Contents

1 Introduction.

The title of this thesis, "Seiberg-Witten-Floer Homology", should be read as "a Floer-type homology for the Seiberg-Witten equations". In this introduction I will try to give a brief description of the two components: The Seiberg-Witten equations and Floer homology.

The Seiberg-Witten equations on a four-manifold are a pair of partial differential equations for a unitary connection on a line bundle and a section in a two-dimensional vector bundle. The geometric setting of the equations is that of a $spin^c$-structure on the manifold - something which always exists ($[15, 4.1(v)]$, where it is attributed to Whitney). The equations are equivariant with respect to the action of gauge transformations of the $spin^c$-structure and one studies the moduli space of solutions to obtain invariants of four-manifolds. These invariants are very strong. Since Seiberg and Witten wrote down there equations in 1994 [33], there has been a huge research activity among both mathematicians and physicists studying the Seiberg-Witten equations. This has resulted in many new theorems and in easier proofs of known results. So much has been achieved that I won't try to give a list. But let me mention that the advantage of the Seiberg-Witten gauge theory compared to e.g. the older Yang-Mills theory is that the moduli space of solutions turns out to be compact.

In classical Morse theory for a compact manifold M one associates to a given Morse function f a chain complex. One ends up computing the singular homology of the manifold, but this is by no means obvious at first sight: The generators of the j'th chain group is the critical points of f of index j. The boundary operator is obtained as follows: First one proves that the set of flow curves γ connecting critical points p and q in the sense that

$$\lim_{t\to-\infty}\gamma(t)=p, \quad \lim_{t\to\infty}\gamma(t)=q,$$

is a smooth oriented manifold of dimension equal to the difference in index between p and q for a generic metric. There is a natural action of $\mathbb{R}$ on the set of flow curves given by translation of the parametrisation. The quotient by the action is a manifold of dimension one less than the above. Thus if p has index j and q has index $j-1$, one can count, with sign, the finitely many unparametrised flow curves between p and q. This integer is the coefficient of q in the image of the boundary operator on p. It is possible to formulate this Morse homology as an axiomatic homology theory and using only relative terms in the sense that the only part of the flow of ∇f used is the above manifolds of flow curves connecting critical points [29].

It is also possible to define a chain complex for a Morse one-form, that is: A one-form, which in a neighbourhood of a zero looks like the derivate of a Morse function near a critical point. This is done in [27] and gives a complex with coefficients in a so-called Novikov ring, similar to the ring of Laurent series, which computes the homology of a covering of the manifold tensor the Novikov ring.

Floer used the relative approach to define a Morse homology in an infinite dimensional setting in symplectic topology $[16, 6.5]$. Given a symplectic manifold (M, ω)

and a one-periodic smooth function $H : M \times \mathbb{R} \to \mathbb{R}$, the generators of Floer's complex are the one-periodic solutions of the differential equation

$$\dot{x} = X_H(x).$$

They are the critical points of an action functional on the set of smooth, contractible loops in M. It is possible to assign an index to these loops - defined modulo an integer - and this index gives the grading of the chain complex. Given generators x and y whose index differ by one, the (x, y)-matrix element of the boundary operator is defined as follows: Count, with sign, the finitely many solutions of a "flow equation", defined for maps $u : \mathbb{R} \times S^1 \to M$, which converge with respect to the first variable to x and y at each end of the axis, respectively. Of course, this is a very vague formulation and the technical details of this theory are very difficult. Again, the homology is the singular homology of M (shifted by the dimension), what Floer proves by showing that for special choices of H, one gets the above classical Morse homology. This new viewpoint allowed Floer to prove a special case of the Arnold conjecture.

Floer used the same approach to define a homology theory for the Yang-Mills equations [11]. His starting observation is that if the four-manifold is $Y \times \mathbb{R}$, the Yang-Mills equations can be written as a "gradient flow" equation for the Chern-Simons functional on Y. Critical points of this functional, which correspond to flat connections on a principal SU_2-bundle over Y, thus plays the role of generators of the chain complex, whereas solutions to the Yang-Mills equations on $Y \times \mathbb{R}$ converging as the real variable goes to $\pm\infty$ to flat connections gives the "connecting orbits" used in the definition of the boundary operator.

The purpose of this thesis is to define a similar homology theory for the Seiberg-Witten equations, but I attemp to use not only the zero-dimensional set of flow curves between critical points of relative index 1, as it is done in the examples above and for the Seiberg-Witten equations in [21], but also the higher dimensional manifolds of flow curves. This has been considered in [7], but the approach here is somewhat different. As would be expected, the resulting chain complex is quite complicated.

As in Floer's approach to Yang-Mills theory the Seiberg-Witten equations on $Y \times \mathbb{R}$ will be rewritten so as to appear as the gradient flow equation of a functional, which will here be denoted the Seiberg-Witten functional. The critical points of this functional corresponds to the solutions of the Seiberg-Witten equations on a three-manifold. This first appeared in [18]. To obtain information in shape of numbers from the higher dimensional manifolds of flow curves, we integrate powers of a canonical characteristic class stemming from the pointwise evaluation homomorphism from the gauge group to S^1 over the even dimensional manifolds.

The approach in this thesis is also inspired by [27] as we will mostly work on a $\mathbb{Z}$-covering of the moduli space where the Seiberg-Witten functional is well defined. We will though work in terms of, not one-forms, but vector fields.

Here is an outline of the thesis. In section 2 there is a brief account of the results needed from *spin*^{c}-geometry. In section 3 the Seiberg-Witten equations on a four-manifold are introduced. The particular case of $Y \times \mathbb{R}$ for Y a three-manifold leads to the definition of the Seiberg-Witten functional and the Seiberg-Witten equations on a three-manifold.

Basic properties of the moduli space are given in section 4 and here we also prove compactness results for the moduli space of solutions. In section 5 it is proved that the latter is (generically) a finite set of oriented points, and in section 6 we define an index of the critical points of the Seiberg-Witten functional using spectral flows of Fredholm operators.

The next sections, 7 to 11, analyses the solutions of the Seiberg-Witten equations on $Y \times \mathbb{R}$ that converges to critical points of the Seiberg-Witten functional as $t \to \pm\infty$. Section 7 contains a technical setup, whereas it is proved section 8 that (generically) the set of the above solutions is a manifold and simple properties of these manifolds are given. Section 9 justifies the analytical setup used in section 8 and in section 10 we prove a theorem on the (lack of) compactness of the manifolds of flow curves. Section 11 deals with a gluing theorem for the solutions on $Y \times \mathbb{R}$.

Finally, in section 12 we enjoy the fruits of the hard analysis, defining the coefficients of the boundary operator ∂ and showing what turns out to be the condition for $\partial \circ \partial$ to be zero. In section 13 the Seiberg-Witten-Floer complex is then defined and we prove that it has a truncation property, thus presenting it as an inverse limit.

There is also three appendices in section 14 to 16: One on Sobolev spaces and differential operators , one on the Dirac operator and one on temporal gauge.

This ph.d.-thesis is the outcome of my work during the last four years, where I have been a ph.d.-student at the Department of Mathematical Sciences at the University of Aarhus. It has been four hard, but also rewarding years. As my time as a student at the University of Aarhus is now coming to an end, there is some people I want to thank: My advisor, Jørgen Tornehave, from whom I have learnt so much. FFB for good company and those cold Ceres Top. And finally, JAI, Handball, where I have always had more than I have given.

2 *Spin*^{c} Geometry.

The definition and the basic properties of the Seiberg-Witten equations are closely related to the geometry of *spin*^{c}-structures on the manifold. Therefore we will start out by giving the results on this subject needed later on. The first subsection 2.1 is purely algebraic and deals with Clifford algebras, spin groups and representations. After the algebra is digested the geometrical constructions in subsection 2.2 follow very easily.

2.1 Clifford Algebras and Spin Groups.

Let V be a vector space of dimension n over $\mathbb{R}$ with an inner product $<,>$ giving a norm denoted by $\|\cdot\|$. The Clifford algebra of the pair $(V, \|\cdot\|)$ is defined as follows: Consider the tensor algebra

$$T(V) = \sum_{j\geq 0} \bigotimes^j V$$

and let I be the ideal generated by the elements of the algebra of the form

$$v \otimes v + \|v\|^2 \mathbf{1}.$$

Define $Cl(V, \|\cdot\|) := T(V)/I$. This is an associative reel algebra with a unit, which furthermore contains V as a subspace as the natural map turns out to be injective $[8, \mathrm{III}]$. It also has a universal property:

Proposition 2.1 (LM I.1.1). *Let $\phi : V \to A$ be a linear map into an associative algebra over $\mathbb{R}$ with unit $\mathbf{1}$, such that for all $v \in V$:*

$$\phi(v) \cdot \phi(v) = -\|v\|^2 \cdot \mathbf{1}.$$

Then ϕ extends uniquely to a homomorphism of $\mathbb{R}$-algebras

$$\phi : Cl(V, \|\cdot\|) \to A.$$

This property characterizes $Cl(V, \|\cdot\|)$ among associative algebras over $\mathbb{R}$ with a unit and containing V as a subspace.

In terms of an orthonormal basis $(e_1, ..., e_n)$ for V, $Cl(V, \|\cdot\|)$ is the algebra generated by $\{e_1, e_2, \ldots, e_n\}$ and 1 satisfying the relations:

$$e_i e_j = -e_j e_i, \qquad e_j^2 = -1.$$

$Cl(V, \|\cdot\|)$ is spanned as a vectorspace by the linearly independent elements

$$e_{i_1} e_{i_2} \ldots e_{i_r}, i_1 < i_2 < \ldots < i_r, 0 \leq r \leq n,$$

so that it's linear dimension is 2^n. The explicit basis also shows that $Cl(V, \|\cdot\|)$ is naturally linearly isomorphic to $\Lambda^* V$ - the exterior algebra of V - through the map $f : Cl(V, \|\cdot\|) \to \Lambda^* V$ given by:

$$f(v_1 v_2 ... v_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge ... \wedge v_{\sigma(k)}.$$

This map is though not a homomorphism of algebras. Actually, considering the multiplication in $Cl(V, \|\cdot\|)$ - Clifford multiplication - as an alternative multiplication on $\Lambda^* V$, it is given by the expression:

$$v \cdot c = v \wedge c - v \angle c,$$

where $v \in V, c \in \Lambda^* V$ and $v\angle : \Lambda^* V \to \Lambda^{*-1} V$ is given by:

$$v\angle(v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_r}) = \sum_{1 \leq j \leq r} (-1)^{j+1} < v, v_{i_j} > v_{i_1} \wedge \ldots \wedge \hat{v}_{i_j} \wedge \ldots \wedge v_{i_r},$$

where $\hat{}$ denotes deletion.

Given another vector space V' with inner product $<,>'$ and a linear isomorphism from V to V' preserving the inner product, 2.1 gives that there is an isomorphism of algebras $Cl(V, \|\cdot\|) \approx Cl(V', \|\cdot\|')$. This allows us to drop the explicit reference to the norm $\|\cdot\|$ in the notation. Notice that this argument also shows that the group $O(V)$ acts on $Cl(V)$ in a way that extends the usual action on V.

$Cl(V)$ carries a natural $\mathbb{Z}_2$-graduation defined as follows: Let α be the endomorphism of $Cl(V)$ extending the linear map α(v) = -v from V to $Cl(V)$ (Proposition 1.1), and let $Cl_0(V)$ and $Cl_1(V)$ be the plus- and minus-eigenspaces of this idempotent homomorphism. Then

$$Cl(V) = Cl_0(V) \oplus Cl_1(V)$$

and the two subspaces are denoted the even and odd part of $Cl(V)$. As $Cl_i(V) \cdot Cl_j(V) = Cl_{i+j}(V)$ (i+j mod 2), $Cl_0(V)$ is a subalgebra.

Lemma 2.2 (LM. I 3.7). *There is an isomorphism of algebras:*

$$Cl(\mathbb{R}^n) \approx Cl_0(\mathbb{R}^{n+1}).$$

Proof: The map $f : \mathbb{R}^n \to Cl_0(\mathbb{R}^{n+1})$ given by $f(e_i) = e_i e_{n+1}, i = 1 \ldots n$ fits into the scheme of proposition 1.1. For the corresponding homomorphism of algebras, also denoted f, a summation of signs gives

$$f(e_I) = \begin{cases} e_I, & \text{if } \mid I \mid \text{ is even} \\ e_I e_{n+1}, & \text{if } \mid I \mid \text{ is odd.} \end{cases}$$

If follows that f is an isomorphism. $\square$

Definition 2.3. *The spin group $Spin(V)$ is the subgroup of the invertible elements of $Cl(V)$ generated by the unit vectors of V, intersected with $Cl_0(V)$.*

$Spin(V)$ acts on $Cl(V)$ by conjugation:

$$Ad_v(c) = vcv^{-1}, v \in Spin(V), c \in Cl(V).$$

This (twisted) adjoint representation of $Spin(V)$ preserves $V \subset Cl(V)$, as is seen from the following calculation:

$$Ad_v(w) = vwv^{-1} = vw(-v) = v(vw + 2 < v, w >) = -w + 2 < v, w > v,$$

where $v, w \in V, \|v\| = 1$. Ad_v is also orthogonal:

$$\|Ad_v(w)\|^2 = -Ad_v(w)^2 = -Ad_v(w^2) = Ad_v(\|w\|^2) = \|w\|^2.$$

This means that Ad restricts to an orthogonal representation of $Spin(V)$ on V, that is: there is a homomorphism of $Spin(V)$ to $O(V)$, whose image is actually in $SO(V)$ as $Spin(V) \subseteq Cl_0(V)$.

Proposition 2.4 (LM. I 2.10). *There is a short exact sequence:*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(V) \xrightarrow{Ad} SO(V) \longrightarrow 0.$$

This proposition implies that $Spin(V)$ is a double-covering of $SO(V)$ and for $n \geq 2$, it is nontrivial. For n$\geq$3, $\pi_1(SO(V)) = \mathbb{Z}_2$ and by general theory of covering spaces, $Spin(V)$ is then the universal covering of $SO(V)$, which in particular means that it is simply connected. Furthermore $Spin(V)$ can be given the structure of a compact Lie group, so that Ad is a local diffeomorphism. The dimension of $Spin(V)$ then equals the dimension of $SO(V)$, which is $\frac{1}{2}$n(n-1), and the Lie algebra of $Spin(V)$ is isomorphic to $\mathfrak{so}(V)$ through the differential of the homomorphism.

Definition 2.5. *The complex spin group $Spin^c(V)$ is the subgroup of the invertible elements of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by $Spin(V)$ and S^1.*

Some facts about $Spin^c(V)$: First of all there is an isomorphism

$$Spin^c(V) \approx Spin(V) \times_{\pm 1} S^1.$$

It follows that $Spin^c(V)$ is a compact Lie group of dimension $\frac{1}{2}$n(n-1) + 1 with a Lie algebra isomorphic to $\mathfrak{so}(V) \oplus i\mathbb{R}$. There are homomorphisms

$$Ad : Spin^c(V) \to SO(V) \quad \text{and} \quad sq : Spin^c(V) \to S^1$$

given by:

$$Ad([g, \lambda]) = Ad(g), \quad sq([g, \lambda]) = \lambda^2.$$

These combine to give a double-covering $Ad \times sq$ from $Spin^c(V)$ to $SO(V) \times S^1$ which is non-trivial on each factor:

$$0 \to \mathbb{Z}_2 \to Spin^c(V) \stackrel{Ad \times sq}{\to} SO(V) \times S^1 \to 0.$$

The Clifford algebras considered here are completely classified and turn out to be direct sums of matrix algebras over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. The first few of them are:

$$Cl(\mathbb{R}) = \mathbb{C}, \qquad Cl(\mathbb{R}^2) = \mathbb{H}, \qquad Cl(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}, \qquad Cl(\mathbb{R}^4) = \mathbb{H}[2],$$

with corresponding complexified versions:

$$Cl(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}, \qquad Cl(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[2],$$
$$Cl(\mathbb{R}^3) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[2] \oplus \mathbb{C}[2], \qquad Cl(\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[4].$$

It holds in general that $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ is a matrix algebra for n even and a sum of two equal matrix algebras for n odd. This is due to the above facts and the isomorphism $Cl(\mathbb{R}^{n+2}) \otimes_{\mathbb{R}} \mathbb{C} \approx (Cl(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (Cl(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C})$ [19, I.4.3].

Explicitly the low dimensional $Spin$ and $Spin^c$ groups are:

$$Spin(1) \approx \{\pm 1\}, Spin(2) \approx S^1, Spin(3) \approx SU_2, Spin(4) \approx SU_2 \times SU_2$$

and

$$Spin^c(1) \approx S^1, Spin^c(2) \approx S^1 \times_{\pm 1} S^1, Spin^c(3) \approx U_2,$$
$$Spin^c(4) \approx \{(g, h) \in U_2 \times U_2 | det(g) = det(h)\}.$$

Assume now that the vector space V is oriented and let $e_1, e_2, \ldots, e_n$ be a positively oriented orthonormal basis for V. Then the complex volume element is

$$\omega_{\mathbb{C}} := i^{[\frac{n+1}{2}]} e_1 e_2 \ldots e_n,$$

where [t] denotes the integer part of t. Because of the demand of positive orientation, this is independent of the choice of basis. $\omega_{\mathbb{C}}$ is central in $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ if n is odd, and if n is even $\omega_{\mathbb{C}}$ commutes with elements of $Cl_0(V)$ and anticommutes with elements of $Cl_1(V)$. Because of the frontfactor, $\omega_{\mathbb{C}}^2 = 1$ in all dimensions [19, I.§5].

Considering the idempotent action of $\omega_{\mathbb{C}}$ on $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ by left-multiplication, one obtains a splitting of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ in ± 1-eigenspaces, denoted $(Cl(V) \otimes_{\mathbb{R}} \mathbb{C})^{\pm}$

Now for the important results on representations of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ (as an algebra!):

Proposition 2.6 (LM. I 5.9 + 5.10). *Let $\rho : Cl(V) \otimes_{\mathbb{R}} \mathbb{C} \to Hom_{\mathbb{C}}(W, W)$ be any irreducible, complex representation. If n is odd then either*

$$\rho(\omega_{\mathbb{C}}) = Id \quad or \quad \rho(\omega_{\mathbb{C}}) = -Id.$$

Both possibilities can occur and the corresponding representations are inequivalent. They are exactly the irreducible representations of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ and are denoted +

and - ,respectively. Their dimension is $2^{\frac{n-1}{2}}$.
If n is even, there is a splitting

$$W = W^+ \oplus W^-,$$

where $W^\pm = (\mathbf{1} \pm \rho(\omega_\mathbb{C}))(W)$. *Each of the subspaces* W^+ *and* W^- *is invariant under the even subalgebra* $Cl_0(V) \otimes_\mathbb{R} \mathbb{C}$ *and under the isomorphism from 2.2:*

$$Cl_0(V^n) \otimes_\mathbb{R} \mathbb{C} \approx Cl(V^{n-1}) \otimes_\mathbb{R} \mathbb{C}$$

they correspond to the two inequivalent irreducible complex representations of $Cl(V^{n-1}) \otimes_\mathbb{R} \mathbb{C}$. *The odd subspace* $Cl_1(V)$ *switches the factors of* W *and the dimension of the representation is* 2^n.

Addition to proposition: For n odd: $Cl_0(V) \otimes_\mathbb{R} \mathbb{C} \approx End(W^\pm)$ and for n even: $Cl(V) \otimes_\mathbb{R} \mathbb{C} \approx End(W)$. Actually, in the latter case one has the following isomorphisms:

$$(Cl_0(V) \otimes_\mathbb{R} \mathbb{C})^+ \approx End(W^+), (Cl_0(V) \otimes_\mathbb{R} \mathbb{C})^- \approx End(W^-),$$
$$(Cl_1(V) \otimes_\mathbb{R} \mathbb{C})^+ \approx Hom(W^+, W^-), (Cl_1(V) \otimes_\mathbb{R} \mathbb{C})^- \approx Hom(W^-, W^+).$$

Definition 2.7. *The complex spin representation of* $Spin^c(V)$ *is the homomorphism*

$$\Delta_n : Spin^c(V) \to Gl_\mathbb{C}(W)$$

given by restricting an irreducible representation of $Cl(V) \otimes_\mathbb{R} \mathbb{C}$ *to* $Spin^c(V) \subseteq Cl_0(V) \otimes_\mathbb{R} \mathbb{C} \subseteq Cl(V) \otimes_\mathbb{R} \mathbb{C}$.

Notice that an irreducible representation of $Cl_0(V) \otimes_\mathbb{R} \mathbb{C}$ restricts to an irreducible representation of $Spin^c(V)$, because $Cl_0(V) \otimes_\mathbb{R} \mathbb{C}$ is the linear span of $Spin^c(V)$. Using this it is not difficult to see:

Proposition 2.8 (LM. I 5.15). *When n is odd, the definition of* Δ_n *is independent of which irreducible representation of* $Cl(V) \otimes_\mathbb{R} \mathbb{C}$ *is used. Furthermore,* Δ_n *is an irreducible representation.*
When n is even, there is a decomposition

$$\Delta_n = \Delta_n^+ \oplus \Delta_n^-$$

into a direct sum of two inequivalent irreducible complex representations of $Spin^c(V)$.

Remark: The above representations of $Spin^c(V)$ are not induced by complex representations of $SO(V)$ as $-\mathbf{1}$ acts as -Id on W because Δ_n is the restriction of a representation of an algebra.

Lemma 2.9 (LM I 5.16). *Let* $\rho : Cl(V) \otimes_\mathbb{R} \mathbb{C} \to End_\mathbb{C}(W', W')$ *be a complex representation of* $Cl(V) \otimes_\mathbb{R} \mathbb{C}$. *There exists an hermitian inner product on* W', *such that the action - Clifford multiplication - of unit vectors in* V *is unitary, that is:*

$$< e \cdot w, e \cdot w' > = < w, w' >,$$

for $w, w' \in W', e \in V, \|e\| = 1$. *Additionally, Clifford multiplication by any element of* V *is a skew hermitian linear transformation of* W'.

Note that if n is even and $W = W^+ \oplus W^-$ is the irreducible representation, $<,>$ may be chosen so that $W^+ \perp W^-$. Also under the isomorphism of the representation spaces in proposition 1.7, $<,>$ can be reused in dimension n-1 as $< e_i e_n \cdot v, e_i e_n \cdot w > = < v, w >$.

The Grassmann inner product on $\bigwedge^* V$ gives an inner product on $Cl(V)$ via the linear isomorphism stated above. Alternatively, in the case where n is even, the map to $End(W)$ is injective and we can pull back the inner product on $End(W)$ given by $< A, B > = tr(AB^*)$. As

$$\begin{aligned}(e_{i_1} e_{i_2} \dots e_{i_p})^* &= (-e_{i_p}) \dots (-e_{i_2})(-e_{i_1}) \\ &= (-1)^{\frac{1}{2}p(p+1)} e_{i_1} e_{i_2} \dots e_{i_p},\end{aligned}$$

one gets:

$$\begin{aligned}tr(e_{i_1} e_{i_2} \dots e_{i_p} (e_{i_1} e_{i_2} \dots e_{i_p})^*) &= tr(e_{i_1} e_{i_2} \dots e_{i_p} (-1)^p e_{i_p} \dots e_{i_2} e_{i_1}) \\ &= tr(Id) = dim(W).\end{aligned}$$

So the normalized trace inner product $< A, B >' = \frac{1}{dim(W)} tr(AB^*)$ equals the Grassmann inner product.

On $\bigwedge^* V$ there is the Hodge operator $* : \bigwedge^* V \to \bigwedge^{*-p} V$ defined by:

$$< *\omega, \tau > vol = \omega \wedge \tau,$$

where $vol = e_1 e_2 \dots e_n$. By [19, II(5.35)]

$$*\phi = (-1)^{\frac{1}{2}p(p+1)} \phi \cdot \omega_{\mathbb{C}}$$

in terms of Clifford multiplication.

We have now stated the algebraic results needed. The next subjects are $spin^c$ and Clifford bundles, connections and the Dirac operator.

2.2 $Spin^c$ and Clifford Bundles.

From this subsection on the real and finite dimensional vector space V equiped with an inner product $<,>$ and a choice of orientation will be the standard case of $\mathbb{R}^n$ with the usual inner product and orientation. Also it is implicit that everything is smooth (maps, bundles, manifolds etc.), unless stated otherwise.

Let M be a n-dimensional manifold. Given a $SO(n)$-bundle $P_{SO(n)}$ over M, there is an associated bundle of Clifford-algebras

$$Cl(P_{SO(n)}) := P_{SO(n)} \times_{SO(n)} Cl(\mathbb{R}^n).$$

The Clifford-bundle contains as a subbundle the metric vector bundle over M corresponding to $P_{SO(n)}$, $\xi = P_{SO(n)} \times_{SO(n)} \mathbb{R}^n$, and is linearly isometric to the exterior algebra of ξ, $\bigwedge^*(\xi)$.

There is a decomposition of bundles of algebras:

$$Cl(P_{SO(n)}) = Cl_0(P_{SO(n)}) \oplus Cl_1(P_{SO(n)}).$$

As the element $\omega_{\mathbb{C}} \in Cl(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ is invariant under the above action of $SO(n)$, there is a section in $Cl(P_{SO(n)}) \otimes_{\mathbb{R}} \mathbb{C}$, also denoted $\omega_{\mathbb{C}}$, that in every fiber equals the algebraic $\omega_{\mathbb{C}}$. This implies that $\omega_{\mathbb{C}}^2 = 1$ (1 of course also defines a section) and so the splitting in $\pm$-subspaces survives the bundle construction:

$$Cl(P_{SO(n)}) \otimes_{\mathbb{R}} \mathbb{C} = (Cl(P_{SO(n)}) \otimes_{\mathbb{R}} \mathbb{C})^+ \oplus (Cl(P_{SO(n)}) \otimes_{\mathbb{R}} \mathbb{C})^-.$$

Again, if n is odd the two subbundles are subbundles of algebras.

Definition 2.10. *A $spin^c(n)$-structure for $P_{SO(n)}$ is a reduction of $P_{SO(n)}$ to a $Spin^c(n)$-bundle. Precisely, it is a $Spin^c(n)$-bundle $P_{Spin^c(n)}$ and a map*

$$\phi : P_{Spin^c(n)} \to P_{SO(n)}$$

commuting with the projections to M and fullfilling

$$\phi(p \cdot g) = \phi(p) \cdot Ad(g)$$

for $p \in P_{Spin^c(n)}$ and $g \in Spin^c(n)$.

Assume that a $spin^c(n)$-structure for $P_{SO(n)}$ exists. This additional structure produces more associated bundles: Define a complex line bundle - the determinant line bundle - by

$$\mathcal{L} = P_{Spin^c(n)} \times_{Spin^c(n)} \mathbb{C},$$

where $Spin^c(n)$ acts on $\mathbb{C}$ through the homomorphism sq: $[g, \lambda] \cdot v = \lambda^2 v$. As the action is unitary, the standard hermitian metric on $\mathbb{C}$ induces an hermitian metric on $\mathcal{L}$. Let P_{S^1} denote the S^1 bundle of unitary linear frames of $\mathcal{L}$. As

$$P_{S^1} \approx P_{Spin^c(n)} \times_{Spin^c(n)} S^1,$$

$P_{Spin^c(n)}$ is a double-covering of the bundle $P_{SO(n)} + P_{S^1}$. This covering is non-trivial on each fiber where it is represented by the covering-homomorphism following definition 2.5.

The $spin^c$ representations give rise to complex vector bundles - the $spin^c$-bundles - over M:

$$S_{\mathbb{C}}(P_{Spin^c(n)}) = P_{Spin^c(n)} \times_{\Delta_n} S_{\mathbb{C}}(\mathbb{R}^n).$$

Again algebra is useful: If n is even, $S_{\mathbb{C}}(P_{Spin^c(n)})$ splits into two subbundles:

$$S_{\mathbb{C}}(P_{Spin^c(n)}) = S^+_{\mathbb{C}}(P_{Spin^c(n)}) \oplus S^-_{\mathbb{C}}(P_{Spin^c(n)}).$$

As the above representations of the $spin^c$ groups were restrictions of representations of $Cl(\mathbb{R}^n)$, there is an action of the Clifford bundle on $S_{\mathbb{C}}(P_{Spin^c(n)})$:

$$c : (Cl(P_{SO(n)}) \otimes_{\mathbb{R}} \mathbb{C}) \otimes S_{\mathbb{C}}(P_{Spin^c(n)}) \to S_{\mathbb{C}}(P_{Spin^c(n)})$$

that is the original Clifford multiplication in each fiber. The hermitian inner product from lemma 2.9 gives an hermitian inner product on $S_{\mathbb{C}}(P_{Spin^c(n)})$, because the $spin^c$ group acts isometrically. Also any unit vector in $Cl(P_{SO(n)})$ acts as an isometry.

Connections on $P_{SO(n)}$ and P_{S^1} with connection one-forms ω and A, respectively, give a connection on $P_{SO(n)} + P_{S^1}$. This connection lifts to a connection $\widetilde{\omega}$ on the double-covering $P_{Spin^c(n)}$ with connection one-form $(Ad_* + sq_*)^{-1} \circ \pi^*(\omega + A)$.

All the associated bundles get induced connections: There is a connection ∇ on $Cl(P_{SO(n)})$ induced from ω on $P_{SO(n)}$. ∇ is a metric connection, as $SO(n)$ acts orthogonally on $Cl(\mathbb{R}^n)$. Also, we have a connection $\widetilde{\nabla}$ on $S_{\mathbb{C}}(P_{Spin^c(n)})$, which is unitary because $Spin^c(n)$ acts by unitary transformations on $S_{\mathbb{C}}(\mathbb{R}^n)$.

∇ fits together with the algebra structure of $Cl(P_{SO(n)})$, as it acts as a derivation on sections:

$$\nabla(\lambda_1\lambda_2) = \nabla(\lambda_1)\lambda_2 + \lambda_1\nabla(\lambda_2),$$

for $\lambda_1, \lambda_2 \in \Omega^0(Cl(P_{SO(n)}))$. Similarly, $\widetilde{\nabla}$ and ∇ are compatible with the action of $Cl(P_{SO(n)})$ on $S_{\mathbb{C}}(P_{Spin^c(n)})$:

$$\widetilde{\nabla}(\lambda \cdot s) = \nabla(\lambda) \cdot s + \lambda \cdot \widetilde{\nabla}(s),$$

for $\lambda \in \Omega^0(Cl(P_{SO(n)}))$ and s $\in \Omega^0(S_{\mathbb{C}}(P_{Spin^c(n)}))$.

As regards the splitting $Cl(P_{SO(n)}) = Cl_0(P_{SO(n)}) \oplus Cl_1(P_{SO(n)})$, it is invariant under ∇. As the section $\omega_{\mathbb{C}}$ is parallel, that is $\nabla_{\mathbb{C}}(\omega_{\mathbb{C}}) = 0$, this is also the case for $(Cl(P_{SO(n)}) \otimes \mathbb{C})^{\pm}$. Finally, in the case where n is even, $S_{\mathbb{C}}(P_{Spin^c(n)}) = S^+_{\mathbb{C}}(P_{Spin^c(n)}) \oplus S^-_{\mathbb{C}}(P_{Spin^c(n)})$ also has this property, now with respect to $\widetilde{\nabla}$.

Definition 2.11. *Two $Spin^c(n)$-structures $P^1_{Spin^c(n)}$ and $P^2_{Spin^c(n)}$ are equivalent, if there is an equivariant map $\varphi : P^1_{Spin^c(n)} \to P^2_{Spin^c(n)}$ commuting with the projections to $P_{SO(n)}$.*

$$\begin{tikzcd}
P^1_{Spin^c(n)} \arrow[rr, "\varphi"] \arrow[dr] & & P^2_{Spin^c(n)} \arrow[dl] \\
& P_{SO(n)} \arrow[d] & \\
& M &
\end{tikzcd}$$

In particular, φ will be an isomorphism of S^1-bundles over $P_{SO(n)}$. So defining $L := P_{Spin^c(n)} \times_{S^1} \mathbb{C}$, a complex line bundle over $P_{SO(n)}$ with the property that $L^2 = \pi^*(\mathcal{L})$, the equivalence of $P^1_{Spin^c(n)}$ and $P^2_{Spin^c(n)}$ implies that $L_1 \approx L_2$, so that $L_1 \otimes L_2^*$ is trivial.

Lemma 2.12. *There is a free and transitive action of $H^2(M,\mathbb{Z})$ on the equivalence classes of $Spin^c(n)$-structures of $P_{SO(n)}$.*

The action is defined as follows: Given $\alpha \in H^2(M,\mathbb{Z})$, let λ_α denote the corresponding complex line bundle over M. Then

$$P_{Spin^c(n)} \cdot \alpha = P_{Spin^c(n)} \times_m P_{S^1}(\lambda_\alpha),$$

where the last expression is to be understood as the $Spin^c(n)$-bundle obtained by multiplying the transition maps of $P_{Spin^c(n)}$ and $P_{S^1}(\lambda_\alpha)$ stemming from a covering of M by neighbourhoods over which both bundles are trivial. The lemma is proved by noting that $L_1 \otimes L_2^*$ is always the pullback of a complex line bundle L_{12} over M and L_{12} is easily seen to satisfy $P^1_{Spin^c(n)} \cdot L_{12} = P^2_{Spin^c(n)}$. Thus $P^1_{Spin^c(n)}$ is equivalent to $P^2_{Spin^c(n)}$ iff L_{12} is trivial. If $P^2_{Spin^c(n)} = P^1_{Spin^c(n)} \cdot \lambda_\alpha$, $L_{12} = \lambda_\alpha$ and thus the action is free.

Here is how the action of $H^2(M,\mathbb{Z})$ affects the bundles associated to a $Spin^c(n)$-structure: For $\alpha \in H^2(M,\mathbb{Z})$,

$$L(P_{Spin^c(n)} \cdot \alpha) = L(P_{Spin^c(n)}) \otimes \pi^*(\lambda_\alpha),$$

whereas

$$\mathcal{L}(P_{Spin^c(n)} \cdot \alpha) = \mathcal{L}(P_{Spin^c(n)}) \otimes \lambda_\alpha^2.$$

Also the representation bundle is transformed in this way:

$$S_{\mathbb{C}}(P_{Spin^c(n)} \cdot \alpha) = S_{\mathbb{C}}(P_{Spin^c(n)}) \otimes \lambda_\alpha^2,$$

where Clifford multiplication takes place in the first factor.

Given connections A_α on $P_{S^1}(\lambda_\alpha)$ and $\widetilde{\omega}$ on P_{Spin^c}, the bundle map

$$m : P_{Spin^c(n)} + P_{S^1}(\lambda_\alpha) \to P_{Spin^c(n)} \cdot \alpha$$

induces a unique connection on $P_{Spin^c(n)} \cdot \alpha$ denoted $\widetilde{\omega} \cdot \alpha$. Similarly, there is an induced connection on $P_{S^1}(\mathcal{L}(P_{Spin^c(n)}) \otimes \lambda_\alpha^2)$ denoted by $A \otimes A_\alpha^2$ and these two connections correspond under the construction of connections on $Spin^c(n)$-bundles described above. The induced connection on $S_{\mathbb{C}}(P_{Spin^c(n)} \cdot \alpha)$ is the tensor product connection $\widetilde{\nabla} \otimes A_\alpha$.

The notion of a $Spin^c(n)$-structure will only be used in the speciel case where $P_{SO(n)}$ is the orthogonal frame bundle of a n-dimensional oriented Riemannian manifold M without boundary and the connection on $P_{SO(n)}$ is the Levi-Cevita connection. The connections on $P_{Spin^c(n)}$ are called the spin connections in this case. As noted in the beginning of this subsection $TM \hookrightarrow Cl(P_{SO(n)})$ and thus, identifying TM and T^*M by means of the Riemannian metric, there is an action of differential forms on the *spin*^{c} bundles $S_{\mathbb{C}}(P_{Spin^c(n)})$. Explicitly:

$$\omega \cdot s(x) = \sum_{1 \le i \le n} \omega(e_i) e_i \cdot s(x),$$

where $\omega \in \Omega^1(TM)$, $s \in \Omega^0(S_{\mathbb{C}}(P_{Spin^c(n)}))$ and $(e_i)_{i=1..n}$ is an orthonormal basis of T_xM. Also notice that one may recover the Levi-Cevita connection from ∇ on $Cl(P_{SO(n)})$ by restricting ∇ to TM.

In the above context, the connection A on P_{S^1} induces a differential operator on $S_{\mathbb{C}}(P_{Spin^c(n)})$, the Dirac operator:

Definition 2.13. *The Dirac operator*

$$\eth_A : \Omega^0(S_{\mathbb{C}}(P_{Spin^c(n)})) \to \Omega^0(S_{\mathbb{C}}(P_{Spin^c(n)}))$$

is defined as $\eth_A = c \circ \widetilde{\nabla}$. *In local terms it is given by:*

$$\eth_A(s)(x) = \sum_{1 \leq i \leq n} e_i \cdot \widetilde{\nabla}_{e_i}(s)(x),$$

where $s \in \Omega^0(S_{\mathbb{C}}(P_{Spin^c(n)}))$ *and* $(e_i)_{i=1..n}$ *is an orthonormal basis of* T_xM.

The Dirac operator has the following properties:

Proposition 2.14. *The Dirac operator is selfadjoint and elliptic [19, I.§5]. Furthermore, it has the unique continuation property from open subsets [5].*

Lemma 2.15 (Mor. II 3.2). *Given a connection* $A' = A + \alpha$, $\alpha \in \Omega^1(M, i\mathbb{R})$, *the Dirac operators satisfy:*

$$\eth_{A'}(s) = \eth_A(s) + \frac{1}{2}\alpha \cdot s.$$

Under the action of $H^2(M, \mathbb{Z})$ the Dirac operator behaves as follows:

$$\eth_{A \otimes A_\alpha} = \eth_A \otimes 1 + c(1 \otimes \nabla^{A_\alpha}).$$

3 The Seiberg-Witten Equations.

The main object of this thesis is the Seiberg-Witten equations on a three-manifold. However, as is well known, this type of equations were first considered on four-manifolds, and the three-dimensional case and the idea of defining a Floer type homology theory in the latter situation originated from properties of the equations in dimension four. So, for motivation and notation, this is where we will start.

Let M be an orientable and Riemannian four-dimensional manifold with a *spin*^{c}-structure P_{Spin^c}. To define the Seiberg-Witten equations we construct a map

$$q : \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c})) \to i\Omega^2_+(M).$$

A section ψ of the spinor bundle $S_{\mathbb{C}}(P_{Spin^c})$ gives, using the hermitian inner product of the bundle, a section ψ^* of the dual $S_{\mathbb{C}}(P_{Spin^c})^*$.

$$\psi^* \otimes \psi \in \Omega^0(S_{\mathbb{C}}(P_{Spin^c}) \otimes S_{\mathbb{C}}(P_{Spin^c})^*) \approx \Omega^0(End(S_{\mathbb{C}}(P_{Spin^c})))$$

and $q(\psi)$ is defined as:

$$q(\psi) = \psi^* \otimes \psi - \frac{1}{2}|\psi|^2 I.$$

If $\psi \in \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))$ then clearly $q(\psi)$ is an endomorphism of the plus-spinor bundle. Clifford multiplication identifies these endomorphisms with $(Cl_0(P_{SO(4)}) \otimes_{\mathbb{R}} \mathbb{C})^+$ (prop. 2.6f). Using the formula in the previous section connecting the Hodge operator and Clifford multiplication by the complex volume form, it is easy to see that under the isomorphism $Cl(P_{SO(4)} \otimes_{\mathbb{R}} \mathbb{C}) \approx \Lambda^*(M, \mathbb{C})$, $(Cl_0(P_{SO(4)}) \otimes_{\mathbb{R}} \mathbb{C})^+$ goes to $\Lambda^2_+(M, \mathbb{C}) \oplus \mathbb{C}(\frac{1}{2}(1 + \omega_{\mathbb{C}}))$. As $q(\psi)$ is trace free it is in the first factor. Finally, as $q(\psi)$ is also selfadjoint, $iq(\psi)$ is pointwise in $\mathbb{H} \subset \mathbb{C}[2]$ that corresponds to the real Clifford algebra. So, $q(\psi)$ is purely imaginary.

Definition 3.1. *For a unitary connection A on $\mathcal{L}_M$ and $\psi \in \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))$ the perturbated Seiberg-Witten equations are:*

$$F^+_A = \frac{1}{2}q(\psi) + h,$$
$$\eth_A(\psi) = 0,$$

where $h \in i\Omega^2_+(M)$ is the perturbation.

