G', similarly defined and normalized.

5 Unitary representations in the principal series of O$(1, n+1)$

The principal series representation $\pi^{\pm}_{p,\lambda}$ is the smooth vectors of a tempered unitary representation if and only if $\lambda \in i\mathbb{R}$ and hence unitarizable (unitary principal series) and we denote its unitary closure by $\hat{\pi}^{\pm}_{p,\lambda}$. But for real parameters the principal series representation might be non-tempered unitarizable and irreducible (complementary series) or contain unitarizible composition factors which might be tempered or not.

5.1 Criterion for unitarizability

Lemma 5.1 (See [KS18] Example 3.32)**.** *$\pi^{\pm}_{p,\lambda}$ is a complementary series representation if and only if*

$$\min(-\rho + p, \rho - p) < \lambda < \max(-\rho + p, \rho - p).$$

The following Proposition can easily be deduced using the calculations in [BÓØ96, Chapter 3.a].

Proposition 5.2. *Let $p \neq 0, \frac{n}{2}, n$. $\pi^{\pm}_{p,\lambda}$ contains a unitarizable composition factor if and only if $\lambda \in \{\rho - p, p - \rho\}$. In this case both the unique submodule as well as the unique quotient are unitarizable. If $p = \frac{n}{2}$, $\pi^{\pm}_{p,\lambda}$ contains a unitarizable composition factor if and only if $\lambda = 0$, In this case both submodules are unitarizable. If $p = 0, n$, $\pi^{\pm}_{p,\lambda}$ contains a unitarizable composition factor if and only if $\lambda \in \{\pm(\rho + j), j \in \mathbb{Z}_{\geq 0}\}$. If $\lambda = \rho + j$, the quotient $\pi^{\pm}_{p,\lambda}/\ker T_{p,\lambda}$ is unitarizable and if $\lambda = -\rho - j$ the submodule $\ker T_{p,\lambda}$ is unitarizable. Only in the special case $\lambda = \pm\rho$, also the other composition factor is unitarizable and one dimensional in this case.*

5.2 Composition factors with non-trivial $(\mathfrak{g}, K)$-cohomology

We introduce notation for the unitarizible composition factors. For $p \neq \frac{n}{2}$, $0 < p < n$ let $\Pi_{p,\pm}$ be the unique proper submodule of $\pi^{\pm}_{p,p-\rho}$ and for $p = \frac{n}{2}$ let

$$\Pi_{\frac{n}{2},\pm} := \ker\left(T_{\frac{n}{2},0} - \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!}\,\mathrm{id}\right) \subseteq \pi^{\pm}_{\frac{n}{2},0}.$$

Moreover let

$$\Pi_{0,+} := \chi_{+,+}, \qquad \Pi_{0,+} := \chi_{-,+}, \qquad \Pi_{n+1,+} := \chi_{+,-}, \qquad \Pi_{n+1,-} := \chi_{-,-}.$$

These four one-dimensional representations correspond to the unique finite dimensional unitarizable composition factors for $p = 0, n$.

The following results are all due to [KS18, Theorem 2.20]

Theorem 5.3. *(i) For $0 \le p \le n$ we have the following exact sequences of G-modules.*

$$0 \to \Pi_{p,\pm} \to \pi^{\pm}_{p,p-\rho} \to \Pi_{p+1,\mp} \to 0,$$

$$0 \to \Pi_{p+1,\mp} \to \pi^{\pm}_{p,\rho-p} \to \Pi_{p,\pm} \to 0.$$

These sequences splits if and only if $p = \frac{n}{2}$.

(ii) The set $\{\Pi_{p,\alpha},\ 0 \le p \le n+1,\ \alpha = +,-\}$ characterizes exactly all irreducible smooth admissible G-representations whose infinitesimal character coincides with the infinitesimal character of the trivial representation (trivial infinitesimal character).

(iii) All irreducible and unitarizable $(\mathfrak{g}, K)$-modules with non-trivial $(\mathfrak{g}, K)$-cohomology are exactly given by the underlying $(\mathfrak{g}, K)$-modules of the elements of $\{\Pi_{p,\alpha},\ 0 \le p \le n+1,\ \alpha = +,-\}$.

(iv) The set of irreducible tempered representations of G with trivial infinitesimal character is given for n even by

$$\{\Pi_{\frac{n}{2},\alpha}, \Pi_{\frac{n}{2}+1,\alpha},\ \alpha = +,-\}$$

and for n odd by

$$\{\Pi_{\frac{n+1}{2},\alpha},\ \alpha = +,-\}.$$

In the odd case both representations are discrete series representations.

(v) We have the following isomorphisms of G-modules

$$\chi_{+,-} \otimes \Pi_{p,\pm} \cong \Pi_{n+1-p,\mp}, \qquad \chi_{-,+} \otimes \Pi_{p,\pm} \cong \Pi_{p,\mp}, \qquad \chi_{-,-} \otimes \Pi_{p,\pm} \cong \Pi_{n+1-p,\pm}.$$

By Lemma 3.1 and Theorem 5.3(v) we immediately obtain the following.

Corollary 5.4. *The restriction $\overline{\Pi}_{p,\pm} := \Pi_{p,\pm}|_{G_0}$ is reducible if and only if $p = \frac{n+1}{2}$. In this case $\overline{\Pi}_{\frac{n+1}{2},\pm}$ decomposes as*

$$\overline{\Pi}_{\frac{n+1}{2},\pm} = \overline{\Pi}^{(+)}_{\frac{n+1}{2},\pm} \oplus \overline{\Pi}^{(-)}_{\frac{n+1}{2},\pm}$$

into two non-isomorphic G_0-representations.

The restriction to the identity component becomes reducible if and only if $\Pi_{p,\pm}$ is a discrete series representation. In this case clearly both $\overline{\Pi}^{(+)}_{\frac{n+1}{2},\pm}$ and $\overline{\Pi}^{(-)}_{\frac{n+1}{2},\pm}$ are discrete series representations of G_0 and are contained in $\bar{\pi}^{\pm}_{\frac{n+1}{2},\frac{1}{2}}$ as submodules.

In the following we adapt the notation for the subgroup G' and denote the representations with non-trivial $(\mathfrak{g}', K')$-cohomology by $\Pi'_{q,\pm}$.

5.3 The additional cases for $p = 0, n$.

We recall some facts about the infinite dimensional unitarizable composition factors in the cases $p = 0, n$. The standard reference here for is [JW77]. For $\lambda = -\rho - j$, $j \in \mathbb{Z}_{>0}$ we denote by $I_{p,j,\pm}$ the unique proper submodule of $\pi^{\pm}_{p,-\rho-j}$ and by $F_{p,j,\pm}$ the unique proper submodule of $\pi^{\pm}_{p,\rho+j}$ which is finite dimensional. Then we have the following non-splitting short exact sequences of G-modules.

$$0 \to I_{p,j,\pm} \to \pi^{\pm}_{p,\rho+j} \to F_{p,j,\pm} \to 0,$$

$$0 \to F_{p,j,\pm} \to \pi^{\pm}_{p,-\rho-j} \to I_{p,j,\pm} \to 0.$$

Similarly we use the notation $I'_{q,j,\pm}$ and $F'_{q,j,\pm}$ for the composition factors of the G'-representations with $q = 0, n-1$.

5.4 Inner products on the complementary series and on unitarizable quotients

Combining the results of the K-spectrum of the Knapp-Stein operator in [KS18, Chapter 8.3.2] with the relations between the scalars acting on K-types in [BÓØ96, Chapter 3.a] we obtain the following.

Proposition 5.5. *Let $\lambda \in [\rho - p, p - \rho] - \{0\}$. If $p < \frac{n}{2}$ the Knapp-Stein operator $T_{p,\lambda}$ acts by non-negative scalars on all K-types in $\pi^{\pm}_{p,\lambda}$ and if $p > \frac{n}{2}$ it acts by non-positive scalars on all K-types in $\pi^{\pm}_{p,\lambda}$. If the Knapp–Stein operator vanishes on a K-type it is contained in the submodule $\ker T_{p,\lambda}$. For $p = 0$ and $\lambda \in (-\rho - 1 - \mathbb{Z}_{\geq 0})$,the Knapp-Stein operator $T_{0,\lambda}$ acts by non-negative scalars on all K-types in $\pi^{\pm}_{0,\lambda}$ and if $p = n$ and $\lambda \in (-\rho - 1 - \mathbb{Z}_{\geq 0})$,the Knapp-Stein operator $T_{n,\lambda}$ acts by non-positive scalars on all K-types in $\pi^{\pm}_{n,\lambda}$.*

In the case $p = \frac{n}{2}$ the only unitarizable composition factors occur at $\lambda = 0$ which is already in the unitary principal series and there is no complementary series. By the proposition above we define the following pairing which is an inner product on the complementary series.

$$\langle \cdot, \cdot \rangle_{p,\lambda} := \begin{cases} \langle \cdot, T_{p,\lambda} \cdot \rangle_{L^2(K)} & \text{if } p < \frac{n}{2}, \\ -\langle \cdot, T_{p,\lambda} \cdot \rangle_{L^2(K)} & \text{if } p > \frac{n}{2}. \end{cases}$$

We denote the corresponding unitary closures by $\hat{\pi}^{\pm}_{p,\lambda}$. Let $\lambda \in -\rho - 1 - \mathbb{Z}_{\geq 0} \cup \{\rho - p, p - \rho\} - \{0\}$ such that $\pi^{\pm}_{p,\lambda}$ contains a unitarizable quotient and let pr_λ be the projection. Since $\ker T_{p,\lambda}$ is the unique proper submodule, we obtain an induced intertwiner

$$_{quo}T_{p,\lambda} : \pi^{\pm}_{p,\lambda} / \ker T_{p,\lambda} \to \pi^{\pm}_{p,-\lambda}$$

which is an isomorphism onto the unique proper submodule of $\pi^{\pm}_{p,-\lambda}$ and which is essentially the Knapp-Stein operator. Then we similarly define the following inner product on the quotients.

$$\langle \cdot, \cdot \rangle_{p,\lambda,quo} := \begin{cases} \langle \cdot, T^{quo}_{p,\lambda} \cdot \rangle_{L^2(K)} & \text{if } p < \frac{n}{2}, \\ -\langle \cdot, T^{quo}_{p,\lambda} \cdot \rangle_{L^2(K)} & \text{if } p > \frac{n}{2}. \end{cases}$$

We remark that by construction for $f \in \pi^{\pm}_{p,\lambda}$ we have

$$\langle \mathrm{pr}_\lambda f, \mathrm{pr}_\lambda f\rangle_{p,\lambda,quo} = \langle f, f\rangle_{p,\lambda}.$$

We denote the corresponding unitary closures by $\widehat{\Pi}_{p,\pm}$ resp. $\hat{I}_{p,j,\pm}$. In the following we use the notation $T'^{quo}_{q,\nu}$ similary for G', $\langle\cdot,\cdot\rangle_{q,\nu}, \langle\cdot,\cdot\rangle_{q,\nu,quo}$ for the inner products and $\widehat{\Pi}'_{q,\pm}$ and $\hat{I}'_{q,j,\pm}$ for the corresponding unitary closures.

6 Classification of symmetry breaking operators

We recall main result of [KS18]. By [KS18, Theorem 1.5] we have that

$$\mathrm{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\pm}_{q,\nu}) \neq \{0\} \Rightarrow q \in \{p-2, p-1, p, p+1\}.$$

We will restrict ourselves to the cases $q = p-1, p$, since the two cases are enough for our purpose of decomposing unitary representations in $\pi^{\pm}_{p,\lambda}$. Let

$$A^{+}_{(p,\lambda),(q,\nu)} \in \mathrm{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\pm}_{q,\nu})$$

be the symmetry breaking operator which is for $\mathrm{Re}(\nu) \ll 0$ and $\mathrm{Re}(\lambda) + \mathrm{Re}(\nu) \gg 0$ given by the distribution kernel

$$|X|^{-2(\nu+\rho')}|X_n|^{\lambda-\rho+\nu+\rho'}\,\mathrm{pr}_{\sigma_p\to\delta_q}(\sigma_p(\psi_n(X)))$$

and let $\tilde{A}^{+}_{(p,\lambda),(q,\nu)}$ be

$$\frac{1}{\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2})\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2})}A^{+}_{(p,\lambda),(q,\nu)}.$$

Let

$$\tilde{A}^{-}_{(p,\lambda),(q,\nu)} \in \mathrm{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\mp}_{q,\nu})$$

be the symmetry breaking operator which is for $\mathrm{Re}(\nu) \ll 0$ and $\mathrm{Re}(\lambda) + \mathrm{Re}(\nu) \gg 0$ given by the distribution kernel

$$|X|^{-2(\nu+\rho')}|X_n|^{\lambda-\rho+\nu+\rho'}\,\mathrm{sgn}(X_n)\,\mathrm{pr}_{\sigma_p\to\delta_q}(\sigma_p(\psi_n(X)))$$

and let $\tilde{A}^{-}_{(p,\lambda),(q,\nu)}$ be

$$\frac{1}{\Gamma(\frac{\lambda+\rho+\nu-\rho'+1}{2})\Gamma(\frac{\lambda+\rho-\nu-\rho'+1}{2})}A^{-}_{(p,\lambda),(q,\nu)}.$$

Then $\tilde{A}^{\pm}_{(p,\lambda),(q,\nu)}$ define families of symmetry breaking operators which are holomorphic in $\lambda, \nu \in \mathbb{C}$ (see [KS18, Theorem 3.10]).

