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Some Results on Optimal Stopping and Skorokhod Embedding with Applications

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Preface

This thesis is the product of my studies over the last three years. It consists of this Preface including a summary, a survey paper, and seven research papers on different topics. Each paper is self-contained and appears as a chapter in the thesis (i.e. a chapter is equivalent to a paper). This means that the chapters can be read separately, even though there is some overlap between them. The chapters are as follows:

Chapter 1. A Survey of Optimal Stopping Problems for Time-Homogeneous Diffusions;
Chapter 2. Computing the expectation of the Azéma-Yor stopping time. Co-author: G. Peskir. *Ann. Inst. H. Poincare Probab. Statist.* Vol. **34**, No. 2, 1998, (265-276);
Chapter 3. Best bounds in Doob's maximal inequality for Bessel processes. *Research Report* No. **373**, 1997, *Dept. Theoret. Statist. Aarhus*, (10 pp). To appear in *J. Multivariate Anal.*;
Chapter 4. Solving non-linear optimal stopping problems by the method of time-change. Co-author: G. Peskir. *Research Report* No. **390**, 1997, *Dept. Theoret. Statist. Aarhus*, (24 pp). To appear in *Stochastic Anal. Appl.*;
Chapter 5. Discounted optimal stopping problems for the maximum process. *Research Report* No. **387**, 1997, *Dept. Theoret. Statist. Aarhus*, (12 pp). (Submitted);
Chapter 6. Azéma-Yor solution to embedding in non-singular diffusion. *Research Report* No. **406**, 1999, *Dept. Theoret. Statist. Aarhus*, (14 pp). (Submitted);
Chapter 7. The minimum maximum of a continuous martingale with given initial and final laws. Co-author: D. G. Hobson;
Chapter 8. Optimal prediction of the ultimate maximum of Brownian motion.

Chapter 1 surveys the existing literature of optimal stopping problems, summarizes the essential facts in this field and presents some examples and applications. Another intention with this chapter is to motivate my work (Chapter 2-8) and relate it to the existing literature. Thus Chapter 1 can be viewed as an introduction (written as an article) to the thesis. The chapters 2-8 contain results I have obtained during my PhD-programme. The main part of the thesis is related to optimal stopping problems and applications (Chapter 2, 3, 4, 5 and 8). Another major part is related to Skorokhod embedding problems (Chapter 6 and 7). It should be noted that there is a certain interplay between these two parts as both focus upon the maximum process of a one-dimensional diffusion and stopping times associated.

Summary

Below is a short summary of the content of each chapter. It may be noted that the summary of each paper almost coincides with the abstract of the paper.

1. A Survey of Optimal Stopping Problems for Time-Homogeneous Diffusions.

The first part of this paper summarizes the essential facts on general optimal stopping theory for time-homogeneous diffusion processes in $\mathbb{R}^n$. The results displayed are stated in a little greater generality, but in such a way that they are neither too restrictive nor too complicated. The second part presents equations for the value function and the optimal stopping boundary as a free-boundary (Stefan) problem and further presents the principle of smooth fit. This part is illustrated by examples where the focus is on optimal stopping problems for the maximum process associated with a one-dimensional diffusion.

2. Computing the expectation of the Azéma-Yor stopping times.
Given the maximum process $(S_t) = (\max_{0 \le r \le t} X_r)$ associated with a diffusion $((X_t), \mathbf{P}_x)$, and a continuous function g satisfying $g(s) < s$, we show how to compute the expectation of the Azéma-Yor stopping time $\tau_g = \inf\{\, t > 0 : X_t \le g(S_t)\,\}$ as a function of x. The method of proof is based upon verifying that the expectation solves a differential equation with two boundary conditions. The third "missing" condition is formulated in the form of a *minimality principle* which states that the expectation is the minimal solution to this system. It enables us to express this solution in a closed form. The minimality principle is the main novelty in this approach. The result is applied in the case the diffusion is a Bessel process and g is a linear function.

3. Best bounds in Doob's maximal inequality for Bessel processes.
The main result of this paper is a sharp maximal inequality of Doob's type for Bessel processes of dimension $\alpha > 0$. The proof is based upon solving an optimal stopping problem by applying the principle of smooth fit and the maximality principle. The constants obtained in the inequality are the best possible. Equality in the inequality is attained in the limit through the explicitly displayed optimal stopping times of the stopping problem.

4. Solving non-linear optimal stopping problems by the method of time-change.
Some non-linear optimal stopping problems can be solved explicitly by using a common method which is based on time-change. The basic idea is to transform the initial (difficult) problem into a new (easier) problem. The method is described and its use illustrated by considering several examples dealing with Brownian motion. In each of these examples explicit formulas are derived for the value functions and the optimal stopping times are displayed. The main emphasis of the paper is on the method of proof and its unifying scope.

5. Discounted optimal stopping problems for the maximum process.
The maximality principle is shown to be valid in some examples of discounted optimal stopping problems for the maximum process. In each of these examples we derive explicit formulas for the value function and display the optimal stopping time. Especially, in the framework of Black-Scholes model we calculate the fair price of two Lookback options with infinite horizon. The main aim of the paper is to show that in each example under consideration the optimal stopping boundary satisfies the maximality principle and that the value function can be determined explicitly.

6. The Azéma-Yor solution to embedding in non-singular diffusions.
Let $(X_t)_{t \ge 0}$ be a non-singular diffusion on $\mathbb{R}$ vanishing at zero which is not necessarily recurrent. Let ν be a probability measure on $\mathbb{R}$ having strictly positive density. Necessary and sufficient conditions are established for ν such that there exists a stopping time τ_* of (X_t) solving the Skorokhod embedding problem, i.e. X_{τ_*} has law ν. Furthermore an explicit construction of τ_* is carried out. The construction is a complement to the Azéma-Yor solution of the recurrent case. In addition, τ_* is characterized uniquely to be the pointwise smallest possible embedding that stochastically maximizes the maximum process of (X_t) up to the time of stopping or stochastically minimizes the minimum process of (X_t) up to the time of stopping, depending on the sign of the mean value of ν.

7. The minimum maximum of a continuous martingale with given initial and terminal laws.
Let (M_t) be a continuous martingale with initial law $M_0 \sim \mu_0$ and terminal law $M_1 \sim \mu_1$ and let $S = \sup_{0 \le t \le 1} M_t$. In this paper we prove that there exists a greatest lower bound with respect to stochastic ordering of probability measures, on the law of S . We give an explicit construction of this bound. Furthermore a martingale is constructed which attains this minimum by solving a Skorokhod embedding problem. The result is applied to the robust hedging of a forward start digital option.

8. Optimal prediction of the ultimate maximum of Brownian motion.
Let $(B_t)_{t \ge 0}$ be a standard Brownian motion vanishing at zero. For fixed $p, \varepsilon > 0$ consider for $T > 0$ the two optimal stopping problems

$$V_*^{(p)}(T) = \inf_\tau \mathbf{E}\big(\max_{0 \le t \le T} B_t - B_\tau\big)^p \quad \text{and} \quad W_*^{(\varepsilon)}(T) = \sup_\tau \mathbf{P}\big(\max_{0 \le t \le T} B_t - B_\tau \le \varepsilon\big)$$

where the infimum and supremum respectively are taken over all stopping times τ for (B_t) satisfying $\tau \le T$. In the first problem an explicit formula is derived for the value functions and the optimal stopping strategy is displayed. In the latter problem we conjecture a theorem and reduce its proof to verifying that a value function is not differentiable over a line. The method of proof is based upon representing the conditional expectation of the gain process $G(\max_{0 \le r \le T} B_r - B_t)$ given $\mathcal{F}_{t+} = \bigcap_{s>t} \sigma(\{B_r \,|\, 0 \le r \le s\})$ as a function of $(\max_{0 \le r \le t} B_r - B_t)$.

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A Survey of Optimal Stopping Problems for Time-Homogenuous Diffusions

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The first part of this paper summarizes the essential facts on general optimal stopping theory for time-homogeneous diffusion processes in $\mathbb{R}^n$. The results displayed are stated in a little greater generality, but in such a way that they are neither too restrictive nor too complicated. The second part presents equations for the value function and the optimal stopping boundary as a free-boundary (Stefan) problem and further presents the principle of smooth fit. This part is illustrated by examples where the focus is on optimal stopping problems for the maximum process associated with a one-dimensional diffusion.

1. Introduction

The purpose of this paper is to review some methodologies used in optimal stopping problems for diffusion processes in $\mathbb{R}^n$. The first aim is to give a quick review of the general optimal stopping theory by introducing the fundamental concepts of excessive and superharmonic functions. The second aim is to introduce the common technique to transform the optimal stopping into a free-boundary (Stefan) problem such that explicit or numerical computations of the value function and the optimal stopping boundary are possible in specific problems.

Problems of optimal stopping have a long history in probability theory and have been widely studied by many authors. The results on optimal stopping were first developed in the discrete case. The first formulations of optimal stopping problems for discrete time stochastic processes were in connection with sequential analysis in mathematical statistics where the number of observations is not fixed in advance (i.e. a random number), but is terminated by the behaviour of the observed data. The results can be found in Wald [30]. Snell [27] obtained the first general results of optimal stopping theory for stochastic processes in discrete time. For a survey of optimal stopping for Markov sequences see Shiryaev [26] and the references therein. For optimal stopping problems for continuous time Markov processes the first general results were obtained by Dynkin [4] using the fundamental concepts of excessive and superharmonic functions. There is an abundance of work in general optimal stopping theory using these concepts, but one of the standard and master reference is the monograph of Shiryaev [26] where the definite results of general optimal stopping theory are stated and it also contains an extensive list of references to this topic. (Another thorough exposition is founded in El Karoui [5]). This method gives results on the existence and uniqueness of an optimal stopping time under very general conditions of the gain function and the Markov process. Generally, for solving a specific problem the method is very difficult to apply. In concrete problem with smooth gain function and continuous Markov process it is a common technique to formulate the optimal stopping

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problem as a free-boundary problem for the value function and the optimal stopping boundary along with the non-trivial boundary condition the principle of smooth fit (also called smooth pasting ([26]) or high contact principle ([31])). The principle of smooth fit says that the first derivatives of the value function and the gain function agree at the optimal stopping boundary (the boundary of the domain of continued observation). The principle was first applied by Mikhalevich [14] (under leadership of Kolmogorov) for concrete problems in sequential analysis and later independently by Chernoff [1] and Lindley [12]. McKean [13] applied the principle to the American option problem. Other important papers in this respect are Grigelionis and Shiryaev [10] and van Moerbeke [29]. For a complete account of the subject and an extensive bibliography see Shiryaev [26]. Peskir and Shiryaev [21] introduced the principle of continuous fit solving sequential testing problems for Poisson processes (processes with jumps).

The background for solving concrete optimal stopping problems is the following. Before and in the seventies the investigated concrete optimal stopping problems were for one-dimensional diffusions where the gain process contained two terms: a function of the time and the process, and a path-dependent integral of the process (see, among others, Taylor [28], Shepp [22] and Davis [2]). In the nineties the maximum process (path-dependent functional) associated with a one-dimensional diffusion was studied in optimal stopping. Jacka [11] treated the case of reflected Brownian motion and later Dubins, Shepp and Shiryaev [3] treated the case of Bessel processes. In both papers the motivation was to obtain sharp maximal inequalities and the problem was solved by guessing the nature of the optimal stopping boundary. Graversen and Peskir [7] formulated the maximality principle for the optimal stopping boundary in the context of geometric Brownian motion. Peskir [20] showed that the maximality principle is equivalent to the superharmonic characterization of the value function from the general optimal stopping theory and led to the solution of the problem for a general diffusion ([20] also contains many references to this subject). Recently Graversen, Peskir and Shiryaev [9] formulated and solved an optimal stopping stopping problem where the gain process was not adapted to the filtration.

Optimal stopping problems appear in many connections. The problems have a wide range of applications from theoretical to applied problems. The following applications illustrate that point.

Mathematical statistics. The Bayesian approach to sequential analysis of problems on testing two statistical hypotheses can be solved by reducing the initial problems to optimal stopping problems. Testing two hypotheses about the mean value of a Wiener process with drift was solved by Mikhalevich [14] and Shiryaev [25]. Peskir and Shiryaev [21] solved the problem of testing two hypotheses about the intensity of a Poisson process. Other problems in this direction is the quickest detection problem (disruption problem). The problem to detect (alarm) when there is a change in the mean value of a Brownian motion with drift with a minimal error (false alarm) was investigated in Shiryaev [24]. Again an thorough exposition of the subject can be found in Shiryaev [26].

Sharp inequalities. Optimal stopping problems are a natural tool to derive sharp versions of known inequalities, as well as to deduce new sharp inequalities. By this method Davis [2] derived sharp inequalities for a reflected Brownian motion. Jacka [11] and Dubins, Shepp and Shiryaev [3] derived sharp maximal inequalities for a reflected Brownian motion and for Bessel processes respectively. In the same direction see Graversen and Peskir [6] and [8] (Doob's inequality for Brownian motion and Hardy-Littlewood inequality, respectively) and Pedersen [16] (Doob's inequality for Bessel processes).

Mathematical finance. The valuation of American options is based on solving optimal stopping problems and is prominent in the modern optimal stopping theory. The most famous result in this direction is of McKean [13] solving the standard American option in the Black-Scholes model. This example can further be interpreted when it is the right time to sell the stocks see Øksendal [31]. In Shepp and Shiryaev [23] the valuation of the Russian option is computed in the Black-Scholes model. The payoff of the option is the maximum value of the asset between the purchase time and exercise time. The literature devoted to pricing American options is extensive and we refer to the survey of Myneni [15] and the references therein for an account of the subject.

Optimal prediction. The development of optimal prediction of an anticipate functional of a continuous time process was recently initiated in Graversen, Peskir and Shiryaev [9]. The general optimal stopping theory can not be applied in this case since the gain process is not adapted to the filtration due to the anticipate variable. The problem under consideration in [9] is to stop a Brownian motion as close as possible to its ultimate maximum. The closeness is measured by a mean-square distance. The problem can be viewed as an optimal decision that should be based on a prediction of the future behaviour of the observable motion. For example a trader predicting the maximum value of an asset in a given period. The argument is also carried over to other applied problems where such a prediction plays a role.

The remainder of this paper is structured as follows. In the next section the formulation of the optimal stopping problem under consideration is done in mathematical terms. The concepts of excessive and superharmonic functions are introduced in Section 3. Then a review of some results on the above concepts is made. In Section 4 the main theorem on optimal stopping of diffusion processes is stated. In Section 5 the optimal stopping problem is transformed into a free-boundary problem. Further the principle of smooth fit is introduced. The paper finishes with three examples in Section 6 where the focus is on optimal stopping problems for the maximum process associated with a diffusion.

2. Formulation of the problem

Let $(X_t)_{t\geq 0}$ be a time-homogeneous diffusion process with state space $\mathbb{R}^n$ associated with the infinitesimal generator

$$\mathbf{L_X} = \sum_{i=1}^{n} \mu_i(x)\,\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^t)_{i,j}(x)\,\frac{\partial^2}{\partial x_i\,\partial x_j} \tag{2.1}$$

for $x \in \mathbb{R}^n$ where $\mu : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous and further $\sigma\sigma^t$ is non-negative definite. See Øksendal [31] for conditions on $\mu(\cdot)$ and $\sigma(\cdot)$ that ensure existence and uniqueness of the diffusion process. Let (Z_t) be a diffusion process depending on both time and space (and hence is not time-homogeneous diffusion) given by $(Z_t) = (t, X_t)$ which under $\mathbf{P}_z$ starts at $z = (t, x)$. Thus (Z_t) is a diffusion process in $\mathbb{R}_+ \times \mathbb{R}^n$ associated with the infinitesimal generator

$$\mathbf{L_Z} = \frac{\partial}{\partial t} + \mathbf{L_X} \tag{2.2}$$

for $z = (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

The optimal stopping problem to be studied in later sections is of the following kind. Let $G : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a *gain function* specified later. Consider the *optimal stopping problem* for the diffusion (Z_t) with the *value function* given by

$$V_*(z) = \sup_\tau \mathbf{E}_z\big(G(Z_\tau)\big) \tag{2.3}$$

where the supremum is taken over all stopping times τ for (Z_t). $G(Z_\tau)$ is set to be $-\infty$ at the points $\omega \in \Omega$ where $\tau(\omega) = \infty$. There is two problems to be solved in connection with the problem (2.3). The first problem is to compute the value function V_* and the second problem is to find an optimal stopping time τ_*, i.e. a stopping time for (Z_t) such that $V_*(z) = \mathbf{E}_z(G(Z_{\tau_*}))$. Note that optimal stopping times may not exist or may not be unique.

3. Excessive and superharmonic functions

In this section we introduce the two fundamental concepts of excessive and superharmonic functions which we shall see in the next section are the basic concepts for a characterization of the value function in (2.3). The facts presented here and a complete account (including proofs) of this subject consult Shiryaev [26].

In the main theorem in the next section we need to assume that the gain function belongs to the following class of functions. Let $\mathcal{L}(Z)$ be the class consisting of all lower semicontinuous functions $H : \mathbb{R}_+ \times \mathbb{R}^n \to (-\infty, \infty]$ satisfying either of the following two conditions

$$\mathbf{E}_z\big(\sup_{s\geq 0} H(Z_s)\big) < \infty \tag{3.1}$$

$$\mathbf{E}_z\big(\inf_{s\geq 0} H(Z_s)\big) > -\infty \tag{3.2}$$

for all $z = (t, x)$. Note that if the function H is bounded from below then condition (3.2) is trivial fulfilled. The following two families of functions are crucial in the sequel presentation of the general optimal stopping theory.

Definition 3.1. (**Excessive functions**). A function $H \in \mathcal{L}(Z)$ is called excessive for (Z_t) if

$$\mathbf{E}_z\big(H(Z_s)\big) \leq H(z)$$

for all $s \geq 0$ and all $z = (t, x)$.

Definition 3.2. (**Superharmonic functions**). A function $H \in \mathcal{L}(Z)$ is called superharmonic for (Z_t) if

$$\mathbf{E}_z\big(H(Z_\tau)\big) \leq H(z)$$

for all stopping times τ for (Z_t) and all $z = (t, x)$.

For basic and useful properties of excessive and superharmonic functions we refer to [26] and [31]. From the two definitions it is clear that a superharmonic function is excessive. Moreover in some cases the converse also holds which is not obvious. The result is stated in the next proposition.

Proposition 3.3. *Let $H \in \mathcal{L}(Z)$ satisfy condition* (3.2). *Then H is excessive for (Z_t) if and only if H is superharmonic for (Z_t).*

The definitions just introduced above play a definite role in describing the structure of the value function in (2.3). The following definition is important in this direction.

Definition 3.4. (**The least superharmonic (excessive) majorant**). Let $G \in \mathcal{L}(Z)$ be finite, i.e. $G < \infty$. A superharmonic (excessive) function H is called a superharmonic (excessive) majorant of G if $H \geq G$. A function $\widehat{G}$ is called the least superharmonic (excessive) majorant of G if

(i) $\widehat{G}$ is a superharmonic (excessive) majorant of G and
(ii) if H is an arbitrary superharmonic (excessive) majorant of G then $\widehat{G} \leq H$.

Before we finish this section we give a general iterative procedure for constructing the least superharmonic majorant under the condition (3.2).

Proposition 3.5. *Let $G \in \mathcal{L}(Z)$ satisfy condition (3.2) and $G < \infty$. Define the operator*

$$Q_j G(z) = G(z) \vee \mathbf{E}_z\big(G(Z_{2^{-j}})\big)$$

and set

$$G_{j,n}(z) = (Q_j^n G)(z)$$

where Q_j^n is the n'te power of the operator Q_j . Then the function

$$\widehat{G}(z) = \lim_{j\to\infty} \lim_{n\to\infty} G_{j,n}(z)$$

is the least superharmonic majorant of G .

There is a simple iterative procedure for the construction of $\widehat{G}$ when the Markov process and the gain function are "nice". It is the following corollary of Proposition 3.5.

Corollary 3.6. *Let (Z_t) be a Feller process and let $G \in \mathcal{L}(Z)$ be continuous and bounded from below. Set*

$$G_j(z) = \sup_{t\geq 0} \mathbf{E}_z\big(G_{j-1}(Z_t)\big)$$

for $j \geq 1$ and $G_0 = G$. Then

$$\widehat{G}(z) = \lim_{j\to\infty} G_j(z)$$

is the least superharmonic majorant of G .

Remark 3.7. Proposition 3.5 and Corollary 3.6 are both valid under condition (3.2) and in this case excessive and superharmonic functions are the same according to Proposition 3.3. When condition (3.2) is violated the least excessive majorant may differ from the least superharmonic majorant. Then the least excessive majorant is smaller than the least superharmonic majorant since there is more excessive functions than superharmonic functions. The construction of the least superharmonic majorant follows a similar pattern but is generally more complicated (see [26]).

Remark 3.8. The iterative procedures to construct the least superharmonic majorant are difficult to apply in concrete problems. This makes it necessary to search for explicit or numerical computations of the least superharmonic majorant.

4. Characterization of the value function

In this section we present the main theorem of general optimal stopping theory of diffusion processes. The result gives existence and uniqueness of an optimal stopping time in problem (2.3). The result could have been stated in a more general setting, but is stated with a minimum of technical assumptions. For instance, the theorem also holds for a larger class of Markov process such as Lévy processes. For details of this and the main theorem consult Shiryaev [26].

Theorem 4.1. *Consider the optimal stopping problem* (2.3) *where the gain function* G *is lower semicontinuous and satisfies either* (3.1) *or* (3.2). *Then we have.*

(**I**). *The value function* V_* *is the least superharmonic majorant of the gain function* G *with respect to the process* $(Z_t)_{t\geq 0}$ *, i.e.*

$$V_*(z) = \widehat{G}(z)$$

for all $z = (t, x)$.

(**II**). *Define the domain of continued observation*

$$C = \{ z \,|\, G(z) < V_*(z) \} \tag{4.1}$$

and let τ_* *be the first exit time of* (Z_t) *from* C *given by*

$$\tau_* = \inf \{ t > 0 \,:\, Z_t \notin C \} \,. \tag{4.2}$$

If $\tau_* < \infty$ $\mathbf{P}_z$*-a.s. for all* z *, then* τ_* *is an optimal stopping time for the problem* (2.3), *at least when* G *is continuous and satisfies both* (3.1) *and* (3.2).

(**III**). *If there exists an optimal stopping time* σ *in problem* (2.3), *then* $\tau_* \leq \sigma$ $\mathbf{P}_z$*-a.s. for all* z *and* τ_* *is also an optimal stopping time for problem* (2.3).

Remark 4.2. Part (**II**) of the theorem gives the existence of an optimal stopping time. The conditions could have been stated a little greater generality, and again we refer to [26] for more details.

Part (**III**) of the theorem says that if there exists an optimal stopping time σ then τ_* is also an optimal stopping time and is the smallest among all optimal stopping times for problem (2.3). This extremal property of the optimal stopping time τ_* characterizes it uniquely.

Remark 4.3. Sometimes it is convenient to consider "approximate" optimal stopping times. An example is given in the setting of Theorem 4.1(**II**), if the stopping time τ_* (4.2) does not satisfies $\tau_* < \infty$ $\mathbf{P}_z$-a.s. Then we have the following approximate stopping times. For $\varepsilon > 0$ let $C_\varepsilon = \{ z \,|\, G(z) < V_*(z) - \varepsilon \}$. Let τ_ε be the first exit time of (Z_t) from C_ε given by $\tau_\varepsilon = \inf \{ t > 0 \,:\, Z_t \notin C_\varepsilon \}$. Then $\tau_\varepsilon < \infty$ $\mathbf{P}_z$-a.s. and τ_ε are approximate optimal in the following sense

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}_z\big(G\big(Z_{\tau_\varepsilon}\big)\big) = V_*(z)$$

for all $z = (t, x)$. Further we have that $\tau_\varepsilon \uparrow \tau_*$ as $\varepsilon \downarrow 0$.

At first glans it seems that the initial setting of the optimal stopping problem (2.3) and Theorem 4.1 only cover the cases where the gain process is a function of time and the state of the process (X_t) . But the next two examples illustrate that problem (4.1) also cover some cases where the gain process contains path-dependent functional of (X_t) where it is a matter of defining (Z_t) properly.

For simplicity, let $n = 1$ in the examples below and moreover assume that (X_t) solves the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t$$

where (B_t) is a standard Brownian motion.

Example 4.4. (**Optimal stopping problems involving an integral**).
Let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}_+$ be continuous functions. Consider the optimal stopping problem

$$W_*(t,x) = \sup_\tau \mathbf{E}_x\Big(F\big(t+\tau, X_\tau\big) - \int_0^\tau c(X_u)\,du\Big)\,. \tag{4.3}$$

The integral term might be interpreted as an accumulated cost. This problem can be reformulated to fit in the setting of problem (2.3) and Theorem 4.1 by the following simple observations.

Set $A_t = \int_0^t c(X_u)\,du$ and define the process (Z_t) by $Z_t = \big(t, X_t, A_t\big)$. Thus (Z_t) is a diffusion process in $\mathbb{R}^3$ associated with the infinitesimal generator

$$\mathbf{L_Z} = \frac{\partial}{\partial t} + \mathbf{L_X} - c(x)\frac{\partial}{\partial a}$$

for $z = (t,x,a)$. Define the gain function $G(z) = F(t,x) - a$ and consider the new optimal stopping problem

$$V_*(z) = \sup_\tau \mathbf{E}\big(G\big(Z_\tau\big)\big)\,.$$

The new problem fits into the setting of Theorem 4.1 and it is clear that $W_*(x) = V_*(0,x,0)$. Note that the gain function G is linear in a.

Another method is by Itô formula to reduce the new problem (4.3) to the setting of the initial problem (2.3). Assume that we have a smooth function $x \mapsto D(x)$ satisfying $\mathbf{L_X}D(x) = c(x)$. Then Itô formula applied to $D(X_t)$ gives

$$D(X_t) = D(x) + \int_0^t \mathbf{L_X}D(X_u)\,du + M_t$$

where $M_t = \int_0^t D'(X_u)\,\sigma(X_u)\,dB_u$ is a continuous local martingale. By optional sampling we have that $\mathbf{E}_x(M_\tau) = 0$ (by localization and some uniform integrability conditions) and hence

$$\mathbf{E}_x\big(D(X_\tau)\big) = D(x) + \mathbf{E}_x\Big(\int_0^\tau c(X_u)\,du\Big)\,.$$

Therefore the problem reduces to solving the initial problem (2.3) where the gain function is given by $\widetilde{G}(t,x) = F(t,x) - D(x)$.

Example 4.5. (**Optimal stopping problems involving the maximum process**).
Peskir [20] made the following observation. Let (S_t) be the maximum process associated with (X_t) given by $S_t = \max_{0\le r\le t} X_r$. Then $(Z_t) = (X_t, S_t)$ is a two-dimensional process with state space $\{\,(x,s) \in \mathbb{R}^2 \mid x \le s\,\}$ (see Figure 1). It can be verified that (Z_t) is a continuous Markov process associated with the infinitesimal generator

$$\begin{aligned} \mathbf{L_Z} &= \mathbf{L_X} \quad && \text{for } x < s \\ \frac{\partial}{\partial s} &= 0 && \text{for } x = s \end{aligned}$$

with $\mathbf{L_X}$ in (2.2). Hence the optimal stopping problem

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\big(G\big(X_\tau, S_\tau\big)\big)$$

for $x \le s$ fits in the setting of Theorem 4.1.

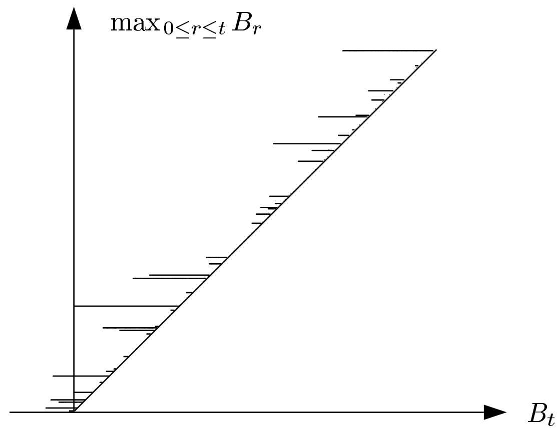

Figure 1. A computer simulation of a path of the two-dimensional process $(B_t(\omega), \max_{0\leq r\leq t} B_r(\omega))$ where (B_t) is a Brownian motion.

5. The free-boundary problem and the principle of smooth fit

For solving a specific optimal stopping problem the superharmonic characterization is not easy to apply. So to carry out explicit computations of the value function we need to use another methodology. In this section we consider the optimal stopping problem as a free-boundary (Stefan) problem. This is also important for computations of the value function from a numerical point of view.

First we need to introduce the notation of characteristic generator (see [31]) which is an extension of the infinitesimal generator. Let (Z_t) be the diffusion process given in Section 2. For any open set $U \subseteq \mathbb{R}_+ \times \mathbb{R}^n$ we associate the first exit time from U of (Z_t) given by

$$\tau_U = \inf\{\, t > 0 \,:\, Z_t \notin U \,\}\,.$$

Definition 5.1. (**Characteristic generator**). The characteristic generator $\mathcal{A}_{\mathbf{Z}}$ of (Z_t) is defined by

$$\mathcal{A}_{\mathbf{Z}} f(z) = \lim_{U \downarrow z} \frac{\mathbf{E}_z\Big(f\big(Z_{\tau_U}\big)\Big) - f(z)}{\mathbf{E}_z\big(\tau_U\big)}$$

where the limit is to be understood in the following sense. The open sets U_j decrease to the point z i.e. $U_{j+1} \subseteq U_j$ and $\cap_{j\geq 1} U_j = \{z\}$. If $\mathbf{E}_z(\tau_U) = \infty$ for all open sets $U \ni z$, we set $\mathcal{A}_{\mathbf{Z}} f(z) = 0$. Let $\mathcal{D}(\mathcal{A}_{\mathbf{Z}})$ be the family of Borel functions f for which the limit exists.

Remark 5.2. As already mentioned above the characteristic generator is an extension of the infinitesimal generator in the following sense.

$$\mathcal{D}(\mathbf{L}_{\mathbf{Z}}) \subseteq \mathcal{D}(\mathcal{A}_{\mathbf{Z}}) \quad \text{and} \quad \mathbf{L}_{\mathbf{Z}} f = \mathcal{A}_{\mathbf{Z}} f$$

for $f \in \mathcal{D}(\mathbf{L}_{\mathbf{Z}})$.

Assume in the sequel that the value function V_* in (2.3) is finite, i.e. $V_*(z) < \infty$. Let $C = \{\, z \in \mathbb{R}_+ \times \mathbb{R}^n \mid V_*(z) > G(z) \,\}$ be the domain of continued observation (see Theorem 4.1). Then the following result gives equations for the value function in the domain of continued observation (see [26]).

Theorem 5.3. *Let the gain function G be continuous and satisfy both conditions* (3.1) *and* (3.2). *Then the value function $V_*(z)$ for $z \in C$ belongs to the domain $\mathcal{D}(\mathcal{A}_\mathbf{Z})$ of the characteristic generator and solves the equation*

$$\mathcal{A}_\mathbf{Z} V_*(z) = 0 \tag{5.1}$$

for $z \in C$.

Remark 5.4. Since the gain function G is continuous and the value function V_* is lower semicontinuous, the domain of continued observation C is an open set in $\mathbb{R}_+ \times \mathbb{R}^n$. If $\tau_C < \infty$ $\mathbf{P}_z$-a.s. then we have from Theorem 4.1 that

$$V_*(z) = \mathbf{E}_z\big(G(Z_{\tau_C})\big) \; . \tag{5.2}$$

From general Markov process theory we have that the function in (5.2) solves the equation (5.1) and Theorem 5.3 follows directly. In other words, we are led to formulate equation (5.1).

By Remark 5.2, if the value function is C^2 in the domain of continued observation, the characteristic generator can be replaced by the infinitesimal generator. This has the advantage that the infinitesimal generator is explicitly given.

Equation (5.1) is referred to as a free-boundary problem. The domain of continued observation C is not known a priori but must be found along with the unknown value function V_* . In general, a free-boundary problem has many solutions and we need to add additional conditions (e.g. the principle of smooth fit) which the value function V_* satisfies. These additional conditions are not always enough to determine V_* . In such a case we either guess or find more sophisticated conditions (e.g. the maximality principle, see Example 6.1 in the next section).

The famous principle of smooth fit is one of the most used non-trivial boundary condition in optimal stopping. The principle is so useful that it is frequently applied in the literature (see, among others, McKean [13], Jacka [11] and Dubins, Shepp and Shiryaev [3]).

The principle of smooth fit. If the gain function G is smooth then a non-trivial boundary condition for the free-boundary problem might be the following

$$\frac{\partial V_*}{\partial t}(z)\Big|_{z\in\partial C} = \frac{\partial G}{\partial t}(z)\Big|_{z\in\partial C} \quad \text{and} \quad \frac{\partial V_*}{\partial x_i}(z)\Big|_{z\in\partial C} = \frac{\partial G}{\partial x_i}(z)\Big|_{z\in\partial C}$$

for $i = 1, \ldots, n$.

A result in Shiryaev [26] gives that the principle of smooth fit holds under fairly general assumptions. The principle of smooth fit is a very fine condition in the sense that the value function often is precisely C^1 at the boundary of the domain of continued observation. This is exactly what happens in the problems in the next section.

The above results can be used to formulate the following method for solving a particular stopping problem.

A recipe to solve optimal stopping problems.

Step 1. First one tries to guess the nature of the optimal stopping boundary. Then by ad hoc arguments one formulates a free-boundary problem with the infinitesimal generator and some boundary conditions. The boundary conditions could be the trivial ones (e.g. the value function is continuous, odd/even, normal reflection condition etc.) and the non-trivial ones such as the principle of smooth fit and the maximality principle.

Step 2. One solves the formulated free-boundary system and maximizes over the family of solutions if there is no unique solution.

Step 3. Finally one verifies that the guessed candidates for the value function and the optimal stopping time indeed are correct (e.g. using Itô formula).

The methodology has been used in, among others, Dubins, Shepp and Shiryaev [3], Graversen and Peskir [7], Pedersen [16] and Shepp and Shiryaev [23].

It is generally difficult to find the appropriate solution of the (partial) differential equation

$$\mathbf{L_Z} V(z) = 0 \, .$$

Therefore it is of most interest to formulate the free-boundary problem such that the dimension is as small as possible. The examples below give some cases where the dimension can be reduced. For simplicity let $n = 1$ and moreover assume that (X_t) solves the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t$$

where (B_t) is a standard Brownian motion.

Example 5.5. (**Integral problem and discounted problem**). In the cases of linear or multiplicative functionals we have from general Markov process theory that the free-boundary problem is of dimension one.

1. Let $F : \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}_+$ be continuous functions and let the gain function be given by $G(x, a) = F(x) - a$ which is linear in a (see Example 4.4). Let $(Z_t) = (X_t, A_t)$ where $A_t = \int_0^t c(X_u)\, du$ and consider the optimal stopping problem

$$V_*(x) = \sup_\tau \mathbf{E}_x \Big(F(X_\tau) - \int_0^\tau c(X_u)\, du \Big) \, .$$

At first glans it seems to be a two-dimensional problem, but from Markov process theory we have that the free-boundary problem can formulated as

$$\mathbf{L_X} V_*(x) = -c(x)$$

for x in the domain of continued observation which is also clear from the last part of Example 4.4. This is a one-dimensional problem.

2. Given the gain function $G(t, x) = e^{-\lambda t} F(x)$ where $\lambda > 0$ is a constant. Let $(Z_t) = (t, X_t)$ and consider the "two-dimensional" optimal stopping problem

$$V_*(x) = \sup_\tau \mathbf{E}_x \big(e^{-\lambda \tau} F(X_\tau) \big) \, .$$

The free-boundary problem can in this case be formulated as

$$\mathbf{L_X} V_*(x) = \lambda\, V_*(x)$$

for x in the domain of continued observation. Again this is a one-dimensional problem.

Example 5.6. (**Deterministic time-change method**). This example uses a deterministic time-change to reduce the problem. The method is described in Pedersen and Peskir [19]. Consider the optimal stopping problem

$$V_*(t, x) = \sup_\tau \mathbf{E}_x \big(\alpha(t + \tau)\, X_\tau \big)$$

where α is a smooth non-linear function. Thus the value function V_* should solve the following partial differential equation

$$\frac{\partial V_*}{\partial t}(t, x) + \mathbf{L_X} V_*(t, x) = 0$$

for (t,x) in the domain of continued observation.

The time-change method is to transform the original problem into a new optimal stopping problem such that the new value function solves an ordinary differential equation. The problem is to find a deterministic time-change $t \mapsto \sigma_t$ satisfying the following two conditions.

(i) $t \mapsto \sigma_t$ is continuous and strictly increasing

(ii) there exists a one-dimensional time-homogeneous diffusion (Y_t) with infinitesimal generator $\mathbf{L_Y}$ such that $\alpha(\sigma_t)\, X_{\sigma_t} = e^{-\lambda t}\, Y_t$ for some $\lambda \in \mathbb{R}$.

The condition (i) ensures that τ is a stopping time for (Y_t) if and only if σ_τ is a stopping time for (X_t) . Substituting (ii) in the problem we obtain the new (time-changed) value function

$$W_*(y) = \sup_\tau \mathbf{E}_y\big(e^{-\lambda\tau} Y_\tau\big) \; .$$

As we saw in Example 5.5 the new problem might solve the ordinary differential equation

$$\mathbf{L_Y} W_*(y) = \lambda\, W_*(y)$$

in the domain of continued observation. Given the diffusion (X_t) the crucial point is to find the process (Y_t) and the time-change σ_t fulfilling the two conditions above. By Itô calculus it can be shown that the time-change given by

$$\sigma_t = \inf \Big\{ \, r > 0 \,\Big|\, \int_0^r \rho(u)\, du > t \Big\}$$

where $\rho(\cdot)$ is such that the two terms

$$\Big(\tfrac{\alpha'(t)}{\alpha(t)}\, y + \alpha(t)\, \mu\Big(\tfrac{y}{\beta(t)}\Big)\Big) \tfrac{1}{\rho(t)} \quad \text{and} \quad \alpha(t)^2\, \sigma^2\Big(\tfrac{y}{\alpha(t)}\Big) \tfrac{1}{\rho(t)}$$

do not depend on t , will fulfill the above two conditions. This clearly imposes the following conditions on α to make the method applicable

$$\mu\Big(\tfrac{y}{\alpha(t)}\Big) = \gamma(t)\, G_1(y) \quad \text{and} \quad \sigma^2\Big(\tfrac{y}{\alpha(t)}\Big) = \tfrac{\gamma(t)}{\alpha(t)}\, G_2(y)$$

where $\gamma(t)$, $G_1(y)$ and $G_2(y)$ are functions required to exist. For more information and remaining details of this method see [19] (see also [9]).

6. Examples and applications

In this section we solve some problems by applying the recipe established in the previous section. The focus will be on optimal stopping problems for the maximum process associated with a one-dimensional diffusion.

Let $n = 1$. Assume that (X_t) is a non-singular diffusion with state space $\mathbb{R}$, i.e. $x \mapsto \sigma(x) > 0$ and moreover that there exists a standard Brownian motion (B_t) such that (X_t) solves the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t \; .$$

Let $S_t = \big(\max_{0 \le r \le t} X_r\big) \vee s$ denote the maximum process associated with (X_t) that starts at $s \ge x$ under $\mathbf{P}_{x,s}$. The scale function and speed measure of (X_t) are given by

$$S(x) = \int_0^x \exp\Big(-2 \int_0^u \frac{\mu(r)}{\sigma^2(r)}\, dr \Big)\, du \quad \text{and} \quad m(dx) = \frac{2}{S'(x)\, \sigma^2(x)}\, dx$$

for $x \in \mathbb{R}$.

The first example is also important from the general optimal stopping theory point of view.

Example 6.1. (**The maximality principle**). The maximality principle introduced in this example was proved by Peskir [20]. Let $x \mapsto c(x) > 0$ be a continuous (cost) function. Consider the optimal stopping problem with the value function

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\Big(S_\tau - \int_0^\tau c(X_u)\,du\Big) \tag{6.1}$$

where the supremum is taken over all stopping times τ for (X_t) satisfying

$$\mathbf{E}_{x,s}\Big(\int_0^\tau c(X_u)\,du\Big) < \infty \tag{6.2}$$

for all $x \leq s$.

1. The process (X_t, S_t) with state space $\{\,(x,s) \in \mathbb{R} \,|\, x \leq s\,\}$ changes only in the second coordinate when it hits the diagonal $x = s$ in $\mathbb{R}^2$ (see Figure 1). It can be shown that it is not optimal to stop on the diagonal. Due to the positive cost function $c(\cdot)$ we might believe that the optimal stopping boundary should be a function staying below the diagonal, i.e. $s \mapsto g_*(s) < s$. Thus the stopping time $\tau_* = \inf\{\,t > 0 : X_t \leq g_*(S_t)\,\}$ should be optimal or the domain of continued observation should be $C = \{\,(x,s) \in \mathbb{R}^2 \,|\, g_*(s) < x \leq s\,\}$. To compute the value function and the optimal stopping boundary it is natural to formulate the following free-boundary problem

$$\mathbf{L_X} V(x,s) = c(x) \qquad \text{for} \quad g(s) < x < s \quad \text{with} \quad s \quad \text{fixed} \tag{6.3}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \qquad \text{(normal reflection)} \tag{6.4}$$

$$V(x,s)\Big|_{x=g(s)+} = s \qquad \text{(instantaneous stopping)} \tag{6.5}$$

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = 0 \quad \text{(smooth fit)}\ . \tag{6.6}$$

Note that (6.3) and (6.4) follow from Example 4.5 and Example 5.5. The condition (6.5) is clear and since the setting is smooth we also believe that the smooth fit condition (6.6) holds. (The theorem below shows that the guessed system indeed is correct).

2. Define the function

$$V_g(x,s) = s + \int_{g(s)}^x \big(S(x) - S(u)\big)c(u)\,m(du) \tag{6.7}$$

for $g(s) \leq x \leq s$ and set $V_g(x,s) = s$ for $x \leq g(s)$. Let the C^1-function $s \mapsto g(s)$ solve the first order non-linear differential equation

$$g'(s) = \frac{\sigma^2(g(s))\,S'(g(s))}{2c\big(g(s)\big)\,\big(S(s) - S(g(s))\big)}\ . \tag{6.8}$$

For a solution $s \mapsto g(s) < s$ of equation (6.8) the corresponding function $V_g(x,s)$ in (6.7) solves the system in the region $g(s) < x < s$.

The problem now is to choose the right optimal stopping boundary $s \mapsto g(s) < s$. To do so we need a new principle, and it will be the maximality principle. The main observations in [20] were the following.

(i) $g \mapsto V_g(x,s)$ is increasing.

(ii) The function $(a,x,s) \mapsto V_g(x,s) - a$ is superharmonic for the Markov process $(Z_t) = (A_t, X_t, S_t)$ (for stopping times τ satisfying (6.2)), where $A_t = \int_0^t c(X_u)\,du + a$.

By the superharmonic characterization of the value function in Theorem 4.1 and the above two observations we are led to formulate the following principle for determining the optimal stopping boundary.

The maximality principle. The optimal stopping boundary $s \mapsto g_*(s)$ for the problem (6.1) is the maximal solution of the differential equation (6.8) which stays strictly below the diagonal in $\mathbb{R}^2$ (and is just called the maximal solution in the sequel).

3. It was proved in [20] that this principle is equivalent to the superharmonic characterization of the value function. The result is formulated in the next theorem and is motivated by Theorem 4.1.

Theorem 6.2. *Consider the optimal stopping problem* (6.1)*. Then we have.*

(**I**). *Let* $s \mapsto g_*(s)$ *denote the maximal solution of* (6.8) *which stays below the diagonal in* $\mathbb{R}^2$*. Then the value function is given by*

$$V_*(x,s) = \begin{cases} s + \displaystyle\int_{g_*(s)}^{x} \big(S(x) - S(u)\big) c(u)\, m(du) & \text{for} \quad g_*(s) < x \le s \\ s & \text{for} \quad x \le g_*(s)\ . \end{cases}$$

(**II**). *The stopping time* $\tau_* = \inf\{\, t > 0 \,:\, X_t \le g_*(S_t)\,\}$ *is optimal whenever it satisfies condition* (6.2)*.*

(**III**). *If there exists an optimal stopping time* σ *in* (6.1) *satisfying* (6.2)*, then* $\tau_* \le \sigma$ $\mathbf{P}_{x,s}$*-a.s. for all* (x,s) *, and* τ_* *is also an optimal stopping time.*

(**IV**). *If there is no maximal solution of* (6.8) *which stays strictly below the diagonal in* $\mathbb{R}^2$*, then* $V_*(x,s) = \infty$ *for all* (x,s) *, and there is no optimal stopping time.*

For more information and details see [20]. A similar method was used in Pedersen and Peskir [18] to compute the expectation of Azéma-Yor stopping times.

The theorem extends to diffusions with other state spaces in $\mathbb{R}$. In particular non-negative diffusion version of the theorem is of interest to derive sharp maximal inequalities which will be applied in the next example.

Peskir [20] conjectured that the maximality principle does holds for the discounted version of problem (6.11). In Shepp and Shiryaev [23] and Pedersen [17] the principle is showed to hold in specific cases. A technical difficult problem in verifying the conjecture is that the corresponding free-boundary problem may have no simple solution and hence defines the (optimal) boundary function implicitly.

Example 6.3. (**Doob's inequality for Brownian motion**). This example is an application of Example 6.2 and the inequality below was derived by Graversen and Peskir [6]. Consider the optimal stopping problem (6.1) with $\mu(x) = \frac{1}{2}p(p-1)\, x^{1-2/p}$, $\sigma^2(x) = p^2\, x^{2-2/p}$ and $c(x) = c\, x^{(p-2)/p}$ for $x > 0$ where $p > 1$ and $c > 0$ are constants. Then (X_t) may be realized as $X_t = |B_t + x|^p$. Then the value function is

$$V_*(x,s) = s - \tfrac{2c}{p-1}\, g_*(s)^{1-1/p}\, x^{1/p} + \tfrac{2c}{p}\, g_*(s) + \tfrac{2c}{p(p-1)}\, x$$

where $s \mapsto g_*(s) < s$ is the maximal solution of the differential equation

$$g'(s) = \frac{p\, g(s)^{1/p}}{2c\big(s^{1/p} - g(s)^{1/p}\big)}\ .$$

The maximal solution can be found to be $g_*(s) = \alpha\, s$ where $0 < \alpha < 1$ is the greater root of the equation

$$\alpha - \alpha^{1-1/p} + p/(2c) = 0 \;. \tag{6.9}$$

One can show that the equation (6.9) admits two roots if and only if $c > p^{p+1}/(2(p-1)^{p-1})$. The stopping times $\tau_*(c) = \inf\{\, t > 0 \,:\, X_t \le \alpha\, S_t \,\}$ satisfy $\mathbf{E}_{x,s}(\tau_*(c)^{p/2}) < \infty$ if and only if $c > p^{p+1}/(2(p-1)^{p-1})$. Then by an extended version of Theorem 6.2 for non-negative diffusions and an observation in Example 5.5 we have by the definition of the value function for $c > p^{p+1}/(2(p-1)^{p-1})$ that

$$\mathbf{E}\big(\max\nolimits_{0\le t\le\tau} |B_t + x|^p\big) \le \tfrac{c}{p(p-1)}\, \mathbf{E}\big(|B_\tau + x|^p\big) + V_*(x,x) - \tfrac{c}{p(p-1)}\, x^p$$

for all stopping times τ for (B_t) satisfying $\mathbf{E}(\tau^{p/2}) < \infty$. Letting $c \downarrow p^{p+1}/(2(p-1)^{p-1})$, we get Doob's inequality.