Let $\mathcal{A}(\mathcal{L}_M)$ denote the affine space of unitary connections on $\mathcal{L}_M$. The Seiberg-Witten equations can be collected in a single map, the Seiberg-Witten map:

$$F : \mathcal{C}(P_{Spin^c}) := \mathcal{A}(\mathcal{L}_M) \times \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c})) \to i\Omega^2_+(M) \oplus \Omega^0(S^-_{\mathbb{C}}(P_{Spin^c}))$$
$$F(A, \psi) := (F^+_A - \frac{1}{2}q(\psi) - h, \eth_A(\psi))$$

There is a symmetry of the Seiberg-Witten equations connected with the action of gauge transformations of P_{Spin^c} on $\mathcal{C}(P_{Spin^c})$:

Definition 3.2. *A gauge transformation on a principal G-bundle Q is an equivariant map $\varphi : Q \to Q$, that satisfies:*

$$\varphi(q) = q \cdot \vartheta(q),$$

where $\vartheta : Q \to G$.

$$\begin{tikzcd} Q \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Q \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

Equivariance of φ is equivalent to ϑ satisfying:

$$\vartheta(q \cdot g) = g^{-1} \cdot \vartheta(q) \cdot g$$

and so a gauge transformation could also be defined as a section of the bundle of groups $Q \times_G G$.

In this context one restricts the set of gauge transformations of the given $Spin^c$-bundle $P_{Spin^c} \xrightarrow{\pi} M$. φ is demanded to commute with the map ϕ to the $SO(4)$ frame bundle of M, $P_{SO(4)}$: $\phi \circ \varphi = \phi$. Such gaugetransformations φ correspond exactly to the maps ϑ with values in the center S^1 of $Spin^c(4)$, which because of the above property are maps from M to S^1. They can also be defined as gauge transformations of the P_{S^1}-bundle associated to the $Spin^c$-structure, lifted to P_{Spin^c}. The set of these special gauge transformations will be denoted $\mathcal{G}_M$.

$$\begin{tikzcd} P_{Spin^c} \arrow[rr, "\varphi"] \arrow[dr, "\phi"'] & & P_{Spin^c} \arrow[dl, "\phi"] \\ & P \arrow[d, "\pi"'] & \\ & M & \end{tikzcd}$$

There is a right action of $\mathcal{G}_M$ on the variables of the Seiberg-Witten map F, using the following two maps, where $\sigma \in \mathcal{G}_M$:

$$det(\sigma) : \mathcal{L}_M \to \mathcal{L}_M,$$
$$det(\sigma)([p, v]) := [\sigma(p), v] = [p, \sigma(\widetilde{\pi}(p))^2 v].$$

$$S^{\pm}(\sigma) : S^{\pm}_{\mathbb{C}}(P_{Spin^c}) \to S^{\pm}_{\mathbb{C}}(P_{Spin^c}),$$
$$S^{\pm}(\sigma)([p, w]) := [\sigma(p), v] = [p, \sigma(\widetilde{\pi}(p)) w].$$

The action $\mathcal{C}(P_{Spin^c}) \times \mathcal{G}_M \to \mathcal{C}(P_{Spin^c})$ is given by:

$$(A, \psi) \cdot \sigma := (det(\sigma)^* A, S(\sigma^{-1})(\psi)).$$

Lemma 3.3 (Mor.III 3.1). *The Seiberg-Witten map F is equivariant under the right action of $\mathcal{G}$:*

$$F((A,\psi)\cdot\sigma) = F(A,\psi)\cdot\sigma,$$

where on the right hand side σ acts by the trivial action on $i\Omega^2_+(M)$ and by $S^-(\sigma^{-1})$ on $S^-_{\mathbb{C}}(P_{Spin^c})$. In particular, the set of solutions of the Seiberg-Witten equations is preserved by the action.

To get three-dimensional manifolds into play, we consider a special case of the above: Let Y be an oriented, Riemannian and closed three-manifold. For later convenience Y will also be assumed to be connected. Let $M = Y \times \mathbb{R}$. M has the product metric and orientation, and a $spin^c$-structure P_{Spin^c} on Y pulls back via the projection $pr : M \to Y$ to a $spin^c$-structure $pr^*(P_{Spin^c})$ on M. Identifying Y and $Y \times \{0\}$, and using the injective map of principal bundles $P_{SO(3)} \to P_{SO(4)}$ given by:

$$[e_1, e_2, e_3]_p \to [e_1, e_2, e_3, \frac{d}{dt}]_{(p,0)},$$

where (e_1, e_2, e_3) is a positively oriented orthonormal basis of T_pY and $[\dots]$ denotes "frame", this fits into a commutative diagram:

$$\begin{array}{ccc} P_{Spin^c} & \hookrightarrow & pr^*(P_{Spin^c}) \\ \downarrow & & \downarrow \\ P_{SO(3)} & \hookrightarrow & P_{SO(4)} \end{array}$$

Here it is used that because of lemma 2.2, there is a natural inclusion of $Spin(3)$ into $Spin(4)$, and therefore of $Spin^c(3)$ into $Spin^c(4)$, commuting with Ad ($SO(3)$ $\subseteq SO(4)$ are the matrices keeping the fourth coordinate fixed. In fact, $Spin(3) \subseteq Spin(4)$ are exactly the elements of $Spin(4)$ commuting with $e_4 = \frac{d}{dt}$).

Because of proposition 2.6 the associated bundles satisfy

$$S^{\pm}_{\mathbb{C}}(pr^*(P_{Spin^c})) \approx pr^*(S^{\pm}_{\mathbb{C}}(P_{Spin^c})), \qquad \mathcal{L}_M \approx pr^*(\mathcal{L}_Y).$$

Furthermore, there is a map:

$$Cl(P_{SO(3)}) \xrightarrow{\cdot\frac{d}{dt}} Cl(P_{SO(4)})$$

given by multiplication by $\frac{d}{dt}$ from the right on the odd part in each fiber and it is thus the isomorphism of lemma 2.2 lifted to the bundle situation. This means that the action of $Cl(P_{SO(3)})$ on $S_{\mathbb{C}}(P_{Spin^c})$ satisfies:

$$pr^*(v\cdot\phi) = pr^*(v)\frac{d}{dt}\cdot pr^*(\phi), \qquad v \in TY.$$

Gauge transformations of the $Spin^c(3)$-bundle P_{Spin^c} are considered under the same restriction as above and are thus identified with maps from Y to S^1. We denote them $\mathcal{G}_Y$. Under pullback by pr, $\mathcal{G}_Y$ acts on $pr^*(P_{Spin^c})$ and they are exactly the gauge transformations of $pr^*(P_{Spin^c})$ that do not depend on the real variable. $\mathcal{G}_Y$ acts exactly as above on $\mathcal{C}(P_{Spin^c}) := \mathcal{A}(\mathcal{L}_Y) \times \Omega^0(S_{\mathbb{C}}(P_{Spin^c}))$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{C}(P_{Spin^c}) \times \mathcal{G}_Y \arrow[r] \arrow[d, "pr^*"'] & \mathcal{C}(P_{Spin^c}) \arrow[d, "pr^*"] \\
\mathcal{C}(pr^*(P_{Spin^c})) \times \mathcal{G}_M \arrow[r] & \mathcal{C}(pr^*(P_{Spin^c}))
\end{tikzcd}$$

Finally, let us define $q : \Omega^0(S^+(P_{Spin^c})) \to i\Omega^2(Y)$ by the same formula as in the four-dimensional situation, but now using the isomorphisms

$$End(S^+(P_{Spin^c})) \approx (Cl(Y) \otimes_{\mathbb{R}} \mathbb{C})^+ \approx \Omega^2(Y, \mathbb{C}),$$

using the projection on the + - part of the Clifford bundle. The relation between the two q's is: $q(pr^*(\psi)) = pr^*(q(\psi))^+$.

We will now derive an expression for the Seiberg-Witten equations on $Y \times \mathbb{R}$ in the special case where the connection $A \in \mathcal{A}(pr^*\mathcal{L}_Y)$ is in temporal gauge, that is, the $\frac{d}{dt}$-component of A is trivial. First we look at the curvature equation: From the formula

$$(F_A)_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j]$$

and the assumption that $A_4 = 0$, we get $F_A = F_{A_t} - \frac{\partial A}{\partial t} \wedge dt$. Using that

$$*_M(pr^*(\omega)) = pr^*(*_Y\omega) \wedge dt \quad \text{for any} \quad \omega \in \Omega^2(Y, \mathbb{C})$$

this gives that:

$$\begin{aligned}
F_A^+ &= (F_{A_t} - \frac{\partial A}{\partial t} \wedge dt)^+ \\
&= \frac{1}{2}(F_{A_t} + *_Y F_{A_t} \wedge dt - \frac{\partial A}{\partial t} \wedge dt - *_Y \frac{\partial A}{\partial t}) \\
&= (F_{A_t} - *_Y \frac{\partial A}{\partial t})^+
\end{aligned}$$

As $q(\psi)_t = q(\psi_t)^+$, the curvature equation is:

$$\frac{\partial A}{\partial t} = - *_Y (\frac{1}{2} q(\psi_t) - F_{A_t} + 2h_t)$$

The Dirac operator behaves as follows:

$$
\begin{aligned}
\eth_A(\psi) &= \sum_{i=1}^{3} e_i \cdot \widetilde{\nabla}_{e_i}(\psi) + \frac{d}{dt} \cdot \widetilde{\nabla}_{\frac{d}{dt}}(\psi) \\
&= \sum_{i=1}^{3} \frac{d}{dt} e_i \frac{d}{dt} \cdot \widetilde{\nabla}_{e_i}(\psi) + \frac{d}{dt} \cdot \frac{\partial \psi}{\partial t} \\
&= \frac{d}{dt} \cdot (\sum_{i=1}^{3} e_i \cdot \widetilde{\nabla}_{e_i}(\psi_t) + \frac{\partial \psi}{\partial t}) \\
&= \frac{d}{dt} \cdot (\eth_{A_t}(\psi_t) + \frac{\partial \psi}{\partial t}).
\end{aligned}
$$

So, the second of the equations is equivalent to

$$
\frac{\partial \psi}{\partial t} = -\eth_{A_t}(\psi_t).
$$

We see that the Seiberg-Witten equations in temporal gauge on the four-manifold $M = Y \times \mathbb{R}$ are equivalent to a flow equation in A and ψ if one interprets these data on M as a curve of similar data on Y. Of course, in the above expressions the notation A_t and ψ_t means restriction to the copy of Y at $Y \times \{t\}$. . Any connection can be gauge transformed into a connection in temporal gauge. The proof of this is given in appendix 16.

The flow equations arise naturally in connection with the following functional defined on $\mathcal{C}(P_{Spin^c})$:

Definition 3.4. *Define the Seiberg-Witten funtional* $C_\mu : \mathcal{C}(P_{Spin^c}) \to \mathbb{R}$ *by:*

$$
\begin{aligned}
C_\mu(A, \psi) &= \int_Y (F_{A_0} + \mu + \frac{1}{2} da) \wedge a + \frac{1}{2} \int_Y \langle \eth_A(\psi), \psi \rangle \\
&= \int_Y (\frac{1}{2}(F_{A_0} + F_A) + \mu) \wedge a + \frac{1}{2} \int_Y \langle \eth_A(\psi), \psi \rangle,
\end{aligned}
$$

where A_0 *is a base connection,* $a = A - A_0$, $\mu \in i\Omega^2(Y)$ *and* $\langle , \rangle$ *denotes the real part of the hermitian inner product on* $S_{\mathbb{C}}(P_{Spin^c})$.

Two remarks on this functional: If one chooses another base connection A_0' the functional is changed by a constant. More precisely, if $A_0' = A_0 + \alpha$ one has:

$$
C_\mu'(A, \psi) = C_\mu(A, \psi) - \int_Y (F_{A_0} + \mu - \frac{1}{2} d\alpha) \wedge \alpha.
$$

Secondly, under the action of $\sigma \in \mathcal{G}_Y$, C_μ behaves as follows:

$$\begin{aligned} C_\mu((A,\psi)\cdot\sigma) - C_\mu(A,\psi) &= \int_Y (F_{A_0} + \mu + \frac{1}{2} d(a + \frac{2d\sigma}{\sigma})) \wedge (a + \frac{2d\sigma}{\sigma}) \\ &\quad - \int_Y (F_{A_0} + \mu + \frac{1}{2} da) \wedge a \\ &\quad + \frac{1}{2} \int_Y (\langle \eth_{A+\frac{2d\sigma}{\sigma}}(\sigma^{-1}\psi), \sigma^{-1}\psi\rangle - \langle \eth_A(\psi),\psi\rangle) \\ &= \int_Y (F_{A_0} + \mu) \wedge \frac{2d\sigma}{\sigma} \qquad \text{(using [23, III.3.2])} \\ &= 8\pi^2 < c_1(\mathcal{L}_Y) \cup c(\sigma), [Y] > +2 \int_Y \mu \wedge \frac{d\sigma}{\sigma}. \end{aligned}$$

Here $c : \mathcal{G}_Y \to H^1(Y, \mathbb{Z})$ is given by pulling back the volume form of S^1:

$$c(\sigma) = \sigma^*(vol_{S^1}) = \frac{1}{2\pi i} \frac{d\sigma}{\sigma}.$$

Let us now find the gradient of C_μ:

$$\begin{aligned} C_\mu(A+b,\psi) - C_\mu(A,\psi) &= \int_Y (F_{A_0} + \mu + \frac{1}{2} db) \wedge b + \frac{1}{2} \int_Y (da \wedge b + db \wedge a) \\ &\quad + \frac{1}{2} \int_Y \langle (\eth_{A+b} - \eth_A)(\psi), \psi\rangle \\ &= \int_Y (F_{A_0} + \mu + \frac{1}{2} db) \wedge b + \int_Y da \wedge b + \frac{1}{4} \int_Y \langle b\cdot\psi, \psi\rangle \end{aligned}$$

$$\begin{aligned} \Rightarrow dC_\mu(A,\psi)(b) &= \int_Y b \wedge (F_{A_0} + \mu) + \int_Y b \wedge da + \frac{1}{4} \int_Y \langle b\cdot\psi,\psi\rangle \\ &= < b, - *_Y (F_A + \mu) + \frac{1}{2} *_Y q(\psi) >, \end{aligned}$$

using $[24, 6.9]$ and lemma 2.15, and as F_A and μ are purely imaginary.

$$\begin{aligned} C_\mu(A,\psi+\eta) - C_\mu(A,\psi) &= \frac{1}{2} \int_Y (\langle \eth_A(\eta),\psi\rangle + \langle \eth_A(\psi),\eta\rangle) + \frac{1}{2} \int_Y \langle \eth_A(\eta),\eta\rangle \\ \Rightarrow dC_\mu(A,\psi)(\eta) &= \int_Y \langle \eta, \eth_A(\psi)\rangle = \langle \eta, \eth_A(\psi)\rangle, \end{aligned}$$

using that the Dirac operator is selfadjoint (Prop. 2.14).
All in all this gives that:

$$\nabla C_\mu(A,\psi) = (- *_Y (F_A + \mu) + \frac{1}{2} *_Y q(\psi), \eth_A(\psi))$$

Hence we see that the four-dimensional Seiberg-Witten equations are the downward gradient flow for the functional C_μ if one interprets the data (A,ψ) on $Y \times \mathbb{R}$ as a curve of data $(A,\psi)_t$ on Y. The perturbation h and μ are related by: $h = -pr^*(\mu)^+$ and $\mu = -2h_t$.

Define a map $\sigma : \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c})) \times \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c})) \to i\Omega^1(Y)$ by:

$$\langle a, \sigma(\psi,\phi)\rangle = \langle a \cdot \psi, \phi\rangle$$

σ is related to q by: $\sigma(\psi,\psi) = 2 *_Y q(\psi)$ [24, 6.9].

This leads to the following definition:

Definition 3.5. *For a unitary connection A on $\mathcal{L}_Y$ and $\psi \in \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))$ the perturbated 3-dimensional Seiberg-Witten equations are:*

$$*_Y F_A = \frac{1}{4}\sigma(\psi,\psi) - *_Y \mu$$
$$\eth_A(\psi) = 0,$$

where $\mu \in i\Omega^2(Y)$ is the perturbation.

It is immediate from the definition that the solutions of the three-dimensional Seiberg-Witten equations are the static solutions of the four-dimensional Seiberg-Witten equations stemming from Y, or equivalently, they are the critical points of the Seiberg-Witten functional C_μ.

Comparing with finite dimensional Morse theory, we have now defined the Morse function which we will study, namely the Seiberg-Witten functional on Y. The Morse flow is connected with the Seiberg-Witten equations on $Y \times \mathbb{R}$ and the critical points of the Morse function are exactly the solutions of the Seiberg-Witten equations on Y. Of course, we still have to verify that C_μ has properties that justifies the label "Morse function".

4 Properties of the moduli space.

4.1 Basics on the full moduli space.

So far all statements have been made in terms of smooth objects. However, the further analysis of the Seiberg-Witten equations naturally leads to the use of differential geometry in an infinite dimensional Banach setting as the varibles are sections of vector bundles. This cannot take place in the smooth category as the set of C^∞ sections of a vector bundle is not a Banach manifold in a natural way and therefore Sobolev spaces are introduced in this section, where the properties of the quotient of $\mathcal{C}(P_{Spin^c})$ by the action of the gauge group $\mathcal{G}_Y$ will be investigated.

In the following the notation "L^p_k" for the Sobolev spaces and the elementary properties of these will be used freely. Also, a basic acquaintance with the bounded maps between these spaces induced by (elliptic) differential operators is assumed. The definitions and results used are shortly summarized in Appendix 14. The basic setup will use only $p = 2, k \geq 1$, but other spaces will come up along the way.

The space of variables, $\mathcal{C}(P_{Spin^c})_{L^2_k}$, is the direct sum of $\Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}$ and $\mathcal{A}(\mathcal{L})_{L^2_k} := A_0 + i\Omega^1(Y)_{L^2_k}$, where A_0 is a smooth, unitary base connection on $\mathcal{L} \equiv \mathcal{L}_Y$. The gauge group, $\mathcal{G}_Y$, is a subset of $\Omega^0(Y, \mathbb{C})$. This space of maps will be completed in the L^2_{k+1}-norm. By the Sobolev embedding theorem the completion injects into $C^0(Y, \mathbb{C})$ and thus the closed subset of maps with values in S^1, $\mathcal{G}_{L^2_{k+1}}$, is well defined. $\mathcal{G}_{L^2_{k+1}}$ is the completed version of the gauge group and is a Hilbert group. The component group of $\mathcal{G}_{L^2_{k+1}}$ is $H^1(Y, \mathbb{Z})$, the correspondence being given by:

$$\pi_0(\mathcal{G}_{L^2_{k+1}}) \ni [\sigma] \mapsto c(\sigma) = \sigma^*(vol_{S^1}) \in H^1(Y, \mathbb{Z}).$$

We define a closed subgroup of $\mathcal{G}_{L^2_{k+1}}$ by

$$\tilde{\mathcal{G}}_{L^2_{k+1}} = \{\sigma \in \mathcal{G}_{L^2_{k+1}} \mid < c_1(\mathcal{L}) \cup c(\sigma), [Y] > = 0\}.$$

Notice that $H := \mathcal{G}_{L^2_{k+1}} / \tilde{\mathcal{G}}_{L^2_{k+1}}$ is isomorphic to either 0 or $\mathbb{Z}$ depending on whether $c_1(\mathcal{L})$ is zero or not.

Assumption: We will assume that $c_1(\mathcal{L}) \neq 0$.

The multiplication theorems for Sobolev spaces imply that the action of $\mathcal{G}_{L^2_{k+1}}$ on $\mathcal{C}(P_{Spin^c})_{L^2_k}$ defined by the same expression as in the previous section is well defined and smooth. Also, choosing $\mu \in i\Omega^1(Y)_{L^2_m}$, where $m \geq k$, the definition of C_μ is extended to a smooth funcional on $\mathcal{C}(P_{Spin^c})_{L^2_k}$ and thus the gradient :

$$\nabla C_\mu : \mathcal{C}(P_{Spin^c})_{L^2_k} \to i\Omega^1(Y)_{L^2_{k-1}} \oplus \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{k-1}}$$

is smooth.

Definition 4.1. *The moduli space of variables, $\mathcal{B}_{L^2_k}(\tilde{\mathcal{B}}_{L^2_k})$, is the orbit space of $\mathcal{C}(P_{Spin^c})_{L^2_k}$ under the action of the gauge group $\mathcal{G}_{L^2_{k+1}}(\tilde{\mathcal{G}}_{L^2_{k+1}})$.*

$\tilde{\mathcal{B}}_{L^2_k}$ is a $\mathbb{Z}$-covering of $\mathcal{B}_{L^2_k}$.

Lemma 4.2 (Mor. III 4.1). *The stabilizer $\mathcal{G}_{(A,\psi)}$ in $\mathcal{G}_{L^2_{k+1}}$ of an element $(A,\psi) \in \mathcal{C}(P_{Spin^c})_{L^2_k}$ is trivial unless $\psi = 0$ in which case the stabilizer consists of the constant maps from Y to S^1, a group naturally identified with S^1.*

As a standard notation, elements of $\mathcal{C}(P_{Spin^c})_{L^2_k}$ with trivial stabilizer are denoted irreducible. All other elements are reducible. Lemma 4.2 gives that the subset of irreducible elements, $\mathcal{C}^*(P_{Spin^c})_{L^2_k}$, is open and thus so is the corresponding subset of $\mathcal{B}_{L^2_k}(\tilde{\mathcal{B}}_{L^2_k})$, $\mathcal{B}^*_{L^2_k}(\tilde{\mathcal{B}}^*_{L^2_k})$. The next lemma states that the action of $\mathcal{G}_{L^2_{k+1}}$ on $\mathcal{C}(P_{Spin^c})_{L^2_k}$ is a proper group action.

Lemma 4.3 (Basic convergence result, Mor. III 4.3).
Suppose that $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ are sequences in $\mathcal{A}(\mathcal{L})_{L^2_k}$ converging to A and B, respectively. Suppose also that for each n we have $\sigma_n \in \mathcal{G}_{L^2_{k+1}}$ with:

$$A_n \cdot \sigma_n = B_n.$$

Then there is a subsequence of $(\sigma_n)_{n=1}^\infty$ that converges to an element $\sigma \in \mathcal{G}_{L^2_{k+1}}$. Furthermore, we have

$$A \cdot \sigma = B.$$

Next item is the obligatory slicing result:

Proposition 4.4 (Mor. III 4.5-4.7). *For every point $(A,\psi) \in \mathcal{C}(P_{Spin^c})_{L^2_k}$, there is an open neighbourhood of (A,ψ) and a smoothly imbedded Hilbert submanifold S of the neighbourhood - the slice - which is invariant under the action of the stabilizer $\mathcal{G}_{(A,\psi)}$ of (A,ψ) such that:*

$$\Phi : S \times_{\mathcal{G}_{(A,\psi)}} \mathcal{G}_{L^2_{k+1}} \to \mathcal{C}(P_{Spin^c})_{L^2_k}$$

given by

$$\Phi([s,\sigma]) = s \cdot \sigma$$

is a diffeomorphism onto an open neighbourhood of the orbit of (A,ψ) in $\mathcal{C}(P_{Spin^c})_{L^2_k}$.

This slicing of the space of parameters implies that $\mathcal{B}^*_{L^2_k}$ has the structure of a Hilbert manifold. The tangent space at $[A,\psi]$ is identified with

$$i\Omega^1(Y)_{L^2_k} \oplus \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}/Im\lambda_\psi \approx Ker\lambda^*_\psi,$$

where

$$\lambda_\psi : i\Omega^0(Y)_{L^2_{k+1}} \to i\Omega^1(Y)_{L^2_k} \oplus \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}$$

is given by $\lambda_\psi = (2d, - \cdot \psi)$.

If $[A, \psi]$ is reducible (so $\psi = 0$) a neighbourhood of $[A, \psi]$ in $\mathcal{B}_{L^2_k}$ is homeomorphic to the quotient of

$$i\Omega^1(Y)_{L^2_k} \oplus \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}/Im\lambda_0 \approx Ker\lambda^*_0$$

by the action of $\mathcal{G}_{(A,\psi)} = S^1$.

The tangent space of $\mathcal{C}^*_{L^2_k}$ at (A, ψ) is $i\Omega^1(Y)_{L^2_k} \oplus \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}$. As $\psi \neq 0$, λ_ψ is injective and thus λ^*_ψ is surjective. This implies that $Ker\lambda^*_.$ defines a vector bundle over $\mathcal{C}^*_{L^2_k}$. Now $\lambda_{\psi\cdot\sigma} = \lambda_\psi \cdot \sigma$ and combined with the slice theorem this gives that

$$T\mathcal{B}^*_{L^2_k} = (Ker\lambda^*_.)/\mathcal{G}_{L^2_{k+1}}$$

Using A to define the L^2_k-Sobolev norm on the spinor part of $Ker\lambda^*_\psi$ over $[A, \psi]$, we get a L^2_k-metric on $T\mathcal{B}^*_{L^2_k}$. This is turn gives a Riemannian L^2_k-metric on $\mathcal{B}^*_{L^2_k}$, which is thus a metric space. Furthermore, as $\mathcal{C}^*_{L^2_k}$ is separable, because all the Sobolev spaces considered here are separable, $\mathcal{B}^*_{L^2_k}$ is separable. This implies that $\mathcal{B}^*_{L^2_k}$ is Lindelöf and (thus) also paracompact [17, 5.35], [9, 5.1, cor.2].

As $\tilde{\mathcal{G}}_{L^2_{k+1}}$ is closed in $\mathcal{G}_{L^2_{k+1}}$ and contains S^1, 4.2 and the basic convergence result 4.3 holds for this group also. Thus as the Lie algebras of $\mathcal{G}_{L^2_{k+1}}$ and $\tilde{\mathcal{G}}_{L^2_{k+1}}$ agree (and equal $i\Omega^0(Y)_{L^2_{k+1}}$), the slice theorem is also true for $\tilde{\mathcal{G}}_{L^2_{k+1}}$ and $\tilde{\mathcal{B}}_{L^2_k}$. Notice that the group of deck transformation of the $\mathbb{Z}$-covering, which is identified with H, acts by isometries on $\tilde{\mathcal{B}}_{L^2_k}$.

Turning to the results of the previous section notice that as

$$\lambda^*_\psi(\alpha, \eta) = 2d^*\alpha + iIm < \psi, \eta >,$$

where $(\alpha, \psi) \in i\Omega^1(Y)_{L^2_k} \oplus \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}$ and $<, >$ denotes the hermitian inner product on $S_{\mathbb{C}}(P_{Spin^c})$, we have

$$\begin{aligned}\lambda^*_\psi(\nabla C_\mu(A, \psi)) &= 2d^*(- *_Y (F_A + \mu) + \frac{1}{4}\sigma(\psi, \psi)) + iIm < \psi, \eth_A\psi > \\ &= \frac{1}{2}d^*\sigma(\psi, \psi) + iIm < \psi, \eth_A\psi > \quad = 0\end{aligned}$$

by a local computation, if μ is closed. Thus under this assumption, which will be made from now on, though C_μ is not well defined on $\mathcal{B}_{L^2_k}$, ∇C_μ does give a vectorfield on $\mathcal{B}^*_{L^2_k}$ - and $\tilde{\mathcal{B}}^*_{L^2_k}$ (Strictly speaking, ∇C_μ is a section of $T\mathcal{B}_{L^2_{k-1}}$).

Unfortunately, the "Morse function" C is not defined on $\mathcal{B}^*_{L^2_k}$. That is why the $\mathbb{Z}$-covering, $\tilde{\mathcal{B}}^*_{L^2_k}$, is introduced because the ambiguity of C under the action of $\mathcal{G}_Y$ - see the calculation after 3.4 - is resolved here. The perturbation of C, C_μ, is not well defined on either space, but it does give a vector field on these infinite dimensional manifolds as shown above. So we are really perturbing ∇C into a vector field, ∇C_μ, which is locally a gradient vector field.

This is parallel to [27], where the given Morse one-form on a manifold M is lifted to a $\mathbb{Z}$-covering of M, where it is the derivative of a function.

4.2 Compactness results.

A strength of the Seiberg-Witten equations is the compactness result described in this section. To prove them one has to be able to bootstrap the equations, what requires some sort of elliptic property. A priori this is not fullfilled, but by working in a particularly nice gauge, equations are obtained that can be bootstrapped. This is the background for the next lemma:

Lemma 4.5 (Gauge fixing lemma). *Given a connection $A \in \mathcal{A}(\mathcal{L})_{L^2_k}$ there exists a gauge transformation $\sigma \in \mathcal{G}_{L^2_{k+1}}$ with the property that the gauge transformed connection equals $A \cdot \sigma = A_0 + \alpha$, where:*

$$d^*\alpha = 0 \qquad \textit{and} \qquad \|H(\alpha)\|_{L^2} \leq K$$

Here H denotes the L^2-projection onto the harmonic one-forms $\mathcal{H}^1$ of Y. The constant C only depends on the Riemannian structure of Y.

Proof: To make α a coboundary we first gauge transform A by $\sigma_1 = \exp(if)$, where $f \in \Omega^0(Y, \mathbb{R})_{L^2_{k+1}}$. The exp is well defined as $\Omega^0(Y, \mathbb{C})_{L^2_{k+1}}$ is a Banach algebra.

$$A \cdot \sigma_1 - A = \sigma_1^{-2} d\sigma_1^2 = 2\frac{d\sigma_1}{\sigma_1} = 2idf.$$

This implies that:

$$d^*\alpha_1 = d^*(A \cdot \sigma_1 - A_0) = d^*(A - A_0) + 2i \triangle (f),$$

so that α is a coboundary if $\triangle(f) = \frac{i}{2} d^*(A - A_0) \in \Omega^0(Y, \mathbb{R})_{L^2_{k-1}}$. A solution of this equation exists by (Sobolev completed) Hodge theory.

To obtain the estimate of the harmonic part of α, we need to perform a second gauge transformation: Choose a cohomology class $\omega \in 4\pi H^1(Y, \mathbb{Z})$ and define $\sigma_2 = \exp(\frac{i}{2}\int_{y_0}^{y} \omega)$, where $H^1(Y, \mathbb{Z}) \subset H^1(Y, \mathbb{R}) \approx \mathcal{H}^1$ and $\int_{y_0}^{y}$ denotes the path integral from y_0 to y. This is well defined by choice of ω and gives a smooth gauge transformation. We get:

$$A \cdot \sigma_1 \cdot \sigma_2 - A_0 = \alpha_1 + i\omega.$$

Denote $\alpha_1 + i\omega$ by α. α is a coboundary as ω is harmonic and if ω is chosen appropriately α lies in the fundamental domain of the lattice $4\pi H^1(Y, \mathbb{Z}) \subset \mathcal{H}^1$. As this domain is a compact torus we get bounds on the norm of $H(\alpha)$ in any norm on the finite dimensional vector space $\mathcal{H}^1$. □

In the gauge of the above lemma the Seiberg-Witten equations look as follows:

$$\begin{aligned} \eth_{A_0}(\psi) &= -\frac{1}{2}\alpha \cdot \psi, \\ d\alpha &= \frac{1}{4} *_Y \sigma(\psi, \psi) - F(A_0) + \mu, \\ d^*\alpha &= 0, \\ \mathcal{H}(\alpha) &\leq K. \end{aligned}$$

Here $A = A_0 + \alpha$ and lemma 2.15 has been used for rewriting the Dirac operator. This will be referred to as the gauge fixed Seiberg-Witten equations.

Theorem 4.6 (Regularity theorem).
Assume that the set up of the above equations is of L^2_k connections and spinor fields, L^2_{k+1} gauge transformations and a L^2_m perturbation ($m \geq k \geq 1$). Then any solution (α, ψ) to these equations satisfies

$$\alpha \in \Omega^1(Y, i\mathbb{R})_{L^2_{m+1}} \quad \text{and} \quad \psi \in \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{m+2}}.$$

Proof: It is used that the following smooth differential operators are elliptic:

$$(d, d^*) : \Omega^1(Y) \to \Omega^0(Y) \oplus \Omega^2(Y)$$
$$\eth_{A_0} : \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c})) \to \Omega^0(S^-_{\mathbb{C}}(P_{Spin^c}))$$

As $L^2_1 \subset L^6$, Hölder's inequality gives $\eth_{A_0}(\psi) \in L^3$. By L^3-regularity of $\eth_{A_0}$, ψ is then L^3_1, so in particular L^{12}. Using Hölder again gives $\eth_{A_0}(\psi) \in L^4$ so that by regularity ψ is L^4_1.

If m $= 1$ $d\alpha$ is now L^2_1 as there is a continous Sobolev multiplication $L^4_1 \times L^4_1 \to L^4_1$. L^2-regularity of the operator (d, d^*) gives $\alpha \in L^2_2 \subset L^4_1$. Using the quoted multiplication and the first equation again gives $\eth_{A_0}(\psi) \in L^4_1 \Rightarrow \psi \in L^4_2 \subset L^2_2$. By the first equation $\eth_{A_0}(\psi)$ is then L^2_2, and finally ψ is L^2_3 by L^2-regularity of $\eth_{A_0}$.

If m ≥ 2 one may proceed from here, shifting between the equations as above. $\square$

An imidiate corollary is that if the perturbation is smooth, any solution is gauge equivalent to a smooth solution.

The proof of the compactness property begins with obtaining a bound on the fiberwise norm of the spinor part of a solution. This bound will depend only on the Riemannian manifold Y and the pertubation μ. The effective tool to work with in a situation like this is a Weizenböck formula:

Proposition 4.7 (Mor. IV 1.5). *Let X be a Riemannian manifold and let $\widetilde{P} \to X$ be a $Spin^c$ structure for X. Let A be a C^1 connection on $\mathcal{L}$ and let $\eth_A$ be the Dirac operator on $S_{\mathbb{C}}(\widetilde{P})$ determined by the Levi-Cevita connection on the orthogonal frame bundle and A on the determinant line bundle. Then for any C^2 section ψ of $S_{\mathbb{C}}(\widetilde{P})$ we have*

$$\eth_A \circ \eth_A(\psi) = \nabla^*_A \nabla_A(\psi) + \frac{\kappa}{4}\psi + \frac{1}{2} F_A \cdot \psi,$$

where ∇_A is the induced connection on $S_{\mathbb{C}}(\widetilde{P})$, κ is the scalar curvature and $\cdot$ denotes Clifford multiplication.

If $m \geq 2$ the above requirements for differentiability of the connection and the spinor are satisfied by a solution to the gauge fixed equations by the regularity theorem, 4.6. As the fiberwise norm of a section in $S_{\mathbb{C}}(P_{Spin^c})$ is invariant under gauge transformations, this is sufficient for the result below. The proof of this result is completely parallel to the proof in [23, IV2.2].

Corollary 4.8. *Every solution (A,ψ) of the Seiberg-Witten equations satisfies:*

$$|\psi|^2 \leq \kappa_Y^- + 2\|\mu\|_\infty$$

Here $\kappa_Y^- := max_Y(max(0,-\kappa))$.

The next convergence result contains the core of the compactness result:

Theorem 4.9. *Let $m \geq k \geq 1, m \geq 2$.*
1. Assume given a set of perturbations $B \subset i\Omega^2(Y)_{L^2_m}$ that is bounded in the L^2_m-norm. Then the set of solutions of the gauge fixed Seiberg-Witten equations perturbed by μ, $\mu \in B$, is bounded in the $L^2_{m+1} \times L^2_{m+2}$-norm on $\mathcal{C}(P_{Spin^c})$.
2. Assume given a sequence of L^2_k connections and spinors (A_n,ψ_n) solving the gauge fixed Seiberg-Witten equations perturbed by μ_n, where μ_n is L^2_m for all n. If the sequence $(\mu_n)_{n=1}^\infty$ converges to μ in the L^2_m-topology, a subsequence of $(A_n,\psi_n)_{n=1}^\infty$ converges with respect to the L^2_{m+1}-norm to a pair (A,ψ) that is a gauge fixed solution of the Seiberg-Witten equations perturbed by μ.

Proof: 1. By the regularity theorem the set of solutions consists in fact of L^2_{m+1} connections and L^2_{m+2} sections. Furthermore, by the above corollary and the assumed boundedness of B, we have a uniform bound on the C^0-norm of the spinors (From now on “a bound” will mean a uniform bound on the sets in question).

This immidiately gives a C^0-bound on $(d,d^*)(\alpha)$, thus in particular a L^2-bound. By appendix 14 this implies that $\alpha - H(\alpha)$ is L^2_1-bounded. Using the uniform bound on the harmonic part of α, we get a L^2_1-bound on α.

As there is a bounded inclusion $L^2_1 \subset L^4$, $\eth_A(\psi)$ is L^4-bounded. By the fundamental elliptic estimate for the Dirac operator, this implies a L^4_1-bound on the spinors, resulting in a L^4_1-bound on $(d,d^*)(\alpha)$. Again using the fundamental elliptic estimate, there is a L^4_2-bound on α, that is, a L^2_2-bound. Applying the first equation again and repeating the argument gives a L^2_2-bound on the spinors also.

Depending on m, one may now continue in this fashion, obtaining the desired bounds. The extra bound on ψ results from putting the L^2_{m+1}-bound back into the first equation in the end.

2. The convergent sequence of perturbations is in particular bounded. By 1. this gives bounds on the sequence $(\alpha_n,\psi_n)_{n=1}^\infty$ as stated above. Using the theorem of Rellich and extracting a subsequence we may assume that $\psi_n \to_n \psi$ in the L^2_{m+1}-norm. This means that $(d,d^*)(\alpha)_{n=1}^\infty$ converges in L^2_m to $(*_Y\sigma(\psi,\psi) - F_{A_0} + \mu, 0)$. Also, by the boundedness of the harmonic part of the sequence $(\alpha_n)_{n=1}^\infty$ in the finite dimensional vector space $\mathcal{H}^1$, we may assume that $H(\alpha_n)$ is convergent. As

$$\|\alpha_n - \alpha_m\|_{L^2_{m+1}} \leq \|H(\alpha_n) - H(\alpha_m)\|_{L^2_{m+1}} + K\|(d,d^*)(\alpha_n - \alpha_m)\|_{L^2_m},$$

$(\alpha_n)_{n=1}^\infty$ is a Cauchy sequence in the L^2_{m+1}-norm and so convergent towards, say, α. Finally $\eth_{A_0}(\psi) = -\frac{1}{2}\alpha \cdot \psi$ by continuity of the Clifford multiplication in these norms. $\square$

Define $HS_{L^2_k} := \{[A,\psi] \in \mathcal{B}_{L^2_k} | \eth_A(\psi) = 0\}$ and consider the map

$$F_1 : HS_{L^2_k} \to i\Omega^1(Y)_{L^2_{k-1}}$$

given by: $F_1([A,\psi]) = - *_Y F_A + \frac{1}{4}\sigma(\psi,\psi)$. F_1 is well defined and continous. Also consider the set

$$\widetilde{HS}_{L^2_k} := \{(A,\psi) \in \mathcal{C}_{L^2_k} | \eth_A(\psi) = 0, d^*(\alpha) = 0, \|H(\alpha)\|_{L^2} \le C\}$$

and the analogous map $\tilde{F}_1$. The quotient map from $\widetilde{HS}_{L^2_k}$ to $HS_{L^2_k}$ is continous and surjective by the gauge fixing lemma, 4.5.

Corollary 4.10. *F_1 is proper.*

Proof: "Proper" means closed with compact fibers. It is enough to show that $\tilde{F}_1$ has these properties. They both follow from the second part of theorem 4.9. □

Corollary 4.11. *Given a perturbation $\mu \in L^2_m, m \ge 2$, the set of solutions of the Seiberg-Witten equations perturbed by μ defined in L^2_k-variables ($m \ge k$) is a compact subset of $\mathcal{B}_{L^2_k}$.*

What if μ is smooth? By the regularity theorem, we know that all critical points are then smooth, up to gauge equivalence. Considering the parallel set up with the C^∞-topology on the spaces HS_∞ and $\tilde{HS}_\infty$, the Sobolev imbedding theorem and an argument using diagonal sequences gives that the second part of theorem 4.9 is still true (The C^∞-topology is precisely the inverse limit of the topologies defined on the set of smooth sections by the L^2_k-norms) and thus also corollaries 4.10 and 4.11 hold.

We state yet another compactness result on the Seiberg-Witten equations formulated in terms of the functional C_μ. Let $\mathcal{M}_\mu$ denote the fiber under F_1 of $*_Y\mu$:

$$\mathcal{M}_\mu = F_1^{-1}(*_Y\mu) = (\nabla C_\mu)^{-1}(0),$$

and $\tilde{\mathcal{M}}_\mu$ the preimage of this in $\tilde{\mathcal{B}}_{L^2_k}$. Assume that $\mathcal{M}_\mu \subseteq \mathcal{B}^*_{L^2_k}$, as is almost always the case (5.2). We then have:

Proposition 4.12 (Palais-Smale Condition, MSzT. 6.10). *The functional C_μ satisfies the Palais-Smale condition, that is: For every $\epsilon > 0$, there exists $\lambda(\epsilon) > 0$, such that if $[A,\psi] \in \mathcal{B}^*_{L^2_k}$ has L^2_k distance from $\mathcal{M}_\mu$ at least ϵ, then $\|\nabla C_\mu\|_{L^2_{k-1}} \ge \lambda$.*

As C_μ is not well defined on $\mathcal{B}^*_{L^2_k}$ the situation is of course not quite the one considered by Palais-Smale. But the formulation of the property is the same as in the classical case.