We define the following subsets of $\mathbb{C}^2$.

$$L^\alpha := \{(-\rho-j, -\rho'-i),\, i,j \in \mathbb{Z},\, 0 \le j \le i \text{ and } i + \frac{1-(\alpha 1)}{2} \equiv j \mod 2\},$$

$$L(p,q)^\alpha := \begin{cases} L^\alpha & \text{if } p = q = 0 \text{ or } p = q+1 = n,\\ (L^+ - \{\nu = -\rho'\}) \cup \{(p-\rho, q-\rho')\} & \text{if } 1 \le p < n,\ q = p \text{ and } \alpha = +,\\ (L^- - \{\nu = -\rho'\}) & \text{if } 1 \le p < n,\ q = p-1, p \text{ and } \alpha = -,\\ (L^+ - \{\nu = -\rho'\}) \cup \{(\rho - p, \rho' - q)\} & \text{if } 1 \le p < n,\ q = q-1 \text{ and } \alpha = +. \end{cases}$$

Theorem 6.1 ([KS18] Theorem 3.18)**.** $\tilde{A}^{\pm}_{(p,\lambda),(q,\nu)} = 0$ *if and only if* $(\lambda,\nu) \in L(p,q)^{\pm}$.

Moreover we define two renormalizations of $A^{\pm}_{(p,\lambda),(q,\nu)}$. Fix ν such that there exists a $\mu \in \mathbb{C}$ so that $(\mu,\nu) \in L(p,q)^{\pm}$. We define

$$\tilde{\tilde{A}}^{+}_{(p,\lambda),(q,\nu)} := \Gamma\left(\frac{\lambda+\rho-\nu-\rho'}{2}\right)\tilde{A}^{+}_{(p,\lambda),(q,\nu)},$$

$$\tilde{\tilde{A}}^{-}_{(p,\lambda),(q,\nu)} := \Gamma\left(\frac{\lambda+\rho-\nu-\rho'+1}{2}\right)\tilde{A}^{-}_{(p,\lambda),(q,\nu)}.$$

Then $\tilde{\tilde{A}}^{\pm}_{(p,\lambda),(q,\nu)}$ defines a non-vanishing family of symmetry breaking operators which is holomoprhic in λ.

For fixed $\lambda+\rho-\nu-\rho' \in -\mathbb{Z}_{\geq 0}$ and $q = p-1, p$ we define the meromorphic functions

$$c_C(p,q,\nu) := \Gamma(\nu+\rho'+1) \times \begin{cases} 1 & \text{if } p \neq 0, n \text{ and } \lambda-\nu \neq -\frac{1}{2}, \\ \frac{1}{\nu+\rho'-p} & \text{if } p = 0 \text{ or } \lambda-\nu = -\frac{1}{2} \text{ and } q = p, \\ \frac{1}{\nu-\rho'+p-1} & \text{if } p = n \text{ or } \lambda-\nu = -\frac{1}{2} \text{ and } q = p-1 \end{cases}$$

and we define the operators

$$C^{+}_{(p,\lambda),(q,\nu)} := c_C(p,q,\nu)\tilde{A}^{+}_{(p,\lambda),(q,\nu)},$$

for $\lambda+\rho-\nu-\rho' \in -2\mathbb{Z}_{\geq 0}$ and

$$C^{-}_{(p,\lambda),(q,\nu)} := c_C(p,q,\nu)\tilde{A}^{-}_{(p,\lambda),(q,\nu)},$$

for $\lambda+\rho-\nu-\rho' \in -1-2\mathbb{Z}_{\geq 0}$. Then $C^{\pm}_{(p,\lambda),(q,\nu)}$ defines a non-vanishing family of symmetry breaking operators which is holomoprhic in ν.

We remark that we set all symmetry breaking operators for $p = 0$ and $q = p-1$ as well as $p = n$ and $q = p$ to zero. That way we can prove many results in the following in a uniform way.

Theorem 6.2 (Classification of symmetry breaking operators, see [KS18] Theorem 3.19 and Theorem 3.26)**.** *For* $(\lambda,\nu) \notin L(p,q)^{+}$ *we have*

$$\mathrm{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\pm}_{q,\nu}) = \mathbb{C}\tilde{A}^{+}_{(p,\lambda),(q,\nu)}$$

and for $(\lambda,\nu) \in L(p,q)^{+}$ *we have*

$$\mathrm{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\pm}_{q,\nu}) = \mathbb{C}\tilde{\tilde{A}}^{+}_{(p,\lambda),(q,\nu)} \oplus \mathbb{C}C^{+}_{(p,\lambda),(q,\nu)}.$$

For $(\lambda,\nu) \notin L(p,q)^{-}$ *we have*

$$\mathrm{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\mp}_{q,\nu}) = \mathbb{C}\tilde{A}^{-}_{(p,\lambda),(q,\nu)}$$

and for $(\lambda,\nu) \in L(p,q)^{-}$ *we have*

$$\mathrm{Hom}_{G'}(\pi^{\pm}_{p,\lambda}|_{G'}, \tau^{\mp}_{q,\nu}) = \mathbb{C}\tilde{\tilde{A}}^{-}_{(p,\lambda),(q,\nu)} \oplus \mathbb{C}C^{-}_{(p,\lambda),(q,\nu)}.$$

7 Functional equations

We recall the following functional equations for the symmetry breaking operators and the Knapp-Stein intertwiners.

Theorem 7.1 ([KS18] Theorem 9.24, Theorem 9.25 and Theorem 9.31)**.**

$$T'^{\pm}_{p-1,\nu} \circ \tilde{A}^{\pm}_{(p,\lambda),(p-1,\nu)} = -\frac{\pi^{\frac{n-1}{2}}}{\Gamma(\nu+\rho'+1)}\tilde{A}^{\pm}_{(p,\lambda),(p-1,-\nu)} \times \begin{cases} (\nu-\rho'+p-1) & \text{if } p \neq \frac{n+1}{2}, \\ 1 & \text{if } p = \frac{n+1}{2}. \end{cases}$$

$$T'^{\pm}_{p,\nu} \circ \tilde{A}^{\pm}_{(p,\lambda),(p,\nu)} = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\nu+\rho'+1)}\tilde{A}^{\pm}_{(p,\lambda),(p,-\nu)} \times \begin{cases} (\nu+\rho'-p) & \text{if } p \neq \frac{n-1}{2}, \\ 1 & \text{if } p = \frac{n-1}{2}. \end{cases}$$

$$\tilde{A}^{\pm}_{(p,-\lambda),(p-1,\nu)} \circ T^{\pm}_{p,\lambda} = \frac{\pi^{\frac{n}{2}}}{\Gamma(-\lambda+\rho+1)}\tilde{A}^{\pm}_{(p,\lambda),(p-1,\nu)} \times \begin{cases} (\lambda+\rho-p) & \text{if } p \neq \frac{n}{2}, \\ 1 & \text{if } p = \frac{n}{2}. \end{cases}$$

$$\tilde{A}^{\pm}_{(p,-\lambda),(p,\nu)} \circ T^{\pm}_{p,\lambda} = -\frac{\pi^{\frac{n}{2}}}{\Gamma(-\lambda+\rho+1)}\tilde{A}^{\pm}_{(p,\lambda),(p,\nu)} \times \begin{cases} (\lambda-\rho+p) & \text{if } p \neq \frac{n}{2}, \\ 1 & \text{if } p = \frac{n}{2}. \end{cases}$$

For the case $\nu = \frac{1}{2}$ and $p = \frac{n}{2}$ we have

$$\tilde{\tilde{A}}^{+}_{(p,0),(p,\nu)} \circ T_{p,0} = \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!}\tilde{\tilde{A}}^{+}_{(p,0),(p,\nu)},$$

$$\tilde{\tilde{A}}^{+}_{(p,0),(p-1,\nu)} \circ T_{p,0} = -\frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!}\tilde{\tilde{A}}^{+}_{(p,0),(p,\nu)}.$$

We remark that the last functional equation is not contained in [KS18] but is proven in the same way as the one before in [KS18, Theorem 9.28]. By the theorem above we define for $q = p, p-1$ the meromorphic functions $t'(p,q,\nu)$ and $t(p,q,\lambda)$ such that

$$T'^{\pm}_{q,\nu} \circ \tilde{A}^{\pm}_{(p,\lambda),(q,\nu)} = t'(p,q,\nu)\tilde{A}^{\pm}_{(p,\lambda),(q,-\nu)}$$

and

$$\tilde{A}^{\pm}_{(p,-\lambda),(q,\nu)} \circ T^{\pm}_{p,\lambda} = t(p,q,\lambda)\tilde{A}^{\pm}_{(p,\lambda),(q,\nu)}.$$

8 Symmetry breaking operators into quotients

Let $\nu \in \mathbb{R} - \{0\}$ such that $\tau^{\pm}_{q,\nu}$ has an unique non-trivial quotient, i.e. $\nu \in (-\rho'-1-\mathbb{Z}_{\geq 0}) \cup (\rho'+1+\mathbb{Z}_{\geq 0}) \cup \{\rho'-q, q-\rho'\} - \{0\}$ and let

$$\mathrm{pr}_\nu : \tau_{q,\nu} \to \tau_{q,\nu}/\ker T_{q,\nu}$$

be the projection on the quotient. Then for a smooth admissible G-representation π, clearly an element $A \in \mathrm{Hom}_{G'}(\pi|_{G'}, \tau^{\pm}_{q,\nu})$ defines by composition with the projection an element of $A^{quo} \in \mathrm{Hom}_{G'}(\pi|_{G'}, \tau^{\pm}_{q,\nu}/\ker T'_{q,\nu})$. Since the unique submodule of $\tau^{\pm}_{q,\nu}$ is $\ker T'_{q,\nu}$, the Knapp-Stein operator $T'_{q,\nu}$ induces an intertwiner

$$T'^{quo}_{q,\nu} : \tau^{\pm}_{q,\nu}/\ker T'_{q,\nu} \to \tau^{\pm}_{q,-\nu},$$

which is an isomorphism onto $\operatorname{im} T'_{q,\nu} = \ker T'_{q,-\nu}$, which is the unique proper sumbodule of $\tau^{\pm}_{q,-\nu}$.

Proposition 8.1. *Let $\nu \in (-\rho' - 1 - \mathbb{Z}_{\geq 0}) \cup (\rho' + 1 + \mathbb{Z}_{\geq 0}) \cup \{\rho' - q, q - \rho'\} - \{0\}$ and $q = p-1, p$.*

(i) For the operator $\tilde{A}^{quo}_{(p,\lambda),(q,\nu)}$ the following functional equations holds.

$$T'^{quo}_{q,\nu} \circ \tilde{A}^{\pm,quo}_{(p,\lambda),(q,\nu)} = t'(p,q,\nu)\tilde{A}^{\pm}_{(p,\lambda),(q,-\nu)}$$

(ii) For $(\lambda_0, -\nu) \in L^{\pm}(p,q)$ the renormalized operator

$$\tilde{\tilde{\tilde{A}}}^{\pm,quo}_{(p,\lambda),(q,\nu)} := \lim_{\lambda \to \lambda_0} \left(\Gamma\left(\frac{\lambda + \rho + \nu - \rho'}{2} - \frac{\pm 1 - 1}{4} \right) \operatorname{pr}_{\nu} \circ \tilde{A}^{\pm}_{(p,\lambda),(q,\nu)} \right)$$

is a well defined symmetry breaking operator and satisfies the functional equation

$$T'^{quo}_{q,\nu} \circ \tilde{\tilde{\tilde{A}}}^{\pm,quo}_{(p,\lambda),(q,\nu)} = t'(p,q,\nu)\tilde{\tilde{A}}^{\pm}_{(p,\lambda_0),(q,-\nu)}.$$

Proof. (i) followis immediately form the functional equations of Theorem 7.1.