Theorem 6.4. *Let (B_t) be a standard Brownian motion started at x under $\mathbf{P}_x$ for $x \ge 0$, let $p > 1$ be given and fixed, and let τ be any stopping time for (B_t) such that $\mathbf{E}_x(\tau^{p/2}) < \infty$. Then the following inequality is sharp*

$$\mathbf{E}\big(\max\nolimits_{0\le t\le\tau} |B_t + x|^p\big) \le \Big(\tfrac{p}{p-1}\Big)^p\, \mathbf{E}\big(|B_\tau + x|^p\big) - \tfrac{p}{p-1}\, x^p \;. \tag{6.10}$$

The constants $(p/(p-1))^p$ and $p/(p-1)$ are the best possible and the equality in (6.10) *is attained through the stopping times $\tau_* = \inf\{\, t > 0 \,:\, X_t \le \alpha\, S_t \,\}$ for $c \downarrow p^{p+1}/(2(p-1)^{p-1})$ where $0 < \alpha < 1$ is the greater root of equation* (6.9).

For details see [6]. The results are extended to Bessel processes in Dubins, Shepp and Shiryaev [3] and Pedersen [16].

Example 6.5. (**Optimal prediction of the ultimate maximum of Brownian motion**). This problem was formulated and solved by Graversen, Peskir and Shiryaev [9]. In this example we shall predict the ultimate maximum of Brownian motion by a mean-square distance in an optimal way. Let $\mathcal{M}$ be the family of all stopping times τ for (B_t) satisfying $\tau \le 1$ and let $S_t = \max_{0\le r\le t} B_r$. Consider the optimal stopping problem with value function

$$V_* = \inf_{\tau\in\mathcal{M}} \mathbf{E}\big(S_1 - B_\tau\big)^2 \;. \tag{6.11}$$

This problem does not fall under the general optimal stopping theory since the gain process is not adapted to the natural filtration of (B_t) . The idea is to transform problem (6.11) into an equivalent problem that can be solved by the method introduced in the previous section.

To follow the above plan we will need the function

$$\Phi(x) = \int_{-\infty}^{x} \tfrac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du$$

for $x \in \mathbb{R}$. Since S_1 is square integrable then by Itô-Clark representation theorem formula we have that

$$S_1 = \mathbf{E} S_1 + \int_0^1 H_u\, dB_u$$

where (H_t) is a unique adapted process satisfying $\mathbf{E}\big(\int_0^1 H_u^2\, du\big) < \infty$. Further it is known that

$$H_t = 2\left(1 - \Phi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right)\right) .$$

If (M_t) denotes the square integrable martingale $M_t = \int_0^t H_u\, dB_u$, we have by martingale theory that $\mathbf{E}(S_1 - B_\tau)^2 = \mathbf{E}(\int_0^\tau (1-2H_u)\, du) + 1$ for all $\tau \in \mathcal{M}$. Hence problem (6.11) can be represented as

$$V_* = \inf_{\tau \in \mathcal{M}} \mathbf{E}\biggl(\int_0^\tau c\Bigl(\frac{S_u - B_u}{\sqrt{1-u}} \Bigr)\, du \biggr) + 1$$

where $c(x) = 4\Phi(x) - 3$. By Lévy's theorem and general optimal stopping theory, we have that problem (6.11) is equivalent to

$$V_* = \inf_{\tau \in \mathcal{M}} \mathbf{E}\biggl(\int_0^\tau c\Bigl(\frac{|B_u|}{\sqrt{1-u}} \Bigr)\, du \biggr) + 1\, .$$

The form of the gain function indicates that the deterministic time-change method introduced in Example 5.6 can be applied successfully. Let $\sigma_t = 1 - e^{-2t}$ be the time-change and let $(Z_t)_{t \geq 0}$ be the time-changed process given by $Z_t = B_{\sigma_t}/\sqrt{1-\sigma_t}$. It can be shown by Itô formula that (Z_t) is the strong solution of the stochastic differential equation

$$dZ_t = Z_t\, dt + \sqrt{2}\, d\beta_t$$

where $(\beta_t)_{t \geq 0}$ is a Brownian motion. Hence (Z_t) is a diffusion with the infinitesimal generator

$$\mathbf{L_Z} = z\, \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}$$

for $z \in \mathbb{R}$. Substituting the time-change we get

$$V_* = \inf_{\sigma} \mathbf{E}\biggl(\int_0^\sigma e^{-2u}\, c(|Z_u|)\, du \biggr) + 1\, .$$

Hence the initial problem (6.11) reduces to solving

$$W_*(z) = \inf_{\sigma} \mathbf{E}_z\biggl(\int_0^\sigma e^{-2u}\, c(|Z_u|)\, du \biggr)$$

where the infimum is taken over all stopping times σ for (Z_t) and $V_* = W_*(0)$. This is a problem that we can solve with the method from Section 5.

We might believe that the domain of continued observation is a symmetric interval around zero, i.e. $C = \{\, z \in \mathbb{R} \,|\, z \in (-z_*, z_*)\,\}$ and the value function is an even C^1-function or equivalent $W_*'(0) = 0$. Therefore we are led to formulate the following free-boundary system

$$\begin{aligned}
&\bigl(\mathbf{L_Z} - 2\bigr) W(z) = -c(|z|) && \text{for} \quad -z_* < z < z_* \\
&W(\pm z_*) = 0 && \text{(instantaneous stopping)} \\
&W'(\pm z_*) = 0 && \text{(smooth fit)} \\
&W'(0) = 0\, .
\end{aligned}$$

The system has a unique solution given by

$$W(z) = \Phi(z_*)\, \bigl(1+z^2\bigr) - 2\Phi'(z) + \bigl(1-z^2\bigr)\, \Phi(z) - 3/2 \tag{6.12}$$

for $z \in [0, z_*]$ where z_* is the unique solution of the equation (6.13). By Itô formula it can be proved that $W(z)$ in (6.12) is the value function and $\sigma_* = \inf\{\, t > 0 \,:\, |B_t| \geq \sqrt{1-t}\,\}$ is an optimal stopping time. Transforming the value function and the optimal strategy back to the initial problem (6.11) we have the following result (for more details see [9]).

Theorem 6.6. *Consider the optimal stopping problem* (6.11). *Then the value* V_* *is given by*

$$V_* = 2\Phi(z_*) - 1 = 0.73\dots$$

where $z_* = 1.12\dots$ *is the unique root of the equation*

$$4\Phi(z_*) - 2z_*\Phi'(z_*) - 3 = 0\ . \tag{6.13}$$

The following stopping time is optimal $\tau_* = \inf\{\, t > 0 : S_t - B_t \ge z_*\sqrt{1-t}\,\}$.

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Computing the Expectation of the Azéma-Yor Stopping Times

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Given the maximum process $(S_t) = (\max_{0 \le r \le t} X_r)$ associated with a diffusion $((X_t), \mathbf{P}_x)$, and a continuous function g satisfying $g(s) < s$, we show how to compute the expectation of the Azéma-Yor stopping time

$$\tau_g = \inf \{ t > 0 : X_t \le g(S_t) \}$$

as a function of x. The method of proof is based upon verifying that the expectation solves a differential equation with two boundary conditions. The third 'missing' condition is formulated in the form of a minimality principle which states that the expectation is the minimal non-negative solution to this system. It enables us to express this solution in a closed form. The result is applied in the case when (X_t) is a Bessel process and g is a linear function.

1. Formulation of the problem

Let $((X_t), \mathbf{P}_x)$ be a non-negative canonical diffusion with the infinitesimal operator on $(0, \infty)$ given by

$$\mathbf{L_X} = \tfrac{1}{2}\sigma^2(x) \frac{\partial^2}{\partial x^2} + \mu(x) \frac{\partial}{\partial x}$$

where σ^2 and μ are continuous functions on $(0, \infty)$ and σ^2 is furthermore strictly positive (see [6]). Assume there exists a standard Wiener process (B_t) such that for every $x > 0$

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t \quad , \quad X_0 = x \quad \mathbf{P}_x\text{-a.s.} \tag{1.1}$$

The main purpose of this paper is to compute the expectation of the Azéma-Yor stopping time (see [1]). More precisely, for any continuous function g on $[0, \infty)$ satisfying $0 < g(s) < s$ for $s > 0$, the Azéma-Yor stopping time is defined as follows

$$\tau_g = \inf \{ t > 0 : X_t \le g(S_t) \}$$

where (S_t) is the maximum process associated with (X_t)

$$S_t = \big(\max_{0 \le r \le t} X_r \big) \vee s \tag{1.2}$$

started at $s > 0$. The main aim of this paper is to present a method for computing the function

$$m(x, s) = \mathbf{E}_{x,s}(\tau_g)$$

for $0 < x \le s$. Here the expectation is taken with respect to the probability measure $\mathbf{P}_x := \mathbf{P}_{x,s}$ under which the process (X_t) starts at x and the process (S_t) starts at s.

The motivation to compute the expectation of such stopping times comes from some optimal stopping problems (see [2], [4], [3] and [7]). In these problems it is of interest to know the expected waiting time for the optimal stopping strategy which is of the form τ_g for some g.

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In view of this application we have assumed that the diffusion (X_t) and the function $s \mapsto g(s)$ are non-negative, but it will be clear from our considerations below that the results obtained are generally valid.

The method of proof relies upon showing that the expectation of the stopping time solves a differential equation with two boundary conditions. The third 'missing' condition is formulated in the form of a minimality principle which states that *the expectation is the minimal non-negative solution to this system* (see Figure 1 below). It enables us to pick up the expectation among all possible candidates in a unique way. The minimality principle is the main novelty in this approach (compare with [2], [4] and [3]).

In Section 2 the minimality principle is formulated, and in Section 3 the existence and uniqueness of the minimal solution is proved. The main theorem is proved in Section 4, and in Section 5 an application of the theorem is given

2. The minimality principle

In the first part of this section we shall observe that the function $x \mapsto m(x,s)$ solves a differential equation with two boundary conditions. In the remaining part of the section we will present the minimality principle as the 'missing' condition, which will enable us to select the expectation of the stopping time in a unique way.

In the sequel we need the following definitions and results. The scale function is for $x > 0$ given by

$$S(x) = \int_1^x \phi(u)\, du$$

where

$$\phi(x) = \exp\Big(-2 \int_1^x \mu(u)/\sigma^2(u)\, du \Big)\ .$$

We define as usual the first exit time from an interval by

$$\tau_{a,b} = \inf\{\, t > 0 \,:\, X_t \notin (a,b)\,\}$$

for $0 < a < b$, and the following formulas for $0 < a < x < b$ are well-known

$$\mathbf{P}_x(X_{\tau_{a,b}} = a) = 1 - \mathbf{P}_x(X_{\tau_{a,b}} = b) = \frac{S(b) - S(x)}{S(b) - S(a)} \tag{2.1}$$

$$\mathbf{E}_x(\tau_{a,b}) = 2 \int_a^x \frac{S(b)-S(x)}{S(b)-S(a)}\, \frac{S(u)-S(a)}{\sigma^2(u)\phi(u)}\, du + 2 \int_x^b \frac{S(x)-S(a)}{S(b)-S(a)}\, \frac{S(b)-S(u)}{\sigma^2(u)\phi(u)}\, du\ . \tag{2.2}$$

Let g be a continuous function satisfying $0 < g(s) < s$ for $s > 0$ such that $m(x,s) = \mathbf{E}_{x,s}(\tau_g)$ is finite for all $0 < x \le s$. We will now state the first result. Whenever $s > 0$ is given and fixed, the function $x \mapsto m(x,s)$ solves the differential equation

$$\mathbf{L_X} m(x,s) = -1 \qquad \text{for} \quad g(s) < x < s \tag{2.3}$$

with the following two boundary conditions

$$m(x,s)\Big|_{x=g(s)+} = 0 \quad \text{(instantaneous stopping)} \tag{2.4}$$

$$\frac{\partial m}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)}\ . \tag{2.5}$$

A first step in the direction of verifying that $(x,s) \mapsto m(x,s)$ satisfies the system above is contained in the following result.

Lemma 2.1. *The function* $s \mapsto m(s,s) := M(s)$ *is* C^1 *and satisfies the equation*

$$M'(s) = \frac{\phi(s)}{S(s) - S(g(s))}\left(M(s) - 2\int_{g(s)}^{s} \frac{S(u) - S(g(s))}{\sigma^2(u)\phi(u)}\, du\right)\ . \tag{2.6}$$

Proof. The proof is essentially contained in Lemma 1 and Lemma 2 in [4]. Alternatively, to obtain a better feeling why (2.6) holds, as well to derive it in another way, one could use (2.7) below with (2.1)+(2.2) above to verify that $(\partial m/\partial x)(s-,s)$ equals the right-hand side in (2.6), thus showing that (2.5) is equivalent to (2.6), and then follow the second part of the proof of Theorem 4.1 below. □

Let $g(s) < x < s$ be given and fixed. It is immediately seen that

$$\tau_g = \tau_{g(s),s} + \tau_g \circ \theta_{\tau_{g(s),s}}\, \mathbf{1}_{\{X_{\tau_{g(s),s}} = s\}} \quad \mathbf{P}_{x,s}\text{-a.s.}$$

and by applying strong Markov property we get

$$m(x,s) = \mathbf{E}_x(\tau_{g(s),s}) + m(s,s)\,\mathbf{P}_x(X_{\tau_{g(s),s}} = s)\ . \tag{2.7}$$

From (2.1) and (2.2) we see that $G(x) = \mathbf{P}_x(X_{\tau_{g(s),s}} = s)$ and $H(x) = \mathbf{E}_x(\tau_{g(s),s})$ solve the following well-known systems respectively

$$\begin{aligned} &\mathbf{L_X} G(x) = 0 \quad \text{for} \quad g(s) < x < s \\ &G(g(s)) = 1 - G(s) = 0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\mathbf{L_X} H(x) = -1 \quad \text{for} \quad g(s) < x < s \\ &H(g(s)) = H(s) = 0\ . \end{aligned} \tag{2.9}$$

Consequently, by (2.6)-(2.9) we easily verify that $x \mapsto m(x,s)$ solves the system (2.3)-(2.5).

Note since τ_g may be viewed as the exit time by diffusion (X_t, S_t) from an open set, the equation (2.3) is well-known and the condition (2.4) is evident. The condition (2.5) is less evident but is known to be satisfied in a similar context (see [6] p. 118-119).

Unfortunately $(x,s) \mapsto m(x,s)$ is not uniquely determined by (2.3) and the two boundary conditions (2.4) and (2.5). Thus we need another condition to determine $(x,s) \mapsto m(x,s)$ uniquely. We formulate the third 'missing' condition in the form of a *minimality principle* (see Figure 1 below): the expectation $m(x,s)$ is the minimal non-negative solution to the system (2.3)-(2.5).

3. Existence and uniqueness of the minimal solution

Since $m(x,s)=0$ for $0<x\le g(s)$ we only need to consider m on $g(s)<x\le s$. Throughout we shall consider the system

$$
\begin{aligned}
&\mathbf{L_X} m(x,s) = -1 \quad \text{for} \quad g(s)<x<s \\
&m(x,s)\Big|_{x=g(s)+} = 0 \\
&\frac{\partial m}{\partial s}(x,s)\Big|_{x=s-} = 0\,.
\end{aligned}
\tag{3.1}
$$

Motivated by the minimality principle, in this section we shall prove the existence (and uniqueness) of a minimal non-negative solution to (3.1). Let us introduce the following notation:

$$
\mathcal{M} = \{\, m : D \mapsto \mathbb{R} \mid m \in C^{2,1}(D^\circ) \cap C(D)\ ,\ m \text{ solves (3.1) and } m \ge 0 \,\} \tag{3.2}
$$

where $D = \{\,(x,s) \mid g(s) \le x \le s\,,\ s>0\,\}$ and D° is the interior of D. The function m belongs to $C^{2,1}(D^\circ)$ if $x \mapsto m(x,s)$ is C^2 and $s \mapsto m(x,s)$ is C^1 on D°.

The main result of this section may be now formulated as follows. If $\mathcal{M}$ is non empty then it contains a minimal element, i.e.

$$
m_* = \inf\{\, m \mid m \in \mathcal{M}\,\} \in \mathcal{M}
$$

where the infimum is taken pointwise. Combined with the results in Section 2 this will be deduced in the proof of Theorem 4.1 below by using Itô calculus. The proof we present here is based upon the uniqueness theorem for the first and second-order differential equations.

For this note that the uniqueness theorem implies that if m_1 and m_2 belong to $\mathcal{M}$ then either $m_1 > m_2$ or $m_1 < m_2$ on D°. Let $(x_0,s_0) \in D$ be given and let $\{m_n\}_{n\ge1}$ be a sequence of functions from $\mathcal{M}$ such that $m_n(x_0,s_0) \downarrow m_*(x_0,s_0)$. Due to the remark just mentioned, the sequence $\{m_n\}_{n\ge1}$ is decreasing, and therefore the limit exists everywhere, i.e.

$$
m_n(x,s) \downarrow \widetilde{m}(x,s)
$$

for all $(x,s) \in D$. If we can show that

$$
\widetilde{m} \in \mathcal{M} \tag{3.3}
$$

then using the uniqueness theorem it follows that $\widetilde{m} = m_*$.

In order to prove (3.3) we first show that $x \mapsto \widetilde{m}(x,s)$ solves the differential equation in (3.1). By the instantaneous stopping condition, and the uniqueness theorem, m_n can be written as

$$
m_n(x,s) = \mathbf{E}(\tau_{g(s),s}) + A_n(s)\,\mathbf{P}_x(X_{\tau_{g(s),s}} = s)
$$

where $s \mapsto A_n(s)$ is a C^1-function. Since $\{m_n\}_{n\ge1}$ is a decreasing sequence of functions, the sequence $\{A_n\}_{n\ge1}$ is also decreasing, and therefore it converges pointwise to a function $\tilde{A}$, i.e.

$$
A_n(s) \downarrow \tilde{A}(s)
$$

for $n \to \infty$. Hence $x \mapsto \widetilde{m}(x,s)$ solves the differential equation in (3.1).

Obviously $x \mapsto \widetilde{m}(x,s)$ satisfies the first boundary condition (instantaneous stopping)

$$
\widetilde{m}(x,s)\Big|_{x=g(s)+} = 0
$$

since each $x \mapsto \widetilde{m}_n(x,s)$ satisfies this condition.

Finally, to verify the second boundary condition (normal reflection), note that straightforward computations based on the normal reflection condition and (2.1)+(2.2) show that $s \mapsto A_n(s)$ solves the following differential equation

$$A_n'(s) = \frac{\phi(s)}{S(s)-S(g(s))}\left(A_n(s) - 2\int_{g(s)}^{s} \frac{S(u)-S(g(s))}{\sigma^2(u)\phi(u)}\,du\right) \tag{3.4}$$

or equivalently

$$A_n(s) - A_n(s_0) = \int_{s_0}^{s} \frac{\phi(u)}{S(u)-S(g(u))}\left(A_n(u) - 2\int_{g(u)}^{u} \frac{S(r)-S(g(u))}{\sigma^2(r)\phi(r)}\,dr\right)du$$

Applying the monotone convergence theorem, we find that $s \mapsto \tilde{A}(s)$ also solves the differential equation (3.4), and we can conclude that $\widetilde{m} \in \mathcal{M}$.

4. The expectation of the Azéma-Yor stopping times

The main result of the paper is contained in the following theorem.

Theorem 4.1. *Let* $((X_t), \mathbf{P}_x)$ *be the non-negative diffusion defined in* (1.1) *and let* (S_t) *be the maximum process associated with* (X_t) *in* (1.2). *Let* g *be a continuous function on* $[0,\infty)$ *satisfying* $0 < g(s) \le s$ *for* $s > 0$ *and let us define the stopping time*

$$\tau_g = \inf\{\, t > 0 \,:\, X_t \le g(S_t)\,\}\,.$$

If $\mathcal{M}$ *from* (3.2) *is non-empty, then* $\mathbf{E}_{x,s}(\tau_g)$ *is finite and is given by*

$$\mathbf{E}_{x,s}(\tau_g) = \begin{cases} m_*(x,s) & \text{for} \quad g(s) < x \le s \\ 0 & \text{for} \quad 0 < x \le g(s) \end{cases}$$

where $(x,s) \mapsto m_*(x,s)$ *is the minimal element in* $\mathcal{M}$. *The converse is also true, and we have the following explicit formula and a criterion for verifying that* $\mathcal{M}$ *is non-empty*

$$\begin{aligned} \mathbf{E}_{x,s}(\tau_g) = {} & 2\frac{S(s)-S(x)}{S(s)-S(g(s))}\int_{g(s)}^{x} \frac{S(u)-S(g(s))}{\sigma^2(u)\phi(u)}du + 2\frac{S(x)-S(g(s))}{S(s)-S(g(s))}\left\{\int_{x}^{s} \frac{S(s)-S(u)}{\sigma^2(u)\phi(u)}du \right.\\ & \left. + \int_{s}^{\infty} \frac{\phi(u)}{S(u)-S(g(u))}\left(\int_{g(u)}^{u} \frac{S(r)-S(g(u))}{\sigma^2(r)\phi(r)}dr\right)\exp\left(-\int_{s}^{u} \frac{\phi(r)}{S(r)-S(g(r))}dr\right)du\right\} \end{aligned} \tag{4.1}$$

for $g(s) < x \le s$ *with* $\mathbf{E}_{x,s}(\tau_g) = 0$ *for* $0 < x \le g(s)$, *which is valid in the usual sense (if the right-hand side in* (4.1) *is finite, then so is the left-hand side, and vice versa)*

Proof. For $(x_0, s_0) \in D$ given and fixed, consider the set

$$G = \{\,(x,s) \,|\, g(s) < x < s+1\,\}$$

Choose bounded open sets $G_1 \subseteq G_2 \subseteq \ldots$ such that

$$(x_0, s_0) \in G_1 \quad \text{and} \quad \bigcup_{n=1}^{\infty} G_n = G\,.$$

Define the exit time of the two-dimensional diffusion (X_t, S_t) from G_n by

$$\sigma_n = \inf\{\, t > 0 \,:\, (X_t, S_t) \notin G_n\,\}$$

Note that $\mathbf{E}_{x_0,s_0}(\sigma_n) < \infty$ and $\sigma_n \uparrow \tau_g$ $\mathbf{P}_{x_0,s_0}$-a.s. as $n \to \infty$.

Let m be any function in $\mathcal{M}$. Note that $m \in C^{2,1}$ and (S_t) is of bounded variation so that Itô formula can be applied (see Remark 1 in [5] p. 139). In this way we get $\mathbf{P}_{x_0,s_0}$-a.s.

$$m(X_{t\wedge\sigma_n}, S_{t\wedge\sigma_n}) = m(x_0, s_0) + \int_0^{t\wedge\sigma_n} \frac{\partial m}{\partial x}(X_u, S_u)\sigma(X_u)\, dB_u + \int_0^{t\wedge\sigma_n} \mathbf{L_X} m(X_u, S_u)\, du + \int_0^{t\wedge\sigma_n} \frac{\partial m}{\partial s}(X_u, S_u)\, dS_u \ .$$

Due to the normal reflection condition the last integral is identically zero. Since the set of those $u > 0$ for which $X_u = S_u$ is of Lebesgue measure zero, and $\mathbf{L_X} m(x,s) = -1$ for $g(s) < x < s$, we can conclude that $\mathbf{P}_{x_0,s_0}$-a.s.

$$\int_0^{\sigma_n} \mathbf{L_X} m(X_u, S_u)\, du = -\sigma_n \ .$$

Let $\{T_k\}_{k\geq 1}$ be a localization for the local martingale

$$\int_0^{t\wedge\sigma_n} \frac{\partial m}{\partial x}(X_u, S_u)\, \sigma(X_u)\, dB_u \ .$$

Then by Fatou's lemma and the optional sampling theorem we get

$$\begin{aligned}\mathbf{E}_{x_0,s_0}(m(X_{\sigma_n}, S_{\sigma_n}) &\leq \liminf_{k\to\infty} \mathbf{E}_{x_0,s_0}\big(m(X_{\sigma_n\wedge T_k}, S_{\sigma_n\wedge T_k})\big)\\ &= m(x_0,s_0) + \liminf_{k\to\infty} \mathbf{E}_{x_0,s_0}\bigg(\int_0^{\sigma_n\wedge T_k} \frac{\partial m}{\partial x}(X_u, S_u)\sigma(X_u)\, dB_u - \big(\sigma_n \wedge T_k\big)\bigg)\\ &\leq m(x_0,s_0) - \mathbf{E}_{x_0,s_0}(\sigma_n) \ .\end{aligned}$$

Thus we have the inequality $\mathbf{E}_{x_0,s_0}(\sigma_n) \leq m(x_0,s_0)$ for all $n \geq 1$, and by monotone convergence it follows $\mathbf{E}_{x_0,s_0}(\tau_g) \leq m(x_0,s_0)$. From this we see that $\mathbf{E}_{x_0,s_0}(\tau_g)$ is finite. By the results in Section 2 we know that the function $(x,s) \mapsto \mathbf{E}_{x,s}(\tau_g)$ satisfies the system (3.1), and since m is arbitrary, hence we obtain

$$\mathbf{E}_{x_0,s_0}(\tau_g) = m_*(x_0,s_0) \ .$$

This completes the first part of the proof.

To derive (4.1) note that (2.6) is a first-order linear differential equation whose general solution is easily found to be

$$\begin{aligned}M(s) = C \exp\bigg(\int_{\cdot}^{s} \frac{\phi(u)}{S(u)-S(g(u))}\, du\bigg) - 2\exp\bigg(\int_{\cdot}^{s} \frac{\phi(u)}{S(u)-S(g(u))}\, du\bigg)\, \cdot \\ \int_{\cdot}^{s} \bigg\{ \frac{\phi(u)}{S(u)-S(g(u))} \bigg(\int_{g(u)}^{u} \frac{S(r)-S(g(u))}{\sigma^2(r)\phi(r)}\, dr\bigg) \exp\bigg(-\int_{\cdot}^{u} \frac{\phi(r)}{S(r)-S(g(r))}\, dr\bigg)\bigg\}\, du\\ = C \exp\bigg(\int_{\cdot}^{s} \frac{\phi(u)}{S(u)-S(g(u))}\, du\bigg) + \int_s^\infty \bigg\{ \frac{\phi(u)}{S(u)-S(g(u))} \bigg(\int_{g(u)}^{u} \frac{S(r)-S(g(u))}{\sigma^2(r)\phi(r)}\, dr\bigg)\, \cdot \\ \exp\bigg(-\int_{s}^{u} \frac{\phi(r)}{S(r)-S(g(r))}\, dr\bigg)\bigg\}\, du\end{aligned}$$

whenever the last integral is finite. Letting $s \to \infty$ we find that $C = 0$ corresponds to the minimal non-negative solution. Combing this with (2.7) and (2.1)+(2.2), we obtain the explicit formula (4.1). The proof of the theorem is complete. $\square$

5. An example

Let $((X_t), \mathbf{P}_x)$ denote the Bessel process of dimension α, where for simplicity we assume that $\alpha > 1$ but $\alpha \neq 2$. (The other cases of α could be treated similarly). Thus (X_t) is a non-negative diffusion with the infinitesimal operator on $(0, \infty)$ given by

$$\mathbf{L_X} = \tfrac{1}{2} \frac{\partial^2}{\partial x^2} + \tfrac{\alpha-1}{2} \frac{1}{x} \frac{\partial}{\partial x} \, .$$

(For more information about Bessel processes see [5].) Let g be a linear function given by

$$g(s)\lambda s$$

where $0 < \lambda < 1$. Denote the stopping time τ_g by

$$\tau_\lambda = \inf \{ \, t > 0 \, : \, X_t \leq \lambda \, S_t \, \} \tag{5.1}$$

It is our aim in this section to present a closed formula for the expectation of the stopping time τ_λ.

More precisely, denote the function

$$m_\lambda(x,s) = \mathbf{E}_{x,s}(\tau_\lambda)$$

for $0 < x \leq s$. Then our main task is to compute explicitly the function m_λ. Instead of using (4.1) directly, we shall rather make use of the minimality principle within the system (3.1).

According to Theorem 4.1 we shall consider the system

$$\mathbf{L_X} m_\lambda(x,s) = -1 \quad \text{for} \quad \lambda s < x < s \tag{5.2}$$

$$m_\lambda(x,s)\Big|_{x=\lambda s+} = 0 \tag{5.3}$$

$$\frac{\partial m_\lambda}{\partial s}(x,s)\Big|_{x=s-} = 0 \, . \tag{5.4}$$

Let $\lambda s < x \leq s$ be given and fixed. The general solution to (5.2) is given by

$$m_\lambda(x,s) = A(s) + B(s) \, x^{2-\alpha} - \tfrac{1}{\alpha} \, x^2 \tag{5.5}$$

where $s \mapsto A(s)$ and $s \mapsto B(s)$ are unknown functions. By (5.3) and (5.4) we find

$$A(s) = \tfrac{\lambda^\alpha}{\alpha \lambda^{\alpha-2}} \, s^2 - C \lambda^{2-\alpha} \, s^{2-\alpha-\Delta} \quad \text{and} \quad B(s) = \Big(\tfrac{\lambda^\alpha}{\alpha} - \tfrac{\lambda^{2\alpha-2}}{\alpha \lambda^{\alpha-2}-2} \Big) \, s^\alpha + C \, S^{-\Delta} \tag{5.6}$$

whenever $(2/\alpha)^{1/(\alpha-2)} < \lambda < 1$, where

$$\Delta = (2-\alpha) \, \tfrac{\lambda^{2-\alpha}}{\lambda^{2-\alpha}-1}$$

and C is an unknown constant.

In order to determine the constant C we shall use the minimality principle. It is easily verified that the minimal non-negative solution corresponds to $C = 0$. Thus by (5.5) and (5.6) with $C = 0$ we have the following candidate for $\mathbf{E}_{x,s}(\tau_\lambda)$:

$$m_\lambda(x,s) = -\tfrac{1}{\alpha} \, x^2 + \Big(\tfrac{\lambda^\alpha}{\alpha} - \tfrac{\lambda^{2\alpha-2}}{\alpha\lambda^{\alpha-2}-2} \Big) \, (s/x)^\alpha \, x^2 + \tfrac{\lambda^\alpha}{\alpha\lambda^{\alpha-2}-2} \, s^2$$

when $\lambda s < x \leq s$. Hence by applying Theorem 4.1 we obtain the following result. Observe that this example is also studied in [2] and [4].

Proposition 5.1. *Let* $((X_t), \mathbf{P}_x)$ *be a Bessel process of dimension* α *started at* $x > 0$ *under* $\mathbf{P}_x$ *, where* $\alpha > 1$ *but* $\alpha \neq 2$ *. Then for the stopping time* τ_λ *defined in* (5.1) *we have*

$$\mathbf{E}_{x,s}(\tau_\lambda) = \begin{cases} -\frac{1}{\alpha}\, x^2 + \left(\frac{\lambda^\alpha}{\alpha} - \frac{\lambda^{2\alpha-2}}{\alpha\lambda^{\alpha-2}-2}\right)(s/x)^\alpha\, x^2 + \frac{\lambda^\alpha}{\alpha\lambda^{\alpha-2}-2}\, s^2 & \text{if } \; (2/\alpha)^{1/(2-\alpha)} < \lambda < 1 \\ \infty & \text{if } \; 0 < \lambda \le (2/\alpha)^{1/(2-\alpha)}\ . \end{cases}$$

(*Note that* $\mathbf{E}_{x,s}(\tau_\lambda) = 0$ *for* $0 < x \le \lambda s$ *.*)

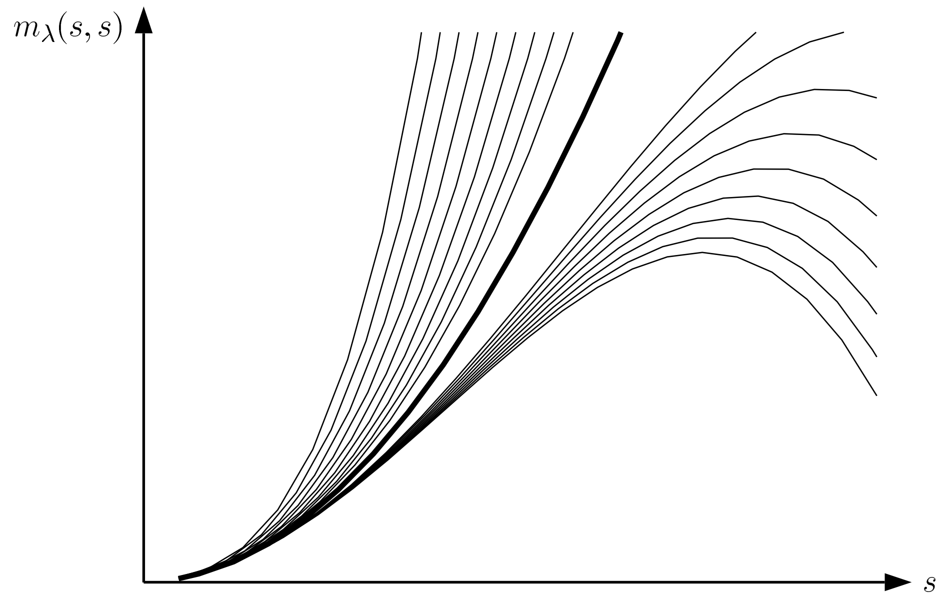

Figure 1. A computer drawing of the solution of the differential equation (2.6) in the case when $\alpha = 4$ and $\lambda = 4/5$. The bold line is the minimal non-negative solution (which never hits zero). By the minimality principle proved above, this solution equals $m_\lambda(s,s) = \mathbf{E}_{s,s}(\tau_\lambda)$ for all $s > 0$.

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Best Bounds in Doob's Maximal Inequality for Bessel Processes

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Let $((Z_t), \mathbf{P}_z)$ be a Bessel process of dimension $\alpha > 0$ started at z under $\mathbf{P}_z$ for $z \geq 0$. Then the following maximal inequality is shown to be satisfied

$$\mathbf{E}_z\Big(\max_{0\leq t\leq \tau} Z_t^p\Big) \leq \Big(\tfrac{p}{p-(2-\alpha)}\Big)^{p/(2-\alpha)} \mathbf{E}_z\big(Z_\tau^p\big) - \tfrac{p}{p-(2-\alpha)}\, z^p$$

for all stopping times τ for (Z_t) with $\mathbf{E}_z(\tau^{p/2}) < \infty$, and all $p > (2-\alpha) \vee 0$. The constants $(p/(p-(2-\alpha)))^{p/(2-\alpha)}$ and $p/(p-(2-\alpha))$ are the best possible. The equality is attained in the limit through the stopping times

$$\tau_{\lambda,p} = \inf\{\, t > 0 \,:\, Z_t^p \leq \lambda \max_{0\leq r\leq t} Z_r^p \,\}$$

when c tends to the best constant $(p/\big(p-(2-\alpha)\big))^{p/(2-\alpha)}$ from above, and λ is the greater root of the equation $\lambda^{1-(2-\alpha)/p} - \lambda = (2-\alpha)/(cp - c(2-\alpha))$. Moreover we show that $\mathbf{E}_z\big(\tau_{\lambda,p}^{q/2}\big) < \infty$ if and only if $\lambda > ((1-(2-\alpha)/q) \vee 0)^{p/(2-\alpha)}$. The proof of the inequality is based upon solving the optimal stopping problem

$$V_*(z) = \sup_\tau \mathbf{E}_z\Big(\max_{0\leq t\leq \tau} Z_t^p - c\, Z_\tau^p\Big)$$

by applying the principle of smooth fit and the maximality principle. In addition, the exact formula for the expected waiting time of the optimal strategy is derived by applying the minimality principle. The main emphasis of the paper is on the explicit expressions obtained.

1. Introduction

The main aim of this paper is to present a sharp maximal inequality of Doob's type for Bessel process of dimension $\alpha > 0$ which may start at any non-negative point.

More precisely, let $((Y_t), \mathbf{P}_z)$ be the square of a Bessel process of dimension $\alpha > 0$ which starts at $z^2 \geq 0$ under $\mathbf{P}_z$. Thus for every $z \geq 0$, (Y_t) is the only non-negative (strong) solution to the stochastic differential equation

$$dY_t = \alpha\, dt + 2\sqrt{|Y_t|}\, dB_t \quad \mathbf{P}_z\text{-a.s.} \tag{1.1}$$

where (B_t) is a standard Brownian motion. (For more information about (the square of) Bessel processes see [9] and [3].) The process (Y_t) is a submartingale. The infinitesimal operator of (Y_t) on $(0, \infty)$ is given by

$$\mathbf{L_Y} = \alpha \frac{\partial}{\partial y} + 2y \frac{\partial^2}{\partial y^2} \tag{1.2}$$

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while the boundary point 0 is an instantaneous reflecting boundary if $0 < \alpha < 2$, and an entrance boundary if $\alpha \geq 2$. The square of a Bessel process of dimension $\alpha = n \in \mathbb{N}$ may be realized as the square of radial part of a n-dimensional Brownian motion $\big(B_t^{(n)}\big) = (B_t^1, \dots, B_t^n)$, i.e. $Y_t = \sum_{k=1}^n (B_t^k)^2$ where $(B_t^1), \dots, (B_t^n)$ are mutually independent Brownian motions. The non-negative process $(Z_t) = (\sqrt{Y_t})$ is called a Bessel process of dimension $\alpha > 0$. It starts at z under $\mathbf{P}_z$. The process (Z_t) is a submartingale if $\alpha \geq 1$, and not a semimartingale if $0 < \alpha < 1$.

Due to Dubins, Shepp and Shiryaev [3] the following maximal inequality for Bessel process of dimension $\alpha > 0$ is known to be valid

$$\mathbf{E}_0\big(\max\nolimits_{0 \leq t \leq \tau} Z_t\big) \leq \gamma(\alpha)\sqrt{\mathbf{E}_0(\tau)} \tag{1.3}$$

for all stopping times τ for (Z_t) , and the constant $\gamma(\alpha)$ is shown to be behave like $\sqrt{\alpha}$ for large α , i.e.

$$\gamma(\alpha)/\sqrt{\alpha} \to 1 \quad \text{for} \quad \alpha \to \infty \, . \tag{1.4}$$

In the paper of Graversen and Peskir [5] one finds results on the rate of convergence in (1.4), after a reformulation of the problem in (1.3) to a more adequate form

$$\mathbf{E}_0\big(\max\nolimits_{0 \leq t \leq \tau} Z_t^2\big) \leq \Gamma(\alpha)^2\, \mathbf{E}_0(\tau)$$

for all stopping times τ for (Z_t) .

Motivated by these results the main aim of this paper is to find explicitly the best constants in Doob's maximal inequality for Bessel process which may start at any given point z . The main result is the following inequality which could be thought of as Doob's maximal inequality for Bessel processes (see Remark 2.2 below)

$$\mathbf{E}_z\Big(\max\nolimits_{0 \leq t \leq \tau} Z_t^p\Big) \leq \Big(\tfrac{p}{p-(2-\alpha)}\Big)^{p/(2-\alpha)} \mathbf{E}_z\big(Z_\tau^p\big) - \tfrac{p}{p-(2-\alpha)}\, z^p \tag{1.5}$$

for all stopping times τ for (Z_t) satisfying $\mathbf{E}_z(\tau^{p/2}) < \infty$, and all $p > (2-\alpha) \vee 0$. Moreover the constants $(p/(p-(2-\alpha)))^{p/(2-\alpha)}$ and $p/(p-(2-\alpha))$ are the best possible. The inequality (1.5) is obtained as a consequence of the following inequality

$$\mathbf{E}_z\Big(\max\nolimits_{0 \leq t \leq \tau} Z_t^p\Big) \leq c\,\mathbf{E}_z\big(Z_\tau^p\big) + \Big(1 + \tfrac{cp-c(2-\alpha)}{2-\alpha}\,\lambda - \tfrac{cp}{2-\alpha}\,\lambda^{1-(2-\alpha)/p}\Big)\, z^p \tag{1.6}$$

which is valid for all stopping times τ for (Z_t) with $\mathbf{E}_z(\tau^{p/2}) < \infty$ whenever $c \geq \big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$ and where λ is the greater root of the equation

$$\lambda^{1-(2-\alpha)/p} - \lambda = (2-\alpha)/\big(cp - c(2-\alpha)\big) \, .$$

The equality in (1.6) is attained when $c > \big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$ at the stopping time

$$\tau_{\lambda,p} = \inf\big\{\, t > 0 \, : \, Z_t^p \leq \lambda \max\nolimits_{0 \leq r \leq \tau} Z_r^p \,\big\} \, .$$

Moreover we show that $\mathbf{E}_z\big(\tau_{\lambda,p}^{q/2}\big) < \infty$ if and only if $\lambda > \big((1-(2-\alpha)/q) \vee 0\big)^{p/(2-\alpha)}$. In addition, an explicit formula for the expectation of $\tau_{\lambda,p}$ is derived. Note that the inequality (1.5) is already known in the case $\alpha = 1$ (see [8]) where (Z_t) may be realized as a reflected Brownian motion.

The method of proof relies upon the principle of smooth fit (see [3] and [4]) and the maximality principle (see [6]). The main emphasis in this paper is on the explicit expressions obtained.

2. The inequality

The main result of the paper is contained in the following theorem.

Theorem 2.1. *Let* $((Z_t), \mathbf{P}_z)$ *be a Bessel process of dimension* $\alpha > 0$ *started at* z *under* $\mathbf{P}_z$ *for* $z \geq 0$. *Then the following inequality is satisfied*

$$\mathbf{E}_z\Big(\max_{0\leq t\leq \tau} Z_t^p\Big) \leq \Big(\tfrac{p}{p-(2-\alpha)}\Big)^{p/(2-\alpha)} \mathbf{E}_z\big(Z_\tau^p\big) - \tfrac{p}{p-(2-\alpha)}\, z^p \tag{2.1}$$

for all stopping times τ *for* (Z_t) *with* $\mathbf{E}_z(\tau^{p/2}) < \infty$, *and all* $p > (2-\alpha) \vee 0$. *The constants* $\big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$ *and* $p/(p-(2-\alpha))$ *are the best possible. The equality in* (2.1) *is attained through the stopping times of the form*

$$\tau_{\lambda,p} = \inf\big\{\, t > 0 \,:\, Z_t^p \leq \lambda \max_{0\leq r\leq t} Z_r^p \,\big\}$$

when c *tends to the best constant* $\big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$ *from above, and* λ *is the greater root in the equation*

$$\lambda^{1-(2-\alpha)/p} - \lambda = (2-\alpha)/\big(cp - c(2-\alpha)\big)\ .$$

Moreover for given $q > 0$ *we have* $\mathbf{E}_z\big(\tau_{\lambda,p}^{q/2}\big) < \infty$ *if and only if* $\lambda > \big((1-(2-\alpha)/q)\vee 0\big)^{p/(2-\alpha)}$. *In the case* $\alpha = 2$ *the inequality* (2.1) *is considered to be of the form*

$$\mathbf{E}_z\big(\max_{0\leq t\leq \tau} Z_t^p\big) \leq e\, \mathbf{E}_z\big(Z_\tau^p\big) - z^p$$

obtained from (2.1) *by passing to the limit as* $\alpha \to 2$.

Proof. Let $\alpha > 0$ be given, and for simplicity assume that $\alpha \neq 2$. (The case $\alpha = 2$ could be treated similarly.) Given $0 \leq x \leq s$, consider the following optimal stopping problem

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\big(S_\tau - c\, X_\tau\big) \tag{2.2}$$

where the supremum is taken over all stopping times τ for (Z_t) satisfying $\mathbf{E}_{x,s}(\tau^{p/2}) < \infty$, and the process (X_t) and the maximum process (S_t) are respectively given by

$$X_t = Z_t^p$$
$$S_t = \big(\max_{0\leq r\leq t} X_r\big) \vee s$$

with $p > (2-\alpha)\vee 0$ and $c > \big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$ given and fixed. The expectation in (2.2) is taken with respect to the probability measure $\mathbf{P}_z := \mathbf{P}_{x,s}$ under which the process (X_t) starts at $x := z^p$ and the process (S_t) starts at s .

By Itô formula it is easily verified that the infinitesimal operator of (X_t) on $(0,\infty)$ is given by

$$\mathbf{L_X} = \tfrac{p\,(p-(2-\alpha))}{2}\, x^{1-2/p}\, \frac{\partial}{\partial x} + \tfrac{p^2}{2}\, x^{2-2/p}\, \frac{\partial^2}{\partial x^2} \tag{2.3}$$

while the boundary point 0 is an instantaneous reflecting boundary if $0 < \alpha < 2$, and an entrance boundary if $\alpha \geq 2$.

If the supremum in (2.2) is attained we know from the general theory of optimal stopping that the following exit time of the Markov process (X_t, S_t) may be optimal

$$\tau_* = \inf\{\, t > 0 \,:\, X_t \leq g_*(S_t)\,\}$$

where $s \mapsto g_*(s) < s$ is the optimal stopping boundary to be found. Thus to compute the value function V_* for $g_*(s) < x < s$ and to determine the optimal stopping boundary g_* it is natural to formulate the following system (see [3])

$$\mathbf{L_X} V(x,s) = 0 \qquad \text{for} \quad g_*(s) < x < s \tag{2.4}$$

$$V(x,s)\Big|_{x=g_*(s)+} = s - c\,g_*(s) \qquad \text{(instantaneous stopping)} \tag{2.5}$$

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g_*(s)+} = -c \qquad \text{(smooth fit)} \tag{2.6}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \qquad \text{(normal reflection)} \tag{2.7}$$

with $\mathbf{L_X}$ in (2.3). The system (2.4)-(2.7) forms a free-boundary problem. The condition (2.6) is imposed since we expect that the principle of smooth fit should hold.

The general solution to (2.4) is given by

$$V(x,s) = A(s)\,x^{(2-\alpha)/p} + B(s) \tag{2.8}$$

where $s \mapsto A(s)$ and $s \mapsto B(s)$ are unknown functions. By (2.5) and (2.6) we find

$$A(s) = -\tfrac{cp}{2-\alpha}\,g_*(s)^{1-(2-\alpha)/p} \quad \text{and} \quad B(s) = s + \tfrac{cp-c(2-\alpha)}{2-\alpha}\,g_*(s)\ . \tag{2.9}$$

Inserting (2.9) into (2.8) we obtain

$$V(x,s) = s + \tfrac{cp-c(2-\alpha)}{2-\alpha}\,g_*(s) - \tfrac{cp}{2-\alpha}\,\big(g_*(s)/x\big)^{1-(2-\alpha)/p}\,x \tag{2.10}$$

for $g_*(s) \le x \le s$. Finally, by the last boundary condition (2.7) we find that $s \mapsto g_*(s)$ is to satisfy the differential equation

$$g'(s) = \tfrac{2-\alpha}{cp-c(2-\alpha)} \Big/ \Big(\big(s/g(s)\big)^{(2-\alpha)/p} - 1\Big)$$

for $s > 0$. This differential equation admits a linear solution $g_*(s) = \lambda s$ for $s > 0$ where the given $0 < \lambda < 1$ is to satisfy the equation

$$\lambda^{1-(2-\alpha)/p} - \lambda = (2-\alpha)/\big(cp - c(2-\alpha)\big)\ . \tag{2.11}$$

By elementary analysis of (2.11) one shows that the conditions $c > \big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$ and $p > (2-\alpha) \vee 0$ ensure that there are exactly two roots. Motivated by the maximality principle (see [6]) we shall choose the greater λ satisfying (2.11). Inserting this into (2.10), our candidate for the value function V_* defined in (2.2) is therefore given by

$$V(x,s) = \begin{cases} s + \tfrac{cp-c(2-\alpha)}{2-\alpha}\,\lambda s - \tfrac{cp}{2-\alpha}\big(\lambda s/x\big)^{1-(2-\alpha)/p}\,x & \text{if} \quad \lambda s < x \le s \\ s - cx & \text{if} \quad 0 \le x \le \lambda s \end{cases} \tag{2.12}$$

where $0 < \lambda < 1$ is the greater root in (2.11). The corresponding candidate for the optimal stopping time τ_* is then to be

$$\tau_* = \inf\{\, t > 0 \,:\, X_t \le \lambda\, S_t \,\} \tag{2.13}$$

In the next step we will show that the candidates for the value function given by (2.12) and the optimal stopping time given by (2.13) are indeed correct.