H acts by isometries on $\tilde{\mathcal{B}}^*_{L^2_k}$ preserving the vector field ∇C_μ and we may thus state a completely analog proposition for the situation on $\tilde{\mathcal{B}}^*_{L^2_k} \supseteq \tilde{\mathcal{M}}_\mu$.

Lemma 4.13. *For every $\epsilon > 0$ there exists $\delta > 0$:*

$$\|\mu - \tau\|_{L^2} < \delta \Rightarrow d_{L^2_1}(\mathcal{M}_\mu, \mathcal{M}_\tau) < \epsilon$$

Proof: The proof will again be by contradiction: Assume given a sequence $(\mu_n)_{n\in\mathbb{N}}$ with $\|\mu - \mu_n\|_{L^2} \to_n 0$ and such that for each $n \in \mathbb{N}$ there exists $(A_n, \psi_n) \in \mathcal{M}_{\mu_n}$ with $d_{L^2_1}((A_n, \psi_n), \mathcal{M}_\mu) \geq \epsilon_0$ for some fixed given $\epsilon_0 > 0$.
Using the Palais-Smale property of C_μ we have that

$$\|\nabla C_\mu(A_n, \psi_n)\|_{L^2} \geq \lambda,$$

for some $\lambda > 0$. But this contradicts that $\nabla C_{\mu_n}(A_n, \psi_n) = 0$ and $\|\mu - \mu_n\|_{L^2} \to_n 0$. □

Lemma 4.14. *For all $\mu \in i\Omega^2(Y)_{L^2_m}$ and for all $\epsilon > 0$, there exists $\delta > 0$ and $\lambda_\mu > 0$:*

$$\|\mu - \tau\|_{L^2} < \delta \Rightarrow$$
$$d_{L^2_1}([A, \psi], \mathcal{M}_\tau) \geq \epsilon \Rightarrow \|\nabla C_\tau(A, \psi)\|_{L^2} \geq \lambda_\mu.$$

Proof: Notice that

$$\|\nabla C_\tau(A, \psi) - \nabla C_\mu(A, \psi)\|_{L^2} = \|\mu - \tau\|_{L^2}.$$

Given $\epsilon > 0$, there exists according to 4.12, $\lambda'_\mu > 0$ such that

$$d_{L^2_1}([A, \psi], \mathcal{M}_\mu) \geq \frac{\epsilon}{2} \Rightarrow \|\nabla C_\mu(A, \psi)\|_{L^2} \geq \lambda'_\mu.$$

Choose $\delta > 0$ belonging to $\frac{\epsilon}{2}$ as specified in the above lemma 4.13 and with the restiction $\delta < \frac{\lambda'_\mu}{2}$. We now have:

$$\begin{aligned} d_{L^2_1}([A, \psi], \mathcal{M}_\tau) \geq \epsilon &\Rightarrow d_{L^2_1}([A, \psi], \mathcal{M}_\mu) \geq \frac{\epsilon}{2} \\ &\Rightarrow \|\nabla C_\mu(A, \psi)\|_{L^2} \geq \lambda'_\mu \\ &\Rightarrow \|\nabla C_\tau(A, \psi)\|_{L^2} \geq \lambda'_\mu - \delta \geq \frac{\lambda'_\mu}{2}. \end{aligned}$$

□

This was the final result on compactness of the set of solutions of the Seiberg-Witten equations.

5 Generic properties of the moduli space of solutions.

The linearization at a point $(A,\psi)\in\mathcal{C}(P_{Spin^c})_{L^2_k}$ of the Seiberg-Witten equations and the action of the gauge group give a short sequence of maps denoted $\mathcal{E}(A,\psi)$:

$$0\to i\Omega^0(Y)_{L^2_{k+1}}\xrightarrow{\lambda_\psi} i\Omega^1(Y)_{L^2_k}\oplus\Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}$$
$$\xrightarrow{H_{(A,\psi)}} i\Omega^1(Y)_{L^2_{k-1}}\oplus\Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{k-1}},$$

where $\lambda_\psi(f)=(2df,-f\psi)$ and $H_{(A,\psi)}$ is given by the matrix

$$\begin{pmatrix} -*_Y d & \frac{1}{2}\sigma(\psi,\cdot)\\ \cdot\frac{1}{2}\psi & \eth_A\end{pmatrix}.$$

This follows from 2.15 and because that as σ is a symmetric, bilinear map it's differential is $D\sigma_{(\psi,\psi)}(\eta)=2\sigma(\psi,\eta)$. This sequence is a complex over (the submanifold [23, III5.1])

$$\mathcal{HS}_{L^2_k}=\{(A,\psi)\in\mathcal{C}(P_{Spin^c})_{L^2_k}|\eth_A(\psi)=0\}.$$

As gauge equivalent pairs in $\mathcal{C}(P_{Spin^c})_{L^2_k}$ give isomorphic sequences, it makes sence to talk of the sequence of maps associated to a point in $\mathcal{B}_{L^2_k}$. The complex over $\mathcal{HS}_{L^2_k}$ is actually elliptic as is most easily seen by deforming it using the following homotopy where we get rid of the zero order terms: The complex $\mathcal{E}_t(A,\psi)$ is for $0\le t\le 1$ given by the maps:

$$\lambda_{\psi,t}(f)=(2df,-tf\psi),\quad H_{(A,\psi),t}=\begin{pmatrix} -*_Y d & \frac{t}{2}\sigma(\psi,\cdot)\\ \cdot\frac{t}{2}\psi & \eth_{A_0}+\frac{t}{2}a\cdot\end{pmatrix}.$$

$\mathcal{E}_0(A,\psi)$ is a direct sum of two elliptic complexes.

The adjoint of $\lambda_{\psi,t}$ is

$$\lambda^*_{\psi,t}(\alpha,\eta)=2d^*\alpha+itIm<\psi,\eta>,$$

where $(\alpha,\eta)\in i\Omega^1(Y)_{L^2_k}\oplus\Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}$ and $<,>$ denotes the pointwise hermitian inner product on $S_{\mathbb{C}}(P_{Spin^c})$. Using this adjoint we can form a single selfadjoint elliptic differential operator for arbitrary $(A,\psi)\in\mathcal{C}(P_{Spin^c})$, containing the information of the complex $\mathcal{E}_t(A,\psi)$ in case $(A,\psi)\in\mathcal{HS}_{L^2_k}$: Define

$$\mathcal{F}_{L^2_k}=i\Omega^0(Y)_{L^2_k}\oplus i\Omega^1(Y)_{L^2_k}\oplus\Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k}$$

and let

$$L_{(A,\psi),t}:\mathcal{F}_{L^2_k}\to\mathcal{F}_{L^2_{k-1}}$$

be given by the matrix $\begin{pmatrix} 0 & \lambda^*_{\psi,t} \\ \lambda_{\psi,t} & H_{(A,\psi),t} \end{pmatrix}$. For $t = 0$ this is a direct sum of the Dirac operator and a twist of the deRham-complex using the Hodge star operator:

$$\begin{pmatrix} 0 & d^* \\ d & *_Y d \end{pmatrix} : i\Omega^0(Y) \oplus i\Omega^1(Y) \circlearrowleft$$

As we are working on an odd dimensional manifold the index of $L_{(A,\psi),0}$ is zero [26, V.prop.8], implying that the index of $L_{(A,\psi),t}$ is zero. As $H_{(A,\psi),t}$ is self-adjoint it is not difficult to see that the kernel of $L_{(A,\psi)}$ is the zero'th and first cohomology groups of the above elliptic complex:

$$\begin{aligned} Ker(L_{(A,\psi)}) &= \mathcal{H}^0 \oplus \mathcal{H}^1 \\ &= Ker(\lambda_\psi) \oplus Ker(H_{(A,\psi)})/Im(\lambda_\psi). \end{aligned}$$

Notice that for $\psi \neq 0$, λ_ψ is injective.

Considering $L_{(A,\psi),t}$ as a continous family of Fredholm operators on the trivial Hilbert bundle $\mathcal{F}_{L^2_k}$ over $\mathcal{C}(P_{Spin^c}) \times I$, we can form the determinant line bundle $det(L)$ of the family. At a point $(A, \psi, t) \in \mathcal{C}(P_{Spin^c}) \times I$ the fiber of $det(L)$ is $\Lambda^{\mathrm{top}}(Ker(L_{(A,\psi),t})) \otimes \Lambda^{\mathrm{top}}(Coker(L_{(A,\psi),t})^*)$. Though the dimension of the kernel and cokernel need not be constant over $\mathcal{C}(P_{Spin^c}) \times I$, there is a way to glue together the above vector spaces to a vector bundle [10, App.].

The family of operators L behaves nicely with respect to gauge transformations: If we define the action of $\mathcal{G}_{L^2_{k+1}}$ on $\mathcal{F}_{L^2_k}$ by

$$(f, \alpha, \eta) \cdot \sigma = (f, \alpha, S(\sigma^{-1})\eta),$$

we have

$$L_{(A,\psi)\cdot\sigma,t}((f, \alpha, \eta) \cdot \sigma) = L_{(A,\psi),t}(f, \alpha, \eta) \cdot \sigma.$$

This means that $\mathcal{G}_{L^2_{k+1}}$ acts on the (co-)kernel of L and therefore on $det(L)$. Because of the slice theorem for the action of $\mathcal{G}_{L^2_{k+1}}$ on $\mathcal{C}^*(P_{Spin^c})$, $det(L)$ descends to a line bundle over $\mathcal{B}^*_{L^2_k} \times I$ (also denoted $det(L)$) by forming the quotient of the action.

Lemma 5.1. *$det(L)$ is a trivial line bundle over $\mathcal{B}^*_{L^2_k}$ and is naturally oriented.*

Proof: By [23, V.6.1] and section 4 it suffices to prove that the restriction of $det(L)$ to $\mathcal{C}^*_{L^2_k} \times \{0\}$ is trivial and is naturally oriented. The proof of this is completely parallel to [23, V.6.2] except for the orientation data. The difference here is in the trivial operator over $\mathcal{C}^*(P_{Spin^c}) \times \{0\}$ which is now

$$D = \begin{pmatrix} 0 & d^* \\ d & *_Y d \end{pmatrix} : i\Omega^0(Y) \oplus i\Omega^1(Y) \circlearrowleft$$

We have that $D = D^*$ and D is elliptic and thus

$$\begin{aligned} Det(D) &= \Lambda^{\text{top}} Ker(D) \otimes (\Lambda^{\text{top}} Coker(D))^* \\ &= \Lambda^{\text{top}} Ker(D) \otimes (\Lambda^{\text{top}} Ker(D))^* \\ &= \mathbb{R}. \end{aligned}$$

Finally notice that the action of $\mathcal{G}_{L^2_{k+1}}$ on $det(L)$ is orientation preserving because it acts as the identity on the trivial part and by multiplication by a complex number on the complex part. $\square$

The main object of this section is to prove that for a generic perturbation the set of gauge equivalence classes of solutions to the perturbed Seiberg-Witten equations is a manifold. This is the object of the next theorem. To make life easier, we would though like to avoid dealing with reducible solutions. The lemma below is concerned with the question of what conditions to put on Y and the perturbation to make sure that we will only obtain irreducible solutions.

Lemma 5.2. *Assume that $b^1(Y) \geq 1$. For any given metric and $Spin^c$-structure on Y there is an open dense subset of $Kerd_{L^2_m} \subseteq i\Omega^2(Y)_{L^2_m}$ such that the Seiberg-Witten equations perturbed by a two-form from this subset has no reducible solutions.*

Proof: A reducible solution to the SW_μ-equations corresponds to a connection A on $\mathcal{L}$ with $F_A = \mu$. In particular, this implies that

$$c_1(\mathcal{L}) = \frac{1}{2\pi i} H(\mu),$$

where H denotes the L^2-projection onto the space of harmonic two-forms. By the assumption $b^1(Y) \geq 1$, the condition that $H(\mu) \neq 2\pi i c_1(\mathcal{L})$ is an open dense condition on the set of two-forms. $\square$

Notice that as $c_1(\mathcal{L})$ was assumed non-zero in section 4, we are already working in the situation where $b^1(Y) \geq 1$.

Theorem 5.3 (Mar. 3.8, Fr. prop.3). *For an open and dense set of perturbations in $i\Omega^2(Y)_{L^2_m}$, the moduli space*

$$\mathcal{M}_\mu = \{[A, \psi] \in \mathcal{B}_{L^2_k} | *_Y F_A = \frac{1}{4}\sigma(\psi, \psi) - *_Y \mu, \eth_A(\psi) = 0\}$$

is a zero-dimensional and compact manifold which contains no reducible solutions and is naturally oriented.

Proof: Pull $T\mathcal{B}^*_{L^2_{k-1}}$ back to the product $\mathcal{B}^*_{L^2_k} \times \Theta_{L^2_m}$, where $\Theta_{L^2_m}$ denotes the open dense subset of 5.2. $\nabla C.$ defines a section in this bundle as noticed in section 4. We now want to prove that $\nabla C.$ is transversal to the zero-section of the pullback of $T\mathcal{B}^*_{L^2_{k-1}}$.

Setting $V_{\psi,L^2_k} := Ker(\lambda^*_\psi)$, the fiberwise derivative of $\nabla C.$ at a zero is given by:

$$D_{((A,\psi),\mu)}\nabla C. : V_{\psi,L^2_k} \times Kerd_{L^2_m} \to V_{\psi,L^2_{k-1}},$$
$$D_{((A,\psi),\mu)}\nabla C.(\alpha,\eta,\nu) = H_{(A,\psi)}(\alpha,\eta) + (*_Y\nu, 0).$$

This map is surjective at a zero $((A,\psi),\mu)$ iff the following perturbed version of the operator L is surjective:

$$\tilde{L}_{A,\psi} : \mathcal{F}_{L^2_k} \times Kerd_{L^2_m} \to \mathcal{F}_{L^2_{k-1}},$$
$$\tilde{L}_{(A,\psi)}(f,\alpha,\eta,\nu) = L_{A,\psi}(f,\alpha,\eta) + (0, *_Y\nu, 0).$$

This follows from the decomposition

$$\Omega^1(Y)_{L^2_k} \times \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k} = \mathcal{H}^1 \times Im\lambda_\psi \times ImH_{(A,\psi)},$$

where $ker\lambda^*_\psi = \mathcal{H}^1 \times ImH_{(A,\psi)}$. This implies that

$$\begin{aligned}
Im\tilde{L}_{A,\psi} &= Im\lambda_\psi \oplus (Im(\lambda^*_\psi, H_{(A,\psi)}) + *_Y Kerd_{L^2_m})\\
&= Im\lambda_\psi \oplus Im\lambda^*_{\psi|Im\lambda_\psi} \oplus (ImH_{(A,\psi)|Ker\lambda^*_\psi} + *_Y Kerd_{L^2_m})\\
&= \mathcal{F}_{L^2_{k-1}}\\
&\Leftrightarrow\\
Ker\lambda^*_\psi &= ImH_{(A,\psi)|Ker\lambda^*_\psi} + *_Y Kerd_{L^2_m}.
\end{aligned}$$

Here it is used that as $\psi \neq 0$, λ_ψ is injective and thus λ^*_ψ is surjective. We choose to work with the operator $\tilde{L}$.

As the statement that $\tilde{L}_{(A,\psi)}$ is surjective is independent of the gauge equivalence class of (A,ψ) (what was implicit in the above), we may by 4.6 assume that A is a L^2_{m+1} connection and that ψ is a L^2_{m+2} section. The image of $L_{(A,\psi)}$ is closed of finite codimension and therefore so is the image of $\tilde{L}_{(A,\psi)}$. Assume that $(g,\beta,\xi) \in \mathcal{F}_{L^2_{-k+1}}$ is L^2-orthogonal to the image of $\tilde{L}_{(A,\psi)}$. We have to show that (g,β,ξ) is zero. Let $(f,\alpha,\eta) \in \mathcal{F}_{L^2_k}$. Now in particular β is L^2-orthogonal to every $*_Y\mu \in *_Y Kerd_{L^2_m}$. Thus β is in the image of d. As λ^*_ψ is surjective even when restricted to $Im\lambda_\psi$, $g = 0$.

Set $\mu = 0, f = 0, \eta = 0$. Then

$$\begin{aligned}
0 = < - *_Y d\alpha, \beta > + < \frac{1}{2}\alpha\cdot\psi, \xi > &= < \alpha, \frac{1}{2}\sigma(\psi,\xi) >\\
&\Rightarrow \sigma(\psi,\xi) = 0.
\end{aligned}$$

Set $\mu = 0, f = 0, \alpha = 0$. Then

$$\begin{aligned}
< \frac{1}{2}\sigma(\psi,\eta), \beta > + < \eth_A\eta, \xi > &= < \frac{1}{2}\beta\cdot\psi + \eth_A\xi, \eta > = 0\\
&\Rightarrow \frac{1}{2}\beta\cdot\psi + \eth_A\xi = 0.
\end{aligned}$$

Finally, as $(0, \beta, \xi)$ is L^2-orthogonal to the image of λ_ψ, it is contained in the kernel of λ_ψ^* and thus:

$$2d^*\beta + iIm < \psi, \xi >= 0.$$

The above equations imply by a bootstrapping argument similar to the one preformed in 4.6 that β is a L^2_{m+3} one-form and that ξ is a section of class L^2_{m+2}. In particular they are both C^2.

The condition $\sigma(\xi, \psi) = 0$ implies that if (e_1, e_2, e_3) is an orthonormal basis of T_xY, where $\psi(x) \neq 0$, then $\langle ie_j \cdot \psi, \xi \rangle = 0, j = 1, 2, 3$. As

$$\langle e_j \cdot \psi, e_k \cdot \psi \rangle = \langle e_j \cdot \psi, \psi \rangle = 0$$

for $j \neq k, j = 1, 2, 3$, this implies that ξ is a purely imaginary multiple of ψ at x. Globalizing we find that there is $h \in i\Omega^0(Y)$ so that $\xi = h\psi$ everywhere on the open set $Y - \psi^{-1}(0)$. This set is dense and connected as $\eth_A$ has the unique continuation property (2.14 and appendix 15). Now

$$\eth_A \xi = \eth_A h\psi = dh \cdot \psi = -\frac{1}{2}\beta \cdot \psi$$

on $Y - \psi^{-1}(0)$, so that $\beta_{|Y - \psi^{-1}(0)} = -2dh$. As $Y - \psi^{-1}(0)$ is connected and β is in the image of d, this means that h may be extended to all of Y where it is of Sobolev class L^2_{m+4}. We now get for h:

$$\Delta h = -\frac{1}{2}d^*\beta = \frac{i}{4}Im < \psi, \xi >= \frac{i}{4}Im < \psi, h\psi >= -\frac{1}{4}h|\psi|^2.$$

This implies that:

$$\|dh\|^2_{L^2} =< \Delta h, h >_{L^2} \leq 0$$

and thus $dh = 0 \Rightarrow h \in i\mathbb{R}$. Then as $\int_Y |h|^2|\psi|^2 = 0$ and $\psi \neq 0, h = 0$. This implies that $\beta = 0$ and $\xi = 0$.

As the section $\nabla C.$ is transversal to the zero section

$$\mathcal{M} := \nabla C_.^{-1}(0)$$

is a submanifold of $\mathcal{B}^*_{L^2_k} \times \Theta_{L^2_m}$. Consider the projection mapping

$$\pi : \mathcal{M} \to \Theta_{L^2_m}.$$

This is a smooth map and it is furthermore Fredholm:

$$\begin{aligned}
Ker(D\pi_{([A,\psi],\omega)}) &= T_{([A,\psi],\omega)}\mathcal{M} \cap T_{[A,\psi]}\mathcal{B}^*_{L^2_k} \\
&= KerD_{([A,\psi],\omega)}\nabla C. \cap T_{[A,\psi]}\mathcal{B}^*_{L^2_k} \\
&= Ker(H_{[A,\psi]}) \cap Ker(\lambda_\psi^*) \\
&= Ker(H_{[A,\psi]})/Im(\lambda_\psi) = Ker(L_{[A,\psi]}). \\
Coker(D\pi_{([A,\psi],\omega)}) &\approx V_{\psi, L^2_k}/Im(H_{[A,\psi]}) = Coker(L_{[A,\psi]}),
\end{aligned}$$

under the mapping $[\omega] \mapsto [*\omega]$ and as λ_ψ^* is surjective.

The Sard-Smale theorem for Fredholm mappings between Lindelöf manifolds [1, 16.2] thus implies that there is a Baire second category set of perturbations that are regular values for the projection mapping. If μ is such a regular value $\pi^{-1}(\mu) = \mathcal{M}_\mu$ is a manifold of dimension the index of π = index of L which is 0 as noticed earlier. We still have to prove that the set of regular values is actually open and that $\mathcal{M}_\mu$ is compact. This follows from the fact that π is proper. This is again a consequence of 4.10 and the commutative diagram below where the upper horizontal map sends $[A,\psi]$ to $([A,\psi], *_Y F_1([A,\psi]))$

$$\begin{tikzcd} HS_{L^2_k} \arrow[r] \arrow[d, "F_1"'] & \mathcal{M} \arrow[d, "\pi"'] \\ i\Omega^1(Y)_{L^2_{k-1}} & \Theta_{L^2_m} \arrow[l, "*_Y"'] \end{tikzcd}$$

For a regular value μ,

$$T_{[A,\psi]}\mathcal{M}_\mu = Ker(D\pi_{[A,\psi]}) = Ker(L_{[A,\psi]}) = det(L)_{[A,\psi]}.$$

By 5.1 this implies that $\mathcal{M}_\mu$ is naturally oriented. $\square$

Proposition 5.4. *For $\mu \in i\Omega^1(Y)_{L^2_m}$, $\mathcal{M}_\mu$ is up to diffeomorphism independent of $k, m \geq k \geq 1, m \geq 2$.*

Proof: If $m \geq l \geq k \geq 1$ there is an obvious inclusion $\mathcal{M}_{\mu,L^2_l} \subseteq \mathcal{M}_{\mu,L^2_k}$. This map is surjective by 4.6 and as for $[A,\psi] \in \mathcal{M}_\mu$ the index of the elliptic operator $L_{[A,\psi]}$ does not depend on the choice of Sobolev norms, the differential of the inclusion map is a linear isomorphism. The inverse function theorem thus gives that the inclusion map is a local diffeomorphism. Finally, the inclusion is injective by the following argument: If $(A,\psi),(B,\phi) \in \mathcal{M}_{\mu,L^2_l}, \sigma \in \mathcal{G}_{L^2_k}$ and $(A,\psi)\cdot\sigma = (B,\phi)$, then $\frac{d\sigma}{\sigma}$ is L^2_l. Thus $d\sigma = \sigma\frac{d\sigma}{\sigma}$ is L^2_k and this means that σ itself is L^2_{k+1}. Continuing in this way we get that σ is in fact L^2_{l+1}. $\square$

In the language of Morse theory the main result of this section is that for a generic perturbation of C there is only finitely many critical points and they each come with a sign. It will be needed in later sections that

$$C(\alpha) \neq C(\beta), \quad \alpha, \beta \in \mathcal{M}_\mu.$$

It seems very plausible to the author that this is an open, dense condition on the set of perturbations and we will assume that it is satisfied.

The inverse image of $\mathcal{M}_\mu$, $\tilde{\mathcal{M}}_\mu$, in $\tilde{\mathcal{B}}_{L^2_k}$ is a $\mathbb{Z}$-covering of $\mathcal{M}_\mu$.

6 Relative indices of critical points.

For a given pair of critical points of C_μ, α and β, we shall define a relative index $i(\alpha,\beta)$ which will eventually give the grading of the chain complex that is the goal of this thesis. This index can be described in several equivalent ways involving indices of Fredholm operators and the notion of spectral flow.

For convenience we introduce a slightly different version of the operator L from section 5:

$$\check{L}_{(A,\psi)} := \begin{pmatrix} 0 & \frac{1}{2}\lambda_\psi^* \\ \frac{1}{2}\lambda_\psi & H_{(A,\psi)} \end{pmatrix} : \mathcal{F}_{L^2_k} \to \mathcal{F}_{L^2_{k-1}}.$$

It follows from the decompostion of $\mathcal{F}_{L^2_k}$ in the proof of 5.3 that $\check{L}_{(A,\psi)}$ has the same image and spectrum as $L_{(A,\psi)}$. We will return to this at the end of the section.

Given two points (A_{t_-},ψ_{t_-}) and (A_{t_+},ψ_{t_+}) in $\mathcal{C}^*(P_{Spin^c})_{L^2_k}$ and a curve (A_t,ψ_t) connecting them, consider the spectral flow of the continous curve of operators $L_{(A_t,\psi_t)} : \mathcal{F}_{L^2_k} \to \mathcal{F}_{L^2_{k-1}}$. This is the numbers of eigenvalues of $L_{(A_t,\psi_t)}$ changing sign as t varies form t_- to t_+, counted with sign $[4,7]$. The spectral flow will be denoted $sf(L_{(A_t,\psi_t)}) = sf(\check{L}_{(A_t,\psi_t)}) \in \mathbb{Z}$. The spectral flow is independent of the curve (A_t,ψ_t) up to homotopy. Also, if $(\sigma_t)_{t_-\le t\le t_+}$ is a continous curve in $\mathcal{G}_{L^2_{k+1}}$, we have

$$sf(L_{(A_t,\psi_t)\cdot\sigma_t}) = sf(L_{(A_t,\psi_t)}) \tag{$*$}$$

as we may first homotop the curve of gauge transformations to the constant curve σ_0 and then use the invariance of the operator L under the action of $\mathcal{G}_{L^2_{k+1}}$ noted in section 5.

Consider an element (A,ψ) of $\mathcal{C}^*_{L^2_{k+1}}(Y\times[t_-,t_+])$ with $r_{t_\pm}(A,\psi) = (A_{t_\pm},\psi_{t_\pm})$, where r_t denotes the restriction map to the copy of Y at t. The curve $(r_t(A,\psi))_{t_-\le t\le t_+}$ is continous and we thus have a spectral flow $sf(L_{r_t(A,\psi)})$. By $[4,7]$ this is the index of the operator

$$\frac{\partial}{\partial t} + \check{L}_{r_t(A,\psi)} : \pi^*\mathcal{F}_{L^2_{k+1}} \to \pi^*\mathcal{F}_{L^2_k}$$

with the boundary conditions described in $[3, 3.10]$:

$$sf(\check{L}_{r_t(A,\psi)_{t_-\le t\le t_+}}) = ind_{b.c.}(\frac{\partial}{\partial t} + \check{L}_{r_t(A,\psi)}).$$

But as the following calculation shows the operator $\frac{\partial}{\partial t} + \check{L}_{r_t(A,\psi)}$ is (almost) the operator obtained from the 4-dimensional Seiberg-Witten equations by twisting the elliptic complex:

$$\begin{array}{ccc} \pi^*(i\Lambda^0(Y)\oplus i\Lambda^1(Y)\oplus S^+_{\mathbb{C}}(P_{Spin^c})) & \xrightarrow{\frac{\partial}{\partial t}+\check{L}_{(A_t,\psi_t)}} & \pi^*(i\Lambda^0(Y)\oplus i\Lambda^1(Y)\oplus S^+_{\mathbb{C}}(P_{Spin^c})) \\ \downarrow\approx & & \approx\downarrow \\ i\Lambda^1(Y\times\mathbb{R})\oplus S^+_{\mathbb{C}}(P_{Spin^c}) & \xrightarrow{L_{(A,\psi)}} & i\Lambda^0(Y\times\mathbb{R})\oplus i\Lambda^2_+(Y\times\mathbb{R})\oplus S^-_{\mathbb{C}}(P_{Spin^c}) \end{array}$$

Here $L_{(A,\psi)} := \lambda^*_\psi \oplus D_{(A,\psi)}F$ with $\lambda^*_\psi(\alpha,\eta) = 2d^*\alpha + iIm < \psi, \eta >$ and

$$D_{(A,\psi)}F = \begin{pmatrix} d^+ & -\frac{1}{2}D_\psi q \\ \cdot\frac{1}{2}\psi & \eth_A \end{pmatrix}.$$

The downward isomorphisms are:

$$\pi^*(\Lambda^0(Y) \oplus \Lambda^1(Y)) \approx \Lambda^1(Y \times \mathbb{R}),$$
$$(f,\tau) \mapsto -\pi^*(f)dt + \pi^*(\tau),$$

$$\pi^*(\Lambda^1(Y)) \approx \Lambda^2_+(Y \times \mathbb{R}),$$
$$\tau \mapsto -(\pi^*(*_Y\tau))^+ = -\frac{1}{2}(\pi^*(\tau) \wedge dt + \pi^*(*_Y\tau)),$$

and $-\frac{d}{dt}\cdot : S^+_{\mathbb{C}}(P_{Spin^c}) \to S^-_{\mathbb{C}}(P_{Spin^c})$.

$$\lambda^*_\psi(-fdt + \tau, \phi) = -2d^*(fdt) + \lambda^*_{\psi_t}(\tau_t, \phi_t) = 2\frac{\partial f}{\partial t} + \lambda^*_{\psi_t}(\tau_t,\phi_t).$$

$$\begin{aligned} d^+(-fdt + \tau) - \frac{1}{2}D_\psi q(\phi) &= (-d_Y f \wedge dt + dt \wedge \frac{\partial \tau}{\partial t} + d_Y\tau)^+ - q(\psi,\phi) \\ &= (-d_Y f \wedge dt - \frac{\partial\tau}{\partial t} \wedge dt + **d_Y\tau - \frac{1}{2}*\sigma(\psi_t,\phi_t))^+. \end{aligned}$$

$$\frac{1}{2}(-fdt + \tau)\cdot\psi + \eth_A(\phi) = \frac{1}{2}\frac{d}{dt}\cdot(-f_t\psi_t + \tau_t\cdot\psi_t) + \frac{d}{dt}\cdot(\frac{\partial\psi}{\partial t} + \eth_{A_t}(\phi_t)).$$

If we assume that $(A,\psi) \in \mathcal{C}^*_{L^2_{k+1}}(Y \times [t_-, t_+])$ is constant near $t_\pm$ we may extend it trivially to $(A,\psi) \in \mathcal{C}^*_{L^2_{k+1,loc}}(Y \times \mathbb{R})$ and consider the operator

$$L_{(A,\psi)} : \pi^*\mathcal{F}_{L^2_{k+1,\delta}} \to \pi^*\mathcal{F}_{L^2_{k,\delta}}.$$

(The weighted Sobolev spaces $L^2_{k,\delta}$ are described in app. 14). We choose δ so small that the operators $L_{(A_{t_\pm},\psi_{t_\pm})}$ have no eigenvalues in $[-\delta,\delta]$. These operators are self-adjoint and elliptic and thus have an L^2-complete orthonormal basis consisting of eigenvectors [26, XI,th.14]. Because (A,ψ) was chosen as above, we can expand an element of $\pi^*\mathcal{F}_{L^2_{k+1}}$ near $t_\pm$ using this basis and it is then a matter of bookkeeping to show that

$$ind_{b.c.}(\frac{\partial}{\partial t} + \check{L}_{r_t(A,\psi)}) = ind_\delta(L_{(A,\psi)}).$$

The index of $L_{(A,\psi)}$ is independent of δ with the restriction above [20, 7.1].

We may now attempt to define a relative index for points $[A_-,\psi_-]$ and $[A_+,\psi_+]$ in $\mathcal{B}^*_{L^2_k}$ as follows: Choose a continous curve $[A_t,\psi_t]$ between the given points and lift this curve to a continous curve (A_t,ψ_t) in $\mathcal{C}^*_{L^2_k}$. Then set

$$i([A_-,\psi_-],[A_+,\psi_+]) = sf(L_{(A_t,\psi_t)}).$$

Of course, we must consider to what extend this is well defined. That is: Is this independent of the chosen curve and lift hereof? As homotopic curves lift to homotopic curves once an initial lift is chosen and the spectral flow only depends on the curve up to homotopy, and as there is not an effect of choosing another initial lift because of $(*)$, this amounts to checking the possible values of the spectral flow along a closed curve in $\mathcal{B}^*_{L^2_k}$. A closed curve will lift to a curve in $\mathcal{C}^*_{L^2_k}$ for which there exists a $\sigma \in \mathcal{G}_{L^2_{k+1}}$ with $(A_{t_-}, \psi_{t_-}) \cdot \sigma = (A_{t_+}, \psi_{t_+})$. The following lemma considers this situation:

Lemma 6.1. *For a curve $(A_t, \psi_t) \in \mathcal{C}^*_{L^2_k}$ with $(A_{t_+}, \psi_{t_+}) = (A_{t_-}, \psi_{t_-}) \cdot \sigma$ for a $\sigma \in \mathcal{G}_{L^2_{k+1}}$, there is the following formula for the spectral flow of $L_{(A_t,\psi_t)}$:*

$$sf(L_{(A_t,\psi_t)}) =< c_1(\mathcal{L}) \cup c(\sigma), [Y] > .$$

Proof: Assume that $[t_-, t_+] = [0, 2\pi]$, that the curve is constant near the boundary of the interval and set

$$\phi : Y \times [0, 2\pi] \to Y \times S^1; \phi(y, t) = (y, \exp(it)).$$

ϕ gives a diffeomorphism $Y \times [0, 2\pi]/((y, 0) \sim (y, 2\pi)) \approx Y \times S^1$. The pullback bundle $p^*\mathcal{L}$ over $Y \times [0, 2\pi]$ gives a line bundle $\tilde{\mathcal{L}}$ over $Y \times S^1$ by gluing the ends of the interval using ϕ and σ so that

$$(p^*\mathcal{L})_{(y,0)} \ni v \sim \sigma^{-2} v \in (p^*\mathcal{L})_{(y,2\pi)}.$$

As $A_{2\pi} = A_0 + \frac{2d\sigma}{\sigma} = (\sigma^2)^*(A_0)$, we get a connection $\tilde{A}$ on $\tilde{\mathcal{L}}$. The first Chern class of $\tilde{\mathcal{L}}$ is calculated using a somewhat special connection

$$A' = \pi^*(A) + \frac{t}{2\pi}\pi^*(\frac{2d\sigma}{\sigma}),$$

where A is a unitary connection on $\mathcal{L}$. Arguing as above this expression gives a connection on $\tilde{\mathcal{L}}$.

$$\begin{aligned} F_{A'} &= \pi^*(d_Y(A + \frac{t}{2\pi}\frac{2d\sigma}{\sigma})) + \frac{\partial}{\partial t}(\frac{t}{2\pi}\pi^*\frac{2d\sigma}{\sigma}) \\ &= \pi^* F_A - \frac{1}{2\pi}\pi^*\frac{2d\sigma}{\sigma} \wedge dt. \end{aligned}$$

So:

$$\begin{aligned} c_1(\tilde{\mathcal{L}}) &= p^* c_1(\mathcal{L}) + [\frac{1}{2\pi i} p^* \frac{2d\sigma}{\sigma} \wedge vol_{S^1}] \\ &= p^* c_1(\mathcal{L}) + 2c(\sigma) \otimes [S^1]^*, \end{aligned}$$

as $\phi^*(vol_{S^1}) = \frac{1}{2\pi}dt$. $H^1(Y \times S^1, \mathbb{Z})$ has no torsion, so $\tilde{\mathcal{L}}$ corresponds uniquely to a $spin^c$-structure, $\tilde{P}_{Spin^c}$, on $Y \times S^1$ with determinant line bundle $\tilde{\mathcal{L}}$ by 2.12. $Y \times S^1$ has the standard metric and as this is trivial in the S^1-factor we get

$$P_{SO(4)}(Y \times S^1) = P_{SO(3)} \times S^1.$$

The line bundle over $P_{SO(4)}(Y \times S^1)$ corresponding to $\tilde{P}_{Spin^c}$ is $\tilde{L}$, where

$$c_1(\tilde{L}) = p^* c_1(L) + \pi^* c(\sigma) \otimes [S^1]^*.$$

Here L is the line bundle over $P_{SO(3)}(Y)$ corresponding to P_{Spin^c}. $\tilde{L}$ can also be described as the line bundle obtained from $\pi^*(L)$ by gluing the fibers over the ends of the interval:

$$(\pi^* L)_{(y,0)} \ni v \sim \sigma^{-1} v \in (\pi^* L)_{(y,2\pi)}.$$

The spinor bundle $S_{\mathbb{C}}(\tilde{P}_{Spin^c}) = \pi^*(S_{\mathbb{C}}(P_{Spin^c}))^\sim$ is obtained by gluing the fibers over the ends of the interval in the same way as for L. We see that ψ gives a section of $S_{\mathbb{C}}(\tilde{P}_{Spin^c})$, $\tilde{\psi}$.

The differential at (A, ψ) of the Seiberg-Witten equations on $Y \times I$ corresponds to the differential at $(\tilde{A}, \tilde{\psi})$ of the Seiberg-Witten equations on $Y \times S^1$ and so using [4, 7] again:

$$\begin{aligned} sf(L_{(A_t,\psi_t)}) &= ind_{b.c.}(L_{(A,\psi)}) \\ &= ind_{Y \times S^1}(L_{(\tilde{A},\tilde{\psi})}). \end{aligned}$$

By [23, III.5] the latter index is given by:

$$\begin{aligned} ind_{Y \times S^1}(L_{(\tilde{A},\tilde{\psi})}) &= \frac{1}{4}(c_1(\tilde{\mathcal{L}})^2 - 2\chi(Y \times S^1) - 3\sigma(Y \times S^1)) \\ &= \frac{1}{4} c_1(\tilde{\mathcal{L}})^2 \\ &= < p^* c_1(\mathcal{L}) \cup c(\sigma) \otimes [S^1]^*, [Y \times S^1] > \\ &= < p^* c_1(\mathcal{L}) \cup c(\sigma), [Y] > . \end{aligned}$$

□

The above lemma allows us to make the following definition:

Definition 6.2. *Given points α and β in $\mathcal{B}^*_{L^2_k}$ ($\tilde{\mathcal{B}}^*_{L^2_k}$) define the relative index $i(\alpha, \beta) \in \mathbb{Z}/N(P_{Spin^c})\mathbb{Z}$ ($\mathbb{Z}$) as the spectral flow of the operators*

$$L_{(A_t,\psi_t)}, \quad t_- \le t \le t_+,$$

where $(A_t, \psi_t)_{t_- \le t \le t_+}$ is a continous curve in $\mathcal{C}^*_{L^2_k}$ with $[A_{t_-}, \psi_{t_-}] = \alpha$ and $[A_{t_+}, \psi_{t_+}] = \beta$. Here $N(P_{Spin^c})$ is given by:

$$Im(< c_1(\mathcal{L}) \cup \cdot, [Y] >: H^1(Y, \mathbb{Z}) \to \mathbb{Z}) \approx N(P_{Spin^c})\mathbb{Z}.$$

Remember that $\pi_1(\mathcal{B}^*_{L^2_k}) = H^1(Y, \mathbb{Z})$. $N(P_{Spin^c})\mathbb{Z}$ is an even, non-zero number as $c_1(\mathcal{L})$ is a non-zero even cohomology class. This follows from the fact that Y is parallizable so that there is a $spin^c$-stucture on Y with trivial determinant line bundle and by the formula from section 2

$$c_1(\mathcal{L}(P_{spin^c} \cdot \alpha)) = c_1(\mathcal{L}(P_{spin^c})) + 2\alpha,$$

which again is a consequence of the way $H^2(Y, \mathbb{Z})$ acts on the equivalence classes of $spin^c$-structures on Y.

If we choose a base point α_0, we can define an absolute index as $i(\alpha) = i(\alpha, \alpha_0)$. We then have:

$$i(\alpha, \beta) = i(\alpha) - i(\beta).$$

Of course, if we choose another base point the index may shift.