Ad (ii). Again by Theorem 7.1 we have

$$t'(p,q,\nu)\tilde{\tilde{A}}^{\pm}_{(p,\lambda_0),(q,-\nu)} = \lim_{\lambda \to \lambda_0} \left(t'(p,q,\nu)\Gamma\left(\frac{\lambda + \rho + \nu - \rho'}{2} - \frac{\pm 1 - 1}{4} \right) \circ \tilde{A}^{\pm}_{(p,\lambda),(q,-\nu)} \right).$$

Then the statement follows from the application of (i). □

We consider the special case $\nu = -\frac{1}{2}$ and $p = \frac{n}{2}$. In this case both $\tau^{\pm}_{p,\nu}$ and $\tau^{\mp}_{p-1,\nu}$ contain the discrete series representation $\Pi'_{p,\pm}$ as a quotient.

Proposition 8.2. *For $\lambda \in i\mathbb{R}$ and $f \in \pi^{\pm}_{\frac{n}{2},\lambda}$*

$$\frac{|\lambda|^2}{4} \| \tilde{A}^{+,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2},-\frac{1}{2})} f \|^2_{\frac{n}{2},-\frac{1}{2},quo} = \| \tilde{A}^{-,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2}-1,-\frac{1}{2})} f \|^2_{\frac{n}{2}-1,-\frac{1}{2},quo},$$

$$\frac{|\lambda|^2}{4} \| \tilde{A}^{+,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2}-1,-\frac{1}{2})} f \|^2_{\frac{n}{2},-\frac{1}{2},quo} = \| \tilde{A}^{-,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2},-\frac{1}{2})} f \|^2_{\frac{n}{2}-1,-\frac{1}{2},quo},$$

and for $\lambda = 0$ and $f \in \pi^{\pm}_{\frac{n}{2},0}$

$$\| \tilde{\tilde{\tilde{A}}}^{+,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2},-\frac{1}{2})} f \|^2_{\frac{n}{2},-\frac{1}{2},quo} = \| \tilde{A}^{-,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2}-1,-\frac{1}{2})} f \|^2_{\frac{n}{2}-1,-\frac{1}{2},quo},$$

$$\|_{quo} \tilde{\tilde{\tilde{A}}}^{+}_{(\frac{n}{2},\lambda),(\frac{n}{2}-1,-\frac{1}{2})} f \|^2_{\frac{n}{2},-\frac{1}{2},quo} = \|_{quo} \tilde{A}^{-}_{(\frac{n}{2},\lambda),(\frac{n}{2},-\frac{1}{2})} f \|^2_{\frac{n}{2}-1,-\frac{1}{2},quo}.$$

Proof. Since $\chi_{-,-} \otimes \Pi'_{p,\pm} = \Pi'_{p,\pm}$, the map

$$\otimes \chi_{-,-} : \tau^{\pm}_{p,\nu} \to \tau^{\mp}_{p-1,\nu}$$

induces an isomorphism of the quotients $\tau^{\pm}_{p,\nu}/\ker T'_{p,\nu}$ and $\tau^{\mp}_{p-1,\nu}/\ker T'_{p-1,\nu}$ which are both isomorphic to $\Pi'_{\frac{n}{2},\pm}$. Then by Theorem 1.2 composition with this isomorphism yields an isomorphism

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau^{\pm}_{p,\nu}/\ker T'_{p,\nu}) \to \mathrm{Hom}_{G'}(\pi|_{G'}, \tau^{\mp}_{p-1,\nu}/\ker T'_{p-1,\nu})$$

for each irreducible G-representation π. Hence for $\lambda \in i\mathbb{R} - \{0\}$

$$\|\tilde{A}^{+,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2},-\frac{1}{2})} f\|^2_{\frac{n}{2},-\frac{1}{2},quo} \text{ and } \|\tilde{A}^{-,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2}-1,-\frac{1}{2})} f\|^2_{\frac{n}{2}-1,-\frac{1}{2},quo}$$

as well as

$$\|\tilde{A}^{+,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2}-1,-\frac{1}{2})} f\|^2_{\frac{n}{2},-\frac{1}{2},quo} \text{ and } \|\tilde{A}^{-,quo}_{(\frac{n}{2},\lambda),(\frac{n}{2},-\frac{1}{2})} f\|^2_{\frac{n}{2}-1,-\frac{1}{2},quo}$$

must be constant positive scalar multiples of each other for every f. Then it is enough to check the eigenvalues of the symmetry breaking operators in question on a K-type contained in the quotient which yields the result for $\lambda \neq 0$ by [KS18, Theorem 9.8]. For $\lambda = 0$ we take the limit $\lambda \to 0$ to obtain the result by Proposition 8.1. □

9 Structure of the open orbit as homogeneous G'-space

The following Lemma is a key point in the decomposition of unitary representations to come. It reduces the problem of decomposing a unitary representation into a problem of harmonic analysis on a homogeneous G'-space.

Lemma 9.1. *(i) Let $\tilde{K}' := \mathrm{Stab}_{G'}(\overline{n}_{e_n}P)$. Then $\tilde{K}' = \mathrm{O}(n)$ and $K'/\tilde{K}' \cong \mathrm{O}(1)$.*

(ii) We have $G' \cdot \overline{n}_{e_n}P = \mathcal{O}_A$ is the open P' orbit in G/P.

Lemma 9.1 implies that $\mathcal{O}_A \cong G'/\tilde{K}' = O(1,n)/O(n)$.

For the prove of this lemma we make use of the explicit action of G' on $G/P \cong K/M$ and of the Iwasawa-decomposition of elements of $\overline{N}$. Therefor consider the map

$$K/M \to S^n \subseteq \mathbb{R}^{n+1}$$

given by

$$k = \begin{pmatrix} a & & \\ & b & * \\ & c & * \end{pmatrix} \mapsto \left(\frac{b}{a}, \frac{c}{a}\right).$$

Lemma 9.2. *(i) The map*

$$K/M \to S^n \subseteq \mathbb{R}^{n+1}$$

$$k = \begin{pmatrix} a & & \\ & b & * \\ & c & * \end{pmatrix} \to \left(\frac{b}{a}, \frac{c}{a}\right)$$

is a G equiavriant isomorphism with the action of G on S^n given by

$$g \cdot \omega = \frac{(g(1,\omega)^t)'}{(g(1,\omega)^t)^1},$$

where $(\,\cdot\,)^1$ is the first and $(\,\cdot\,)'$ the remainign coordinates of the vector.

(ii) We have $\overline{n}_X = \kappa(\overline{n}_X)e^{H(\overline{n}_X)}n \in KAN$*, with*

$$\kappa(\overline{n}_X) = \begin{pmatrix} a & & \\ & b & * \\ & c & * \end{pmatrix},$$

with

$$a = \frac{1+|X|^2}{\sqrt{(1+|X|^2)^2}}, \qquad b = \frac{1-|X|^2}{\sqrt{(1+|X|^2)^2}}$$

and

$$c = \frac{2X^T}{\sqrt{(1+|X|^2)^2}},$$

and

$$H(\overline{n}_X) = \log(1+|X|^2)H.$$

Using the KAN decomposition of the Lemma above,we obtain a map

$$\bar{\mathfrak{n}} \to K/M \cong S^n,$$

by multiplying with $\operatorname{diag}(a^{-1}, a^{-1}, \mathbf{1}_n) \in M$ from the right:

$$\overline{n}_X \mapsto \left(\frac{1-|X|^2}{1+|X|^2}, \frac{2X}{1+|X|^2}\right). \tag{9.1}$$

Proof. This is easily checked by computing the corresponding matrix decomposition. □

Proof of Lemma 9.1. Ad (i): From Lemma 9.2(i) and (9.1) it follows immediately that $K_0' = \mathrm{O}(n)$, embedded in K' in the bottom right corner.

Ad(ii): By Proposition 2.2(iv) we have $\overline{n}_{e_n}P = \tilde{w}_0\overline{n}_{-e_n}P$ such that by Corollary 2.3 we have $P' \cdot \overline{n}_{e_n}P = (\overline{N} - \overline{N}')P$, since w_0 fixes $(\overline{N} - \overline{N}')P$ again by Proposition 2.2(iv). By the Bruhat-decompositon we have $G' = P' \sqcup N'\tilde{w}_0P'$ and $N'\tilde{w}_0 = \tilde{w}_0\overline{N}'$ obviously fixes $(\overline{N} - \overline{N}')P$. □

By Lemma 9.1 we can define an G'-equivariant map

$$\Phi : \pi^{\pm}_{p,\lambda}|_{G'} \to C^\infty(G'/\tilde{K}')$$

given by

$$f \mapsto f|_{\mathcal{O}_A}(\,\cdot\,\overline{n}_{e_n}).$$

In fact this map is up to inner-automorphism onto the smooth sections

$$C^\infty(G'/\tilde{K}', \textstyle\bigwedge^p(\mathbb{C}^n))$$

of the G'-bundle over $G'/\tilde{K}'$, corresponding to the representation $\bigwedge^p(\mathbb{C}^n)$ of $\tilde{K}' = \mathrm{O}(n)$. In the following let for $g \in G$, $g = \overline{n}e^{\overline{H}(g)}\overline{\kappa}(g) \in \overline{N}AK$ be the $\overline{N}AK$ Iwasawa-decomposition.

Lemma 9.3. *The map* Φ *defines a linear continuous* G'*-equivariant map*

$$\pi^{\pm}_{p,\lambda}|_{G'} \to C^\infty(G'/\tilde{K}', \textstyle\bigwedge^p(\mathbb{C}^n)).$$

Proof. We proof first that the map Φ is onto $C^\infty(G'/\tilde{K}', \bigwedge^p(\mathbb{C}^n))$ since it is clearly onto $C^\infty(G'/\tilde{K}')$. Therefore let $k \in \mathrm{O}(n)$. Then it is easily verified that

$$\operatorname{diag}(1, k, 1)\overline{n}_{e_n} = \overline{n}_{e_n} \operatorname{diag}(1, 1, wkw^{-1}),$$

where

$$w = \begin{pmatrix} & -\mathbf{1}_{n-1} \\ -1 & \end{pmatrix} \in \mathrm{O}(n).$$

This implies the claim immediately. $\square$

By Mackey theory, the restriction to an open subset carries enough information for our purpose.

Lemma 9.4. *For* $\mathrm{Re}(\lambda) > -\frac{1}{2}$*, the map* Φ *extends to an* G'*-equivariant map*

$$\pi^{\pm}_{p,\lambda}|_{G'} \to L^2\left(G'/\tilde{K}', \bigwedge^p(\mathbb{C}^n)\right)$$

which is unitary for $\lambda \in i\mathbb{R}$*.*

Recall that by the Iwasawa decomposition the following integral formula holds

$$\int_{G'/K'} f(g)dg = \int_{\overline{N}' \times \mathfrak{a}} f(\overline{n}e^X)e^{2\rho'(X)}d\overline{N}'dX. \tag{9.2}$$

Moreover, let $\omega = \pm 1$ and $\Xi = \operatorname{diag}(\omega, \omega, \mathbf{1}_n) \in M'$. Then by Lemma 2.1(ii),

$$\Xi\overline{n}_{e_n} = \overline{n}_{\omega e_n}\Xi. \tag{9.3}$$

Proof of Lemma 9.4. Let $f \in \pi^{\pm}_{p,\lambda}$. We choose representatives Ξ of $K'/\tilde{K}' \cong \mathrm{O}(1)$. By (9.2)

$$\begin{aligned}
\|\Phi f\|_{L^2(G'/\tilde{K}')} &= \int_{G'} |f(g\overline{n}_{e_n})|^2 dg \\
&= \int_{\overline{N}' \times \mathfrak{a} \times \mathrm{O}(1)} |f(\overline{n}_{(X',0)}e^{rH}\Xi\overline{n}_{e_n})|^2 e^{2\rho' r} dX'\, dr\, d\omega \\
&= \int_{\overline{N}' \times \mathfrak{a} \times \mathrm{O}(1)} |f(\overline{n}_{(X', e^{-r}\omega)})|^2 e^{-2r(\mathrm{Re}(\lambda)+\rho-\rho')} dX'\, dr\, d\omega \\
&= \int_{\overline{N}' \times \mathbb{R}_+ \times \mathrm{O}(1)} |f(\overline{n}_{(X', t\omega)})|^2 t^{2(\mathrm{Re}(\lambda)+\rho-\rho')-1} d(X', Z)\, dt\, d\omega \\
&= \int_{\mathcal{O}_A} |f(\overline{n}_{(X', X_n)})|^2 |X''|^{2\,\mathrm{Re}(\lambda)} dX.
\end{aligned} \tag{9.4}$$

Now as above we have

$$\overline{n}_{(X', X_n)} = ke^{\log(1+|X'|^2+|X_n|^2)H} n \in KAN,$$

such that there exists a non-negative constant c_f such that

$$|f(\overline{n}_{(X', X_n)})|^2 \le c_f((1 + |X'|^2 + |X_n|^2)^2)^{-(\mathrm{Re}(\lambda)+\rho)}.$$