First, we verify that the stopping time τ_* fulfills the integrability condition $\mathbf{E}_{x,s}(\tau^{p/2}) < \infty$. Denote the stopping time $\tau_{\lambda,p}$ given by

$$\tau_{\lambda,p} = \inf\{\, t > 0 \,:\, X_t \le \lambda\, S_t \,\} \tag{2.14}$$

where $0 < \lambda < 1$. Let $q > 0$ be given, then $\mathbf{E}_{x,s}\big(\tau_{\lambda,p}^{q/2}\big)$ is finite if and only if $\lambda > \big((1-(2-\alpha)/q)\vee 0\big)^{p/(2-\alpha)}$. Indeed by Burkholder-Davis-Gundy inequality for Bessel processes (see [2]) we see that $\mathbf{E}_{x,s}\big(\tau_{\lambda,p}^{q/2}\big) < \infty$ if and only if $\mathbf{E}_{x,s}\big(S_{\tau_{\lambda,p}}^{q/p}\big) < \infty$. By a result in [1] we have

$$\mathbf{P}_x\big(S_{\tau_{\lambda,p}} \geq r\big) = \exp\Big(-\int_x^r \frac{S'(u)}{S(u)-S(\lambda u)}\,du\Big)$$

for $r \geq s$, where $S(x) = p\,(x^{(2-\alpha)/p}-1)/(2-\alpha)$ is the scale function for (X_t) . From this it is easily seen that $\mathbf{E}_{x,s}\big(S_{\tau_{\lambda,p}}^{q/p}\big) < \infty$ if and only if $\lambda > \big((1-(2-\alpha)/q)\vee 0\big)^{p/(2-\alpha)}$. Furthermore in the case $q=p$ when $\lambda > \big((1-(2-\alpha)/p)\vee 0\big)^{p/(2-\alpha)}$ we have that

$$\mathbf{E}_{x,s}\big(S_{\tau_{\lambda,p}}\big) = \Bigg(\tfrac{1}{1-\lambda^{(2-\alpha)/p}}\Big(1-\tfrac{(2-\alpha)\lambda^{(2-\alpha)/p}}{(2-\alpha)-p(1-\lambda^{(2-\alpha)/p})}\Big) + \tfrac{1}{1-\lambda^{(2-\alpha)/p}}\Big(\tfrac{2-\alpha}{(2-\alpha)-p(1-\lambda^{(2-\alpha)/p})}-1\Big)\Big(\frac{x}{s}\Big)^{(2-\alpha)/p}\Bigg)\,s\,. \tag{2.15}$$

Next, we verify that that the formula (2.12) is correct. The function V in (2.12) depends on x through $y = x^{2/p}$ and therefore we define the function U such that $U(y,s) = V(x,s)$, $y = x^{2/p}$, i.e.

$$U(y,s) = \begin{cases} s + \frac{cp-c(2-\alpha)}{2-\alpha}\,\lambda s - \frac{cp}{2-\alpha}(\lambda s)^{1-(2-\alpha)/p}\,y^{1-\alpha/2} & \text{if} \quad \lambda s < y^{p/2} \leq s \\ s - c\,y^{p/2} & \text{if} \quad 0 \leq y^{p/2} \leq \lambda s\,. \end{cases}$$

The square of a Bessel process $(Y_t) = (X_t^{2/p})$ is a semimartingale for all α , thus applying Itô-Tanaka formula (two-dimensionally) to $U(Y_t,S_t)$ we get $\mathbf{P}_{x,s}$-a.s.

$$\begin{aligned} V(X_t,S_t) = U(Y_t,S_t) = U(y^{p/2},s) + \int_0^t \frac{\partial U}{\partial y}(Y_u,S_u)\,dY_u \\ + \int_0^t \frac{\partial U}{\partial s}(Y_u,S_u)\,dS_u + \frac{1}{2}\int_0^t \frac{\partial^2 U}{\partial y^2}(Y_u,S_u)\,d[Y]_u \end{aligned}$$

where $\mathbf{L_Y}$ is given in (1.2), and $(\partial^2 U/\partial y^2)(\lambda s, s)$ is defined to be zero. Since the increment dS_u equals zero outside the diagonal $y^{p/2} = s$ and U satisfies the normal reflection condition

$$\frac{\partial U}{\partial s}(y,s)\Big|_{y^{p/2}=s-} = 0$$

we have that

$$\int_0^t \frac{\partial U}{\partial s}(Y_u,S_u)\,dS_u = 0\,.$$

Moreover by (1.1) and (1.2) we have

$$V(X_t,S_t) = V(x,s) + \int_0^t \mathbf{L_Y}U(Y_u,S_u)\,du + M_t \tag{2.16}$$

where (M_t) is a continuous local martingale given by

$$M_t = 2\int_0^t \frac{\partial U}{\partial y}(Y_u,S_u)\,\sqrt{Y_u}\,dB_u\,.$$

Since the set of those $u > 0$ for which ${Y_u}^{p/2} = S_u$ is of Lebesgue measure zero, and $\mathbf{L_Y} U(y,s) \le 0$ for $0 \le y^{p/2} < s$, we get $\int_0^t \mathbf{L_Y} U(Y_u, S_u)\, du \le 0$. Hence we have the following inequality

$$V(X_t, S_t) \le V(x,s) + M_t \tag{2.17}$$

Let τ be any stopping time for (Z_t) satisfying $\mathbf{E}_{x,s}(\tau^{p/2}) < \infty$ and let $\{\tau_k\}_{k \ge 1}$ be a localization sequence of bounded stopping times for (M_t) . By Doob's optional sampling theorem and the inequality (2.17) we get $\mathbf{E}_{x,s}\big(V(X_{\tau\wedge\tau_k}, S_{\tau\wedge\tau_k})\big) \le V(x,s)$ and letting $k \to \infty$ and using Fatou's lemma we have that

$$\mathbf{E}_{x,s}\big(V(X_\tau, S_\tau)\big) \le V(x,s)\ .$$

Since $V(x,s) \ge s - cx$ for all $0 \le x \le s$ we get

$$\mathbf{E}_{x,s}\big(S_\tau - c\,X_\tau\big) \le \mathbf{E}_{x,s}\big(V(X_\tau, S_\tau)\big) \le V(x,s)$$

and taking supremum over all stopping times τ for (Z_t) satisfying $\mathbf{E}_{x,s}(\tau^{p/2}) < \infty$ we obtain

$$V_*(x,s) \le V(x,s)\ . \tag{2.18}$$

Finally, to prove that the equality in (2.18) is attained, and that the stopping time (2.13) is optimal, it is enough to verify that

$$V(x,s) = \mathbf{E}_{x,s}\big(S_{\tau_*} - c\,X_{\tau_*}\big)\ . \tag{2.19}$$

We have by the definition of the stopping time τ_* that $X_{\tau_*} = \lambda S_{\tau_*}$ in law and since λ is a root in the equation (2.11) we have by (2.15) that

$$\mathbf{E}_{x,s}\big(S_{\tau_*} - c\,X_{\tau_*}\big) = (1 - c\lambda)\,\mathbf{E}_{x,s}\big(S_{\tau_*}\big) = V(x,s)\ .$$

In particular, since $V = V_*$ we have from (2.12) that

$$V_*(x,x) = \Big(1 + \tfrac{cp - c(2-\alpha)}{2-\alpha}\,\lambda - \tfrac{cp}{2-\alpha}\,\lambda^{1-(2-\alpha)/p}\Big)\,x$$

for $c > \big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$. Letting $c \downarrow \big(p/(p-(2-\alpha))\big)^{p/(2-\alpha)}$ and thereby $\lambda \downarrow \big(1-(2-\alpha)/p\big)^{p/(2-\alpha)}$ we get by the definition of the value function (2.2) that

$$\mathbf{E}_x\big(S_\tau\big) \le \Big(\tfrac{p}{p-(2-\alpha)}\Big)^{p/(2-\alpha)}\,\mathbf{E}_x\big(X_\tau\big) - \tfrac{p}{p-(2-\alpha)}\,x$$

for all stopping times τ for (Z_t) with $\mathbf{E}_x(\tau^{p/2}) < \infty$. The sharpness of the inequality follows from the definition of the value function (2.2). The proof is complete. □

Remark 2.2. Note that from the above remarks about the Bessel process (Z_t) it follows that (Z_t^r) is a submartingale whenever $r \ge 2$ for $\alpha > 0$, as well as $r \ge 1$ for $\alpha \ge 1$. In the case $\alpha \ge 2$ Itô formula can be applied to $({Y_t}^{r/2}) = (Z_t^r)$ since the boundary point 0 is an entrance boundary, and one easily verifies that (Z_t^r) is a submartingale for $r > 0$. In the case $0 < \alpha < 2$ the inequality (2.1) indicates that (Z_t^r) might be a submartingale if $r \ge 2 - \alpha$ because it satisfies Doob's maximal inequality for non-negative submartingales. In particular, note if $r = 2 - \alpha$ then the first constant in (2.1) is equal to the best constant in Doob's maximal inequality in the following sense

$$\mathbf{E}_0\Big(\max\nolimits_{0\le t\le\tau}(Z_t^r)^p\Big) \le \Big(\tfrac{(2-\alpha)p}{(2-\alpha)p-(2-\alpha)}\Big)^{(2-\alpha)p/(2-\alpha)}\,\mathbf{E}_0\big((Z_\tau^r)^p\big) = \Big(\tfrac{p}{p-1}\Big)^p\,\mathbf{E}_0\big((Z_\tau^r)^p\big)$$

where $p > 1$. This is already known in the case $\alpha = 1$ (see [8]).

3. The expected waiting time

In this section we compute the expectation of the optimal stopping time τ_* constructed in the proof of Theorem 2.1 . Let $\tau_{\lambda,p}$ be the stopping time defined in (2.14) for $0 < \lambda < 1$. Then our task is to derive a closed formula for the function

$$m_{\lambda,p}(x,s) = \mathbf{E}_{x,s}(\tau_{\lambda,p})$$

for $0 \leq x \leq s$. If the minimal non-negative solution to the system

$$\mathbf{L_X} m_{\lambda,p}(x,s) = -1 \quad \text{for} \quad \lambda s < x < s \tag{3.1}$$

$$m_{\lambda,p}(x,s)\Big|_{x=\lambda s+} = 0 \quad \text{(instantaneous stopping)} \tag{3.2}$$

$$\frac{\partial m_{\lambda,p}}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)} \tag{3.3}$$

exists then by the minimality principle $m_{\lambda,p}$ equals this solution (see [7]), where $\mathbf{L_X}$ is given in (2.3). Assume again that $\alpha \neq 2$. (The case $\alpha = 2$ could be treated similarly.) The general solution to (3.1) is given by

$$m_{\lambda,p}(x,s) = A(s) + B(s)\, x^{(2-\alpha)/p} - \tfrac{1}{\alpha}\, x^{2/p}$$

where $s \mapsto A(s)$ and $s \mapsto B(s)$ are unknown functions. By (3.2) and (3.3) we find

$$A(s) = -\tfrac{\lambda^{2/p}}{2\lambda^{(2-\alpha)/p}-\alpha}\, s^{2/p} - C\,\lambda^{(2-\alpha)/p} s^{(2-\alpha)/p-\Delta} \quad \text{and} \quad B(s) = \tfrac{2\lambda^{2/p}}{\alpha\,(2\lambda^{(2-\alpha)/p}-\alpha)}\, s^{\alpha/p} + Cs^{-\Delta}$$

whenever $(\alpha/2)^{p/(2-\alpha)} < \lambda < 1$, where

$$\Delta = \tfrac{(2-\alpha)\,\lambda^{(2-\alpha)/p}}{p\,(\lambda^{(2-\alpha)/p}-1)}$$

and C is an unknown constant. It is now easily verified that the minimal non-negative solution corresponds to $C = 0$. Hence we have the following result.

Proposition 3.1. *Let $((Z_t), \mathbf{P}_z)$ be a Bessel process of dimension $\alpha > 0$ started at $z > 0$ under $\mathbf{P}_z$. Let $p > 0$ be given. Then for the stopping time $\tau_{\lambda,p}$ defined in (2.14) we have*

$$\mathbf{E}_{x,s}(\tau_{\lambda,p}) = \begin{cases} \tfrac{2\lambda^{2/p}}{\alpha\,(2\lambda^{(2-\alpha)/p}-\alpha)}\,(s/x)^{\alpha/p} x^{2/p} - \tfrac{\lambda^{2/p}}{2\lambda^{(2-\alpha)/p}-\alpha}\, s - \tfrac{1}{\alpha}\, x^{2/p} & \text{if} \quad (\alpha/2)^{p/(2-\alpha)} < \lambda < 1 \\ \infty & \text{if} \quad 0 < \lambda \leq (\alpha/2)^{p/(2-\alpha)} \end{cases}$$

for $\lambda s < x \leq s$, where $x := z^p$. (Note that $\mathbf{E}_{x,s}(\tau_{\lambda,p}) = 0$ for $0 < x \leq \lambda s$.)
In the case $\alpha = 2$, the formula is considered to be of the form

$$\mathbf{E}_{x,s}(\tau_{\lambda,p}) = \tfrac{\lambda^{2/p}}{p\log(\lambda^{2/p})+p}\, \log\big(s/x\big)\, s^{2/p} - \tfrac{\lambda^{2/p}}{2\log(\lambda^{2/p})+2}\, s^{2/p} - \tfrac{1}{2}\, x^{2/p}$$

if $e^{-p/2} < \lambda < 1$ obtained by passing to the limit as $\alpha \to 2$.

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Solving Non-Linear Optimal Stopping Problems by the Method of Time-change

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Some non-linear optimal stopping problems can be solved explicitly by using a common method which is based on time-change. We describe this method and illustrate its use by considering several examples dealing with Brownian motion. In each of these examples we derive explicit formulas for the value function and display the optimal stopping time. The main emphasis of the paper is on the method of proof and its unifying scope.

1. Introduction

The main goal of this paper is to present a deterministic time-change method which enables one to solve some non-linear optimal stopping problems explicitly. The basic idea is to transform the original (difficult) problem into a new (easier) problem. The method is illustrated through several examples with applications in the next section.

1. To explain the ideas in more detail, let $((X_t), \mathbf{P}_x)$ be a one-dimensional time-homogeneous diffusion associated with the infinitesimal generator

$$\mathbf{L_X} = \mu(x)\,\frac{\partial}{\partial x} + \sigma^2(x)\,\frac{1}{2}\,\frac{\partial^2}{\partial x^2}$$

where $x \mapsto \sigma(x) > 0$ and $x \mapsto \mu(x)$ are continuous. Assume moreover that there exists a standard Brownian motion (B_t) such that (X_t) solves the stochastic differential equation

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t$$

with $X_0 = x$ under $\mathbf{P}_x$. The typical optimal stopping problem which appears under consideration below has the value function given by

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(\alpha\big(t+\tau\big)\,X_\tau\Big)$$

where the supremum is taken over a class of stopping times τ for (X_t) and α is a smooth but *non-linear* function. This forces us to take (t, X_t) as the underlying diffusion in the problem, and thus by general optimal stopping theory we know that the value function V_* should solve the following partial differential equation

$$\frac{\partial V}{\partial t}(t,x) + \mathbf{L_X} V(t,x) = 0$$

in the domain of continued observation (see [7]). However, it is generally difficult to find the appropriate solution of the partial differential equation, and the basic idea of the time-change method is to transform the original problem into a new optimal stopping problem such that the new value function solves an ordinary differential equation.

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2. To do so one is naturally led to find a deterministic time-change $t \mapsto \sigma_t$ satisfying the following two conditions:

(i) $t \mapsto \sigma_t$ is continuous and strictly increasing
(ii) there exists a one-dimensional time-homogeneous diffusion (Z_t) with infinitesimal generator $\mathbf{L_Z}$ such that $\alpha(\sigma_t)\,X_{\sigma_t} = e^{-rt}\,Z_t$ for some $r \in \mathbb{R}$.

From general optimal stopping theory we know that the new (time-changed) value function

$$W_*(z) = \sup_\tau \mathbf{E}_z\Big(e^{-r\tau} Z_\tau\Big)$$

where the supremum is taken over a class of stopping times τ for (Z_t) , should solve the ordinary differential equation

$$\mathbf{L_Z} W_*(z) = r\,W_*(z)$$

in the domain of continued observation. Note that under condition (i) there is a one-to-one correspondence between the original problem and the new problem, i.e. if τ is a stopping time for (Z_t) then σ_τ is a stopping time for (X_t) and vice versa.

3. Given the diffusion (X_t) the crucial point is to find the process (Z_t) and the time-change σ_t fulfilling conditions (i) and (ii) above. Itô formula offers an answer to these questions.

Setting $(Y_t) = (\beta(t)X_t)$ where $\beta \neq 0$ is a smooth function, by Itô formula we get

$$Y_t = Y_0 + \int_0^t \left(\frac{\beta'(u)}{\beta(u)}\,Y_u + \beta(u)\,\mu\!\left(\frac{Y_u}{\beta(u)}\right)\right) du \,+ \int_0^t \beta(u)\,\sigma\!\left(\frac{Y_u}{\beta(u)}\right) dB_u$$

and hence (Y_t) has the infinitesimal generator

$$\mathbf{L_Y} = \left(\frac{\beta'(t)}{\beta(t)}\,y + \beta(t)\,\mu\!\left(\frac{y}{\beta(t)}\right)\right)\frac{\partial}{\partial y} \,+\, \beta^2(t)\,\sigma^2\!\left(\frac{y}{\beta(t)}\right)\frac{1}{2}\,\frac{\partial^2}{\partial y^2}\,. \tag{1.1}$$

The time-changed process $(Z_t) = (Y_{\sigma_t})$ has the infinitesimal generator (see [4] p.175)

$$\mathbf{L_Z} = \frac{1}{\rho(t)}\,\mathbf{L_Y} \tag{1.2}$$

where σ_t is the time-change given by

$$\sigma_t = \inf\left\{\, r > 0 \,:\, \int_0^r \rho(u)\,du > t \,\right\}$$

for some $u \mapsto \rho(u) > 0$ (to be found) such that $\sigma_t \to \infty$ as $t \to \infty$.

The process (Z_t) and the time-change σ_t will be fulfilling conditions (i) and (ii) above if the infinitesimal generator $\mathbf{L_Z}$ *does not depend on* t . In view of (1.1) this clearly imposes conditions on β (and α above) which make the method applicable:

$$\mu\!\left(\frac{y}{\beta(t)}\right) = \gamma(t)\,G_1(y) \tag{1.3}$$

$$\sigma^2\!\left(\frac{y}{\beta(t)}\right) = \frac{\gamma(t)}{\beta(t)}\,G_2(y) \tag{1.4}$$

where $\gamma = \gamma(t)$, $G_1 = G_1(y)$ and $G_2 = G_2(y)$ are functions required to exist.

4. In our examples below the diffusion (X_t) is given as Brownian motion $(B_t + x)$ started at x under $\mathbf{P}_x$, and thus its infinitesimal generator is given by

$$\mathbf{L_X} = \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,.$$

By the foregoing observations we shall find a time-change σ_t and a process (Z_t) satisfying conditions (i) and (ii) above. With the notation introduced above we see from (1.1) that the infinitesimal generator of (Y_t) in this case is given by

$$\mathbf{L_Y} = \frac{\beta'(t)}{\beta(t)}\, y\, \frac{\partial}{\partial y} + \beta^2(t)\, \frac{1}{2}\, \frac{\partial^2}{\partial y^2}\ .$$

Observe that conditions (1.3) and (1.4) are easily realized with $\gamma(t)=\beta(t)$, $G_1=0$ and $G_2=1$. Thus if β solves the differential equation $\beta'(t)/\beta(t) = -\beta^2(t)/2$, and we set $\rho = \beta^2/2$, then from (1.2) we see that $\mathbf{L_Z}$ does not depend on t . Noting that $\beta(t) = 1/\sqrt{1+t}$ solves this equation, and putting $\rho(t) = 1/2(1+t)$, we find that

$$\sigma_t = \inf\left\{ r > 0 \ : \ \int_0^r \rho(u)\, du > t \right\} = e^{2t} - 1\ . \tag{1.5}$$

Thus the time-changed process (Z_t) has the infinitesimal generator given by

$$\mathbf{L_Z} = -z\, \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}$$

and hence (Z_t) is an Ornstein-Uhlenbeck process. While this fact is well-known, the technique described may be applied in a similar context involving other diffusions (Example 2.15).

5. We believe that the time-change arguments described above are well-known to the specialists in the field, although we could not find it in the literature on optimal stopping. In the next section we shall apply this method and present solutions to several optimal stopping problems some of which were already treated earlier and solved by means of other techniques. Apart from the time-change arguments just described, the method of proof makes also use of Brownian scaling and the principle of smooth fit in a free-boundary problem. Once the guess is performed, Itô calculus is used as a verification tool. The main emphasis of the paper is on the method of proof and its unifying scope.

2. Examples and applications

In this section we explicitly solve some non-linear optimal stopping problems by applying the time-change method described in the first section.

Throughout (B_t) denotes a standard Brownian motion started at zero under $\mathbf{P}$, and the diffusion (X_t) is given as the Brownian motion $B_t + x$ started at x under $\mathbf{P}_x$.

Given the time-change $\sigma_t = e^{2t} - 1$ from (1.5), we know that the time-changed process

$$Z_t = X_{\sigma_t}/\sqrt{1+\sigma_t} \tag{2.1}$$

is an Ornstein-Uhlenbeck process satisfying

$$dZ_t = -Z_t\, dt + \sqrt{2}\, dB_t \tag{2.2}$$

$$\mathbf{L_Z} = -z\, \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}\ . \tag{2.3}$$

With this notation we may now enter into the first example.

Example 2.1. Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(|X_\tau| - c\sqrt{t+\tau}\Big) \tag{2.4}$$

where the supremum is taken over all stopping times τ for (X_t) satisfying $\mathbf{E}_x(\sqrt{\tau}) < \infty$ and $c > 0$ is given and fixed. We shall solve this problem in three steps.

1. In the first step we shall apply Brownian scaling and note that $\tilde{\tau} = \tau/t$ is a stopping time for the Brownian motion $s \mapsto t^{-1/2} B_{ts}$. If we now rewrite (2.4) as

$$V_*(t,x) = \sup_{\tau} \mathbf{E}\Big(|B_\tau + x| - c\sqrt{t+\tau}\Big) = \sqrt{t}\, \sup_{\tau/t} \mathbf{E}\Big(\big|t^{-1/2} B_{t(\tau/t)} + x/\sqrt{t}\,\big| - c\sqrt{1+\tau/t}\Big)$$

we clearly see that

$$V_*(t,x) = \sqrt{t}\, V_*(1, x/\sqrt{t}\,) \tag{2.5}$$

and therefore we only need to look at $V_*(1,x)$ in the sequel. By using (2.5) we can also make the following observation on the optimal stopping boundary for the problem (2.4).

Remark 2.2. In the problem (2.4) the gain function equals $g(t,x) = |x| - c\sqrt{t}$ and the diffusion is identified with $\big(t{+}r, X_r\big)$. If a point (t_0, x_0) belongs to the boundary of the domain of continued observation, i.e. (t_0, x_0) is an instantaneously stopping point ($\tau \equiv 0$ is an optimal stopping time), then we get from (2.5) that $V_*(t_0, x_0) = |x_0| - c\sqrt{t_0} = \sqrt{t_0}\, V_*(1, x_0/\sqrt{t_0}\,)$. Hence $V_*(1, x_0/\sqrt{t_0}\,) = |x_0|/\sqrt{t_0} - c$ and therefore the point $(1, x_0/\sqrt{t_0}\,)$ is also instantaneously stopping. Set now $\gamma_0 = |x_0|/\sqrt{t_0}$ and note that if (t,x) is any point satisfying $|x|/\sqrt{t} = \gamma_0$, then this point is also instantaneously stopping. This offers a heuristic argument that the optimal stopping boundary should be $|x| = \gamma_0 \sqrt{t}$ for some $\gamma_0 > 0$ to be found.

2. In the second step we shall apply the time-change $t \mapsto \sigma_t$ from (1.5) to the problem $V_*(1,x)$ and transform it into a new problem. From (2.1) we get

$$|X_{\sigma_\tau}| - c\sqrt{1+\sigma_\tau} = \sqrt{1+\sigma_\tau}\, \big(|Z_\tau| - c\big) = e^\tau \big(|Z_\tau| - c\big) \tag{2.6}$$

and the problem to determine $V_*(1,x)$ therefore reduces to compute

$$V_*(1,x) = W_*(x) \tag{2.7}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_{\tau} \mathbf{E}_z\Big(e^\tau \big(|Z_\tau| - c\big)\Big) \tag{2.8}$$

the supremum being taken over all stopping times τ for (Z_t) for which $\mathbf{E}_z(e^\tau) < \infty$. Observe that this problem is one-dimensional.

3. In the third step we shall show how to solve the problem (2.8). From general optimal stopping theory we know that the following stopping time should be optimal

$$\tau_* = \inf\, \big\{\, t > 0 \,:\, |Z_t| \ge z_* \,\big\} \tag{2.9}$$

where $z_* \ge 0$ is the optimal stopping point to be found. Observe that this guess agrees with Remark 2.2. Note that the domain of continued observation $C = (-z_*, z_*)$ is assumed symmetric around zero since the Ornstein-Uhlenbeck process is symmetric, i.e. the process $(-Z_t)$ is also an Ornstein-Uhlenbeck process started at $-z$. By using the same argument we may also argue that the value function W_* should be *even*.

To compute the value function W_* for $z \in (-z_*, z_*)$ and to determine the optimal stopping point z_*, in view of (2.8)+(2.9) it is natural to formulate the following system

$$\mathbf{L_Z} W(z) = -W(z) \quad \text{for } z \in (-z_*, z_*) \tag{2.10}$$

$$W(\pm z_*) = z_* - c \quad \text{(instantaneous stopping)} \tag{2.11}$$

$$W'(\pm z_*) = \pm 1 \quad \text{(smooth fit)} \tag{2.12}$$

with $\mathbf{L_Z}$ in (2.3). The system (2.10)-(2.12) forms a *free-boundary problem.* The condition (2.12) is imposed since we expect that *the principle of smooth fit* should hold.

It is known (see Section 3 below) that the equation (2.10) admits the even solution (3.3) and the odd solution (3.4) as two linearly independent solutions. Since the value function should be even, we can forget the odd solution and from (3.3) we see that

$$W(z) = -A\, M(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{z^2}{2})$$

for some $A > 0$ to be found.

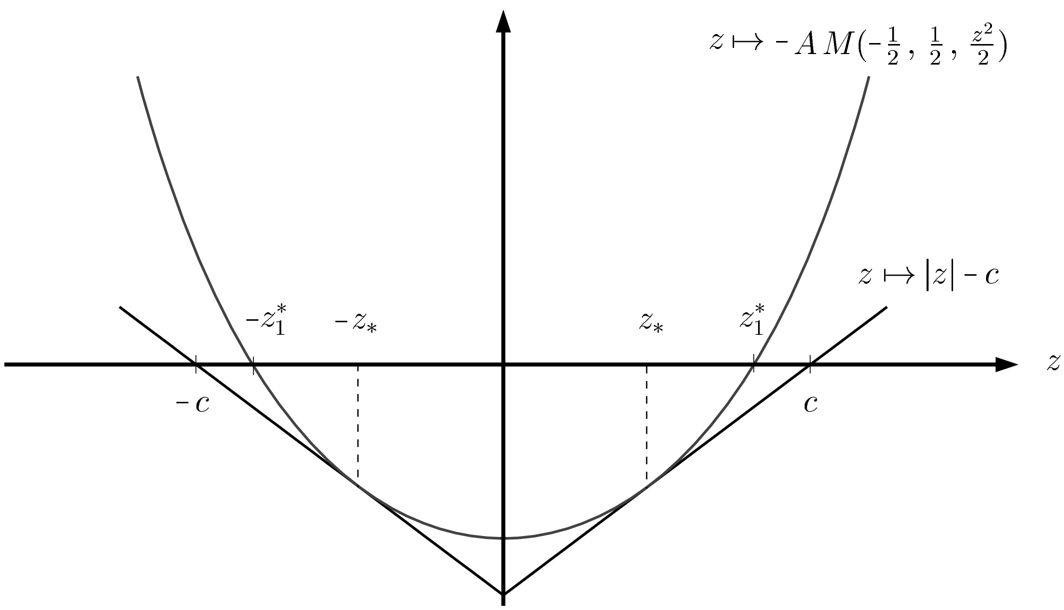

Figure 1. A computer drawing of the solution of the free-boundary problem (2.10)-(2.12). The solution equals $z \mapsto -A\,M(-1/2, 1/2, z^2/2)$ for $|z| < z_*$ and $z \mapsto |z| - c$ for $|z| \geq z_*$. The constant A is chosen (and z_* is obtained) such that the smooth fit holds at $\pm z_*$ (the first derivative of the solution is continuous at $\pm z_*$).

From Figure 1 we clearly see that only for $c \geq z_1^*$ the two boundary conditions (2.11)+(2.12) can be fulfilled, where z_1^* is the unique positive root of $M(-1/2\,, 1/2\,, z^2/2) = 0$. Thus by (2.11)+(2.12) and (3.5) when $c \geq z_1^*$ we find that $A = z_*^{-1}/M(1/2\,, 3/2\,, z_*^2/2)$ and that $z_* \leq z_1^*$ is the unique positive root of the equation

$$z^{-1} M(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{z^2}{2}) = (c - z)\, M(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{z^2}{2})\,. \tag{2.13}$$

Note that for $c < z_1^*$ the equation (2.13) have no solution.

In this way we have obtained the following candidate for the value function W_* in the problem (2.8) when $c \geq z_1^*$

$$W(z) = \begin{cases} -z_*^{-1} M(-\frac{1}{2}, \frac{1}{2}, \frac{z^2}{2}) / M(\frac{1}{2}, \frac{3}{2}, \frac{z_*^2}{2}) & \text{if } |z| < z_* \\ |z| - c & \text{if } |z| \geq z_* \end{cases} \tag{2.14}$$

and the following candidate for the optimal stopping time τ_* when $c > z_1^*$

$$\tau_{z_*} = \inf \{ \, t > 0 \, : \, |Z_t| \geq z_* \, \} \, . \tag{2.15}$$

In the proof below we shall see that $\mathbf{E}_z(e^{\tau_{z_*}}) < \infty$ when $c > z_1^*$ (and thus $z_* < z_1^*$). For $c = z_1^*$ (and thus $z_* = z_1^*$) the stopping time τ_{z_*} fails to satisfy $\mathbf{E}_z(e^{\tau_{z_*}}) < \infty$, but clearly τ_{z_*} are approximately optimal if we let $c \downarrow z_1^*$ (and hence $z_* \uparrow z_1^*$) . For $c < z_1^*$ we have $W(z) = \infty$ and it is never optimal to stop.

4. To verify that these formulas are correct (with $c > z_1^*$ given and fixed) we shall apply Itô formula to the process $(e^t \, W(Z_t))$. For this note that $z \mapsto W(z)$ is C^2 everywhere but at $\pm z_*$. However, since Lebesgue measure of those u for which $Z_u = \pm z_*$ is zero, the values $W''(\pm z_*)$ matter little in the sequel whatever set to be. In this way by (2.2) we obtain

$$e^t \, W(Z_t) = W(z) + \int_0^t e^u \big(\mathbf{L_Z} W(Z_u) + W(Z_u) \big) \, du + M_t$$

where (M_t) is a continuous local martingale given by

$$M_t = \sqrt{2} \int_0^t e^u \, W'(Z_u) \, dB_u \, .$$

Using that $\mathbf{L_Z} W(z) + W(z) \leq 0$ for $z \neq \pm z_*$, hence we get

$$e^t \, W(Z_t) \leq W(z) + M_t \tag{2.16}$$

for all t . Let τ be any stopping time for (Z_t) satisfying $\mathbf{E}_z(e^\tau) < \infty$. Choose a localization sequence (σ_n) of bounded stopping times for (M_t) . Clearly $W(z) \geq |z| - c$ for all z , and hence from (2.16) we find

$$\mathbf{E}_z \Big(e^{\tau \wedge \sigma_n} \big(|Z_{\tau \wedge \sigma_n}| - c \big) \Big) \leq \mathbf{E}_z \Big(e^{\tau \wedge \sigma_n} \, W(Z_{\tau \wedge \sigma_n}) \Big) \leq W(z) + \mathbf{E}_z \big(M_{\tau \wedge \sigma_n} \big) = W(z)$$

for all $n \geq 1$. Letting $n \to \infty$ and using Fatou's lemma, and then taking supremum over all stopping times τ satisfying $\mathbf{E}_z(e^\tau) < \infty$, we obtain

$$W_*(z) \leq W(z) \, . \tag{2.17}$$

Finally, to prove that equality in (2.17) is attained, and that the stopping time (2.15) is optimal, it is enough to verify that

$$W(z) = \mathbf{E}_z \Big(e^{\tau_{z_*}} \big(|Z_{\tau_{z_*}}| - c \big) \Big) = (z_* - c) \, \mathbf{E}_z \big(e^{\tau_{z_*}} \big) \, . \tag{2.18}$$

However, from general Markov process theory we know that $w(z) = \mathbf{E}_z(e^{\tau_{z_*}})$ solves (2.10), and clearly it satisfies $w(\pm z_*) = 1$. Thus (2.18) follows immediately from (2.14) and definition of z_* (see also Remark 2.7 below).

5. In this way we have established that the formulas (2.14) and (2.15) are correct. Recalling by (2.5) and (2.7) that

$$V_*(t,x) = \sqrt{t} \, W_*(x/\sqrt{t})$$

we have therefore proved the following result.

Theorem 2.3. *Let z_1^* denote the unique positive root of $M(-1/2\,,1/2\,,z^2/2)=0$. The value function of the optimal stopping problem* (2.4) *for $c \ge z_1^*$ is given by*

$$V_*(t,x) = \begin{cases} -\sqrt{t}\, z_*^{-1}\, M(-\frac{1}{2}\,,\frac{1}{2}\,,\frac{x^2}{2t})/M(\frac{1}{2}\,,\frac{3}{2}\,,\frac{z_*^2}{2}) & \text{if } |x|/\sqrt{t} < z_* \\ |x| - c\sqrt{t} & \text{if } |x|/\sqrt{t} \ge z_* \end{cases}$$

where z_* is the unique positive root of the equation

$$z^{-1} M(-\tfrac{1}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = (c-z)\, M(\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})$$

satisfying $z_* \le z_1^*$. The optimal stopping time in (2.4) *for $c > z_1^*$ is given by* (see Figure 2)

$$\tau_* = \inf\left\{\, r>0 \,:\, |X_r| \ge z_*\sqrt{t+r}\,\right\}. \tag{2.19}$$

For $c = z_1^*$ the stopping times τ_* are approximately optimal if we let $c \downarrow z_1^*$. For $c < z_1^*$ we have $V_*(t,x) = \infty$ and it is never optimal to stop.

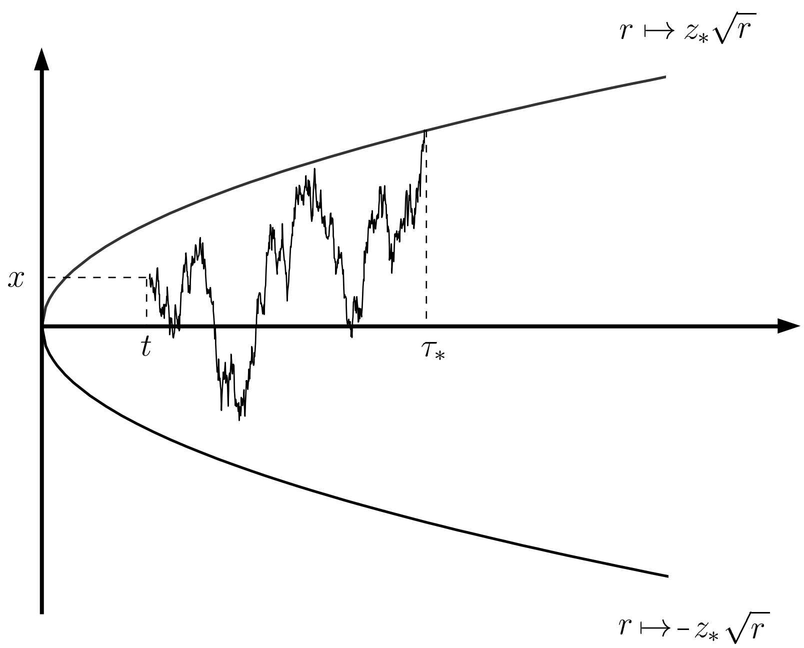

Figure 2. A computer simulation of the optimal stopping time τ_* in the problem (2.4) for $c > z_1^*$ as defined in (2.19). The process above is a standard Brownian motion which at time t starts at x . The optimal time τ_* is obtained by stopping the process as soon as it hits the area above or below the parabolic boundary $r \mapsto \pm z_*\sqrt{r}$.

Using $\sqrt{t+\tau} \le \sqrt{t} + \sqrt{\tau}$ in (2.4) it is easily verified that $V_*(t,0) \to V_*(0,0)$ as $t \downarrow 0$. Hence we see that $V_*(0,0) = 0$ with $\tau_* \equiv 0$. Note also that $V_*(0,x) = |x|$ with $\tau_* \equiv 0$.

6. Let τ be any stopping time for (B_t) satisfying $\mathbf{E}(\sqrt{\tau}) < \infty$. Then from Theorem 2.3 we see that $\mathbf{E}(|X_\tau|) \le c\,\mathbf{E}(\sqrt{t+\tau}\,) + V_*(t,0)$ for all $c > z_1^*$. Letting first $t \downarrow 0$, and then $c \downarrow z_1^*$, we obtain the following sharp inequality which was first derived by Davis [2].

Corollary 2.4. *Let (B_t) be a standard Brownian motion started at 0 , and let τ be any stopping time for (B_t) . Then the following inequality is satisfied*

$$\mathbf{E}\big(|B_\tau|\big) \le z_1^*\, \mathbf{E}\big(\sqrt{\tau}\big)$$

with z_1^* *being the unique positive root of* $M(-1/2\,,1/2\,,z^2/2)=0$. *The constant* z_1^* *is best possible. The equality is attained through the stopping times*

$$\tau_* = \inf\,\big\{\, r>0 \,:\, |B_r| \ge z_*\sqrt{t+r}\,\big\}$$

when $t\downarrow 0$ *and* $c\downarrow z_1^*$, *where* z_* *is the unique positive root of the equation*

$$z^{-1}\,M(-\tfrac{1}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = (c-z)\,M(\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})$$

satisfying $z_* < z_1^*$. *(Numerical calculations show that* $z_1^* = 1.30693\dots$*)*

7. The optimal stopping problem (2.4) can naturally be extended from the power 1 to all other $p>0$. For this consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(|X_\tau|^p - c\,(t+\tau)^{p/2}\Big) \tag{2.20}$$

where the supremum is taken over all stopping times τ for (X_t) satisfying $\mathbf{E}_x(\tau^{p/2})<\infty$ and $c>0$ is given and fixed.

Note that the case $p=2$ is easily solved directly, since we have

$$V_*(t,x) = \sup_\tau\Big((1-c)\,\mathbf{E}(\tau) + x^2 - c\,t\Big)$$

due to $\mathbf{E}|B_\tau|^2 = \mathbf{E}(\tau)$ whenever $\mathbf{E}(\tau)<\infty$. Hence we see that $V_*(t,x)=+\infty$ if $c<1$ (and it is never optimal to stop), and $V_*(t,x) = x^2 - ct$ if $c\ge 1$ (and it is optimal to stop instantly). Thus below we concentrate most to the cases when $p\ne 2$ (although the results formally extend to the case $p=2$ by passing to the limit).

The following extension of Theorem 2.3 and Corollary 2.4 is valid. (Note that in the second part of the results we make use of parabolic cylinder functions $z\mapsto D_p(z)$ which are introduced in Section 3 below.)

Theorem 2.5. (I): *For* $0<p<2$ *given and fixed, let* z_p^* *denote the unique positive root of* $M(-p/2\,,1/2\,,z^2/2)=0$. *The value function of the optimal stopping problem* (2.20) *for* $c\ge(z_p^*)^p$ *is given by*

$$V_*(t,x) = \begin{cases} -t^{p/2}\,z_*^{p-2}\,M(-\frac{p}{2}\,,\frac{1}{2}\,,\frac{x^2}{2t})/M(1-\frac{p}{2}\,,\frac{3}{2}\,,\frac{z_*^2}{2}) & \text{if } |x|/\sqrt{t} < z_* \\ |x|^p - c\,t^{p/2} & \text{if } |x|/\sqrt{t}\ge z_* \end{cases}$$

where z_* *is the unique positive root of the equation*

$$z^{p-2}\,M(-\tfrac{p}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = (c-z^p)\,M(1-\tfrac{p}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})$$

satisfying $z_*\le z_p^*$. *The optimal stopping time in* (2.20) *for* $c>(z_p^*)^p$ *is given by*

$$\tau_* = \inf\,\big\{\, r>0 \,:\, |X_r|\ge z_*\sqrt{t+r}\,\big\}\;.$$

For $c=(z_p^*)^p$ *the stopping times* τ_* *are approximately optimal if we let* $c\downarrow(z_p^*)^p$. *For* $c<(z_p^*)^p$ *we have* $V_*(t,x)=\infty$ *and it is never optimal to stop.*

(II): *For* $2<p<\infty$ *given and fixed, let* z_p *denote the largest positive root of* $D_p(z)=0$. *The value function of the optimal stopping problem* (2.20) *for* $c\ge(z_p)^p$ *is given by*

$$V_*(t,x) = \begin{cases} t^{p/2}\,z_*^{p-1}\,e^{(x^2/4t)-(z_*^2/4)}\,D_p(|x|/\sqrt{t})/D_{p-1}(z_*) & \text{if } |x|/\sqrt{t} > z_* \\ |x|^p - c\,t^{p/2} & \text{if } |x|/\sqrt{t}\le z_* \end{cases}$$

where z_* *is the unique root of the equation*

$$z^{p-1}\,D_p(z) = (z^p - c)\,D_{p-1}(z)$$

satisfying $z_* \geq z_p$. *The optimal stopping time in* (2.20) *for* $c > (z_p)^p$ *is given by*

$$\tau_* = \inf \left\{ r > 0 \ : \ |X_r| \leq z_* \sqrt{t+r} \ \right\} .$$

For $c = (z_p)^p$ *the stopping times* τ_* *are approximately optimal if we let* $c \downarrow (z_p)^p$. *For* $c < (z_p)^p$ *we have* $V_*(t,x) = \infty$ *and it is never optimal to stop.*

Proof. The proof is an easy extension of the proof of Theorem 2.3, and we only present a few steps with differences for convenience.

By Brownian scaling we have

$$\begin{aligned} V_*(t,x) &= \sup_\tau \mathbf{E}_x \Big(|B_\tau + x|^p - c\left(t+\tau\right)^{p/2} \Big) \\ &= t^{p/2} \sup_{\tau/t} \mathbf{E}_x \Big(\big| t^{-1/2} B_{t(\tau/t)} + x/\sqrt{t}\, \big|^p - c\,(1+\tau/t)^{p/2} \Big) \end{aligned}$$

and hence we see that

$$V_*(t,x) = t^{p/2}\, V_*(1, x/\sqrt{t}\,) \ . \tag{2.21}$$

By the time-change $t \mapsto \sigma_t$ from (1.5) we find

$$|X_{\sigma_\tau}|^p - c\,(1+\sigma_\tau)^{p/2} = (1+\sigma_\tau)^{p/2} \left(|Z_\tau|^p - c \right) = e^{p\tau} \left(|Z_\tau|^p - c \right)$$

and the problem to determine $V_*(1,x)$ therefore reduces to compute

$$V_*(1,x) = W_*(x) \tag{2.22}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z \Big(e^{p\tau} \left(|Z_\tau|^p - c \right) \Big) \tag{2.23}$$

the supremum being taken over all stopping times τ for (Z_t) for which $\mathbf{E}_z(e^{p\tau}) < \infty$.

To compute W_* we are naturally led to formulate the following free-boundary problem

$$\mathbf{L_Z} W(z) = -p\, W(z) \qquad \text{for } z \in C \tag{2.24}$$

$$W(z) = |z|^p - c \qquad \text{for } z \in \partial C \quad \text{(instantaneous stopping)} \tag{2.25}$$

$$W'(z) = \operatorname{sign}(z)\, p |z|^{p-1} \qquad \text{for } z \in \partial C \quad \text{(smooth fit)} \tag{2.26}$$

where C is the domain of continued observation. Observe again that W_* should be even.