For (A, ψ) non-degenerate

$$\Delta_{(A,\psi)} = \lambda_\psi \lambda_\psi^* + H^2_{(A,\psi)} : i\Omega^1(Y) \times \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c})) \circlearrowleft$$

is an elliptic, self-adjoint differential operator with $Ker\Delta = 0$, so that it gives an isomorphism on the Sobolev completed spaces. Using the decomposition

$$\Omega^1(Y)_{L^2_k} \oplus \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_k} = Im\lambda_\psi \times ImH_{(A,\psi)}$$

we get that $H_{(A,\psi)} : V_\psi \to V_\psi$ and $\lambda_\psi \lambda_\psi^* : Im\lambda_\psi \to Im\lambda_\psi$ are isomorphisms and thus that $L_{(A,\psi)}$ splits in two isomorphisms:

$$L_{(A,\psi)} = \begin{pmatrix} 0 & \lambda_\psi^* \\ \lambda_\psi & 0 \end{pmatrix} \oplus H_{(A,\psi)}.$$

In particular, $sp(H_{(A,\psi)}) \subseteq sp(L_{(A,\psi)})$. $sp(L_{(A,\psi)}) = sp(L_{(A,\psi)})^-_a$ by [26, XI,th.14], [28, 4.5.13]. Let $\delta_{(A,\psi)}$ denote the numerically smallest eigenvalue of $L_{(A,\psi)}$. We see in particular, that $\|H^{-1}_{(A,\psi)}\| \leq \delta^{-1}_{(A,\psi)}$. Finally, for μ in the generic set of 5.3, set

$$\delta_\mu = min_{(A,\psi)\in\mathcal{M}_\mu} \delta_{(A,\psi)}.$$

The considerations at the end of appendix 14 show that $L_{(A,\psi)}$ is a Fredholm operator on the $L^2_{k,\delta}$-completed spaces for $|\delta| < \delta_\mu$, if the end values $(A_{t_\pm}, \psi_{t_\pm})$ are in $\mathcal{M}_\mu$.

7 Analysis of flow curves.

This section is the most technical part of this thesis. In subsection 7.1 there is a setup for analysing the Seiberg-Witten equations on $Y \times \mathbb{R}$ using exponentially weighted Sobolev spaces and the construction of a perturbation to the four-dimensional equations that will be used in the next section to obtain manifolds of solutions on $Y \times \mathbb{R}$. Furthermore, in subsection 7.2 there is results on the behaviour of the perturbed solutions considering them as gradient flow lines for the Seiberg-Witten functional approaching critical points at infinity.

7.1 Analytical setup.

Definition 7.1. *Define the gauge group $\mathcal{G}_{L^2_{l+1,\delta}}$ by*

$$\begin{aligned}\mathcal{G}_{L^2_{l+1,\delta}} :=&\{\sigma \in \Omega^0(Y \times \mathbb{R}, \mathbb{C})_{L^2_{l+1,loc}} | Im(\sigma) \subseteq S^1, \exists \sigma_{\pm\infty} \in \mathcal{G}_{L^2_{l+1}} : \\ &\sigma\sigma_{\pm\infty}^{-1} - 1 \in \Omega^0(Y \times \mathbb{R}, \mathbb{C})_{L^2_{l+1,\delta}(\mathbb{R}_\pm)}\}\end{aligned}$$

and the space of variables $\mathcal{C}_{L^2_{l,\delta}}$ by

$$\mathcal{C}_{L^2_{l,\delta}} := \mathcal{A}_{L^2_{l,\delta}} \oplus \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}},$$

where

$$\begin{aligned}\mathcal{A}_{L^2_{l,\delta}} &:= \{A \in A'_0 + i\Omega^1(Y \times \mathbb{R})_{L^2_{l,loc}} | \exists A_{\pm\infty} \in \mathcal{A}_{L^2_l} : A - A_{\pm\infty} \in i\Omega^1(Y \times \mathbb{R})_{L^2_{l,\delta}(\mathbb{R}_\pm)}\}, \\ \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}} &:= \{\psi \in \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,loc}} | \exists \psi_{\pm\infty} \in \Omega^0(S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_l} : \\ &\qquad \psi - \psi_{\pm\infty} \in \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}(\mathbb{R}_\pm)}\}.\end{aligned}$$

We make here the somewhat unconventional choice: $\mathbb{R}_- = (-\infty; 1]$ and $\mathbb{R}_+ = [-1; \infty)$. A'_0 denotes the pullback to $pr^*\mathcal{L}_Y$ of A_0. The definition and elementary properties of the $L^2_{l,\delta}$-spaces are given in appendix 14. We assume $l \geq 3$. The above sets will be given a structure of Hilbert manifold. Details are given for $\mathcal{G}_{L^2_{l+1,\delta}}$:

For $\tau \in \mathcal{G}_{L^2_{l+1,\delta}}, V_\pm \in \mathcal{O}(\tau_{\pm\infty})$ and $\epsilon > 0$ set

$$U(\tau, V_\pm, \epsilon) := \{\sigma \in \mathcal{G}_{L^2_{l+1,\delta}} | \sigma_{\pm\infty} \in V_\pm, \|\sigma\sigma_{\pm\infty}^{-1} - \tau\tau_{\pm\infty}^{-1}\|_{L^2_{l+1,\delta}(\mathbb{R}_\pm)} < \epsilon\}.$$

The set of all such U form a basis for a topology on $\mathcal{G}_{L^2_{l+1,\delta}}$. Furthermore, there is a bijection:

$$\begin{aligned}U(\tau, V_\pm, \epsilon) \approx \Delta(V_- \times V_+ \times B_\epsilon(\tau\tau_{-\infty}^{-1}) \times B_\epsilon(\tau\tau_\infty^{-1})), \\ \sigma \mapsto (\sigma_{-\infty}, \sigma_\infty, \sigma\sigma_{-\infty}^{-1}, \sigma\sigma_\infty^{-1}),\end{aligned}$$

where $\Delta(\ldots)$ is the inverse image of the diagonal under the map

$$\begin{aligned}V_- \times V_+ \times B_\epsilon(\tau\tau_{-\infty}^{-1}) \times B_\epsilon(\tau\tau_\infty^{-1}) \to \Omega^0(Y \times [-1; 1], \mathbb{C})^2_{L^2_{l+1}}, \\ (\kappa, \vartheta, \pi, \nu) \mapsto (\pi\kappa, \nu\vartheta).\end{aligned}$$

This inverse image is a manifold, and as the intersection of two sets of U-type is mapped onto an open subset of $\Delta(\dots)$ and as the transition maps turn out to be trivial, the above bijections give a manifold structure on $\mathcal{G}_{L^2_{l+1,\delta}}$. It is easy to verify that $\mathcal{G}_{L^2_{l+1,\delta}}$ is a Hilbert group with Lie algebra

$$\mathfrak{g}_{L^2_{l+1,\delta}} = \{f \in i\Omega^0(Y \times \mathbb{R})_{L^2_{l+1,loc}} | \exists f_{\pm\infty} \in i\Omega^0(Y)_{L^2_{l+1}} : f - f_{\pm\infty} \in i\Omega^0(Y \times \mathbb{R})_{L^2_{l+1,\delta}(\mathbb{R}_\pm)}\}$$

and that $\mathcal{G}_{L^2_{l+1,\delta}}$ acts smoothly on $\mathcal{C}_{L^2_{l,\delta}}$ by the usual expression:

$$\begin{gathered}\mathcal{G}_{L^2_{l+1,\delta}} \times \mathcal{C}_{L^2_{l,\delta}} \to \mathcal{C}_{L^2_{l,\delta}}, \\ (\sigma, (A, \psi)) \mapsto (A, \psi) \cdot \sigma = (A + \frac{2d\sigma}{\sigma}, \sigma^{-1}\psi).\end{gathered}$$

Here it is used that there is bounded multiplication maps

$$\begin{gathered}L^2_{l+1,\delta} \times L^2_{l,\delta} \to L^2_{l,\delta}, \\ L^2_{l+1} \times L^2_{l,\delta} \to L^2_{l,\delta}.\end{gathered}$$

It should be clear from the construction that there are smooth and equivariant endpoint maps $\pi_{\pm\infty} : \mathcal{G}_{L^2_{l+1,\delta}} \to \mathcal{G}_{L^2_{l+1}}$ and likewise for $\mathcal{C}_{L^2_{l,\delta}}$. Clearly, these are submersive so that e.g.

$$\tilde{\mathcal{G}}_{L^2_{l+1,\delta}} := (\pi_{-\infty} \times \pi_\infty)^{-1}(\tilde{\mathcal{G}}_{L^2_{l+1}} \times \tilde{\mathcal{G}}_{L^2_{l+1}})$$

is a smooth Hilbert subgroup of $\mathcal{G}_{L^2_{l+1,\delta}}$.
There is also smooth and equivariant restriction maps r_t for $t \in \mathbb{R}$.

Definition 7.2. *The moduli space of variables, $\mathcal{B}_{L^2_{l,\delta}}(\tilde{\mathcal{B}}_{L^2_{l,\delta}})$, is the orbit space of $\mathcal{C}_{L^2_{l,\delta}}$ under the action of the gauge group $\mathcal{G}_{L^2_{l+1,\delta}}(\tilde{\mathcal{G}}_{L^2_{l+1,\delta}})$.*

The above maps, $\pi_{\pm\infty}$ and $r_t, t \in \mathbb{R}$, induce smooth maps on the moduli spaces. We denote elements of $\mathcal{C}_{L^2_{l,\delta}}$ with $\psi \neq 0$ irreducible. All other elements are reducible. The subset of irreducible elements, $\mathcal{C}^*_{L^2_{l,\delta}}$, is open in $\mathcal{C}_{L^2_{l,\delta}}$. The corresponding open subset of $\mathcal{B}_{L^2_{l,\delta}}$ is denoted $\mathcal{B}^*_{L^2_{l,\delta}}$. The stabilizers under the action is as in section 4 and 4.2 can be restated verbatim in this new situation.

Lemma 7.3 (Basic convergence result). *Suppose that $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are sequences in $\mathcal{A}_{L^2_{l,\delta}}$ converging to A and B, respectively. Suppose also that for all n, there exists $\sigma_n \in \mathcal{G}_{L^2_{l+1,\delta}}$ such that:*

$$A_n \cdot \sigma_n = B_n.$$

Then there is a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$, which converges to an element $\sigma \in \mathcal{G}_{L^2_{l+1,\delta}}$ with

$$A \cdot \sigma = B.$$

Proof: As

$$A_{n,\pm\infty} \cdot \sigma_{n,\pm\infty} = B_{n,\pm\infty}$$

and

$$A_{n,\pm\infty} \to_n A_{\pm\infty}, B_{n,\pm\infty} \to_n B_{\pm\infty},$$

we may by the basic convergence result, 4.3, extract a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_{n,\pm\infty}$ converges in $\mathcal{G}_{L^2_{l+1}}$ to, say, $\sigma_{\pm\infty}$. Furthermore, as

$$A_n - A_{n,\pm\infty} + \frac{2d(\sigma_n\sigma_{n,\pm\infty}^{-1})}{\sigma_n\sigma_{n,\pm\infty}^{-1}} = B_n - B_{n,\pm\infty},$$

we see that $2d(\sigma_n\sigma_{n,\pm\infty}^{-1})(\sigma_n\sigma_{n,\pm\infty}^{-1})^{-1}$ converges to, say, $\omega_\pm$ in $i\Omega^1(Y\times\mathbb{R})_{L^2_{l,\delta}(\mathbb{R}_\pm)}$.

Now define for $t \geq -1$:

$$\sigma(y,t) = \sigma_\infty(y)\exp(\frac{1}{2}\int_{\gamma_y}\omega_+(t)).$$

Here γ_y denotes a curve in $Y = Y\times\{t\}$ from y_0 to y, where y_0 is a fixed base point in Y. This is independent of the choice of curve in Y as convergence in $L^2_{l,\delta}$ implies uniform convergence over compact subsets ($l \geq 3$) and thus

$$\exp(\int_{\gamma_y}\omega_+) = \exp(\int_{\gamma_y}\lim_n \frac{2d(\sigma_n\sigma_{n,\pm\infty}^{-1})}{\sigma_n\sigma_{n,\pm\infty}^{-1}}) = \lim_n \exp(\int_{\gamma_y}\frac{2d(\sigma_n\sigma_{n,\pm\infty}^{-1})}{\sigma_n\sigma_{n,\pm\infty}^{-1}}).$$

We now want to prove that $\sigma\sigma_\infty^{-1} - 1 \in L^2_\delta$. First we have:

$$\begin{aligned}|\int_{\gamma_y}\omega_+| &= |\int_0^1 \omega_+(\gamma(s))(\gamma'(s))dt| \\ &\leq \int_0^1 \|\omega_+(t)\|_0|\gamma'(s)|ds \leq diam(Y)\|\omega_+(t)\|_{L^2_l}.\end{aligned}$$

Thus

$$\begin{aligned}|\frac{\sigma}{\sigma_\infty} - 1| &= |\exp(\frac{1}{2}\int_{\gamma_y}\omega_+) - 1| \\ &\lesssim \frac{1}{2}|\int_{\gamma_y}\omega_+| \\ &\leq \frac{1}{2}diam(Y)\|\omega_+(t)\|_{L^2_l}\end{aligned}$$

and so

$$\|\frac{\sigma}{\sigma_\infty} - 1\|^2_{L^2_\delta} \leq \frac{1}{2}diam(Y)^2\int_0^\infty e^{2\delta t}\|\omega_+(t)\|^2_{L^2_l}dt < \infty.$$

As $\sigma^{-1}d\sigma = \sigma_\infty^{-1}d\sigma_\infty + \frac{1}{2}\omega_+$ on $Y \times \mathbb{R}_+$ a bootstrapping argument gives that $\sigma \in \Omega^0(Y \times \mathbb{R}_+, \mathbb{C})_{L^2_{l+1,loc}}$ with $\sigma\sigma_\infty^{-1} - 1 \in \Omega^0(Y \times \mathbb{R}_+, \mathbb{C})_{L^2_{l+1,\delta}}$. We are thus done if we can prove the above definition of σ on the positive halfaxis and the similar one on the negative halfaxis have the same restiction to $Y \times [-1;1]$. This is so as they are both the limit of σ_n on this subset as is seen from the following estimation:

$$\begin{aligned} \|\sigma - \sigma_n\|_{L^2_{l+1}(Y\times[-1;1])} \leq{} & \|\sigma - \sigma\sigma_{n,\infty}^{-1}\sigma_\infty\|_{L^2_{l+1}(Y\times[-1;1])} \\ & + \|\sigma_n\sigma_{n,\infty}^{-1} - \sigma\sigma_\infty^{-1}\|_{L^2_{l+1}(Y\times[-1;1])}\|\sigma_{n,\infty}\|_{L^2_{l+1}(Y)}. \end{aligned}$$

□

The slice theorem, 4.4, can also be generalised to the present situation if L^2_k-Sobolev spaces are replaced with their $L^2_{k,\delta}$ counterparts everywhere.

For μ in the generic set of two-forms from 5.3 define

$$\tilde{\mathcal{B}}_{L^2_{l,\delta}}(\mu) := (\pi_{-\infty} \times \pi_\infty)^{-1}(\tilde{\mathcal{M}}_\mu)$$

and

$$\mathcal{C}_{L^2_{l,\delta}}(\mu) := \pi^{-1}(\tilde{\mathcal{B}}_{L^2_{l,\delta}}(\mu)).$$

These are submanifolds of $\tilde{\mathcal{B}}^*_{L^2_{l,\delta}}$ and $\mathcal{C}_{L^2_{l,\delta}}$, respectively. The same holds for $\mathcal{C}(\alpha,\beta)_{L^2_{l,\delta}}$ and $\tilde{\mathcal{B}}(\alpha,\beta)_{L^2_{l,\delta}}$ for $\alpha, \beta \in \tilde{\mathcal{M}}_\mu$, defined in the analogous way. Also define the Seiberg-Witten map:

$$\begin{aligned} F_\mu : \mathcal{C}_{L^2_{l,\delta}} &\to i\Omega^2_+(Y \times \mathbb{R})_{L^2_{l,\delta}} \times \Omega^0(pr^* S^-_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}} \\ F_\mu(A,\psi) &= ((F_A + pr^*\mu)^+ - \frac{1}{2}q(\psi), \eth_A\psi) \end{aligned}$$

Below we will constuct a perturbation of the Seiberg-Witten map F_μ. This perturbation is technically difficult to work with but it has some very nice properties, which will be described along the way. And most importantly, it allows me to give a precise analytical description of the moduli spaces of solutions considered in the next section. The construction is inspired by [13, 2].

$C(\tilde{\mathcal{M}}_\mu)$ is a discrete subset of $\mathbb{R}$ as $\tilde{\mathcal{M}}_\mu$ is a $\mathbb{Z}$-covering of a finite set and C is changed by a fixed number under the action of the group of deck transformations H, namely

$$8\pi^2 < c_1(\mathcal{L}) \cup c(\sigma), [Y] > = 8\pi^2 k(\sigma)N \ \in 8\pi^2\mathbb{Z} \quad \text{for } \sigma \in H.$$

Choose disjoint open intervals of length ϵ centered around each element of $C(\tilde{\mathcal{M}}_\mu)$ for some small $\epsilon > 0$. Denote the complement of this set in $\mathbb{R}$ by Ξ.

For $[A,\psi] \in \tilde{\mathcal{B}}_{L^2_{l,\delta}}$ let the C^∞ function $h_{[A,\psi]} : \mathbb{R} \to \mathbb{R}$ be given by:

$$h_{[A,\psi]}(t) = \textstyle\int_{\mathbb{R}} \eta(s-t)C(A_s,\psi_s)ds,$$

where $\eta : \mathbb{R} \to \mathbb{R}_+$ is a smooth map with support in $[-1;1]$ and $\int_{\mathbb{R}} \eta(t)dt = 1$. $h_{[A,\psi]}$ is smooth as we can differentiate under the integral and this is possible as $t \mapsto C(A_t, \psi_t)$ is a bounded map for any $[A,\psi] \in \tilde{\mathcal{B}}_{L^2_{l,\delta}}$. Furthermore, all $h_{[A,\psi]}$ are bounded in all C^p-norms on $\mathbb{R}$ and the map

$$\tilde{\mathcal{B}}_{L^2_{l,\delta}} \ni [A,\psi] \mapsto h_{[A,\psi]} \in C^p$$

is C^∞ for all p. The derivative is:

$$D_{[A,\psi]}h(\alpha,\zeta) = \int_{\mathbb{R}} \eta(s-t) < \nabla C(A_s,\psi_s), (\alpha_s,\zeta_s) >_{L^2} ds.$$

$h_{[A,\psi]}$ is equivariant under translations:

$$h_{[A,\psi]_\tau}(t) = h_{[A,\psi]}(t+\tau),$$

where $[A,\psi]_\tau$ denotes the translation by $\tau \in \mathbb{R}$ of $[A,\psi]$. It also fullfills

$$h_{[A,\psi]\cdot\sigma}(T) = h_{[A,\psi]}(T) + 8\pi^2 k(\sigma) N \quad \text{for} \quad \sigma \in H.$$

For the space of perturbations we use a construction of Floer in [12, lemma 5.1]. Let

$$iC^\epsilon(J^l\Lambda^2(Y \times \mathbb{R})^*)$$

denote the space of purely imaginary sections of the l-jet bundle of 2-forms over $Y \times \mathbb{R}$ for which the sum

$$\|\omega\|_{C^\epsilon} := \sum_{r=0}^{\infty} \epsilon_r \|\omega\|_{C^r}$$

is finite. This is a Banach space consisting entirely of smooth sections and for an appropriate choice of the sequence $\epsilon = (\epsilon_r)_{r=0}^\infty$ it is dense in $i\Omega(J^l\Lambda^2(Y \times \mathbb{R})^*)_{L^2}$. Define a closed subspace Ω_l of $iC^\epsilon(J^l\Lambda^2(Y \times \mathbb{R})^*)$ by the condition

$$\omega_{|Y \times \Xi^c} = 0.$$

Ω_l is dense in $i\Omega(J^l\Lambda^2(Y \times \Xi))^*)_{L^2}$.

Now define

$$Q : \tilde{\mathcal{B}}_{L^2_{l,\delta}}(\mu) \times \Omega_l \to i\Omega^0(J^l\Lambda^2(Y \times \mathbb{R}))_{L^2_{l,\delta}},$$
$$Q([A,\psi],\omega) := h^*_{[A,\psi]}(\omega).$$

Here $h^*_{[A,\psi]}(\omega)$ denotes the somewhat unusual pullback:

$$h^*_{[A,\psi]}(\omega)(y,t) := \omega(y, h_{[A,\psi]}(t)).$$

Notice that as

$$h_{[A,\psi]}(t) \to C([A,\psi]_{\pm\infty}) \in \Xi^c, \quad \text{for} \quad t \to \pm\infty,$$

$Q([A,\psi],\omega)(y,t) = 0$ for $|t|$ big, or more precisely for $t \in h^{-1}_{[A,\psi]}(\Xi^c)$. This gives that $Q([A,\psi],\omega) \in i\Omega^0(J^l\Lambda^2(Y\times\mathbb{R}))_{L^2_\delta}$. Note that actually $Q([A,\psi],\omega)$ is smooth and in $i\Omega^0(J^l\Lambda^2(Y\times\mathbb{R}))_{L^2_{p,\delta}}$ for all p.

The image of $Q_{[A,\psi]}$ is dense in $i\Omega^0(J^l\Lambda^2(Y\times C(A,\psi)^{-1}(\Xi)))_{L^2}$. This follows from the above property of Ω_l, $[2,3.35]$ and lemma 7.4 where we will prove that $h'_{[A,\psi]} < 0$ for $C(A,\psi) \in \Xi$.

Q is a smooth map whose differential is given by:

$$D_{([A,\psi],\omega)}Q = h^*_{[A,\psi]}(\frac{d\omega}{dt})D_{[A,\psi]}h.$$

$DQ_{[A,\psi]}$ is a compact map as it can be factored e.g. in the following way:

$$\Omega_l \to C^{l+1}_{Y\times[-T;T]} \overset{compact}{\to} C^l_{Y\times[-T;T]} \to L^2_{l-1,\delta}$$

for some large T depending on $[A,\psi]$. Q is translation equivariant:

$$\begin{aligned} Q([A,\psi]_\tau,\omega)(y,t) &= \omega(y,h_{[A,\psi]_\tau}(t)) \\ &= \omega(y,h_{[A,\psi]}(t+\tau)) = Q([A,\psi],\omega)_\tau(y,t) \end{aligned}$$

and Q behaves as follows under the action of H:

$$\begin{aligned} Q([A,\psi]\cdot\sigma,\omega)(y,t) &= \omega(y,h_{[A,\psi]\cdot\sigma}(t)) \\ &= \omega(y,h_{[A,\psi]}(t) + 8\pi^2k(\sigma)N) \\ &= \omega(y,h_{[A,\psi]}(t))_{8\pi^2k(\sigma)N} \qquad = Q([A,\psi],\omega_{8\pi^2k(\sigma)N})(y,t) \end{aligned}$$

$i\Omega^0(J^l\Lambda^2(Y\times\mathbb{R}))_{L^2_\delta}$ contains as a closed subspace $i\Omega^2(Y\times\mathbb{R})_{L^2_{l,\delta}}$. Let

$$P : i\Omega^0(J^l\Lambda^2(Y\times\mathbb{R}))_{L^2_\delta} \to i\Omega^2(Y\times\mathbb{R})_{L^2_{l,\delta}}$$

denote the orthogonal projection. The image of $PQ_{[A,\psi]}$ is dense in $i\Omega^2(Y\times C(A,\psi)^{-1}(\Xi))_{L^2_{l,\delta}}$.

The perturbed version of F_μ is:

$$\begin{aligned} &\tilde{F}_\mu : \mathcal{C}_{L^2_{k,\delta}}(\mu)\times\Omega_l \to i\Omega^2_+(Y\times\mathbb{R})_{L^2_{l,\delta}} \times \Omega^0(pr^*S^-_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}}, \\ &\tilde{F}_\mu(A,\psi,\omega) = ((F_A + pr^*\mu + PQ([A,\psi],\omega))^+ - \frac{1}{2}q(\psi), \eth_A\psi). \end{aligned}$$

Notice that $\tilde{F}_\mu$ is $\tilde{\mathcal{G}}_{L^2_{k+1,\delta}}$ equivariant.

7.2 The decrease of functionals along flow lines.

From section 3 we have that if (A,ψ) is in temporal gauge, $\tilde{F}_{\mu,\omega}(A,\psi)=0$ is equivalent to the perturbed gradient flow equation

$$\frac{d}{dt}(A,\psi) = -\nabla C_\mu(A_t,\psi_t) + PQ([A,\psi],\omega)'_t.$$

Lemma 7.4 (Fr. lemma 4). *For any $(A,\psi)\in\mathcal{C}_{L^2_{l,loc}}$ in temporal gauge and $\omega\in\Omega_l$ with $\tilde{F}_{\mu,\omega}(A,\psi)=0$, we have:*
1) For $\|\omega\|_{C^\epsilon}$ sufficiently small, either

$$\frac{d}{dt}(C_\mu(A_t,\psi_t)-C_\mu(A_0,\psi_0))<0$$

or $[A_t,\psi_t]\equiv\alpha\in\mathcal{M}_\mu$ for all t.
2) For $\|\omega\|_{C^\epsilon}$ and $\|\mu\|_{L^2_m}$ sufficiently small, there is a constant $\lambda>0$, so that

$$\frac{d}{dt}C(A_t,\psi_t)\le-\lambda<0$$

and

$$\frac{d}{dt}h_{[A,\psi]}<0$$

for $t\in C(A,\psi)^{-1}(\Xi)$.

Proof: Ad 1): We first compute

$$\begin{aligned}\frac{d}{dt}(C_\mu(A_t,\psi_t)-C_\mu(A_0,\psi_0)) &= <\nabla C_\mu(A_t,\psi_t), -\nabla C_\mu(A_t,\psi_t)+PQ([A,\psi],\omega)'_t>_{L^2}\\ &= -\|\nabla C_\mu(A_t,\psi_t)\|^2_{L^2}+<\nabla C_\mu(A_t,\psi_t),PQ([A,\psi],\omega)'_t>_{L^2}.\end{aligned}$$

Assume now that the statement above is wrong. Then there exists a sequence of perturbations $(\omega_n)_{n\in\mathbb{N}}$ converging to zero and a corresponding sequence of varibles $(A^n,\psi^n)_{n\in\mathbb{N}}$ with $\tilde{F}_{\mu,\omega_n}(A^n,\psi^n)=0$ such that for all $n\in\mathbb{N}$, there is a $t_n\in\mathbb{R}$ with

$$\frac{d}{dt}(C_\mu(A^n_{t_n},\psi^n_{t_n})-C_\mu(A^n_0,\psi^n_0))\ge 0.$$

and there exists $t\in\mathbb{R}:[A^n_t,\psi^n_t]\notin\mathcal{M}_\mu$. This implies that

$$\begin{aligned}\|\nabla C_\mu(A^n_{t_n},\psi^n_{t_n})\|_{L^2} &\le \|h^*_{[A^n,\psi^n]}(\omega_n)'_{t_n}\|_{L^2}\le vol(Y)\|\omega_n\|_0\\ &\le vol(Y)\epsilon_0^{-1}\|\omega_n\|_{C^\epsilon}\longrightarrow 0, n\to\infty.\end{aligned}$$

By $[18,\text{lemma }4]$ and the analogue of 4.9 we may assume that

$$[A^n,\psi^n]_{-t_n}\to_n[A,\psi]\in\mathcal{B}_{L^2_l}(Y\times[-1;1]),$$

as $PQ([A^n, \psi^n], \omega_n)_{Y \times [t_n - 1; t_n + 1]}$ is bounded in $\Omega^2(Y \times [t_n - 1; t_n + 1])_{L^2_{l+1}}$. By the above estimate $\nabla C_\mu((A, \psi)_0) = 0$ and by L^2_l-convergence

$$\frac{d}{dt}(A, \psi) = -\nabla C_\mu(A, \psi).$$

Thus $(A_t, \psi_t) \equiv (A_0, \psi_0)$ for $t \in [-1; 1]$. This implies

$$h_{[A^n, \psi^n]}(t_n) \to C([A_0, \psi_0]) \in \Xi^c, n \to \infty,$$

which again gives that $Q([A^n, \psi^n], \omega_n)_{t_n} = 0$, for n big. Now

$$\nabla C_\mu(A^n_{t_n}, \psi^n_{t_n}) = 0$$

for n big and as $\mathcal{M}_\mu$ is discrete, we can assume that $[A^n_{t_n}, \psi^n_{t_n}] = [A, \psi] \in \mathcal{M}_\mu$. This will hold on the maximal open interval around t_n on which $Q([A^n, \psi^n], \omega)_t = 0$. But this must be all of $\mathbb{R}$.

Ad 2): We begin with a simple lemma:

Lemma 7.5. *Let K be a compact subset of a metric space X and $f : X \to \mathbb{R}$ a continous function on X. Then for any $\epsilon > 0$, there exists $\delta > 0$, such that*

$$d(x, K) < \delta \Rightarrow |f(x) - f(y_x)| < \epsilon,$$

where y_x denotes a point in K with $d(x, y_x) = d(x, K)$.

Proof: Assume by contradiction that there is an $\epsilon_0 > 0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ with $d(x_n, K) \to 0, n \to \infty$, whereas $|f(x_n) - f(y_{x_n})| \geq \epsilon_0$ for all $n \in \mathbb{N}$. By compactness of K we may assume that $y_{x_n} \to y \in K, n \to \infty$. But then also $x_n \to y, n \to \infty$ and the assumed estimate above now contradicts the continuity of f. $\square$

Let $\Xi(\mu, \epsilon)$ denote set constructed in subsection 7.1. We now claim that:

$$\forall \epsilon > 0 \; \exists \delta > 0, \kappa > 0 : \|\mu\|_{L^2} < \kappa \wedge d_{L^2_1}([A, \psi], \tilde{\mathcal{M}}_\mu) < \delta \Rightarrow C([A, \psi]) \in \Xi(\mu, \epsilon).$$

Let $\epsilon > 0$ be given. Notice that $\tilde{\mathcal{M}}_0 \cap C^{-1}([0; 16\pi^2 N])$ is compact as $\mathcal{M}_0$ is compact and the bound on C implies that each point in $\mathcal{M}_0$ has a finite number of preimages in $\tilde{\mathcal{M}}_0 \cap C^{-1}([0; 16\pi^2 N])$. Choose $\delta > 0$ as described in the above lemma with $K = \tilde{\mathcal{M}}_0 \cap C^{-1}([0; 16\pi^2 N])$ and $f = C$ and "$\epsilon = \frac{\epsilon}{6}$". By uniform continuity of C on $\tilde{\mathcal{M}}_0$, which follows form the compactness of a "finite" part of the covering and the transitive action of H, we may also assume that:

$$d_{L^2_1}(y_1, y_2) < \delta \Rightarrow |C(y_1) - C(y_2)| < \frac{\epsilon}{6},$$

for $y_1, y_2 \in \tilde{\mathcal{M}}_0$. By 4.13 there exists $\kappa > 0$ such that

$$\|\mu\|_{L^2} < \kappa \Rightarrow d_{L^2_1}(\mathcal{M}_\mu, \mathcal{M}_0) < \frac{\delta}{4}.$$

Now assume that $\|\mu\|_{L^2} < \kappa$ and that $z \in \tilde{\mathcal{B}}_{L^2_1}$ has $d_{L^2_1}(z, \tilde{\mathcal{M}}_\mu) < \frac{1}{4}\delta$. Choose $x \in \tilde{\mathcal{M}}_\mu$ with $d_{L^2_1}(z,x) < \frac{1}{4}\delta$. By assumption on μ we get: $d_{L^2_1}(x, y_x) < \frac{1}{4}\delta$, and thus $d_{L^2_1}(z, y_z) < \frac{1}{2}\delta$. This again means that we can estimate:

$$d_{L^2_1}(y_x, y_z) \le d_{L^2_1}(y_x, x) + d_{L^2_1}(x,z) + d_{L^2_1}(z, y_z) < \delta.$$

By choice of δ we get:

$$|C(z) - C(x)| \le |C(z) - C(y_z)| + |C(y_z) - C(y_x)| + |C(y_x) - C(x)| < \frac{\epsilon}{2}$$

and this proves the claim.

Remember 4.14 which in particular gives that:

$$\forall \delta > 0 \; \exists \kappa' > 0, \lambda_0 > 0 : \text{For all } \mu \text{ with } \|\mu\|_{L^2} < \kappa' :$$
$$d_{L^2_1}([A,\psi], \tilde{\mathcal{M}}_\mu) \ge \delta \Rightarrow \|\nabla C_\mu(A,\psi)\|_{L^2} \ge \lambda_0.$$

Choosing $\|\mu\|_{L^2} < min(\kappa, \kappa')$, we have:

$$C(A,\psi) \in \Xi(\mu,\epsilon)^c \Rightarrow \|\nabla C_\mu(A,\psi)\|_{L^2} \ge \lambda_0.$$

We now calculate for $(A,\psi) \in \mathcal{C}_{L^2_{k,loc}}$ with $\tilde{F}_{\mu,\omega}(A,\psi) = 0$:

$$\begin{aligned}
\frac{d}{dt} C(A_t,\psi_t) &= < \nabla C(A_t,\psi_t), -\nabla C_\mu(A_t,\psi_t) + PQ([A,\psi],\omega)'_t >_{L^2} \\
&= -\|\nabla C(A_t,\psi_t)\|^2_{L^2} + < \nabla C(A_t,\psi_t), - *_Y \mu + PQ([A,\psi],\omega)'_t >_{L^2} \\
&\le -\|\nabla C(A_t,\psi_t)\|^2_{L^2} + \|\nabla C(A_t,\psi_t)\|_{L^2}(\|\mu\|_{L^2} + \|Q([A,\psi],\omega)_t\|_{L^2}).
\end{aligned}$$

Thus if we restrict $\|\mu\|_{L^2}, vol(Y)\epsilon_0^{-1}\|\omega\|_{C^\epsilon} < \frac{1}{4}\lambda_0$, we get that $\frac{d}{dt}C(A_t,\psi_t) \le -\frac{1}{2}\lambda_0^2$.

If $C(A_T,\psi_T) \in \Xi$ there must be a smallest $t_0 \in [T-1, T+1]$ such that $C(A_t,\psi_t) \in \Xi$ for $t \ge t_0$, as the derivative is negative for $C(A_t,\psi_t) \in \Xi$. We think here of $C(A_t,\psi_t)$ as leaving Ξ^c. A similar argument applies when $C(A_t,\psi_t)$ is entering Ξ^c. By the above result and as η is increasing on the negative axis and as $t_0 \le T$, we have:

$$\begin{aligned}
h'_{[A,\psi]}(T) &= \int_{\mathbb{R}} \eta(s) \frac{d}{dt} C(A_{s+T}, \psi_{s+T}) ds \\
&= \int_{-1 \le s \le t_0 - T} \eta(s) \frac{d}{dt} C(A_{s+T}, \psi_{s+T}) ds + \int_{t_0 - T \le s \le 1} \eta(s) \frac{d}{dt} C(A_{s+T}, \psi_{s+T}) ds \\
&< 0.
\end{aligned}$$

□

Remarks:
1) If $(A,\psi) \in \mathcal{C}(\mu)_{L^2_{k,\delta}}$ the above result implies that $C(A_t,\psi_t)$ can never exceed $C(A_{-\infty},\psi_{-\infty}) + \frac{\epsilon}{2}$ and also that when $C(A_t,\psi_t)$ has left the ϵ-interval centered on $C(A_{-\infty},\psi_{-\infty})$, it can never return. Corresponding statements hold at the other end.
2) If $[A,\psi] \in \tilde{\mathcal{B}}_{L^2_{l,\delta}}(\mu)$ the expression $C_\mu(A_t,\psi_t) - C_\mu(A_0,\psi_0)$ is independent of the lift of $[A,\psi]$ to $\mathcal{C}_{L^2_{l,\delta}}(\mu)$.
3) Any $L^2_{l,\delta}$ connection can be gauge transformed into temporal gauge. This is proved in appendix 16.

8 Manifolds of flow curves.

We will now write the equation $F_{\mu,\omega}(A,\psi)=0$ in a way suitable for bootstrapping. Given $(A,\psi)\in\mathcal{C}_{L^2_{l,\delta}}(\mu)$, $\pi_{\pm\infty}[A,\psi]\in\mathcal{M}_\mu$ and if we assume that $\mu\in i\Omega^2(Y)_{L^2_m}, m\geq l$, by 4.6 there exists gauge transformations $\sigma_{\pm\infty}\in\mathcal{G}_{L^2_{l+1}}$ such that $\pi_{\pm\infty}(A,\psi)\cdot\sigma_{\pm\infty}$ consists of a L^2_{m+1}-connection and a L^2_{m+2}-spinor. Notice furthermore that $\sigma_{\pm\infty}$ by the proof of the gauge fixing lemma, 4.5, can be assumed to lie in the connected component of the identity - the condition on the norm of the harmonic part of the one-forms is not used in the proof of 4.6. Thus there exists a gauge transformation $\sigma\in\tilde{\mathcal{G}}_{L^2_{l+1,\delta}}$ transforming (A,ψ) into a solution of the perturbed equations with $\pi_{\pm\infty}(A,\psi)$ as differentiable as stated above.

Choose a base curve $(A_0,\psi_0)\in L^2_{m+1,loc}\times L^2_{m+2,loc}$ which is asymptotically constant and has $\pi_{\pm\infty}(A_0,\psi_0)=\pi_{\pm\infty}(A,\psi)$ and write

$$A=A_0+a_\delta \quad\text{and}\quad \psi=\psi_0+\psi_\delta,$$

where $a_\delta\in i\Omega^1(Y\times\mathbb{R})_{L^2_{k,\delta}}$ and $\psi_\delta\in\Omega^0(pr^*S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{k,\delta}}$. The equations for these new variables are:

$$d^+a_\delta=\frac{1}{2}q(\psi_\delta)+q(\psi_0,\psi_\delta)+PQ([A,\psi],\omega)^+-((F_{A_0}+\mu)^+-\frac{1}{2}q(\psi_0)),$$
$$\eth_{A_0}\psi_\delta=-\frac{1}{2}a_\delta\cdot(\psi_0+\psi_\delta)-\eth_{A_0}\psi_0.$$

Note that the last term in each equation has compact support by assumption on (A_0,ψ_0). $\eth_{A_0}$ is an elliptic operator. As far as d^+ goes the operator (d^*,d^+) is elliptic. Thus if we can assume that $d^*a_\delta=0$, $F_{\mu,\omega}(A,\psi)=0$ is transformed into an elliptic set of equations. Gauge transformation of the form

$$\sigma=\exp(if)\quad\text{for}\quad f\in i\Omega^0(Y\times\mathbb{R})_{L^2_{l+1,\delta}}$$

changes a_δ into $a'_\delta=a_\delta+2idf$. Thus

$$d^*a'_\delta=0\Leftrightarrow\Delta f=\frac{i}{2}d^*a_\delta.$$

It is possible to solve this equation by Hodge theory for the $L^2_{l,\delta}$-spaces. As $\mathcal{H}^0_\delta(Y\times\mathbb{R})=0$, [14, th.3.1], we need not gauge fix any further as in 4.5.

Lemma 8.1. *If μ is $L^2_m, m\geq 3$, a solution (A,ψ) to the above gauge fixed equations consists of a $L^2_{m+1,\delta}$-connection and a $L^2_{m+2,\delta}$-spinor.*

Proof: As $L^2_{l,\delta}, l\geq 3$ is already an algebra, this is an easy argument following the lines of 4.6. $\square$

A point $(A,\psi)\in\mathcal{C}_{L^2_{k,\delta}}$ gives rice to a short sequence of maps by linearizing the four-dimensional Seiberg-Witten equations keeping the points at infinity fixed:

$$0\to i\Omega^0(Y\times\mathbb{R})_{L^2_{l+1,\delta}}\stackrel{\lambda_\psi}{\to} i\Omega^1(Y\times\mathbb{R})_{L^2_{l,\delta}}\oplus\Omega^0(pr^*S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}}\stackrel{DF_{(A,\psi)}}{\to}$$
$$i\Omega^2_+(Y\times\mathbb{R})_{L^2_{l-1,\delta}}\oplus\Omega^0(pr^*S^-_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l-1,\delta}}\to 0.$$

Here $\lambda_\psi(f) = (2df, -f\psi)$ and $DF_{(A,\psi)}$ is given by the matrix $\begin{pmatrix} P_+d & -\frac{1}{2}Dq_\psi \\ \cdot\frac{1}{2}\psi & \eth_A \end{pmatrix}$. As in the 3-dimensional case this is a complex whenever $\eth_A\psi = 0$ and gauge equivalent pairs of variables give isomorphic sequences. Thus it makes sense to talk of the sequence associated to a point $[A,\psi] \in \tilde{\mathcal{B}}_{L^2_{l,\delta}}$. Again, we may homotop the sequence defining

$$\lambda_{\psi,t}(f) = (2df, -tf\psi), DF_{(A,\psi),t}(\alpha,\eta) = \begin{pmatrix} P_+d & -\frac{t}{2}Dq_\psi \\ \cdot\frac{t}{2}\psi & \eth_A \end{pmatrix}$$

for $0 \le t \le 1$. The symbol sequence of the short sequence is exact and defining

$$\mathcal{J}_{L^2_{l,\delta}} = i\Omega^1(Y \times \mathbb{R})_{L^2_{l,\delta}} \oplus \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}}$$

and

$$\mathcal{K}_{L^2_{l,\delta}} = i\Omega^0(Y \times \mathbb{R})_{L^2_{l,\delta}} \oplus i\Omega^2_+(Y \times \mathbb{R})_{L^2_{l,\delta}} \oplus \Omega^0(pr^* S^-_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}}$$

we thus obtain an elliptic differential operator

$$L_{(A,\psi),\delta,t} : \mathcal{J}_{L^2_{l,\delta}} \to \mathcal{K}_{L^2_{l-1,\delta}}$$

given by $L_{(A,\psi),\delta,t} = (\lambda^*_{\psi,t}, DF_{(A,\psi),t})$. For $t = 0$, this is a direct sum of two elliptic operators - the Dirac operator $\eth_A$ and the deRham-operator (d^*, P_+d).