Hence

$$\begin{aligned}\|\Phi f\|_{L^2(G'/\tilde{K}')} \leq& c_f \int_{\overline{N}} (1+|X'|^2+|X_n|^2)^{-2(\operatorname{Re}(\lambda)+\rho)}|X_n|^{2\operatorname{Re}(\lambda)}\, dX \\ =&\tilde{c}_f \int_{(\mathbb{R}_+)^2} (1+r^2+s^2)^{-2(\operatorname{Re}(\lambda)+\rho)} r^{n-2} s^{2\operatorname{Re}(\lambda)} dr\, ds,\end{aligned}$$

where $\tilde{c}_f = 2\operatorname{Vol}(S^{n-2})c_f$. Using polar coordinates on $(\mathbb{R}_+)^2$ we find

$$\|\Phi f\|_{L^2(G'/\tilde{K}')} \leq \frac{\tilde{c}_f}{4}\int_0^{\frac{\pi}{2}} \cos^{n-2}\phi \sin^{2\operatorname{Re}(\lambda)}\phi\, d\phi \int_0^\infty x^{(\operatorname{Re}(\lambda)+\rho)-1}(1+x)^{-2(\operatorname{Re}(\lambda)+\rho)}\, dx,$$

which converges for $\operatorname{Re}\lambda > -\frac{1}{2}$. That the map is a unitary one for the unitary principal series follows from the equation (9.4). □

Clearly the bundle $C^\infty\left(G'/\tilde{K}', \bigwedge^p(\mathbb{C}^n)\right)$ fibers over $\widehat{\mathrm{O}}(1)$ such that it decomposes as

$$C^\infty\left(G'/\tilde{K}', \bigwedge^p(\mathbb{C}^n)\right) \cong C^\infty\left(G'/K', \bigwedge^p(\mathbb{C}^n)\right) \oplus \left(\chi_{-,+}\otimes C^\infty\left(G'/K', \bigwedge^p(\mathbb{C}^n)\right)\right).$$

Concretely this map is given for $f \in \pi^{\pm}_{p,\lambda}$ by

$$\Phi f = \Phi_+ f + \Phi_- f$$

with

$$\Phi_+ f(g) = \frac{1}{2}(f(g\overline{n}_{e_n}) \pm f(g\tilde{w}_0\overline{n}_{e_n}))$$

and

$$\Phi_- f(g) = \frac{1}{2}(f(g\overline{n}_{e_n}) \mp f(g\tilde{w}_0\overline{n}_{e_n})).$$

Then by restriction to G'_0 we obtain the following.

Corollary 9.5. *As G'_0-representations there is a G'_0-equivariant linear continuous map*

$$\bar{\pi}^{\pm}_{p,\lambda}|_{G'_0} \to C^\infty\left(G'_0/K'_0, \bigwedge^p(\mathbb{C}^n)\right) \oplus C^\infty\left(G'_0/K'_0, \bigwedge^p(\mathbb{C}^n)\right),$$

which extends for $\operatorname{Re}(\lambda) > -\frac{1}{2}$ *to*

$$\bar{\pi}^{\pm}_{p,\lambda}|_{G'_0} \to L^2\left(G'_0/K'_0, \bigwedge^p(\mathbb{C}^n)\right) \oplus L^2\left(G'_0/K'_0, \bigwedge^p(\mathbb{C}^n)\right),$$

which is a unitary map for $\lambda \in i\mathbb{R}$.

Let $\mathrm{pr}_{\mathrm{O}(1)}, \mathrm{pr}_{\mathrm{O}(n)}$ denote the projections of $K' \cong M \cong \mathrm{O}(1)\times \mathrm{O}(n)$ to the $\mathrm{O}(1)$ and $\mathrm{O}(n)$ factors. Moreover denote $m(g)$ the M-factor in the $\overline{N}MAN$ decomposition.

Corollary 9.6. *(i) We have*

$$H(e^{-rH}\overline{n}_{(-X,0)}) = rH + \log(N((X', e^{-r}))^2)H.$$

(ii) We have

$$\mathrm{pr}_{\mathrm{O}(n)}(\kappa(g^{-1}) = w^{-1}\,\mathrm{pr}_{\mathrm{O}(n)}(m(\tilde{w}_0 g\overline{n}_{e_n}))w.$$

(iii) We have for $X_n \in \mathbb{R}^\times$ and $g \in G'$ with $\overline{n}_{(X',X_n)} \in g\overline{n}_{e_n}P$,

$$\mathrm{pr}_{\mathrm{O}(1)}(\kappa(g^{-1})) = \operatorname{sgn} X_n.$$

Proof. Ad(i): This follows immediately from Lemma 9.2(ii).

Ad (ii): As in the proof of Lemma 9.3 we have

$$\mathrm{diag}(1,k,1)\overline{n}_{e_n} = \overline{n}_{e_n}\,\mathrm{diag}(1,1,wkw^{-1}),$$

where

$$w = \begin{pmatrix} & -\mathbf{1}_{n-1} \\ -1 & \end{pmatrix} \in \mathrm{O}(n),$$

which implies by the $\overline{N}MAN$ decomposition that for all $g \in G$,

$$\mathrm{pr}_{\mathrm{O}(n)}(\overline{\kappa}(g)) = w^{-1}\,\mathrm{pr}_{\mathrm{O}(n)}(m(g\overline{n}_{e_n}))w.$$

Moreover $\kappa(g) = \overline{\kappa}((\tilde{w}_0 g)^{-1})\tilde{w}_0$ and since $\mathrm{pr}_{\mathrm{O}(n)}(m(\tilde{w}_0)) = \mathbf{1}_n$ this implies the statement.

Ad (iii): We have

$$\mathrm{diag}\left(\frac{X_n}{|X_n|}, \frac{X_n}{|X_n|}, \mathbf{1}_n\right)\kappa(g^{-1}) = \kappa\left(e^{\log|X_n|H}\overline{n}_{(-X',0)}\right) = \kappa\left(\overline{n}_{(-|X_n|^{-1}X',0)}\right).$$

Then the statement follows from Lemma 9.2(ii). ☐

10 A Plancherel formula for $L^2(G_0'/K_0', \bar{\sigma}_p)$

We introduce the notation $\bar{\sigma} = \sigma|_{\mathrm{SO}(n)}$ for admissible representations of $\mathrm{O}(n)$ and similarly for representations of $\mathrm{O}(n-1)$ restricted to $\mathrm{SO}(n-1)$. In [Cam18] a Plancherel formula for vector bundles over Riemannian symmetric spaces is established and the example of $L^2(\mathrm{SO}_0(1,n)/\mathrm{SO}(n), \bigwedge^p(\mathbb{C}^n))$ carried out in great detail. We recall this example in this section. Let $\phi : G_0' \to \mathrm{End}(\bar{\sigma}_p)$ be a spherical function, i.e. satisfying

$$\int_{K_0'} \phi(gkh)\,dk = \phi_\tau(g)\phi(h), \qquad \phi(kgk') = \sigma_p(k)\phi(g)\sigma_p(k'). \tag{10.1}$$

and normalized to $\phi_\tau(\mathbf{1}_{n+1})) = \mathbf{1}$.

10.1 The Plancherel measure

Recall that as $\mathrm{SO}(n)$ resp. $\mathrm{SO}(n-1)$-representations we have the isomorphism

$$\bar{\sigma}_p \cong \bar{\sigma}_{n-p}, \qquad \bar{\delta}_q \cong \bar{\delta}_{n-1-q}$$

and that for n even, $\bar{\sigma}_{\frac{n}{2}}$ is reducible and decomposes into two non-isomorphic irreducibles as

$$\bar{\sigma}_{\frac{n}{2}} = \bar{\sigma}_{\frac{n}{2}}^{(+)} \oplus \bar{\sigma}_{\frac{n}{2}}^{(-)},$$

as well as for n odd, $\bar{\delta}_{\frac{n-1}{2}}$ is reducible and decomposes into two non-isomorphic irreducibles as

$$\bar{\delta}_{\frac{n-1}{2}} = \bar{\delta}_{\frac{n-1}{2}}^{(+)} \oplus \bar{\delta}_{\frac{n-1}{2}}^{(-)}.$$

Lemma 10.1. *(i) For $p \neq \frac{n}{2}, \frac{n\pm 1}{2}$ we have*

$$\bar\sigma_p|_{\mathrm{SO}(n-1)} = \bar\delta_{p-1} \oplus \bar\delta_p.$$

(ii) For $p = \frac{n}{2}$ we have

$$\bar\sigma^{(\pm)}_{\frac{n}{2}}|_{\mathrm{SO}(n-1)} = \bar\delta_{\frac{n}{2}}.$$

(iii) For $p = \frac{n-1}{2}$ we have

$$\bar\sigma_{\frac{n-1}{2}}|_{\mathrm{SO}(n-1)} = \bar\delta_{\frac{n-3}{2}} \oplus \bar\delta^{(+)}_{\frac{n-1}{2}} \oplus \bar\delta^{(+)}_{\frac{n-1}{2}}.$$

Then in the case $p = \frac{n}{2}$ also the bundle $L^2(G_0'/K_0', \bar\sigma_{\frac{n}{2}})$ is reducible

$$L^2(G_0'/K_0', \bar\sigma_{\frac{n}{2}}) \cong L^2(G_0'/K_0', \bar\sigma^{(+)}_{\frac{n}{2}}) \oplus L^2(G_0'/K_0', \bar\sigma^{(-)}_{\frac{n}{2}}).$$

The following Plancherel formula holds

$$L^2(G_0'/K_0') \cong \int^{\oplus}_{\hat G_0'(\bar\sigma_p)} m_{\bar\sigma_p}(\tau)\tau \, d\mu_{\bar\sigma_p}(\tau), \tag{10.2}$$

with a Plancherel measure $d\mu_{\bar\sigma_p}$, $\hat G_0'(\bar\sigma_p) \subseteq \hat G_0'$ being the support of the measure and $m_{\bar\sigma_p}$ the multiplicities. We denote the the corresponding Plancherel measures for $p = \frac{n}{2}$ as

$$\mu_{\bar\sigma_{\frac{n}{2}}} = \mu_{\bar\sigma^{(+)}_{\frac{n}{2}}} + \mu_{\bar\sigma^{(-)}_{\frac{n}{2}}}.$$

We recall the support and normalization of the Plancerel measure $d\mu_{\sigma_p}$ from [Cam18, Section 4]. Let P_0' be a minimal parabolic of G_0', for example $P' \cap G_0'$. Consistent with the notation of Section 3.2

$$\bar\tau_{q,\nu} = \mathrm{Ind}_{P_0'}^{G_0'}(\bar\delta_q \otimes e^\nu \otimes \mathbf{1})$$

is the principal series representation and we denote for n odd

$$\bar\tau^{(\pm)}_{\frac{n-1}{2},\nu} = \mathrm{Ind}_{P_0'}^{G_0'}(\bar\delta^{(\pm)}_{\frac{n-1}{2}} \otimes e^\nu \otimes \mathbf{1}).$$

Proposition 10.2. *(i) The continuous part of the support of $d\mu_{\bar\sigma_p}$ is for $p \neq \frac{n}{2}, \frac{n\pm1}{2}$ given by all $\bar\tau_{q,\nu}$ with $q \in \{p-1, p\} \cap \mathbb{Z}_{\geq 0}$ and $\nu \in i\mathbb{R}$ and all multiplicities are one.*

(ii) The continuous part of the support of $d\mu_{\bar\sigma^{(\pm)}_{\frac{n}{2}}}$ is given by all $\bar\tau_{\frac{n}{2},\nu}$ with $\nu \in i\mathbb{R}$ and all multiplicities are one in each case respectively.

(iii) The continuous part of the support of $d\mu_{\bar\sigma_{\frac{n-1}{2}}}$ is given by all $\bar\tau^{(\pm)}_{\frac{n-1}{2},\nu}$ and all $\bar\tau_{\frac{n-3}{2},\nu}$ with $\nu \in i\mathbb{R}$ and all multiplicities are one.

(iv) The discrete part of the support of $d\mu_{\bar\sigma_p}$ is empty if and only if $p \neq \frac{n}{2}$. If $p = \frac{n}{2}$, the discrete part of the support of $d\mu_{\bar\sigma_p^{(\pm)}}$ is given by $\overline{\Pi}'^{(\pm)}_{\frac{n}{2},+}$. The discrete series representation occurs with multiplicity one in each case respectively.

Proposition 10.2 gives an explicit description of the Plancherel formula (10.2). For our purposes we are further interested in the explicit inversion formula. We therefore define the $\mathrm{End}(\bar{\sigma}_p)$-valued function $\bar{\phi}_{p,\nu}$ given by

$$\bar{\phi}_{p,\nu}(g) = \int_{K_0'} \bar{\sigma}_p(\kappa(gk)k^{-1})e^{(\nu-\rho')H(gk)}\,dk$$

which is a spherical function (see e.g. [OS17, (3.7)]).