In the case $0 < p < 2$ we have $C = (-z_*, z_*)$ and the stopping time

$$\tau_* = \inf \left\{ t > 0 \ : \ |Z_t| \geq z_* \right\}$$

is optimal. The proof in this case can be carried out along exactly the same lines as above when $p = 1$. However, in the case $2 < p < \infty$ we have $C = (-\infty, -z_*) \cup (z_*, \infty)$ and thus the following stopping time

$$\tau_* = \inf \left\{ t > 0 \ : \ |Z_t| \leq z_* \right\}$$

is optimal. The proof in this case requires a small modification of the previous argument. The main difference is that the solution of (2.24) used above does not have the power of smooth fit (2.25)+(2.26) any longer. It turns out, however, that the solution $z \mapsto e^{z^2/4} D_p(z)$ has this power (see Figure 3 and Figure 4), and once this being understood, the proof is again easily completed along the same lines as above (see also Remark 2.7 below). □

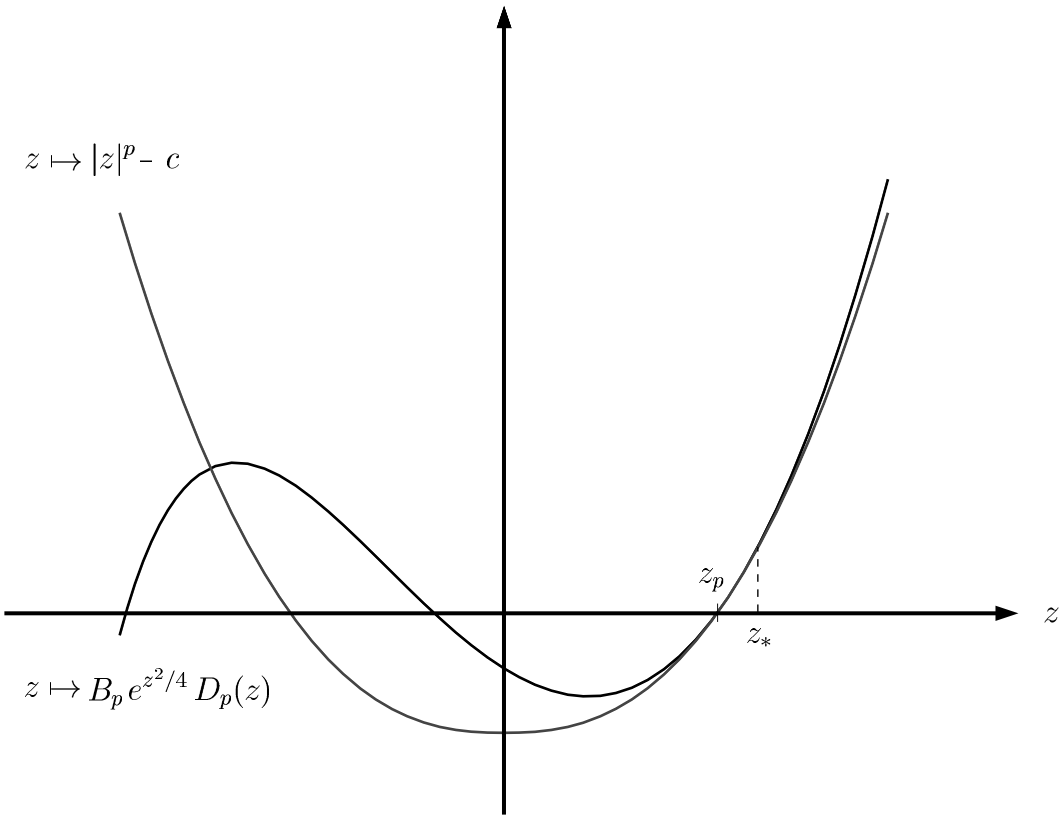

Figure 3. A computer drawing of the solution of the free-boundary problem (2.24)-(2.26) for positive z when $p = 2.5$. The solution equals $z \mapsto B_p \exp(z^2/4)\, D_p(z)$ for $z > z_*$ and $z \mapsto z^p - c$ for $0 \le z \le z_*$. The solution extends to negative z by mirroring to an even function. The constant B_p is chosen (and z_* is obtained) such that the smooth fit holds at z_* (the first derivative of the solution is continuous at z_*). A similar picture holds for all other $p > 2$ which are not even integers.

Corollary 2.6. *Let (B_t) be a standard Brownian motion started at zero, and let τ be any stopping time for (B_t) .*

(I): *For $0 < p \le 2$ the following inequality is satisfied*

$$\mathbf{E}\big(|B_\tau|^p\big) \le (z_p^*)^p\, \mathbf{E}\big(\tau^{p/2}\big)$$

with z_p^* being the unique positive root of $M(-p/2\,, 1/2\,, z^2/2) = 0$. The constant $(z_p^*)^p$ is best possible. The equality is attained through the stopping times

$$\tau_* = \inf\,\big\{\, r > 0 \,:\, |B_r| \ge z_* \sqrt{t+r}\,\big\}$$

when $t \downarrow 0$ and $c \downarrow (z_p^*)^p$, where z_* is the unique positive root of the equation

$$z^{p-2}\, M(-\tfrac{p}{2}\,, \tfrac{1}{2}\,, \tfrac{z^2}{2}) = (c - z^p)\, M(1 - \tfrac{p}{2}\,, \tfrac{3}{2}\,, \tfrac{z^2}{2})$$

satisfying $z_* < z_p^*$.

(II): *For $2 \le p < \infty$ the following inequality is satisfied*

$$\mathbf{E}\big(|B_\tau|^p\big) \le (z_p)^p\, \mathbf{E}\big(\tau^{p/2}\big)$$

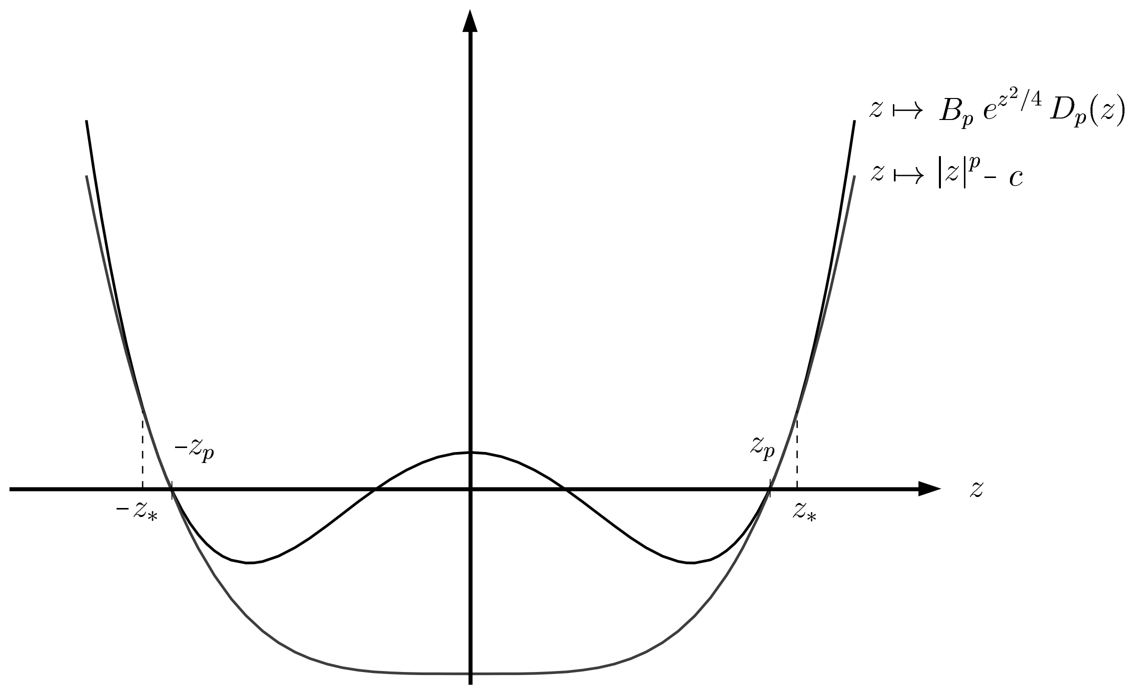

Figure 4. A computer drawing of the solution of the free-boundary problem (2.24)-(2.26) when $p = 4$. The solution equals $z \mapsto B_p \exp(z^2/4)\, D_p(z)$ for $|z| > z_*$ and $z \mapsto |z|^p - c$ for $|z| \leq z_*$. The constant B_p is chosen (and z_* is obtained) such that the smooth fit holds at $\pm z_*$ (the first derivative of the solution is continuous at $\pm z_*$). A similar picture holds for all other $p > 2$ which are even integers.

with z_p *being the largest positive root of* $D_p(z) = 0$. *The constant* $(z_p)^p$ *is best possible. The equality is attained through the stopping times*

$$\sigma_* = \inf\{\, r > 0 \, : \, |B_r + x| \leq z_* \sqrt{r}\, \}$$

when $x \downarrow 0$ *and* $c \downarrow (z_p)^p$, *where* z_* *is the unique root of the equation*

$$z^{p-1}\, D_p(z) = (z^p - c)\, D_{p-1}(z)$$

satisfying $z_* > z_p$.

Remark 2.7. The argument used above to verify (2.18) extends to the general setting of Theorem 2.5 and leads to the following explicit formulas for $0 < p < \infty$. (Note that these formulas are also valid for $-\infty < p < 0$ upon setting $z_p^* = +\infty$ and $z_p = -\infty$.)

1. For $a > 0$ define the following stopping times

$$\tau_a = \inf\left\{\, r > 0 \, : \, |Z_r| \geq a \,\right\}$$
$$\gamma_a = \inf\left\{\, r > 0 \, : \, |X_r| \geq a \sqrt{t + r}\,\right\} .$$

By Brownian scaling and the time-change (1.5) it is easily verified that

$$\mathbf{E}_x\Big((\gamma_a + t)^{p/2} \Big) = t^{p/2}\, \mathbf{E}_{x/\sqrt{t}}\big(e^{p\tau_a} \big) . \tag{2.27}$$

The argument quoted above for $|z| < a$ then gives

$$\mathbf{E}_z\big(e^{p\tau_a}\big) = \begin{cases} M(-\frac{p}{2}, \frac{1}{2}, \frac{z^2}{2})/M(-\frac{p}{2}, \frac{1}{2}, \frac{a^2}{2}) & \text{if } 0 < a < z_p^* \\ \infty & \text{if } a \geq z_p^* \,. \end{cases}$$

Thus by (2.27) for $|x| < a\sqrt{t}$ we obtain

$$\mathbf{E}_x\Big(\big(\gamma_a + t\big)^{p/2}\Big) = \begin{cases} t^{p/2}\, M(-\frac{p}{2}, \frac{1}{2}, \frac{x^2}{2t})/M(-\frac{p}{2}, \frac{1}{2}, \frac{a^2}{2}) & \text{if } 0 < a < z_p^* \\ \infty & \text{if } a \geq z_p^* \,. \end{cases}$$

This formula is also derived in [5].

2. For $a > 0$ define the following stopping times

$$\widetilde{\tau}_a = \inf\,\big\{\, r > 0 \,:\, Z_r \leq a \,\big\}$$
$$\widetilde{\gamma}_a = \inf\,\big\{\, r > 0 \,:\, X_r \leq a\sqrt{t+r}\,\big\}\,.$$

By precisely the same arguments for $z > a$ we get

$$\mathbf{E}_z\big(e^{p\widetilde{\tau}_a}\big) = \begin{cases} e^{(z^2/4)-(a^2/4)}\, D_p(z)/D_p(a) & \text{if } a > z_p \\ \infty & \text{if } a \leq z_p \end{cases}$$

and for $x > a\sqrt{t}$ we thus obtain

$$\mathbf{E}_x\Big(\big(\widetilde{\gamma}_a + t\big)^{p/2}\Big) = \begin{cases} t^{p/2}\, e^{(x^2/4t)-(a^2/4)}\, D_p(x/\sqrt{t})/D_p(a) & \text{if } a > z_p \\ \infty & \text{if } a \leq z_p \,. \end{cases}$$

This formula is also derived in [3].

Example 2.8. Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_{\tau} \mathbf{E}_x\Big(X_\tau/(t+\tau)\Big) \tag{2.28}$$

where the supremum is taken over all stopping times τ for (X_t) . This problem was first solved by Shepp [6] and Taylor [8], and it was later extended by Walker [10] and Van Moerbeke [9]. To compute (2.28) we shall use the same arguments as in the proof of Theorem 2.3 above.

1. In the first step we rewrite (2.28) as

$$V_*(t,x) = \sup_{\tau} \mathbf{E}\Big((B_\tau + x)/(t+\tau)\Big) = \tfrac{1}{\sqrt{t}} \sup_{\tau/t} \mathbf{E}\Big(\big(t^{-1/2}\, B_{t(\tau/t)} + x/\sqrt{t}\big)/(1+\tau/t)\Big)$$

and note by Brownian scaling that

$$V_*(t,x) = \tfrac{1}{\sqrt{t}}\, V_*(1, x/\sqrt{t}\,) \tag{2.29}$$

so that we only need to look at $V_*(1,x)$ in the sequel. In exactly the same way as in Remark 2.2 above, from (2.29) we can heuristically conclude that the optimal stopping boundary should be $x = \gamma_0\sqrt{t}$ for some $\gamma_0 > 0$ to be found.

2. In the second step we apply the time-change $t \mapsto \sigma_t$ from (1.5) to the problem $V_*(1,x)$ and transform it into a new problem. From (2.1) we get

$$X_{\sigma_\tau}/(1+\sigma_\tau) = Z_\tau/\sqrt{1+\sigma_\tau} = e^{-\tau} Z_\tau$$

and the problem to determine $V_*(1,x)$ therefore reduces to compute

$$V_*(1,x) = W_*(x) \tag{2.30}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z\Big(e^{-\tau} Z_\tau\Big) \tag{2.31}$$

the supremum being taken over all stopping times τ for (Z_t) .

3. In the third step we solve the problem (2.31). From general optimal stopping theory we know that the following stopping time should be optimal

$$\tau_* = \inf\big\{\, t > 0 \,:\, Z_t \geq z_* \,\big\} \tag{2.32}$$

where z_* is the optimal stopping point to be found.

To compute the value function W_* for $z < z_*$ and to determine the optimal stopping point z_* , it is natural to formulate the following free-boundary problem

$$\mathbf{L_Z} W(z) = W(z) \quad \text{for } z < z_* \tag{2.33}$$

$$W(z_*) = z_* \quad \text{(instantaneous stopping)} \tag{2.34}$$

$$W'(z_*) = 1 \quad \text{(smooth fit)} \tag{2.35}$$

with $\mathbf{L_Z}$ in (2.3).

The equation (2.33) is of the same type as the equation from Example 2.1. Since the present problem is not symmetrical, we choose its general solution in accordance with (3.6)+(3.7)

$$W(z) = A\, e^{z^2/4} D_{-1}(z) + B\, e^{z^2/4} D_{-1}(-z)$$

where A and B are unknown constants.

To determine A and B the following observation is crucial. Letting $z \to -\infty$ above, we see by (3.9) that $e^{z^2/4} D_{-1}(z) \to \infty$ and $e^{z^2/4} D_{-1}(-z) \to 0$. Hence we find that $A > 0$ would contradict the clear fact that $z \mapsto W_*(z)$ is increasing, while $A < 0$ would contradict the fact that $W_*(z) \geq z$ (by observing that $e^{z^2/4} D_{-1}(z)$ converges to ∞ faster than a polynomial). Therefore we must have $A = 0$. Moreover, from (3.9) we easily find that

$$e^{z^2/4}\, D_{-1}(-z) = e^{z^2/2} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

and hence $W'(z) = z\, W(z) + B$. The boundary condition (2.35) implies that $1 = W'(z_*) = z_*\, W(z_*) + B = z_*^2 + B$, and hence we obtain $B = 1 - z_*^2$ (see Figure 5). Setting this into (2.34), we find that z_* is the root of the equation

$$z = (1 - z^2)\, e^{z^2/2} \int_{-\infty}^{z} e^{-u^2/2}\, du\, .$$

In this way we have obtained the following candidate for the value function W_*

$$W(z) = \begin{cases} (1 - z_*^2)\, e^{z^2/2} \displaystyle\int_{-\infty}^{z} e^{-u^2/2}\, du & \text{if } z < z_* \\ z & \text{if } z \geq z_* \end{cases} \tag{2.36}$$

and the following candidate for the optimal stopping time

$$\widehat{\tau}_{z_*} = \inf\big\{\, t > 0 \,:\, Z_t \geq z_* \,\big\}\, . \tag{2.37}$$

4. To verify that these formulas are correct, we can apply Itô formula to $(e^{-t}\, W(Z_t))$, and in exactly the same way as in the proof of Theorem 2.3 above we can conclude

$$W_*(z) \leq W(z)\, .$$

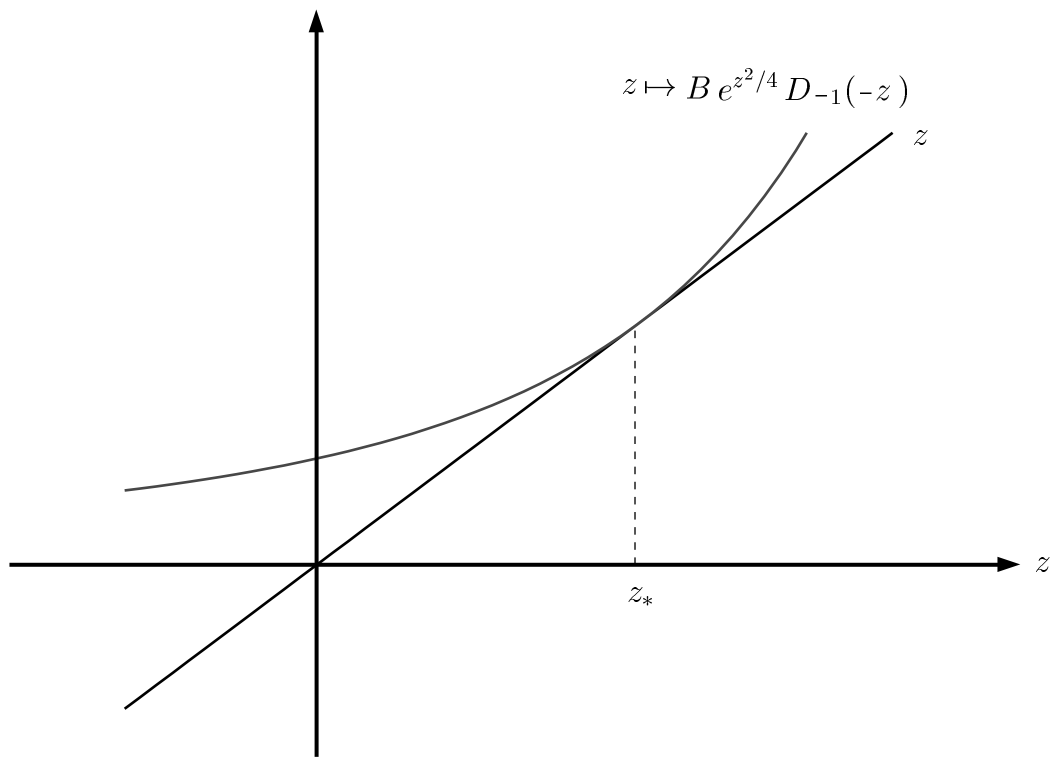

Figure 5. A computer drawing of the solution of the free-boundary problem (2.33)-(2.35). The solution equals $z \mapsto B\,\exp(z^2/4)\,D_{-1}(-z)$ for $z < z_*$ and $z \mapsto z$ for $z \ge z_*$. The constant B is chosen (and z_* is obtained) such that the smooth fit holds at $\pm z_*$ (the first derivative of the solution is continuous at $\pm z_*$).

To prove that equality is attained at $\widehat{\tau}_{z_*}$ from (2.37), it is enough to show that

$$W(z) = \mathbf{E}_z\Big(e^{-\hat{\tau}_{z_*}} Z_{\hat{\tau}_{z_*}}\Big) = z_*\, \mathbf{E}_z\big(e^{-\hat{\tau}_{z_*}}\big)\ . \tag{2.38}$$

However, from general Markov process theory we know that $w(z) = \mathbf{E}_z(e^{-\hat{\tau}_{z_*}})$ solves (2.33), and clearly it satisfies $w(z_*) = 1$ and $w(-\infty) = 0$. Thus (2.38) follows from (2.36).

5. In this way we have established that formulas (2.36) and (2.37) are correct. Recalling by (2.29) and (2.30) that

$$V_*(t,x) = \tfrac{1}{\sqrt{t}}\, W_*(x/\sqrt{t}\,)$$

we have therefore proved the following result.

Theorem 2.9. *The value function of the optimal stopping problem* (2.28) *is given by*

$$V_*(t,x) = \begin{cases} \frac{1}{\sqrt{t}}\,(1-z_*^2)\,e^{x^2/2t} \displaystyle\int_{-\infty}^{x/\sqrt{t}} e^{-u^2/2}\,du & \text{if } x/\sqrt{t} < z_* \\ x/t & \text{if } x/\sqrt{t} \ge z_* \end{cases}$$

where z_* *is the unique root of the equation*

$$z = (1-z^2)\,e^{z^2/2} \int_{-\infty}^{z} e^{-u^2/2}\,du\ . \tag{2.39}$$

The optimal stopping time in (2.28) *is given by* (see Figure 6)

$$\tau_* = \inf \left\{ r > 0 \; : \; X_r \geq z_* \sqrt{t+r} \, \right\} . \tag{2.40}$$

(*Numerical calculations show that* $z_* = 0.83992\dots$)

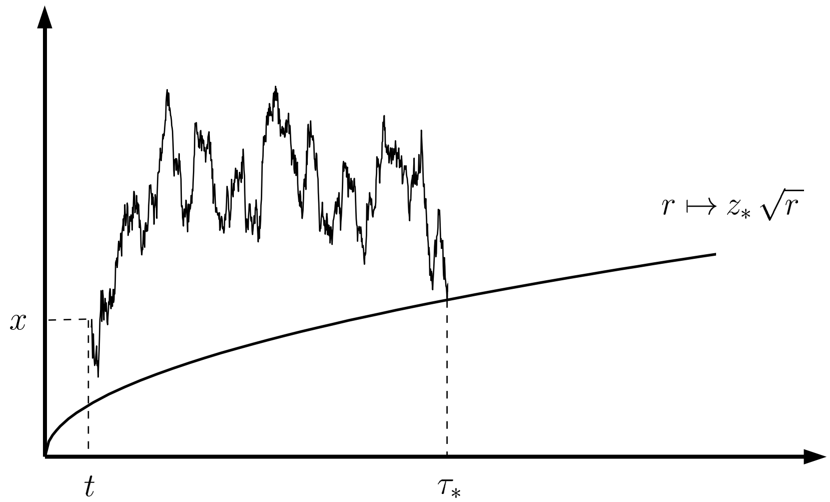

Figure 6. A computer simulation of the optimal stopping time τ_* in the problem (2.28) as defined in (2.40). The process above is a standard Brownian motion which at time t starts at x . The optimal time τ_* is obtained by stopping this process as soon as it hits the area below the parabolic boundary $r \mapsto z_* \sqrt{r}$.

6. Since the state space of (X_t) is $\mathbb{R}$ the most natural way to extend the problem (2.28) is to take (X_t) to the power of an odd integer (such that the state space again is $\mathbb{R}$). Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x \Big(X_\tau^{2n-1} / (t+\tau)^q \Big) \tag{2.41}$$

where the supremum is taken over all stopping times τ for (X_t) , and $n \geq 1$ and $q > 0$ are given and fixed. This problem was solved by Walker [10] in the case $n = 1$ and $q > 1/2$. We may now further extend Theorem 2.9 as follows.

Theorem 2.10. *Let* $n \geq 1$ *and* $q > 0$ *be taken to satisfy* $q > n - \frac{1}{2}$. *Then the value function of the optimal stopping problem* (2.41) *is given by*

$$V_*(t,x) = \begin{cases} z_*^{2n-1} \, t^{n-q-1/2} \, e^{(x^2/4t)-(z_*^2/4)} \, D_{2(n-q)-1}(-x/\sqrt{t}) / D_{2(n-q)-1}(-z_*) & \text{if } x/\sqrt{t} < z_* \\ x^{2n-1}/t^q & \text{if } x/\sqrt{t} \geq z_* \end{cases}$$

where z_* *is the unique root of the equation*

$$(2n-1) \, D_{2(n-q)-1}(-z) = z \, \big(2(q-n)+1\big) \, D_{2(n-q-1)}(-z) \, .$$

The optimal stopping time in (2.41) *is given by*

$$\tau_* = \inf \left\{ r > 0 \; : \; X_r \geq z_* \sqrt{t+r} \, \right\} .$$

(*Note that in the case* $q \leq n - 1/2$ *we have* $V_*(t,x) = \infty$ *and it is never optimal to stop.*)

Proof. The proof will only be sketched, since the arguments are the same as for the proof of Theorem 2.9. By Brownian scaling and the time-change we find

$$V_*(t,x) = t^{n-q-1/2}\, W_*(x/\sqrt{t}) \tag{2.42}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_{\tau} \mathbf{E}_z\Big(e^{(2(n-q)-1)\tau}\, Z_\tau^{2n-1}\Big) \tag{2.43}$$

the supremum being taken over all stopping times τ for (Z_t) .

Again the optimal stopping time should be of the form

$$\tau_* = \inf\,\big\{\, t>0 \,:\, Z_t \geq z_* \,\big\} \tag{2.44}$$

and therefore the value function W_* and the optimal stopping point z_* should solve the following free-boundary problem

$$\mathbf{L_Z} W(z) = \big(1-2(n-q)\big)\, W(z) \quad \text{for } z<z_* \tag{2.45}$$

$$W(z_*) = z_*^{2n-1} \quad \text{(instantaneous stopping)} \tag{2.46}$$

$$W'(z_*) = (2n-1)\, z_*^{2(n-1)} \quad \text{(smooth fit).} \tag{2.47}$$

Arguing like in the proof of Theorem 2.9 we find that the following solution of (2.45) should be taken into consideration

$$W(z) = A\, e^{z^2/4}\, D_{2(n-q)-1}(-z)$$

where A is an unknown constant. The two boundary conditions (2.46)+(2.47) with (3.8) imply that $A = z_*^{2n-1} e^{-z_*^2/4}/D_{2(n-q)-1}(-z_*)$ where z_* is the root of the equation

$$(2n-1)\, D_{2(n-q)-1}(-z) = z\,(2(q-n)+1)\, D_{2(n-q-1)}(-z)\ .$$

Thus the candidate guessed for W_* is

$$W(z) = \begin{cases} z_*^{2n-1}\, e^{(z^2/4)-(z_*^2/4)}\, D_{2(n-q)-1}(-z)/D_{2(n-q)-1}(-z_*) & \text{if } z<z_* \\ z^{2n-1} & \text{if } z \geq z_* \end{cases}$$

and the optimal stopping time is given by (2.44). By applying Itô formula like in the proof of Theorem 2.9 one can verify that these formulas are correct. Finally, inserting this back into (2.42) one obtains the result. $\square$

Remark 2.11. By exactly the same arguments as in Remark 2.7 above, we can extend the verification of (2.38) to the general setting of Theorem 2.10, and this leads to the following explicit formulas for $0<p<\infty$.

For $a>0$ define the following stopping times

$$\widehat{\tau}_a = \inf\,\big\{\, r>0 \,:\, Z_r \geq a \,\big\}$$

$$\widehat{\gamma}_a = \inf\,\big\{\, r>0 \,:\, X_r \geq a\sqrt{t+r} \,\big\}\ .$$

Then for $z<a$ we get

$$\mathbf{E}_z\big(e^{-p\widehat{\tau}_a}\big) = e^{(z^2/4)-(a^2/4)}\, D_{-p}(-z)/D_{-p}(-a)$$

and for $x<a\sqrt{t}$ we thus obtain

$$\mathbf{E}_x\Big(\big(\widehat{\gamma}_a + t\big)^{-p/2}\Big) = t^{-p/2}\, e^{(x^2/4t)-(a^2/4)}\, D_{-p}(-x/\sqrt{t})/D_{-p}(-a)\ .$$

Example 2.12. Consider the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(|X_\tau|/(t+\tau)\Big) \tag{2.48}$$

where the supremum is taken over all stopping times τ for (X_t). This problem is a natural extension of the problem (2.28) and can be solved likewise.

By Brownian scaling and the time-change we find

$$V_*(t,x) = \tfrac{1}{\sqrt{t}}\, W_*(x/\sqrt{t}) \tag{2.49}$$

where W_* is the value function of the new (time-changed) optimal stopping problem

$$W_*(z) = \sup_\tau \mathbf{E}_z\Big(e^{-\tau}\,|Z_\tau|\Big) \tag{2.50}$$

the supremum being taken over all stopping times for (Z_t).

The problem (2.50) is symmetrical (recall the discussion about (2.9) above), and therefore the following stopping time should be optimal

$$\tau_* = \inf\,\big\{\, t>0 \,:\, |Z_t| \ge z_* \,\big\}\,. \tag{2.51}$$

Thus it is natural to formulate the following free-boundary problem

$$\mathbf{L_Z} W(z) = W(z) \quad \text{for } z \in (-z_*, z_*) \tag{2.52}$$

$$W(\pm z_*) = |z_*| \quad \text{(instantaneous stopping)} \tag{2.53}$$

$$W'(\pm z_*) = \pm 1 \quad \text{(smooth fit)}. \tag{2.54}$$

From the proof of Theorem 2.3 we know that the equation (2.52) admits an even and an odd solution which are linearly independent. Since the value function should be even, we can forget the odd solution, and therefore we must have

$$W(z) = A\, M(\tfrac{1}{2}\,, \tfrac{1}{2}\,, \tfrac{z^2}{2})$$

for some $A>0$ to be found. Note from Section 3 below that $M(1/2\,,1/2\,,z^2/2) = \exp(z^2/2)$. The two boundary conditions (2.53) and (2.54) imply that $A = 1/\sqrt{e}$ and $z_* = 1$, and in this way we obtain the following candidate for the value function

$$W(z) = e^{(z^2/2)-(1/2)}$$

for $z \in (-1,1)$, and the following candidate for the optimal stopping time

$$\tau = \inf\,\big\{\, t>0 \,:\, |Z_t| \ge 1 \,\big\}\,.$$

By applying Itô formula (as in Example 2.8) one can prove that these formulas are correct. Inserting this back into (2.49) we obtain the following result.

Theorem 2.13. *The value function of the optimal stopping problem* (2.48) *is given by*

$$V_*(t,x) = \begin{cases} \frac{1}{\sqrt{t}}\, e^{(x^2/2t)-(1/2)} & \text{if } |x| < \sqrt{t} \\ |x|/t & \text{if } |x| \ge \sqrt{t}\,. \end{cases}$$

The optimal stopping time in (2.48) *is given by*

$$\tau_* = \inf\,\big\{\, r>0 \,:\, |X_r| \ge \sqrt{t+r} \,\big\}\,.$$

As in Example 2.8 above, we can further extend (2.48) by considering the optimal stopping problem with the value function

$$V_*(t,x) = \sup_\tau \mathbf{E}_x\Big(|X_\tau|^p/(t+\tau)^q\Big) \tag{2.55}$$

where the supremum is taken over all finite stopping times τ for (X_t), and $p\,,\,q>0$ are given and fixed. The arguments used to solve the problem (2.48) can be repeated, and in this way we obtain the following result.

Theorem 2.14. *Let $p\,,\,q>0$ be taken to satisfy $q>p/2$. Then the value function of the optimal stopping problem (2.55) is given by*

$$V_*(t,x) = \begin{cases} z_*^p\, t^{p/2-q}\, M(q-\frac{p}{2}\,,\frac{1}{2}\,,\frac{x^2}{2t})/M(q-\frac{p}{2}\,,\frac{1}{2}\,,\frac{z_*^2}{2}) & \text{if } |x|/\sqrt{t} < z_* \\ |x|^p/t^q & \text{if } |x|/\sqrt{t} \ge z_*\,. \end{cases}$$

where z_* is the unique root of the equation

$$p\,M(q-\tfrac{p}{2}\,,\tfrac{1}{2}\,,\tfrac{z^2}{2}) = z^2\,(2q-p)\,M(q+1-\tfrac{p}{2}\,,\tfrac{3}{2}\,,\tfrac{z^2}{2})\,.$$

The optimal stopping time in (2.55) is given by

$$\tau_* = \inf\big\{\,r>0\,:\,X_r \ge z_*\sqrt{t+r}\,\big\}\,.$$

(Note that in the case $q\le p/2$ we have $V_*(t,x)=\infty$ and it is never optimal to stop.)

Example 2.15. In this example we indicate how the problem and the results in Example 2.1 and Example 2.12 above can be extended from reflected Brownian motion to *Bessel processes* of arbitrary dimension $\alpha\ge 0$. The avoid the computational complexity which arises, we shall only indicate the essential steps towards solution.

1. *The case* $\alpha>1$. The Bessel process of dimension $\alpha>1$ is a unique (non-negative) strong solution of the stochastic differential equation

$$dX_t = \frac{\alpha-1}{2X_t}\,dt + dB_t \tag{2.56}$$

satisfying $X_0=x$ for some $x\ge 0$. The boundary point 0 is *instantaneously reflecting* if $\alpha<2$, and is an *entrance* boundary point if $\alpha\ge 2$. (When $\alpha\in\mathbb{N}$ the process (X_t) may be realized as the *radial part* of the α-dimensional Brownian motion.)

In the notation of Section 1 consider the process $(Y_t)=(\beta(t)X_t)$ and note that $\mu(x)=(\alpha-1)/2x$ and $\sigma(x)=1$. Thus conditions (1.3) and (1.4) may be realized with $\gamma(t)=\beta(t)$, $G_1(y)=(\alpha-1)/2y$ and $G_2(y)=1$. Noting that $\beta(t)=1/\sqrt{1+t}$ solves $\beta'(t)/\beta(t)=-\beta^2(t)/2$ and setting $\rho=\beta^2/2$, we see from (1.2) that

$$\mathbf{L_Z} = \Big(-z+\frac{\alpha-1}{z}\Big)\frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}$$

where $(Z_t)=(Y_{\sigma_t})$ with $\sigma_t=e^{2t}-1$. Thus (Z_t) solves the equation:

$$dZ_t = \Big(-Z_t+\frac{\alpha-1}{Z_t}\Big)\,dt + \sqrt{2}\,dB_t\,.$$

Observe that the diffusion (Z_t) may be seen as *the Euclidean velocity* of the α-dimensional Brownian motion whenever $\alpha\in\mathbb{N}$, and thus may be interpreted as the Euclidean velocity of the Bessel process (X_t) of any dimension $\alpha>1$.

The Bessel process (X_t) of any dimension $\alpha \geq 0$ satisfies the *Brownian scaling* property $Law\big((c^{-1}X_{c^2t}) \,|\, \mathbf{P}_{x/c}\big) = Law\big((X_t) \,|\, \mathbf{P}_x\big)$ for all $c > 0$ and all x. Thus the initial arguments used in Example 2.1 and Example 2.12 can be repeated, and the crucial point in the formulation of the corresponding free-boundary problem is the analogue of the equations (2.10) and (2.52)

$$\mathbf{L_Z} W(z) = \rho W(z)$$

where $\rho \in \mathbb{R}$. In comparison with the equation (3.1) this reads as follows

$$y''(x) - \Big(x - \tfrac{\alpha-1}{x}\Big)\, y'(x) - \rho\, y(x) = 0 \tag{2.57}$$

where $\rho \in \mathbb{R}$. By substituting $y(x) = x^{-(\alpha-1)/2} \exp(x^2/4)\, u(x)$ the equation (2.57) reduces to the following equation

$$u''(x) - \Big(\tfrac{x^2}{4} + \big(\rho - \tfrac{\alpha}{2}\big) + \tfrac{\alpha-1}{2}\big(\tfrac{\alpha-1}{2} - 1\big)\tfrac{1}{x^2}\Big)\, u(x) = 0\,. \tag{2.58}$$

The unpleasant term in this equation is $1/x^2$, and the general solution is not immediately found in the list of special functions in [1]. Motivated by our considerations below when $0 \leq \alpha \leq 1$, we may substitute $\bar{y}(x^2) = y(x)$ and observe that the equation (2.57) is equivalent to:

$$4z\, \bar{y}\,''(z) + 2(\alpha - z)\, \bar{y}\,'(z) - \rho\, \bar{y}(z) = 0 \tag{2.59}$$

where $z = x^2$. This equation now can be reduced to the *Whittaker's equation* (see [1]) as described in (2.60) and (2.61) below. The general solution of the Whittaker's equation is given by Whittaker's functions which are expressed in terms of Kummer's functions. This establishes a basic fact about the extension of the free-boundary problem from the reflected Brownian motion to the Bessel process of the dimension $\alpha > 1$. The problem then can be solved in exactly the same manner as before. It is interesting to observe that if the dimension α of the Bessel process (X_t) equals 3, then the equation (2.58) is of the form (3.2), and thus the optimal stopping problem is solved immediately by using the corresponding closed form solution given in Example 2.1 and Example 2.12 above.

2. *The case* $0 \leq \alpha \leq 1$. The Bessel process of dimension $0 \leq \alpha \leq 1$ does not solve a stochastic differential equation in the sense of (2.56), and therefore it is convenient to look at the *squared Bessel process* $(\bar{X}_t)$ which is a unique (non-negative) strong solution of the stochastic differential equation

$$d\bar{X}_t = \alpha\, dt + 2\sqrt{\bar{X}_t}\, dB_t$$

satisfying $\bar{X}_0 = \bar{x}$ for some $\bar{x} \geq 0$. (This is true for all $\alpha \geq 0$.) The Bessel process (X_t) is then defined as the square root of $(\bar{X}_t)$. Thus

$$X_t = \sqrt{\bar{X}_t}\,.$$

The boundary point 0 is *instantaneously reflecting* if $0 < \alpha \leq 1$, and is a *trap* if $\alpha = 0$. (The Bessel process (X_t) may be realized as a reflected Brownian motion when $\alpha = 1$.)

In the notation of Section 1 consider the process $(\bar{Y}_t) = (\beta(t)\bar{X}_t)$ and note that $\mu(x) = \alpha$ and $\sigma(x) = 2\sqrt{x}$. Thus conditions (1.3) and (1.4) may be realized with $\gamma(t) = 1$, $G_1(y) = \alpha$ and $G_2(y) = 4y$. Noting that $\beta(t) = 1/(1+t)$ solves $\beta'(t)/\beta(t) = -\beta(t)$ and setting $\rho = \beta/2$, we see from (1.2) that

$$\mathbf{L}_{\bar{\mathbf{Z}}} = 2\big(-z + \alpha\big)\, \frac{\partial}{\partial z} + 4z\, \frac{\partial^2}{\partial z^2}$$

where $(\bar{Z}_t) = (\bar{Y}_{\sigma_t})$ with $\sigma_t = e^{2t}-1$. Thus $(\bar{Z}_t)$ solves the equation:

$$d\bar{Z}_t = 2\big(-\bar{Z}_t + \alpha\big)\, dt + 2\sqrt{2\bar{Z}_t}\, dB_t \; .$$

It is interesting to observe that

$$\bar{Z}_t = \bar{Y}_{\sigma_t} = \frac{\bar{X}_{\sigma_t}}{1+\sigma_t} = \left(\frac{X_{\sigma_t}}{\sqrt{1+\sigma_t}}\right)^2$$

and thus the process $\big(\sqrt{\bar{Z}_t}\,\big)$ may be seen as *the Euclidean velocity* of the α-dimensional Brownian motion for $\alpha \in [0,1]$.

This enables us to reformulate the initial problem about (X_t) in terms of $(\bar{X}_t)$ and then after Brownian scaling and time-change $t \mapsto \sigma_t$ in terms of the diffusion $(\bar{Z}_t)$. The pleasant fact is hidden in the formulation of the corresponding free-boundary problem for $(\bar{Z}_t)$:

$$\mathbf{L}_{\bar{\mathbf{Z}}} W = \rho W$$

which in comparison with the equation (3.1) reads as follows

$$4x\, y''(x) + 2(\alpha - x)\, y'(x) - \rho\, y(x) = 0 \; . \tag{2.60}$$

Observe that this equation is of the same type as the equation (2.59). By substituting $y(x) = x^{-\alpha/4} \exp(x/4)\, u(x)$ the equation (2.60) reduces to

$$u''(x) + \left(-\frac{1}{16} + \frac{1}{4}\Big(\rho + \frac{\alpha}{2}\Big)\frac{1}{x} + \frac{\alpha}{4}\Big(1 - \frac{\alpha}{4}\Big)\frac{1}{x^2}\right) u(x) = 0 \tag{2.61}$$

which may be recognized as a *Whittaker's equation* (see [1]). The general solution of the Whittaker's equation is given by Whittaker's functions which are expressed in terms of Kummer's functions. This again establishes a basic fact about the extension of the free-boundary problem from the reflected Brownian motion to the Bessel process of the dimension $0 \le \alpha < 1$. The problem then can be solved in exactly the same manner as before. Note also that the arguments about the passage to the squared Bessel process just presented are valid for all $\alpha \ge 0$. When $\alpha > 1$ it is a matter of taste which way to choose.

Example 2.16. In this example we show how to solve some *path-dependent* optimal stopping problems (i.e. problems with the gain function depending on the entire path of the underlying process up to the time of observation).

Given an Ornstein-Uhlenbeck process (Z_t) satisfying (2.2), started at z under $\mathbf{P}_z$, consider the optimal stopping problem with the value function

$$\widetilde{W}_*(z) = \sup_{\tau} \mathbf{E}_z\left(\int_0^{\tau} e^{-u} Z_u\, du\right) \tag{2.62}$$

where the supremum is taken over all stopping time τ for (Z_t) . This problem is motivated by the fact that the integral appearing above may be viewed as a measure of the accumulated gain (up to the time of observation) which is assumed proportional to the velocity of the Brownian particle being discounted. We will first verify by Itô formula that this problem is in fact equivalent to the one-dimensional problem (2.31). Then by using the time-change σ_t we shall show that these problems are also equivalent to yet another path-dependent optimal stopping problem which is given in (2.64) below.

1. Applying Itô formula to the process $(e^{-t}Z_t)$, we find by using (2.2) that

$$e^{-t} Z_t = z + M_t - 2\int_0^t e^{-u} Z_u\, du$$

where (M_t) is a continuous local martingale given by

$$M_t = \sqrt{2} \int_0^t e^{-u}\, dB_u \; .$$

If τ is a bounded stopping time for (Z_t), then by the optional sampling theorem we get

$$\mathbf{E}_z\Big(\int_0^\tau e^{-u} Z_u\, du \Big) = \frac{1}{2}\Big(z + \mathbf{E}_z\Big(e^{-\tau}(-Z_\tau)\Big)\Big) \; .$$

Taking supremum over all bounded stopping times τ for (Z_t), and using that $(-Z_t)$ is an Ornstein-Uhlenbeck process starting from $-z$ under $\mathbf{P}_z$, we obtain

$$\widetilde{W}_*(z) = \frac{1}{2}\Big(z + W_*(-z)\Big) \tag{2.63}$$

where W_* is the value function from (2.31). The explicit expression for W_* is given in (2.36), and inserting it in (2.63), we immediately obtain the following result.

Corollary 2.17. *The value function of the optimal stopping problem* (2.62) *is given by*

$$\widetilde{W}_*(z) = \begin{cases} \frac{1}{2}\Big(z + (1-z_*^2)\, e^{z^2/2} \displaystyle\int_z^\infty e^{-u^2/2}\, du \Big) & \text{if } z > -z_* \\ 0 & \text{if } z \le -z_* \end{cases}$$

where $z_* > 0$ *is the unique root of* (2.39). *The optimal stopping time in* (2.62) *is given by*

$$\tau_* = \inf\, \big\{\, t > 0 \,:\, Z_t \le -z_* \,\big\} \; .$$

2. Given the Brownian motion $X_t = B_t + x$ started at x under $\mathbf{P}_x$, consider the optimal stopping problem with the value function

$$\widetilde{V}_*(t,x) = \sup_\tau \mathbf{E}_x \Big(\int_0^\tau \frac{X_u}{(t+u)^2}\, du \Big) \tag{2.64}$$

where the supremum is taken over all stopping times τ for (X_t). It is easily verified by Brownian scaling that we have

$$\widetilde{V}_*(t,x) = \tfrac{1}{\sqrt{t}}\, \widetilde{V}_*(1, x/\sqrt{t}) \; . \tag{2.65}$$

Moreover, by time-change (1.5) we get

$$\begin{aligned} \int_0^{\sigma_\tau} X_u/(1+u)^2\, du &= \int_0^\tau X_{\sigma_u}/(1+\sigma_u)^2\, d\sigma_u \\ &= 2 \int_0^\tau e^{2u} (1+\sigma_u)^{-3/2} Z_u\, du = 2 \int_0^\tau e^{-u} Z_u\, du \end{aligned}$$

and the problem to determine $\widetilde{V}_*(1,x)$ therefore reduces to compute

$$\widetilde{V}_*(1,x) = \widetilde{W}_*(x) \tag{2.66}$$

where $\widetilde{W}_*$ is given by (2.62). From (2.65) and (2.66) we thus obtain the following result as an immediate consequence of Corollary 2.17.

Corollary 2.18. *The value function of the optimal stopping problem* (2.64) *is given by*

$$\widetilde{V}_*(t,x) = \begin{cases} \frac{x}{t} + \frac{1}{\sqrt{t}}(1-z_*^2)\, e^{x^2/2t} \displaystyle\int_{x/\sqrt{t}}^{\infty} e^{-u^2/2}\, du & \text{if } x/\sqrt{t} > -z_* \\ 0 & \text{if } x/\sqrt{t} \le -z_* \end{cases}$$

where $z_* > 0$ *is the unique root of* (2.39). *The optimal stopping time in* (2.64) *is given by*

$$\tau_* = \inf\left\{ r > 0 \,:\, X_r \le -z_* \sqrt{t+r}\, \right\}.$$

3. The optimal stopping problem (2.62) can be naturally extended by considering the optimal stopping problem with the value function

$$\widetilde{W}_*(z) = \sup_{\tau} \mathbf{E}_z \Big(e^{-p\tau} He_n(Z_u)\, du \Big) \tag{2.67}$$

where the supremum is taken over all stopping times τ for (Z_t) and $x \mapsto He_n(x)$ is the Hermite polynomial given by (3.10), with $p > 0$ given and fixed. The crucial fact is that $x \mapsto He_n(x)$ solves the differential equation (3.1), and by Itô formula and (2.2) this implies

$$\begin{aligned} e^{-pt} He_n(Z_t) &= He_n(z) + M_t + \int_0^t e^{-pu} \Big(\mathbf{L_Z}\big(He_n\big)(Z_u) - p He_n(Z_u) \Big)\, du \\ &= He_n(z) + M_t - (n+p) \int_0^t e^{-pu} He_n(Z_u)\, du \end{aligned}$$

where (M_t) is a continuous local martingale given by

$$M_t = \sqrt{2} \int_0^t e^{-pu} (He_n)'(Z_u)\, du\,.$$

Again as above we find that

$$\widetilde{W}_*(z) = \tfrac{1}{n+p} \big(He_n(z) + W_*(z) \big)$$

with W_* being the value function of the optimal stopping problem

$$W_*(z) = \sup_{\tau} \mathbf{E}_z \Big(e^{-p\tau} \big(-He_n(Z_u) \big) \Big)$$

where the supremum is taken over all stopping times τ for (Z_t). This problem is one-dimensional and can be solved by the method used in Example 2.1.

4. Observe that the problem (2.67) with the arguments just presented can be extended from the Hermite polynomial to any solution of the differential equation (3.1).

3. Appendix: Auxiliary results

In the examples above we need the general solution of the second-order differential equation

$$y''(x) - x\, y'(x) - \rho\, y(x) = 0 \tag{3.1}$$

where $\rho \in \mathbb{R}$. By substituting $y(x) = \exp(x^2/4)\, u(x)$ the equation (3.1) reduces to

$$u''(x) - \Big(\tfrac{x^2}{4} + \big(\rho - \tfrac{1}{2}\big) \Big)\, u(x) = 0\,. \tag{3.2}$$

The general solution of (3.2) is well-known, and in the text above we make use of the following two pairs of linearly independent solutions (see [1]).