The above operator $L_{(A,\psi),\delta,t}$ is Fredholm if we restrict (A,ψ) to $\mathcal{C}_{L^2_{k,\delta}}(\mu)$ and $|\delta| < \delta_\mu$. This follows from $[20, 6.1]$ and 6. Furthermore, the index of this operator is given by the spectral flow between $[A,\psi]_{-\infty}$ and $[A,\psi]_\infty$:

$$ind(L_{(A,\psi),\delta,t}) = i([A,\psi]_{-\infty}) - i([A,\psi]_\infty).$$

Because, as we saw in the section on relative indices, if we change (A,ψ) using a smooth partition function into $(A,\psi)'$ which is asymptotically constant, the new operator $L_{(A,\psi)',\delta,t}$ will be Fredholm for the specified values of δ and have an index given by the spectral flow above. But this is independent of where (A,ψ) is cut off and as (A_t,ψ_t) converges exponentially to $(A,\psi)_{\pm\infty}$, if we cut off for sufficiently large values of t, the operators will be so close in operator norm, that they have the same index.

Considering $L_{(A,\psi),\delta,t}$ as a continous family of Fredholm operators between the trivial Hilbert bundles $\mathcal{J}_{L^2_{l,\delta}}$ and $\mathcal{K}_{L^2_{l-1,\delta}}$ over $\mathcal{C}_{L^2_{l,\delta}} \times I$, we can form the determinant line bundle $det(L_\delta)$ of the family. As in the 3-dimensional case $L_{(A,\psi),\delta,t}$ commutes with the action of $\tilde{\mathcal{G}}_{L^2_{l+1,\delta}}$ on $\mathcal{J}_{L^2_{l,\delta}}$ and $\mathcal{K}_{L^2_{l-1,\delta}}$

$$L_{(A,\psi)\cdot\sigma}((f,\alpha,\eta)\cdot\sigma) = L_{(A,\psi)}(f,\alpha,\eta)\cdot\sigma,$$

and $\tilde{\mathcal{G}}_{L^2_{l+1,\delta}}$ thus acts on $det(L_\delta)$. Because of the slice theorem $det(L_\delta)$ descends to a line bundle - again denoted $det(L_\delta)$ - over $\tilde{\mathcal{B}}_{L^2_{l,\delta}} \times I$ by taking the quotient.

Lemma 8.2. *$det(L_\delta)$ is a trivial line bundle over $\tilde{\mathcal{B}}_{L^2_{l,\delta}} \times I$ and it is canonically oriented.*

Proof: The proof of triviality is exactly as in 5.1. As for the orientation the trivial line bundle over $\mathcal{B}_{L^2_{l,\delta}} \times \{0\}$ is now a direct sum of the continously varying Dirac operators and the deRham-operator above. The factors of the determinant line bundle determined by the Dirac operators has a complex structure and is thus naturally oriented. The deRham-factor looks as follows

$$\Lambda^{top}\mathcal{H}^1_\delta(Y \times \mathbb{R}) \otimes \Lambda^{top}\mathcal{H}^0_\delta(Y \times \mathbb{R})^* \otimes \Lambda^{top}\mathcal{H}^2_{+,\delta}(Y \times \mathbb{R})^*$$

But the involved spaces are all zero because of the demand of exponential convergence to zero along the $\mathbb{R}$-axis. We have already seen this for the harmonic functions [14, th.3.1]. As for the one-forms, writing $\omega = \alpha + \beta \wedge dt$, $d\omega = 0, d^*\omega = 0$ implies the equation:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha \\ *\beta\end{pmatrix} = D\begin{pmatrix}\alpha \\ *\beta\end{pmatrix},$$

where

$$D = \begin{pmatrix}-i*d & d* \\ d* & 0\end{pmatrix}.$$

$D^* = -D$ and thus $\|\omega\|_{L^2}$ is constant if ω is harmonic. For the self-dual two-forms my reference is [7, prop.2.16]. □

Theorem 8.3. *For a generic set of perturbations $\omega \in \Omega_l$ and for all $\alpha, \beta \in \tilde{\mathcal{M}}_\mu$*

$$\mathcal{M}(\alpha,\beta) := \{(A,\psi) \in \mathcal{C}_{L^2_{l,\delta}}(\mu) | \tilde{F}_{\mu,\omega}(A,\psi) = 0\}/\tilde{\mathcal{G}}_{L^2_{l+1,\delta}} \subseteq \tilde{\mathcal{B}}_{L^2_{l,\delta}}(\mu)$$

is a smooth and oriented manifold of dimension $i(\alpha,\beta) = i(\alpha) - i(\beta)$.

Proof: As $\tilde{F}_\mu$ is $\tilde{\mathcal{G}}_{L^2_{k+1,\delta}}$ equivariant, by the slice theorem it is enough to show that $\tilde{F}^{-1}_{\mu,\omega}(0)$ is a smooth manifold for a generic perturbation. The key step is to show that $\tilde{F}^{-1}_\mu(0)$ is a smooth submanifold of $\mathcal{C}_{L^2_{l,\delta}}(\mu) \times \Omega_l$. So assume that $(A,\psi,\omega) \in \tilde{F}^{-1}_\mu(0)$. We claim that the differential of $\tilde{F}_\mu$ at (A,ψ,ω) is surjective from the "slice":

$$\begin{aligned} D_{(A,\psi,\omega)}\tilde{F}_\mu : i\Omega^1(Y \times \mathbb{R})_{L^2_{l,\delta}} \oplus \Omega^0(pr^*S^+_\mathbb{C}(P_{Spin^c}))_{L^2_{l,\delta}} \oplus \Omega_l \\ \to i\Omega^2_+(Y \times \mathbb{R})_{L^2_{l-1,\delta}} \oplus \Omega^0(pr^*S^-_\mathbb{C}(P_{Spin^c}))_{L^2_{l-1,\delta}}, \end{aligned}$$

$$D_{(A,\psi,\omega)}\tilde{F}_\mu(\alpha,\phi,\kappa) = (d^+\alpha + h^*_{[A,\psi]}(\kappa)^+ + PD_{([A,\psi],\omega)}Q(\alpha,\phi)^+ - \frac{1}{2}D_\psi q(\phi), \eth_A\phi + \frac{1}{2}\alpha\cdot\psi).$$

Surjectivity of the differential of $\tilde{F}_\mu$ will follow from surjectivity of

$$\tilde{L}_{(A,\psi)} = \lambda^*_\psi \oplus D_{(A,\psi,\omega)}\tilde{F}_\mu : \mathcal{J}_{L^2_{l,\delta}} \oplus \Omega_l \to \mathcal{K}_{L^2_{l-1,\delta}}.$$

As $\tilde{L}_{(A,\psi)}$ commutes with the action of the gauge group $\tilde{\mathcal{G}}_{L^2_{l+1,\delta}}$, we can assume that (A,ψ) is in temporal gauge and is as regular as described in 8.1.

$L_{(A,\psi)}$ is Fredholm and so has a closed image of finite codimension. This is thus also true for $\tilde{L}_{(A,\psi)}$. Consider now a triple $(g,\tau,\eta) \in \mathcal{K}_{L^2_{-l+1,-\delta}}$ L^2-orthogonal to this image.

g is in particular orthogonal to the image of

$$\lambda^*_\psi \lambda_\psi(f) = 4\Delta(f) + f|\psi|^2.$$

This is a selfadjoint Fredholm operator and by the maximum principle it has no kernel. This means that it is an isomorphism and thus $g = 0$. As the image of $PQ_{[A,\psi]}$ is dense in $\Omega^2(Y \times C(A,\psi)^{-1}(\Xi))_{L^2_l}$, varying κ alone we get that τ must be zero on $Y \times C(A,\psi)^{-1}(\Xi)$.

Setting $\alpha = 0, \kappa = 0$ and assuming $supp(\phi) \subseteq Y \times C(A,\psi)^{-1}(\Xi)$, we get:

$$< \eth_A\phi, \eta > = < \phi, \eth_A\eta > = 0 \Rightarrow \eth_A\eta_{|Y\times C(A,\psi)^{-1}(\Xi)} = 0.$$

By assumption on the regularity of A, this means that η is $L^2_{m+2,loc}$ on $Y \times h^{-1}_{[A,\psi]}(\Xi)$ and η is thus in particular continous. We now argue as in [23, V.2.1]: As $\psi \neq 0$, $\eth_A\psi = 0$ and $\eth_A$ has the unique continuation property by appendix 15, ψ does not vanish on an open set. The same is true for $\eta_{|Y\times C(A,\psi)^{-1}(\Xi)}$ - if η is not identically zero here. Assuming that this is not the case, there is a point $(y_0,t_0) \in Y \times C(A,\psi)^{-1}(\Xi)$ where $\psi(y_0,t_0), \eta(y_0,t_0) \neq 0$. If $C(A,\psi)^{-1}(\Xi) \supset (t_-;t_+)$, we assume that $t_0 \in (t_-+1; t_+-1)$. The projection $\mathbb{R}^4\otimes_{\mathbb{R}}\mathbb{C} \to (Cl_1(\mathbb{R}^4)\otimes_{\mathbb{R}}\mathbb{C})^+$ and Clifford multiplication induce an isomorphism

$$\mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C} \approx (Cl_1(\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C})^+ \approx Hom(S^+_{\mathbb{C}}(\mathbb{R}^4), S^-_{\mathbb{C}}(\mathbb{R}^4)).$$

As $i\mathbb{R}^4 \to S^-_{\mathbb{C}}(\mathbb{R}^4), v \mapsto v \cdot w, w \neq 0$, is surjective because

$$\langle e_i \cdot w, e_j \cdot w\rangle = 0, i \neq j,$$

there exists a purely imaginary one-form a supported in a small neighbourhood of (y_0,t_0) with

$$\langle a \cdot \psi, \eta\rangle \neq 0.$$

This contradicts the assumption that (g,τ,η) is L^2-orthogonal to the image of $D_{(A,\psi,\omega)}\tilde{F}_\mu$, and thus $\eta_{|Y\times C(A,\psi)^{-1}(\Xi)} = 0$.

By the assumption that A is in temporal gauge and the calculations in section 6, we see that (τ,η) corresponds to an element in the kernel of

$$(\frac{d}{dt} + \check{L}_{r_t(A,\psi)})^* = -\frac{d}{dt} + \check{L}_{r_t(A,\psi)}.$$

But by [5] and arguments similar to those in appendix 15, this operator has the unique continuation property and so (τ,η) is zero as it vanishes on an open set.

So we see that $\tilde{F}_\mu^{-1}(0)$ is a smooth submanifold of $\mathcal{C}_{L^2_{l,\delta}}(\mu)$ and using the slice theorem so is it's reduction moduli the gauge group, $[\tilde{F}_\mu^{-1}(0)]$.
Consider the projection

$$\pi : [\tilde{F}_\mu^{-1}(0)] \to \Omega_l.$$

π is Fredholm:
$Ker D_{(A,\psi,\omega)}\pi = Ker D_{(A,\psi,\omega)}\tilde{F}_\mu \cap Ker\lambda^*_\psi \cap T_{(A,\psi)}\mathcal{C}_{L^2_{k,\delta}}(\mu) = Ker L'_{(A,\psi)}$,
$Coker D_{(A,\psi,\omega)}\pi \approx Coker L'_{(A,\psi)}$,
using the map $[\kappa] \mapsto [h^*_{[A,\psi]}(\kappa)^+]$ and where $L'_{(A,\psi)} := \tilde{L}(A,\psi)_{|\mathcal{J}_{L^2_{l,\delta}}}$. Thus by the Sard-Smale theorem for Lindelöf Banach manifolds, [1, 16.2], there is a generic set of perturbations in Ω_l that are regular values of π. The inverse image of such a regular value is exactly $\coprod_{\alpha,\beta\in\tilde{\mathcal{M}}_\mu}\mathcal{M}(\alpha,\beta)$ and the dimension of $\mathcal{M}(\alpha,\beta)$ is the index of π at $[A,\psi]\in\mathcal{M}(\alpha,\beta)$, which by the above equals $ind(L'_{(A,\psi)})$. As $L'_{(A,\psi)}$ equals $L_{(A,\psi)}$ up to a compact perturbation, $ind(L'_{(A,\psi)}) = ind(L_{(A,\psi)})$ and the remarks preceeding the theorem gives the result on dimensions.

As regards the orientation of $\mathcal{M}(\alpha,\beta)$, we have

$$T_{[A,\psi]}\mathcal{M}(\alpha,\beta) = Ker L'_{(A,\psi)} = Det(L')_{[A,\psi]}.$$

Homotoping the compact perturbation away, gives that $Det(L') \cong Det(L)$, where the isomorphism is well defined up to multiplication by a positive function [23, V.6.1]. By 8.2 the determinant line bundles are oriented and we get an orientation of $\mathcal{M}(\alpha,\beta)$. □

Notice that the arguments of the proof of lemma 7.4 gives that the "lazy curves" constantly equal to a critical point is in fact a solution of the perturbed gradient flow of ∇C_μ. From lemma 7.4 we see that actually for $\alpha=\beta$:

$$\mathcal{M}(\alpha,\alpha) = \{\alpha\}.$$

Under the action of $\sigma\in H$, the perturbation Q transformed by a translation on ω:

$$Q([A,\psi]\cdot\sigma,\omega) = Q([A,\psi],\omega_{8\pi^2 k(\sigma)N}).$$

We see that $[F^{-1}_{\mu,\omega}(0)] \subseteq \mathcal{B}(\alpha,\beta)_{L^2_{l,\delta}}$ is mapped to $[F^{-1}_{\mu,\omega_{8\pi^2k(\sigma)N}}(0)] \subseteq \mathcal{B}(\alpha\cdot\sigma,\beta\cdot\sigma)_{L^2_{l,\delta}}$ by the action of σ. Also,

$$\begin{aligned} L'_{(A,\psi)\cdot\sigma} &= L_{(A,\psi)\cdot\sigma} + (PD_{[A,\psi]\cdot\sigma}Q, 0) \\ &= (Id,\sigma)\circ L_{(A,\psi)}\circ(Id,\sigma^{-1}) + (PD_{[A,\psi]},0) \\ &= (Id,\sigma)\circ L'_{(A,\psi)}\circ(Id,\sigma^{-1}) \end{aligned}$$

from what we get that $L'_{(A,\psi)\cdot\sigma}$ is surjective iff $L'_{(A,\psi)}$ is. Or in other words, ω is a regular value of

$$\pi : [F_\mu^{-1}(0)] \subseteq \mathcal{B}(\alpha,\beta)_{L^2_{l,\delta}} \to \Omega_l$$

iff $\omega_{8\pi^2 k(\sigma)N}$ is a regular value of

$$\pi : [F_\mu^{-1}(0)] \subseteq \mathcal{B}(\alpha\cdot\sigma, \beta\cdot\sigma)_{L^2_{l,\delta}} \to \Omega_l.$$

In section 11 it will be convienient that the same perturbation is used for all manifolds of flow curves. As there is no reason why ω should be $8\pi^2 N$-periodic, fulfilling this demand means that there is no imidiate diffeomorphism

$$\mathcal{M}(\alpha,\beta) \approx \mathcal{M}(\alpha\cdot\sigma, \beta\cdot\sigma),$$

as would seem natural as the unperturbed Seiberg-Witten equations are gauge equivariant. I claim though that the manifolds $\mathcal{M}(\alpha,\beta)$ and $\mathcal{M}(\alpha\cdot\sigma,\beta\cdot\sigma)$ corresponding to the same ω are even diffeotopic. To prove this one could consider the variation of ω to $\omega_{8\pi^2 k(\sigma)N}$ and prove that there is a $i(\alpha)-i(\beta)+1$-dimensional manifold with boundary $\mathcal{M}(\alpha,\beta)\coprod\mathcal{M}(\alpha\cdot\sigma,\beta\cdot\sigma)$. I won't try to prove this, but simply assume the above claim.

We will now examine the structure of $\coprod_{\alpha,\beta\in\tilde{\mathcal{M}}_\mu}\mathcal{M}(\alpha,\beta)$ with respect to the action of H on $\tilde{\mathcal{B}}_{L^2_{l,\delta}}$. First notice that by 6.1

$$i(\beta\cdot\sigma) = i(\beta\cdot\sigma,\beta) + i(\beta,\alpha_0) = i(\beta) - < c_1(\mathcal{L})\cup c(\sigma), [Y] > .$$

This means that

$$i(\alpha,\beta\cdot\sigma) = i(\alpha) - i(\beta) + < c_1(\mathcal{L})\cup c(\sigma), [Y] > .$$

Thus the manifolds of flow curves can be organised in families: If we choose a lift of $\mathcal{M}_\mu$ to $\tilde{\mathcal{M}}_\mu$ and let $t\cdot$ denote action by a generator σ_0 of H with

$$< c_1(\mathcal{L})\cup c(\sigma_0), [Y] > = N(P_{Spin^c}).$$

and also mod out the diagonal action of H, we get

$$\coprod_{\alpha,\beta\in\mathcal{M}_\mu;k\in\mathbb{Z}}\mathcal{M}(\alpha,t^k\beta) = \coprod_{\alpha,\beta\in\tilde{\mathcal{M}}_\mu}\mathcal{M}(\alpha,\beta).$$

As the equations are translation invariant, there is an $\mathbb{R}$-action on $\mathcal{M}(\alpha,\beta)$ for all $\alpha,\beta\in\tilde{\mathcal{M}}_\mu$ defined by

$$\gamma\cdot s := \gamma_s, \quad \gamma_s(t) = \gamma(t+s), t\in\mathbb{R}$$

for $\gamma\in\mathcal{M}(\alpha,\beta)$ and $s\in\mathbb{R}$. As the translation map

$$(\tau, s)\mapsto\tau_s$$

on $\mathcal{C}_{L^2_{k,\delta}}\times\mathbb{R}$ is continous, [2, p.32], and as

$$\frac{d}{ds}\gamma_s = \dot{\gamma}_s$$

this action is l times differentiable. For $\alpha \neq \beta$, it is also free, as we know that C is strictly decreasing along part of the curve by lemma 7.4.
Define a map $\phi : \mathcal{M}(\alpha, \beta) \to \mathbb{R}$ by

$$\phi(\gamma) := C(\gamma(0)).$$

ϕ is well defined and smooth and

$$D_\gamma \phi(\tau) =< \nabla C(\gamma(0)), \omega(0) >_{L^2} .$$

Choose a regular value ϵ for ϕ in the interval $\Xi \cap]C(\beta); C(\alpha)[$. Then

$$\mathcal{M}(\alpha, \beta)_0 := \phi^{-1}(\epsilon)$$

is a smooth submanifold of $\mathcal{M}(\alpha, \beta)$ of dimension $i(\alpha) - i(\beta) - 1$. We claim that the map

$$\Psi^0 : \mathcal{M}(\alpha, \beta)_0 \times \mathbb{R} \to \mathcal{M}(\alpha, \beta),$$
$$\Psi^0(\gamma, s) := \gamma \cdot s,$$

is a C^l-diffeomorphism. By the above it is l times differentiable and it is also equivariant with respect to the $\mathbb{R}$-action:

$$\Psi^0(\gamma, s + t) = \gamma \cdot (s + t) = (\gamma \cdot s) \cdot t = \Psi^0(\gamma, s) \cdot t.$$

Ψ^0 is injective: Assume that $\gamma \cdot s = \gamma' \cdot s'$. The map

$$t \mapsto C(\gamma(t))$$

has derivative strictly negative for $C(\gamma(t)) \in \Xi$ by lemma 7.4 and thus we see that it attains the value ϵ precisely once. But γ and γ' are in $\mathcal{M}(\alpha, \beta)_0$ and so $s = s'$, implying that $\gamma = \gamma'$.
Ψ^0 is surjective: For every $\gamma \in \mathcal{M}(\alpha, \beta)$, there exists a unique $t \in \mathbb{R}$ with $C(\gamma(t)) = \epsilon$ by the above. Translate by $-t$ to get an element of $\mathcal{M}(\alpha, \beta)_0, \gamma \cdot (-t)$. Now $\Psi^0(\gamma \cdot (-t), t) = \gamma$.
The differential of Ψ^0 is injective: As Ψ^0 is equivariant it is enough to show this at $s = 0$. Here we have:

$$D_{(\gamma,0)} \Psi^0(\omega, t) = t\dot{\gamma} + \omega.$$

Assume that (ω, t) is in the kernel of this map. As $\omega \in T_\gamma \mathcal{M}(\alpha, \beta)_0$ we have $< \nabla C(\gamma(0)), \omega(0) >= 0$ and

$$\frac{d}{dt} C(\gamma(t))_{|t=0} =< \nabla C(\gamma(0)), \dot{\gamma}(0) >_{L^2} < 0,$$

implies that $t = 0$ and thus that $\omega = 0$.
The inverse function theorem now gives the conclusion.

Define

$$\hat{\mathcal{M}}(\alpha,\beta) := \mathcal{M}(\alpha,\beta)/\mathbb{R}.$$

This quotient is homeomorphic to $\mathcal{M}(\alpha,\beta)_0$ and as the diffeomorphism class of this manifold is independent of the choice of regular value and time of evaluation - by the above diffeomorphism - $\hat{\mathcal{M}}(\alpha,\beta)$ gets a well defined structure of differentiable manifold such that the above homeomorphism is a diffeomorphism.

As translations act orientation preserving on $\mathcal{M}(\alpha,\beta)$, we see that $\hat{\mathcal{M}}(\alpha,\beta)$ has a natural orientation.

The tangentspace of $\mathcal{M}(\alpha,\beta)_0$ at γ is

$$Ker(< \nabla C(\gamma(0)), \cdot(0) >_{L^2}).$$

As $< \nabla C\gamma(0), \dot{\gamma}(0) >_{L^2} < 0$ by choice of ϵ we thus get

$$T\mathcal{M}(\alpha,\beta) = T\mathcal{M}(\alpha,\beta)_0 \oplus \mathbb{R}\frac{\partial}{\partial t}(\cdot).$$

9 Properties of the functional C.

The previous section used the $L^2_{l,\delta}$ Sobolev spaces as a model for the manifolds of flow curves. A natural question is whether this is reasonable. Do all flow curves converge exponentially towards critical points - and if so by what rate? This section will be concerned with answering these questions. Also, we state a regularity theorem analogous to 5.4.

The first lemma analyses the local behaviour of C_μ near a non-degenerate critical point, that is: A critical point where the Hessian H from section 5 is an isomorphism.

Lemma 9.1. *Assume that $\alpha \in \mathcal{B}^*_{L^2_1}$ is a non-degenerate critical point of C_μ. Then there exists a neighbourhood U of α and a constant $D_\alpha \geq 1$ such that if $u \in U$:*

$$|C_\mu(u) - C_\mu(\alpha)| \leq D_\alpha \delta_\alpha^{-1} \|\nabla(C_\mu)(u)\|^2_{L^2},$$

where when evaluating C_μ we work in a particular slice so that the difference above is well defined.

Proof: Let U' be a neighbourhood of α so small that it is contained in a slice and so that it can be identified with an open, convex neighbourhood of zero in $Ker\lambda^*_\psi$ where α corresponds to zero. The L^2-tangent vector bundle, $T\mathcal{B}^*_{L^2}$, is trivial over U' and ∇C_μ is a section in this bundle. By assumption $\nabla C_\mu(\alpha) = 0$ and $H_\alpha : V_{\alpha, L^2_1} \to V_{\alpha, L^2}$ is an isomorphism. Working in this local picture the inverse function theorem implies that there is a neighbourhood of zero, U, such that the map

$$\nabla C_\mu : U \to V \subseteq V_{\alpha, L^2}$$

is an diffeorphism with inverse $\Psi : V \to U$. We may assume that $\|H_u\| \leq K$ for $u \in U$ and $\|D_v \Psi\| \leq 2\delta_\alpha^{-1}$ for $v \in V$ by the remark at the end of section 6. We then have for $u \in U$:

$$\begin{aligned}
|C_\mu(u) - C_\mu(\alpha)| &= |\int_0^1 \frac{d}{dt} C_\mu(u + t(\alpha - u))dt| \\
&= |\int_0^1 < \nabla C_\mu(u + t(\alpha - u)), \alpha - u >_{L^2} dt| \\
&= |\int_0^1 < \nabla C_\mu(u + t(\Psi(0) - \Psi(\nabla C_\mu(u)))), \\
&\quad \Psi(0) - \Psi(\nabla C_\mu(u)) >_{L^2} dt| \\
&\leq \int_0^1 \|\nabla C_\mu(u + t(\Psi(0) - \Psi(\nabla C_\mu(u))))\|_{L^2} dt \\
&\quad \cdot \|\Psi(0) - \Psi(\nabla C_\mu(u))\|_{L^2} \\
&\leq (\|\nabla C_\mu(u)\|_{L^2} + K\|\Psi(0) - \Psi(\nabla C_\mu(u))\|_{L^2_1}) \\
&\quad \cdot \|\Psi(0) - \Psi(\nabla C_\mu(u))\|_{L^2_1} \\
&= 2(1 + 2K\delta_\alpha^{-1})\delta_\alpha^{-1} \|\nabla C_\mu(u)\|^2_{L^2}.
\end{aligned}$$

As the inequality is independent of the chosen slice, we are done. □

Let $u : [t_0; \infty) \to \mathcal{B}^*_{L^2_k}$ be a flow curve of the vector field $-\nabla C_\mu$, so that:

$$\dot{u}(t) = -\nabla C_\mu(u(t)).$$

Assume furthermore, that $u(t) \to \alpha, t \to \infty$ in the L^2_k-topology. This implies that for $t \geq t_0$:

$$\begin{aligned} C_\mu(u(t)) - C_\mu(\alpha) &= -\int_t^\infty \frac{d}{dt} C_\mu(u(s)) ds \\ &= -\int_t^\infty < \nabla C_\mu(u(s)), \dot{u}(s) >_{L^2} ds \\ &= \int_t^\infty \|\nabla C_\mu(u(s))\|^2_{L^2} ds. \end{aligned}$$

Using 9.1 (for $t \geq t_0$, say) we thus see that

$$\begin{aligned} C_\mu(u(t)) - C_\mu(\alpha) &\leq D_\alpha \delta_\alpha^{-1} \|\nabla C_\mu(u(t))\|^2_{L^2} \\ &= -D_\alpha \delta_\alpha^{-1} \frac{d}{dt}(C_\mu(u(t)) - C_\mu(\alpha)). \end{aligned}$$

Thus

$$\begin{aligned} &\frac{d}{dt} log(C_\mu(u(t)) - C_\mu(\alpha)) \leq -D_\alpha^{-1} \delta_\alpha \\ &\quad \Rightarrow C_\mu(u(t)) - C_\mu(\alpha) \leq K \exp(-D_\alpha^{-1} \delta_\alpha (t - t_0)), \end{aligned}$$

so that $C_\mu(u(t))$ converges exponentially towards it's limiting value. This has further implications:

$$\begin{aligned} -\frac{d}{dt}(C_\mu(u(t)) - C_\mu(\alpha))^{\frac{1}{2}} &= -\frac{1}{2}(C_\mu(u(t)) - C_\mu(\alpha))^{-\frac{1}{2}} \frac{d}{dt}(C_\mu(u(t)) - C_\mu(\alpha)) \\ &= \frac{1}{2}(C_\mu(u(t)) - C_\mu(\alpha))^{-\frac{1}{2}} \|\nabla C_\mu(u(t))\|_{L^2} \|\dot{u}(t)\|_{L^2} \\ &\geq \frac{\delta_\alpha^{\frac{1}{2}}}{2 D_\alpha^{\frac{1}{2}}} \|\dot{u}(t)\|_{L^2} \\ \Rightarrow \|\dot{u}(t)\|_{L^2} &\leq -2 D_\alpha^{\frac{1}{2}} \delta^{-\frac{1}{2}} \frac{d}{dt}(C_\mu(u(t)) - C_\mu(\alpha))^{\frac{1}{2}}. \end{aligned}$$

We can now estimate the L^2-distance from $u(t)$ to α:

$$\begin{aligned} d_{L^2}(u(t), \alpha) &\leq \int_t^\infty \|\dot{u}(s)\|_{L^2} ds \\ &\leq \int_t^\infty -2 D_\alpha^{\frac{1}{2}} \delta_\alpha^{-\frac{1}{2}} \frac{d}{dt}(C_\mu(u(s)) - C_\mu(\alpha))^{\frac{1}{2}} ds \\ &= 2 D_\alpha^{\frac{1}{2}} \delta_\alpha^{-\frac{1}{2}} (C_\mu(u(t)) - C_\mu(\alpha))^{\frac{1}{2}} \\ &\leq 2 D_\alpha^{\frac{1}{2}} \delta_\alpha^{-\frac{1}{2}} K^{\frac{1}{2}} \exp(-\frac{1}{2} D_\alpha^{-1} \delta_\alpha (t - t_0)). \end{aligned}$$

So the L^2-distance decreaces exponentially. The proofs of the above inequalities are taken from $[30, 2\&3]$, where a more general situation is considered. Of course, there is a completely analogous result if u is defined on an interval infinite to the left.

Lemma 9.2. *Given an element (A,ψ) of $\mathcal{C}_{L^2_{l,loc}}(Y\times\mathbb{R})$ with $\tilde{F}_\mu(A,\psi)=0$ we have:*
1) If (A,ψ) has a finite variation of C and $\|\mu\|_{L^2_m}$ and $\|\omega\|_{C^\epsilon}$ are sufficiently small, (A,ψ) has a finite variation of C_μ.
2) If (A,ψ) has a finite variation of C and of C_μ, $[A_t,\psi_t]$ approaches critical points of C_μ in $\tilde{\mathcal{B}}$ in the L^2_l-topology for $t\to\pm\infty$.

Proof: Ad 1): By appendix 16 we may assume that (A,ψ) is in temporal gauge. Using lemma 7.4 and the assumption on μ and ω, for t sufficiently big, say $t\geq T$, $C(A_t,\psi_t)\in\Xi^c$ and thus

$$\frac{d}{dt}(A_t,\psi_t)=-\nabla C_\mu(A_t,\psi_t).$$

As

$$\int_T^\infty \frac{d}{dt}C(A_t,\psi_t)dt=\int_T^\infty <\nabla C(A_t,\psi_t),-\nabla C_\mu(A_t,\psi_t)>_{L^2}dt$$

is finite, the integrand cannot be bounded away from zero. Thus there exists a sequence $(t_n)_{n=1}^\infty$ in $\mathbb{R}$ with $t_n\to_n\infty$ and

$$\begin{aligned}<\nabla C(A_{t_n},\psi_{t_n}),-\nabla C_\mu(A_{t_n},\psi_{t_n})>_{L^2}&=-\|\nabla C_\mu(A_{t_n},\psi_{t_n})\|^2_{L^2}\\ &\quad-<(\nabla C_\mu(A_{t_n},\psi_{t_n}))_1,*\mu>_{L^2}\to_n 0.\end{aligned}$$

Now

$$\begin{aligned}\|\nabla C_\mu(A_{t_n},\psi_{t_n})\|^2_{L^2}&=-(-\|\nabla C_\mu(A_{t_n},\psi_{t_n})\|^2_{L^2}-<(\nabla C_\mu(A_{t_n},\psi_{t_n}))_1,*\mu>_{L^2})\\ &\quad-<(\nabla C_\mu(A_{t_n},\psi_{t_n}))_1,*\mu>_{L^2}\\ &\leq-(-\|\nabla C_\mu(A_{t_n},\psi_{t_n})\|^2_{L^2}-<(\nabla C_\mu(A_{t_n},\psi_{t_n}))_1,*\mu>_{L^2})\\ &\quad+\|(\nabla C_\mu(A_{t_n},\psi_{t_n}))_1\|_{L^2}\|\mu\|_{L^2}\end{aligned}$$

implies that $\|\nabla C_\mu(A_{t_n},\psi_{t_n})\|_{L^2}$ is bounded. Arguing as in the proof of 4.12 gives a bound on $\|A_{t_n}\|_{L^2_1}$ and $\|\psi_{t_n}\|_{L^2_1}$. As

$$C_\mu(A_t,\psi_t)=C(A_t,\psi_t)+\int_Y\mu\wedge a_t,$$

this gives that $C_\mu(A_{t_n},\psi_{t_n})$ is bounded. By lemma 7.4 $C_\mu(A_t,\psi_t)$ is strictly decreasing or constant and is thus bounded for $t\to\infty$. A similar argument applies at the other end.
Ad 2): By lemma 7.4 we may assume that

$$\begin{aligned}\frac{d}{dt}(C_\mu(A_t,\psi_t)-C_\mu(A_0,\psi_0))&=-\|\nabla C_\mu(A_t,\psi_t)\|^2_{L^2}\\ &\quad+<\nabla C_\mu(A_t,\psi_t),PQ([A,\psi],\omega)>_{L^2}<0.\end{aligned}$$

This is by assumption integrable over $\mathbb{R}$. By lemma 7.4, the derivative of $C(A_t, \psi_t)$ is negative and bounded away from zero, if $C(A_t, \psi_t) \in \Xi$. Thus there must exist T_0 such that $C(A_t, \psi_t) \in \Xi^c$ for $|t| \geq T_0$. By construction this implies that $Q([A, \psi], \omega)_t = 0$ for $|t| \geq T_0$.

For $|t|$ big we thus have

$$\frac{d}{dt}(C_\mu(A_t, \psi_t) - C_\mu(A_0, \psi_0)) = -\|\nabla C_\mu(A_t, \psi_t)\|_{L^2}^2$$

and as this was integrable, we can find a sequence $(s_n)_{n\in\mathbb{N}}$, $s_n \to_n \infty$ with

$$\|\nabla C_\mu(A_{s_n}, \psi_{s_n})\|_{L^2} \to_n 0.$$

By the Palais-Smale property and the finite variation of C, $[A_{s_n}, \psi_{s_n}]$ can be assumed to converge to an element α' of $\tilde{\mathcal{M}}_\mu$. As $C_\mu(A_t, \psi_t) - C_\mu(A_0, \psi_0)$ has a limiting value, we get by [25, §12, prop.2] that $[A_t, \psi_t]$ converges in $\mathcal{B}_{L_l^2}$ towards β' for $t \to \infty$. Similarly, we find that (A_t, ψ_t) approaches a critical point α' of C_μ for $t \to -\infty$. $\square$

Let $(A, \psi) \in \mathcal{C}_{L_{l,loc}^2}(Y \times \mathbb{R})$ with $F_{\mu,\omega}(A, \psi) = 0$ and $C(A, \psi)$ bounded. Then by the above lemma $[A_t, \psi_t]$ converges to elements of $\tilde{\mathcal{M}}_\mu$, α and β, for $t \to \pm\infty$, respectively. By the first part of this section it follows that $[A, \psi] \in \tilde{\mathcal{B}}_{L_\delta^2}(\mu) \cap \tilde{\mathcal{B}}_{L_{l,loc}^2}$. A bootstrapping argument will give that $[A, \psi] \in \tilde{\mathcal{B}}_{L_{l,\delta}^2}(\mu)$.

We see that all gradient flow curves of C_μ with a finite variation of C converges exponentially towards critical points of C_μ under suitable conditions on the Sobolev-norms used. Also, the rate of convergence depends on the critical points through the constant

$$\delta'_\alpha = \frac{1}{2} D_\alpha \delta_\alpha,$$

Define

$$\delta(\mu)' := min_{\alpha \in \mathcal{M}_\mu} \delta'_\alpha.$$

This is smaller than $\delta(\mu)$ defined at the end of section 6 but of the same order. All statements of the previous sections should be reformulated in terms of this new constant, though probably it is possible to do somewhat better, that is, get a bigger rate of decay than in the above argument.

Lemma 9.3. *If μ is $L_m^2, m \geq 3$, $\mathcal{M}(\alpha, \beta)$ is up to diffeomorphism independent of $l, m \geq l \geq 3$ and $\delta, 0 \leq \delta < \delta'_\mu$.*

Proof: The proof follows that of 5.4. The obvious inclusion map

$$\mathcal{M}(\alpha, \beta)_{L_{p,\delta}^2} \hookrightarrow \mathcal{M}(\alpha, \beta)_{L_{l,\delta'}^2}$$

for $p \geq l$ and $\delta \geq \delta'$ is surjective by 8.1 and the results of this section, and by an argument similar to the one in 5.4 it is injective. As the index of the Fredholm operators involved does not depend on the Sobolev-index l and the decay rate δ by [20, 7.1], the differential of the inclusion map is a linear isomorphism. The inverse function theorem now gives the inclusion. $\square$

10 Compactness properties of the moduli space of flow curves.

The statement of the theorem in this section is that although $\hat{\mathcal{M}}(\alpha,\beta)$ is not compact in general, a sequence of points in this space does have a subsequence with at least some local convergence properties.

Theorem 10.1. *$\hat{\mathcal{M}}(\alpha,\beta)$ has the following compactness property: A sequence $([A_n,\psi_n])_{n\in\mathbb{N}}$ in $\hat{\mathcal{M}}(\alpha,\beta)$ either has a subsequence converging to an element of $\hat{\mathcal{M}}(\alpha,\beta)$ or there exists*

$$\alpha=\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_{m-1},\alpha_m=\beta$$

elements in $\tilde{\mathcal{M}}_\mu$ with

$$i(\alpha)>i(\alpha_1)>i(\alpha_2)>\ldots>i(\alpha_{m-1})>i(\beta)$$

and $[A^j,\psi^j]\in\hat{\mathcal{M}}(\alpha_{j-1},\alpha_j)$, together with a sequence of real numbers $(t_n^j)_{n\in\mathbb{N}}$, $j=1,\dots,m$, such that after passing to a subsequence

$$[A_n,\psi_n]\longrightarrow_{L^2_{l,loc}}([A^j,\psi^j])_{j=1}^m, n\to\infty.$$

Here convergence in $L^2_{l,loc}$ means that

$$[A_n,\psi_n]_{t_n^j|Y\times I}\longrightarrow_n[A^j,\psi^j]_{|Y\times I}$$

in $\tilde{\mathcal{B}}_{L^2_l}(Y\times I)$ for all finite intervals I and all j, where $[A,\psi]_t$ denotes the translation of $[A,\psi]$ by $t\in\mathbb{R}$.

Proof: Choose a lift $\hat{\mathcal{M}}(\alpha,\beta)\approx\mathcal{M}(\alpha,\beta)_0^\kappa$, where $\kappa\in\Xi\cap[C(\beta);C(\alpha)]$. Also choose representatives in $\mathcal{C}^*_{L^2_l}$ for α and β, which will also be denoted α and β, and given a sequence $([A_n,\psi_n])_{n\in\mathbb{N}}$ in $\hat{\mathcal{M}}(\alpha,\beta)$ choose corresponding lifts in $\mathcal{C}^*_{L^2_{l,\delta}}(\mu)$, $(A_n,\psi_n)_{n\in\mathbb{N}}$, with

$$\pi_{-\infty}(A_n,\psi_n)=\alpha,\quad \pi_\infty(A_n,\psi_n)=\beta.$$

By assumption we have:

$$(F_{A_n}+\mu+P([A_n,\psi_n],\omega))^+-\frac{1}{2}q(\psi_n,\psi_n)=0,$$
$$\eth_{A_n}\psi_n=0.$$

Let $T>1$. As $(P([A_n,\psi_n],\omega)_{|Y\times[-T;T]})_{n\in\mathbb{N}}$ is bounded in L^2_l-norm, we can assume by extracting a subsequence that

$$P([A_n,\psi_n],\omega)^+\to_n w\in i\Omega(Y\times[-T;T])^+_{L^2_{l-1}}.$$

By the analogue of 4.9 in the 4-dimensional case [18, lemma 4], there exists a sequence of gauge transformations $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{G}_{L^2_{l+1}}(Y \times [-T;T])$ and an element $(A_T, \psi_T) \in \mathcal{C}_{L^2_l}(Y \times [-T;T])$ such that

$$(A_n, \psi_n)_{|Y\times[-T;T]} \cdot \sigma_n \to_n (A_T, \psi_T).$$

This in particular means that

$$C((A_n(0), \psi_n(0)) \cdot \sigma_n(0)) = C(A_n(0), \psi_n(0)) + 8\pi^2 k(\sigma_n) N$$

is convergent. As $C(A_n(0), \psi_n(0))$ is constant, we see that $(k(\sigma_n))_{n\in\mathbb{N}}$ is an eventually stable sequence in $\mathbb{Z}$. In other words, the equivalence classes of the gauge transformations in H, $[\sigma_n]$, can be assumed stable. Choose a gauge transformation $\sigma \in \mathcal{G}_{L^2_{l+1}}(Y \times [-T;T])$ with $[\sigma] = \lim_n [\sigma_n]$. Then after passing to a subsequence:

$$[A_n, \psi_n]_{|Y\times[-T;T]} = [(A_n, \psi_n) \cdot \sigma_n \sigma^{-1}]_{|Y\times[-T;T]}.$$

$$(A_n, \psi_n) \cdot \sigma_n \sigma^{-1} \to_n (A_T, \psi_T) \cdot \sigma^{-1} \in \mathcal{C}_{L^2_l}(Y \times [-T;T])$$

This implies that

$$P([A_n, \psi_n], \omega)_{|Y\times[-T+1;T-1]} \to_n P([(A_T, \psi_T) \cdot \sigma^{-1}], \omega)$$

in $i\Omega^2_+(Y \times [-T+1; T-1])^+_{L^2_{l-1}}$, as $h_{[A_n,\psi_n]}$ converges uniformly to $h_{[(A_T,\psi_T)\cdot\sigma^{-1}]}$ on this interval. This gives that w equals $P([(A_T, \psi_T) \cdot \sigma^{-1}], \omega)$ on $Y \times [-T+1; T-1]$ and thus the limit $[(A_T, \psi_T) \cdot \sigma^{-1}]$ is a solution to the perturbed Seiberg-Witten equations on $Y \times [-T+1; T-1]$.