Lemma 10.3.

$$\bar{\phi}_{p,\nu}(g^{-1}h) = \int_{K_0'} \bar{\sigma}_p(\kappa(h^{-1}k))e^{(\nu-\rho')H(h^{-1}k)}\bar{\sigma}_p(\kappa(g^{-1}k)^{-1})e^{-(\nu+\rho')H(g^{-1}k)}\,dk.$$

Proof. First note that $g^{-1}hk = g^{-1}\kappa(hk)e^{H(hk)}n$, and since A normalizes N' we have

$$\kappa(g^{-1}hk) = \kappa(g^{-1}\kappa(hk)), \qquad H(g^{-1}hk) = H(hk) + H(g^{-1}\kappa(hk)), \tag{10.3}$$

such that

$$\begin{aligned}&\bar{\phi}_{p,\nu}(g^{-1}h)\\ &\quad= \int_{K_0'} \bar{\sigma}_p(\kappa(h^{-1}\kappa(gk))\bar{\sigma}_p(\kappa(g^{-1}\kappa(gk))^{-1})e^{(-\nu+\rho')H(g^{-1}\kappa(gk))}e^{(\nu-\rho')H(h^{-1}\kappa(gk))}\,dk.\end{aligned}$$

By the formula

$$\int_{K_0'} F(\kappa(gk))\,dk = \int_{K_0'} F(k)e^{-2\rho H(g^{-1}k)}\,dk$$

we obtain the Lemma. □

According to Lemma 10.3 we have for $f \in C_0^\infty(G_0'/K_0', \bar{\sigma}_p)$,

$$\bar{\phi}_{p,\nu} * f(h) = \int_{K_0'} \bar{\sigma}_p(\kappa(h^{-1}k))e^{(\nu-\rho')H(h^{-1}k)} \int_{G_0'} \bar{\sigma}_p(\kappa(g^{-1}k)^{-1})e^{-(\nu+\rho')H(g^{-1}k)}f(g)\,dg\,dk$$

and we define the corresponding Fourier transform by

$$\tilde{f}(k,p,\nu) := \int_{G_0'} \bar{\sigma}_p(\kappa(g^{-1}k)^{-1})e^{-(\nu+\rho')H(g^{-1}k)}f(g)\,dg.$$

Then clearly for $f \in C_0^\infty(G_0'/K_0', \bar{\sigma}_p)$ and $man \in M'AN'$,

$$\tilde{f}(kman,p,\nu) = \bar{\sigma}_p(m^{-1})a^{-(\nu+\rho')}\tilde{f}(k,p,\nu),$$

such that the Fourier transform defines a G_0'-intertwining operator

$$C_0^\infty(G_0'/K_0', \bar{\sigma}_p) \to \mathrm{Ind}_{P_0'}^{G_0'}(\bar{\sigma}_p|_{M_0'} \otimes e^\nu \otimes \mathbf{1}).$$

Now $\bar{\sigma}_p|_{M_0'}$ is reducible and decomposes into $\mathrm{SO}(n-1)$-representations according to Lemma 10.1. And on the one hand if $\bar{\delta}$ is a M_0'-representation occuring in $\bar{\sigma}_p|_{M_0'}$ and $\mathrm{Ind}_{P_0'}^{G_0'}(\bar{\delta} \otimes e^\nu \otimes \mathbf{1}) \in \hat{G}_0'(\bar{\sigma}_p)$, every other principal series $\mathrm{Ind}_{P_0'}^{G_0'}(\bar{\delta}' \otimes e^\nu \otimes \mathbf{1}) \in \hat{G}_0'(\bar{\sigma}_p)$ for all other $\bar{\delta}'$ occurring in $\bar{\sigma}_p|_{M_0'}$ according to Proposition 10.2. Applying these results to the Plancherel formula (10.2) and the corresponding inversion formula [Cam18, (39)] we obtain the following Theorem.

Theorem 10.4 (Inversion formula)**.** *We have for $p \neq \frac{n}{2}$*

$$L^2(G_0'/K_0', \bar{\sigma}_p) \cong \int_{i\mathbb{R}}^{\oplus} L^2 - \mathrm{Ind}_{P_0'}^{G_0'}(\bar{\sigma}_p|_{M_0'} \otimes e^{\nu} \otimes \mathbf{1})\, d\mu_{\bar{\sigma}_p}(\nu)$$

and for all $f \in C_0^\infty(G_0'/K_0', \bar{\sigma}_p)$

$$f(g) = \int_{i\mathbb{R}} \bar{\phi}_{p,\nu} * f(g)\, d\mu_{\bar{\sigma}_p}(\nu).$$

For $p = \frac{n}{2}$ we have

$$L^2(G_0'/K_0', \bar{\sigma}_p^{(\pm)}) \cong \int_{i\mathbb{R}}^{\oplus} L^2 - \mathrm{Ind}_{P_0'}^{G_0'}(\bar{\sigma}_p^{(\pm)}|_{M_0'} \otimes e^{\nu} \otimes \mathbf{1})\, d\mu_{\bar{\sigma}_p^{(\pm)}}(\nu) \oplus \widehat{\overline{\Pi}}{}'^{(\pm)}_{\frac{n}{2},+}$$

and for all $f \in C_0^\infty(G_0'/K_0', \bar{\sigma}_p^{(\pm)})$

$$f(g) = \int_{i\mathbb{R}} \bar{\phi}_{p,\nu} * f(g)\, d\mu_{\bar{\sigma}_p^{(\pm)}}(\nu) + c_p \bar{\phi}_{p,\frac{1}{2}} * f(g)$$

with $c_p \in \mathbb{C}$ a constant.

Following [Cam18, Example 4.4] we have explicitly for $p \neq \frac{n}{2}$, $p \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$,

$$d\mu_{\bar{\sigma}_p}(\nu) = \binom{n-1}{p} \frac{d\nu}{c(p,\nu)c(p,-\nu)},$$

with c-function

$$c(p,\nu) = 2^{-n+2} \frac{\Gamma(\frac{n}{2})(\nu + \rho' - p)\Gamma(\nu)}{\Gamma(\nu + \rho' + 1)}$$

and for $p = \frac{n}{2}$

$$d\mu_{\bar{\sigma}_p^{(\pm)}}(\nu) = \frac{1}{2} \binom{n-1}{p} \frac{d\nu}{c(p,\nu)c(p,-\nu)},$$

with c-function as above and discrete constant

$$c_{\frac{n}{2}} = 2^{-n} \frac{n!}{(\frac{n}{2})!} \prod_{s=1}^{\frac{n}{2}-1} (2s)!.$$

11 The Plancherel formula for $L^2(G'/K', \sigma_p)$

In this section we lift the results of the previous section to the disconnected group G'. we choose representatives $\tilde{v}_0, \tilde{w}_0 \in K'$ generating the component group G'/G_0' given by

$$\tilde{v}_0 = \mathrm{diag}(-1, \mathbf{1}_{n+1}), \qquad \tilde{w}_0 = \mathrm{diag}(-1, \tilde{m}),$$

with $\tilde{m} = \mathrm{diag}(-1, \mathbf{1}_n)$. For $f \in L^2(G_0'/G_0', \bar{\sigma}_p)$ we define for $g \in G_0'$

$$f(\tilde{v}_0 g) := f(\tilde{v}_0 g \tilde{v}_0^{-1})$$

and

$$f(\tilde{w}_0 g) := \sigma(\tilde{m}^{-1}) f(\tilde{w}_0 g \tilde{m}_0^{-1}),$$

where $\sigma \in \{\sigma_p, \sigma_{n-p}\}$, such that $\sigma|_{\mathrm{SO}(n)} = \bar{\sigma}_p$.

Moreover we define the $\mathrm{End}(\sigma_p)$-valued function on G'

$$\phi_{p,\nu}(g) = \int_{K'} \sigma_p(\kappa(gk)k^{-1}) e^{(\nu - \rho')H(gk)}\, dk.$$

Theorem 11.1. *We have for* $p \neq \frac{n}{2}$

$$L^2(G'/K', \sigma_p) \cong \int_{i\mathbb{R}}^{\oplus} L^2 - \mathrm{Ind}_{P'}^{G'_0}(\bar{\sigma}_p|_{M'} \otimes e^{\nu} \otimes \mathbf{1})\, d\mu_{\sigma_p}(\nu)$$

and for all $f \in C_0^\infty(G'/K', \sigma_p)$

$$f(g) = \int_{i\mathbb{R}} \phi_{p,\nu} * f(g)\, d\mu_{\sigma_p}(\nu).$$

For $p = \frac{n}{2}$ *we have*

$$L^2(G'/K', \sigma_p) \cong \int_{i\mathbb{R}}^{\oplus} L^2 - \mathrm{Ind}_{P'}^{G'}(\sigma_p|_{M'} \otimes e^{\nu} \otimes \mathbf{1})\, d\mu_{\sigma_p}(\nu) \oplus \Pi'_{\frac{n}{2},+}$$

and for all $f \in C_0^\infty(G'/K', \sigma_p)$

$$f(g) = \int_{i\mathbb{R}} \phi_{p,\nu} * f(g)\, d\mu_{\sigma_p}(\nu) + c_p \phi_{p,\frac{1}{2}} * f(g)$$

with $c_p \in \mathbb{C}$ *as before and*

$$d\mu_{\sigma_p}(\nu) = d\mu_{\bar{\sigma}_{\min(p,n-p)}}(\nu)$$

in the notation of the last section.

Proof. Let $p \neq \frac{n}{2}$ and w.l.o.g. $n - p < p$. Let $f \in C_0^\infty(G'/K', \sigma_p)$ and $h = h_c h_0 \in G'$ with $h_0 \in G'_0$ and $h_c \in G'/G'_0$. Then by the construction above

$$\begin{aligned} f(h) &= \sigma_p(h_c^{-1}) \int_{i\mathbb{R}} \bar{\phi}_{p,\nu} * f(h_c h_0 h_c^{-1})\, d\mu_{\bar{\sigma}_p}(\nu) \\ &= \sigma_p(h_c^{-1}) \int_{i\mathbb{R}} \int_{K'_0} \bar{\sigma}_p(\kappa(h_c h_0^{-1} h_c^{-1} k)) e^{(\nu-\rho')H(h_c h_0^{-1} h_c^{-1} k)} \\ &\qquad\qquad \times \int_{G'_0} \bar{\sigma}_p(\kappa(g^{-1}k)^{-1}) e^{-(\nu+\rho')H(g^{-1}k)} f(g)\, dg\, dk\, d\mu_{\bar{\sigma}_p}(\nu). \end{aligned}$$

The K'_0-integral is right M'_0-invariant and the G'_0-integral is right K'_0-invariant. Moreover $K'_0/M'_0 = K'/M'$ and $G'_0/K'_0 = G'/K'$ and since $\bar{\sigma}_p = \sigma_p|_{\mathrm{SO}(n)}$, if we replace $\bar{\sigma}_p$ by σ_p we obtain right M' and right K' invariant integrals

$$\begin{aligned} &= \int_{i\mathbb{R}} \int_{K'} \sigma_p(\kappa(h^{-1}k)) e^{(\nu-\rho')H(h^{-1}k)} \\ &\qquad\qquad \times \int_{G'} \sigma_p(\kappa(g^{-1}k)^{-1}) e^{-(\nu+\rho')H(g^{-1}k)} f(g)\, dg\, dk\, d\mu_{\sigma_p}(\nu) \\ &= \int_{i\mathbb{R}} \phi_{p,\nu} * f(h)\, d\mu_{\bar{\sigma}_p}(\nu). \end{aligned}$$

For $p = \frac{n}{2}$ the proof works in the same way using the direct sum $\bar{\sigma}_{\frac{n}{2}} = \bar{\sigma}_{\frac{n}{2}}^{(+)} \oplus \bar{\sigma}_{\frac{n}{2}}^{(-)}$ and carrying the discrete summand through the calculation. If $p > n - p$ we have to apply the $\mathrm{SO}_0(1, n)$-Plancherel and inversion formula for $\bar{\sigma}_{n-p}$ which concludes the argument. □

Similarly we define the corresponding Fourier-transform for $f \in C^\infty(G'/K', \sigma_p^w)$ by

$$\tilde{f}(k,p,\nu) = \int_{G'} \sigma_p^w(\kappa(g^{-1}k)^{-1}) e^{(-\nu-\rho')H(g^{-1}k)} f(g)\, dg.$$

Then clearly for $\nu \in i\mathbb{R}$

$$\langle \phi_{p,\nu} * f, f\rangle_{L^2(G')} = \|\tilde{f}(\cdot,p,\nu)\|^2_{L^2(K')}. \tag{11.1}$$

We remark that we use the equivalent representation σ_p^w which is twisted by w for convenience in the following.

12 Branching laws for unitary representations

We further lift the results of the section before to $\pi^{\pm}_{p,\lambda}$ and proof the main theorems.