1. The *Kummer confluent hypergeometric* function is defined by

$$M(a,b,x) = 1 + \tfrac{a}{b}\,x + \tfrac{a(a+1)}{b(b+1)}\,\tfrac{x^2}{2!} + \,\ldots$$

Two linearly independent solutions of (3.2) can be expressed as

$$u_1(x) = e^{-x^2/4}\, M(\tfrac{\rho}{2}\,,\,\tfrac{1}{2}\,,\,\tfrac{x^2}{2}) \quad \text{and} \quad u_2(x) = x\, e^{-x^2/4}\, M(\tfrac{\rho}{2}+\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{x^2}{2})$$

and therefore two linearly independent solutions of (3.1) are given by

$$y_1(x) = M(\tfrac{\rho}{2}\,,\tfrac{1}{2}\,,\,\tfrac{x^2}{2}) \tag{3.3}$$

$$y_2(x) = x\, M(\tfrac{\rho}{2}+\tfrac{1}{2}\,,\,\tfrac{3}{2}\,,\tfrac{x^2}{2})\;. \tag{3.4}$$

Observe that y_1 is *even* and y_2 is *odd.* Note also that

$$M'(a,b,x) = \tfrac{a}{b}\, M(a+1, b+1, x)\;. \tag{3.5}$$

2. The *parabolic cylinder* function is defined by

$$D_\nu(x) = A_1\, e^{-x^2/4}\, M(-\tfrac{\nu}{2}\,,\tfrac{1}{2}\,,\tfrac{x^2}{2}) + A_2\, x\, e^{-x^2/4}\, M(-\tfrac{\nu}{2}+\tfrac{1}{2}\,,\tfrac{3}{2}\,,\tfrac{x^2}{2})$$

where $A_1 = 2^{\nu/2}\,\pi^{-1/2}\cos(\nu\pi/2)\,\Gamma((1+\nu)/2)$ and $A_2 = 2^{(1+\nu)/2}\,\pi^{-1/2}\sin(\nu\pi/2)\,\Gamma(1+\nu/2)$. Two linearly independent solutions of (3.2) can be expressed as

$$\widetilde{u}_1(x) = D_{-\rho}(x) \quad \text{and} \quad \widetilde{u}_2(x) = D_{-\rho}(-x)$$

and therefore two linearly independent solutions of (3.1) are given by

$$\widetilde{y}_1(x) = e^{x^2/4}\, D_{-\rho}(x) \tag{3.6}$$

$$\widetilde{y}_2(x) = e^{x^2/4}\, D_{-\rho}(-x) \tag{3.7}$$

whenever $-\rho \notin \mathbb{N}\cup\{0\}$. Note that $\widetilde{y}_1$ and $\widetilde{y}_2$ are not symmetric around zero unless $-\rho \in \mathbb{N}\cup\{0\}$. Note also that

$$\frac{d}{dx}\big(\, e^{x^2/4} D_\nu(x)\,\big) = \nu\, e^{x^2/4} D_{\nu-1}(x)\;. \tag{3.8}$$

Moreover, the following integral representation is valid

$$D_\nu(x) = \frac{e^{-x^2/4}}{\Gamma(-\nu)} \int_0^\infty u^{-\nu-1}\, e^{-xu-u^2/2}\, du \tag{3.9}$$

whenever $\nu < 0$.

3. To identify zero points of the solutions above, it is useful to note that

$$M(-n\,,\tfrac{1}{2}\,,\tfrac{x^2}{2}) = He_{2n}(x)/He_{2n}(0)$$
$$e^{x^2/4} D_n(x) = He_n(x)$$

where $x \mapsto He_n(x)$ is the *Hermite polynomial*

$$He_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\Big(e^{-x^2/2}\Big) \tag{3.10}$$

for $n \geq 0$. For more information on the facts presented in this section we refer to [1].

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Discounted Optimal Stopping Problems for the Maximum Process

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The maximality principle [6] is shown to be valid in some examples of discounted optimal stopping problems for the maximum process. In each of these examples we derive explicit formulas for the value function and display the optimal stopping time. Especially, in the framework of Black-Scholes model we calculate the fair price of two Lookback options with infinite horizon. The main aim of the paper is to show that in each example under consideration the optimal stopping boundary satisfies the maximality principle and that the value function can be determined explicitly.

1. Introduction

The main purpose of the paper is to illustrate by examples that *the maximality principle* [6] remains valid for discounted optimal stopping problems involving the maximum process associated with a one-dimensional time-homogeneous diffusion. This is done by solving some problems explicitly (see Remark 2.3 and 2.9 below). The main interest for such a class of optimal stopping problems comes from option pricing theory in Mathematical Finance, and an example below is related to that. The motivation of this problem was the conjecture by Peskir [6], that the maximality principle holds in discounted optimal stopping problems for the maximum process (see also the paper of Graversen and Peskir [4]). For completeness the method and the ideas are recalled here.

Let $((X_t), \mathbf{P}_x)$ denote a non-negative diffusion associated with the infinitesimal generator on $(0, \infty)$ given by

$$\mathbf{L_X} = \mu(x)\,\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2(x)\,\frac{\partial^2}{\partial x^2} \tag{1.1}$$

where $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ are assumed to be continuous for $x > 0$. Denote by (S_t) the maximum process associated with (X_t) given by

$$S_t = \big(\max_{0 \le r \le t} X_r\big) \vee s$$

started at $s \ge x$ under $\mathbf{P}_{x,s} := \mathbf{P}_x$. Let the discounting rate $x \mapsto \lambda(x) \ge 0$ be a continuous function, and define the functional

$$\Lambda_t = \int_0^t \lambda(X_u)\,du\ .$$

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The optimal stopping problems which appear in the next section have the value function given by

$$V_*(x,s) = \sup_{\tau} \mathbf{E}_{x,s}\Big(e^{-\Lambda_\tau}\big(S_\tau - D(X_\tau)\big)^+\Big) \tag{1.2}$$

where the supremum is taken over all finite stopping times τ for (X_t) , and the cost function $x \mapsto D(x)$ is a non-negative C^1-function. An example (the Russian option) of problem (1.2) was solved in the framework of option pricing theory by Shepp and Shiryaev [9] and [10], when the diffusion (X_t) is a geometric Brownian motion, $\lambda(x)$ is a positive constant and $D(x) \equiv 0$ (see also [1] and [3]).

Let $(\widehat{X}_t)$ be the killed diffusion at rate $\lambda(\cdot)$ of (X_t) (see [8]). If a new point Δ is adjoined to the state space $I = [0,\infty)$, and we set $I_\Delta = [0,\infty) \cup \{\Delta\}$, the (homogeneous) transition function of the process $(\widehat{X}_t)$ is given by

$$\widehat{P}_t(x,A) = \mathbf{E}_x\big(e^{-\Lambda(t)}\mathbf{1}_A(X_t)\big)$$

and the probability that $(\widehat{X}_t)$ started at x gets killed at time t is $\widehat{P}_t(x,\{\Delta\}) = 1 - \mathbf{E}_x\big(e^{-\Lambda(t)}\big)$. The killed process $(\widehat{X}_t)$ corresponds to the killing of the paths of (X_t) at the rate λ , and at the time of killing, the process $(\widehat{X}_t)$ takes the value Δ and stays in Δ . The infinitesimal operator of $(\widehat{X}_t)$ is given by

$$\mathbf{L}_{\widehat{\mathbf{X}}} = \mu(x)\,\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2(x)\,\frac{\partial^2}{\partial x^2} - \lambda(x) = \mathbf{L}_{\mathbf{X}} - \lambda(x)\,.$$

Note that in the special case of constant killing rate $\lambda(x) \equiv \lambda > 0$, the killing time is a random variable T independent of (X_t) and has the exponential distribution of parameter λ .

By the foregoing, it follows that the problem (1.2) reduces to the problem

$$V_*(x,s) = \sup_{\tau} \mathbf{E}_{x,s}\Big(\big(\widehat{S}_\tau - D(\widehat{X}_\tau)\big)^+\Big) \tag{1.3}$$

and since the point Δ cannot affect the value of X_t and due to the specific form of S_t , we may take $\widehat{S}_t = S_t$. From the reduced problem (1.3) we are led to think that the value function V_* solves the equation (see [6])

$$\mathbf{L}_{\widehat{\mathbf{X}}} V(x,s) = 0 \quad \text{for} \quad g_*(s) < x < s$$

where $s \mapsto g_*(s) < s$ is the optimal stopping boundary and that the stopping time

$$\tau_{g_*} = \inf\{\, t > 0 \,:\, X_t \le g_*(S_t)\,\}$$

is optimal, i.e. $V_*(x,s) = \mathbf{E}_{x,s}\big(\exp\big(-\Lambda_{\tau_{g_*}}\big)\big(S_{\tau_{g_*}} - D(X_{\tau_{g_*}})\big)^+\big)$. For this reason it is natural to think that the value function V_* and the optimal stopping boundary g_* solve the following system

$$\mathbf{L}_{\mathbf{X}} V(x,s) = \lambda(x) V(x,s) \qquad \text{for} \quad g(s) < x < s \tag{1.4}$$

$$V(x,s)\Big|_{x=g(s)+} = s - D(g(s)) \quad \text{(instantaneous stopping)} \tag{1.5}$$

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = -D'(g(s)) \quad \text{(smooth fit)} \tag{1.6}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \qquad \text{(normal reflection)} \tag{1.7}$$

with $\mathbf{L}_{\mathbf{X}}$ in (1.1) and where (1.5) and (1.6) are only to be understood when $g(s) > 0$.

The system (1.4)-(1.7) is a free-boundary problem, and has not a unique solution. But the maximality principle enables us to pick up the optimal stopping boundary g_* among all possible ones in a unique way, i.e. the optimal stopping boundary $s \mapsto g_*(s)$ is the maximal solution which stays below and never hits the diagonal in $\mathbb{R}^2$ (see Figure 1 and 2 below). This solution will be called the maximal solution in the sequel. Note that in general this system may have no simple solution. Thus, the system defines the boundary function $s \mapsto g(s)$ implicitly and this is the main technical difficulty to verify the maximality principle for the problem (1.2) in full generality.

Example 2.1 is a simple example with past-depending discounting where the diffusion is a reflected Brownian motion. In Example 2.4 the diffusion is the square of the Bessel process and the optimal stopping problem is of the same type as Example 2.1. The main emphasis in these examples is on the explicit expressions obtained. The fair price of the perpetual Lookback option with fixed/floating strike is calculated in Example 2.7 in the framework of Black-Scholes model. The optimal stopping boundary for the perpetual Lookback option with fixed strike is rather nontrivial, thus showing the full power of the maximality principle.

2. Examples

In this section we explicitly solve some discounted optimal stopping problems for the maximum process by applying the technique described in the first section.

Throughout (B_t) denotes a standard Brownian motion started at zero under $\mathbf{P}$.

Example 2.1. Reflected Brownian motion

The main emphasis of this example (and the next example) is on the closed formulas obtained for the value function and the optimal stopping time.

Let the diffusion (X_t) be a reflected Brownian motion, i.e. $(X_t) = (|B_t + x|)$ started at $x \geq 0$ under $\mathbf{P}_x$. The infinitesimal operator of (X_t) on $(0, \infty)$ is given by

$$\mathbf{L_X} = \tfrac{1}{2} \frac{\partial^2}{\partial x^2} \; . \tag{2.1}$$

In the setting of section 1 with discounting rate $\lambda(x) = x^{-2}$ and cost function $D(x) \equiv 0$ the optimal stopping problem (1.2) is given by

$$V_*(x, s) = \sup_\tau \mathbf{E}_{x,s}\big(e^{-\Lambda_\tau}\, S_\tau\big) \tag{2.2}$$

for $0 < x \leq s$ where the functional Λ_t is given by

$$\Lambda_t = \int_0^t |X_u|^{-2}\, du \; .$$

We shall now solve the problem (2.2).

The first step is to solve the system (1.4)-(1.7) with $\mathbf{L_X}$ in (2.1). The particular choice of the discounting rate $\lambda(\cdot)$ makes the equation (1.4) of a Cauchy-type, and the general solution is

$$V(x, s) = A(s)\, x^2 + B(s)\, x^{-1} \quad \text{for} \quad g(s) < x < s$$

where $s \mapsto A(s)$ and $s \mapsto B(s)$ are unknown functions. The instantaneous stopping condition (1.5) and the smooth fit condition (1.6) imply that

$$A(s) = \tfrac{1}{3}\, g(s)^{-2}\, s \quad \text{and} \quad B(s) = \tfrac{2}{3}\, g(s)\, s$$

so that $V(x,s) = \frac{1}{3}\, g(s)^{-2}\, s\, x^2 + \frac{2}{3}\, g(s)\, s\, x^{-1}$ for $g(s) < x < s$. Finally, the normal reflection condition (1.7) implies that $s \mapsto g(s)$ satisfies the differential equation

$$(2.3) \qquad g'(s) = \left[\frac{1}{2} \left(\frac{s}{g(s)} \right)^2 + \left(\frac{s}{g(s)} \right)^{-1} \right] \Big/ \left[\left(\frac{s}{g(s)} \right)^3 - 1 \right] .$$

Instead of a long analysis of the first-order nonlinear differential equation (2.3), we observe that $g(s) = \beta s$ with $\beta = (1/4)^{1/3}$ is a solution. g will be our candidate for the optimal stopping boundary, i.e. g should be the maximal solution (see figure 1 below). Thus, the guessed candidate for the value function V_* in (2.2) is

$$V(x,s) = \tfrac{1}{3}\beta^{-2}\, s^{-1}\, x^2 + \tfrac{2}{3}\beta\, s^2 x^{-1}$$

for $\beta s < x \le s$ and the candidate for the optimal stopping time τ_* is

$$\tau = \inf\{\, t > 0 \,:\, X_t \le \beta S_t \,\} .$$

Formulating the guessed formulas in the following proposition, the last step is to apply Itô formula to prove the correctness of the proposition.

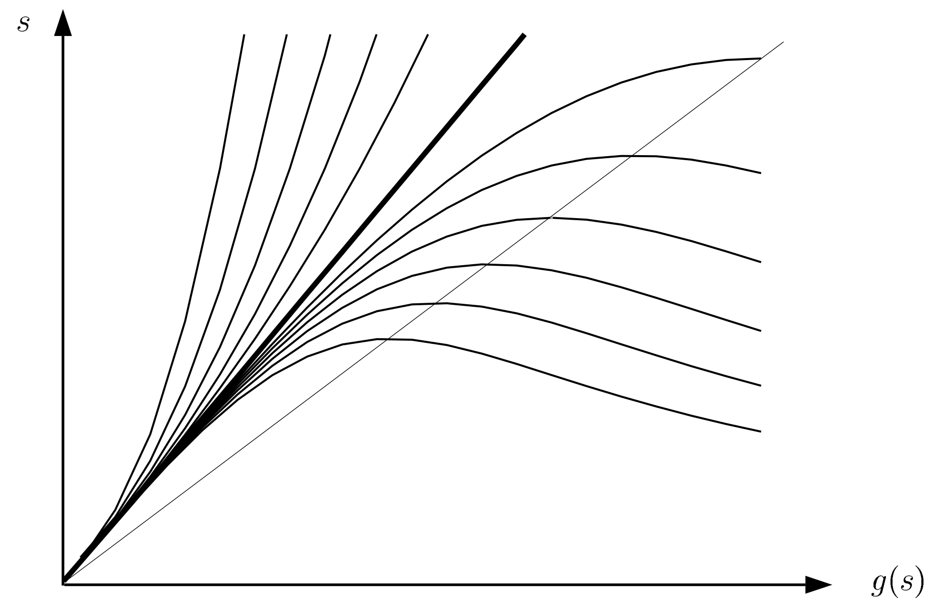

Figure 1. A computer drawing of solutions of the differential equation (2.3). The bold line is the maximal solution which stays below and never hits the diagonal in $\mathbb{R}^2$. By the maximality principle, this solution equals g_* .

Theorem 2.2. *Consider the optimal stopping problem* (2.2). *Then the value function* V_* *is given by*

$$(2.4) \qquad V_*(x,s) = \begin{cases} \frac{1}{3}\beta^{-2}\, s^{-1}\, x^2 + \frac{2}{3}\beta\, s^2 x^{-1} & \text{if } \beta s < x \le s \\ s & \text{if } 0 < x \le \beta s \end{cases}$$

and the optimal stopping time τ_* *is given by*

$$(2.5) \qquad \tau_* = \inf\{\, t > 0 \,:\, X_t \le \beta S_t \,\}$$

where $\beta = (1/4)^{1/3}$.

Proof. The following computations are under $\mathbf{P}_{x,s}$. Denote the function on the right-hand side in (2.4) by $V(x,s)$ and put $g(s)=\beta s$. Applying Itô formula to the process $e^{-\Lambda_t}V(X_t,S_t)$ we obtain

$$e^{-\Lambda_t}V(X_t,S_t)=V(x,s)-\int_0^t \lambda(X_u)e^{-\Lambda_u}\,V(X_u,S_u)\,du+\int_0^t e^{-\Lambda_u}\,\frac{\partial V}{\partial x}(X_u,S_u)\,dB_u$$
$$+\int_0^t e^{-\Lambda_u}\,\frac{\partial V}{\partial s}(X_u,S_u)\,dS_u+\frac{1}{2}\int_0^t e^{-\Lambda_u}\,\frac{\partial^2 V}{\partial x^2}(X_u,S_u)\,du\;.$$

The integral with respect to dS_u is identically zero, since the increment ΔS_u is zero outside the diagonal in $\mathbb{R}^2$, while at the diagonal V satisfies the normal reflection condition (1.7). Thus we have

$$e^{-\Lambda_t}V(X_t,S_t)=V(x,s)+M_t+\int_0^t e^{-\Lambda_u}\Big(\mathbf{L_X}V(X_u,S_u)-\lambda(X_u)V(X_u,S_u)\Big)\,du$$

where (M_t) is a continuous local martingale given by

$$M_t=\int_0^t e^{-\Lambda_u}\,\frac{\partial V}{\partial x}(X_u,S_u)\,dB_u\;.$$

Using that $\mathbf{L_X}V(x,s)-\lambda(x)V(x,s)\le 0$ for $0<x<g(s)$ and $\mathbf{L_X}V(x,s)-\lambda(x)V(x,s)=0$ for $g(s)<x<s$, and the fact that the set of all $u>0$ for which X_u is either $g(S_u)$ or S_u is of Lebesgue measure zero, we have the equality

$$e^{-\Lambda_{\tau_*\wedge t}}V(X_{\tau_*\wedge t},S_{\tau_*\wedge t})=V(x,s)+M_{\tau_*\wedge t} \tag{2.6}$$

and the inequality

$$e^{-\Lambda_t}V(X_t,S_t)\le V(x,s)+M_t\;. \tag{2.7}$$

Let τ be any stopping time for (X_t) . Choose a localization $\{\sigma_k\}_{k\ge 1}$ for (M_t) of bounded stopping times. Clearly $V(x,s)\ge s$ for all $0<x\le s$ and from (2.7) we get

$$\mathbf{E}_{x,s}\big(e^{-\Lambda_{\tau\wedge\sigma_k}}S_{\tau\wedge\sigma_k}\big)\le \mathbf{E}_{x,s}\big(e^{-\Lambda_{\tau\wedge\sigma_k}}V(X_{\tau\wedge\sigma_k},S_{\tau\wedge\sigma_k})\big)$$
$$\le V(x,s)+\mathbf{E}_{x,s}\big(M_{\tau\wedge\sigma_k}\big)=V(x,s)$$

for all $k\ge 1$. Letting $k\to\infty$, it is immediately seen by Fatou's lemma that

$$\mathbf{E}_{x,s}\big(e^{-\Lambda_\tau}S_\tau\big)\le V(x,s)\;.$$

Taking supremum over all stopping times τ for (X_t) we obtain

$$V_*(x,s)\le V(x,s)\;. \tag{2.8}$$

Finally, to prove equality in (2.8) and that the value function V_* and the optimal stopping time τ_* are given by (2.4) and (2.5) respectively, it is enough to prove

$$V(x,s)=\mathbf{E}_{x,s}\big(e^{-\Lambda_{\tau_*}}S_{\tau_*}\big)\;. \tag{2.9}$$

By (2.6) and the definition of the stopping time τ_* we have

$$e^{-\Lambda_{\tau_*}}S_{\tau_*}=e^{-\Lambda_{\tau_*}}V(X_{\tau_*},S_{\tau_*})=V(x,s)+M_{\tau_*}$$

so the proof will be completed, if we show that

$$\mathbf{E}_{x,s}\big(M_{\tau_*}\big)=0\;. \tag{2.10}$$

By Doob's optional sampling theorem and Burkholder-Davis-Gundy's inequality for continuous local martingales, in order to prove (2.10) it is enough to show that

$$\mathbf{E}_{x,s}\left(\sqrt{\int_0^{\tau_*}\left(e^{-\Lambda_u}\,\frac{\partial V}{\partial x}(X_u,S_u)\right)^2 du}\;\right) := J < \infty\,. \tag{2.11}$$

For this we compute

$$\frac{\partial V}{\partial x}(x,s) = \tfrac{2}{3}\beta^{-1}\Big(\frac{x}{\beta s} - \Big(\frac{\beta s}{x}\Big)^2\Big) \le \tfrac{2}{3}(\beta^{-2}-\beta)$$

for $\beta s < x < s$. Inserting this into (2.11) we get

$$\begin{aligned} J &= \mathbf{E}_{x,s}\left(\sqrt{\int_0^{\tau_*}\left(e^{-\Lambda_u}\,\frac{\partial V}{\partial x}(X_u,S_u)\right)^2 du}\;\right) \\ &\le \tfrac{2}{3}(\beta^{-2}-\beta)\,\mathbf{E}_{x,s}\big(\sqrt{\tau_*}\big) < \infty \end{aligned}$$

provided that $\mathbf{E}_{x,s}\big(\sqrt{\tau_*}\big) < \infty$ which is known to be true (see [5]). The proof is completed. $\square$

Remark 2.3. In the proof up to (2.8) we did not make any use of the specific form of the optimal stopping boundary $s \mapsto g_*(s)$ and the corresponding value function V_* . Let g solve the equation (2.3) and stay below the diagonal in $\mathbb{R}^2$. Let V_g be the corresponding function which solve the system (1.4)-(1.7). With exactly the same arguments as in the proof (2.8) it follows that $V_* \le V_g$. We also have that $g \mapsto V_g$ is (strictly) decreasing. Therefore $s \mapsto g_*(s)$ is the maximal solution, and thus this example illustrate the validity of the maximality principle.

Example 2.4. Bessel process

This example is of the same type as Example 2.1. Thus the results in this example will only be postulated since the computations and proofs are almost the same as in Example 2.1.

Let (X_t) be the square of a Bessel process of dimension $\alpha > 0$ (see [7]) satisfying the stochastic differential equation

$$dX_t = \alpha\,dt + 2\sqrt{X_t}\,dB_t\,.$$

The infinitesimal operator of (X_t) on $(0,\infty)$ is given by

$$\mathbf{L_X} = \alpha\,\frac{\partial}{\partial x} + 2x\,\frac{\partial^2}{\partial x^2}\,. \tag{2.12}$$

With discounting rate $\lambda(x) = rx^{-1}$ where $r > 0$ is a constant and cost function $D(x) \equiv 0$, the optimal stopping problem (1.2) is given by

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\big(e^{-\Lambda_\tau} S_\tau\big) \tag{2.13}$$

for $0 < x \le s$, where the functional Λ_t is given by

$$\Lambda_t = r\int_0^t X_u^{-1}\,du\,.$$

Again by the choice of the discounting rate $\lambda(\cdot)$ the equation (1.4) with $\mathbf{L_X}$ in (2.12) is of Cauchy-type, and it is possible to find the solution to the system (1.4)-(1.7) which satisfies the maximality principle. It turns out that the optimal stopping boundary $s \mapsto g_*(s)$ is a linear function (see Remark 2.6 below). The result is stated in the following theorem.

Theorem 2.5. *Consider the optimal stopping problem* (2.13). *Let* $\gamma_1 < \gamma_2$ *be the two roots of the quadratic equation*

$$2\gamma^2 - (2-\alpha)\gamma - r = 0$$

i.e.

$$\gamma_{1,2} = \frac{(2-\alpha) \mp \sqrt{(2-\alpha)^2 + 8r}}{4}\ .$$

If $r > \alpha$, *then* $\gamma_1 < 0$ *and* $\gamma_2 > 1$ *and the value function* V_* *is given by*

$$V_*(x,s) = \begin{cases} \dfrac{\gamma_2}{\gamma_2-\gamma_1}\left(\dfrac{x}{\beta s}\right)^{\gamma_1} s - \dfrac{\gamma_1}{\gamma_2-\gamma_1}\left(\dfrac{x}{\beta s}\right)^{\gamma_2} s & \text{if} \quad \beta s < x \le s \\ s & \text{if} \quad 0 < x \le \beta s \end{cases}$$

and the optimal stopping time τ_* *is given by*

$$\tau_* = \inf\{\, t > 0 \,:\, X_t \le \beta S_t \,\}$$

where β *is a constant given by*

$$\beta = \left(\frac{1-1/\gamma_2}{1-1/\gamma_1}\right)^{1/(\gamma_2-\gamma_1)}\ .$$

For $r \le \alpha$ *we have* $V_*(x,s) = \infty$ *and it is never optimal to stop.*

Remark 2.6. The boundary function $s \mapsto g(s)$ in the system (1.4)-(1.7) satisfies the differential equation

$$g'(s) = \left[\frac{1}{\gamma_2}\left(\frac{s}{g(s)}\right)^{\gamma_2} - \frac{1}{\gamma_1}\left(\frac{s}{g(s)}\right)^{\gamma_1}\right] \Big/ \left[\left(\frac{s}{g(s)}\right)^{\gamma_2+1} - \left(\frac{s}{g(s)}\right)^{\gamma_1+1}\right]$$

and the optimal stopping boundary $g_*(s) = \beta s$ is the maximal solution. This shows the validity of the maximality principle in this example.

Example 2.7. Perpetual Lookback options

In the framework of the standard Black-Scholes model under the equivalent martingale measure we shall consider two examples of pricing an American option with infinite horizon. Thus, the diffusion (X_t) is a geometric Brownian motion satisfying the stochastic differential equation

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t \tag{2.14}$$

where $r > 0$ and $\sigma > 0$ are two given constants. The infinitesimal operator of (X_t) on $(0,\infty)$ is given by

$$\mathbf{L_X} = rx\,\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2 x^2\,\frac{\partial^2}{\partial x^2}\ . \tag{2.15}$$

Let us consider the following two Lookback options.

1. The payment function of the *Lookback option with fixed strike* (called 'option on extrema' in [2]) is given by

$$f_t = e^{-\lambda t}(S_t - K)^+$$

where $\lambda > 0$ and $K \geq 0$ are two constants. For $K = 0$ it is the Russian option [9]. Under these assumptions the fair price of the perpetual Lookback option with fixed strike is according to the general option pricing theory the value of the optimal stopping problem

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\big(e^{-r\tau} f_\tau\big) \tag{2.16}$$

for $0 < x \leq s$. With the notation of section 1 with discounting rate $\lambda(x) \equiv r + \lambda$ and cost function $D(x) \equiv K$, we can rewrite (2.16) as follows

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\Big(e^{-\Lambda_\tau}\,(S_\tau - K)^+\Big)\ . \tag{2.17}$$

for $0 < x \leq s$ where the functional Λ_t is given by

$$\Lambda_t = (r+\lambda)\,t\ .$$

The first step in solving problem (2.17) is to find all solutions to the system (1.4)-(1.7) with $\mathbf{L_X}$ in (2.15), and straightforward computations give that the solutions are

$$V(x,s) = \frac{s-K}{\gamma_2 - \gamma_1}\left[\gamma_2\left(\frac{x}{g(s)}\right)^{\gamma_1} - \gamma_1\left(\frac{x}{g(s)}\right)^{\gamma_2}\right] \tag{2.18}$$

for $g(s) < x < s$ where $s \mapsto g(s)$ satisfies the differential equation

$$g'(s) = \left[\frac{1}{\gamma_2}\left(\frac{s}{g(s)}\right)^{\gamma_2} - \frac{1}{\gamma_1}\left(\frac{s}{g(s)}\right)^{\gamma_1}\right] \Big/ \left[\frac{s-K}{g(s)}\left(\left(\frac{s}{g(s)}\right)^{\gamma_2} - \left(\frac{s}{g(s)}\right)^{\gamma_1}\right)\right] \tag{2.19}$$

and $\gamma_1 < 0$ and $\gamma_2 > 1$ are the two roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\,\gamma^2 + (r - \tfrac{1}{2}\sigma^2)\,\gamma - (r+\lambda) = 0$$

i.e.

$$\gamma_{1,2} = \left(\frac{1}{2} - \frac{r}{\sigma^2}\right) \mp \sqrt{\left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}}\ . \tag{2.20}$$

If $K = 0$ we see from Proposition 2.10 below (with $\kappa = 0$) that $g_*(s) = \beta s$ is the maximal solution to the equation (2.19) which stays below and never hits the diagonal in $\mathbb{R}^2$, where β is given by

$$\beta = \left(\frac{1 - 1/\gamma_2}{1 - 1/\gamma_1}\right)^{1/(\gamma_2 - \gamma_1)}\ .$$

If $K > 0$ by Picard's method of successive approximations we can establish the existence of the solution g_* to the equation (2.19) such that $g_*(s) < s$ and $|g_*(s) - \beta s| \to 0$ for $s \to \infty$. The proof of this statement is technical and will be omitted (see [4] for a similar proof). The maximal solution should be g_* (see Figure 2 below). Thus, the guessed candidate for the optimal stopping time τ_* is

$$\tau_* = \inf\{\, t > 0 \,:\, X_t \leq g_*(S_t)\,\}$$

and the candidate for the value function V_* is given in (2.18) for $K < x \leq s$. If the process starts at $0 < x \leq s < K$, and $\tau_* = \tau_K + \tau_* \circ \theta_{\tau_K}$ is optimal, we have by strong Markov property that

$$V_*(x,s) = \mathbf{E}_x\big(\exp(-\Lambda_{\tau_K})\big)\, V_*(K,K)$$

where

$$\tau_K = \inf\{\, t > 0 \,:\, X_t = K\,\}\ .$$

It is well-known (see [7]) that

$$\mathbf{E}_x\big(\exp(-\Lambda_{\tau_K})\big) = \big(x/K\big)^{\gamma_2} .$$

The guessed formulas are formulated in the following theorem.

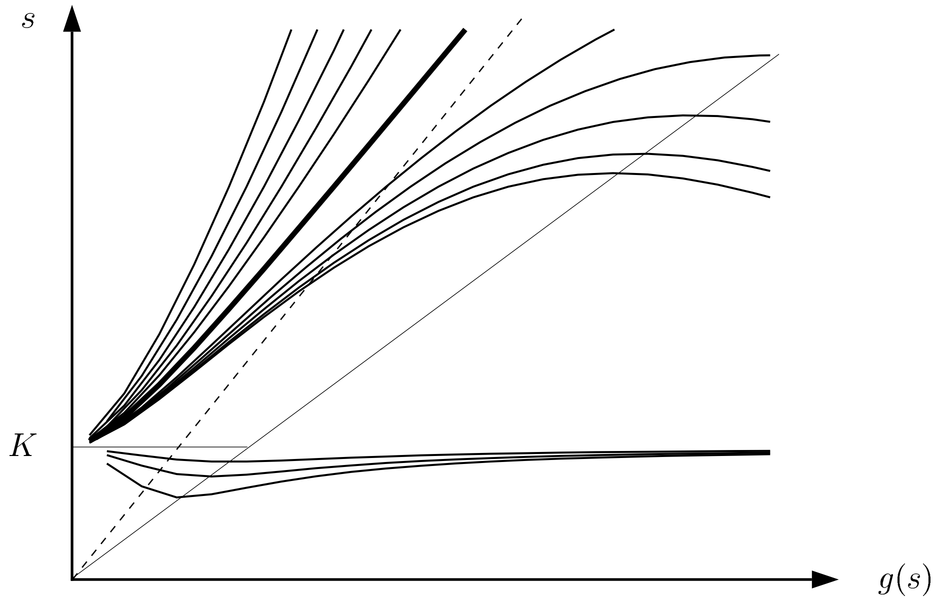

Figure 2. A computer drawing of solutions of the differential equation (2.19). The bold line is the maximal solution which stays below and never hits the diagonal in $\mathbb{R}^2$. By the maximality principle, this solution equals g_* . The stipple line is the function $s \mapsto \beta s$.

Theorem 2.8. *The fair price of the perpetual Lookback option with fixed strike defined by* (2.17) *is given by*

$$V_*(x,s) = \begin{cases} \dfrac{s-K}{\gamma_2-\gamma_1}\bigg(\gamma_2\Big(\dfrac{x}{g_*(s)}\Big)^{\gamma_1} - \gamma_1\Big(\dfrac{x}{g_*(s)}\Big)^{\gamma_2}\bigg) & \textit{if} \quad s > K \quad \textit{and} \quad g_*(s) < x \le s \\ \big(x/K\big)^{\gamma_2}\, V_*(K,K) & \textit{if} \quad 0 < x \le s \le K \\ s-K & \textit{if} \quad s > K \quad \textit{and} \quad 0 < x \le g_*(s) \end{cases}$$

where $V_*(K,K) = \lim_{s \downarrow K} V_*(K,s)$. *The optimal stopping time is given by*

$$\tau_* = \inf\{\, t > 0 \,:\, X_t \le g_*(S_t) \,\}$$

where $s \mapsto g_*(s)$ *is the solution of the differential equation*

$$g'(s) = \bigg[\frac{1}{\gamma_2}\Big(\frac{s}{g(s)}\Big)^{\gamma_2} - \frac{1}{\gamma_1}\Big(\frac{s}{g(s)}\Big)^{\gamma_1}\bigg] \Big/ \bigg[\frac{s-K}{g(s)}\bigg(\Big(\frac{s}{g(s)}\Big)^{\gamma_2} - \Big(\frac{s}{g(s)}\Big)^{\gamma_1}\bigg)\bigg]$$

such that $g_*(s) < s$ *for all* $s > 0$ *and*

$$|g_*(s) - \beta s| \to 0 \quad \textit{for} \quad s \to \infty$$

where β *is a constant given by*

$$\beta = \bigg(\frac{1-1/\gamma_2}{1-1/\gamma_1}\bigg)^{1/(\gamma_2-\gamma_1)}$$

and γ_1 *and* γ_2 *are defined in* (2.20).

Proof. For $K = 0$ it follows directly from Proposition 2.10 below (with $\kappa = 0$). Assume that $K > 0$ and that the following computations are under $\mathbf{P}_{x,s}$. Denote the function on the right-hand side in the theorem by $V(x,s)$. Applying Itô formula to the process $e^{-\Lambda_t}V(X_t,S_t)$, by the same arguments as in the proof of Theorem 2.2, we obtain

$$e^{-\Lambda_t}V(X_t,S_t) = V(x,s) + M_t + \int_0^t e^{-\Lambda_u}\Big(\mathbf{L_X}V(X_u,S_u) - \lambda(X_u)V(X_u,S_u)\Big)\,du$$

by means of (2.14), where the process (M_t) is the continuous local martingale given by

$$M_t = \sigma \int_0^t e^{-\Lambda_u}\,\frac{\partial V}{\partial x}(X_u,S_u)X_u\,dB_u\ .$$

Similarly as in the proof of Theorem 2.2 we have the equality

$$e^{-\Lambda_{\tau_*\wedge t}}V(X_{\tau_*\wedge t},S_{\tau_*\wedge t}) = V(x,s) + M_{\tau_*\wedge t} \tag{2.21}$$

and the inequality

$$V_*(x,s) \le V(x,s)\ . \tag{2.22}$$

It is enough to show by (2.21) that

$$\mathbf{E}_{x,s}\big(M_{\tau_*}\big) = 0 \tag{2.23}$$

to prove equality in (2.22). It is easily seen by (2.21) that $(M_{\tau_*\wedge t})$ is bounded from below by $-V(x,s)$. Let $\epsilon > 0$ be given and hence there exists s' such that $\tau_* \le \tau_{s'} + \tau'\circ\theta_{\tau_{s'}}$ where τ' is a stopping time given by

$$\tau' = \inf\{\, t>0 \,:\, X_t \le (\beta S_t - \epsilon)\,\}\ .$$

The upper bound for $(M_{\tau_*\wedge t})$ is then

$$\begin{aligned}
M_{\tau_*\wedge t} &\le e^{-\Lambda_{\tau_*\wedge t}}\,V(X_{\tau_*\wedge t},S_{\tau_*\wedge t}) \le e^{-\Lambda_{\tau_*\wedge t}}\,V(S_{\tau_*\wedge t},S_{\tau_*\wedge t}) \le \sup_{t\le\tau_*} e^{-\Lambda_t}\,V(S_t,S_t)\\
&\le \sup_{t\le\tau_{s'}+\tau'\circ\theta_{\tau_{s'}}} e^{-\Lambda_t}\,V(S_t,S_t) \le \Big\{\max_{t\le\tau_{s'}} e^{-\Lambda_t}\,V(S_t,S_t)\Big\} \vee \Big\{\sup_{\tau_{s'}\le t\le\tau'\circ\theta_{\tau_{s'}}} e^{-\Lambda_t}\,V(S_t,S_t)\Big\}\\
&\le k_1 \vee \Big\{\sup_{\tau_{s'}\le t\le\tau'\circ\theta_{\tau_{s'}}} e^{-\Lambda_t}\,\frac{S_t-K}{\gamma_2-\gamma_1}\Big(\gamma_2\Big(\frac{S_t}{g_*(S_t)}\Big)^{\gamma_1} - \gamma_1\Big(\frac{S_t}{g_*(S_t)}\Big)^{\gamma_2}\Big)\Big\}\\
&\le k_1 \vee \Big\{\sup_{\tau_{s'}\le t\le\tau'\circ\theta_{\tau_{s'}}} e^{-\Lambda_t}\,\frac{S_t}{\gamma_2-\gamma_1}\Big(\gamma_2 - \gamma_1\Big(\frac{s'}{\beta s'-\epsilon}\Big)^{\gamma_2}\Big)\Big\} \le k_1 + k_2\sup_{t>0}\{e^{-\Lambda_t}\,S_t\}
\end{aligned}$$

where k_1 and k_2 are two constants. The variable $\sup_{t>0} e^{-\Lambda_t}\,S_t$ is integrable (see [9]) and therefore is $(M_{\tau_*\wedge t})$ uniformly integrable and hence we can conclude that $\mathbf{E}_{x,s}\big(M_{\tau_*}\big) = 0$. The proof is complete. □

Remark 2.9. By the same arguments as in Remark 2.3 we see that $s \mapsto g_*(s)$ is the maximal solution. Moreover we do not see how the problem could be solved without the maximality principle.

2. The payment function of the *Lookback option with floating strike* (called ' partial Lookback' in [2]) is given by

$$h_t = e^{-\lambda t}\big(S_t - \kappa X_t\big)^+$$

where $\lambda > 0$ and $\kappa \geq 0$ are constants. For $\kappa = 0$ it is the Russian option [9]. The fair price of the perpetual Lookback option with floating strike is the value of the optimal stopping problem

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\big(e^{-r\tau}h_\tau\big) \ .$$

for $0 < x \leq s$. The only change from the Lookback option with fixed strike is the cost function $D(x) = \kappa x$ and we have the optimal stopping problem

$$V_*(x,s) = \sup_\tau \mathbf{E}_{x,s}\Big(e^{-\Lambda_\tau}\big(S_\tau - \kappa X_\tau\big)^+\Big) \tag{2.24}$$

The problem (2.24) was solved in [10] in the special case $\kappa = 0$ (see also [9] and [3]) and in the general case the following proposition was proved in [1].

Proposition 2.10. *The fair price of the perpetual Lookback option with floating strike defined by* (2.24) *is given by*

$$V_*(x,s) = \begin{cases} \dfrac{x/\beta}{\gamma_2 - \gamma_1}\left[\Big(\gamma_2 - \kappa\beta(\gamma_2 - 1)\Big)\Big(\dfrac{x}{\beta s}\Big)^{\gamma_1 - 1} - \Big(\gamma_1 - \kappa\beta(\gamma_1 - 1)\Big)\Big(\dfrac{x}{\beta s}\Big)^{\gamma_2 - 1}\right] & \text{if } \ \beta s < x \leq s \\ s - \kappa x & \text{if } \ 0 < x \leq \beta s \end{cases}$$

and the optimal stopping time τ_* *is given by*

$$\tau_* = \inf\{\, t > 0 \, : \, X_t \leq \beta S_t \,\}$$

where β *is the unique solution to the equation*

$$\beta^{\gamma_2 - \gamma_1} = \frac{1 - 1/\gamma_2}{1 - 1/\gamma_1}\,\frac{1 - \kappa\beta(1 - 1/\gamma_1)}{1 - \kappa\beta(1 - 1/\gamma_2)}$$

and γ_1 *and* γ_2 *are defined in* (2.20)*.*

Remark 2.11. It is easily checked that the value function V_* and the optimal stopping boundary $s \mapsto g_*(s)$ solve the system (1.4)-(1.7) with $\mathbf{L_X}$ in (2.15) which satisfies the maximality principle. However, in this case it is not the most natural method to solve the problem (2.24). Instead by Girsanov's change of measure the 2-dimensional problem can be reduced to a 1-dimensional problem. The 1-dimensional problem can be solved by different methods, but in any of the methods it is crucial that the process (S_t/X_t) is a new diffusion which is a special property of the geometric Brownian motion (see [10], [3] and [1]).

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The Azéma-Yor Solution to Embedding in Non-Singular Diffusions

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Let $(X_t)_{t \geq 0}$ be a non-singular diffusion on $\mathbb{R}$ vanishing at zero and is not necessarily recurrent. Let ν be a probability measure on $\mathbb{R}$ having strictly positive density. Necessary and sufficient conditions on ν are given such that there exists a stopping time τ_* of (X_t) solving the Skorokhod embedding problem, i.e. X_{τ_*} has the law ν. Furthermore an explicit construction of τ_* is carried out which is an extension of the Azéma-Yor solution when the process is a recurrent diffusion. In addition, τ_* is characterized uniquely to be the pointwise smallest possible embedding that stochastically maximizes the maximum process of (X_t) up to the time of stopping or stochastically minimizes the minimum process of (X_t) up to the time of stopping.

1. Introduction

Consider a probability measure ν on $\mathbb{R}$ and a non-singular time-homogeneous diffusion $(X_t)_{t \geq 0}$ vanishing at zero. In this paper we consider the problem of embedding the given law ν in the process (X_t) by construction of a stopping time τ_* of (X_t), i.e. by finding a stopping time τ_* of (X_t) satisfying $X_{\tau_*} \sim \nu$ and determining conditions on ν which make this possible. The problem is known as *Skorokhod embedding problem.*

The proof (see below) leads naturally to explicit construction of an extremal embedding of ν in the following sense. The embedding is an extension of the Azéma-Yor construction [1] that is pointwise the smallest possible embedding that stochastically maximizes $\max_{0 \leq t \leq \tau_*} X_t$ or stochastically minimizes $\min_{0 \leq t \leq \tau_*} X_t$ over all embeddings τ_*.

The Skorokhod embedding problem has been investigated by many authors and was initiated in Skorokhod [16] when (X_t) is Brownian motion. In this case Azéma and Yor [1] (see Rogers [13] for an excursion argument) and Perkins [9] yield two different explicit extremal solutions of the Skorokhod embedding problem in the natural filtration. An extension of the Azéma-Yor embedding when the Brownian motion has an initial law was given in Hobson [6]. The existence of an embedding in a general Markov process was characterized by Rost [15], but no explicit construction of the stopping time was given. Bertoin and Le Jan [3] constructed a new class of embeddings when the process (X_t) is a Hunt process starting at a regular recurrent point. Furthermore Azéma and Yor [1] give an explicit solution when the process (X_t) is a recurrent diffusion. The case where the process (X_t) is Brownian motion with drift (non-recurrent diffusion) was studied in Grandits [5] and Peskir [10] and in the latter paper a necessary and sufficient condition on ν is given such that an explicit embedding (an extension of the Azéma-Yor embedding) is possible. More general embedding problems for martingales are considered in Rogers [14] and Brown, Hobson and Rogers [4].

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Applications of Skorokhod embedding problems have gained some interest to option pricing theory. How to design an option given the law of the risk is studied in [11], and bounds on the prices of Lookback options obtained by robust hedging are studied in [7].

This paper was motivated by the works of Grandits [5] and Peskir [10] where they show that an extension of the Azéma-Yor construction is an embedding for the non-recurrent diffusion: Brownian motion with drift. In this paper we extend this solution to general non-recurrent non-singular diffusions. The approach of finding a solution to the Skorokhod problem is the following. First, the initial problem is transformed by composing (X_t) with its scale function into an analogous embedding problem for a continuous local martingale. Secondly, by the time-change given in the construction of the Dambis, Dubins-Schwarz Brownian motion (see [12]) the martingale embedding is shown to be equivalent to embedding in Brownian motion. When (X_t) is Brownian motion we have the embedding given in [1]. This method is well-known (see [1]) and we believe that the results of this paper are known to the specialists in the field, although we could not find it in the literature on Skorokhod embedding problems. The embedding problem for a continuous local martingale has some novelty since the martingale is convergent when the initial diffusion is non-recurrent. Also some properties of the constructed embedding mentioned above are given so to characterize the embedding uniquely. The main emphasis of the paper is on the explicit construction of the embeddings and simplicity of proofs.

2. Formulation of the problem

Let $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ be two Borel functions such that $1/\sigma^2(\cdot)$ and $|\mu(\cdot)|/\sigma^2(\cdot)$ are locally integrable at every point in $\mathbb{R}$. Let $(X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ be the unique weak solution up to an explosion time e of the one-dimensional time-homogeneous stochastic differential equation

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t \quad , \quad X_0 = 0 \tag{2.1}$$

where (B_t) is a standard Brownian motion and $e = \inf\{\, t > 0 : X_t \notin \mathbb{R} \,\}$. In Section 5 the definition, existence and uniqueness of solutions to the stochastic differential equation (2.1) are recalled together with some basic facts on the solutions. For simplicity, the state space of (X_t) is taken to be $I = \mathbb{R}$, but it will be clear that the considerations are generally valid for any state space which is an interval $I = (l, r)$ (see also Section 5).

The scale function of (X_t) is given by

$$S(x) = \int_0^x \exp\Big(-2\int_0^u \frac{\mu(r)}{\sigma^2(r)}\,dr \Big)\,du$$

for $x \in \mathbb{R}$. The scale function $S(\cdot)$ has a strictly positive continuous derivative and the second derivative exists almost everywhere. Thus $S(\cdot)$ is strictly increasing with $S(0) = 0$. Define the open interval $J = \big(S(-\infty), S(\infty)\big)$. If $J = \mathbb{R}$ then (X_t) is recurrent and if J is bounded from below or above then (X_t) is non-recurrent (see Proposition 5.3).

Let ν be in the class of probability measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} |S(u)|\,\nu(du) < \infty$$

and having a strictly positive density F' where F is the distribution function associated with ν . The assumption that ν has a strictly positive density is made for simplicity. The main problem under consideration in this paper is the following. Given the probability measure ν find a stopping time τ_* of (X_t) satisfying

$$X_{\tau_*} \sim \nu \tag{2.2}$$

and determine necessary and sufficient conditions on ν which make such a construction possible.