The gauge transformations $\sigma_n \sigma^{-1}$ can be extended to gauge transformations on $Y \times \mathbb{R}$ in $\tilde{\mathcal{G}}_{L^2_{l+1,\delta}}$. Starting out with the representatives $(A_n, \psi_n)_{|Y\times[-2T,2T]} \cdot \sigma_n \sigma^{-1}$ on $Y \times [-2T, 2T]$ and repeting the above procedure gives a solution to the perturbed Seiberg-Witten equations on $Y \times [-2T+1, 2T-1]$, (A_{2T}, ψ_{2T}), and a sequence of gauge transformations $(\rho_n)_{n\in\mathbb{N}}$ on $Y \times [-2T, 2T]$ representing the identity in H such that:

$$(A_n, \psi_n)_{|Y\times[-2T;2T]} \cdot \sigma_n \sigma^{-1} \cdot \rho_n \to_n (A_{2T}, \psi_{2T})$$

in $\mathcal{C}_{L^2_l}(Y \times [-2T; 2T])$.

By the fundamental convergence result on the four-manifold $Y \times [-T;T]$, we may assume that $\rho_{n|Y\times[-T;T]} \to_n \rho \in \mathcal{G}_{L^2_{l+1}}(Y \times [-T;T])$ representing the identity in H and thus

$$(A_{2T}, \psi_{2T})_{|Y\times[-T;T]} \cdot \rho^{-1} = (A_T, \psi_T) \cdot \sigma^{-1}.$$

The new solution can thus be assumed to agree with the previous one on $Y \times [-T;T]$.

Continuing in this way and arguing by diagonal sequences we get an element (A,ψ) of $\mathcal{C}_{L^2_{l,loc}}(Y\times\mathbb{R})$ which is a solution of the perturbed Seiberg-Witten equations on $Y\times\mathbb{R}$ such that for all finite intervals I we have

$$[A_n,\psi_n]_{|Y\times I}\to_n [A,\psi]_{|Y\times I}$$

in $\tilde{\mathcal{B}}_{L^2_l}(Y\times I)$. Notice that this local convergence implies that $C([A_0,\psi_0])=\kappa$, and that (A,ψ) has a finite variation of C.

By lemma 9.2 and the remarks following it there exists $\alpha',\beta'\in\tilde{\mathcal{M}}_\mu$ such that $[A,\psi]\in\hat{\mathcal{M}}(\alpha',\beta')$. $\alpha'\neq\beta'$ by 4.9 as $C([A_0,\psi_0])\notin C(\tilde{\mathcal{M}}_\mu)$. So actually $[A,\psi]\in\hat{\mathcal{M}}(\alpha',\beta')^\kappa_0$. As this manifold has dimension $i(\alpha')-i(\beta')-1$, $i(\alpha')>i(\beta')$.

By the local convergence $C(\alpha)\geq C(\alpha')$ and $C(\beta)\leq C(\beta')$. If, say, $\alpha\neq\alpha'$, we must have $C(\alpha)-C(\alpha')>0$. In that case choose another lift of $\hat{\mathcal{M}}(\alpha,\beta)$, $\mathcal{M}(\alpha,\beta)^{\kappa'}_0$, where $\kappa'\in\Xi\cap[C(\alpha');C(\alpha)]$. By this different choice of lift the original sequence will be changed by translation by a sequence $(t_n)_{n\in\mathbb{N}}$ by the end of section 8. Repeat the above procedure. This gives a new flow curve travelling between critical points α'' and β''. Possibly, now $\alpha=\alpha''$ and $\alpha'=\beta''$. If not, we must repeat the above once more - but as there is only finitely many critical point with values of C between $C(\alpha)$ and $C(\beta)$, this is a finite process.

Finally consider the case where in the first step $\alpha=\alpha'$ and $\beta'=\beta$. We must show that $([A_n,\psi_n])_{n\in\mathbb{N}}$ converges to $[A,\psi]$ in $\hat{\mathcal{M}}(\alpha,\beta)$. It is enough to show that the exponential convergence of $(A_{n,t},\psi_{n,t})$ towards α and β is uniform in n, that is:

$$\forall\epsilon>0\;\exists T_0>0:|t|\geq T_0\Rightarrow$$
$$\|D^j_y\frac{d^k}{dt^k}((A_{n,t},\psi_{n,t})-\alpha)\|_{L^2},\|D^j_y\frac{d^k}{dt^k}((A_{n,t},\psi_{n,t})-\beta)\|_{L^2}\leq K\exp(-\delta'_\mu(|t|-T))$$

for all $n\in\mathbb{N}, j+k\leq l$. Because then, by our choice of δ, we get for $T\geq T_0$:

$$\int_T^\infty e^{2\delta s}\|D^j_y\frac{d^k}{ds^k}((A_{n,s},\psi_{n,s})-\beta)\|^2_{L^2}ds\leq K\int_T^\infty e^{2(\delta-\delta'_\mu)s}ds$$
$$\leq K'\exp(2(\delta-\delta'_\mu)T)$$

and similarly on the negative end for all $n\in\mathbb{N}$. We can also assume that the same estimates hold for (A,ψ) with the same T_0. Then given $\epsilon>0$, choose $T\geq T_0$ so big that

$$\sum_{j+k\leq l}\int_T^\infty e^{2\delta s}\|D^j_y\frac{d^k}{ds}((A_{n,s},\psi_{n,s})-(A_s,\psi_s))\|^2_{L^2}ds\leq\frac{1}{3}\epsilon$$

for all $n\in\mathbb{N}$ and likewise on the interval $(-\infty;-T]$. On the finite interval $[-T,T]$ the local convergence gives that there exists $N\in\mathbb{N}$ such that

$$\sum_{j+k\leq l}\int_{-T}^T e^{2\delta t}\|D^j_y\frac{d^k}{dt^k}((A_{n,t},\psi_{n,t})-(A_t,\psi_t))\|^2_{L^2}dt\leq\frac{1}{3}\epsilon$$

for $n \geq N$, and these three estimates imply the convergence in $\hat{\mathcal{M}}(\alpha, \beta)$.

We still have to prove the uniform convergence of the sequence towards the limiting critical points: There exists a $T > 0$, such that

$$t \geq T \Rightarrow C[A_t, \psi_t] \in \Xi(\beta, \frac{\epsilon}{2})^c$$

and there exists a neighbourhood U of $[A_T, \psi_T]$ such that $z \in U \Rightarrow C(z) \in \Xi(\beta, \epsilon)^c$. Now for n sufficiently big, by the local convergence $[A_{n,T}, \psi_{n,T}] \in U$ and thus $C([A_{n,T}, \psi_{n,T}]) \in \Xi(\beta, \epsilon)^c$. But this means that $C([A_{n,T}, \psi_{n,T}]) \in \Xi(\beta, \epsilon)^c$ for $t \geq T$ by 7.4. Thus we can find T' so that

$$t \geq T' \Rightarrow C([A_{n,t}, \psi_{n,t}]) \in \Xi(\beta, \epsilon)^c$$

for all $n \in \mathbb{N}$.
Choose a neighbourhood V of β such that 9.1 works with $U = V$. For $t \geq T'$ the perturbation is zero and we are dealing with a pure gradient flow. By [25, §12, prop.2] we may choose $V' \subseteq V$ such that a flow curve starting in V' will never leave V. There exists $T'' \geq T'$ such that $(A_{T''}, \psi_{T''}) \in V'$ and by the local convergence for n big enough $[A_{n,T''}, \psi_{n,T''}] \in V'$. Thus there exists a $S \geq T''$ with the property that for all $n \in \mathbb{N}$ and for $t \geq S$ we have $[A_{n,t}, \psi_{,t}n] \in V$. By the results of the previous section this means that we have the uniform exponential decay demanded above. $\square$

A m-tuple $([A^j, \psi^j])_{j=1}^m$ as in the theorem above will be called a broken trajectory of order m between α and β.

11 Gluing broken trajectories.

The results of this section will give a more detailed information on the ends of $\hat{\mathcal{M}}(\alpha,\beta)$ and a sort of converse to the result on divergence towards broken trajectories. Given a broken trajectory and a gluing parameter it will be shown that one can glue together the components of the broken trajectory to an actual flow curve in $\hat{\mathcal{M}}(\alpha,\beta)$. The gluing constructed will be a smooth embedding over compact sets of broken trajectories. This detailed understanding of the moduli space of flow curves is important for the definition of the boundary operator in the forthcoming sections. The strategy of the proofs in this section owes a lot (if not all) to [29, sec.2.5].

To keep the exposition (relatively) simple, we will only consider the case where the order of the broken trajectory is two. So given critical points α,β and γ with $C(\gamma)\in]C(\beta);C(\alpha)[$ and given elements

$$[A^1,\psi^1]\in\hat{\mathcal{M}}(\alpha,\gamma),\ [A^2,\psi^2]\in\hat{\mathcal{M}}(\gamma,\beta),$$

the goal is to produce a flow curve $[A^1,\psi^1]\#[A^2,\psi^2]$ in $\hat{\mathcal{M}}(\alpha,\beta)$ by gluing the two given curves in a neighbourhood of γ.

First we define a pre-gluing map $\#^0$ on a given compact subset K of $\hat{\mathcal{M}}(\alpha,\gamma)\times\hat{\mathcal{M}}(\gamma,\beta)$:

$$\#^0 : K\times[\rho_0,\infty)\to\tilde{\mathcal{B}}(\alpha,\beta)_{L^2_{l,\delta}}.$$

This map is easy to define and will give an approximate solution to the flow equation. The rest of the argument deals with the proof that there exists a correction to the pre-gluing map on the given compact set K, giving actual solutions.

Choose once and for all a chart on a neigbourhood U of γ

$$\Psi : U\to V,$$

where $V\subseteq V_{\gamma,L^2_l}$ is a convex neigbourhood of zero, $\Psi(\gamma)=0$, and choose lifts $\hat{\mathcal{M}}(\alpha,\gamma)\approx\mathcal{M}(\alpha,\gamma)_0^{\kappa_1}, \hat{\mathcal{M}}(\gamma,\beta)\approx\mathcal{M}(\gamma,\beta)_0^{\kappa_2}$. By compactness of K there exists $\rho_0(K)$ such that for $(u_1,u_2)\in K$, $u_1(\rho-1),u_2(-\rho+1)\in U$ for $\rho\geq\rho_0$. Now define

$$\#^0 : K\to\tilde{\mathcal{B}}(\alpha,\beta)_{L^2_{l,\delta}},$$

$$\#^0(u_1,u_2,\rho) := \begin{cases} (u_1)_\rho & \text{for } t\leq -1\\ \Psi^{-1}(\beta^-\Psi((u_1)_\rho)+\beta^+\Psi((u_2)_{-\rho})) & \text{for } -1\leq t\leq 1\\ (u_2)_{-\rho} & \text{for } t\geq 1.\end{cases}$$

Here u_ρ denotes translation with ρ and $\beta^+:\mathbb{R}\to[0;1]$ is a smooth bump function with

$$\beta^+(t) = \begin{cases} 0 & \text{for } t\leq 0\\ 1 & \text{for } t\geq 1.\end{cases}$$

We set $\beta^-(t) = \beta^+(-t)$.

Define vector bundles over $\tilde{\mathcal{B}}(\alpha,\gamma)_{L^2_{l,\delta}}$ as the quotient of the action of $\tilde{\mathcal{G}}_{L^2_{l+1,\delta}}$ on $i\Omega^1(Y \times \mathbb{R})_{L^2_{l,\delta}} \oplus \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}}$ and $i\Omega^2_+(Y \times \mathbb{R})_{L^2_{l,\delta}} \oplus \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}}$ given by

$$(\alpha,\eta)\cdot\sigma = (\alpha, \sigma^{-1}\eta)$$

for $\alpha \in i\Omega^1(Y \times \mathbb{R})_{L^2_{l,\delta}}/i\Omega^2_+(Y \times \mathbb{R})_{L^2_{l,\delta}}$ and $\eta \in \Omega^0(pr^* S^+_{\mathbb{C}}(P_{Spin^c}))_{L^2_{l,\delta}}$. Denote these bundles by $\mathcal{P}^1_{L^2_{l,\delta}}$ and $\mathcal{P}^+_{L^2_{l,\delta}}$, respectively. Then define the Seiberg-Witten bundle map by

$$\begin{aligned} F : T\tilde{\mathcal{B}}(\alpha,\gamma)_{L^2_{l,\delta}} \subseteq \mathcal{P}^1_{L^2_{l,\delta}} &\to \mathcal{P}^+_{L^2_{l-1,\delta}}, \\ F_{[A,\psi]}(\alpha,\eta) &:= \tilde{F}_{\mu,\omega}(A+a, \psi+\eta). \end{aligned}$$

This is well defined by the equivariance properties of $\tilde{F}_{\mu,\omega}$ stated in section 7 and clearly smooth. The idea is now to pull back the above bundles and F to $K \times [\rho_0;\infty)$ and find a smooth section

$$\phi : K \times [\rho_0;\infty) \to T\tilde{\mathcal{B}}(\alpha,\gamma)_{L^2_{l,\delta}}$$

that satisfies $F_{\#^0(u_1,u_2,\rho)}(\phi(u_1,u_2,\rho)) = 0$. Then the gluing map will be

$$\begin{aligned} \hat{\#} : K \times [\rho_0;\infty) &\to \hat{\mathcal{M}}(\alpha,\beta), \\ \hat{\#}(u_1,u_2,\rho) &= [\#^0(u_1,u_2,\rho) + \phi(u_1,u_2,\rho)]. \end{aligned}$$

The existence of such a section will follow from the existence of a section ϕ of $(\#^0)^*\mathcal{P}^1_{L^2_{l,\delta}}$ with $(\lambda^*_\cdot, F)\circ\phi = 0$ as the condition $\lambda^*_{[A,\psi]}(\alpha,\eta) = 0$ exactly ensures that $(\alpha,\eta) \in T_{[A,\psi]}\tilde{\mathcal{B}}(\alpha,\gamma)_{L^2_{l,\delta}}$. Thus we will work in the picture below:

$$\begin{tikzcd} (\#^0)^*\mathcal{P}^1_{L^2_{l,\delta}} \arrow[rr, "(\lambda^*_\cdot, F)"] \arrow[dr] & & (\#^0)^*\mathcal{P}^+_{L^2_{l-1,\delta}} \arrow[dl] \\ & K \times [\rho_0;\infty) & \end{tikzcd}$$

We will prove the existence of the desired section by applying the following analytical lemma in each fiber of the bundles over $K \times [\rho_0;\infty)$:

Lemma 11.1 (FM,prop. 24). *Assume that f is a smooth map*

$$f : E \to F$$

between Banach spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ with a Taylor expansion

$$f(\xi) = f(0) + Df(0)\xi + N(\xi)$$

where the kernel of $Df(0)$ splits and $Df(0)$ has a right inverse G satisfying

$$\|GN(\xi) - GN(\zeta)\|_E \leq C_N(\|\xi\|_E + \|\zeta\|_E)\|\xi - \zeta\|_E$$

for some constant C_N. If

$$\|Gf(0)\|_E \leq (8C_N)^{-1},$$

then the zero-set of f in $B_\epsilon = \{\xi \in E | \|\xi\|_E < \epsilon\}$ with $\epsilon = \frac{1}{4C_N}$ is a smooth submanifold of dimension equal to the dimension of $Ker(Df(0))$. In fact, if

$$K_\epsilon = \{\xi \in Ker(Df(0)) | \|\xi\|_E < \epsilon\}$$

and

$$K_\epsilon^\perp = \{\xi \in GF \|\xi\|_E < \epsilon\}$$

then there exists a smooth function $\theta : K_\epsilon \to K_\epsilon^\perp$ so that $f(\xi + \theta(\xi)) = 0$, and all zeroes of f in B_ϵ are of the form $\xi + \theta(\xi)$. Moreover, we have the estimates

$$\|\theta(\xi)\|_E \leq 2\|Gf(0)\|_E,$$
$$\|D\theta(\xi)\| \leq 8C_N\|Gf(0)\|_E.$$

Let us identify what goes into the lemma above: Given $[A, \psi] \in \tilde{\mathcal{B}}_{L^2_{l,\delta}}$

$$D_{[A,\psi]}(\lambda^*_\cdot, F) = L_{[A,\psi]} \oplus (0, PD_{[A,\psi]}Q) =: L'_{[A,\psi]}.$$

As for N we have:

$$\begin{aligned}(\lambda^*_{[A,\psi]}, F)(\alpha, \eta) &= (\lambda^*_{[A,\psi]}(\alpha, \eta), (F_{A+\alpha} + \mu + PQ_{([A+\alpha,\psi+\eta],\omega)})^+ - \frac{1}{2}q(\psi + \eta), \\ &\qquad \eth_{A+a}(\psi + \eta)) \\ &= (0, F(A, \psi)) + L'_{[A,\psi]}(\alpha, \eta) + (0, N_{Q,[A,\psi]}(\alpha, \eta) - \frac{1}{2}q(\eta), \frac{1}{2}\alpha \cdot \eta),\end{aligned}$$

where

$$N_{Q,[A,\psi]}(\alpha, \eta) = PQ_{([A+\alpha,\psi+\eta],\omega)} - PQ_{([A,\psi],\omega)} - PD_{([A,\psi],\omega)}Q(\alpha, \eta).$$

This gives that

$$N_{[A,\psi]}(\alpha, \eta) = (N_{Q,[A,\psi]}(\alpha, \eta) - \frac{1}{2}q(\eta), \frac{1}{2}\alpha \cdot \eta).$$

The operator $L'_{[A,\psi]}$ is Fredholm (section 8) and thus $L'_{[A,\psi]}$ has a finite dimensional kernel and a right inverse, $G_{[A,\psi]}$.

There is the following estimate of $N_{[A,\psi]}$:

$$\|N_{[A,\psi]}(\alpha, \eta) - N_{[A,\psi]}(\alpha', \eta')\|_{L^2_{l-1,\delta}} \leq C(\|\alpha, \eta\|_{L^2_{l,\delta}} + \|\alpha', \eta'\|_{L^2_{l,\delta}})\|(\alpha - \alpha', \eta' - \eta)\|_{L^2_{l,\delta}}.$$

For the part of N not depending on P this follows from the following general estimations for a bounded, bilinear map $B : E \times E \to F$ between Banach spaces E and F:

$$
\begin{aligned}
B(x,y) - B(x',y') &= B(x-x',y) + B(x',y-y') \Rightarrow \\
\|B(x,y) - B(x',y')\| &\leq C(\|x-x'\|\|y\| + \|x'\|\|y-y'\|) \\
&\leq C(\|(x,y)\| + \|(x',y')\|)\|(x-x',y-y')\|.
\end{aligned}
$$

For the bilinear part of N the constant C thus does not depend on $[A,\psi]$. For $N_{Q,[A,\psi]}$ an estimation using Taylors formula, the translational invariance of $h.$ and the compactness of K gives that in the L^2_l-norm there is an estimate of as required in lemma 11.1 over $K \times [\rho_0;\infty)$. The proof uses that the L^2_l-norm is translation invariant. Translations are still continous for $\delta > 0$, but their norms are not uniformly bounded because of the exponential factor and this is a property that will also be needed below - most importantly in the proof of lemma 11.2.

By translational invariance of the Seiberg-Witten map $\tilde{F}_{\mu,\omega}$ we get for $u_1 \in \hat{\mathcal{M}}(\alpha,\gamma)$ and $u_2 \in \hat{\mathcal{M}}(\gamma,\beta)$:

$$
\begin{aligned}
\|F_{\#^0(u_1,u_2,\rho)}(0)\|_{L^2_{l-1,\delta}} &= \|\tilde{F}_{\mu,\omega}(\#^0(u_1,u_2,\rho))\|_{L^2_{l-1,\delta}} \\
&= \|\tilde{F}_{\mu,\omega}(\#^0(u_1,u_2,\rho))\|_{L^2_{l-1,\delta}(Y\times[-1,1])} \leq C_{K,\delta}e^{-\delta\rho}
\end{aligned}
$$

by the exponential convergence of u_1 and u_2 towards γ.

If G is uniformly bounded over $K \times [\rho;\infty)$ for ρ sufficiently large, we have the estimations needed in the analytical lemma. To prove that this is in fact the case we construct a suitable complement to the kernel of $L'_{[A,\psi]}$ and show that $L'_{[A,\psi]}$ is surjective if the gluing parameter ρ is sufficiently large.

We first define a linear model for the gluing map:

$$
\begin{gathered}
\# : T\hat{\mathcal{M}}(\alpha,\gamma) \times T\hat{\mathcal{M}}(\gamma,\beta) \times [\rho_0;\infty) \to (\#^0)^*\mathcal{P}^1_{L^2_{l,\delta}}, \\
\#(\xi,\tau,\rho) = \#_\rho(\xi,\tau) = \xi\#_\rho\tau = \beta_2^-\xi_\rho + \beta_{-2}^+\tau_{-\rho}.
\end{gathered}
$$

Remember, that $T_{[A,\psi]}\hat{\mathcal{M}}(\alpha,\gamma) = Ker(L'_{[A,\psi]})$.

Lemma 11.2 (Sch. 2.50). *Let P be the orthogonal L^2-projection to $Ker(L')$ in $(\#^0)^*\mathcal{P}^1_{L^2_{l,\delta}}$. For any compact set $K \subseteq \hat{\mathcal{M}}(\alpha,\gamma) \times \hat{\mathcal{M}}(\gamma,\beta)$, there exists a $\rho' \geq \rho_0$ such that for $\rho \geq \rho'$*

$$
P \circ \#_\rho : T\hat{\mathcal{M}}(\alpha,\gamma) \times T\hat{\mathcal{M}}(\gamma,\beta) \to Ker(L_{\#^0(\cdot,\cdot,\rho)})
$$

is an isomorphism and $L_{\#^0(\cdot,\cdot,\rho)}$ is surjective for every point in K.

Before the proof this lemma, let us note a few consequences: In particular, for $\rho \geq \rho'$, $\#_\rho$ is injective and so $Im(\#_\rho)$ is a finite dimensional subbundle of $(\#^0)^*\mathcal{P}^1_{L^2_{l,\delta}}$. We can then define the L^2-orthogonal complement of $Im(\#_\rho)$:

$$
(\mathcal{P}^\perp_{L^2_{l,\delta}})_\chi := \{v \in ((\#^0)^*\mathcal{P}^1_{L^2_{l,\delta}})_\chi | < v, \xi\#_\rho\tau >_{L^2} = 0, \forall \xi \in T_{u_1}\hat{\mathcal{M}}(\alpha,\gamma), \tau \in T_{u_2}\hat{\mathcal{M}}(\gamma,\beta)\}
$$

for $\chi = \#^0(u_1, u_2, \rho)$, $\rho \geq \rho'$. This is a vector bundle over $K \times [\rho', \infty)$ that satisfies

$$Im(\#) \oplus \mathcal{P}^{\perp}_{L^2_{l,\delta}} = (\#^0)^* \mathcal{P}^1_{L^2_{l,\delta}}.$$

We will prove the lemma in the case $\delta = 0$, but by $[20, 7.1]$ the kernels of $L'_{[A,\psi]}$ are the same for δ in the allowed interval, so that the lemma is true for $\delta > 0$ also.

Proof: Note that by additivity of the index it is enough to show that $P \circ \#$ is surjective, because then:

$$\begin{aligned} dim(Ker(L'_\chi)) &\leq dim(Ker(L'_{u_1})) + dim(Ker(L'_{u_2})) \\ &= ind(L'_{u_1}) + ind(L'_{u_2}) \\ &= (i(\alpha) - i(\gamma)) + (i(\gamma) - i(\beta)) \\ &= i(\alpha) - i(\beta) = ind(L'_\chi) \\ &\leq dim(Ker(L'_\chi)) \end{aligned}$$

Remember that for $u_1 \in \hat{\mathcal{M}}(\alpha, \gamma)$, L'_{u_1} is surjective, and likewise for $u_2 \in \hat{\mathcal{M}}(\gamma, \beta)$.

Furthermore, it is enough to show that the following inequality holds for ρ bigger than some ρ':

$$\exists c(K) > 0 \, \forall (u_1, u_2) \in K : \|L'_\chi \xi\|_{L^2_{l-1}} \geq c\|\xi\|_{L^2_l} \quad \text{for} \quad \xi \in (\mathcal{P}^{\perp}_{L^2_l})\chi.$$

Because, if $P \circ \#_\rho$ is not surjective, there is a non-zero vector in $Ker(L'_\chi) \cap (\mathcal{P}^{\perp}_{L^2_l})_\chi$. But by the above inequality any such vector is zero.

The proof of the inequality will be by contradiction. So assume given a sequence of gluing parameters $(\rho_n)_{n\in\mathbb{N}}$ with $\rho_n \to_n \infty$, a sequence of flow curves from K, $(u_n, v_n)_{n\in\mathbb{N}}$ and vectors $(\xi_n)_{n\in\mathbb{N}}$, $\xi_n \in (\mathcal{P}^{\perp}_{L^2_l})_{\chi_n}$ which satisfies:

$$\|L'_{\chi_n}\xi_n\|_{L^2_{l-1}} \to_n 0, \quad \|\xi_n\|_{L^2_l} = 1.$$

We will reach a contradiction in 3 steps:

1) $\xi_n \to_n 0$ in $L^2_{l,loc}$.

Choose a smooth bump function $\upsilon : \mathbb{R} \to [0, 1]$ with

$$\upsilon(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{1}{2} \\ 0 & \text{for } |t| \geq 1. \end{cases}$$

and set $\upsilon^{\frac{1}{2}\rho_n}(t) = \upsilon(\frac{2}{\rho_n}t)$. The operator

$$L_\gamma : (\#^0)^* \mathcal{P}^1_{L^2_l} \to (\#^0)^* \mathcal{P}^+_{L^2_{l-1}}$$

corresponding to the constant flow curve γ is an isomorphism $[20, \text{p.}419-420]$. As $\rho_n \to_n \infty$, it will thus be enough to show that $L_\gamma(\upsilon^{\frac{1}{2}\rho_n}\xi_n) \to_n 0$. We estimate:

$$\begin{aligned} \|L_\gamma(\upsilon^{\frac{1}{2}\rho_n}\xi_n)\|_{L^2_{l-1}} &= \|\frac{d}{dt}(\upsilon^{\frac{1}{2}\rho_n})\xi_n + \upsilon^{\frac{1}{2}\rho_n} L_\gamma \xi_n\|_{L^2_{l-1}} \\ &\leq \frac{2}{\rho_n}\|\dot{\upsilon}\|_\infty + K(\upsilon)\|L_\gamma\xi_n\|_{L^2_{l-1}(Y\times[-\frac{1}{2}\rho_n;\frac{1}{2}\rho_n])} \\ &\leq \frac{2}{\rho_n}\|\dot{\upsilon}\|_\infty + K(\upsilon)(\|L'_{\chi_n}\xi_n\|_{L^2_{l-1}} + \|L_\gamma - L'_{\chi_n}\|_{Y\times[-\frac{1}{2}\rho_n;\frac{1}{2}\rho_n]}), \end{aligned}$$

where $K(v)$ depends on the $l-1$ first derivatives of v. As K is compact we may assume that $(u_n, v_n) \to_n (u, v) \in K$. The map $(A, \psi) \mapsto L'_{(A,\psi)}$ is continous and the norms are translational invariant, so that in the limit we may replace χ_n with $\#(u, v, \rho_n)$. On the negative halfaxis we see that:

$$\|\gamma - u_{\rho_n}\|_{L^2_l(Y\times[-\frac{1}{2}\rho_n;0])} = \|\gamma - u\|_{L^2_l(Y\times[\frac{1}{2}\rho_n;\rho_n])} \leq \|\gamma - u\|_{L^2_{l,\delta}(Y\times\mathbb{R}_+)} e^{-\frac{1}{2}\rho_n\delta}.$$

This and the similar estimate on the positive halfaxis implies that $\|L_\gamma(v^{\frac{1}{2}\rho_n}\xi_n)\|_{L^2_{l-1}}$ converges to zero, as n goes to infinity.
2) ξ_n converges to zero on both ends of $Y \times \mathbb{R}$.
We first prove: $\|L'_u(\beta^-_{1-\rho_n}\xi_{n,-\rho_n})\|_{L^2_{l-1}} \to_n 0$. This follows from the following estimates:

$$\begin{aligned}
\|L'_{u_n}(\beta^-_{1-\rho_n}\xi_{n,-\rho_n})\|_{L^2_{l-1}} &= \|\dot{\beta}^-_{1-\rho_n}\xi_{n,-\rho_n} + \beta^-_{1-\rho_n}L'_{u_n}\xi_{n,-\rho_n}\|_{L^2_{l-1}} \\
&\leq \|\dot{\beta}^-_1\xi_n\|_{L^2_{l-1}} + \|\beta^-_1 L_{u_n,\rho_n}\xi_n\|_{L^2_{l-1}} \\
&\leq K(\beta)\|\xi_n\|_{L^2_{l-1}(Y\times[-2;-1])} + \|\beta^-_1 L_{\chi_n}\xi_n\|_{L^2_{l-1}} \\
&\leq K(\beta)\|\xi_n\|_{L^2_{l-1}(Y\times[-2;-1])} + K'(\beta)\|L_{\chi_n}\xi_n\|_{L^2_{l-1}}.
\end{aligned}$$

As the sequence $(\beta^-_{1-\rho_n}\xi_{n,-\rho_n})_{n\in\mathbb{N}}$ is bounded in L^2_{l-1} and $u_n \to_n u$, the claim above follows. L'_u is a Fredholm operator and thus has a right inverse on the complement of $Ker(L'_u)$. We see that the component of $\beta^-_{1-\rho_n}\xi_{n,-\rho_n}$ in this complement converges to zero.

We now claim that the projection of $\beta^-_{1-\rho_n}\xi_{n,-\rho_n}$ to $Ker(L'_u)$ also converges to zero: Let $\tau \in Ker(L'_u)$. As $u_n \to_n u$ and as the kernels of $L'_.$ form a finite dimensional vector bundle over $K \times [\rho_0; \infty)$, there exists a sequence $(\tau^n)_{n\in\mathbb{N}}$ with $\tau^n \in Ker(L'_{u_n})$ such that $\tau^n \to_n \tau$ in L^2_l. As $\xi_n \in (\mathcal{P}^\perp_{L^2_l})_{\chi_n}$ we get:

$$< \beta^-_1\xi_n, \beta^-_2\tau^n_{\rho_n} >_{L^2} = 0.$$

Thus as $(\beta^-_1\xi_n)_{n\in\mathbb{N}}$ is bounded sequence:

$$\begin{aligned}
< \beta^-_{1-\rho_n}\xi_{n,-\rho_n}, \tau >_{L^2} &= < \beta^-_1\xi_n, \tau_{\rho_n} >_{L^2} \\
&= \lim_{n\to\infty} < \beta^-_1\xi_n, \tau^n_{\rho_n} >_{L^2} \\
&= \lim_{n\to\infty} < \beta^-_1\xi_n, \beta^-_2\tau^n_{\rho_n} >_{L^2} = 0,
\end{aligned}$$

as $\xi_n \to_n 0$ in L^2-norm over $Y \times [-3; -1]$ by 1). We now have:

$$\|\beta^-_1\xi_n\|_{L^2_l} = \|\beta^-_{1-\rho_n}\xi_{n,-\rho_n}\|_{L^2_l} \to_n 0.$$

A similar argument gives $\|\beta^+_{-1}\xi_n\|_{L^2_l} \to_n 0$.
3) The contradiction:

$$\|\xi\|_{L^2_l} \leq \|\xi\|_{L^2_l(Y\times[-2;2])} + \|\beta^-_1\xi_n\|_{L^2_l} + \|\beta^+_{-1}\xi_n\|_{L^2_l} \to_n 0.$$

$\square$

We now return to the assumptions in the analytical lemma. From the inequality in the proof of the above lemma, we see that for $\delta = 0$ and $\rho \geq \rho'$ we have a uniform bound on the norm of the right inverse G_χ of the surjective linear map L'_χ: $\|G_\chi\| \leq c^{-1}$.

Thus for $\delta = 0$ we get for $(u, v, \rho) \in K \times [\rho', \infty)$, using the notation of lemma 11.1, a map

$$\theta : K_{(u,v,\rho),\epsilon} \to Im(G_\chi) = (\mathcal{P}^\perp_{L^2_l})_\chi$$

with $(\lambda^*_\chi, F)(\zeta + \theta(\zeta)) = 0, \zeta \in K_{(u,v,\rho),\epsilon}$. In particular, we can define the correction term to the pregluing map by:

$$\phi : K \times [\rho', \infty) \to \mathcal{P}^\perp_{L^2_l},$$
$$\phi(u, v, \rho) = \theta_{(u,v,\rho)}(0).$$

This section ϕ is smooth: This will follow from the implicit function theorem if we can show that the fiber derivative of $(\lambda^*_\cdot, F)$ at $\phi(u, v, \rho)$ is an isomorphism. At the zero points of the fibers this is the case - notice that ϕ is a section of the bundle of orthogonal complements, $\mathcal{P}^\perp_{L^2_l}$. We also know that the norm of the inverse here is bounded - this is the bound on the norm of G obtained above - and by the analytical lemma we have

$$\|\phi(u, v, \rho)\|_{L^2_l} \leq 2\|G_\chi(\lambda^*_\cdot, F)(0)\|_{L^2_l} \leq 2c^{-1}\|(\lambda^*_\cdot, F)(0)\|_{L^2_l},$$

which we have seen converges uniformly to zero over K for $\rho \to \infty$. Thus by choosing ρ' sufficiently big we get that the norm distance between the fiber derivative of $(\lambda^*_\cdot, F)$ at zero and at $\phi(u, v, \rho)$ is less than c^{-1}, and we can thus conclude that the fiber derivative of $(\lambda^*_\cdot, F)$ at $\phi(u, v, \rho)$ is an isomorphism.

We still have to consider the case $\delta > 0$: As we saw in section 9, for δ in the allowed interval the diffeomorphism type of the manifold of flow curves does not depend on the choice of δ. Thus we get a gluing map also in this case and we can state the following theorem:

Theorem 11.3 (Gluing theorem. Sch.ch.2,th.2, CW.th.2.20).
Let $\alpha, \beta, \gamma \in \tilde{\mathcal{M}}_\mu$ with $C(\alpha) > C(\gamma) > C(\beta)$ be given. Then for any compact set $K \subseteq \hat{\mathcal{M}}(\alpha, \gamma) \times \hat{\mathcal{M}}(\gamma, \beta)$, there exists a $\rho' > 0$ and a smooth map:

$$\hat{\#} : K \times [\rho', \infty) \to \hat{\mathcal{M}}(\alpha, \beta),$$
$$\hat{\#}(u, v, \rho) = [\#^0(u, v, \rho) + \phi(u, v, \rho)],$$

where $\|\phi(u, v, \rho)\|_{L^2_l} < C_K e^{-\delta\rho}$ and $\|D\phi(u, v, \rho)\| < C'_K e^{-\delta\rho}$.

Notice that the map constructed actually factors though $\mathcal{M}(\alpha, \beta)$. We will use the notation $\#$ for this lift of $\hat{\#}$.

As claimed already at the beginning of this section, the above result can be made a lot stronger. Not only is it possible to glue together broken trajectories, the above gluing map is, if we choose the gluing parameter sufficiently big, actually an embedding:

Theorem 11.4 (Embedding theorem. Sch.ch.2,th.2, CW.th.2.20).
Let $\alpha, \beta, \gamma \in \tilde{\mathcal{M}}_\mu$ with $C(\alpha) > C(\gamma) > C(\beta)$ be given. Then for any compact set $K \subseteq \hat{\mathcal{M}}(\alpha,\gamma) \times \hat{\mathcal{M}}(\gamma,\beta)$, there exists a $\rho'' \geq \rho$ such that the gluing map from the gluing theorem 11.3 *is an orientation preserving embedding:*

$$\hat{\#} : K \times [\rho'', \infty) \hookrightarrow \hat{\mathcal{M}}(\alpha, \beta).$$

Proof: We prove the theorem in three steps:
1): $\exists \rho'' \geq \rho' : \rho \geq \rho'' \Rightarrow$ the differential of $\hat{\#}$ is injective (and thus is an isomorphism by counting dimensions).
2): $\exists \rho'' \geq \rho' : \rho \geq \rho'' \Rightarrow$ $\hat{\#}$ is injective.
3): $\hat{\#}$ is orientation preserving.

Ad 1): Choose lifts $\hat{\mathcal{M}}(\alpha,\gamma) = \mathcal{M}(\alpha,\gamma)_0^\kappa$ and $\hat{\mathcal{M}}(\gamma,\beta) = \mathcal{M}(\gamma,\beta)_0^{\kappa'}$ for for some $\kappa \in]C(\gamma); C(\alpha)[\cap \Xi, \kappa' \in]C(\beta); C(\gamma)[\cap \Xi$. We work as before in the L_l^2-topology. Write

$$D\hat{\#} = [D\#^0 + D\phi].$$

The differential of the pre-gluing map is:

$$\begin{aligned} D\#^0_{(u,v,\rho)}(\xi,\tau,t) &= D\#^0_\rho(u,v)(\xi,\tau) + tD\#^0_{(u,v)}(\rho)(\dot{u}, -\dot{v}) \\ &= \beta^-\xi_\rho + \beta^+\tau_{-\rho} + t(\beta^-\dot{u}_\rho - \beta^+\dot{v}_{-\rho}) \\ &= \#^0_\rho(\xi + t\dot{u}, \tau - t\dot{v}). \end{aligned}$$

The proof is by contradiction, so assume given a sequence of gluing parameters ρ_n converging to $+\infty$, a sequence of points in K, $(u_n, v_n)_{n\in\mathbb{N}}$, and for each $n \in \mathbb{N}$ vectors $(\xi_n, \tau_n, t_n) \in T_{(u_n,v_n)}(\mathcal{M}(\alpha,\gamma)_0^\kappa \times \mathcal{M}(\gamma,\beta)_0^{\kappa'}) \times \mathbb{R}$ with

$$D\hat{\#}_{(u_n,v_n,\rho_n)}(\xi_n, \tau_n, t_n) = 0.$$

The identification of the normal bundle of $\mathcal{M}(\alpha,\gamma)_0^\kappa$ at the end of section 8 implies that there is a sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ such that:

$$(D\#^0 + D\phi)_{(u_n,v_n,\rho_n)}(\xi_n, \tau_n, t_n) = s_n \frac{\partial}{\partial t}(u_n \#_{\rho_n} v_n).$$

We may assume that $\|\xi\|^2_{L^2_l} + \|\tau\|^2_{L^2_l} + |t_n|^2 = 1$ and by compactness of K that $(u_n, v_n) \to_n (u,v) \in K$.