Theorem 12.1. *Let* $\mathrm{Re}(\lambda) > -\frac{1}{2}$ *and* $f \in \pi^{\pm}_{p,\lambda}$. *We have*

$$2\tilde{\Phi_\pm f}(\cdot,p,\nu) = A^{\mp}_{(p,\lambda),(p-1,\nu)} f + A^{\pm}_{(p,\lambda),(p,\nu)} f.$$

Proof. We carry out the proof for $\alpha = \mathbf{1}$ since the other case works analogously. Let $f \in \pi^+_{p,\lambda}$. By Lemma 9.4 $\Phi_+ f \in L^2(G'/K', \sigma_p^w)$ such that we can apply the Fourier transform. Clearly the integrand is right K'-invariant in g such that we have by the $\overline{N}'AK'$ Iwasawa decomposition and by the integral formula

$$\int_{G'/K'} f(g)\, dg = \int_{\overline{N}\times\mathfrak{a}} e^{2\rho' X} f(\overline{n}e^X)\, d\overline{n}\, dX,$$

$$\begin{aligned}
&\int_{G'} \sigma_p{}^w(\mathrm{pr}_{\mathrm{O}(n)}(\kappa(g^{-1}h)) e^{-(\nu+\rho')H(g^{-1}h)} \Phi_+ f(g)\, dg \\
=& \int_{\mathbb{R}^{n-1}\times\mathbb{R}} \sigma_p{}^w(\mathrm{pr}_{\mathrm{O}(n)}(\kappa(e^{-rH}\overline{n}_{(-X',0)})) e^{-(\nu+\rho')H(e^{-rH}\overline{n}_{(-X',0)})} \Phi_+ f(h\overline{n}_{(X',0)}e^{rH})\, dX'\, dr
\end{aligned}$$

which is by Corollary 9.6

$$\begin{aligned}
=& \int_{\mathbb{R}^{n-1}\times\mathbb{R}} \sigma_p(\mathrm{pr}_{\mathrm{O}(n)}(m(\tilde{w}_0\overline{n}_{(X',0)}e^{rH}\overline{n}_{e_n})) \\
&\qquad\qquad \times e^{(-\nu+\rho')r}|e^{-2r} + |X'|^2|^{\frac{-\nu-\rho'}{2}} \Phi_+ f(h\overline{n}_{(X',0)}e^{rH})\, dX'\, dr
\end{aligned}$$

which is by Lemma 2.1

$$\begin{aligned}
=&\frac{1}{2}\int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} \sigma_p(\psi_n(X',X_n))|X_n|^{\lambda-\rho-\nu-\rho'}||X_n|^2 + |X'|^2|^{\frac{-\nu-\rho'}{2}} \\
&\qquad\qquad \times (f(h\overline{n}_{(X',|X_n|)}) + f(h\overline{n}_{(X',-|X_n|)}\operatorname{diag}(\mathbf{1}_{n+1},-1))\, dX'\, dX_n \\
=&\frac{1}{2}\int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} \sigma_p(\psi_n(X',X_n))|X_n|^{\lambda-\rho-\nu-\rho'}||X_n|^2 + |X'|^2|^{\frac{-\nu-\rho'}{2}} \\
&\qquad \times (f(h\overline{n}_{(X',|X_n|)}) + (-1)^p\sigma_p(\operatorname{diag}(-\mathbf{1}_{n-1},1))f(h\overline{n}_{(X',-|X_n|)})\, dX'\, dX_n
\end{aligned}$$

Since $\sigma_P|_{M'} = \delta_{p-1} \oplus \delta_p$ we project to the two subspaces separately. Since $\delta_q(-\mathbf{1}_{n-1}) = (-1)^q$ we obtain

$$\begin{aligned}
=&\frac{1}{2}\int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} \mathrm{pr}_{\sigma_p\to\delta_p}\circ\sigma_p(\psi_n(X',X_n))|X_n|^{\lambda-\rho-\nu-\rho'}||X_n|^2+|X'|^2|^{\frac{-\nu-\rho'}{2}}\\
&\qquad\qquad\times\left(f(h\overline{n}_{(X',|X_n|)})+f(h\overline{n}_{(X',-|X_n|)})\right)dX'\,dX_n\\
&+\frac{1}{2}\int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} \mathrm{pr}_{\sigma_p\to\delta_{p-1}}\circ\sigma_p(\psi_n(X',X_n))|X_n|^{\lambda-\rho-\nu-\rho'}||X_n|^2+|X'|^2|^{\frac{-\nu-\rho'}{2}}\\
&\qquad\qquad\times\left(f(h\overline{n}_{(X',|X_n|)})-f(h\overline{n}_{(X',-|X_n|)})\right)dX'\,dX_n\\
=&\frac{1}{2}\int_{\mathbb{R}^n}\mathrm{pr}_{\sigma_p\to\delta_p}\circ\sigma_p(\psi_n(X))|X_n|^{\lambda-\rho-\nu-\rho'}|X|^{-\nu-\rho'}f(h\overline{n}_X)\,dX\\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^n}\mathrm{pr}_{\sigma_p\to\delta_{p-1}}\circ\sigma_p(\psi_n(X))|X_n|^{\lambda-\rho-\nu-\rho'}|X|^{-\nu-\rho'}\operatorname{sgn}(X_n)f(h\overline{n}_X)\,dX.\ \square
\end{aligned}$$

Combining this result with Lemma 9.4 and Theorem 11.1 we immediately obtain the unitary branching law and Plancherel formula for the unitary principal series. Therefore we define the following functions which depend meromorphically on λ and ν.

$$c(p,\lambda,\nu)^\pm := \frac{c(\min\{p,n-p\},-\nu)c(\min\{p,n-p\},\nu)}{\Gamma(\frac{-\lambda+\rho-\nu-\rho'}{2}-\frac{\pm1-1}{4})\Gamma(\frac{-\lambda+\rho+\nu-\rho'}{2}-\frac{\pm1-1}{4})\Gamma(\frac{\lambda+\rho-\nu-\rho'}{2}-\frac{\pm1-1}{4})\Gamma(\frac{\lambda+\rho+\nu-\rho'}{2}-\frac{\pm1-1}{4})},$$

$$c(p,\lambda,q,k)^\pm_{Res} := \pi\,\mathrm{Res}_{\mu=\lambda+1-(\pm\frac{1}{2})+2k}\left(\frac{1}{c(p,\lambda,\mu)^\pm t'(p,q,\mu)c_C(p,q,\mu)^2}\right),$$

$$c(\lambda)_d := \frac{c_p\Gamma(\rho)\Gamma(\frac{-\lambda+1}{2})\Gamma(\frac{-\lambda+2}{2})\Gamma(\frac{\lambda+1}{2})\Gamma(\frac{\lambda+2}{2})}{2\pi^{\frac{n-1}{2}}}.$$

Lemma 12.2. *For $\lambda\in i\mathbb{R}$ and $p\neq\frac{n}{2}$ we have*

$$\hat{\pi}^\pm_{p,\lambda}|_{G'}\cong\bigoplus_{\alpha=\pm}\bigoplus_{q=p-1,p}\int^\oplus_{i\mathbb{R}}\hat{\tau}^\alpha_{q,\nu}\,d\mu_{\sigma_p}(\nu)$$

and for $f\in\pi^\pm_{p,\lambda}$

$$\|f\|^2_{L^2(K)}=\frac{1}{4}\sum_{\alpha=+,-}\sum_{q=p-1,p}\int_{i\mathbb{R}}\|\tilde{A}^\alpha_{(p,\lambda),(q,\nu)}f\|^2_{L^2(K')}\frac{d\nu}{c(p,\lambda,\nu)^\alpha}.$$

For $\lambda\in i\mathbb{R}$ and $p=\frac{n}{2}$ we have

$$\hat{\pi}^\pm_{p,\lambda}|_{G'}\cong\widehat{\Pi}'_{\frac{n}{2},+}\oplus\widehat{\Pi}'_{\frac{n}{2},-}\oplus\bigoplus_{\alpha=\pm}\bigoplus_{q=p-1,p}\int^\oplus_{i\mathbb{R}}\hat{\tau}^\alpha_{q,\nu}\,d\mu_{\sigma_p}(\nu)$$

and for $f\in\pi^\pm_{p,\lambda}$

$$\begin{aligned}
\|f\|^2_{L^2(K)}=&\frac{1}{4}\sum_{\alpha=+,-}\sum_{q=p-1,p}\int_{i\mathbb{R}}\|\tilde{A}^\alpha_{(p,\lambda),(q,\nu)}f\|^2_{q,\nu}\frac{d\nu}{c(p,\lambda,\nu)^\alpha}\\
&+c(\lambda)_d\|\tilde{A}^{-,quo}_{(p,\lambda),(p,-\frac{1}{2})}f\|^2_{p,-\frac{1}{2},quo}+c(\lambda)_d\|\tilde{A}^{-,quo}_{(p,\lambda),(p-1,-\frac{1}{2})}f\|^2_{p-1,-\frac{1}{2},quo}.
\end{aligned}$$

Proof. Let $\lambda \in i\mathbb{R}$ and $f \in \pi^{\alpha}_{p,\lambda}$. Then by Lemma 9.4 $\Phi = \Phi_+ + \Phi_-$ is a unitary map such that by orthogonality

$$\|f\|^2_{L^2(K)} = \|\Phi f\|^2_{L^2(G')} = \|\Phi_+ f\|^2_{L^2(G')} + \|\Phi_- f\|^2_{L^2(G')}.$$

Let $p \neq \frac{n}{2}$. Then applying the inversion formula of Theorem 11.1 and Theorem 12.1 and (11.1) we obtain

$$\begin{aligned}\|\Phi_\pm f\|^2_{L^2(G')} &= \frac{1}{4}\int_{i\mathbb{R}} \left(\|A^{\mp}_{(p,\lambda),(p-1,\nu)} f\|^2_{L^2(K')} + \|A^{\pm}_{(p,\lambda),(p,\nu)} f\|^2_{L^2(K')}\right) d\mu_{\sigma_p}(\nu)\\ &= \frac{1}{4}\left(\int_{i\mathbb{R}} \|\tilde{A}^{\mp}_{(p,\lambda),(p-1,\nu)} f\|^2_{L^2(K')} \frac{d\nu}{c(p,\lambda,\nu)^{\mp}} + \int_{i\mathbb{R}} \|\tilde{A}^{\pm}_{(p,\lambda),(p,\nu)} f\|^2_{L^2(K')} \frac{d\nu}{c(p,\lambda,\nu)^{\pm}}\right)\end{aligned}$$

by renormalization to holomorphic families. For $p = \frac{n}{2}$ the argument for the continuous part of the Plancherel formula is the same. For the discrete summands we reformulate for $\lambda \neq 0$

$$\begin{aligned}\langle \phi_{p,\frac{1}{2}} * \Phi_+ f, \Phi_+ f\rangle_{L^2(G')} &= \left\langle \tilde{\Phi_+ f}\left(\cdot, p, \frac{1}{2}\right), \tilde{\Phi_+ f}\left(\cdot, p, -\frac{1}{2}\right)\right\rangle_{L^2(K')}\\ &= \langle A^{-}_{(p,\lambda),(p-1,\frac{1}{2})} f, A^{-}_{(p,\lambda),(p-1,-\frac{1}{2})} f\rangle_{L^2(K')} + \langle A^{+}_{(p,\lambda),(p,\frac{1}{2})} f, A^{+}_{(p,\lambda),(p,-\frac{1}{2})} f\rangle_{L^2(K')}.\end{aligned}$$

Then

$$\begin{aligned}&\langle A^{+}_{(p,\lambda),(p,\frac{1}{2})} f, A^{+}_{(p,\lambda),(p,-\frac{1}{2})} f\rangle_{L^2(K')}\\ &\quad = \Gamma\left(\frac{-\lambda}{2}\right)\Gamma\left(\frac{-\lambda+1}{2}\right)\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda+1}{2}\right)\langle \tilde{A}^{+}_{(p,\lambda),(p,\frac{1}{2})} f, \tilde{A}^{+}_{(p,\lambda),(p,-\frac{1}{2})} f\rangle_{L^2(K')}\end{aligned}$$

Since the image of $\tilde{A}^{+}_{(p,\lambda),(p,\frac{1}{2})}$ is $\Pi'_{p,\alpha}$, we have

$$\langle \tilde{A}^{+}_{(p,\lambda),(p,\frac{1}{2})} f, \tilde{A}^{+}_{(p,\lambda),(p,-\frac{1}{2})} f\rangle_{L^2(K')} = \langle \tilde{A}^{+}_{(p,\lambda),(p,\frac{1}{2})} f, \tilde{A}^{+,quo}_{(p,\lambda),(p,-\frac{1}{2})} f\rangle_{L^2(K')}.$$