1. The first step is to introduce the continuous local martingale $(M_t)_{t\geq 0}$ which shall be used in transforming the original problem into an analogous Skorokhod problem. Let (M_t) be the continuous local martingale given by composing (X_t) with the scale function $S(\cdot)$, i.e.

$$M_t = S(X_t)\ . \tag{2.3}$$

Since $x \mapsto S(x)$ is strictly increasing then Proposition 5.3 ensures that $S(-\infty) < M_t < S(\infty)$ for $t < e$ and if J is bounded from below or above M_t converges to the boundary of J for $t \to e$ and $M_t = M_e$ on $\{e < \infty\}$ for $t \geq e$. By Itô-Tanaka formula it follows that (M_t) is a solution to the stochastic differential equation

$$dM_t = \tilde{\sigma}(M_t)\, dB_t$$

where

$$\tilde{\sigma}(x) = \begin{cases} S'\big(S^{-1}(x)\big)\, \sigma\big(S^{-1}(x)\big) & \text{for} \quad x \in J \\ 0 & \text{else}\ . \end{cases}$$

The quadratic variation process is therefore given by

$$\langle M, M\rangle_t = \int_0^t \tilde{\sigma}^2(M_u)\, du = \int_0^{t\wedge e} \Big(S'(X_u)\, \sigma(X_u)\Big)^2 du$$

and it is immediately seen that $t \mapsto \langle M, M\rangle_t$ is strictly increasing for $t < e$. If J is bounded from below or above then $\langle M, M\rangle_e < \infty$, If $J = \mathbb{R}$ the local martingale (M_t) is recurrent, or equivalent $\langle M, M\rangle_e = \infty$ and $e = \infty$. The process (M_t) does not explode, but the explosion time e for (X_t) can be expressed as $e = \inf\{\, t > 0 : M_t \notin J\,\}$.

Let G be the distribution function given by

$$G(x) = F\big(S^{-1}(x)\big) \tag{2.4}$$

for $x \in J$ with $G\big(S(-\infty)\big) = 0$ and $G\big(S(\infty)\big) = 1$. Then $x \mapsto G(x)$ is continuous, differentiable and strictly increasing on J . For a stopping time τ_* of (X_t) it is not difficult to see that $X_{\tau_*} \sim F$ if and only if $M_{\tau_*} \sim G$. Therefore the initial problem (2.2) is analogous to the problem of finding a stopping time τ_* of (M_t) satisfying

$$M_{\tau_*} \sim G\ . \tag{2.5}$$

Moreover if τ_* is an embedding for (M_t) then by the above observations it follows that $S(-\infty) < M_{\tau_*} < S(\infty)$ and hence $\tau_* < e$.

2. The second step is to apply time-change and verify that the embedding problem of the continuous local martingale (2.5) is equivalent to an embedding problem of Brownian motion. Let (T_t) be the time-change given by

$$T_t = \inf\{\, s > 0 : \langle M, M\rangle_s > t\,\} = \langle M, M\rangle_t^{-1} \tag{2.6}$$

for $t < \langle M, M\rangle_e$. Define the process $(W_t)_{t\geq 0}$ by

$$W_t = \begin{cases} M_{T_t} & \text{if} \quad t < \langle M, M\rangle_e \\ M_e & \text{if} \quad t \geq \langle M, M\rangle_e\ . \end{cases} \tag{2.7}$$

Since $t \mapsto T_t$ is strictly increasing for $t < \langle M, M\rangle_e$ we have that $\left(\mathcal{F}^M_{T_t}\right) = \left(\mathcal{F}^W_t\right)$. This implies that, if $\tau < \langle M, M\rangle_e$ is a stopping time for (W_t) then T_τ is a stopping time for (M_t) , and vice versa if $\tau < e$ is a stopping time for (M_t) then $\langle M, M\rangle_\tau$ is a stopping time for (W_t) . The process (W_t) is a Brownian motion stopped at $\langle M, M\rangle_e$ according to Dambis, Dubins-Schwarz theorem (see [12]). By the definition of (W_t) it is clear that $\langle M, M\rangle_e = \inf\{\, t > 0 \,:\, W_t \notin J \,\}$ and hence the two processes $(W_t)_{t\geq 0}$ and $(B_{\tau_{S(\infty),S(\infty)}\wedge t})_{t\geq 0}$ have the same law where $\tau_{S(-\infty),S(\infty)} = \inf\{\, t > 0 \,:\, B_t \notin J \,\}$.

From the above observation we deduce that the embedding problem for the continuous local martingale is equivalent to embedding in the stopped Brownian motion, i.e the martingale case (2.5) is equivalent to find a stopping time τ_* of (W_t) satisfying $W_{\tau_*} \sim G$.

The method just described will be applied below in Section 4 to find a solution to the initial problem (2.2).

3. Skorokhod embedding in Brownian motion

The above observations show that a construction of an embedding in the initial problem (2.2) can be obtained from an embedding in Brownian motion. Therefore an outline of the Azéma-Yor [1] construction of embedding in Brownian motion will be recalled in this section together with some facts of the embedding. These results and facts will be applied in the next section where the construction of the embedding in the initial problem will be carried out.

Let G be the distribution function given in (2.4) i.e. $x \mapsto G(x)$ is continuous, differentiable and strictly increasing on the open interval $J = (\alpha, \beta)$ with $G(\alpha) = 0$ and $G(\beta) = 1$ where $\alpha = S(-\infty)$ and $\beta = S(\infty)$. Furthermore G has finite mean and denote it by

$$m = \int_{\mathbb{R}} u\, dG(u)\ . \tag{3.1}$$

Thus we want to find a stopping time τ_* of (B_t) satisfying

$$B_{\tau_*} \sim G\ . \tag{3.2}$$

Define the two functions

$$c(x) = \int_{\mathbb{R}} (u - x)^+\, dG(u) \quad \text{and} \quad p(x) = \int_{\mathbb{R}} (x - u)^+\, dG(u) \tag{3.3}$$

for $x \in \mathbb{R}$.

It is now possible to present the construction of the Azéma-Yor embedding which is a solution to problem (3.2). If $m \geq 0$, define the increasing function $s \mapsto b_+(s)$ as follows. For $m < s < \beta$ set $b_+(s)$ as the value $z < s$ which minimizes

$$\frac{c(z)}{s - z}$$

and set $b_+(s) = -\infty$ for $s \leq m$ and $b_+(s) = s$ for $s \geq \beta$ (see [4] that $b_+(\cdot)$ is well-defined). Note that $\lim_{s\downarrow m} b_+(s) = \alpha$. The left inverse of $b_+(\cdot)$ is given by

$$b_+^{-1}(x) = \frac{1}{1 - G(x)} \int_x^\infty u\, dG(u) \tag{3.4}$$

for $x < \beta$ and $b_+^{-1}(x) = x$ for $x \geq \beta$. The function $x \mapsto b_+^{-1}(x)$ is the barycentre function of G . Define the stopping time τ_{b_+} by (see Figure 1)

$$\tau_{b_+} = \inf\left\{\, t > 0 \,:\, B_t \leq b_+\big(\max\nolimits_{0\leq r\leq t} B_r\big) \,\right\}\ . \tag{3.5}$$

Observe that the stopping time τ_{b_+} can be described by $\tau_{b_+} = \tau_m + \tau_{b_+} \circ \theta_{\tau_m}$ where $\tau_m = \inf\{t > 0 : B_t = m\}$ since $b_+(\cdot)$ for $s \le m$ is defined to be $-\infty$. Similarly, if $m \le 0$, define the increasing function $s \mapsto b_-(s)$ as follows. For $\alpha < s < m$ set $b_-(s)$ as the value of $z > s$ which minimizes

$$\frac{p(z)}{z - s}$$

and set $b_-(s) = \infty$ for $s \ge m$ and $b_-(s) = s$ for $s \le \alpha$. Note that $\lim_{s \uparrow m} b_-(s) = \beta$. The left inverse of $b_-(\cdot)$ is given by

$$b_-^{-1}(x) = \frac{1}{G(x)} \int_{-\infty}^{x} u \, dG(u) \tag{3.6}$$

for $x > \alpha$ and $b_-^{-1}(x) = x$ for $x \le \alpha$. Define the stopping time τ_{b_-} by

$$\tau_{b_-} = \inf\left\{ t > 0 : B_t \ge b_-\big(\min_{0 \le r \le t} B_r\big) \right\}. \tag{3.7}$$

The stopping time τ_{b_-} can be described by $\tau_{b_-} = \tau_m + \tau_{b_-} \circ \theta_{\tau_m}$.

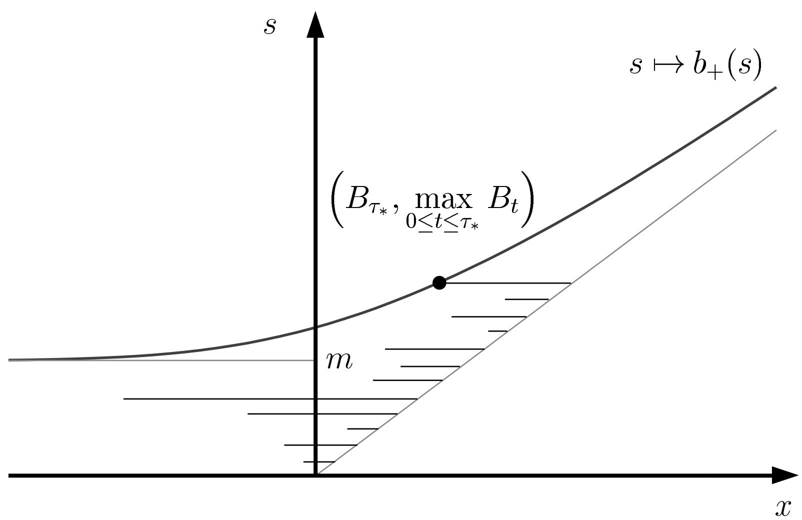

Figure 1. A computer drawing of the map $s \mapsto b_+(s)$ where the inverse is given in (3.4) when G is the distribution function of a $N(1,1)$-variable. The above process is $\big(B_t, \max_{0 \le r \le t} B_r\big)$. The process can increase in the second component only after hitting the diagonal $x = s$. The stopping time τ_{b_+} given in (3.5) is obtained by stopping the process as soon it hits the boundary $s \mapsto b_+(s)$.

One more observation is needed before stating the result. If τ_* is an embedding of the centered distribution function $x \mapsto G(m + x)$ then strong Markov property ensures that the stopping time $\tau_m + \tau_* \circ \theta_{\tau_m}$ is an embedding of G where $\tau_m = \inf\{t > 0 : B_t = m\}$. The proposition below follows from [1], the above observation and the fact that $(-B_t)$ is Brownian motion.

Proposition 3.1. *Let the distribution function* G *be given as above. For* $m \ge 0$ *set* $\tau_* = \tau_{b_+}$ *given in* (3.5) *and for* $m \le 0$ *set* $\tau_* = \tau_{b_-}$ *given in* (3.7). *Then* $B_{\tau_*} \sim G$.

The embedding given in Proposition 3.1 has some extremal properties given in the proposition below. Loosely speaking, the proposition says that for $m \ge 0$ the embedding τ_* is pointwise the smallest embedding that stochastically maximizes the maximum process $\max_{0 \le t \le \tau_*} B_t$.

These properties were observed in [11] for $m = 0$ and in [10] (with drift tending to zero) for non-centered distribution functions. This characterizes τ_* uniquely (called the minimax property in [11] and [10]). For $m \leq 0$ a vice versa result holds for the embedding.

Proposition 3.2. *Under the assumptions of Proposition 3.1, let* τ *be any stopping time of* (B_t) *satisfying* $B_\tau \sim G$. *Then we have.*

(I). *If* $m \geq 0$ *and* $\mathbf{E}\big(\max_{0\leq t\leq \tau} B_t\big) < \infty$ *then the following inequality holds*

$$\mathbf{P}\big(\max_{0\leq t\leq \tau} B_t \geq s\big) \leq \mathbf{P}\big(\max_{0\leq t\leq \tau_*} B_t \geq s\big) \tag{3.8}$$

for all $s > 0$. *If furthermore* G *satisfies*

$$\int_0^\infty u \log(u)\, dG(u) < \infty \tag{3.9}$$

and the stopping time τ *satisfies* $\max_{0\leq t\leq \tau} B_t \sim \max_{0\leq t\leq \tau_*} B_t$ (*i.e. there is equality in* (3.8) *for all* $s > 0$) *then*

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

(II). *If* $m \leq 0$ *and* $\mathbf{E}\big(\min_{0\leq t\leq \tau} B_t\big) > -\infty$ *then the following inequality holds*

$$\mathbf{P}\big(\min_{0\leq t\leq \tau} B_t \leq s\big) \leq \mathbf{P}\big(\min_{0\leq t\leq \tau_*} B_t \leq s\big) \tag{3.10}$$

for all $s < 0$. *If furthermore* G *satisfies*

$$\int_{-\infty}^0 u \log(-u)\, dG(u) > -\infty \tag{3.11}$$

and the stopping time τ *satisfies* $\min_{0\leq t\leq \tau} B_t \sim \min_{0\leq t\leq \tau_*} B_t$ (*i.e. there is equality in* (3.10) *for all* $s < 0$) *then*

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

Remark 3.3. The conditions (3.9) and (3.11) are respectively equivalent to

$$\mathbf{E}\big(\max_{0\leq t\leq \tau_*} B_t\big) < \infty \quad \text{and} \quad \mathbf{E}\big(\min_{0\leq t\leq \tau_*} B_t\big) > -\infty\ .$$

Remark 3.4. For $m \geq 0$ we have that

$$\mathbf{P}\big(\max_{0\leq t\leq \tau_*} B_t \geq s\big) = \inf_{z<s} \frac{c(z)}{s-z} = \exp\left(-\int_0^s \frac{dr}{r - b_+(r)}\right)$$

for $s > 0$ and for $m \leq 0$ we have that

$$\mathbf{P}\big(\min_{0\leq t\leq \tau_*} B_t \leq s\big) = \inf_{z>s} \frac{p(z)}{z-s} = \exp\left(-\int_s^0 \frac{dr}{b_-(r) - r}\right)$$

for $s < 0$.

Remark 3.5. For $m = 0$, Perkins [9] construction of an embedding σ_* is another extremal embedding which stochastically minimizes $\max_{0\leq t\leq \sigma_*} B_t$ over all embeddings. The construction of the embedding is the following. Define the decreasing function $s \mapsto a_+(s)$ as follows. For $0 < s < \beta$ set $a_+(s)$ as the value $z < s$ which maximizes

$$\frac{c(s) - p(z)}{s - z}$$

and set $a_+(s) = -s$ for $s \geq \beta$. Still for $0 < s < \beta$ the function $a_+(s)$ is the unique root to the equation

$$\frac{c(s) - p(z)}{s - z} = G(z)$$

satisfying $a_+(s) < s$. Define the decreasing function $s \mapsto a_-(s)$ as follows. For $\alpha < s < 0$ set $a_-(s)$ as the value of $z > s$ which maximizes

$$\frac{p(s) - c(z)}{z - s}$$

and set $a_-(s) = -s$ for $s \leq \alpha$. For $\alpha < s < 0$ the function $a_-(s)$ is the unique root to the equation

$$\frac{p(s) - c(z)}{z - s} = 1 - G(z)$$

satisfying $a_-(s) > s$. Define the two stopping times

$$\begin{aligned}
\sigma_{a_+} &= \inf \big\{ \, t > 0 \, : \, B_t \leq a_+\big(\max_{0 \leq r \leq t} B_r\big) \, \big\} \\
\sigma_{a_-} &= \inf \big\{ \, t > 0 \, : \, B_t \geq a_-\big(\min_{0 \leq r \leq t} B_r\big) \, \big\} \, .
\end{aligned}$$

For the stopping time σ_* for (B_t) given by $\sigma_* = \sigma_{a_+} \wedge \sigma_{a_-}$ we have that $B_{\sigma_*} \sim G$.

The embedding σ_* can be characterized uniquely in the following way. Let τ be given as in Proposition 3.2 then

$$\mathbf{P}\big(\max_{0 \leq t \leq \tau} B_t \geq s\big) \geq \mathbf{P}\big(\max_{0 \leq t \leq \sigma_*} B_t \geq s\big) \tag{3.12}$$

for $s > 0$. If there is equality in (3.12) for all $s > 0$ then $\tau = \sigma_*$. Finally we have that

$$\begin{aligned}
\mathbf{P}\big(\max_{0 \leq t \leq \sigma_*} B_t \geq s\big) &= 1 - G(s) + \sup_{z < s} \frac{c(s) - p(z)}{s - z} \\
&= \exp\left(- \int_0^s \frac{dr}{r - a_+(r)} \right) - \int_0^s \exp\left(- \int_u^s \frac{dr}{r - a_+(r)} \right) dG(u)
\end{aligned}$$

for $s > 0$. Thus for any embedding τ given in Proposition 3.2 there is a lower and upper bound of the distribution function of the maximum process $\max_{0 \leq t \leq \tau} B_t$ and the bounds can be attained. There are similar results for the minimum process.

4. Skorokhod embedding in non-singular diffusions

In this section we shall translate the results of the two previous sections into the case of a non-singular diffusion. The main result of this paper is contained in the theorem below.

Let ν be the probability measure on $\mathbb{R}$ with strictly positive density function F' where F is the distribution function associated with ν introduced in Section 2. We shall use the same notation as in Section 2 and 3. Let G be the distribution function given in (2.4). Then m in (3.1) can be rewritten as

$$m = \int_{\mathbb{R}} S(u) \, dF(u)$$

and the two functions $c(\cdot)$ and $p(\cdot)$ in (3.3) can be rewritten as

$$c(x) = \int_{\mathbb{R}} \big(S(u) - x\big)^+ \, dF(u) \quad \text{and} \quad p(x) = \int_{\mathbb{R}} \big(x - S(u)\big)^+ \, dF(u)$$

for $x \in \mathbb{R}$.

The construction of an embedding which is a solution to the initial problem (2.2) is the following. If $m \geq 0$, define the increasing function $s \mapsto h_+(s)$ as follows. For $s > S^{-1}(m)$ set $h_+(s)$ as the value of $z < s$ which minimizes

$$\frac{c(S(z))}{S(s) - S(z)}$$

and set $h_+(s) = -\infty$ for $s \leq S^{-1}(m)$. Note that $\lim_{s \downarrow S^{-1}(m)} h_+(s) = -\infty$. The left inverse of $h_+(\cdot)$ is given by

$$h_+^{-1}(x) = S^{-1}\Big(\frac{1}{1-F(x)} \int_x^\infty S(u)\, dF(u)\Big)$$

for all $x \in \mathbb{R}$. It is not difficult to see the following connection between $h_+^{-1}(\cdot)$ and $b_+^{-1}(\cdot)$ from (3.4) is valid

$$h_+^{-1}(\cdot) = (S^{-1} \circ b_+^{-1} \circ S)(\cdot) \ . \tag{4.1}$$

Define the stopping time τ_{h_+} by

$$\tau_{h_+} = \inf \big\{\, t > 0 \,:\, X_t \leq h_+\big(\max_{0 \leq r \leq t} X_r\big) \,\big\} \ . \tag{4.2}$$

Thus by (4.1) and the definition of (M_t) (see (2.3)) it is clear that

$$\tau_{h_+} = \inf \big\{\, t > 0 \,:\, M_t \leq b_+\big(\max_{0 \leq r \leq t} M_r\big) \,\big\} \ . \tag{4.3}$$

If $m \leq 0$, define the increasing function $s \mapsto h_-(s)$ as follows. For $s < S^{-1}(m)$ set $h_-(s)$ as the value of $z > s$ which minimizes

$$\frac{p(S(z))}{S(z) - S(s)}$$

and set $h_-(s) = \infty$ for $s \geq S^{-1}(m)$. Note that $\lim_{s \uparrow S^{-1}(m)} h_-(s) = \infty$. The left inverse of $h_-(\cdot)$ is given by

$$h_-^{-1}(x) = S^{-1}\Big(\frac{1}{F(x)} \int_{-\infty}^x S(u)\, dF(u)\Big)$$

for all $x \in \mathbb{R}$. Note that the connection between $h_-^{-1}(\cdot)$ and $b_-^{-1}(\cdot)$ from (3.6) is the same as in (4.1). Define the stopping time τ_{h_-} by

$$\tau_{h_-} = \inf \big\{\, t > 0 \,:\, X_t \geq h_-\big(\min_{0 \leq r \leq t} X_r\big) \,\big\} \ . \tag{4.4}$$

Again it is clear that

$$\tau_{h_-} = \inf \big\{\, t > 0 \,:\, M_t \geq b_-\big(\min_{0 \leq r \leq t} M_r\big) \,\big\} \ .$$

The following theorem is an extension of Proposition 3.1 and states that the above stopping times are solutions to the Skorokhod embedding problem (2.2).

Theorem 4.1. *Let (X_t) be a non-singular diffusion vanishing at zero. Let ν be a probability measure on $\mathbb{R}$ having a strictly positive density F' such that*

$$\int_{\mathbb{R}} |S(u)|\, \nu(du) < \infty \quad \textit{and set} \quad m = \int_{\mathbb{R}} S(u)\, \nu(du) \ .$$

Then there exists a stopping time τ_* for (X_t) such that $X_{\tau_*} \sim \nu$ if and only if one of the following four cases holds

(i) $S(-\infty) = -\infty$ *and* $S(\infty) = \infty$

(ii) $S(-\infty) = -\infty$, $S(\infty) < \infty$ *and* $m \geq 0$
(iii) $S(-\infty) > -\infty$, $S(\infty) = \infty$ *and* $m \leq 0$
(iv) $S(-\infty) > -\infty$, $S(\infty) < \infty$ *and* $m = 0$.
Moreover, if $m \geq 0$ *then* τ_* *is given by* (4.2), *and if* $m \leq 0$ *then* τ_* *is given by* (4.4).

Proof. First to verify that the conditions in cases (i)-(iv) are sufficient, let G be the distribution function given in (2.4). Assume that $m \geq 0$ and let the inverse of $s \mapsto b_+(s)$ be given as in (3.4). Let (W_t) be the process given in (2.7) and define the stopping time $\tilde{\tau}_*$ for (W_t) by

(4.5) $$\tilde{\tau}_* = \inf \big\{ \, t > 0 \, : \, W_t \leq b_+\big(\max_{0 \leq r \leq t} W_r\big) \, \big\} \, .$$

As observed in Section 2 that $\langle M, M \rangle_e = \inf \{ \, t > 0 \, : \, W_t \notin (S(-\infty), S(\infty)) \, \}$ and by the definition of $b_+(\cdot)$ we see that $\tilde{\tau}_* < \langle M, M \rangle_e$ if either $S(-\infty) = -\infty$, or $m = 0$ with $S(-\infty) > -\infty$ and $S(\infty) < \infty$. Therefore in the cases (i), (ii) and (iv) we have that $\tilde{\tau}_* < \langle M, M \rangle_e$. Note that $\tilde{\tau}_* < \langle M, M \rangle_e$ fails in the other cases. The process (W_t) is a Brownian motion stopped at $\langle M, M \rangle_e$ and hence from Proposition 3.1 we have that $W_{\tilde{\tau}_*} \sim G$. Again by an observation in Section 2 the stopping time τ_* for (M_t) given by

$$\tau_* = T_{\tilde{\tau}_*} = \inf \big\{ \, t > 0 \, : \, M_t \leq b_+\big(\max_{0 \leq r \leq t} M_r\big) \, \big\}$$

satisfies $M_{\tau_*} = W_{\tilde{\tau}_*} \sim G$ where (T_t) is the time change given in (2.6). From (4.3) we see that τ_* is given in (4.2) and it clearly fulfills $X_{\tau_*} \sim F$. The same arguments hold for $m \leq 0$.

The conditions in the cases (i)-(iv) are necessary as well. Indeed, case (i) is trivial because there is no restriction on the class of probability measures we are considering. In case (ii) let τ_* be a stopping time for (X_t) satisfying $X_{\tau_*} \sim F$ or equivalently $M_{\tau_*} \sim G$. Then the process $(M_{\tau_* \wedge t})$ is a continuous local martingale which is bounded from above by $S(\infty) < \infty$. Let $\{\gamma_n\}_{n \geq 1}$ be a localization for the local martingale. Applying Fatou's lemma and the optional sampling theorem we have that

$$m = \mathbf{E}\big(M_{\tau_*}\big) \geq \liminf_n \mathbf{E}\big(M_{\tau_* \wedge \gamma_n}\big) = 0 \, .$$

The cases (iii) and (iv) are proved in exactly in the same way. Note that (M_t) is a bounded martingale in case (iv). $\square$

Next we explore the properties stated in Proposition 3.2 in the context of a diffusion.

Proposition 4.2. *Under the assumptions of Theorem 4.1, let* τ *be any stopping time of* (X_t) *satisfying* $X_\tau \sim \nu$. *Then we have that.*

(I). *If* $m \geq 0$ *and* $\mathbf{E}\big(\max_{0 \leq t \leq \tau} S(X_t)\big) < \infty$ *then the following inequality holds*

(4.6) $$\mathbf{P}\big(\max_{0 \leq t \leq \tau} X_t \geq s\big) \leq \mathbf{P}\big(\max_{0 \leq t \leq \tau_*} X_t \geq s\big)$$

for all $s \geq 0$. *If furthermore* ν *satisfies*

(4.7) $$\int_0^\infty S(u) \log(S(u)) \, \nu(du) < \infty$$

and the stopping time τ *satisfies* $\max_{0 \leq t \leq \tau} X_t \sim \max_{0 \leq t \leq \tau_*} X_t$ (*i.e. there is equality in* (4.6) *for all* $s > 0$) *then we have*

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

(II). *If* $m \leq 0$ *and* $\mathbf{E}\big(\min_{0 \leq t \leq \tau} S(X_t)\big) > -\infty$ *then the following inequality holds*

(4.8) $$\mathbf{P}\big(\min_{0 \leq t \leq \tau} X_t \leq s\big) \leq \mathbf{P}\big(\min_{0 \leq t \leq \tau_*} X_t \leq s\big)$$

for all $s \leq 0$ *. If furthermore* ν *satisfies*

$$\int_{-\infty}^{0} S(u)\log(-S(u))\,\nu(du) > -\infty \tag{4.9}$$

and the stopping time τ *satisfies* $\min_{0\leq t\leq\tau} X_t \sim \min_{0\leq t\leq\tau_*} X_t$ *(i.e. there is equality in* (4.8) *for all* $s < 0$ *) then we have*

$$\tau = \tau_* \quad \mathbf{P}\text{-}a.s.$$

Proof. The two cases (I) and (II) are proved with precisely the same arguments. Therefore we will concentrate on case (I). Let τ be the stopping time given in the proposition. Then we have that $M_\tau \sim G$ and $\mathbf{E}\big(\max_{0\leq t\leq\tau} M_t\big) < \infty$. Since τ and τ_* are two embeddings we have from Section 2 that the two stopping times $\tilde\tau$ and $\tilde\tau_*$ for (W_t) given by

$$\tilde\tau = \langle M,M\rangle_\tau \quad \text{and} \quad \tilde\tau_* = \langle M,M\rangle_{\tau_*}$$

satisfy

$$W_{\tilde\tau} \sim W_{\tilde\tau_*} \sim G\,.$$

Note that $\tilde\tau_*$ is given in (4.5) and that

$$\mathbf{E}\big(\max_{0\leq t\leq\tilde\tau} W_t\big) = \mathbf{E}\big(\max_{0\leq t\leq\tau} M_t\big) < \infty\,.$$

Then Proposition 3.2 gives that

$$\mathbf{P}\big(\max_{0\leq t\leq\tilde\tau} W_t \geq s\big) \leq \mathbf{P}\big(\max_{0\leq t\leq\tilde\tau_*} W_t \geq s\big)$$

for $s > 0$ and going back we obtain (4.6). The second part is verified with the same arguments. $\square$

Remark 4.3. For $m \geq 0$ we have that

$$\mathbf{P}\big(\max_{0\leq t\leq\tau_*} X_t \geq s\big) = \inf_{z<s} \frac{c(S(z))}{S(s)-S(z)} = \exp\Big(-\int_0^s \frac{dS(u)}{S(u)-S(h_+(u))}\Big)$$

for $s > 0$ and the condition (4.7) is trivial when $S(\cdot)$ is bounded from above. For $m \leq 0$ we have that

$$\mathbf{P}\big(\min_{0\leq t\leq\tau_*} X_t \leq s\big) = \inf_{z>s} \frac{p(S(z))}{S(z)-S(s)} = \exp\Big(-\int_s^0 \frac{dS(u)}{S(h_-(u))-S(u)}\Big)$$

for $s < 0$ and the condition (4.9) is trivial when $S(\cdot)$ is bounded from below.

Remark 4.4. If $m = 0$ we have another extremal embedding σ_* of Perkins [9] which stochastically minimizes $\max_{0\leq t\leq\tau_*} X_t$. The construction of the embedding is the following. Define the decreasing function $s \mapsto g_+(s)$ as follows. For $s > 0$ set $g_+(s)$ as the value of $z < s$ which maximizes

$$\frac{c(S(s)) - p(S(z))}{S(s)-S(z)}$$

For $s > 0$ the function $g_+(s)$ is the unique root to the equation

$$\frac{c(S(s)) - p(S(z))}{S(s)-S(z)} = F(z)$$

satisfying $g_+(s) < s$. Define the decreasing function $s \mapsto g_-(s)$ as follows. For $s < 0$ set $s \mapsto g_-(s)$ as the value of $z > s$ which maximizes

$$\frac{p(S(s)) - c(S(z))}{S(z)-S(s)}$$

For $s < 0$ the function $g_-(s)$ is the unique root to the equation

$$\frac{p(S(s)) - c(S(z))}{S(z) - S(s)} = 1 - F(z)$$

satisfying $g_-(s) > s$. Define the two stopping times

$$\sigma_{g_+} = \inf \left\{ t > 0 \,:\, X_t \le a_+\bigl(\max_{0 \le r \le t} X_r\bigr) \right\}$$
$$\sigma_{g_-} = \inf \left\{ t > 0 \,:\, X_t \ge a_-\bigl(\min_{0 \le r \le t} X_r\bigr) \right\} .$$

For the stopping time σ_* for (X_t) given by $\sigma_* = \sigma_{g_+} \wedge \sigma_{g_-}$ we have that $X_{\sigma_*} \sim F$.

The embedding σ_* can be characterized uniquely in the following way. If τ is given as in Proposition 4.2 then

$$\mathbf{P}\bigl(\max_{0 \le t \le \tau} X_t \ge s\bigr) \ge \mathbf{P}\bigl(\max_{0 \le t \le \sigma_*} X_t \ge s\bigr) \tag{4.10}$$

for $s > 0$. If there is equality in (4.10) for all $s > 0$ then $\tau = \sigma_*$. Finally we have that

$$\mathbf{P}\bigl(\max_{0 \le t \le \sigma_*} X_t \ge s\bigr) = 1 - G(s) + \sup_{z<s} \frac{c(S(s)) - p(S(z))}{S(s) - S(z)}$$
$$= \exp\left(- \int_0^s \frac{dS(r)}{S(r) - S(g_+(r))} \right) - \int_0^s \exp\left(- \int_u^s \frac{dS(r)}{S(r) - S(g_+(r))} \right) dF(u)$$

for $s > 0$. There is similar results for the minimum process.

5. Appendix: Stochastic differential equation

This Section presents well-known results on existence and uniqueness and various aspects of solutions of the one-dimensional time-homogeneous stochastic differential equation (2.1). For a survey and proofs of these results see Karatzas and Shreve [8].

Let $I = (l, r)$ with $-\infty \le l < r \le \infty$. Consider the non-singular stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t \tag{5.1}$$

where $\mu : I \to \mathbb{R}$ and $\sigma : I \to (0, \infty)$ are Borel functions.

Definition 5.1. A weak solution in the interval I up to an explosion time e of the one-dimensional time-homogeneous stochastic differential equation (5.1) is a triple $((X_t), (B_t))_{t \ge 0}$, $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\mathcal{F}_t)$ satisfying the following three conditions.

(i) $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and $(\mathcal{F}_t)$ is a filtration of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions.

(ii) $(X_t)_{t \ge 0}$ is a continuous $(\mathcal{F}_t)$-adapted, $[l, r]$-valued process with $X_0 \in [l, r]$ **P**-a.s. and $(B_t)_{t \ge 0}$ is a standard $(\mathcal{F}_t)$-Brownian motion.

(iii) For all $n \ge 1$ we have

$$\int_0^{e_n \wedge t} \Bigl(\bigl|\mu(X_u)\bigr| + \sigma^2(X_u) \Bigr)\, du < \infty \quad \mathbf{P}\text{-a.s.}$$

for all $0 \le t < \infty$ and

$$\bigl(X_{e_n \wedge t}\bigr)_{t \ge 0} = \Bigl(X_0 + \int_0^t \mu(X_u) \mathbf{1}_{\{u \le e_n\}}\, du + \int_0^t \sigma(X_u) \mathbf{1}_{\{u \le e_n\}}\, dB_u \Bigr)_{t \ge 0} \quad \mathbf{P}\text{-a.s.}$$

where $e_n = \inf \{ t > 0 \,:\, X_t \notin (l_n, r_n) \}$ and $l < l_n < r_n < r$ with $l_n \downarrow l$ and $r_n \uparrow r$ for $n \to \infty$.

The explosion time e for the process (X_t) is defined as

$$e := \inf\{\, t > 0 \,:\, X_t \notin (l,r)\,\} = \lim_{n\to\infty} e_n \,.$$

Since $\sigma(\cdot)$ is strictly positive (non-singular) we have the following sharp sufficient conditions for existence and uniqueness of solutions of (5.1).

Theorem 5.2. *If for all $x \in I$ there exists $\epsilon > 0$ such that*

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1+|\mu(u)|}{\sigma^2(u)}\, du < \infty \tag{5.2}$$

then for every initial distribution of X_0, the stochastic equation (5.1) has a weak solution in I up to an explosion time e, and this solution is unique in the sense of probability law.

Assume that $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the condition (5.2) and let $\mathbf{P}_x$ denote the probability measure when $X_0 = x$. The scale function of (X_t) is given by

$$S(x) = \int_{x_0}^{x} \exp\Big(-2\int_{x_0}^{u} \frac{\mu(r)}{\sigma^2(r)}\, dr\Big)\, du$$

for $x \in I$ and some $x_0 \in I$. The next proposition states necessary and sufficient condition for the process (X_t) to be recurrent.

Proposition 5.3. *Let (X_t) be a weak solution in the interval I of the stochastic differential equation (5.1). We distinguish four cases.*

(i) *If $S(l+) = -\infty$ and $S(r-) = \infty$, then*

$$\mathbf{P}_x\big(e = \infty\big) = \mathbf{P}_x\big(\limsup_{t\uparrow\infty} X_t = r\big) = \mathbf{P}_x\big(\liminf_{t\uparrow\infty} X_t = l\big) = 1$$

for all $x \in I$. In particular, the process (X_t) is recurrent, i.e. $\mathbf{P}_x(\tau_y < \infty) = 1$ for all $x, y \in I$ where $\tau_y = \inf\{\, t > 0 \,:\, X_t = y\,\}$.

(ii) *If $S(l+) > -\infty$ and $S(r-) = \infty$, then $\lim_{t\uparrow e} X_t$ exists $\mathbf{P}_x$-a.s. and*

$$\mathbf{P}_x\big(\lim_{t\uparrow e} X_t = l\big) = \mathbf{P}_x\big(\sup_{0\le t<e} X_t < r\big) = 1$$

for all $x \in I$.

(iii) *If $S(l+) = -\infty$ and $S(r-) < \infty$, then $\lim_{t\uparrow e} X_t$ exists $\mathbf{P}_x$-a.s. and*

$$\mathbf{P}_x\big(\lim_{t\uparrow e} X_t = r\big) = \mathbf{P}_x\big(\inf_{0\le t<e} X_t > l\big) = 1$$

for all $x \in I$.

(iv) *If $S(l+) > -\infty$ and $S(r-) < \infty$, then $\lim_{t\uparrow e} X_t$ exists $\mathbf{P}_x$-a.s. and*

$$\mathbf{P}_x\big(\lim_{t\uparrow e} X_t = l\big) = 1 - \mathbf{P}_x\big(\lim_{t\uparrow e} X_t = r\big) = \frac{S(r-) - S(x)}{S(r-) - S(l+)}$$

for all $x \in I$.

The process (X_t) is non-recurrent in cases (ii), (iii) *and* (iv)

For completeness, to give necessary and sufficient conditions for non-explosion we need to introduce the following function

$$\kappa(x) = 2\int_{x_0}^{x} \frac{S(x) - S(u)}{S'(u)\sigma^2(u)}\, du$$

for $x \in I$.

Proposition 5.4. (*Feller's Test for Explosions.*) *Let* (X_t) *be a weak solution in the interval* I *of the stochastic differential equation* (5.1). *We distinguish three cases.*

(i) *If* $\kappa(l+) = \kappa(r-) = \infty$ *then* $\mathbf{P}_x\big(e = \infty\big) = 1$ *for all* $x \in I$.

(ii) *If* $\kappa(l+) < \infty$ *or* $\kappa(r-) < \infty$ *then* $\mathbf{P}_x\big(e = \infty\big) < 1$ *for all* $x \in I$.

(iii) *We have* $\mathbf{P}_x\big(e < \infty\big) = 1$ *if and only if one of the following conditions holds.*

  (a) $\kappa(r-) < \infty$ *and* $\kappa(l+) < \infty$ (*in this case* $\mathbf{E}_x(e) < \infty$).

  (b) $\kappa(r-) < \infty$ *and* $S(l+) = -\infty$.

  (c) $\kappa(l+) < \infty$ *and* $S(r-) = \infty$.

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The Minimum Maximum of a Continuous Martingale with Given Initial and Terminal Laws

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Let (M_t) be a continuous martingale with initial law $M_0 \sim \mu_0$ and terminal law $M_1 \sim \mu_1$ and let $S = \sup_{0 \le t \le 1} M_t$. In this paper we prove that there exists a greatest lower bound with respect to stochastic ordering of probability measures, on the law of S. We give an explicit construction of this bound. Furthermore a martingale is constructed which attains this minimum by solving a Skorokhod embedding problem. The result is applied to the robust hedging of a forward start digital option.

1. Introduction

Let μ_0 and μ_1 be probability measures on $\mathbb{R}$, let $\mathcal{M} \equiv \mathcal{M}(\mu_0, \mu_1)$ be the space of all martingales $(M_t)_{0 \le t \le 1}$ with initial law μ_0 and terminal law μ_1 and let $\mathcal{M}_C \equiv \mathcal{M}_C(\mu_0, \mu_1)$ be the subspace of $\mathcal{M}$ consisting of the *continuous* martingales. For a martingale $(M_t) \in \mathcal{M}$ let $S \equiv \sup_{0 \le t \le 1} M_t$ and denote the law of S by ν_M. In this article we are interested in the sets $\mathcal{P} \equiv \mathcal{P}(\mu_0, \mu_1) \equiv \{ \nu_M \,|\, (M_t) \in \mathcal{M} \}$ and $\mathcal{P}_C \equiv \mathcal{P}_C(\mu_0, \mu_1) \equiv \{ \nu_M \,|\, (M_t) \in \mathcal{M}_C \}$ of possible laws ν. In particular we find a greatest lower bound for $\mathcal{P}_C$. Here comparisons of measures are made in the sense of stochastic domination. The fact that (M_t) is a martingale with no jumps imposes quite restrictive conditions on the law of the maximum ν.

Clearly $\mathcal{M}$ is empty unless the random variables corresponding to the laws μ_i have the same finite mean, and henceforth we will assume without loss of generality that this mean is zero. Moreover a simple application of Jensen's inequality shows that a further necessary condition for the space to be non-empty is that

$$\int_{(x,\infty)} (u - x)\, \mu_0(du) \le \int_{(x,\infty)} (u - x)\, \mu_1(du) \tag{1.1}$$

for all $x \in \mathbb{R}$. This condition is also sufficient, see for example Strassen [17, Theorem 2] or Meyer [10, Chapter XI]. It follows from the construction in Chacon and Walsh [4] that this is also a necessary and sufficient condition for $\mathcal{M}_C$ to be non-empty. Henceforth we assume that (1.1) holds.

Consider first the problem of determining bounds on $\mathcal{P}(\delta_0, \mu_1)$ where δ_0 is the unit mass at 0. This problem is a special case of a problem first considered in Blackwell and Dubins [2] and Dubins and Gilat [6]. Let $\preceq$ denote stochastic ordering on probability measures, (so that $\rho \preceq \pi$ if and only if $\rho((-\infty, x)) \ge \pi((-\infty, x))$ for all $x \in \mathbb{R}$), and let ρ^* denote the Hardy transform of a probability measure ρ. Then it follows from [2], [6] and Azéma and Yor [1] that

$$\delta_0 \vee \mu_1 \preceq \nu \preceq \mu_1^* . \tag{1.2}$$

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Kertz and Rösler [9] have shown that the converse to (1.2) also holds: for any probability measure ρ satisfying $\delta_0 \vee \mu_1 \preceq \rho \preceq \mu_1^*$, there is a martingale with terminal distribution μ_1 whose maximum has law ρ . (See also Rogers [15] for a proof of these results based on excursion theory which will motivate many of our arguments). Thus

$$\mathcal{P}(\delta_0, \mu_1) \equiv \{ \nu \,|\, \delta_0 \vee \mu_1 \preceq \nu \preceq \mu_1^* \} .$$

Note that the lower bound is attained by a martingale which consists of a single jump at time 1 where the jump is chosen to have law μ_1 .

Now consider $\mathcal{P}_C(\delta_0, \mu_1)$. Then the least upper bound is unchanged since there is a continuous martingale whose maximum has law μ_1^* , as can be seen from the example in Rogers [15]. Moreover Perkins [11] gives an expression for the greatest lower bound, which to be consistent with future notation we shall label $\nu^{\#}(\delta_0, \mu_1)$. This lower bound will arise as a special case of the construction we give below for general initial conditions. See the remarks in Example 2.2. In particular $\mathcal{P}_C(\delta_0, \mu_1) \subseteq \{ \nu \,|\, \nu^{\#}(\delta_0, \mu_1) \preceq \nu \preceq \mu_1^* \}$ and both $\nu^{\#}(\delta_0, \mu_1)$ and μ_1^* are elements of $\mathcal{P}_C$.

We are interested in the problem with a general initial condition. As Kertz and Rösler [9, Remark 3.3] observe,

$$\mathcal{P}_C(\mu_0, \mu_1) \subseteq \mathcal{P}(\mu_0, \mu_1) \subseteq \{ \nu \,|\, \mu_0 \vee \mu_1 \preceq \nu \preceq \mu_1^* \} .$$

Further Hobson [7] derives a least upper bound $\nu_{0,1}^*$ for both of the sets $\mathcal{P}_C(\mu_0, \mu_1)$ and $\mathcal{P}(\mu_0, \mu_1)$. Since there is a continuous martingale with the correct marginal distributions whose maximum has law $\nu_{0,1}^*$, the least upper bound is attained in each case.

The main result of this article is that there is a greatest lower bound $\nu^{\#} \equiv \nu^{\#}(\mu_0, \mu_1)$ for $\mathcal{P}_C$, and that this bound is attained, i.e. there exists $\nu^{\#} \in \mathcal{P}_C$ such that $\nu^{\#} \preceq \nu$ for all $\nu \in \mathcal{P}_C$. The measure $\nu^{\#}$ is difficult to characterize but we give a simple pictorial representation in Figure 1 and 2 below. It turns out that it is simple to show that $\nu^{\#}$ is a lower bound, but surprisingly difficult to show that it is attained.

If the continuity restriction is dropped then it is easy to define a lower bound $\nu_{\#}$ for $\mathcal{P}(\mu_0, \mu_1)$ non-constructively via

$$\nu_{\#}((-\infty, x]) \equiv \sup_{M \in \mathcal{M}} (\nu_M(-\infty, x]) .$$

However there is a simple example in Hobson [7] to show that for general initial measures this lower bound for $\mathcal{P}$ is not attained. Again any minimal element of $\mathcal{P}$ corresponds to a martingale with a single jump at time 1. These two factors explain why it is more interesting to restrict attention to continuous martingales, a restriction that we now make.

The problem of characterising the greatest lower bound for the maximum of a martingale constrained to have given initial and terminal laws has an application to the pricing of derivative securities in mathematical finance. The derivatives in question are forward start barrier options and lookbacks. This idea has been explored in Hobson [8] and Brown, Hobson and Rogers [3].

The remainder of this paper is constructed as follows. In the next section we give the construction of the measure $\nu^{\#}(\mu_0, \mu_1)$ and state the main theorem of the paper. The construction is illustrated by three examples. In Section 3 we show that $\nu^{\#}$ is stochastically dominated by every measure in $\mathcal{P}_C(\mu_0, \mu_1)$. Further we briefly outline the connection between this result and a problem in the robust hedging of financial derivatives. Some preliminary lemmas are stated in Section 4. Finally in section 5 we show that $\nu^{\#}$ is an element of $\mathcal{P}_C$ and hence that it is a greatest lower bound. In the Appendix there is proofs of some lemmas from Section 4.

2. The main result

The main result is contained in the next theorem. Let μ_0 and μ_1 be two centered probability measures on $\mathbb{R}$ satisfying the inequality (1.1) (i.e. $\mathcal{M}_C(\mu_0,\mu_1)$ is then non-empty). For $i=0,1$ we set

$$c_i(x)=\mathbf{E}(M_i-x)^+=\int_{(x,\infty)}(u-x)\,\mu_i(du) \tag{2.1}$$

for $x\in\mathbb{R}$ and from (1.1) it follows that $c_1(x)\geq c_0(x)$. Hence the function

$$c(x)=c_1(x)-c_0(x) \tag{2.2}$$

is non-negative. Define

$$\Gamma(x)=\mu_1((-\infty,x))-\sup_{y<x}\frac{c(x)-c(y)}{x-y}\ . \tag{2.3}$$

Theorem 2.1. Γ *is a left continuous distribution function. Further for any continuous martingale* $(M_t)_{0\leq t\leq 1}\in\mathcal{M}_C(\mu_0,\mu_1)$ *we have that* $\mathbf{P}(S<x)\geq\Gamma(x)$ *for* $x\in\mathbb{R}$. *Moreover there exists a continuous martingale* $(M_t^{\#})\in\mathcal{M}_C(\mu_0,\mu_1)$ *with maximum* $S^{\#}$ *for which* $\mathbf{P}(S^{\#}<x)=\Gamma(x)$ *for* $x\in\mathbb{R}$.

Let $\nu^{\#}$ be the associated measure of Γ i.e

$$\nu^{\#}((\infty,x))=\Gamma(x) \tag{2.4}$$

for all $x\in\mathbb{R}$.

Before we prove the theorem in later sections we will first describe the construction of $(M_t^{\#})$ and look at some examples to make the construction clearer. For this we need some notation. Let F_i be the distribution function associated with μ_i . For $x\in\mathbb{R}$ we define

$$\gamma(x)=\sup_{y<x}\frac{c(x)-c(y)}{x-y}\ . \tag{2.5}$$

The two functions $c_i(x)$ are convex and hence the left-hand derivate of $c(x)$ exists and is given by $c'_-(x)=F_1(x-)-F_0(x-)$. If the supremum in (2.5) is not attained then $\gamma(x)=c'_-(x)$. We define the function $x\mapsto g(x)$ as follows. For $x\in\mathbb{R}$, let $g(x)\leq x$ be the maximal value where the supremum in (2.5) is attained and if the supremum is not attained $g(x)=x$. Note that in the cases $\gamma(x)=c'_-(x)$ then $g(x)=x$. In orther terms (see Figure 1), $g(x)$ is the point where the supporting (generalize) tangent to $c(\cdot)$ which hits c(x) and $\gamma(x)$ is the slope of the tangent.