The derivate with respect to time of the glued flow curves is:

$$\begin{aligned} \frac{\partial}{\partial t}(u\#_\rho v) &= D\#^0_\rho(\dot{u}, \dot{v}) + \frac{\partial}{\partial t}\phi(u,v,\rho) \\ &= \#^0_\rho(\dot{u}, \dot{v}) + \frac{\partial}{\partial t}\phi(u,v,\rho) \end{aligned}$$

As $\|D\#^0_{(u,v,\cdot)}\|$, $\|\phi\|$ and $\|D\phi\|$ are all bounded independently of ρ and $\frac{\partial}{\partial t}(u_n\#_{\rho_n}v_n)$ is bounded away from zero by the assumption of convergence of (u_n, v_n), $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence. Now remember that both $\|\phi\|_{L^2_l}$ and $\|D\phi\|$ converge uniformly to zero over K when $\rho \to \infty$ by 11.3. This implies that

$$\#^0_{\rho_n}(\xi_n + (2t_n - s_n)\dot{u}_n, \tau_n - (2t_n + s_n)\dot{v}_n) \to_n 0.$$

Thus in particular both

$$\|(\xi_n + (2t_n - s_n)\dot{u}_n)_{\rho_n}\|_{L^2_l(Y\times(-\infty;-1])} = \|\xi_n + (2t_n - s_n)\dot{u}_n\|_{L^2_l(Y\times(-\infty;\rho_n-1])}$$

and

$$\|(\tau_n - (2t_n + s_n)\dot{v}_n)_{-\rho_n}\|_{L^2_l(Y\times[1;\infty))} = \|\tau_n - (2t_n + s_n)\dot{v}_n\|_{L^2_l(Y\times[-\rho_n+1;\infty))}$$

converge to zero. As the manifolds are finite dimensional, as we have convergence of the base points and as $\rho_n \to_n \infty$, this gives

$$\begin{aligned} \xi_n + (2t_n - s_n)\dot{u}_n &\to_n 0, \\ \tau_n - (2t_n + s_n)\dot{v}_n &\to_n 0. \end{aligned}$$

Because $T\mathcal{M}_0 \cap \mathbb{R}\frac{\partial}{\partial t} = \{0\}$, this implies that

$$\begin{aligned} \xi_n &\to_n 0, \quad & \tau_n &\to_n 0, \\ (2t_n - s_n)\dot{u}_n &\to_n 0, \quad & (2t_n + s_n)\dot{v}_n &\to_n 0. \end{aligned}$$

But as $\dot{u}_n \to_n \dot{u}, \dot{v}_n \to_n \dot{v}$, where the limits are both non-zero, we must have $2t_n \pm s_n \to_n 0$ and thus finally $t_n, s_n \to_n 0$. This contradicts the normalization assumed above.

Ad 2): The proof will again be by contradiction: Assume given sequences of gluing parameters $\rho_{i,n} \to_n \infty, i = 1,2$ and corresponding sequences of points in K, $(u_{1,n}, v_{1,n})_{n\in\mathbb{N}}, (u_{2,n}, v_{2,n})_{n\in\mathbb{N}}$ such that

$$u_{1,n}\hat{\#}_{\rho_{1,n}}v_{1,n} = u_{2,n}\hat{\#}_{\rho_{2,n}}v_{2,n}.$$

Choose a lift of $\hat{\mathcal{M}}(\alpha,\beta)$, $\mathcal{M}(\alpha,\beta)^{\kappa''}_0$. The matching lifts of the glued sequences are $(u_{i,n}\#_{\rho_{i,n}}v_{i,n})_{\tau_{i,n}}, i = 1,2$ for some real sequences $(\tau_{i,n})_{n\in\mathbb{N}}$. Again using that the correction section ϕ converges uniformly to zero over K gives that

$$\|(u_{1,n}\#^0_{\rho_{1,n}}v_{1,n})_{\tau_{1,n}} - (u_{2,n}\#^0_{\rho_{2,n}}v_{2,n})_{\tau_{2,n}}\|_{L^2_l} \to_n 0.$$

By the compactness of K we can assume that $(u_{i,n}, v_{i,n}) \to_n (u_i, v_i), i = 1,2$. Then by definition of the pre-gluing map:

$$\|u_{i,n}\#^0_{\rho_{i,n}}v_{i,n} - u_i\#^0_{\rho_{i,n}}v_i\|_{L^2_l} \to_n 0, i = 1,2.$$

With a sloppy notation this implies that for n big:

$$\begin{aligned} u_1(t) = u_1\#^0_{\rho_{1,n}}v_1(t-\rho_n) &\cong u_{1,n}\#^0_{\rho_{1,n}}v_{1,n}(t-\rho_{1,n}) \\ &\cong u_{2,n}\#^0_{\rho_{2,n}}v_{2,n}(t-\rho_{1,n}+\tau_{2,n}-\tau_{1,n}) \end{aligned}$$

We now argue that $\tau_{2,n} - \tau_{1,n} - \rho_{1,n} \to_n -\infty$: Remember that the lifts were chosen so that $C(u_i(0)) = \kappa > C(\gamma) + \frac{1}{2}\epsilon$ and $C(v_i(0)) = \kappa' < C(\gamma) - \frac{1}{2}\epsilon$. Thus if there is a subsequence with $\tau_{2,n_k} - \tau_{1,n_k} - \rho_{1,n_k} \geq -c - 1$, we get

$$\liminf_k C(u_{2,n_k} \#^0_{\rho_{2,n_k}} v_{2,n_k}(c - \rho_{n_k} + \tau_{2,1,n_k} - \tau_{1,n_k})) \leq C(\gamma) + \frac{1}{2}\epsilon,$$

what contradicts the above for $t = c$. Thus we get:

$$u_1(t) \cong u_2(t + \rho_{2,n} - \rho_{1,n} + \tau_{2,n} - \tau_{1,n})$$

As $C(u_2(t))$ assumes the value κ precisely once, this implies

$$\rho_{2,n} - \rho_{1,n} + \tau_{2,n} - \tau_{1,n} \to_n 0.$$

Thus: $u_1 = u_2$.

Similarly, we get $v_1 = v_2$. But the differential of the gluing map is by 1) an isomorphism at $(u, v) = (u_1, v_1) = (u_2, v_2)$ and ρ sufficiently big, and by the inverse function theorem this implies in particular that # is locally injective. This contradiction ends the proof of 2).

Ad 3): We sketch the proof: The idea of the proof is contained in [29, 3.1] and the key observation here is a gluing theorem analogous to lemma 11.2. Remember from 8 that $\mathcal{M}(\alpha, \beta)$ got an orientation from the orientation of a determinant line bundle $det(L.)$, which was again oriented by homotoping the Fredholm operator $L.$ to a single, trivial operator whose determinant line bundle was canonically oriented.

Using a gluing map for Fredholm operators in spirit similar to $\#^0$ and the linear model for the gluing map, #, it is possible to obtain a result parallel to [29, prop.3.6] with a proof analogous to 11.2 that states that for sufficiently large gluing parameters, there is a linear isomorphism:

$$\begin{aligned} Det(L_{(A',\psi')}) \otimes Det(L_{(A,\psi)}) &\xrightarrow{\approx} Det(L_{(A',\psi')} \# L_{(A,\psi)}) \\ &\approx Det(L_{(A',\psi')\#(A,\psi)}), \end{aligned}$$

where the last isomorphism is due to a homotopi of the operators involved. The proof extends to give an isomorphism over the homotopy to the trivial operators above, thus giving that the gluing of determinant line bundles is orientation preserving. Analogous to [29, 3.13] we should then prove that $D\#_\rho$ induces the same orientation as the gluing of the determinant line bundles and thus is orientation preserving. $\square$

Given α and β in $\tilde{\mathcal{M}}_\mu$, there is only finitely many $\gamma \in \tilde{\mathcal{M}}_\mu$ with $C(\alpha) > C(\gamma) > C(\beta)$ and thus there is also only finitely many gluing maps with image in $\hat{\mathcal{M}}(\alpha, \beta)$. A natural question is now: What is the complement of the images of all possible gluing maps? The next lemma will help answer these questions.

Lemma 11.5. *Let $(u, v) \in \hat{\mathcal{M}}(\alpha, \gamma) \times \hat{\mathcal{M}}(\gamma, \beta)$ and assume that the gluing map $\#_{(u,v)}$ is defined for $\rho \geq \rho'$. We then have:*

$$u \#_\rho v \to (u, v), \rho \to \infty.$$

Proof: Choose lifts as in the proof of the theorem above, but this time choose $\kappa'' = \kappa$. Then for $\rho > 1$, the lifts of $u\hat{\#}_\rho v$ is $(u\#_\rho v)_{-\rho}$. Obviously, this will converge against u in $L^2_{l,loc}$ as $(u\#_\rho v)_{-\rho}(t) = u(t)$, for $t \leq \rho - 1$. On the other hand the translation of the lift by 2ρ, $(u\#_\rho v)_\rho$ equals v for $t \leq -\rho + 1$.

□

I claim that the converse is also true: If $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\hat{\mathcal{M}}(\alpha, \beta)$ that diverges to $(u, v) \in \hat{\mathcal{M}}(\alpha, \gamma) \times \hat{\mathcal{M}}(\gamma, \beta)$, then z_n will eventually be in the image of some gluing map defined on $K \times [\rho_0; \infty) \subset \hat{\mathcal{M}}(\alpha, \gamma) \times \hat{\mathcal{M}}(\gamma, \beta) \times \mathbb{R}$, $(u, v) \in K^\circ$.

After having proved the gluing and embedding theorems in the case of broken flow curves with two components, we now state the general result:

Theorem 11.6 (General gluing theorem). *Given*

$$\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \alpha_m = \beta$$

elements of $\hat{\mathcal{M}}_\mu$ *with*

$$\iota(\alpha) > \iota(\alpha_1) > \ldots > \iota(\alpha_{m-1}) > \iota(\beta),$$

and a compact subset

$$K \subseteq \hat{\mathcal{M}}(\alpha, \alpha_1) \times \hat{\mathcal{M}}(\alpha_1, \alpha_2) \times \ldots \times \hat{\mathcal{M}}(\alpha_{m-1}, \beta),$$

there exists a $\rho' > 0$ *such that there is a gluing map*

$$\hat{\#} : K \times [\rho'; \infty)^{m-1} \to \hat{\mathcal{M}}(\alpha, \beta).$$

For ρ' *sufficiently big* $\hat{\#}$ *is an embedding and furthermore:*
1) $u_1\#_{\rho_1}u_2\#_{\rho_2}\ldots\#_{\rho_{m-1}}u_{m-1} \to (u_1, u_2, \ldots, u_{m-1})$ *for* $min_i(\rho_i) \to \infty$.
2) If a sequence of flow curves from $\hat{\mathcal{M}}(\alpha, \beta)$ *diverges to* $(u_1, u_2, \ldots, u_{m-1})$, *it will eventually be contained in the image of the gluing map* $\hat{\#}_{(u_1,u_2,\ldots,u_{m-1})}$.

Notice, that although this is not indicated in the notation above, there is no reason why the gluing of broken flow curves should be associative.

We now again specialize to the case of broken flow curves with two components. We want to define the gluing map on a subset of $\hat{\mathcal{M}}(\alpha, \gamma) \times \hat{\mathcal{M}}(\gamma, \beta) \times \mathbb{R}$ which projects onto $\mathcal{M} := \hat{\mathcal{M}}(\alpha, \gamma) \times \hat{\mathcal{M}}(\gamma, \beta)$. This will be done by a partition of unity-like argument: Choose a ascending sequence of compacts subsets of $\mathcal{M}$, $(K_n)_{n\in\mathbb{N}}$, with $\bigcup_{n=1}^\infty K_n = \mathcal{M}$ and $K_n \subseteq K^\circ_{n+1}$. Also, choose smooth functions $\Psi_n : \mathcal{M} \to [0; 1]$ with $\Psi_{n|K_{n-2}} = 0$, $\Psi_{n|K^c_{n-1}} = 1$ ($K_{-1} = K_0 = \varnothing$). Finally choose $\rho_n \geq \rho_{n-1}$ according to the embedding theorem such that the gluing map is defined on $K_n \times [\rho_n; \infty)$. Set $\Psi' = \bigvee_{n=1}^\infty \Psi_n(\rho_n + 1)$. Ψ' is continous and we can find a smooth approximation Ψ to Ψ' with $\Psi > \rho_n$ on K_n. Then the gluing map is defined on

$$\mathcal{D}(\alpha, \gamma, \beta, \hat{\#}, \Psi) := \{(u, v, \rho) \in \mathcal{M} \times \mathbb{R} | t \geq \Psi(u, v)\}.$$

I claim that if the values of Ψ are sufficiently big, $\hat{\#}$ embeds $\mathcal{D}$ into $\hat{\mathcal{M}}(\alpha, \beta)$. $\mathcal{M}$ is embedded in $\mathcal{D}(\alpha, \gamma, \beta, \hat{\#}, \Psi)$ by $\iota_\Psi(u_1, u_2) = (u_1, u_2, \Psi(u_1, u_2))$. Of course, ι_Ψ is not only an embedding, it is also a homotopy equivalence.

By the compactness result of section 10 and the above general version of the gluing & embedding theorem, we see that the complement of the images of the finitely many gluing maps into $\hat{\mathcal{M}}(\alpha, \beta)$ is a compact set.

12 Coefficients of the boundary operator.

We are now almost ready for the definition of the Seiberg-Witten-Floer chain complex. Most of this section will deal with the definition of the coefficients of the boundary operator. We start out with a simple lemma which is an extension and a detailed proof of [23, III.4.8]:

Lemma 12.1. *The 3 gauge groups* $(S^1)^Y, \mathcal{G}_{L^2_{k+1}}, k \geq 1,$ *and* $\mathcal{G}_{L^2_{l+1,\delta}}, l \geq 2,$ *are all homotopic to*

$$S^1 \times H^1(Y, \mathbb{Z})$$

The homotopy is induced by the maps:

$$\sigma \mapsto (\sigma(y_0), [\sigma^*(vol_{S^1})])$$

and

$$\sigma \mapsto (\sigma(y_0, t_0), [\sigma_t^*(vol_{S^1})]),$$

respectively.

Proof: A theorem of Hopf [6, V.11.6 + 11.9] states that

$$\pi_0((S^1)^Y) = [Y, S^1] = H^1(Y, \mathbb{Z}),$$

the isomorphism being given by pullback:

$$[\phi] \mapsto \phi^*(vol_{S^1}).$$

We thus have to prove that the connected component of the identity, $(S^1)^Y_0$, is homotopic to S^1. Choose a point in Y, y_0, and let

$$ev_{y_0} : (S^1)^Y \to S^1$$

denote the evaluation at y_0. This group homomorphism gives an exact sequence of continous maps:

$$1 \to (S^1, 1)_0^{(Y, y_0)} \to (S^1)^Y_0 \to S^1 \to 1,$$

which again gives the group homeomorphism

$$(S^1)^Y_0 \approx S^1 \times (S^1, 1)_0^{(Y, y_0)}.$$

$(S^1, 1)_0^{(Y, y_0)}$ is contractible: This is a consequence of $(S^1, 1)_0^{(Y, y_0)}$ being homeomorphic to $(\mathbb{R}, 0)_0^{(Y, y_0)}$, which definitely is contractible as $\mathbb{R}$ is. The homeomorphism is defined as follows: Given a map $f \in (S^1, 1)_0^{(Y, y_0)}$, we map it into the lift of f, $\tilde{f}$:

$$\begin{tikzcd} & (\mathbb{R}, 0) \arrow[d, "\exp(i\cdot)"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (S^1, 1) \end{tikzcd}$$

Continuity of $f \mapsto \tilde{f}$: Convergence in $(\mathbb{R}, 0)_0^{(Y,y_0)}$ with the compact-open topology is equivalent to uniform convergence over compact sets [6, VII.2.13]. Thus the following estimate where $K \subseteq Y$ is compact and contractible suffices:

$$\begin{aligned}\|\tilde{f} - \tilde{g}\|_{K,\infty} &= \|\log(f) - \log(g)\|_{K,\infty} = \|\log(f/g)\|_{K,\infty} \\ &\leq \|f/g - 1\|_{K,\infty} = \|f - g\|_{K,\infty},\end{aligned}$$

for $\|f - g\|_{K,\infty} < 1$. As $f = \exp(i\tilde{f})$ the continuity of the inverse map follows from [6, VII.2.8].

For the other two gauge groups first note that they have the same group of connected components as $(S^1)^Y \cong (S^1)^{Y \times \mathbb{R}}$. Because any element of $(S^1)^Y$ is homotopic to a smooth map and a homotopy between two smooth maps can be choosen smooth - in the case of $\mathcal{G}_{L^2_{k+1,\delta}}$ furthermore choose the maps to be constant in the $\mathbb{R}$-direction. Thus, arguing as above we again have to prove that the map sending f to it's realvalued lift $\tilde{f}$ and the inverse mapping $\tilde{f} \mapsto \exp(i\tilde{f})$ are continous in the respective topologies. The latter is easy; in both cases use the series expansion of the exponential map. This is the exponential map from the Lie algebras of the groups.

$\mathcal{G}_{L^2_{k+1}}$: If $(f_n)_{n\in\mathbb{N}}$ converges to f in the L^2_{k+1}-topology, we in particular have uniform convergence. By the result above this implies uniform convergence of $(\tilde{f}_n)_{n\in\mathbb{N}}$ towards $\tilde{f}$ and thus also L^2-convergence. We have $d\tilde{f} = \bar{f}df \in L^2_k$, which is obviously continous in f.

$\mathcal{G}_{L^2_{l+1,\delta}}$: Estimating as above we get

$$\|\tilde{f} - \tilde{f}_\infty\|_{L^2_\delta(K)} \leq \|f/f_\infty - 1\|_{L^2_\delta(K)} < \infty,$$

where of course $\tilde{f}_\infty = (f_\infty)^\sim$. By the above argument $f \mapsto (f_\infty)^\sim$ is continous and $f \mapsto \tilde{f} - \tilde{f}_\infty \in L^2_\delta$ is continous by the estimate:

$$\|(\tilde{f} - \tilde{f}_\infty) - (\tilde{g} - \tilde{g}_\infty)\|_{L^2_\delta(K)} = \|(f/g)^\sim - (f_\infty/g_\infty)^\sim\|_{L^2_\delta(K)} \leq \|f/f_\infty - g/g_\infty\|_{L^2_\delta(K)}.$$

Furthermore,

$$d\tilde{f} - d\tilde{f}_\infty = f^{-1}df - f_\infty^{-1}df_\infty = (ff_\infty^{-1})^{-1}d(ff_\infty^{-1}) \in L^2_{l,\delta},$$

an expression which is again continous in f. $\square$

Notice that in particular the maps $\iota : \mathcal{G}_{L^2_{k+1}} \to (S^1)^Y$ and $r : \mathcal{G}_{L^2_{l+1,\delta}} \to \mathcal{G}_{L^2_{l+1}}$ are homotopy equivalences by the proof of the above lemma. The theorem also holds for the one-sided gauge groups $\mathcal{G}_{L^2_{l+1,\delta}(\mathbb{R}_\pm)}$. This means that also the restriction mappings

$$r_\pm : \mathcal{G}_{L^2_{l+1,\delta}} \to \mathcal{G}_{L^2_{l+1,\delta}(\mathbb{R}_\pm)}$$

are homotopy equivalences. The homotopy inverse of $r_\pm$ is in the case of $(S^1)^{Y\times\mathbb{R}_\pm}$ the crude extension map, e.g.

$$u_+(\sigma) = \begin{cases} \sigma_0, & \text{on } Y\times\mathbb{R}_- \\ \sigma, & \text{on } Y\times\mathbb{R}_+. \end{cases}$$

By the homotopy equivalences above similar maps, which we will also denote $u_\pm$, are induced on the Sobolev completed spaces.

Similar statements hold for the gauge groups $\tilde{\mathcal{G}}_{L^2_{k+1}}$ and $\tilde{\mathcal{G}}_{L^2_{k+1,\delta}}$ that are homotopic to $S^1\times Ker(<c_1(\mathcal{L})\cup\cdot,[Y]>\to\mathbb{Z})$.

$\mathcal{C}^*_{L^2_k}$ is the product of an affine space and an infinite dimensional Banach space minus a point. It is thus a CW-complex and thus contractible by [6, XI.10.11]. By the slice theorem and [31, 19.4], $\mathcal{B}^*_{L^2_{l+1,\delta}}$ is a classifying space for $\mathcal{G}_{L^2_{l+1,\delta}}$, and likewise for the other Sobolev completed gauge groups and moduli spaces. This means that the above restriction and extension mappings induce mappings $Br_\pm, Bi, Br$, that are all homotopy equivalences. We claim that these mappings are the natural restriction and extension mappings between the various moduli spaces, and we delete the "B" hereafter.

By lemma 12.1 the classifying spaces are homotopic to

$$\mathbb{C}P^\infty\times K(H^1(Y,\mathbb{Z}),1),$$

where $K(H^1(Y,\mathbb{Z}),1)$ is an Eilenberg-McLane space of type $(H^1(Y,\mathbb{Z}),1)$. As $K(H^1(Y,\mathbb{Z}),1)$ is connected this implies that $H^*(\mathcal{B}^*,\mathbb{Q})$ contains a copy of

$$H^*(\mathbb{C}P^\infty,\mathbb{Q})\approx\mathbb{Q}[x], \deg(x)=2,$$

and of $H^*(K(H^1(Y,\mathbb{Z}),1),\mathbb{Q})$. As $\pi_1(K(H^1(Y,\mathbb{Z}),1)) = H^1(Y,\mathbb{Z})$ is abelian, we have $H_1(K(H^1(Y,\mathbb{Z}),1),\mathbb{Z}) = H^1(Y,\mathbb{Z})$. This implies that

$$H^1(K(H^1(Y,\mathbb{Z}),1),\mathbb{Q})\approx(H_1(K(H^1(Y,\mathbb{Z}),1),\mathbb{Q}))^*\approx(H^1(Y,\mathbb{Q}))^*\approx H_1(Y,\mathbb{Q}).$$

So there is a map:

$$H^*(\mathbb{C}P^\infty,\mathbb{Q})\oplus\Lambda^*H_1(Y,\mathbb{Q})\to H^*(\mathcal{B}^*,\mathbb{Q}).$$

Likewise $\tilde{\mathcal{B}}^*$ is homotopy equivalent to a product of $\mathbb{C}P^\infty$ with an Eilenberg-MacLane space $K(Ker(<c_1(\mathcal{L})\cup\cdot,[Y]>\to\mathbb{Z}),1)$, such that we get a map

$$H^*(\mathbb{C}P^\infty,\mathbb{Q})\oplus\Lambda^*Ker(<c_1(\mathcal{L})\cup\cdot,[Y]>\to\mathbb{Z})^*\to H^*(\tilde{\mathcal{B}}^*,\mathbb{Q}).$$

The maps between the gauge groups commute with the evaluation at a point and thus they preserve the polynomial algebra in cohomology. The same is true for the exterior algebra that came from the group of components. Thus the image of the above map is preserved by the pullbacks of the maps $r_\pm, u_\pm, r$ and ι.

The next proposition is important for the definition of the coefficients of the boundary map. Let $\mathrm{B} = B(S^1)^Y$.

Proposition 12.2. *Given α, β and γ in $\tilde{\mathcal{M}}_\mu$ with $C(\alpha) > C(\gamma) > C(\beta)$, and a gluing map defined on $\mathcal{D}(\alpha, \gamma, \beta, \#)$ as constructed section 11, the following diagram commutes up to homotopy:*

$$\begin{tikzcd}
\hat{\mathcal{M}}(\alpha,\gamma) \times \hat{\mathcal{M}}(\gamma,\beta) \arrow[r, "\iota\Psi"] \arrow[d, "\iota_1 \times \iota_2"'] & \mathcal{D}(\alpha,\gamma,\beta,\#) \arrow[r, "\#"] & \hat{\mathcal{M}}(\alpha,\beta) \arrow[d, "\iota"] \\
\tilde{\mathcal{B}}^*_{L^2_{l,\delta}} \times \tilde{\mathcal{B}}^*_{L^2_{l,\delta}} & & \tilde{\mathcal{B}}^*_{L^2_{l,\delta}} \arrow[ll, "(u_- r_-, u_+ r_+)"']
\end{tikzcd}$$

Proof: We first choose lifts of the $\mathbb{R}$-quotients $\hat{\mathcal{M}}$ of the manifolds of flow curves as done several times in the previous section, so that they are submanifolds of $\mathcal{M}$ and $\mathcal{B}^*$. We again work in the L^2_l-topology. Then the gluing map $\#$ can be written as

$$\#(u_1, u_2, \rho) = (u_1 \#_\rho u_2)_{t(u_1,u_2,\rho)}.$$

We first homotop the translational correction parameter away:

$$F(u_1, u_2, s) = (u_1 \#_{\Psi(u_1,u_2)} u_2)_{s \cdot t(u_1,u_2,\Psi(u_1,u_2))}.$$

We then restrict to e.g. the negative half axis, which is here $\mathbb{R}_- = (-\infty; -1]$. We then have

$$r_- F(u_1, u_2, 0) = (u_1)_{\Psi(u_1,u_2)} + \phi(u_1, u_2, \Psi(u_1, u_2))_{|Y \times \mathbb{R}_-}.$$

The correction section ϕ is homotoped away, together with the translation by the gluing parameter:

$$H(u_1, u_2, s) = (u_1)_{s \cdot \Psi(u_1,u_2)} + s \cdot \phi(u_1, u_2, \Psi(u_1, u_2))_{|Y \times \mathbb{R}_-}.$$

We end up with the restriction of u_1 to $Y \times \mathbb{R}_-$. We use here the following expanded diagram, where all the maps in the lower rectangel are homotopy equivalences:

$$\begin{tikzcd}
\hat{\mathcal{M}}(\alpha,\gamma) \times \hat{\mathcal{M}}(\gamma,\beta) \arrow[r] \arrow[d, "\iota_1 \times \iota_2"'] & \cdots \arrow[d] \\
\tilde{\mathcal{B}}^*_{L^2_{l,\delta}} \times \tilde{\mathcal{B}}^*_{L^2_{l,\delta}} \arrow[d] & \tilde{\mathcal{B}}^*_{L^2_{l,\delta}} \arrow[l, dotted] \arrow[d, "\iota"'] \\
\mathrm{B} \times \mathrm{B} & \mathrm{B} \arrow[l, "(u_- r_-, u_+ r_+)"']
\end{tikzcd}$$

$\square$

Let z be an element of the image of the above map to the cohomology of $\tilde{\mathcal{B}}^*$. We then have:

$$\iota^*(z) = \iota^*(u_- r_-)^*(z) = \iota^*(u_+ r_+)^*(z)$$

This gives using 12.1 and the map ι_Ψ:

$$\begin{aligned}
\iota_\Psi^* \#^* \iota^*(z) &= \iota_\Psi^* \#^* \iota^* d^*(u_- r_-, u_+ r_+)^* \frac{1}{2}(1 \times z + z \times 1) \\
&= (\iota_1 \times \iota_2)^* \frac{1}{2}(1 \times z + z \times 1) \\
&= \frac{1}{2}(\iota_1^* z \times 1 + 1 \times \iota_2^* z).
\end{aligned}$$

In the definition of the boundary map of the Seiberg-Witten-Floer complex we will use the above property in the case of $z = x$, the generator of the polynomial algebra inside the cohomology of the moduli spaces. First we give two consequences of the formula:

Let $\hat{\mathcal{M}}(\alpha, \beta)$ be of dimension $i(\alpha) - i(\beta) = 2n$. We want to integrate the cohomology class x^n over this manifold. At the face of it this is not possible, since as seen in section 10, $\hat{\mathcal{M}}(\alpha, \beta)$ is not compact in general. But as shown in section 10, the ends of this manifold are exactly the images of the gluing maps arising from intermidiate critical points. The restriction of x to the image of, say,

$$\hat{\#} : \mathcal{D}(\alpha, \gamma, \beta) \to \hat{\mathcal{M}}(\alpha, \beta),$$

can be found by further pullback with the homotopy equivalence ι_Ψ. Using the formula above we get:

$$\begin{aligned}
\iota_\Psi^* \#^*(x^n) &= \frac{1}{2^n}(x \times 1 + 1 \times x)^n \\
&= \frac{1}{2^n} \sum_{j=1}^{n} \binom{n}{j} x^j \times x^{n-j} \\
&= 0,
\end{aligned}$$

where we have used that x is of even degree and that the dimension of both $\hat{\mathcal{M}}(\alpha, \gamma)$ and $\hat{\mathcal{M}}(\gamma, \beta)$ is strictly less than $\frac{1}{2}n$. This implies that x^n has compact support on $\hat{\mathcal{M}}(\alpha, \beta)$: $x^n \in H^{2n}_{comp}(\hat{\mathcal{M}}(\alpha, \beta), \mathbb{Z})$ and the integral

$$\varepsilon_{\alpha,\beta} := \int_{\hat{\mathcal{M}}(\alpha,\beta)} x^n$$

is thus well defined.

Now assume that the dimension of $\hat{\mathcal{M}}(\alpha, \beta)$ is odd and write it as $2n + 1$. As seen in the previous section the set

$$K = \hat{\mathcal{M}}(\alpha, \beta) / \bigcup_\gamma Im \hat{\#}_\gamma,$$

where γ runs through the intermidiate critical points between α and β for which a gluing map can be defined as described in section 11, is compact. Furthermore, it can be arranged that K is a smooth manifold with boundary $\partial K = \coprod_\gamma Im(\hat{\#}_\gamma \iota_\Psi)$.

Write the dimension of $\hat{\mathcal{M}}(\alpha, \gamma)$ as $2k_\gamma$ and the dimension of $\hat{\mathcal{M}}(\gamma, \beta)$ as $2(n - k_\gamma)$. We then get:

$$\begin{aligned}
0 = \int_K d(x^n) = \int_{\partial K} x^n &= \sum_\gamma \int_{\hat{\mathcal{M}}(\alpha,\gamma)\times\hat{\mathcal{M}}(\gamma,\beta)} x^n \\
&= \frac{1}{2^n} \sum_\gamma \binom{n}{k_\gamma} \int_{\hat{\mathcal{M}}(\alpha,\gamma)} x^{k_\gamma} \int_{\hat{\mathcal{M}}(\gamma,\beta)} x^{n-k_\gamma} \\
&= \frac{1}{2^n} \sum_\gamma \binom{n}{k_\gamma} \varepsilon_{\alpha,\gamma}\varepsilon_{\gamma,\beta}.
\end{aligned}$$

We will prove the above two relations for the integers $\varepsilon_{\alpha,\beta}, \alpha, \beta \in \tilde{\mathcal{M}}_\mu$ in another, more geometric way. This is the approach of [7] and the arguments will be more sketchy. The evaluation of gauge transformations at a fixed base point (y_0, t_0) gives an exact sequence of groups:

$$1 \to \tilde{\mathcal{G}}^0_{L^2_{l+1,\delta}} \to \tilde{\mathcal{G}}_{L^2_{l+1,\delta}} \to S^1 \to 1,$$

where $\sigma(y_0, t_0) = 1$ for $\sigma \in \tilde{\mathcal{G}}^0_{L^2_{l+1,\delta}}$. This again gives a S^1-bundle

$$\mathcal{B}^0_{L^2_{l,\delta}} := \mathcal{C}_{L^2_{l,\delta}} / \tilde{\mathcal{G}}^0_{L^2_{l+1,\delta}}$$

over $\mathcal{B}^*_{L^2_{l,\delta}}$. The corresponding complex line bundle is denoted

$$L := \mathcal{B}^0_{L^2_{l,\delta}} \times_{S^1} \mathbb{C}.$$

We see from 12.1 that the first Chern class of L equals x: $c_1(L) = x$. It also holds that

$$\#^*(L \otimes L) \approx \pi_1^*(L) \otimes \pi_2^*(L),$$

as we are really just moving around the same line bundle. The formula derived above may be reformulated as

$$2\iota_\Psi^* \hat{\#}^*(c_1(L)) = c_1(L) \times 1 + 1 \times c_1(L).$$

Given $\hat{\mathcal{M}}(\alpha, \beta)$ of dimension $2n$ we now want to define sections of $L^{\oplus n}$ with a finite set of zeroes. Had $\hat{\mathcal{M}}(\alpha, \beta)$ been compact, this would be automatically satisfied for a transversal section with a zero set which is a zero-dimensional submanifold of $\hat{\mathcal{M}}(\alpha, \beta)$. This does not hold in general but by defining "allowed" sections over the moduli spaces of flow curves by induction after the dimension of the manifold, we can secure that there is no zeroes over the ends of the moduli space. This requires that we also choose sections in $L^{\oplus n}$ over $\hat{\mathcal{M}}(\alpha, \beta)$ of odd dimension $2n + 1$, where the zero set of a transversal section is 1-dimensional.

$\dim\hat{\mathcal{M}}(\alpha,\beta)=0$: We set $L^{\oplus 0}=\mathbb{C}$, the trivial line bundle. As $\hat{\mathcal{M}}(\alpha,\beta)$ is in fact compact - there is no room for any gluing - we define the (only) allowed section to be simply the zero section.

$\dim\hat{\mathcal{M}}(\alpha,\beta)=1$: Here we again have $L^{\oplus 0}=\mathbb{C}$ and we again choose the zero section as the only allowed section over these moduli space.

Now for the inductive step: In the even dimensional case, working over an end where broken trajectories passing through the intermediate critical point γ are glued in, we can without loss of generality assume the dimension of $\hat{\mathcal{M}}(\alpha,\gamma)$ to be even, say $2k$, and the dimension of $\hat{\mathcal{M}}(\gamma,\beta)$ to be odd, say $2m+1$, where $n=k+m+1$. Choose n allowable sections of L over each of the two components, where "allowable" includes that all intersecions of zero sets of these sections should be submanifolds of the respective base manifold. Denote these sections $s_1,s_2,\dots,s_n$ and $t_1,t_2,\dots,t_n$, respectively. Then

$$v:=\oplus_{j=1}^n s_j\otimes t_j,$$

is a section of $(L\otimes L)^{\oplus n}$ over the end $Im\hat{\#}_\gamma$. We have:

$$\begin{aligned}
v^{-1}(0) &= \bigcap_{j=1}^{n}(s_j\otimes t_j)^{-1}(0)\\
&= \bigcap_{j=1}^{n} s_j^{-1}(0)\times\hat{\mathcal{M}}(\gamma,\beta)\times\mathbb{R}\cap\mathcal{D}(\alpha,\gamma,\beta)\\
&\qquad \cup\hat{\mathcal{M}}(\alpha,\gamma)\times t_j^{-1}(0)\times\mathbb{R}\cap\mathcal{D}(\alpha,\gamma,\beta)\\
&=\emptyset
\end{aligned}$$

by counting dimensions. Thus v has no zeroes over $Im\hat{\#}_\gamma$. Define v similarly over the other ends of $\hat{\mathcal{M}}(\alpha,\beta)$ and then extend it to a section over the whole manifold transversal to the zero section of $(L\otimes L)^{\oplus n}$. This section has a zero dimensional zero set consisting of finitely many points by compactness of the complement of the gluing-images. Thus we can define

$$\varepsilon'_{\alpha,\beta}:=\#v^{-1}(0),$$

the oriented sum of the points in $v^{-1}(0)$. This number is independent of the choices made: Had we choosen another gluing map, other sections of L over the ends and another extension to the rest of $\hat{\mathcal{M}}(\alpha,\beta)$, we would have a section of $(L\otimes L)^{\oplus n}$ over the boundary of $\hat{\mathcal{M}}(\alpha,\beta)\times[0;1]$. This section could be extended to a section transversal to the zero section of $(L\otimes L)^{\oplus n}$ on all of $\hat{\mathcal{M}}(\alpha,\beta)\times[0;1]$, and the 1-dimensional zero set of this section would give an oriented cobordism between the two zero sets over the boundary. The existence of such a cobordism implies that the oriented sum of the zero sets at each boundary component is the same.

If the dimension of $\hat{\mathcal{M}}(\alpha,\beta)$ is odd, say $2n+1$, we use the same procedure: The dimensions of $\hat{\mathcal{M}}(\alpha,\gamma)$ and $\hat{\mathcal{M}}(\gamma,\beta)$ are now even, say $2k$ and $2l$, respectively, where

$k+l=n$. As above choose n allowable sections of L over each of these manifolds and combine them to a section v of $(L\otimes L)^{\oplus n}$ defined as above. We get:

$$v^{-1}(0)=\bigcup_{j_1<j_2<\ldots<j_m}(\bigcap_{i=1}^{m}s_{j_i}^{-1}(0)\times\bigcap_{k=1}^{l}t_{j_k}^{-1}(0)\times\mathbb{R})\cap\mathcal{D}(\alpha,\gamma,\beta),$$

where j_k runs through the indices not contained in the list $(j_1,j_2,\ldots,j_m)$. By the induction assumption each of

$$\bigcap_{j=1}^{m}s_{j_i}^{-1}(0)\quad\text{and}\quad\bigcap_{k=1}^{l}t_{j_k}^{-1}(0)$$

are finite, oriented sets. Thus the zero set of v is half lines over finitely many points:

$$v^{-1}(0)=\bigcup_{j_1<j_2<\ldots<j_m}\bigcap_{i=1}^{m}s_{j_i}^{-1}(0)\times\bigcap_{k=1}^{l}t_{j_k}^{-1}(0)\times\{\text{half lines}\}.$$

We do a similar construction over the other ends of $\hat{\mathcal{M}}(\alpha,\beta)$ and extend the resulting section to a section v defined on all of $\hat{\mathcal{M}}(\alpha,\beta)$ that is transversal to the zero section of $(L\otimes L)^{\oplus n}$. $v^{-1}(0)$ is a 1-dimensional submanifold of $\hat{\mathcal{M}}(\alpha,\beta)$ and gives a cobordism between the zero sets over the different ends of the manifold. The total oriented count of the points in the boundary of this cobordism is zero. We thus get

$$\sum_{\gamma}\binom{n}{k_\gamma}\#s^{-1}(0)\#t^{-1}(0)=\sum_{\gamma}\binom{n}{k_\gamma}\varepsilon'_{\alpha,\gamma}\varepsilon'_{\gamma,\beta}=0,$$

as each of the components in the expression above for $v^{-1}(0)$ in the end determined by γ gives the same contribution.

As

$$\begin{aligned}e((L\otimes L)^{\oplus n})&=c_n((L\otimes L)^{\oplus n})=(c(L\otimes L)^n)_{2n}\\&=((1+2c_1(L))^n)_{2n}=2^nc_1(L)^n,\end{aligned}$$

we have $\varepsilon'_{\alpha,\beta}=2^n\varepsilon_{\alpha,\beta}$.

We also give a geometric description of the B-maps induced by the homotopy equivalence φ from 12.1:

$$\begin{aligned}\varphi&:\mathcal{G}\to S^1\times H^1(Y,\mathbb{Z}),\\\varphi(\sigma)&=(\sigma(y_0),[\sigma^*(vol_{S^1})]).\end{aligned}$$

We begin by defining holonomy maps on the moduli spaces. Given a smooth closed curve $\gamma:S^1\to Y$, we can define maps

$$\begin{aligned}\varphi_\gamma&:\mathcal{B}\to S^1,\\\varphi_\gamma([A,\psi])&=Hol_\gamma(A),\end{aligned}$$

the holonomy along γ of the unitary connection A. This is well defined because the holonomy is invariant under the action of gauge transformations as S^1 is Abelian. For the same reason it is independent of the point in the fiber over y_0 to which we lift the curve. Furthermore, φ_γ only depends on the homotopy class of γ and it is independent of the chosen base point y_0. So we get for each element γ of $H_1(Y,\mathbb{Z})$ a map φ_γ.

The derivative of φ_γ is

$$\varphi_\gamma^{-1} D_{[A,\psi]}\varphi_\gamma(\alpha) = -\int_\gamma \alpha.$$

It is a closed one-form on $\mathcal{B}^*$ and thus gives an element of $H^1(\mathcal{B}^*,\mathbb{R})$.

We next observe that the decomposition

$$H^1(Y,\mathbb{Z}) = \mathbb{Z}^n \oplus \mathrm{Tor},$$

implies $K(H^1(Y,\mathbb{Z}),1) = T^n \times K(\mathrm{Tor},1)$. If $(\gamma_i)_{i=1}^n$ generates the free part of $H^1(Y,\mathbb{Z})$ in this splitting, we claim that the map

$$\begin{gathered} B\varphi_2 : \mathcal{B}^* \to T^n, \\ B\varphi_2([A,\psi]) = (Hol_{\gamma_i}(A))_{i=1}^n, \end{gathered}$$

is exactly the B-map of the second coordinate of φ. This will imply that

$$B\varphi_2^*([\gamma_i]) = \varphi_{\gamma_i}^*(vol_{S^1}) = -\frac{1}{2\pi i}\int_{\gamma_i} \cdot\ ,$$

for $i = 1,\ldots,n$ and thus for all $[\gamma]$ in the free part of $H_1(Y,\mathbb{Z})$. We also claim that $B\varphi_1$ is the map - uniquely defined up to homotopy - which by pullback of the canonical line bundle $H \downarrow \mathbb{C}P^\infty$ gives the line bundle L over $\mathcal{B}^*$.