Then applying Proposition 8.1 we obtain

$$\begin{aligned}\langle \tilde{A}^{+}_{(p,\lambda),(p,\frac{1}{2})} f, \tilde{A}^{+,quo}_{(p,\lambda),(p,-\frac{1}{2})} f\rangle_{L^2(K')} &= -\frac{\Gamma(\rho)}{\pi^{\frac{n-1}{2}}}\langle T'^{quo}_{p,-\frac{1}{2}} \circ \tilde{A}^{+,quo}_{(p,\lambda),(p,-\frac{1}{2})} f, \tilde{A}^{+,quo}_{(p,\lambda),(p,-\frac{1}{2})} f\rangle_{L^2(K')}\\ &= \frac{\Gamma(\rho)}{\pi^{\frac{n-1}{2}}}\|\tilde{A}^{+,quo}_{(p,\lambda),(p,-\frac{1}{2})} f\|^2_{p,-\frac{1}{2},quo} = \frac{4\Gamma(\rho)}{\lambda(-\lambda)\pi^{\frac{n-1}{2}}}\|\tilde{A}^{-,quo}_{(p,\lambda),(p-1,-\frac{1}{2})} f\|^2_{p-1,-\frac{1}{2},quo}\end{aligned}$$

by Proposition 8.2. Similarly

$$\begin{aligned}&\langle A^{-}_{(p,\lambda),(p-1,\frac{1}{2})} f, A^{-}_{(p,\lambda),(p-1,-\frac{1}{2})} f\rangle_{L^2(K')}\\ &\qquad = \frac{\Gamma(\rho)\Gamma(\frac{-\lambda+1}{2})\Gamma(\frac{-\lambda+2}{2})\Gamma(\frac{\lambda+1}{2})\Gamma(\frac{\lambda+2}{2})}{\pi^{\frac{n-1}{2}}}\|\tilde{A}^{-,quo}_{(p,\lambda),(p-1,-\frac{1}{2})} f\|^2_{p-1,-\frac{1}{2},quo}.\end{aligned}$$

For $\Phi_- f$ the argument works analogously. $\square$

13 Analytic continuation

Since we have by Lemma 9.4 good behavior for $\mathrm{Re}(\lambda) > -\frac{1}{2}$ we can extend this Plancherel formula onto the real axis.

Corollary 13.1. *For $\lambda \in (-\frac{1}{2}, \frac{1}{2})$, $p \neq \frac{n}{2}$ and $f \in \pi^{\pm}_{p,\lambda}$ we have*

$$\|f\|^2_{p,\lambda} = \frac{1}{4} \sum_{\alpha=+,-} \sum_{q=p-1,p} \int_{i\mathbb{R}} \|\tilde{A}^{\alpha}_{(p,\lambda),(q,\nu)} f\|^2_{L^2(K')} \frac{|t(p,q,\lambda)|}{c(p,\lambda,\nu)^{\alpha}} d\nu.$$

We note that we abuse notation here to denote the paring $\langle f, f\rangle_{p,\lambda}$ by $\|f\|^2_{p,\lambda}$ also in the case where it is not a norm and similarly for the pairings of G'.

Proof. In the following we abbreviate the integral pairings

$$\int_* f(h)g(h)\, dh$$

by $(f,g)_*$ for Lie-gropus $*$. First by the same calculation as in the proof of Lemma 9.4 we have for $\lambda \in \mathbb{R}$ and $f \in \pi^{\pm}_{p,\lambda}$ that

$$(\Phi f, \Phi \circ T_{p,\lambda}\overline{f})_{G'} = \|f\|^2_{p,\lambda},$$

for $p < \frac{n}{2}$ and

$$(\Phi f, \Phi \circ -T_{p,\lambda}\overline{f})_{G'} = \|f\|^2_{p,\lambda},$$

for $p > \frac{n}{2}$, since $T_{p,\lambda}\overline{f} \in \pi^{\pm}_{p,-\lambda}$. Moreover by Lemma 9.4 both $\Phi f \in L^2(G'/K', \sigma_p)$ and $\Phi \circ T_{p,\lambda} f \in L^2(G'/K', \sigma_p)$ for $\mathrm{Re}(\lambda) \in (-\frac{1}{2}, \frac{1}{2})$ such that we can apply the inversion formula of Theorem 11.1 to f and exchange orders of integrals. For $p \neq \frac{n}{2}$ we obtain

$$(f, T_{p,\lambda}\overline{f})_K = \frac{1}{4} \sum_{\alpha=+,-} \sum_{q=p-1,p} \int_{i\mathbb{R}} (\tilde{A}^{\alpha}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{\alpha}_{(p,-\lambda),(q,-\nu)} \circ T_{p,\lambda}\overline{f})_{K'} \frac{d\nu}{c(p,\lambda,\nu)^{\alpha}}$$

in the same way as in Lemma 12.2. Applying the functional equation for $T_{p,\lambda}$ of Theorem 7.1 this proves the statement since $t(p,q,\alpha) \geq 0$ for all $p < \frac{n}{2}$ and $t(p,q,\alpha) \leq 0$ for all $p > \frac{n}{2}$. □

Rewriting the parings of the Plancherel formula in the Corollary above as integral pairings we obtain an equality for $\lambda \in (-\frac{1}{2}, \frac{1}{2})$

$$(f, T_{p,\lambda}\overline{f})_K = \frac{1}{4} \sum_{\alpha=+,-} \sum_{q=p-1,p} \int_{i\mathbb{R}} (\tilde{A}^{\alpha}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{\alpha}_{(p,-\lambda),(q,-\nu)} \circ T_{p,\lambda}\overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{\alpha}} d\nu. \quad (13.1)$$

In this sense the left hand side of this equation is holomorphic in λ if we consider f as a function in the compact picture, i.e. as a function on K/M. The right hand side on the other hand is meromorphic in λ and has its meromorphic structure governed by the function $c(p,\lambda,\nu)^{\pm}$ since $t(p,q,\lambda)$ is a regular function for $\lambda \leq \frac{1}{2}$. Hence we can analytically continue the right hand side towards $\lambda \in (-\infty, 0)$, where the left hand side is essentially $\|f\|_{p,\lambda}$, to obtain plancherel formulas on the whole complementary series and on unitarizable quotients.

Proposition 13.2. *For* $\lambda \in (-\infty, \frac{1}{2})$ *and* $p \neq \frac{n}{2}$

$$
\begin{aligned}
(f, T_{p,\lambda}\overline{f})_K = \frac{1}{4} \sum_{\alpha=+,-} \sum_{q=p-1,p} \Bigg(&\int_{i\mathbb{R}} \|\tilde{A}^{\alpha}_{(p,\lambda),(q,\nu)} f\|^2_{L^2(K')} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{\alpha}} d\nu \\
&+ \sum_{k \in [0, \frac{-\lambda-1+(\alpha\frac{1}{2})}{2}) \cap \mathbb{Z}} t(p,q,\lambda) c(p,\lambda,q,k)^{\alpha}_{Res} \\
&\times (C^{\alpha}_{(p,\lambda),(q,\lambda+1-(\alpha\frac{1}{2})+2k)} f, T'_{q,\lambda+1-(\alpha\frac{1}{2})+2k} \circ C^{\alpha}_{(p,\lambda),(q,\lambda+1-(\alpha\frac{1}{2})+2k)} \overline{f})_{K'} \Bigg).
\end{aligned}
$$

Proof. We prove the statement by induction. Conisder the statement holding for $\lambda \in (-\frac{1}{2} - 2k, -2k)$ and consider the integral for $\alpha = +$ of (13.1)

$$
\int_{i\mathbb{R}} (\tilde{A}^{+}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{+}} \, d\nu. \tag{13.2}
$$

Then for $\nu \in (0, \frac{1}{2})$, and $\lambda \in (-\frac{1}{2} - 2k, 2k)$ $c(p,\lambda,\nu)^{+}$ vanishes at $\nu = \lambda + \frac{1}{2} + 2k$ such that the integral has a simple pole. Then moving the contour of integration we obtain

$$
\begin{aligned}
&\int_{i\mathbb{R}+\frac{1}{2}} (\tilde{A}^{+}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{+}} \, d\nu \\
&\qquad + 2\pi \operatorname{Res}_{\mu=\lambda+\frac{1}{2}+2k} \left((\tilde{A}^{+}_{(p,\lambda),(q,\mu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\mu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\mu)^{+}} \right).
\end{aligned}
$$

Now for $\operatorname{Re}\nu = \frac{1}{2}$, $c(p,\lambda,\nu)^{+}$ does not vanish for $\lambda \in (-1-2k, -2k)$ such that for $\lambda \in (-1-2k, -\frac{1}{2} - 2k]$ this defines an analytic continouation. On the other hand for $\nu \in (0, \frac{1}{2})$ and $\lambda \in (-1-2k, -\frac{1}{2} - 2k]$, $c(p,\lambda,\nu)^{+}$ vanishes only at $\nu = -\lambda - \frac{1}{2} - 2k$ such that moving the contour of integration back towards $\operatorname{Re}\nu = 0$ we have for $\lambda = -\frac{1}{2} - 2k$

$$
\begin{aligned}
&\int_{i\mathbb{R}+\frac{1}{2}} (\tilde{A}^{+}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{+}} \, d\nu \\
&\qquad = \int_{i\mathbb{R}} (\tilde{A}^{+}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{+}} \, d\nu
\end{aligned}
$$

such that (13.2) is in fact defined for $\lambda \in [-\frac{1}{2} - 2k, -2k)$ and for $\lambda \in (-1-2k, -\frac{1}{2} - 2k)$ we obtain

$$
\begin{aligned}
&\int_{i\mathbb{R}+\frac{1}{2}} (\tilde{A}^{+}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{+}} \, d\nu \\
&\qquad = \int_{i\mathbb{R}} (\tilde{A}^{+}_{(p,\lambda),(q,\nu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^{+}} \, d\nu \\
&\qquad\quad - 2\pi \operatorname{Res}_{\mu=-\lambda-\frac{1}{2}-2k} \left((\tilde{A}^{+}_{(p,\lambda),(q,\mu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\mu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\mu)^{+}} \right).
\end{aligned}
$$

Since

$$
\begin{aligned}
&- \operatorname{Res}_{\mu=-\lambda-\frac{1}{2}-2k} \left((\tilde{A}^{+}_{(p,\lambda),(q,\mu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\mu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\mu)^{+}} \right) \\
&\qquad = \operatorname{Res}_{\mu=\lambda+\frac{1}{2}+2k} \left((\tilde{A}^{+}_{(p,\lambda),(q,\mu)} f, \tilde{A}^{+}_{(p,\lambda),(q,-\mu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\mu)^{+}} \right),
\end{aligned}
$$

we obtain

$$\int_{i\mathbb{R}} (\tilde{A}^+_{(p,\lambda),(q,\nu)} f, \tilde{A}^+_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^+} \, d\nu \\ + 4\pi \operatorname{Res}_{\mu=\lambda+\frac{1}{2}+2k} \left((\tilde{A}^+_{(p,\lambda),(q,\mu)} f, \tilde{A}^+_{(p,\lambda),(q,-\mu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\mu)^+} \right)$$

as the analytic continuation of (13.2) on $\lambda \in (-1-2k, -\frac{1}{2}-2k]$. Since $c(p,\lambda,\nu)^+$ is a regular function on $\lambda \in (-\frac{1}{2}-2(k+1), -\frac{1}{2}-2k)$ for $\operatorname{Re}\nu = 0$ this even defines an analytic continuation on $(-\frac{1}{2}-2(k+1), -\frac{1}{2}-2k]$. For

$$\int_{i\mathbb{R}} (\tilde{A}^-_{(p,\lambda),(q,\nu)} f, \tilde{A}^-_{(p,\lambda),(q,-\nu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\nu)^-} \, d\nu$$

the argument works in the same way. Then by inserting the operator $T'_{q,\nu}$ and renormalizing we have

$$\operatorname{Res}_{\mu=\lambda+1-(\alpha\frac{1}{2})+2k} \left((\tilde{A}^\alpha_{(p,\lambda),(q,\mu)} f, \tilde{A}^\alpha_{(p,\lambda),(q,-\mu)} \overline{f})_{K'} \frac{t(p,q,\lambda)}{c(p,\lambda,\mu)^\alpha} \right) = t(p,q,\lambda) c(p,\lambda,q,k)^\alpha_{Res} \\ \times (C^\alpha_{(p,\lambda),(q,\lambda+1-(\alpha\frac{1}{2})+2k)} f, T'_{q,\lambda+1-(\alpha\frac{1}{2})+2k} \circ C^\alpha_{(p,\lambda),(q,\lambda+1-(\alpha\frac{1}{2})+2k)} \overline{f})_{K'}. \quad \square$$

To state the main result we formulate the residues $c(p,\lambda,q,k)^\pm_{Res}$ more explicitly. The following can be deduced by simple calculations using the definitions of the functions $c(p,\lambda,\nu)^\pm$, $t'(p,q,\nu)$ and $c_C(p,q,\nu)^\pm$.