With the above notation we can describe the martingale $(M_t^{\#})$. On some suitable sample space define the three elements

- A random variable B_0 with law μ_0 .
- A random variable G with law

  $$\mathbf{P}(G\geq s\,|\,B_0=r)=\exp\left(-\int_{(r,s)}\frac{F_1^c(du)}{F_0(u-)-\Gamma(u)}\right)\prod_{u\in[r,s)}\left(1-\frac{\Delta F_1(u)}{F_0(u-)-\Gamma(u)}\right)$$

  for $s>r$, where F_1^c is the non-atomic part of F_1 .
- A Brownian motion $(W_t)_{t\geq 0}$ independent of B_0 and G .

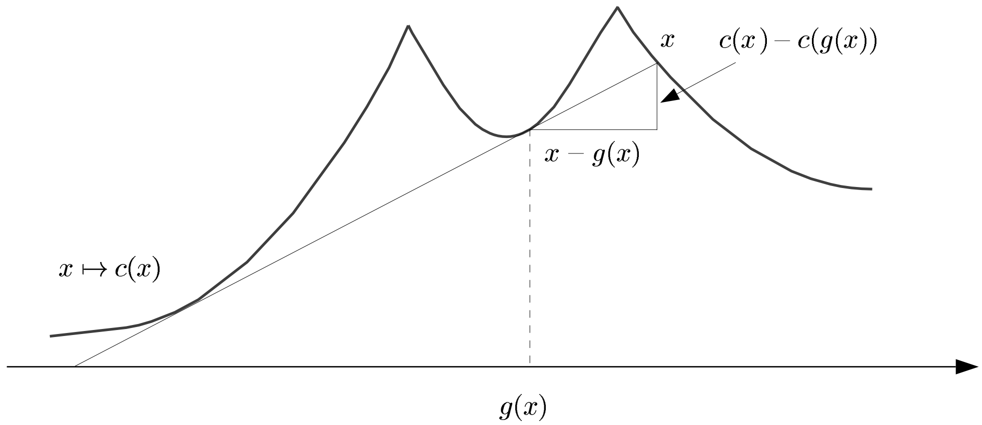

Figure 1. The construction of $\gamma(x)$ involves finding a tangent to $c(\cdot)$ which crosses $c(\cdot)$ at x . $\gamma(x)$ is the slope of the tangent and $g(x)$ is the point supporting the tangent.

Then $B_t = B_0 + W_t$ is a Brownian motion with initial law μ_0 . Let $S_t = \max_{0 \le r \le t} B_r$ and define the stopping times

$$\begin{aligned}\tau_G &= \inf\{\, t > 0 \,:\, S_t \ge G \,\}\\ \tau_g &= \inf\{\, t > 0 \,:\, B_t \le g(S_t) \,\}\\ \tau &= \tau_G \wedge \tau_g \;.\end{aligned}$$

In later sections we will prove that B_τ has law μ_1 and S_τ has law Γ . See Figure 2 for a picture of the stopping times. Then $M_t^{\#}$ is a time change of $B_{t\wedge\tau}$ and is given by

$$M_t^{\#} = B_{\frac{t}{1-t} \wedge \tau} \tag{2.6}$$

for $t \le 1$. We first give some examples of this construction.

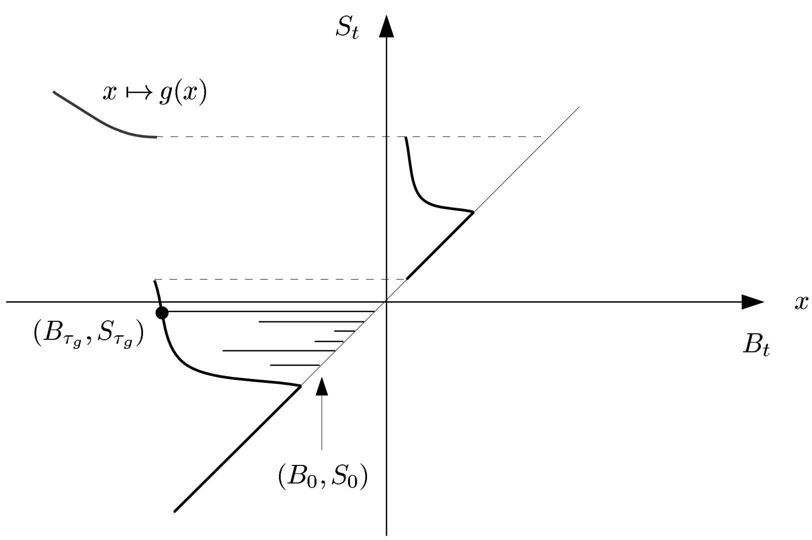

Figure 2. Describing stopping times in (B_t, S_t) plane. Excursions down from the maximum.

Example 2.2. Let $\mu_0 = \delta_0$ and μ_1 is the uniform distribution on $[-1,1]$. Then we compute that

$$c(x) = \left(\tfrac{1}{4}(1-x)^2 + x\right)\mathbf{1}_{(-1,0)}(x) + \tfrac{1}{4}(1-x)^2\,\mathbf{1}_{[0,1)}(x)$$

and $g(x) = x - 2\sqrt{x}$ for $0 < x \le 1$ and $g(x) = x$ elsewhere (see Figure 3). Further we compute that $\Gamma(x) = \sqrt{x}\,\mathbf{1}_{[0,1)}(x) + \mathbf{1}_{[1,\infty)}(x)$ (see Figure 4).

This example is also studied in Perkins [11] and illustrated the difference between the two constructions. In Perkins construction the stopping time τ_G is replaced by a stopping time of the form $\tau_h = \inf\{\, t > 0 : B_t \ge h(\min_{0\le u\le t} B_u)\,\}$ where h is a positive decreasing function. Perkins construction only need an extra random variable in the case that μ_1 has an atom at 0 (i.e. δ_0 and μ_1 have a simultaneous atom). The construction in this paper the stopping time τ is not adapted to the natural filtration of the Brownian motion. The extra random variable G is used to stop the process (B_t, S_t) at the diagonal by τ_G. This construction works in all cases, also when μ_0 and μ_1 have simultaneous atoms. Although we believe that Perkins construction can be extended to the cases when μ_0 and μ_1 have no simultaneous atoms and hence have a stopping time adapted to the natural filtration.

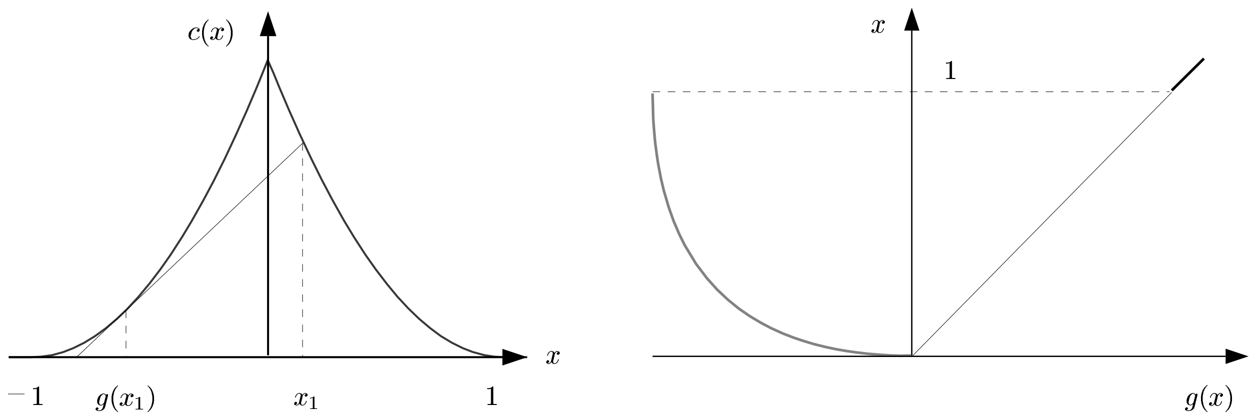

Figure 3. To the left a drawing of $c(x)$ in Example 2.2 and the slop of the tangent is $\gamma(x)$. To the right a drawing of $g(x)$ in Example 2.2.

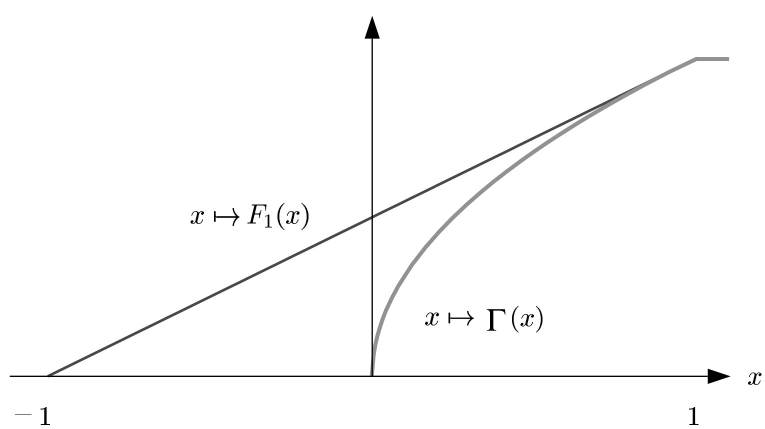

Figure 4. A drawing of $\Gamma(x)$ and $F_1(x-)$ in Example 2.2.

Example 2.3. Let μ_0 be the uniform measure on $\{-1,1\}$ and let μ_1 have atoms at $-2,0,2$ with probability $p, 1-2p, p$ respectively, where $\frac{1}{4} < p < \frac{1}{2}$. Then we compute that

$$\begin{aligned} c(x) =& p(2+x)\,\mathbf{1}_{(-2,-1)}(x) + \big(2p - \tfrac{1}{2} - (\tfrac{1}{2} - p)x\big)\,\mathbf{1}_{[-1,0)}(x) \\ &+ \big(2p - \tfrac{1}{2} + (\tfrac{1}{2} - p)x\big)\,\mathbf{1}_{[0,1)}(x) + p(2-x)\,\mathbf{1}_{[1,2)}(x)\,. \end{aligned}$$

If $\frac{3}{8} \le p < \frac{1}{2}$ then $g(x) = -2$ if $-1 < x \le 2$ and $g(x) = x$ elsewhere. If $\frac{1}{4} < p < \frac{3}{8}$ then

$$g(x) = \begin{cases} -2 & \text{if } \quad -1 < x \le 0 \quad \text{and} \quad \frac{2}{8p-1} < x \le 2 \\ 0 & \text{if } \quad 1 < x \le \frac{2}{8p-1} \\ x & \text{elsewhere}\,. \end{cases}$$

The cases are illustrated in Figure 5 and 6 and see Figure 7 for a pictorial representation of the distribution functions Γ , F_0 and F_1 .

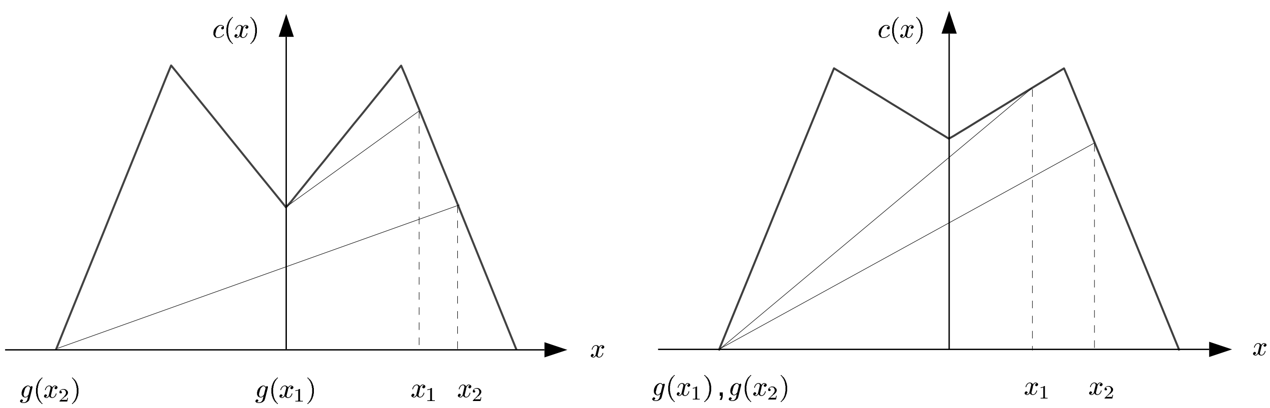

Figure 5. Two drawings of c for different parameter values in Example 2.3. The left picture is for $p = 1/3$ and the right picture represents $p = 4/10$.

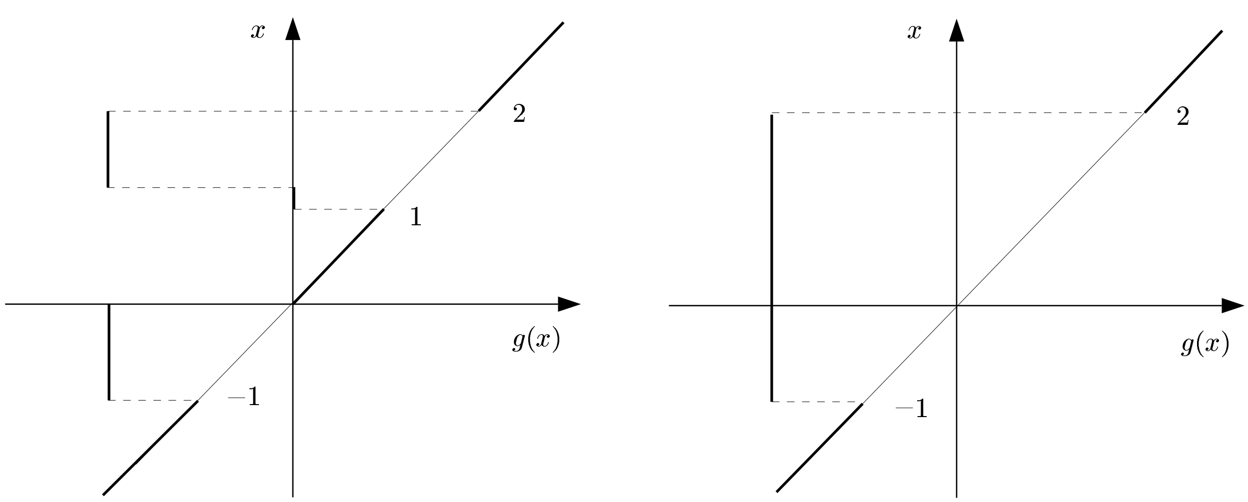

Figure 6. Two drawings of g for different parameter values in Example 2.3. The left picture is for $p = 1/3$ and the right picture represents $p = 4/10$.

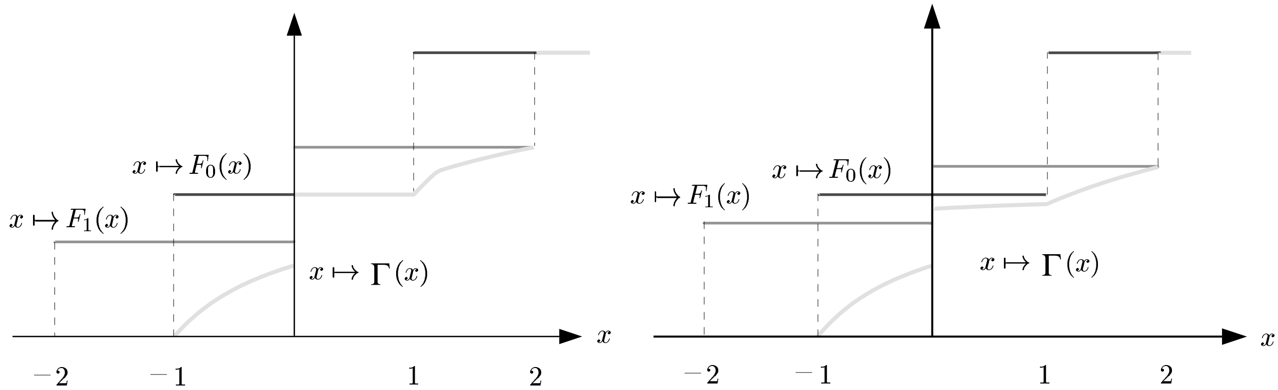

Figure 7. Two drawings of $\Gamma(x)$ and $F_i(x-)$ $(i = 0, 1)$ for different parameter values in Example 2.3. The left picture is for $p = 1/3$ and the right picture represents $p = 4/10$.

Example 2.4. This is an example to show that the function g can get complicated with even simple expressions for μ_0 and μ_1 . Let μ_0 be the uniform measure on $[-2,-1] \cup [1,2]$ and μ_1 be the uniform measure on $[-3,-2] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [2,3]$. The functions c and g are illustrated in Figure 8. Note that for x values in the range of $[\frac{1}{8}, 1]$ we have that $g(x) = x$. See Figure 9 for a pictorial representation of the distribution functions Γ , F_0 and F_1 .

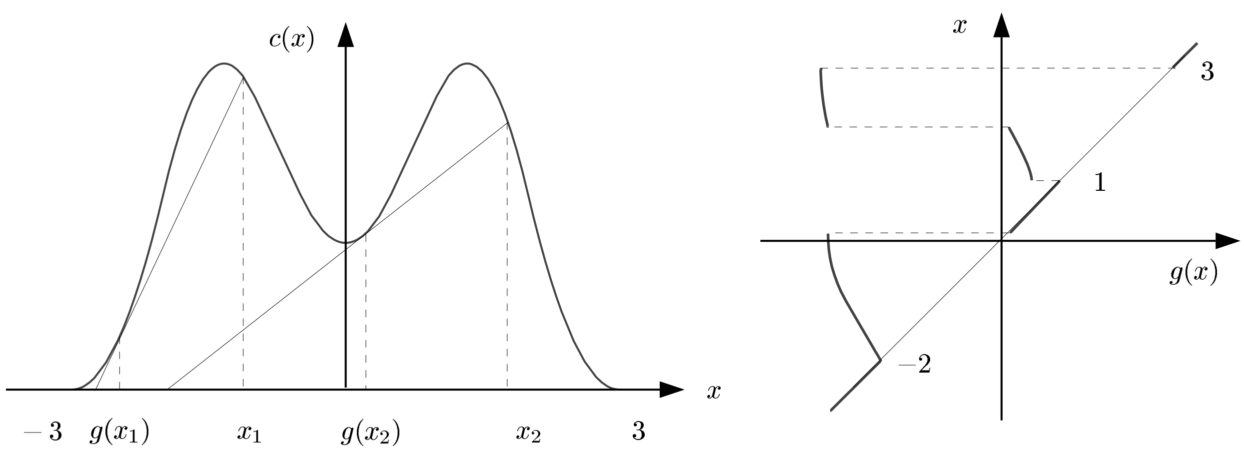

Figure 8. To the left a drawing of $c(x)$ in and to the right a drawing of $g(x)$ in Example 2.4.

3. The lower bound

The first step in the proof of Theorem 2.1 is to verify that Γ is indeed a lower bound.

Lemma 3.1. *For any* $(M_t) \in \mathcal{M}_C(\mu_0, \mu_1)$ *we have that* $\mathbf{P}(S \geq x) \geq 1 - \Gamma(x)$ *for* $x \in \mathbb{R}$ *, where* Γ *is given in* (2.3).

Proof. Let x be fixed. Suppose that $y < x$ then we have the inequality

$$\begin{aligned}\mathbf{1}_{\{S \geq x\}} \geq \mathbf{1}_{\{M_1 \geq x\}} &+ \frac{(M_1 - x)^+}{x - y} - \frac{(M_0 - x)^+}{x - y} - \frac{(M_1 - y)^+}{x - y} + \frac{(M_0 - y)^+}{x - y} \\ &+ \mathbf{1}_{\{y < M_0 < x\}} \frac{M_1 - M_0}{x - y} + \mathbf{1}_{\{S \geq x\}} \mathbf{1}_{\{y < M_0 < x\}} \frac{x - M_1}{x - y}\end{aligned} \tag{3.1}$$

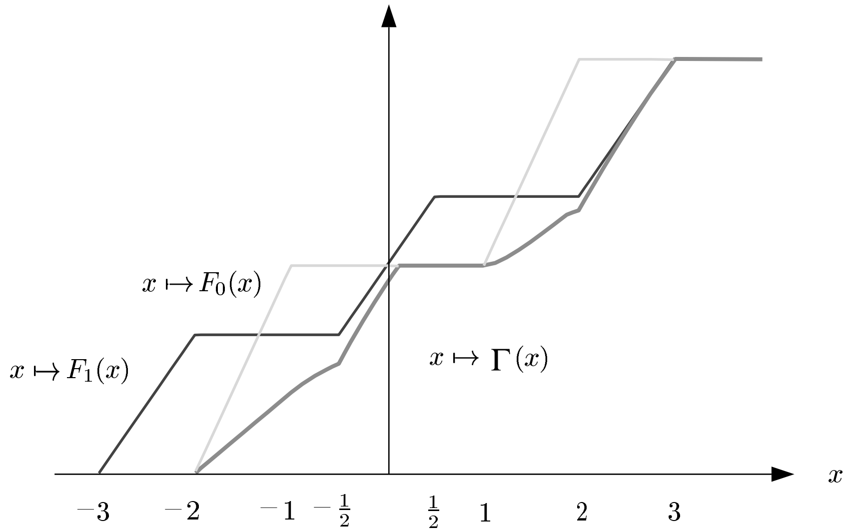

Figure 9. A drawing of $\Gamma(x)$ and $F_i(x-)$ ($i = 0, 1$) in Example 2.4 .

which can be verified on a case by case basis. Since (M_t) is a continuous martingale we have equality in Doob's submartingale inequality and hence

$$\mathbf{E}\left(\frac{x - M_1}{x - y}\,;\, S \geq x, y < M_0 < x\right) = 0\,.$$

By taking expectation in (3.1) and using martingale property we have that

$$\mathbf{P}(S \geq x) \geq \mathbf{P}(M_1 \geq x) + \frac{c(x) - c(y)}{x - y}$$

for any $y < x$ and the result follows. □

Remark 3.2. The above proof has a financial interpretation in the pricing of a forward start digital option (see [8] and [3] for greater details). Let (M_t) denotes the price process of an asset and suppose for simplicity that there are zero-interest rates and no transaction costs. From the general theory of mathematical finance it follows that the fair price of an European call option with strike x and maturity T is $\mathbf{E}((M_T - x)^+)$, where the expectation is taken with respect to the martingale measure. Thus for pricing purpose we may assume that M is a martingale. Suppose we know the call prices at times one and two for this asset. Then we can derive the laws μ_1 and μ_2 of M_1 and M_2 respectively, under the pricing measure.

Consider the digital option on sale at 0 which pays one unit if the value of the asset is above the barrier x at any time in the period $[1, 2]$, i.e. the payoff is given by

$$\mathbf{1}_{\{\max_{1 \leq t \leq 2} M_t \geq x\}}\,.$$

If we assume that the price process is continuous then from the above lemma we have that

$$\mathbf{P}(\,\max_{1 \leq t \leq 2} M_t \geq x) \geq \mathbf{P}(M_2 \geq x) + \sup_{y<x} \frac{[c_2(x) - c_1(x)] - [c_2(y) - c_1(y)]}{x - y}$$

where $c_i(x) = \mathbf{E}((M_i - x)^+)$ is the price of a call option with strike x and maturity i .

The inequality (3.1) can be used to motivate a hedging strategy. Initially we fix any $y < x$ and buy a digital option with payoff $\mathbf{1}_{\{M_2 \geq x\}}$, buy $1/(x - y)$ maturity 2 calls with strike x, sell $1/(x-y)$ maturity 1 calls with strike x, sell $1/(x-y)$ maturity 2 calls with strike y, and buy $1/(x-y)$ maturity 1 calls with strike y. This is the static part of the hedge and costs $\mu_2([x,\infty)) + \big([c_2(x)-c_1(x)] - [c_2(y)-c_1(y)]\big)/(x-y)$. For the dynamic part of the hedge we proceed as follows. If the underlying at time 1 is lower or equal y or greater or equal x we do nothing. If the underlying at time 1 is between y and x we buy $1/(x-y)$ units of the underlying and if the underlying reaches the level x we sell $1/(x-y)$ units of the underlying.

From the inequality (3.1) we have that for each y this is a sub-replicative strategy. The cost of the strategy is $\mu_2([x,\infty)) + \big([c_2(x)-c_1(x)] - [c_2(y)-c_1(y)]\big)/(x-y)$ which is a lower bound on the price of a digital option. Since $y<x$ is arbitrary the greatest lower bound on the price of a digital option is

$$\mu_2([x,\infty)) + \sup_{y<x} \frac{[c_2(x)-c_1(x)] - [c_2(y)-c_1(y)]}{x-y} .$$

If the digital option is offered for sale below this price, then arbitrage profits can be made. If the ask price is above this bound then it is not possible to create riskless profits unless the dynamics of the price process are known (for instance, the price process is geometric Brownian motion).

4. Some preliminary lemmas

In this section we state some technical results which will be required in the sequel. Some of the proofs are relegated to an appendix, although we try to explain intuitively why they must be true.

Recall the definitions of Γ, γ and g:

$$\Gamma(x) = \mu_1((-\infty,x)) - \gamma(x)$$

$$\gamma(x) = \sup_{y<x} \frac{c(x)-c(y)}{x-y} \tag{4.1}$$

and $g(x)$ is the value of y where the supremum in (4.1) is attained. If the supremum is not attained then we set $g(x)=x$, it follows that $\gamma(x) \geq c'_-(x)$ where c'_- is the left derivative of c. If the supremum is attained at more than one value of y then we choose the largest (or more precisely the supremum) of the candidate values.

Lemma 4.1. *The function $x \mapsto \gamma(x)$ is positive, left-continuous and has no downward jumps.*

Proof. This is a standard piece of analysis given the fact that the left derivative of c exists, is bounded, and indeed equals $F_1(x-) - F_0(x-)$. □

Now we prove one of the statements in Theorem 2.1, namely that the candidate law Γ is indeed (a left-continuous version of) a distribution function.

Proposition 4.2. *$x \mapsto \Gamma(x)$ is a left-continuous distribution function, i.e. Γ is increasing, left-continuous satisfying $\Gamma(-\infty) = 1 - \Gamma(+\infty) = 0$.*

Proof. From Lemma 4.1 it follows that Γ is left-continuous. It is clear that $\gamma(\pm\infty)=0$ so we only need to verify that Γ is increasing. Note that $\gamma(x) \geq 0 \vee (F_1(x-) - F_0(x-))$ and hence $\Gamma(x) \le F_0(x-) \wedge F_1(x-)$. Further by (2.1) and (2.2) we have the following expression for Γ ,

$$\Gamma(x) = F_0(x-) - \sup_{y<x} \int_{(y,x)} \frac{u-y}{x-y} (\mu_0(du) - \mu_1(du)) . \tag{4.2}$$

Fix $y > x$. With the above observations we have the following:
Case 1: $\gamma(y) = c'_-(y)$. Then $\Gamma(y) = F_0(y-) \ge F_0(x-) \ge \Gamma(x)$.

Case 2: $\gamma(y) > c'_-(y)$ and $g(y) \le x$. Then from (4.2) we have that

$$\begin{aligned}
\Gamma(y) &= F_0(y-) - \int_{(g(y),y)} \frac{u-g(y)}{y-g(y)} \mu_0(du) + \int_{(g(y),y)} \frac{u-g(y)}{y-g(y)} \mu_1(du) \\
&\ge F_0(y-) - \int_{(g(y),x)} \frac{u-g(y)}{y-g(y)} \big(\mu_0(du) - \mu_1(du)\big) - \int_{[x,y)} \mu_0(du) \\
&= F_0(x-) - \int_{(g(y),x)} \frac{u-g(y)}{y-g(y)} (\mu_0(du) - \mu_1(du)) \\
&\ge F_0(x-) - \sup_{z<x} \int_{(z,x)} \frac{u-z}{x-z} (\mu_0(du) - \mu_1(du)) = \Gamma(x) .
\end{aligned}$$

Case 3: $\gamma(y) > c'_-(y)$ and $x < g(y)$. From the first line in the previous case

$$\begin{aligned}
\Gamma(y) &\ge F_0(y-) - \int_{(g(y),y)} \frac{u-g(y)}{y-g(y)} \mu_0(du) \\
&\ge F_0(y-) - \int_{(g(y),y)} \mu_0(du) \\
&= F_0(g(y)) \ge F_0(x-) \ge \Gamma(x) .
\end{aligned}$$

Hence Γ is increasing. □

The conclusion is that $\nu^{\#}$ given in (2.4) is a probability measure which, by Lemma 3.1 is a lower bound for $\mathcal{P}_C$. We summarise this in a proposition.

Proposition 4.3. *Let (M_t) be a continuous martingale with initial law μ_0 and terminal law μ_1 . Let ν be the law of the maximum process S . Then $\nu^{\#} \preceq \nu$, i.e. for all $\nu \in \mathcal{P}_C(\mu_0, \mu_1)$ we have that $\nu^{\#} \preceq \nu$.*

It remains to show that $\nu^{\#} \in \mathcal{P}_C(\mu_0, \mu_1)$. This is the subject of the next section. For the remainder of this section we state further lemmas, beginning with one on the properties of g .

Lemma 4.4. *The function $x \mapsto g(x)$ has the following properties.*
(i) *If $z \ge x$, then either $g(z) \le g(x)$ or $g(z) \ge x$.*
(ii) *If $g(x) < x$, then $c'_-(g(x)) \le \gamma(x) \le c'_+(g(x))$.*
(iii) *If $g(x) = x$ then $F_0(x-) = \Gamma(x)$.*

Proof. These statements are best understood using a picture; recall Figure 1. (ii) follows from interpretation of γ as the gradient of the tangent to c at $g(x)$, and (iii) is true by l'Hôpital's rule. □

It follows from the lemma that the typical behaviour of g is that either $g(x) = x$, or $g(x) < x$ and g is decreasing. In fact if g increases then it must increase to the diagonal.

Lemma 4.5.

(i) *If* $x_n \downarrow x$, *with* $g(x_n) \geq x$ *then* $\Gamma(x+) = F_0(x)$.
(ii) *If* $g(x) < x$ *over an interval* (y, z) , *and if* $g(z-) \equiv \lim_{u \uparrow z} g(u)$, *then* $g(z) = g(z-)$.

Proof. (i): By Lemma 4.4(ii) we have $c'_-(x_n) \leq \gamma(x_n) \leq \sup_{y \in [x, x_n]} c'_+(y) \vee c'_-(x_n)$ and so $\gamma(x_n) \to c'_+(x) = F_1(x) - F_0(x)$.
(ii): g is decreasing over the interval (y, z) and so $g(z-)$ exists. Further, either $g(z) = z$ or $g(z) \leq g(z-)$, and

$$\gamma(z) = \gamma(z-) = \lim_{x \uparrow z} \frac{c(x) - c(g(x))}{x - g(x)} = \frac{c(z) - c(g(z-))}{z - g(z-)} .$$

Thus $g(z-)$ attains the supremum in (4.1) and by maximality $g(z) = g(z-)$. □

We set $A^+ = \{ x \in \mathbb{R} \,|\, \Gamma(x+) = F_0(x) \}$ and $A^- = \{ x \in \mathbb{R} \,|\, \Gamma(x) = F_0(x-) \}$ and let $A = A^+ \cup A^-$. A will play a special role in the next section where we show that for the optimal martingale, if $M_0 < x \in A$ then necessarily $S \leq x$ also.

Lemma 4.6. *If* $x \notin A^+$ *then* γ *is continuous and decreasing at* x . *Hence* $\Delta\Gamma(x) = \Delta F_1(x)$. *More generally,* $\Delta\Gamma(x) \leq \Delta F_1(x)$.

Proof. The proof is only sketched and some technical details are omitted. If $x \notin A^+$ then by the previous lemma for all y in some interval $(x, x + \delta)$ we have $g(y) < x < y$. By Lemma 4.4(i), g is non-increasing on this interval. Let $g(x+) = \lim_{y \downarrow x} g(y) \leq g(x)$.

Further, since $x \notin A^+$, we must have $c'_+(x) < \gamma(x)$, and for y in some new smaller interval $(x, x + \delta')$ we have $c(y) < c(x) + (y - x)\gamma(x)$. Then

$$\begin{aligned}
\gamma(y) &\leq \frac{c(x) + (y - x)\,\gamma(x) - c(g(y))}{y - g(y)} \\
&= \frac{c(x) - c(g(y))}{x - g(y)} \, \frac{x - g(y)}{y - g(y)} + \frac{y - x}{y - g(y)} \gamma(x) \\
&\leq \frac{x - g(y)}{y - g(y)} \gamma(x) + \frac{y - x}{y - g(y)} \gamma(x) = \gamma(x) .
\end{aligned}$$

Thus γ is decreasing to the right of x and to prove that it is decreasing from the left of x can be done in the same manner.

Suppose first that $g(x+) < x$. Then

$$\gamma(y) = \frac{c(y) - c(g(y))}{y - g(y)} \to \frac{c(x) - c(g(x+))}{x - g(x+)} \leq \gamma(x) .$$

But also

$$\gamma(y) \geq \frac{c(y) - c(g(x))}{y - g(x)} \to \gamma(x) .$$

Thus γ is right continuous and hence continuous at x .

Now suppose $g(x+) = x$, so that $g(x) = x$ and $\gamma(x) = c'_-(x)$. Then $c'_-(g(y)) \leq \gamma(y) \leq c'_+(g(y))$, and so $\gamma(y) \to c'_-(x)$, and again γ is continuous at x .

The final statements about $\Delta\Gamma(x)$ follow from the representation $\Gamma(x) = F_1(x-) - \gamma(x)$. □

Lemma 4.7.

(i) *If* I *is an open interval disjoint from* A *then* $\displaystyle\int_I \frac{du}{u-g(u)} = -\int_I \frac{d\gamma(u)}{F_0(u-)-\Gamma(u)}$.

(ii) *Further* γ *satisfies* $\displaystyle\gamma(x) = \int_{\{u>x\,,\,g(u)<g(x)\}} \frac{F_0(u-)-\Gamma(u)}{u-g(u)}\,du$.

Intuition. γ is decreasing, and so $d\gamma$ exists. If F_1 and F_0 are continuous random variables, so that c is differentiable everywhere, and if g is differentiable on I then γ is also differentiable and the result follows by differentiation of $\gamma(v) = (c(v) - c(g(v))/(v - g(v))$.

If we consider the differential version of the first expression, and multiply both sides by $F_0(y-) - \Gamma(y)$ then the second expression will follow if we integrate over suitable intervals. For full proofs see the appendix.

We close this section with a couple of lemmas concerning distribution functions, the proofs of which are in the appendix. We denote by $\Delta\pi(u)$ the atom of π at u , and by π^c the continuous part of the distribution.

Lemma 4.8. *Let* π, ρ *be two measures on* $\mathbb{R}$ *satisfying* $\rho \preceq \pi$. *Let* $J(x) = \pi((\infty, x)) - \rho((\infty, x))$. *If* π *and* ρ *have no simultaneous jumps on the interval* $[y, x)$, *then*

$$\frac{1}{J(x)} \exp\left(-\int_y^x \frac{\rho^c(du)}{J(u)}\right) \prod_{u\in[y,x)} \left(1 - \frac{\Delta\rho(u)}{J(u)}\right) =$$

$$\frac{1}{J(y)} \exp\left(-\int_y^x \frac{\pi^c(du)}{J(u)}\right)\left(\prod_{v\in[y,x)} \left(1 + \frac{\Delta\pi(u)}{J(u)}\right)\right)^{-1} .$$

Lemma 4.9. *Let* π, ρ *be as above. Then, for all* $y \in \mathbb{R}$,

$$\int_{(-\infty,y)} \frac{\pi(du)}{F_\pi(u-) - F_\rho(u-)} \exp\left(-\int_u^y \frac{\pi^c(dv)}{F_\pi(v) - F_\rho(v)}\right) \prod_{v\in[u,y)} \left(1 + \frac{\Delta\pi(v)}{F_\pi(v-) - F_\rho(v-)}\right)^{-1} = 1 .$$

5. The minimum maximum is attained

In this section we construct a martingale $(M_t^\#)$ which is an element of $\mathcal{M}_C(\mu_0, \mu_1)$ and has the property that it's maximum S has the law $\nu^\#$. Thus, not only is $\nu^\#$ a lower bound for $\mathcal{P}_C(\mu_0, \mu_1)$ but also $\nu^\# \in \mathcal{P}_C(\mu_0, \mu_1)$.

The key idea in the construction of $(M_t^\#)$ is to exhibit the martingale as the solution of a Skorokhod embedding problem (see [7] and [15]). Let $(B_t)_{t\ge 0}$ be a Brownian motion with initial law μ_0 . The problem is to find a stopping time τ satisfying B_τ has the law μ_1 and $\sup_{0\le t\le\tau} B_t$ has the law $\nu^\#$. Then we can define $(M_t^\#)$ as a time change of (B_t) by

$$M_t^\# = B_{\frac{t}{1-t}\wedge\tau} . \tag{5.1}$$

$(M_t^\#)$ is a true martingale and not just a local martingale provided that $(B_{t\wedge\tau})_{t\ge 0}$ is uniform integrable. To display the stopping time the function g will be important for us.

If F is a distribution function, we denote by ΔF the jumps of F , and by F^c the continuous part of F. Recall that in Section 3 we defined a Brownian motion B with initial law μ_0 and a random variable G which depended on B only through the initial value B_0 :

$$\mathbf{P}(G \geq s \,|\, B_0 = r) = \exp\Big(- \int_{(r,s)} \frac{F_1^c(du)}{F_0(u-) - \Gamma(u)} \Big) \prod_{u \in [r,s)} \Big(1 - \frac{\Delta F_1(u)}{F_0(u-) - \Gamma(u)} \Big) . \tag{5.2}$$

Let $S_t = \max_{0 \leq s \leq t} B_r$ and define the stopping times

$$\tau_G = \inf\{ t > 0 \,:\, S_t \geq G \} \quad \text{and} \quad \tau_g = \inf\{ t > 0 \,:\, B_t \leq g(S_t) \} .$$

Set $\tau = \tau_G \wedge \tau_g$.

We have to prove two identities in law, namely that $B_\tau \sim \mu_1$ and $S_\tau \sim \nu^{\#}$. We consider the second identity first, but we begin with a useful lemma. Recall the definitions of the sets A^+, A^- and A before Lemma 4.6.

Lemma 5.1.

(i) *Suppose* $x \in A^-$. *If* $B_0 < x$ *then* $S_\tau < x$.

(ii) *Suppose* $x \in A^+$. *If* $B_0 < x$ *then* $S_\tau \leq x$. *Further* $\mathbf{P}(S_\tau > x, B_0 \leq x) = 0$.

Proof. We shall omit some technical details in the proof. Suppose $x \in A$, $B_0 = r < x$ and let H_z be the first hitting time by B of level $z > r$ i.e.

$$H_z = \inf\{ t > 0 \,:\, B_t = z \} .$$

If there exists $z \in (r,x)$ with $g(z) = z$ then $\tau \leq \tau_g \leq H_z$ and $S_\tau \leq z < x$.

Otherwise $g(z) < z$ on the interval (r,x) . Hence g is decreasing on this interval, and by Lemma 4.5 $g(x) < x$.

(i): If $x \in A^-$ we show

$$\int_{(r,x)} \frac{F_1^c(du)}{F_0(u-) - \Gamma(u)} = \infty \tag{5.3}$$

so that $\mathbf{P}(G \geq x \,|\, B_0 = r) = 0$ and $\tau_G < H_x$.

By considering

$$-\infty = \int^x d[\ln(F_0(u) - \Gamma(u))] = \int^x \frac{dF_0(u)}{F_0(u-) - \Gamma(u)} - \int^x \frac{d\Gamma(u)}{F_0(u-) - \Gamma(u)}$$

we deduce that this final integral must be infinite. Also

$$\int^x \frac{du}{u - g(u)} = - \int^x \frac{\gamma(du)}{F_0(u-) - \Gamma(u)}$$

is finite so that

$$\int^x \frac{dF_1(u)}{F_0(u-) - \Gamma(u)} = \int^x \frac{d\Gamma(u)}{F_0(u-) - \Gamma(u)} + \int^x \frac{\gamma(du)}{F_0(u-) - \Gamma(u)} = \infty .$$

(ii): Finally suppose $x \in A^+$. If $\Delta F_1(x) > 0$ then $F_0(x-) - \Gamma(x) \leq F_0(x) - \Gamma(x+) + \Delta\Gamma(x) \leq \Delta F_1(x)$ so that $\mathbf{P}(G > x \,|\, B_0 = r) = 0$ and $S_\tau \leq x$. Otherwise F_1 is continuous at x and then $\Delta\Gamma(x) = 0$ so that $F_0(x-) \geq \Gamma(x) = \Gamma(x+) = F_0(x)$. In particular $\Delta F_0(x) = \mathbf{P}(B_0 = x) = 0$. □

Lemma 5.2. *Suppose the open interval* (u,y) *is disjoint from* A . *Then*

$$\mathbf{P}(S_{\tau_g} \geq y \,|\, B_0 = u) = \exp\Big(- \int_{(u,y)} \frac{dv}{v - g(v)} \Big)$$

Proof. For a Brownian motion the rate of excursions below the maximum (at s) which get down to $g(s)$ is given by $(s-g(s))^{-1}$. See Rogers [14] or Revuz and Yor [12]. $\square$

Proposition 5.3. *We have that* $S_\tau \sim \nu^{\#}$.

Proof. For $y \in A$ we have $\mathbf{P}(S_\tau < y) = \mathbf{P}(B_0 < y)$, or $\mathbf{P}(S_\tau \le y) = \mathbf{P}(B_0 \le y)$.

Otherwise, consider $y \notin A$ and define $z^{\#} = z^{\#}(y) = \sup_{z<y}\{z \in A\}$. If $z^{\#} = y$ then by left continuity $\mathbf{P}(S_\tau < y) = \mathbf{P}(B_0 < y)$. So suppose $z^{\#}(y) < y$. Then

$$
\begin{aligned}
\mathbf{P}(S_\tau \ge y) &= \int_{\mathbb{R}} \mathbf{P}(S_\tau \ge y \,|\, B_0 = u)\, \mu_0(du) \\
&= \mathbf{P}(B_0 \ge y) + \int_{[z^{\#},y)} \mathbf{P}(S_{\tau_g} \ge y \,|\, B_0 = u)\, \mathbf{P}(G \ge y \,|\, B_0 = u)\, \mu_0(du) \\
&= \mathbf{P}(B_0 \ge y) + \int_{[z^{\#},y)} \mu_0(du) \exp\left(-\int_{(u,y)} \frac{dv}{v-g(v)}\right) \\
&\quad \times \exp\left(-\int_{(u,y)} \frac{F_1^c(dv)}{F_0(v-)-\Gamma(v)}\right) \prod_{v\in[u,y)} \left(1-\frac{\Delta F_1(v)}{F_0(v-)-\Gamma(v)}\right) \\
&= \mathbf{P}(B_0 \ge y) + \int_{[z^{\#},y)} \mu_0(du) \exp\left(-\int_{(u,y)} \frac{\Gamma^c(dv)}{F_0(v-)-\Gamma(v)}\right) \\
&\qquad \times \prod_{v\in[u,y)} \left(1-\frac{\Delta\Gamma(v)}{F_0(v-)-\Gamma(v)}\right)
\end{aligned}
$$

where in the last equality we have used Lemma 4.7(i), $\Gamma^c = F_1^c - \gamma^c$ and $\Delta F_1 = \Delta\Gamma$.

If we now apply Lemma 4.8 with $F_\rho(x) = (\Gamma(x+) - F_0(z^{\#}-))\,\mathbf{1}_{[z^{\#},\infty)}(x)$ and $F_\pi(x) = (F_0(x) - F_0(z^{\#}-))\,\mathbf{1}_{[z^{\#},\infty)}(x)$ then this becomes

$$
\begin{aligned}
\mathbf{P}(S_\tau \ge y) &= \mathbf{P}(B_0 \ge y) + (F_0(y-) - \Gamma(y)) \\
&\quad \times \int_{[z^{\#},y)} \frac{\mu_0(du)}{F_0(u-)-\Gamma(u-)} \exp\left(-\int_{(u,y)} \frac{F_0^c(dv)}{F_0(v-)-\Gamma(v)}\right) \\
&\qquad \times \prod_{v\in[u,y)} \left(1+\frac{\Delta F_0(v)}{F_0(v-)-\Gamma(v)}\right)^{-1} .
\end{aligned}
$$

Finally, applying Lemma 4.9, now with $\pi = \mu_0$ and $\rho = \Gamma$ we get that

$$
\mathbf{P}(S_\tau \ge y) = \mathbf{P}(B_0 \ge y) + (F_0(y-) - \Gamma(y)) = 1 - \Gamma(y)
$$

and the result follows. $\square$

Proposition 5.4. *For the above construction we have that* $B_\tau \sim \mu_1$.

Proof. From the construction we have that if $B_\tau < S_\tau$, then $B_\tau = g(S_\tau) < S_\tau$. This happens if the Brownian motion has an excursion down below the maximum (at s) which reaches $g(s)$. Results from excursion theory give that this happens at rate $(s-g(s))^{-1}$. Then

$$\begin{aligned}\mathbf{P}(B_\tau < y) &= \mathbf{P}(S_\tau < y) + \mathbf{P}(S_\tau \geq y\,,\, B_\tau < y)\\ &= \Gamma(y) + \int_{\{u\geq y,\, g(u)<y\}} \mathbf{P}(B_0 < u, S_\tau \geq u)\,\frac{du}{u-g(u)}\\ &= \Gamma(y) + \int_{\{u\geq y,\, g(u)<y\}} \frac{F_0(u-)-\Gamma(u)}{u-g(u)}\,du\\ &= \Gamma(y) - \gamma(y) = F_1(y-)\end{aligned}$$

this last line following from Lemma 4.7(ii). □

For $(M_t^{\#})$ from (2.6) we have as a corollary of the above proposition the main result of the paper.

Theorem 5.5. *$(M_t^{\#})_{0\leq t\leq 1} \in \mathcal{M}_C(\mu_0,\mu_1)$ and for any $\nu \in \mathcal{P}_C(\mu_0,\mu_1)$ stochastically dominates $\nu^{\#}$.*

Proof. From Lemma 2.4 in [15] it follows that $(B_{(t/(1-t))\wedge\tau})_{t\geq 0}$ is uniform integrable. Hence $(M_t^{\#})$ is a martingale. □

6. Appendix: Proofs of some lemmas from Section 4

Proof. (**Lemma 4.1**). Since $\lim_{y\downarrow-\infty} c(y) = 0$ we have that γ is positive so we begin by showing that $x \mapsto \gamma(x)$ is left continuous. Let $x_n \uparrow x$, then we wish to verify that $\limsup_{x_n\uparrow x}\gamma(x_n) \leq \gamma(x) \leq \liminf_{x_n\uparrow x}\gamma(x_n)$.

Consider first the second inequality. If $\gamma(x) = F_1(x-)-F_0(x-)$ then, since more generally $\gamma(y) \geq c'_-(y) = F_1(y-) - F_0(y-)$, we have that

$$\gamma(x_n) \geq F_1(x_n-) - F_0(x_n-) \to F_1(x-) - F_0(x-)\,.$$

Conversely if $\gamma(x) > F_1(x-) - F_0(x-)$ then $g(x) < x$ and for $x_n > g(x)$ we have that

$$\gamma(x_n) \geq \frac{c(x_n)-c(g(x))}{x_n-g(x)} \to \frac{c(x)-c(g(x))}{x-g(x)} = \gamma(x)\,.$$

Hence the inequality $\liminf_{x_n\uparrow x}\gamma(x_n) \geq \gamma(x)$ is proved.