13 Chain complex and boundary operator.

This section contains the definition of the Seiberg-Witten-Floer complex and an elementary truncation property of the complex and it's homology.

First choose the lift of $\mathcal{M}_\mu$ to $\tilde{\mathcal{M}}_\mu$ for which the indices of the lifted critical points is in the interval $[0; N-1]$. Recall from section 8 that $t\cdot$ denotes the action of the generator σ_0 of H for which

$$< c_1(\mathcal{L}) \cup c(\sigma_0), [Y] > = N.$$

Then, in a slightly ambigious notation, we have $\pi^{-1}(\alpha) = \{t^k\alpha\}_{k\in\mathbb{Z}} \subseteq \tilde{\mathcal{M}}_\mu$ for every $\alpha \in \mathcal{M}_\mu$.

For $j \in \mathbb{Z}$ define the j'th chain group, C_j, in the Seiberg-Witten-Floer complex:

$$\begin{aligned} C_j &= \bigoplus_{\substack{\alpha\in\tilde{\mathcal{M}}_\mu : i(\alpha)\equiv j \quad (2) \\ i(\alpha)\leq j}} \mathbb{Z}\alpha \\ &= \bigoplus_{\substack{\alpha\in\mathcal{M}_\mu \\ i(\alpha)\equiv j \quad (2)}} \mathbb{Z}[[t]]_{\{k\geq\frac{1}{N}(i(\alpha)-j)\}}\alpha \end{aligned}$$

Remember that $i(t^k\alpha) = i(\alpha) - kN$, so that the condition $k \geq \frac{1}{N}(i(\alpha) - j)$ is equivalent to $i(t^k\alpha) \leq j$.

Define numbers δ and Δ by:

$$\delta(j, t^k\alpha) := \frac{1}{2}(j - i(t^k\alpha)) = \frac{1}{2}(j - i(\alpha) + kN)$$

and

$$\Delta(t^k\alpha, t^l\beta) := \frac{1}{2}(i(t^k\alpha) - i(t^l\beta) - 1) = \frac{1}{2}(i(\alpha) - i(\beta) - 1 + (l-k)N),$$

for $\alpha, \beta \in \mathcal{M}_\mu, j, k, l \in \mathbb{Z}$. Notice that

$$\delta(j-1, t^l\beta) - \delta(j, t^l\alpha) = \Delta(t^k\alpha, t^l\beta).$$

Define the boundary map of the Seiberg-Witten-Floer complex by:

$$\begin{aligned} &\partial_j : C_j \to C_{j-1}, \\ \partial_j(t^k\alpha) := \sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} &\sum_{l: i(\alpha)-kN > i(\beta)-lN} \binom{\delta(j-1, t^l\beta)}{\delta(j, t^k\alpha)} \varepsilon_{t^k\alpha, t^l\beta} t^l\beta. \end{aligned}$$

This map is extended additively to C_j. This is well defined as there is only finitely many α and k for which $j \geq i(\alpha) - kN > i(\beta) - lN$ for given $t^l\beta \in C_{j-1}$.

Theorem 13.1. *∂_j is a boundary map, that is: $\partial_{j-1}\partial_j = 0$ for all $j \in \mathbb{Z}$.*

Proof: A calculation gives:

$$\begin{aligned}
\partial_{j-1}\partial_j t^k\alpha &= \partial_{j-1}\Big(\sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> \\ i(\beta)-lN}} \binom{\delta(j-1, t^l\beta)}{\delta(j, t^k\alpha)} \varepsilon_{t^k\alpha, t^l\beta} t^l\beta\Big) \\
&= \sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> \\ i(\beta)-lN}} \binom{\delta(j-1, t^l\beta)}{\delta(j, t^k\alpha)} \varepsilon_{t^k\alpha, t^l\beta} \\
&\quad \sum_{\substack{\gamma \\ i(\gamma)\equiv j \quad (2)}} \sum_{\substack{m: i(\beta)-lN> \\ i(\gamma)-mN}} \binom{\delta(j-2, t^m\gamma)}{\delta(j-1, t^l\beta)} \varepsilon_{t^l\beta, t^m\gamma} t^m\gamma \\
&= \sum_{\substack{\gamma \\ i(\gamma)\equiv j \quad (2)}} \sum_{m: i(\gamma)-mN\leq j-2} \sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> \\ i(\beta)-lN> i(\gamma)-mN}} \\
&\quad \binom{\delta(j-1, t^l\beta)}{\delta(j, t^k\alpha)} \binom{\delta(j-2, t^m\gamma)}{\delta(j-1, t^l\beta)} \varepsilon_{t^k\alpha, t^l\beta} \varepsilon_{t^l\beta, t^m\gamma} t^m\gamma.
\end{aligned}$$

Thus $\partial_{j-1}\partial_j t^k\alpha = 0$ iff for all $\gamma \in \mathcal{M}_\mu : i(\gamma) \equiv j \ (2)$ and all $m : i(\alpha) - kN > i(\gamma) - mN$:

$$\sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> i(\beta)-lN \\ > i(\gamma)-mN}} \binom{\delta(j-1, t^l\beta)}{\delta(j, t^k\alpha)} \binom{\delta(j-2, t^m\gamma)}{\delta(j-1, t^l\beta)} \varepsilon_{t^k\alpha, t^l\beta} \varepsilon_{t^l\beta, t^m\gamma} = 0.$$

We have that

$$\begin{aligned}
\binom{\delta(j-1, t^l\beta)}{\delta(j, t^k\alpha)} \binom{\delta(j-2, t^m\gamma)}{\delta(j-1, t^l\beta)} &= \frac{1}{\delta(j, t^k\alpha)!\Delta(t^k\alpha, t^l\beta)!} \frac{\delta(j-2, t^m\gamma)!}{\Delta(t^l\beta, t^m\gamma)!} \\
&= \binom{\delta(j-2, t^m\gamma)}{\delta(j, t^k\alpha)} \binom{\frac{1}{2}(i(t^k\alpha) - i(t^m\gamma)) - 1}{\Delta(t^k\alpha, t^l\beta)}.
\end{aligned}$$

Up to the common factor $\binom{\delta(j-2, t^m\gamma)}{\delta(j, t^k\alpha)}$, we see that the identity above is exactly the identity derived in the previous section for the manifold $\hat{\mathcal{M}}(t^k\alpha, t^m\gamma)$ of dimension $i(t^k\alpha) - i(t^m\gamma) - 1 = 2n + 1$. □

Thus we have now defined the Seiberg-Witten-Floer chain complex. The corresponding Seiberg-Witten-Floer homology is denoted

$$H_j^{SW}(Y, P_{Spin^c}) := Ker(\partial_j)/Im(\partial_{j-1})$$

for $j \in \mathbb{Z}$. In the rest of this section we will define a truncation of the chain complex that will represent it as an inverse limit in a natural way. Also, we will define a version of the chain complex over $\mathbb{Q}$ where the boundary operator commutes with the multiplication by t.

Set

$$C_j^n := \{\sum_k q_k t^k \alpha \in C_j | k \leq n\}.$$

$C_j^n = 0$ for -n big and the chain complex $(C_j^n)_{j\in\mathbb{Z}}$ is thus finite in the negative direction. Denote by

$$\pi_n : C_j \to C_j^n$$

the projection onto this subgroup. We also set $\partial_j^n := \pi_n \partial_j$ and claim that this is a boundary operator on $(C_j^n)_{j\in\mathbb{Z}}$. This will follow from the formula

$$\pi_n \partial_j \pi_n = \pi_n \partial_j,$$

as it then follows that:

$$\partial_{j-1}^n \partial_j^n = \pi_n \partial_{j-1} \pi_n \partial_j = \pi_n \partial_{j-1} \partial_j = 0.$$

Again we calculate:

$$\begin{aligned}\pi_n\partial_j(\sum_k q_k t^k\alpha) &= \pi_n(\sum_k q_k \sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> \\ i(\beta)-lN}} \begin{pmatrix}\delta(j-1,t^l\beta)\\ \delta(j,t^k\alpha)\end{pmatrix} \varepsilon_{t^k\alpha,t^l\beta} t^l\beta) \\ &= \sum_k q_k \sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> \\ i(\beta)-lN, l\leq n}} \begin{pmatrix}\delta(j-1,t^l\beta)\\ \delta(j,t^k\alpha)\end{pmatrix} \varepsilon_{t^k\alpha,t^l\beta} t^l\beta.\end{aligned}$$

Working on the other side of the equation we get:

$$\begin{aligned}\pi_n\partial_j\pi_n(\sum_k q_k t^k\alpha) &= \pi_n\partial(\sum_{n\geq k} q_k t^k\alpha) \\ &= \pi_n(\sum_{n\geq k} q_k \sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> \\ i(\beta)-lN}} \begin{pmatrix}\delta(j-1,t^l\beta)\\ \delta(j,t^k\alpha)\end{pmatrix} \varepsilon_{t^k\alpha,t^l\beta} t^l\beta) \\ &= \sum_{n\geq k} q_k \sum_{\substack{\beta \\ i(\beta)\equiv j-1 \quad (2)}} \sum_{\substack{l: i(\alpha)-kN> \\ i(\beta)-lN, l\leq n}} \begin{pmatrix}\delta(j-1,t^l\beta)\\ \delta(j,t^k\alpha)\end{pmatrix} \varepsilon_{t^k\alpha,t^l\beta} t^l\beta.\end{aligned}$$

These two sums are equal as the inequality $i(\alpha) - kN > i(\beta) - lN$ implies $l \geq k$.

The projection π_n is a chain map:

$$\pi_n\partial_j = \pi_n\partial_j\pi_n = \partial_j^n\pi_n.$$

Defining $\pi_{n+1,n} : C_j^{n+1} \to C_j^n$ as the natural projection, we again get a chain map:

$$\begin{aligned}\pi_{n+1,n}\partial_j^{n+1} &= \pi_{n+1,n}\pi_{n+1}\partial_j \\ &= \pi_n\partial_j \\ &= \pi_n\partial_j\pi_{n+1,n} \\ &= \partial_j^n\pi_{n+1,n},\end{aligned}$$

where the third equality is shown by writing out the sums as above.

We see that the truncation of the Seiberg-Witten-Floer complex gives a tower of chain complexes, $(C_j^n)_{j\in\mathbb{Z}}$, where the maps are $\pi_{n+1,n}: C_j^{n+1} \to C_j^n$.

$$\ldots \to C_*^{n+2} \overset{\pi_{n+2,n+1}}{\to} C_*^{n+1} \overset{\pi_{n+1,n}}{\to} C_*^n \to \ldots \to C_*^1 \overset{\pi_{1,0}}{\to} C_*^0$$

This tower satisfies the Mittag-Leffler condition as the images of the naturally defined projections $\pi_{m,n}: C_j^m \to C_j^n, m \geq n$, are all surjective. As we also have $C_* = \varprojlim C_*^n$, we get [32, th.3.5.8] for each $q \in \mathbb{Z}$:

$$0 \to \varprojlim{}^1 H_{q+1}(C_*^n) \to H_q^{SW} \to \varprojlim H_q(C_*^n) \to 0.$$

A natural question to ask is now whether the $\varprojlim^1$-term vanishes, so that the Seiberg-Witten-Floer homology can be expressed as the inverse limit of the homologies of the truncated complexes. This would be the case if the homology groups themselves satisfy the Mittag-Leffler condition [32, prop.3.5.7]. In the case of countable groups, such as this, it is know that the Mittag-Leffler condition is equivalent to the vanishing of $\varprojlim^1 H_{q+1}(C_*^n)$. Furthermore, if the $\varprojlim^1$-term is not zero, it is an uncountable group [22, th.2]. Thus the above exact sequence tells us that either H_*^{SW} is computabel as the inverse limit of the more tractable truncated homologies or it is absolutely horrible.

I have not been able to verify the Mittag-Leffler condition - or to find a counter example. But if we instead define the Seiberg-Witten-Floer homology over the rationals as below, we do get such a result.

Let $C_{j,\mathbb{Q}} := C_j \otimes_{\mathbb{Z}} \mathbb{Q}$ and define the boundary operator

$$\partial_{j,\mathbb{Q}} : C_{j,\mathbb{Q}} \to C_{j-1,\mathbb{Q}}$$

by

$$\partial_{j,\mathbb{Q}}(t^k\alpha) := \sum_{\substack{\beta\in\mathcal{M}_\mu \\ i(\beta)\equiv j \quad (2)}} \sum_{l: i(\alpha)-kN > i(\beta)-lN} \frac{1}{\Delta(t^k\alpha, t^l\beta)!} \varepsilon_{t^k\alpha, t^l\beta} t^l\beta.$$

That this is in fact a boundary operator is shown in the same way as for ∂_j, but now the binomial expressions are a little simpler. As above one can define the truncated versions of the chain complex by $C_{j,\mathbb{Q}}^n := C_j^n \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\partial_{j,\mathbb{Q}}^n := \pi_n \partial_{j,\mathbb{Q}}$. The natural projections π_n and $\pi_{m.n}$ are again chain maps and we get a tower of chain complexes where each chain group is now a finite dimensional vector space over $\mathbb{Q}$. Again, the tower obviously satisfies the Mittag-Leffler condition and we get the same exact sequence as above for the new homology groups $H_*^{SW,\mathbb{Q}}$ and $H_*(C_{*,\mathbb{Q}}^n)$. But now the tower of homology groups consists also of finite dimensional vector spaces over $\mathbb{Q}$ and thus the $\varprojlim^1$-term vanishes in this case [32, ex.3.5.2]. Thus:

$$H_*^{SW,\mathbb{Q}} \approx \varprojlim H_*(C_{*,\mathbb{Q}}^n).$$

The connection between H_*^{SW} and $H_*^{SW,\mathbb{Q}}$ is given by the map

$$\varphi_j : C_j \to C_{j,\mathbb{Q}},$$
$$\varphi(\sum_k q_k t^k \alpha) := \sum_k \frac{q_k}{\delta(j, t^k\alpha)!} t^k \alpha.$$

This is a chain map:

$$\begin{aligned}
\partial_{j,\mathbb{Q}} \varphi_j(t^k\alpha) &= \partial_{j,\mathbb{Q}}(\frac{1}{\delta(j, t^k\alpha)!} t^k \alpha) \\
&= \sum_{\substack{\beta \\ i(\beta)\equiv \quad (2)}} \sum_{\substack{l: i(\beta)-lN \\ < i(\alpha)-kN}} \frac{1}{\delta(j, t^k\alpha)! \Delta(t^k\alpha, t^l\beta)!} \varepsilon_{t^k\alpha, t^l\beta} t^l \beta \\
&= \varphi_{j-1}(\sum_{\substack{\beta \\ i(\beta)\equiv \quad (2)}} \binom{\delta(j-1, t^l\beta)}{\delta(j, t^k\alpha)} \varepsilon_{t^k\alpha, t^l\beta} t^l \beta) \\
&= \varphi_{j-1} \partial_j (t^k \alpha).
\end{aligned}$$

φ_* has an obvious inverse and thus induces an isomorphism of chain complexes

$$(C_* \otimes_{\mathbb{Z}} \mathbb{Q}, \partial_* \otimes \mathbb{Q}) \approx (C_{*,\mathbb{Q}}, \partial_{*,\mathbb{Q}}).$$

Künneth's formula gives isomorphims for the homologies:

$$H_*^{SW,\mathbb{Q}} \approx H_*(C_* \otimes_{\mathbb{Z}} \mathbb{Q}) \approx H_*^{SW} \otimes \mathbb{Q}.$$

Similarly, there is maps φ_j^n for $j, n \in \mathbb{Z}$ that induce the corresponding isomorphisms in homology for the truncated chain complexes.

The Seiberg-Witten-Floer homology over $\mathbb{Q}$ has another special feature, namely that the boundary operator commutes with multiplication with t:

$$\begin{aligned}
t^m \partial_{j,\mathbb{Q}}(t^k\alpha) &= \sum_{\substack{\beta \\ i(\beta)\equiv j \quad (2)}} \sum_{l: i(\beta)-lN < i(\alpha)-kN} \frac{1}{\Delta(t^k\alpha, t^l\beta)!} \varepsilon_{t^k\alpha, t^l\beta} t^{l+m} \beta \\
&= \sum_{\substack{\beta \\ i(\beta)\equiv j \quad (2)}} \sum_{l: i(\beta)-pN < i(\alpha)-(k+m)N} \frac{1}{\Delta(t^{k+m}\alpha, t^p\beta)!} \varepsilon_{t^{k+m}\alpha, t^p\beta} t^p \beta \\
&= \partial_{j,\mathbb{Q}}(t^{k+m}\alpha),
\end{aligned}$$

where it is used that for $p = l + m$:

$$\varepsilon_{t^k\alpha, t^l\beta} = \varepsilon_{t^{k+m}\alpha, t^p\beta},$$
$$\Delta(t^k\alpha, t^l\beta) = \Delta(t^{k+m}\alpha, t^p\beta),$$
$$i(\beta) - lN < i(\alpha) - kN \Leftrightarrow i(\beta) - pN < i(\alpha) - (k+m)N.$$

This implies that $H_*^{SW,\mathbb{Q}}$ is independent of the chosen lift of $\mathcal{M}_\mu$ to $\tilde{\mathcal{M}}_\mu$ as long as the lift is chosen so that all critical points have indices within one period of length N. Also, $(C_{*,\mathbb{Q}}, \partial_{*,\mathbb{Q}})$ is chain complex of modules over $\mathbb{Q}[t]$. The $\mathbb{Z}$-version does not have this property.

This ends the construction of a Seiberg-Witten-Floer homology using higher dimensional manifolds of flow curves.

14 Appendix on Sobolev spaces and differential operators.

Sobolev completions of the space of sections of a vector bundle and the maps differential operators induce on these spaces appear throughout the thesis. This appendix briefly lists the results used. For Sobolev spaces of functions on an open subset of $\mathbb{R}^n$ my standard reference is [2]. Below $p \geq 1, k \geq 0$.

Assume that X is a compact manifold of dimension n and let ξ be a vector bundle over X of dimension m. The following three equivalent definitions of Sobolev spaces of sections of ξ have all been used more or less implicitly:

Definition 14.1.
*1)**[DK,App.AII;Pal.X.§4]** Choose a finite covering $(U_\alpha, \varphi_\alpha)_{\alpha\in I}$ of X by open coordinate neighbourhoods over which ξ is trivial with trivializations ψ_α and a partition of unity $(\rho_\alpha)_{\alpha\in I}$ subordinate to this covering. Then define the L^p_k-Sobolev norm of a smooth section $s : X \to \xi$ by*

$$\|s\|_{L^p_k} := \sum_{\alpha\in I} \|\psi_\alpha(\rho_\alpha s)\|_{L^p_k},$$

where on the right hand side $\|\cdot\|_{L^p_k}$ denotes the Sobolev norm on vector valued map on an open subset of $\mathbb{R}^n$.
*2)**[DK,App.A;He.2.1]** Choose a metric and a metric connection on ξ and a strictly positive, smooth measure on X. Then set*

$$\|s\|'_{L^p_k} := (\sum_{1\leq i\leq k} \int |\nabla^i(s)|^p)^{\frac{1}{p}}.$$

*3)**[Pal.IX.§3]** Choose a strictly positive, smooth measure on X, a metric on ξ and let $J^k\xi$ denote the k-jet bundle of ξ. Then define*

$$\|s\|''_{L^p_k} := \|j_k(s)\|_{L^p},$$

where $\|\cdot\|_{L^p}$ denotes the L^p-norm on sections of $J^k\xi$ and $j_k : \Omega^0(\xi) \to \Omega^0(J^k\xi)$ is the k-jet bundle map.

In all three cases define $\Omega^0(\xi)_{L^p_k}$ to be the completion of $\Omega^0(\xi)$ in the respective norms. Furthermore, define $\Omega^0(\xi)_{L^p_{-k}} := \overline{(\Omega^0(\xi)_{L^p_k})^*}$.
4) With the notation of 3), set

$$\|s\|_{C^r} := \|j_k(s)\|_{C^0} = sup_{x\in X}|j_k(s(x))|$$

and denote the completion of $\Omega^0(\xi)$ in this norm by $\Omega^0(\xi)_{C^r}$.

The above defintions 1)-3) do not give the same norm on the smooth sections of ξ, but the different norms are all equivalent and thus give the completion of $\Omega^0(\xi)$ a well defined structure of Banachable space.

Theorem 14.2 (Embedding theorems).
1) ***[DK,App.A.5;He.3.5]*** *If $k \geq l$ and $k - \frac{n}{p} \geq l - \frac{n}{q}$, there is a continous embedding*

$$\Omega^0(\xi)_{L^p_k} \to \Omega^0(\xi)_{L^q_l}.$$

2) ***(Sobolev embedding theorem) [DK,App.A.5;Pal.X.§.th.4]***
If $k - \frac{n}{2} > r$, there is a continous embedding

$$\Omega^0(\xi)_{L^2_k} \to \Omega^0(\xi)_{C^r}.$$

3) ***(Rellich's theorem) [DK,App.A.4+A.9;Pal.X.§.4.th.4]***
In 1), if there is strict inequality $k - \frac{n}{p} > l - \frac{n}{q}$, the embedding is compact. This is always the case in 2).

Of course, $\Omega^0(\xi)_{C^k}$ maps into $\Omega^0(\xi)_{L^2_k}$.

Multiplication theorems for Sobolev spaces are used numerous times in the body of the text. Mostly they are shown by hand using Leibniz's rule and Hölder's inequality several times, but there is one important general result:

Lemma 14.3 (Ad.5.23). *If $kp > n$, $\Omega^0(\xi)_{L^p_k}$ is an algebra.*

We also have the following:

Theorem 14.4 (Restriction theorem. Pal.X§4,th.7). *Let Y be a smooth and compact submanifold of X of dimension m and set $t = \frac{1}{2}(n - m)$. Then for $k \geq t$, there is a surjective continous map*

$$\Omega^0(\xi)_{L^2_k} \to \Omega^0(\xi_{|Y})_{L^2_{k-t}}.$$

Let D denote a differential operator of degree r acting on sections of ξ and having values in the space of sections of the vector bundle η over X. Then D gives a bounded map

$$D : \Omega^0(\xi)_{L^p_k} \to \Omega^0(\eta)_{L^p_{k-r}}.$$

by e.g. [26, XI.th.6]. If the differential operator is assumed to be elliptic there is the following nice results on the extension to the Sobolev spaces of sections:

Theorem 14.5 (DK,App.A.7+A.8+A.16;Pal.XI.th.5-7).
1) D is Fredholm, that is: D has a finite dimensional kernel and cokernel and the image of D is closed.
2) For $p = 2$: $Im(D)^{\perp_{L^2}} \approx Ker(D^*)$.
3) ***(Regularity)*** *If $s \in \Omega^0(\xi)_{L^p_k}$ for some k and $Ds \in \Omega^0(\xi)_{L^p_l}$, then $s \in \Omega^0(\xi)_{L^p_{l+r}}$.*
4) ***(Fundamental elliptic inequality)*** *For $s \in \Omega^0(\xi)_{L^p_{k+r}}$:*

$$\|s\|_{L^p_{k+r}} \leq C(\|s\|_{L^p} + \|Ds\|_{L^p_k})$$

for some constant C.

The index of the elliptic diffential operator D is the index of the Fredholm map D. This is independent of the Sobolev norm used [26, XI.th.8].

If the sequence of symbols corresponding to the sequence of differential operators

$$\xi_0 \xrightarrow{D_1} \xi_1 \xrightarrow{D_2} \xi_2$$

is exact, the operator $\Delta = D_1 D_1^* + D_2^* D_2$ is elliptic. If follows that

$$\begin{aligned} E_1 &= Ker(\Delta) \oplus Im(\Delta) \\ &= (Ker(D_1^*) \cap Ker(D_2)) \oplus Im(D_1) \oplus Im(D_2^*). \end{aligned}$$

We now turn to exponentially weighted Sobolev spaces of sections of vector bundles over manifolds of type $X \times \mathbb{R}$, where X is as above. The standard reference to this is [20] where a more general situation is considered. For the convinience of the reader we summarize some of the results of that paper here.

Definition 14.6. *The Banach space $\Omega(\xi)_{L^p_{k,\delta}}$ is defined using the approach from* 14.1, *1) with coordinate charts of form $U_\gamma \times \mathbb{R}$, smooth sections of compact support and the norm*

$$\|s\|_{L^p_{k,\delta}} = \sum_{|\alpha| \le k} \| \exp(\delta t) D^\alpha s\|_{L^p(\varphi(U_\gamma)\times\mathbb{R})}$$

on maps from $\varphi(U_\gamma) \times \mathbb{R}$ to $\mathbb{R}^m$.

Notice that if the exponential factor is moved to the right of D^α, we get a norm equivalent to the one above. This implies [20, p.421] that $\Omega(\xi)_{L^p_{k,\delta}}$ is isomorphic as a Banach space to $\Omega(\xi)_{L^p_k}$. This again gives that there is the following embedding results:

Proposition 14.7 (He.2.7+2.8+3.14). *If $k \ge l \ge 0, q > p \ge 1$ and $k - \frac{n}{p} \ge l - \frac{n}{q}$, there is a bounded embedding:*

$$\Omega(\xi)_{L^p_{k,\delta}} \to \Omega(\xi)_{L^q_{l,\delta}}.$$

In [20, lemma 7.2] there is more embedding results. Notice though that there is no versions of Rellich's theorem for the unbounded domains.

Also, if one can establish multiplication maps for the Sobolev spaces with $\delta = 0$ they are valid for the weighted spaces also - and even maps into spaces with a stronger weight:

$$\begin{aligned} \|s_1 s_2\|_{L^p_{k,\delta_1+\delta_2}} = \| \exp((\delta_1 + \delta_2)t) s_1 s_2\|_{L^p_k} &\le K\| \exp(\delta_1 t) s_1\|_{L^{p_1}_{k_1}} \| \exp(\delta_2 t) s_2\|_{L^{p_2}_{k_2}} \\ &= K\|s_1\|_{L^{p_1}_{k_1,\delta_1}} \|s_2\|_{L^{p_2}_{k_2,\delta_2}} \end{aligned}$$

using the multiplication

$$L^{p_1}_{k_1} \times L^{p_2}_{k_2} \to L^p_k.$$

Translation invariant operators give bounded maps between the weighted Sobolev spaces as above (Also notice [20, 6], though). The main theorem of [20] is the following, with the notation adapted to the setup of this appendix:

Theorem 14.8. *If D is elliptic and translation invariant there is a discrete set $\mathcal{D}(D) \subset \mathbb{R}$ such that*

$$D : \Omega(\xi)_{L^p_{k,\delta}} \to \Omega(\eta)_{L^p_{k-r,\delta}}$$

is Fredholm if and only if $\delta \in \mathbb{R} - \mathcal{D}(D)$.

$\lambda \in \mathcal{C}(D) \Leftrightarrow$ there exists a solution to $Du = 0$ on the form

$$u(y,t) = \exp(i\lambda t)p(y,t),$$

where p is a polynomial in t with smooth coefficients in $\Omega^0(\xi)$. If $D = \frac{\partial}{\partial t} + P$ a calculation shows that the top term of the polynomial p is an eigenvector for P belonging to the eigenvalue $-i\lambda$. If P is selfadjoint this implies that $\lambda \in i\mathbb{R}$ and $-i\lambda = im\lambda$. By definition

$$\mathcal{D}(D) = \{im\lambda | \lambda \in \mathcal{C}(D)\}.$$

15 Appendix on the Dirac operator.

The purpose of this appendix is to give a proof of two properties of the Dirac operator used in the proof of 5.3.

In section 1 we noted that the Dirac operators have the unique continuation property from open subsets: If $\eth_A(\psi) = 0$ and ψ vanishes on an open subset of Y, ψ is identically zero. In the litterature this theorem is stated in the case where $\eth_A$ is a smooth differential operator, so in particular the connection A is assumed to be smooth. This is not necessarily the case in our setup, actually we have only assumed that A is of Sobolev-class L_k^2, where $k \geq 1$, so the question is whether this is enough to secure that $\eth_A$ still has the unique continuation property from open subsets. An investigation of the proof in [5] shows that it is sufficient that A is of Sobolev class L_3^2. Notice that in the above argument we could assume that A was a L_{m+1}^2-connection, $m \geq 2$. So the use of the unique continuation property from open subsets in the above argument is okay.

We also used in the proof of 5.3 that the complement of the zero-set of a harmonic spinor is connected.

Proposition 15.1. *Let A be a connection of Sobolev class L_k^2, where $k - \frac{1}{2}n > 0$, on a Clifford bundle over the compact manifold Y^n. Assume that the corresponding Dirac operator $\eth_A$ has the unique continuation property. Then if ψ is a non-zero C^1-section of the bundle with $\eth_A\psi = 0$, the complement of the zero set of ψ, $Y - \psi^{-1}(0)$, is connected.*

Proof: Let U be a component of $Y - \psi^{-1}(0)$. U is open in Y and $\psi_{|\partial U} = 0$, $\psi(y) \neq 0$ for $y \in U$. Assume now that $U \neq Y - \psi^{-1}(0)$. Then define ϕ by:

$$\phi(y) = \begin{cases} \psi(y) & \text{if } y \in U \\ 0 & \text{if } y \in U^c. \end{cases}$$

We want to prove that $\eth_A\phi = 0$. Then the unique continuation property will give that $\phi = 0$, contradicting that $\psi \neq 0$.

ϕ is continous and thus L^2 on Y. It will be enough to show that:

$$< \phi, \eth_A\eta >= 0,$$

for all smooth sections η. Now

$$< \phi, \eth_A\eta >= \int_U < \phi, \eth_A\eta >= \int_U div(V),$$

where V is a section of $TY \otimes \mathbb{C}$ given by:

$$< V, W > = < \phi, W \cdot \eta >$$

pointwise on Y [23, p.48]. V is C^1 on U and continous on Y.

Choose a smooth map $\rho : \mathbb{R} \to [0;1]$ with $\rho(t) = 1, t \geq 1$ and $\rho(t) = 0$, $t \leq \frac{1}{2}$ and define $\rho_n : Y \to \mathbb{R}$ by $\rho_n(y) = \rho(n|\phi(y)|^2)$. ρ_n is C^1 on U and continous on Y. Also $\lim_{n\to\infty} \rho_n(t) = 1, \forall y \in U$ and $supp_Y \rho_n = supp_U \rho_n$ is compact. ρ_n is used to split $div(V)$:

$$\int_U div(V) = \int_U div(\rho_n V) + \int_U div((1-\rho_n)V).$$

The first term is zero: Approximate $|\phi|^2$ by a smooth map $f : Y \to \mathbb{R}$ so that $\||\phi|^2 - f\|_\infty < \frac{1}{2\sqrt{2n}}$ and f is zero outside an open neighbourhood of $supp_Y \rho_n$ in U. Pick a regular value δ of f, $0 < \delta < \frac{1}{2\sqrt{2n}}$. Then $f^{-1}([\delta;\infty))$ is a smooth submanifold of Y with $supp_Y \rho_n V \subseteq f^{-1}[\delta;\infty)$ and $\rho_n V = 0$ on the boundary $f^{-1}(\delta)$. This gives:

$$\int_U div(\rho_n V) = \int_{f^{-1}[\delta;\infty)} div(\rho_n V) = \int_{f^{-1}(\delta)} \tilde{\rho_n V} = 0.$$

As for the second term:

$$\begin{aligned} div((1-\rho_n)V) &= *d*((1-\rho_n)V)^\sim = *(d\rho_n \wedge *\tilde{V}) + (1-\rho_n)div(V) \\ &= < d\rho_n, \tilde{V} > + (1-\rho_n)div(V) = d\rho_n(V) + (1-\rho_n)div(V) \\ &= < \nabla\rho_n, V > + (1-\rho_n)div(V). \end{aligned}$$

By Lebesgue's theorem on dominated convergence:

$$\int_U (1-\rho_n)div(V) \to 0, n \to \infty.$$

As $\nabla\rho_n$ is zero outside of the set $(|\phi|^2)^{-1}(\frac{1}{\sqrt{2n}}; \frac{1}{\sqrt{n}})$, it is enough to consider the integrand of $\int_U < \nabla\rho_n, V >$ on this set. We have:

$$| < \nabla\rho_n, V > | \leq |\nabla\rho_n||V| \leq C|\nabla\rho_n||\phi||\eta|.$$

As $\nabla\rho_n = n\rho'(n|\phi|^2)\nabla|\phi|^2$ and $d|\phi|^2 = 2 < \nabla\phi, \phi >$ for a metric connection ∇ on the Clifford bundle, we get that $| < \nabla\rho_n, V > |$ is integrable over U and using Lebesgue's theorem on dominated convergence again gives

$$\int_U < \nabla\rho_n, V > \to 0, n \to \infty.$$

$\square$

16 Appendix on temporal gauge.

In this appendix we will prove the existence of temporal gauge, that is: For any reasonable given connection A on $pr^*\mathcal{L}$ over $Y\times\mathbb{R}$ there exists a gauge transformation σ such that the gauge transformed connection $A\cdot\sigma$ is trivial in the $\mathbb{R}$-direction. We will examine two cases:
1) $A \in \mathcal{A}(Y\times I)_{L^2_k}$, $k \geq 2$, I an interval of finite length.
Use $A_0' = pr^* A_0$ as base connection for some smooth connection A_0 on $\mathcal{L}$ and write $a = A - A_0'$. Then

$$A\cdot\sigma - A_0' = a + 2\sigma^{-1}d\sigma.$$

Thus setting $a_t := a(\frac{d}{dt})$, $A\cdot\sigma$ is in temporal gauge iff

$$a_t + 2\sigma^{-1}\frac{d}{dt}\sigma = 0.$$

This differential equation for σ is solved by the expression

$$\sigma(y,t) = \sigma_0(y)\exp(-\frac{1}{2}\int_0^t a_t(y,s)ds),$$

where $\sigma_0 : Y \to S^1$ is arbitrary of Sobolev class L^2_{k+1}. As the interval I was assumed finite, σ is of Sobolev class L^2_k on $Y\times I$.

2) $A \in \mathcal{A}(Y\times\mathbb{R})_{L^2_{k,\delta}}$, $k \geq 2$.
Define σ as above. Let $0 < \delta' < \delta$. We will prove that σ is in $\mathcal{G}_{L^2_{k,\delta'}}$. First define

$$\sigma_\infty(y) := \sigma_0(y)\exp(-\frac{1}{2}\int_0^\infty a_t(y,s)ds).$$

This is well defined and an element of $\mathcal{G}_{L^2_k}$. We now estimate:

$$\begin{aligned}|\frac{\sigma}{\sigma_\infty}(y,t) - 1| &= |\exp(\frac{1}{2}\int_t^\infty a_t(y,s)ds)| \\ &\lesssim |\frac{1}{2}\int_t^\infty a_t(y,s)ds| \\ &\leq \frac{1}{2}\int_t^\infty |a_t(y,s)|ds.\end{aligned}$$

Thus using Hölder's inequality:

$$\begin{aligned}|\frac{\sigma}{\sigma_\infty}(y,t) - 1|^2 &\leq \frac{1}{2}\int_t^\infty e^{-2(\delta-\delta')s}ds\int_t^\infty e^{2(\delta-\delta')s}|a_t(y,s)|^2ds \\ &= K_{\delta'}\int_t^\infty e^{2(\delta-\delta')s}|a_t(y,s)|^2ds.\end{aligned}$$

This implies that:

$$
\begin{aligned}
\int_0^\infty e^{2\delta' t}\left|\frac{\sigma}{\sigma_\infty}(y,t)-1\right|^2 dt &\leq \int_0^\infty e^{2\delta' t} K_{\delta'} \int_t^\infty e^{2(\delta-\delta')s}|a_t(y,s)|^2 ds \\
&= K_{\delta'} \int_0^\infty e^{2(\delta-\delta')s}|a_t(y,s)|^2 \int_0^s e^{2\delta' t} dt\, ds \\
&= K_{\delta'} \int_0^\infty e^{2(\delta-\delta')s}|a_t(y,s)|^2 \frac{1}{2\delta'}(e^{2\delta' s}-1) ds \\
&\leq K'_{\delta'} \|a_t\|_{L^2_\delta}.
\end{aligned}
$$

The derivatives with respect to t and the variables of Y are treated in a similar manner. Thus working in temporal gauge costs a little bit in the exponential decay. Also notice that the above gauge transformations are not in general of Sobolev class one higher than the connections given. Thus they really don't fit into the scheme otherwise used in this thesis. This could be delt with by shifting all lower Sobolev indices by one ...

References

[1] Ralph Abraham, Joel Robbin. *Transversal Mappings and Flows.* W.A. Benjamin, Inc., 1967.

[2] Robert A. Adams. *Sobolev Spaces.* Academic Press, 1975.

[3] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry. I. *Mathematical Proceedings of the Cambridge Philosofical Society*, 77:43–69, 1975.

[4] M.F. Atiyah, V.K. Patodi, I.M. Singer. Spectral asymmetry and Riemannian geometry. III. *Mathematical Proceedings of the Cambridge Philosofical Society*, 79:71–99, 1976.

[5] Bernhelm Booss-Bavnbek. Boundary Reduction of Spectral Invariants and Unique Continuation Property. Tekst nr. 367, IMFUFA, Roskilde Universtity.

[6] Glen E. Bredon. *Topology and Geometry*, volume 139 of *GTM*. Springer-Verlag, 1993.

[7] Alan L. Carey, Bai-Ling Wang. Seiberg-Witten-Floer Homology and Gluing Formula. Preprint available at http:www.maths.adelaide.edu.au/pure/bwang/preprints.html, 1999.

[8] C. Chevalley. *The construction and study of certain important algebras.* Publications of the Mathematical Society of Japan, 1955.

[9] R. Engelking. *General Topology*, volume 44. Polish Scientific Publishers, 1977.

[10] A. Floer, H. Hofer. Coherent orientation for periodic orbit problems in symplectic geometry. *Mathematische Zeitschrift*, 212:13–38, 1993.

[11] Andreas Floer. An Instanton-Invariant for 3-Manifolds. *Communications in Mathematical Physics*, 118:215–240, 1988.

[12] Andreas Floer. The Unregularized Gradient Flow of the Symplectic Action. *Communications on Pure and Applied Mathematics*, 41:775–813, 1988.

[13] Kim A. Frøyshov. The Seiberg-Witten equations and four-manifolds with boundary. *Mathematical Research Letters*, 3:373–390, 1996.

[14] D. Gilbarg, N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order, 2.edition.* Number 224 in Grundlehren der Mathematischen Wissenschaften. Springer Verlag, 1983.

[15] F. Hirzebruch, H. Hopf. Felder von flachenelementen. *Mathematische Annalen*, 136:156–172, 1958.

[16] Helmut Hofer, Eduard Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser Advanced Texts. Birkhäuser Verlag, 1994.

[17] J. Kelley. *General Topology*. D. van Nostrand Co. Inc., 1955.

[18] Peter Kronheimer, Tom Mrowka. The genus of embedded surfaces in the projective plane. *Mathematical Research Letters*, 1:797–808, 1994.

[19] H.B. Lawson, M.-L. Michelsohn. *Spin Geometry*. Princeton University Press, 1989.

[20] Robert B. Lockhardt, Robert C. McOwen. Elliptic Differential Operators on Noncompact Manifolds. *Annali della Scuola Normale Superiore di Pisa*, serie. IV,XII(3):409–447, 1985.

[21] Matilde Marcolli. Seiberg-Witten-Floer Homology and Heegaard Splittings. *International Journal of Mathematics*, 7(5):671–696, 1996.

[22] C.A. McGibbon, J.M. Møller. On spaces with the same n-type for all n. *Topology*, 31:177–201, 1992.

[23] John W. Morgan. *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*. Princeton University Press, 1996.

[24] John W. Morgan, Soltan Szabo, Clifford Henry Taubes. A Product Formula for the Seiberg-Witten Invariants and the Generalized Thom Conjecture. *Journal of Differential Geometry*, 44:706–788, 1996.

[25] Richard S. Palais. Morse Theory on Hilbert Manifolds. *Topology*, 2:299–340, 1963.

[26] Richard S. Palais, editor. *Seminar on the Atiyah-Singer index theorem*, number 57 in Annals of Mathematical Study. Princeton University Press, 1965.

[27] A. V. Pazhitnov. On the Novikov complex for rational Morse forms. *Ann. Fac. Sci. Toulouse Math. (6)*, 4(2):297–338, 1995.

[28] Gert K. Pedersen. *Analysis Now*. Number 118 in GTM. Springer Verlag, 1989.

[29] Matthias Schwarz. *Morse Homology*, volume 111 of *Progress in Mathematics*. Birkhauser Verlag, 1993.

[30] Leon Simon. Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. *Annals of Mathematics*, 118:525–571, 1983.

[31] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, 1951.

[32] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in advanced mathematics*. Cambridge University Press, 1994.

[33] E. Witten. Monopoles and Four-Manifolds. *Mathematical Researh Letters*, 1:764–796, 1994.