Lemma 13.3. *(i) Let $p = 0$ and $\lambda \in (-\infty, 0)$. For $k \in [0, \frac{-\lambda-\frac{1}{2}}{2}) \cap \mathbb{Z}$ the function $c(0,\lambda,0,k)^+_{Res}$ is strictly positive and for $k \in [0, \frac{-\lambda-\frac{3}{2}}{2}) \cap \mathbb{Z}$ the function $c(0,\lambda,0,k)^-_{Res}$ is strictly positive.*

(ii) Let $0 < p < \frac{n-1}{2}$, $q = p-1, p$ and $\lambda \in [p-\rho, 0)$. For $k \in [0, \frac{-\lambda-\frac{1}{2}}{2}) \cap \mathbb{Z}$ the function $c(p,\lambda,q,k)^+_{Res}$ is strictly positive and for $k \in [0, \frac{-\lambda-\frac{3}{2}}{2}) \cap \mathbb{Z}$ the function $c(p,\lambda,q,k)^-_{Res}$ is strictly positive.

(iii) Let $\frac{n+1}{2} < p < n$, $q = p-1, p$ and $\lambda \in [\rho-p, 0)$. For $k \in [0, \frac{-\lambda-\frac{1}{2}}{2}) \cap \mathbb{Z}$ the function $c(p,\lambda,q,k)^+_{Res}$ is strictly negative and for $k \in [0, \frac{-\lambda-\frac{3}{2}}{2}) \cap \mathbb{Z}$ the function $c(p,\lambda,q,k)^-_{Res}$ is strictly negative.

(iv) Let $p = n$ and $\lambda \in (-\infty, 0)$. For $k \in [0, \frac{-\lambda-\frac{1}{2}}{2}) \cap \mathbb{Z}$ the function $c(n,\lambda,n-1,k)^+_{Res}$ is strictly negative and for $k \in [0, \frac{-\lambda-\frac{3}{2}}{2}) \cap \mathbb{Z}$ the function $c(n,\lambda,n-1,k)^-_{Res}$ is strictly negative.

Remark 13.4. For example we have

$$c(0,\lambda,0,k)^+_{Res} = \frac{2^{2n-6}(-\lambda-\frac{1}{2}-2k)\Gamma(-\lambda+\rho-1-2k)\Gamma(-\lambda-k)\Gamma(k+\frac{1}{2})}{\pi^{\frac{n-3}{2}}\Gamma(\frac{n}{2})^2 k! \Gamma(-\lambda-k+\frac{1}{2})}$$

and

$$c(0,\lambda,0,k)^-_{Res} = \frac{2^{2n-6}(-\lambda-\frac{1}{2}-2k)\Gamma(-\lambda+\rho-1-2k)\Gamma(-\lambda-k+1)\Gamma(k+\frac{3}{2})}{\pi^{\frac{n-3}{2}}\Gamma(\frac{n}{2})^2k!\Gamma(-\lambda-k+\frac{1}{2})}.$$

We don't give the explicit expressions in every case for the sake of the length and readability of this article.

By Lemma 13.3 the formula of Proposition 13.2 immediately yields Plancherel formulas for the unitarizable representations occurring in $\pi^{\pm}_{p,\lambda}$.

Corollary 13.5. *(i) Let $p=0$. For $\lambda \in (-\rho,0)\cup -\rho-\mathbb{Z}_{\geq 0}$ we have for $f \in \pi^{\pm}_{0,\lambda}$*

$$\|f\|^2_{0,\lambda} = \frac{1}{4}\sum_{\alpha=+,-}\int_{i\mathbb{R}}\|\tilde{A}^{\alpha}_{(0,\lambda),(0,\nu)}f\|^2_{L^2(K')}\frac{|t(0,0,\lambda)|}{c(p,\lambda,\nu)^{\alpha}}d\nu$$
$$+\sum_{k\in[0,\frac{-\lambda-1+(\alpha\frac{1}{2})}{2})\cap\mathbb{Z}}|t(0,0,\lambda)|c(0,\lambda,0,k)^{\alpha}_{Res}\|C^{\alpha}_{(0,\lambda),(0,\lambda+1-(\alpha\frac{1}{2})+2k)}f\|^2_{0,\lambda+1-(\alpha\frac{1}{2})+2k}\Bigg).$$

(ii) For $0<p<n$. For $\lambda\in[p-\rho,0)$ we have for $f\in\pi^{\pm}_{p,\lambda}$

$$\|f\|^2_{p,\lambda} = \frac{1}{4}\sum_{\alpha=+,-}\sum_{q=p-1,p}\int_{i\mathbb{R}}\|\tilde{A}^{\alpha}_{(p,\lambda),(q,\nu)}f\|^2_{L^2(K')}\frac{|t(p,q,\lambda)|}{c(p,\lambda,\nu)^{\alpha}}d\nu$$
$$+\sum_{k\in[0,\frac{-\lambda-1+(\alpha\frac{1}{2})}{2})\cap\mathbb{Z}}|t(p,q,\lambda)|c(p,\lambda,q,k)^{\alpha}_{Res}\|C^{\alpha}_{(p,\lambda),(q,\lambda+1-(\alpha\frac{1}{2})+2k)}f\|^2_{q,\lambda+1-(\alpha\frac{1}{2})+2k}\Bigg).$$

(iii) Let $p=n$. For $\lambda \in (-\rho,0)\cup -\rho-\mathbb{Z}_{\geq 0}$ we have for $f \in \pi^{\pm}_{n,\lambda}$

$$\|f\|^2_{n,\lambda} = \frac{1}{4}\sum_{\alpha=+,-}\int_{i\mathbb{R}}\|\tilde{A}^{\alpha}_{(n,\lambda),(n-1,\nu)}f\|^2_{L^2(K')}\frac{|t(n,n-1,\lambda)|}{c(p,\lambda,\nu)^{\alpha}}d\nu$$
$$+\sum_{k\in[0,\frac{-\lambda-1+(\alpha\frac{1}{2})}{2})\cap\mathbb{Z}}|t(n,n-1,\lambda)|c(n,\lambda,n-1,k)^{\alpha}_{Res}$$
$$\times\|C^{\alpha}_{(n,\lambda),(n-1,\lambda+1-(\alpha\frac{1}{2})+2k)}f\|^2_{n-1,\lambda+1-(\alpha\frac{1}{2})+2k}\Bigg).$$

We remark that for $p\neq\frac{n}{2}$, $t(p,p-1,p-\rho)=t(p,p,\rho-p)=0$ while $t(p,q,\lambda)\neq 0$ in all other cases. By the corollary above we immeditelly obtain unitary branching laws for the complementary series.

Theorem 13.6. *(i) For $p=0$ and $\lambda\in(-\rho,0)$ we have*

$$\hat{\pi}^{\pm}_{0,\lambda}|_{G'} \cong \bigoplus_{\alpha=+,-}\left(\int^{\oplus}_{i\mathbb{R}}\hat{\tau}^{\alpha}_{0,\nu}\,d\nu \oplus \bigoplus_{k\in[0,\frac{-\lambda-1+(\alpha\frac{1}{2})}{2})\cap\mathbb{Z}}\hat{\tau}^{\alpha}_{0,\lambda+\frac{1}{2}+2k}\right).$$

(ii) for $0 < p < n$ *and* $\lambda \in (-|\rho - p|, 0)$ *we have*

$$\hat{\pi}^{\pm}_{p,\lambda}|_{G'} \cong \bigoplus_{\alpha=+,-} \bigoplus_{q=p-1,p} \left(\int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{q,\nu}\, d\nu \oplus \bigoplus_{k \in [0, \frac{-\lambda-1+(\alpha\frac{1}{2})}{2}) \cap \mathbb{Z}} \hat{\tau}^{\alpha}_{q,\lambda+\frac{1}{2}+2k} \right).$$

(iii) For $p = n$ *and* $\lambda \in (-\rho, 0)$ *we have*

$$\hat{\pi}^{\pm}_{n,\lambda}|_{G'} \cong \bigoplus_{\alpha=+,-} \left(\int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{n-1,\nu}\, d\nu \oplus \bigoplus_{k \in [0, \frac{-\lambda-1+(\alpha\frac{1}{2})}{2}) \cap \mathbb{Z}} \hat{\tau}^{\alpha}_{n-1,\lambda+\frac{1}{2}+2k} \right).$$

Similarly we can deduce unitary branching laws for the unitary quotients with non-trivial $(\mathfrak{g}, K)$-cohomology.

Theorem 13.7. *(i) For the one dimensional unitary quotients we have*

$$\widehat{\Pi}_{0,\pm}|_{G'} \cong \widehat{\Pi}'_{0,\pm}, \qquad \widehat{\Pi}_{n+1,\pm}|_{G'} \cong \widehat{\Pi}'_{n,\pm}.$$

(ii) For $0 < p \leq \frac{n}{2}$ *we have*

$$\widehat{\Pi}_{p,\pm}|_{G'} \cong \widehat{\Pi}'_{p,\pm} \oplus \bigoplus_{k \in (0, \rho'-p+1) \cap \mathbb{Z}} \tau^{\pm(-1)^k}_{p-1,p-1-\rho'+k} \oplus \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{p-1,\nu}\, d\nu.$$

(iii) For $p = \frac{n+1}{2}$ *we have*

$$\widehat{\Pi}_{\frac{n+1}{2},\pm}|_{G'} \cong \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{\frac{n-1}{2},\nu}\, d\nu.$$

(iv) For $\frac{n+1}{2} < p \leq n$ *we have*

$$\widehat{\Pi}_{p,\pm}|_{G'} \cong \widehat{\Pi}'_{p-1,\pm} \oplus \bigoplus_{k \in (0, p-1-\rho') \cap \mathbb{Z}} \tau^{\pm(-1)^k}_{p-1,\rho'-p+1+k} \oplus \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{p-1,\nu}\, d\nu.$$

Proof. The statement for the one-dimensional representations is clear. For $0 \leq p \leq \frac{n-1}{2}$ the statement follows from Corollary 13.5 in the following way. Since $\|f\|_{p,p-\rho} = \|\mathrm{pr}_{p-\rho} f\|_{p,p-\rho,quo}$ for all $f \in \hat{\pi}^{\pm}_{p,p-\rho}$ the Plancherel formula of Corollay 13.5 is essentially the Plancherel formula for the quotient $\widehat{\Pi}_{p+1,\mp}$. Then the statement follows since $t(p, p-1, p-\rho) = 0$ and $t(p, p, p-\rho) \neq 0$. Similarly for $\frac{n+1}{2} \leq p \leq n$, and the quotient $\widehat{\Pi}_{p,\pm}$ of $\hat{\pi}^{\pm}_{p,\rho-p}$ since $t(p, p, \rho-p) = 0$ and $t(p, p-1, \rho-p) \neq 0$. For $p = \frac{n}{2}$ we have that

$$\Pi_{\frac{n}{2},\pm} := \ker\left(T_{\frac{n}{2},0} - \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!}\,\mathrm{id}\right) \subseteq \pi^{\pm}_{\frac{n}{2},0}$$

and

$$\Pi_{\frac{n}{2}+1,\mp} := \ker\left(T_{\frac{n}{2},0} + \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!}\,\mathrm{id}\right) \subseteq \pi^{\pm}_{\frac{n}{2},0}.$$

Then the statement follows from the functional equations Theorem 7.1. $\square$

In the same way as above we obtain branching laws for the unitarizable representations $I_{p,j,\pm}$ for $p = 0, n$.

Theorem 13.8. *(i) For $p = 0$ we have*

$$\hat{I}_{0,j,\pm}|_{G'} \cong \widehat{\Pi}'_{1,\mp} \oplus \bigoplus_{k=1}^{j} \hat{I}'_{0,k,\pm(-1)^{k+j}} \oplus \bigoplus_{k \in (0,\rho') \cap \mathbb{Z}} \hat{\tau}^{\pm(-1)^{k+j}}_{0,-\rho'+k} \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{0,\nu} \, d\nu.$$

(ii) For $p = n$ we have

$$\hat{I}_{n,j,\pm}|_{G'} \cong \widehat{\Pi}'_{n-1,\pm} \oplus \bigoplus_{k=1}^{j} \hat{I}'_{n-1,k,\pm(-1)^{k+j}} \oplus \bigoplus_{k \in (0,\rho') \cap \mathbb{Z}} \hat{\tau}^{\pm(-1)^{k+j}}_{n-1,-\rho'+k} \bigoplus_{\alpha=+,-} \int_{i\mathbb{R}}^{\oplus} \hat{\tau}^{\alpha}_{n-1,\nu} \, d\nu.$$

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