Consider now the first inequality, let $0<\varepsilon<1$ be given. Choose $0<\delta<1$ such that $F_1(x-) - F_1(x-\delta) < \varepsilon$ and $F_0(x-) - F_0(x-\delta) < \varepsilon$. By the identity $c'_-(y) - c'_-(x) = (F_0(x-)-F_0(y-)) - (F_1(x-)-F_1(y-))$ we see that $|c'_-(y)-c'_-(x)| < \varepsilon$ for $y \in (x-\delta, x)$. Note that $|c'_-(x)| \leq 1$, and hence $0 \leq \gamma(x) - c'_-(x) \leq 2$. Choose $\alpha < \varepsilon\delta/5$ and fix $y > x-\alpha$. If $g(y) > x-\delta$ then

$$\begin{aligned}\gamma(y) &\leq \sup_{z\in(x-\delta,y)} c'_+(z) \leq F_1(y) - F_0(x-\delta)\\ &\leq (F_1(x-)-F_0(x-)) + (F_0(x-)-F_0(x-\delta)) \leq \gamma(x)+\varepsilon\,.\end{aligned}$$

Conversely if $g(y) \leq x-\delta$ then, since

$$c(x) = c(y) + \int_y^x c'_-(z)\,dz \geq c(y) + \int_y^x (c'_-(x)-\varepsilon)\,dz = c(y) + (x-y)(c'_-(x)-\varepsilon)$$

we have that

$$
\begin{aligned}
\gamma(y) = \frac{c(y)-c(g(y))}{y-g(y)} &\le \frac{c(x)-c(g(y))-(x-y)(c'_-(x)-\varepsilon)}{y-g(y)} \\
&\le \frac{\gamma(x)(x-g(y))-(x-y)c'_-(x)+(x-y)\varepsilon}{y-g(y)} \\
&= \gamma(x)+(x-y)\frac{\gamma(x)-c'_-(x)+\varepsilon}{y-g(y)} \\
&\le \gamma(x)+\alpha\frac{3+\varepsilon}{\delta-\alpha} \le \gamma(x)+\varepsilon
\end{aligned}
$$

since $4\alpha/(\delta-\delta/5) \le 5\alpha/\delta$. Hence in both cases we have the inequality $\limsup_{x_n \uparrow x} \gamma(x_n) \le \gamma(x)$, and γ is left continuous.

It remains to show that γ has no downward jumps. Recall that γ is left-continuous, so we are interested in. If $g(x) < x$ and $\varepsilon > 0$ then

$$
\gamma(x+\varepsilon) \ge \sup_{y<x} \frac{c(x+\varepsilon)-c(y)}{x+\varepsilon-y} \ge \frac{c(x+\varepsilon)-c(g(x))}{x+\varepsilon-g(x)} .
$$

Letting $\varepsilon \downarrow 0$, we obtain $\gamma(x+) \ge \gamma(x)$.
If $g(x) = x$ and fix $\delta > 0$ then

$$
\gamma(x+\varepsilon) \ge \frac{c(x+\varepsilon)-c(x-\delta)}{\varepsilon+\delta} .
$$

Letting $\varepsilon \downarrow 0$, we see that

$$
\gamma(x+) \ge \frac{c(x)-c(x-\delta)}{\delta}
$$

for all δ . Hence by taking supremum over δ we have that

$$
\gamma(x+) \ge \sup_{\delta>0} \frac{c(x)-c(x-\delta)}{\delta} = \gamma(x) .
$$

The proof is complete. □

Proof. (**Lemma 4.7**). (i): On I we must have $v < g(v)$ and $\Gamma(v) < F_0(v-)$. Further γ is decreasing so that $\gamma(dv)$ must exist. We prove that Lebesgue almost surely, γ is differentiable on I with derivative

$$
\frac{\gamma(dv)}{dv} = \frac{F_0(v-)-\Gamma(v)}{v-g(v)} .
$$

Suppose v is such that F_0, F_1, Γ and the decreasing function g are all continuous at v . Then, for $v < y \in I$

$$
\begin{aligned}
\gamma(y)-\gamma(v) &\ge \frac{c(y)-c(g(v))}{y-g(v)} - \frac{c(v)-c(g(v))}{v-g(v)} \\
&\ge \frac{c(v)-c(g(v))+(y-v)(F_1(y)-F_0(y))}{v-g(v)}\left[1-\frac{y-v}{y-g(v)}\right] - \frac{c(v)-c(g(v))}{v-g(v)}
\end{aligned}
$$

and so

$$
\liminf_{y\downarrow v} \frac{\gamma(y)-\gamma(v)}{y-v} \ge \frac{c'_+(v)-\gamma(v)}{v-g(v)} .
$$

We can obtain the reverse inequality by considering

$$\gamma(y) - \gamma(v) \le \frac{c(y) - c(g(y))}{y - g(y)} - \frac{c(v) - c(g(y))}{v - g(y)}$$

and the left derivative follows by similar arguments.

(ii): Some technical details are omitted. Fix x . Then there exist monotonic functions $p = p_x(u), q = q_x(u)$ such that $p(u) \le g(x) < x \le q(u)$, $p(u) = g(q(u))$ and $\gamma(q(u)) = u$. Then

$$\begin{aligned}\int_{\{u>x,\, g(u)<g(x)\}} \frac{F_0(u-) - \Gamma(u)}{u - g(u)}\, du &= \int_{\{u>x,\, g(u)<g(x)\}} d\gamma(u) \\ &= \int_{\{u \,|\, q(u) \ge x\}} d\gamma(q(u)) = \int_0^{\gamma(x)} du = \gamma(x) \ .\end{aligned}$$

□

Proof. Lemma 4.8. Denote $J^c(x) = F_\pi^c(x) - F_\rho^c(x)$. Then we have that

$$\begin{aligned} d[\log J(x)] &= \frac{dJ^c(x)}{J(x)} + \log \frac{J(x+)}{J(x)} \\ &= \frac{dJ^c(x)}{J(x)} + \log\left(1 + \frac{\Delta\pi(x)}{J(x)}\right) + \log\left(1 - \frac{\Delta\rho(x)}{J(x)}\right) .\end{aligned}$$

If we integrate over the set $[y, x)$ we obtain that

$$\begin{aligned}\log \frac{J(x)}{J(y)} &= \int_{(y,x)} \frac{\pi^c(du)}{J(u)} - \int_{(y,x)} \frac{\rho^c(du)}{J(u)} \\ &\quad + \sum_{u\in[y,x)} \log\left(1 + \frac{\Delta\pi(u)}{J(u)}\right) + \sum_{u\in[y,x)} \log\left(1 - \frac{\Delta\rho(x)}{J(x)}\right)\end{aligned}$$

and the result follows easily. □

Proof. (**Lemma 4.9**). Define the function

$$K(x) \equiv K_y(x) = -\int_x^y \frac{F_\pi^c(dv)}{F_\pi(v) - F_\rho(v)} - \sum_{v\in[x,y)} \log\left(1 + \frac{\Delta F_\pi(v)}{F_\pi(v-) - F_\rho(v-)}\right)$$

Then we have $K(y) = 0$ and

$$K(x) = -\int_x^y d(\log F_\pi(dv))$$

so that $K(-\infty) = -\infty$. Then we have

$$\begin{aligned} d[e^{K(u)}] &= e^{K(u)} \frac{F_\pi^c(du)}{F_\pi(u-) - F_\rho(u-)} + e^{K(u)}\left(e^{K(u+)-K(u)} - 1\right) \\ &= e^{K(u)} \frac{F_\pi^c(du)}{F_\pi(u-) - F_\rho(u-)} + e^{K(u)} \frac{\Delta F_\pi(u)}{F_\pi(u-) - F_\rho(u-)} .\end{aligned}$$

Integrate over the set $(-\infty, y)$ we get that

$$1 = e^{K(y)} - e^{K(-\infty)} = \int_{(-\infty,y)} \frac{\pi(du)}{F_\pi(u-) - F_\rho(u-)}\, e^{K(u)} .$$

□

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Optimal Prediction of the Ultimate Maximum of Brownian Motion

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Let $(B_t)_{t\geq 0}$ be a standard Brownian motion vanishing at zero. For fixed $p, \varepsilon > 0$ consider for $T > 0$ the two optimal stopping problems

$$V_*^{(p)}(T) = \inf_\tau \mathbf{E}\big(\max_{0\leq t\leq T} B_t - B_\tau\big)^p \quad \text{and} \quad W_*^{(\varepsilon)}(T) = \sup_\tau \mathbf{P}\big(\max_{0\leq t\leq T} B_t - B_\tau \leq \varepsilon\big)$$

where the infimum and supremum respectively are taken over all stopping times τ for (B_t) satisfying $\tau \leq T$. In the first problem an explicit formula is derived for the value functions and the optimal stopping strategy is displayed. In the latter problem we conjecture a theorem and reduce its proof to verifying that a value function is not differentiable over a line. The method of proof is based upon representing the conditional expectation of the gain process $G(\max_{0\leq r\leq T} B_r - B_t)$ given $\mathcal{F}_{t+} = \bigcap_{s>t} \sigma(\{B_r \mid 0 \leq r \leq s\})$ as a function of $(\max_{0\leq r\leq t} B_r - B_t)$.

1. Introduction

In a time interval based only on the information of the past and present of a Brownian motion, the problem of stopping the Brownian motion as close to its unknown ultimate maximum in p-mean and probability distance is solved in this paper. The ultimate maximum is a variable depending of the entire path of the motion over the time interval and its value is first known at the terminal time. Thus the problem is to find a stopping time such that the distance between the stopped Brownian motion and the ultimate maximum is as close as possible. The interpretation of the problems is that the unknown ultimate maximum of the observable motion is predict in an optimal way.

In mathematical terms the problems are formulate the following way. Let $(B_t)_{t\geq 0}$ be a Brownian motion vanishing at zero defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by (B_t). Let (S_t) given by

$$S_t = \max_{0\leq r\leq t} B_r \tag{1.1}$$

denote the maximum process associated with (B_t). The stopping times for (B_t) are referred to $(\mathcal{F}_{t+})$-stopping times. For $T > 0$, let $\mathcal{M}_T$ be the family of all stopping times τ for (B_t) satisfying $\tau \leq T$ **P**-a.s. For fixed $p > 0$ and $\varepsilon > 0$ the problems are to compute the two value functions given by

$$V_*^{(p)}(T) = \inf_{\tau\in\mathcal{M}_T} \mathbf{E}\big(S_T - B_\tau\big)^p \tag{1.2}$$

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and

$$W_*^{(\varepsilon)}(T) = \sup_{\tau \in \mathcal{M}_T} \mathbf{P}\big(S_T - B_\tau \le \varepsilon\big) \tag{1.3}$$

and to find an optimal stopping time in each of the problems, i.e. a stopping time for which the optimum is attained. The general optimal stopping theory (see [6]) can not be applied to solve the two optimal stopping problems since both gain processes $\big(G(S_T - B_t)\big)_{0\le t\le T}$ are not adapted to the filtration $(\mathcal{F}_t)_{0\le t\le T}$.

This paper was motivated by the work of Graversen, Peskir and Shiryaev [2] where the optimal stopping problem (1.2) is solved in the case the closeness is measured in square-mean distance ($p = 2$). The idea in [2] is the following. The ultimate maximum is represented as a stochastic integral by Itô-Clark representation theorem and the initial problem is then transformed into a new equivalent path-dependent integral optimal stopping problem which can be solved by known methods. The main idea and novelty of this paper is that the conditional expectation of the gain process $G(S_T - B_t)$ given $\mathcal{F}_{t+}$ is represented as a function of $(S_t - B_t)$ and the initial problem is then transformed into an equivalent one-dimensional optimal stopping problem.

As indicated above to solve the problems (1.2) and (1.3) the following result is useful. Denote the density and distribution function of a standard normal variable by

$$\varphi(y) = \tfrac{1}{\sqrt{2\pi}}\, e^{-y^2/2} \quad \text{and} \quad \Phi(y) = \int_{-\infty}^{y} \varphi(u)\, du \tag{1.4}$$

for $y \in \mathbb{R}$. Let (W_t) be a $(\mathcal{G}_t)$-Brownian motion (see [3]) where $(\mathcal{G}_t)_{t\ge 0}$ is a filtration.

Proposition 1.1. *Let* $G : \mathbb{R}_+ \to \mathbb{R}_+$ *be a strictly increasing continuous function. Then for* $0 \le t \le T$ *the conditional expectation* $\mathbf{E}\big(G\big(\max_{0\le r\le T} W_r - W_t\big) \,\big|\, \mathcal{G}_t\big)$ *is given by the following formula*

$$\begin{aligned} &\mathbf{E}\big(G\big(\max_{0\le r\le T} W_r - W_t\big) \,\big|\, \mathcal{G}_t\big) \\ &= G\big(\max_{0\le r\le t} W_r - W_t\big) + \int_{G(\max_{0\le r\le t} W_r - W_t)}^{\infty} \Big(1 - F_{T-t}\big(G^{-1}(u)\big)\Big)\, du \end{aligned}$$

where F_{T-t} *is the distribution function for* $\max_{0\le r\le T-t} W_r$ *given by*

$$F_{T-t}(y) = 2\Phi\left(\frac{y}{\sqrt{T-t}}\right) - 1 \tag{1.5}$$

for $y \ge 0$.

Proof. Let $t \le T$ be given. The stationary independent increments of (W_t) give that

$$\begin{aligned} &\mathbf{E}\big(G\big(\max_{0\le r\le T} W_r - W_t\big) \,\big|\, \mathcal{G}_t\big) \\ &= \mathbf{E}\big(G\big(\{\max_{0\le r\le t} W_r\} \vee \{\max_{t\le r\le T}(W_r - W_t) + W_t\} - W_t\big) \,\big|\, \mathcal{G}_t\big) \\ &= \mathbf{E}\big(G\big(\{s\} \vee \{\max_{0\le r\le T-t} W_r + x\} - x\big)\big)\Big|_{s=\max_{0\le r\le t} W_r}^{x=B_t} . \end{aligned}$$

The above expectation is for $x \leq s$ given by

$$\begin{aligned}\mathbf{E}\big(G\big(\{s\} \vee \{\max_{0\leq r\leq T-t} W_r + x\} - x\big)\big) &= \int_0^\infty \mathbf{P}\big(G\big(\{s\} \vee \{\max_{0\leq r\leq T-t} W_r + x\} - x\big) > u\big)\, du \\ &= G(s-x) + \int_{G(s-x)}^\infty \mathbf{P}\big(\max_{0\leq r\leq T-t} W_r > G^{-1}(u)\big)\, du\end{aligned}$$

and thus

$$\begin{aligned}&\mathbf{E}\big(G\big(\max_{0\leq r\leq T} W_r - W_t\big) \,\big|\, \mathcal{G}_t\big) \\ &= G\big(\max_{0\leq r\leq t} W_r - W_t\big) + \int_{G(\max_{0\leq r\leq t} W_r - W_t)}^\infty \mathbf{P}\big(\max_{0\leq r\leq T-t} W_r > G^{-1}(u)\big)\, du\ .\end{aligned}$$

Since $\max_{0\leq r\leq T-t} W_r \sim |W_{T-t}|$, the distribution function of $\max_{0\leq r\leq T-t} W_r$ is F_{T-t} given in (1.5), and the proof is complete. □

Remark 1.2. It is only the property of stationary independent increments of (W_t) that is used to prove Proposition 1.1. Thus the result extend to processes (X_t) with stationary independent increments vanishing at zero and the natural filtration $(\mathcal{F}_t^X)$ with $F_{T-t}(y) = \mathbf{P}(\max_{0\leq r\leq T-t} X_r \leq y)$.

Remark 1.3. Note that (B_t) is a $(\mathcal{F}_{t+})$-Brownian motion (see [3]).

2. Prediction in p-mean distance

The first main result of this paper is contained in the next theorem and is the solution of the optimal stopping problem (1.2). The problem is explicitly solved by applying the idea introduced in the first section.

For $p = 1$ the problem (1.2) is trivial since for any stopping time $\tau \in \mathcal{M}_T$ the optional sampling theorem implies that $\mathbf{E}(S_T - B_\tau) = \mathbf{E}(S_T)$. Hence any stopping time $\tau \in \mathcal{M}_T$ is optimal and the case $p = 1$ therefore is excluded from the theorem.

For $a, b \in \mathbb{R}$ let

$$M(a\,,b\,,x) = 1 + \tfrac{a}{b}\,x + \tfrac{1}{2!}\tfrac{a(a+1)}{b(b+1)}\,x^2 + \cdots$$

denote the Kummer confluent hypergeometric function (see [1]). Note that $M'(a\,,b\,,x) = \frac{a}{b}\,M(a+1\,,b+1\,,x)$.

Theorem 2.1. *Consider the optimal stopping problem* (1.2) *where* $(B_t)_{t\geq 0}$ *is a standard Brownian motion vanishing at zero. For* $p > 0$, $p \neq 1$ *and* $T > 0$ *the value function* $V_*^{(p)}(T)$ *is given by*

$$V_*^{(p)}(T) = \frac{e^{z_p^2/2}\,H(z_p)}{M\big(\frac{p+1}{2}\,,\frac{1}{2}\,,\frac{1}{2}\,z_p^2\big)}\,T^{p/2}$$

where z_p *is the unique strictly positive root of the equation*

$$\frac{H'(z)}{H(z)} + z = (p+1)z\,\frac{M\big(\frac{p+3}{2}\,,\frac{3}{2}\,,\frac{1}{2}\,z^2\big)}{M\big(\frac{p+1}{2}\,,\frac{1}{2}\,,\frac{1}{2}\,z^2\big)} \tag{2.1}$$

and $z \mapsto H(z)$ *is given by*

$$H(z) = z^p + 2p\int_z^\infty u^{p-1}\,\big(1 - \Phi(u)\big)\,du \tag{2.2}$$

for $z \geq 0$ *. The optimal stopping time is given by* (*see* Figure 1 *and* 2)

$$\tau_* = \inf \left\{ 0 \leq t \leq T \, : \, S_t - B_t \geq z_p \sqrt{T-t} \right\} \tag{2.3}$$

where (S_t) *is given by* (1.1).

Proof. The proof falls in five steps where the problem is reduced to an equivalent problem that can be solved by known methods from optimal stopping theory.

1. The first step is by applying Brownian scaling and τ/T is a stopping time for the Brownian motion $t \mapsto T^{-1/2} B_{Tt}$ we get

$$\inf_{\tau \in \mathcal{M}_T} \mathbf{E}\big(\max_{0 \leq t \leq T} B_t - B_\tau \big)^p = T^{p/2} \inf_{\tau/T \in \mathcal{M}_1} \mathbf{E}\big(\max_{0 \leq t/T \leq 1} T^{-1/2} B_{T(t/T)} - T^{-1/2} B_{T(\tau/T)} \big)^p$$

and it is clear that

$$V_*^{(p)}(T) = T^{p/2} \, V_*^{(p)}(1) \, . \tag{2.4}$$

In the sequel it is enough to look at $V_*^{(p)}(1)$.

2. By Proposition 1.1 we have that

$$\mathbf{E}\big((S_1 - B_t)^p \, \big| \, \mathcal{F}_{t+} \big) = \widetilde{G}\big(t, S_t - B_t\big)$$

where

$$\widetilde{G}(t,z) = z^p + 2 \int_{z^p}^{\infty} \left(1 - \Phi\left(\frac{u^{1/p}}{\sqrt{1-t}} \right) \right) du = z^p + 2p\,(1-t)^{p/2} \int_{z/\sqrt{1-t}}^{\infty} u^{p-1} \big(1 - \Phi(u)\big) \, du \, .$$

With this representation we obtain

$$V_*^{(p)}(1) = \inf_{\tau \in \mathcal{M}_1} \mathbf{E}\Big(\mathbf{E}\big((S_1 - B_\tau)^p \, \big| \, \mathcal{F}_{\tau_+} \big) \Big) = \inf_{\tau \in \mathcal{M}_1} \mathbf{E}\big(\widetilde{G}\big(\tau, S_\tau - B_\tau\big) \big) \, .$$

As a consequence of Lévy's theorem (see [5]) that $(S_t - B_t)_{t \geq 0} \sim (|B_t|)_{t \geq 0}$ we have

$$\begin{aligned} V_*^{(p)}(1) &= \inf_{\tau \in \mathcal{M}_1} \mathbf{E}\big(\widetilde{G}\big(\tau, |B_\tau|\big) \big) \\ &= \inf_{\tau \in \mathcal{M}_1} \mathbf{E}\Big(|B_\tau|^p + 2p\,(1-\tau)^{p/2} \int_{|B_\tau|/\sqrt{1-\tau}}^{\infty} u^{p-1} \big(1 - \Phi(u)\big) \, du \Big) \, . \end{aligned} \tag{2.5}$$

3. The form of the gain function $\widetilde{G}$ indicates that the method of deterministic time-change (see [4]) can be applied successfully. Let $\sigma_t = 1 - e^{-2t}$ be the time-change and let $(Z_t)_{t \geq 0}$ be the time-changed process given by

$$Z_t = B_{\sigma_t} / \sqrt{1 - \sigma_t} = e^t \, B_{1 - e^{-2t}} \, .$$

By Itô formula we see that (Z_t) is the strong solution of the stochastic differential equation

$$dZ_t = Z_t \, dt + \sqrt{2} \, d\beta_t \tag{2.6}$$

where $(\beta_t)_{t \geq 0}$ is the Brownian motion given by

$$\beta_t = \frac{1}{\sqrt{2}} \int_0^{\sigma_t} \frac{1}{\sqrt{1-u}} \, dB_u = \frac{1}{\sqrt{2}} \int_0^t \frac{1}{\sqrt{1 - \sigma_u}} \, dB_{\sigma_u} \, .$$

For greater details see [2]. Hence (Z_t) is a diffusion with the infinitesimal generator

$$\mathbf{L_Z} = z \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} \tag{2.7}$$

for $z \in \mathbb{R}$. Observe that (Z_t) is a non-recurrent diffusion i.e. $|Z_t| \to \infty$ $\mathbf{P}$-a.s. for $t \to \infty$. Since the time-change $t \mapsto \sigma_t$ is strictly increasing the stopping time τ for (Z_t) (i.e. a $(\mathcal{F}^Z_{t+})$-stopping time where $(\mathcal{F}^Z_t)$ is the natural filter generated by (Z_t)) if and only if $\sigma_\tau \in \mathcal{M}_1$. Hence by the foregoing we obtain that

$$\begin{aligned}\widetilde{G}\big(\sigma_\tau, |B_{\sigma_\tau}|\big) &= |B_{\sigma_\tau}|^p + 2p\,(1-\sigma_\tau)^{p/2} \int_{|B_{\sigma_\tau}|/\sqrt{1-\sigma_\tau}}^\infty u^{p-1}\,\big(1-\Phi(u)\big)\,du \\ &= e^{-p\tau}\bigg(|Z_\tau|^p + 2p\int_{|Z_\tau|}^\infty u^{p-1}\,\big(1-\Phi(u)\big)\,du\bigg) = e^{-p\tau}H(|Z_\tau|)\end{aligned}$$

where $H(\cdot)$ is defined in (2.2). The problem (2.5) reduces to compute

$$V_*^{(p)}(1) = \inf_\sigma \mathbf{E}\Big(e^{-p\sigma}H(|Z_\sigma|)\Big) \tag{2.8}$$

where the infimum is taken over all stopping times σ for (Z_t) . This is a one-dimensional problem that can be solved by well known method from the general theory of optimal stopping.

4. To compute (2.8) is quit straightforward (see e.g. [4]) but for completeness the details are presented here. Therefore we introduce the problem

$$\widetilde{V}_*(z) = \inf_\sigma \mathbf{E}_z\big(e^{-p\sigma}H(|Z_\sigma|)\big) \tag{2.9}$$

for $z \in \mathbb{R}$, where $Z_0 = z$ under $\mathbf{P}_z$ and the infimum is taken as in (2.8). From the general theory of optimal stopping, the stopping time

$$\sigma_* = \inf\{\, t > 0 \,:\, |Z_t| > z_p \,\} \tag{2.10}$$

should be optimal where $z_p > 0$ is the optimal stopping point to be found. By the stochastic differential equation (2.6) and the domain of continuous observation $(-z_p, z_p)$ the value function $\widetilde{V}_*$ should be even.

To compute the value function $z \mapsto \widetilde{V}_*(z)$ and to determine z_p in view of (2.9) and (2.10) it is natural to formulate the following system

$$\mathbf{L_Z} V(z) = p\,V(z) \qquad \text{for } z \in (-z_p, z_p) \tag{2.11}$$

$$V(\pm z_p) = H(z_p) \qquad \text{(instantaneous stopping)} \tag{2.12}$$

$$V'(\pm z_p) = \pm H'(z_p) \quad \text{(smooth fit)} \tag{2.13}$$

with $\mathbf{L_Z}$ in (2.7). The system (2.11)-(2.13) forms a free-boundary (Stefan) problem.

The general solution to (2.11) is given by

$$V(z) = C_1\, e^{-z^2/2}\, M\big(\tfrac{p+1}{2}\,, \tfrac{1}{2}\,, \tfrac{1}{2}\,z^2\big) + C_2\, z\, e^{-z^2/2}\, M\big(\tfrac{p+2}{2}\,, \tfrac{3}{2}\,, \tfrac{1}{2}\,z^2\big)$$

where C_1 and C_2 are unknown constants. Since the value function should be even, we can forget the odd solution. Set $C_2 = 0$ and we have that

$$V(z) = C_1\, e^{-z^2/2} M\big(\tfrac{p+1}{2}\,, \tfrac{1}{2}\,, \tfrac{1}{2}\,z^2\big) \tag{2.14}$$

for some C_1 to be found. The two conditions (2.12) and (2.13) determine z_p and C_1 uniquely. Taking log on both sides of (2.14) and use the conditions (2.12) and (2.13) it is easily verified that z_p is the strictly positive root to equation (2.1) and $C_1 = e^{z_p^2/2}H(z_p)/M\big(\tfrac{p+1}{2}\,, \tfrac{1}{2}\,, \tfrac{1}{2}\,z_p^2\big)$.

Thus we have the following candidate for the value function

$$(2.15)\qquad V(z) = \begin{cases} H(z_p)\, e^{(z_p^2-z^2)/2}\, \dfrac{M\left(\frac{p+1}{2}, \frac{1}{2}, \frac{1}{2} z^2\right)}{M\left(\frac{p+1}{2}, \frac{1}{2}, \frac{1}{2} z_p^2\right)} & \text{if} \quad |z| < z_p \\ H(|z|) & \text{if} \quad |z| \geq z_p \end{cases}$$

and the candidate for the optimal stopping time

$$(2.16)\qquad \sigma_{z_p} = \inf\{\, t > 0 \,:\, |Z_t| \geq z_p \,\}\ .$$

Note that $\sigma_{z_p} < \infty$ $\mathbf{P}$-a.s. since $|Z_t| \to \infty$ $\mathbf{P}$-a.s. for $t \to \infty$.

5. To verify that these candidates indeed are correct, note that $z \mapsto V(z)$ is C^2 everywhere but at $\pm z_p$ it is C^1 . Extend V'' at $\pm z_p$ to any value and note that Lebesgue measure of those $t > 0$ for which $Z_t = \pm z_p$ is zero, we obtain by Itô-Tanaka formula and (2.6) that

$$(2.17)\qquad e^{-pt}\, V(Z_t) = V(z) + M_t + \int_0^t e^{-pu} \Big(\mathbf{L_Z} V(Z_u) - pV(Z_u)\Big)\, du$$

where (M_t) is a continuous local martingale given by

$$M_t = \sqrt{2} \int_0^t e^{-pu}\, V'(Z_u)\, d\beta_u\ .$$

Hence by (2.17) and $\mathbf{L_Z} V(z) - pV(z) \geq 0$ for $|z| \geq z_p$ we get

$$e^{-pt}\, V(Z_t) \geq V(z) + M_t$$

for all $t \geq 0$. Let σ be any stopping time for (Z_t) and choose a localization sequence $\{\gamma_n\}$ of bounded stopping times for (M_t) . Further more $V(z) \leq H(|z|)$ for all z hence we find that

$$\mathbf{E}_z\big(e^{-p(\sigma\wedge\gamma_n)}\, H(Z_{\sigma\wedge\gamma_n})\big) \geq \mathbf{E}_z\big(e^{-p(\sigma\wedge\gamma_n)}\, V(Z_{\sigma\wedge\gamma_n})\big) \geq V(z) + \mathbf{E}_z\big(M_{\sigma\wedge\gamma_n}\big) = V(z)$$

for all $n \geq 1$. Letting $n \to \infty$ and using Fatou's lemma, and then taking infimum over all stopping times we have that

$$(2.18)\qquad V_*(z) \geq V(z)\ .$$

Finally, to prove equality in (2.18) and that the stopping time in (2.16) is optimal it is enough to verify that

$$(2.19)\qquad V(z) = \mathbf{E}_z\big(e^{-p\sigma_{z_p}}\, H(Z_{\sigma_{z_p}})\big)\ .$$

Again from (2.17) we have that

$$e^{-p\sigma_{z_p}}\, H(z_p) = e^{-p\sigma_{z_p}}\, V(Z_{\sigma_{z_p}}) = V(z) + M_{\sigma_{z_p}}$$

and taking expectation we see that

$$V(z) = H(z_p)\, \mathbf{E}_z\big(e^{-p\sigma_{z_p}}\big)$$

because $\mathbf{E}_z\big(M_{\sigma_{z_p}}\big) = 0$. Indeed by Burkholder-Davis-Gundy inequality and that

$$\mathbf{E}_z \sqrt{\int_0^{\sigma_{z_p}} \Big(e^{-pu} V'(Z_u)\Big)^2\, du} \leq C\, \mathbf{E}_z\big(e^{-p\sigma_{z_p}}\big) < \infty$$

it follows that $\mathbf{E}_z\big(M_{\sigma_{z_p}}\big) = 0$ where C is a constant (see also Remark 2.2 below).

Putting all these computation together we have from (2.4), (2.8) and (2.9) that

$$V_*^{(p)}(T) = T^{p/2}\,\widetilde{V}_*(0) = \frac{e^{z_p^2/2}\,H(z_p)}{M\big(\frac{p+1}{2}\,,\frac{1}{2}\,,z_p^2/2\big)}\,T^{p/2}\,.$$

Transforming σ_* from (2.16) back to the original problem we get that τ_* from (2.3) is optimal. □

Remark 2.2. The argument to verify (2.19) extends to a more general setting and leads to the following explicit formulas for the Laplace transform of the stopping time $\sigma_{z_p} = \inf\{\,t > 0 \,:\, |Z_t| \geq z_p\,\}$ from (2.16). For $\lambda > 0$ define the function $l_\lambda(z) = \mathbf{E}(e^{-\lambda\sigma_{z_p}})$. From general Markov process theory gives that $z \mapsto l_\lambda(z)$ solves (2.11) (with $p = \lambda$) and satisfies $l_\lambda(\pm z_p) = 1$. The argument quoted above gives

$$\mathbf{E}_z\big(e^{-\lambda\sigma_{z_p}}\big) = \begin{cases} e^{(z_p^2-z^2)/2}\,\dfrac{M\big(\frac{2\lambda+1}{2}\,,\frac{1}{2}\,,\frac{1}{2}\,z^2\big)}{M\big(\frac{2\lambda+1}{2}\,,\frac{1}{2}\,,\frac{1}{2}\,z_p^2\big)} & \text{if} \quad |z| < z_p \\ 1 & \text{if} \quad |z| \geq z_p \end{cases}$$

for $\lambda > 0$. Since $(|B_t|)_{t\geq 0} \sim (S_t - B_t)_{t\geq 0}$ the stopping time τ_* from (2.3) is identically distributed as the stopping time $\tilde{\tau} = \inf\{\,0 \leq t \leq T \,:\, |B_t| \geq z_p\sqrt{T-t}\,\}$. Together with this observation and by Brownian scaling and the time-change it follows that $\mathbf{E}\big(T-\tau_*\big)^{\lambda/2} = T^{\lambda/2}\,\mathbf{E}_0\big(e^{-\lambda\sigma_{z_p}}\big)$. Thus we immediately get that

$$\mathbf{E}\big(T-\tau_*\big)^{\lambda/2} = \frac{e^{z_p^2/2}}{M\big(\frac{2\lambda+1}{2}\,,\frac{1}{2}\,,\frac{1}{2}\,z_p^2\big)}\,T^{\lambda/2}\,.$$

For the special cases $\lambda = 2$ and $\lambda = 4$ the formula reads as

$$\mathbf{E}\big(T-\tau_*\big) = \frac{T}{1+z_p^2} \quad \text{and} \quad \mathbf{E}\big(T-\tau_*\big)^2 = \frac{T^2}{1+2z_p^2+\frac{1}{3}z_p^4}$$

and it is easy to calculate

$$\mathbf{E}\big(\tau_*\big) = \frac{z_p^2}{1+z_p^2}\,T \quad \text{and} \quad \mathrm{Var}\,\big(\tau_*\big) = \frac{2z_p^4}{(1+z_p^2)^2\,(3+6z_p^2+z_p^4)}\,T^2\,.$$

The expectation and variance of τ_* is also calculated in [2] by a different method.

Example 2.3. It is of some interest to compute the value function in specific cases. Let $p = 1/2$ and $T = 1$. Then $z_p = 0.966\ldots$ is the unique positive root of the equation (2.1) and the value function has the value

$$V_*^{(p)}(1) = \inf_{\tau\in\mathcal{M}_1} \mathbf{E}\sqrt{S_1 - B_\tau} = \frac{e^{z_p^2/2}\,H(z_p)}{M\big(\frac{3}{4}\,,\frac{1}{2}\,,\frac{1}{2}\,z_p^2\big)} = 0.744\ldots$$

The optimal strategy is illustrated in Figure 1 and 2 and we further more have that

$$\mathbf{E}\big(\tau_*\big) = 0.48\ldots \quad \text{and} \quad \mathrm{Var}\,\big(\tau_*\big) = 0.05\ldots$$

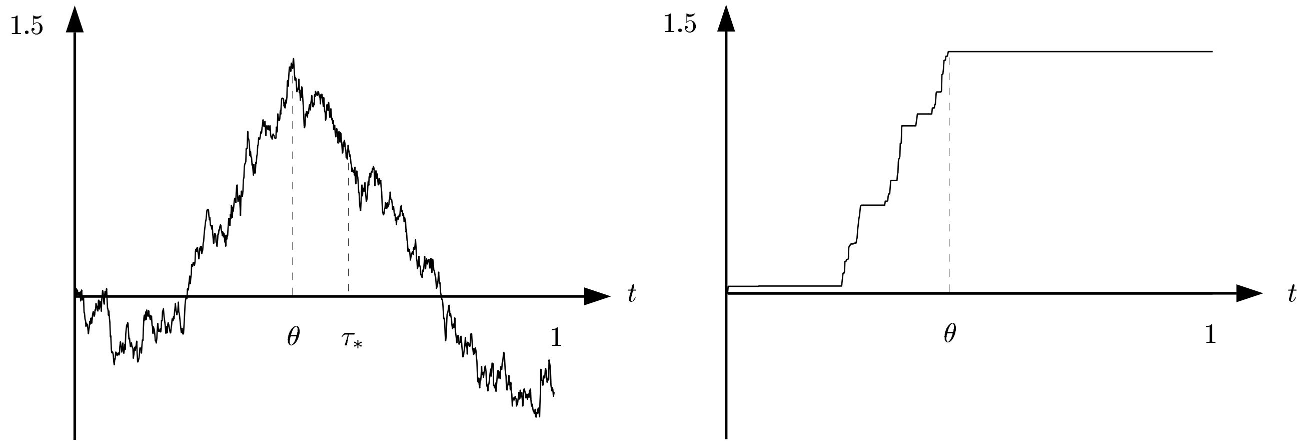

Figure 1. The left drawing is a computer simulation of a Brownian path $B_t(\omega)$ for $t \in [0,1]$. The right drawing is the maximum process $S_t(\omega)$ associated with the Brownian path. The maximum is attained at $\theta = 0.46$ and τ_* is the optimal stopping strategy (2.3) (see Figure 2).

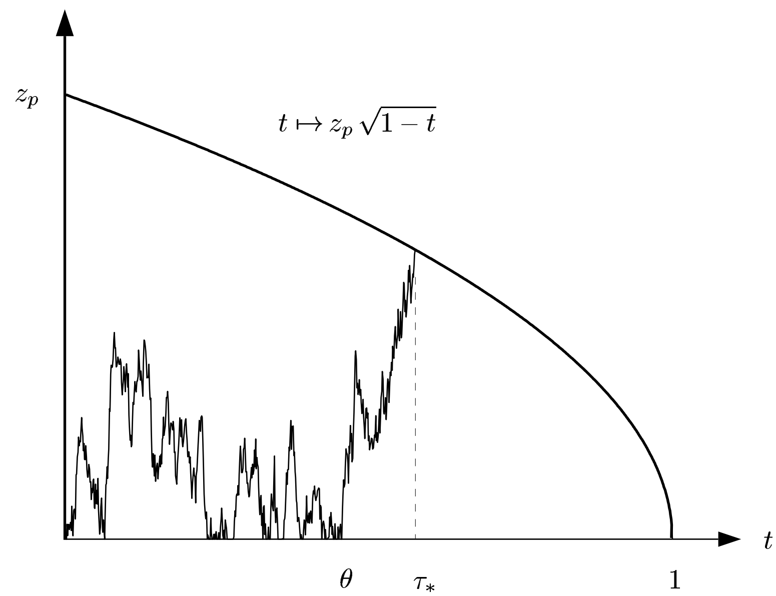

Figure 2. A drawing of the optimal stopping strategy (2.3) for the Brownian path from Figure 1 when $p = 1/2$ and $T = 1$ with $z_p = 0.966\ldots$ In this case $\tau_* = 0.58$

3. Prediction in probability distance

The second main "result" of this paper is a conjecture stated as a theorem below and its proof is reduced to verifying that a value function is not differentiable over a line (see the proof below).

Theorem 3.1. *Consider the optimal stopping problem* (1.3) *where* $(B_t)_{t\geq 0}$ *is a standard Brownian motion vanishing at zero. For* $T > 0$ *and* $\varepsilon > 0$ *the value function* $W_*^{(\varepsilon)}(T)$ *is given by*

$$W_*^{(\varepsilon)}(T) = 1 - 2\int_0^T \Phi\left(\frac{-\varepsilon}{\sqrt{T-y}}\right) f(y)\, dy$$

where

$$f(y) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{(2k+1)\varepsilon}{\sqrt{2\pi}\, y^{3/2}}\, e^{-((2k+1)\varepsilon)^2/2y}$$

for $y > 0$. *The optimal stopping time is given by* (*see* Figure 1 *and* 3)

$$\tau_* = \inf\{\, 0 < t \le T \,:\, S_t - B_t = \varepsilon \,\} \qquad (\inf \emptyset = T) \tag{3.1}$$

where (S_t) *is given by* (1.1).

Proof. The theorem will be proved up to a conjecture. The main idea is to apply Proposition 1.1 as in the proof Theorem 2.1.

1. By applying Brownian scaling as in the proof of Theorem 2.1 we get

$$W_*^{(\varepsilon)}(T) = W_*^{(\varepsilon/\sqrt{T})}(1)$$

and therefore it is enough to look at $W_*^{(\varepsilon)}(1)$.

2. By Proposition 1.1 we have for $0 \le t \le 1$ that

$$\mathbf{E}\Big(\mathbf{1}_{[0,\varepsilon]}(S_1 - B_t) \,\Big|\, \mathcal{F}_{t+}\Big) = \mathbf{1}_{[0,\varepsilon]}(S_t - B_t)\left(2\Phi\left(\frac{\varepsilon}{\sqrt{1-t}}\right) - 1\right) \tag{3.2}$$

where we have used that $\mathbf{P}\big(S_{1-t} \le \varepsilon\big) = 2\Phi\big(\varepsilon/\sqrt{1-t}\big) - 1$. Thus we have that

$$\begin{aligned} W_*^{(\varepsilon)}(1) &= \sup_{\tau \in \mathcal{M}_1} \mathbf{E}\Big(\mathbf{E}\Big(\mathbf{1}_{[0,\varepsilon]}(S_1 - B_\tau) \,\Big|\, \mathcal{F}_{\tau+}\Big)\Big) \\ &= \sup_{\tau \in \mathcal{M}_1} \mathbf{E}\left(\mathbf{1}_{[0,\varepsilon]}(S_\tau - B_\tau)\left(2\Phi\left(\frac{\varepsilon}{\sqrt{1-\tau}}\right) - 1\right)\right) . \end{aligned}$$

Then Lévy's theorem implies as in the proof of Theorem 2.1 that

$$W_*^{(\varepsilon)}(1) = \sup_{\tau \in \mathcal{M}_1} \mathbf{E}\left(\mathbf{1}_{[0,\varepsilon]}(|B_\tau|)\left(2\Phi\left(\frac{\varepsilon}{\sqrt{1-\tau}}\right) - 1\right)\right) . \tag{3.3}$$

3. Let the process (X_t) defined by

$$X_t = B_t + x$$

starts at x under $\mathbf{P}_x$. Denote the stopping time

$$\tau_\varepsilon = \inf\{\, t > 0 \,:\, |X_t| = \varepsilon \,\} \tag{3.4}$$

for $\varepsilon > 0$. Define the optimal stopping problem

$$W(t,x) = \sup_{\tau \in \mathcal{M}_{1-t}} \mathbf{E}_x\left(\mathbf{1}_{[0,\varepsilon]}(|X_\tau|)\left(2\Phi\left(\frac{\varepsilon}{\sqrt{1-t-\tau}}\right) - 1\right)\right)$$

for $0 \le t \le 1$ and $x \ge 0$. Let $\tau \in \mathcal{M}_{1-t}$ be an arbitrary stopping time and define the stopping time

$$\tilde{\tau} = (\tau + \tau_\varepsilon \circ \theta_\tau) \wedge (1-t)$$

where τ_ε is the stopping time from (3.4). It is easily seen that

$$\mathbf{E}_x\left(\mathbf{1}_{[0,\varepsilon]}(|X_\tau|)\left(2\Phi\left(\frac{\varepsilon}{\sqrt{1-t-\tau}}\right) - 1\right)\right) \le \mathbf{E}_x\left(\mathbf{1}_{[0,\varepsilon]}(|X_{\tilde{\tau}}|)\left(2\Phi\left(\frac{\varepsilon}{\sqrt{1-t-\tilde{\tau}}}\right) - 1\right)\right)$$

and the conclusion is that it is only optimal to stop if $|X_\tau| = \varepsilon$ on the set $\{\tau < 1-t\}$. Therefore the optimal stopping is on the form:

$$\tau = \inf\{\, 0 \le s \le 1-t \,:\, |X_s| = \varepsilon\, b_D(t+s) \,\}$$

where D (the domain of stopping) is an open subset of the interval $[0,1]$ and $b(s)=1$ if $s \in D$ and $b(s)=\infty$ elsewhere. Then from Markov process theory it follows that the function $(t,x) \mapsto W_*(t,x)$ is continuous. Furthermore for $x<\varepsilon$ we have that

$$\begin{aligned}
&\mathbf{E}_x\Big(\mathbf{1}_{[0,\varepsilon]}\big(|X_{\tau_\varepsilon \wedge (1-t)}|\big)\Big(2\Phi\Big(\frac{\varepsilon}{\sqrt{1-t-\tau_\varepsilon \wedge (1-t)}}\Big)-1\Big)\Big)\\
&= \mathbf{E}_x\Big(2\Phi\Big(\frac{\varepsilon}{\sqrt{1-t-\tau_\varepsilon \wedge (1-t)}}\Big)-1\,,\ \tau_\varepsilon < 1-t\Big) + \mathbf{P}_x(\tau_\varepsilon \geq 1-t)\\
&= \mathbf{E}_x\Big(\mathbf{P}_0(|X_{1-t-s}| \leq \varepsilon)\Big|_{s=\tau_\varepsilon} < 1-t\Big) + \mathbf{E}_x(1\,,\ \tau_\varepsilon \geq 1-t)\\
&\geq \mathbf{P}_0(|X_{1-t}| \leq \varepsilon) = \mathbf{1}_{[0,\varepsilon]}(x)\Big(2\Phi\Big(\frac{\varepsilon}{\sqrt{1-t}}\Big)-1\Big)
\end{aligned}$$

and the inequality is trivial for $x \geq \varepsilon$. These observations indicate that $\tau_\varepsilon \wedge (1-t)$ is an optimal stopping time, i.e. $D=[0,1]$.

4. If for fixed t the function $x \mapsto W_*(t,x)$ is not differentiable at $x=\varepsilon$ then $D=[0,1]$, or else $W_*(t,\cdot)$ is differentiable where there is a “hole” in D. We conjecture that $x \mapsto W_*(t,x)$ is not differentiable at ε.

5. Note that $f(\cdot)$ is the density of τ_ε under $\mathbf{P}_0$. Thus we have that

$$W(0,0) = \mathbf{E}\Big(2\Phi\Big(\frac{\varepsilon}{\sqrt{1-t}}\Big)-1\,;\tau_\varepsilon<1\Big)\,.$$

Transforming this back to the initial problem we obtain the result. □

Example 3.2. An example of Theorem 3.1 when $T=1$ and $\varepsilon=\frac{1}{2}$. The optimal strategy is illustrated in Figure 1 and 3 and numerical computation gives that

$$W_*^{(1/2)}(1) = \sup_{\tau \in \mathcal{M}_1} \mathbf{P}\big(S_1 - B_\tau \leq \tfrac{1}{2}\big) = 0.756\ldots$$

For comparison note that from (3.2) and Lévy's theorem that

$$\begin{aligned}
W_0^{(\varepsilon)}(T) &= \sup_{0\leq t\leq T} \mathbf{P}\big(S_T - B_t \leq \varepsilon\big) = \mathbf{P}\big(S_t - B_t \leq \varepsilon\big)\Big(2\Phi\Big(\frac{\varepsilon}{\sqrt{T-t}}\Big)-1\Big)\\
&= \Big(2\Phi\Big(\frac{\varepsilon}{\sqrt{t}}\Big)-1\Big)\Big(2\Phi\Big(\frac{\varepsilon}{\sqrt{T-t}}\Big)-1\Big)\,.
\end{aligned}$$

With T and ε given as above we have that $W_0^{(1/2)}(1)=0.386\ldots$ with the supremum being attained at $t=0.027\ldots$

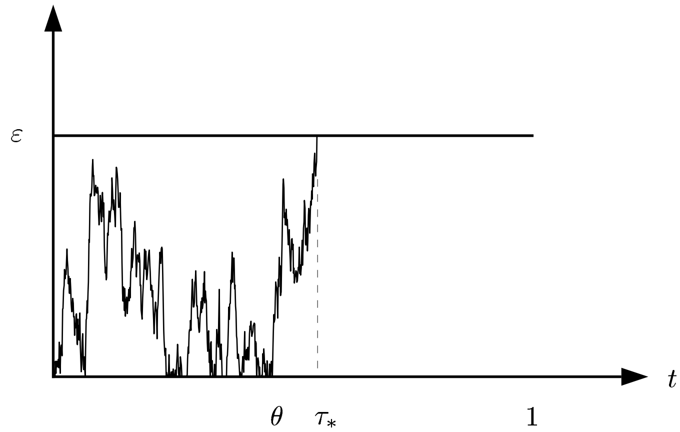

Figure 3. A drawing of the optimal stopping strategy (3.1) for the Brownian path from Figure 1 when $\varepsilon = 1/2$ and $T = 1$. In this case $\tau_* = 0.55$

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