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SECOND CELL TILTING MODULES

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Contents

Preface

This thesis presents the results of my Ph.D. project at the Department of Mathematical Sciences, University of Aarhus. The results presented in Chapters 6 and 8 appear also in a paper accepted for publication in the Journal of Algebra (Rasmussen n.d.).

I express my gratitude to my adviser Henning Haahr Andersen. He has been a source of new ideas and inspiration. I have been privileged to have his direction and guidance.

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CHAPTER 1

Introduction

This thesis is concerned with the representation theory of an almost simple group over an algebraically closed field k of prime characteristic p. The structure of the tilting modules poses a highly interesting unsolved problem. The notion of a tilting module was originally introduced by Ringel (1991) in the setting of quasi hereditary algebras; but modules with the properties of tilting modules had been studied before this: see Collingwood and Irving (1989). Later Donkin (1993) adapted the machinery of tilting modules to reductive algebraic groups. In this setting, a tilting module is a module with a filtration of Weyl modules and a filtration of dual Weyl modules. The tilting modules form a family of modules with very interesting properties: It is closed under tensor products, and any summand of a tilting module is tilting. For each dominant weight λ there is an indecomposable tilting module $T(\lambda)$ with highest weight λ; this accounts for all indecomposable tilting modules. A tilting module is uniquely determined by its character, but the characters of the indecomposable tilting modules are in general unknown.

Knowledge of the characters of all indecomposable tilting modules would in fact allow us to deduce the characters of the simple modules, see (Donkin 1993) and (Andersen 1998). The characters of the simple modules is the basic goal within the representation theory of our group; though progress have been made in later years, and though much is known in special cases, the characters of the simple modules still present an open problem. This stresses the importance as well as the difficulty of identifying the indecomposable tilting modules.

The indecomposable tilting modules may be determined by an account of the Weyl factors in $T(\lambda)$. We write $[T(\lambda) : V(\mu)]$ for the number of times the Weyl module $V(\mu)$ appears in a filtration of $T(\lambda)$. The decomposition numbers $[T(\lambda) : V(\mu)]$ for all dominant μ is a convenient way to express the characters of the indecomposable tilting modules, since the characters of the Weyl modules are known. However, apart from the information obtained through the construction of tilting modules (see Chapter 2), the decomposition numbers $[T(\lambda) : V(\mu)]$ are effectively unknown.

A second way to reveal the structure of the tilting modules is to obtain the multiplicities of $T(\lambda)$ in any tilting module with known character. The multiplicity of $T(\lambda)$ in a tilting module M is the number of times $T(\lambda)$ appears as a summand of M, and we denote this number by $[M : T(\lambda)]$. Formulae for these multiplicities $[M : T(\lambda)]$ is enough indeed to determine the characters of the indecomposable tilting modules. Some progress have been made along this line. If λ belongs to the first alcove (see Chapter 2 for a more detailed account of the notation), the answer is well known, due to Georgiev and Mathieu (1994) and Andersen and Paradowski (1995).

$$[Q : T(\lambda)] = \sum_{x\in W,\ x.\lambda\in X^+} (-1)^{l(x)}[Q : V(x.\lambda)] \tag{1.1}$$

Little seems to be known if λ does not belong to the first alcove.

A third way to examine the structure of tilting modules is via quantizations: For each modular tilting module M there is a quantum tilting module (meaning a tilting module of the corresponding quantum group at a p'th root of unity) M_q with the same character. As the characters of the indecomposable quantum tilting modules are known, we may compute the multiplicities $[M_q : T_q(\lambda)]$ (see Chapter 3 for the notation) if we know the character of M. Also, knowledge of the quantum multiplicities $[T(\lambda)_q : T_q(\mu)]$ for all μ will determine the characters of the indecomposable tilting modules. The characters of quantum tilting modules are expressed in terms of Hecke algebra combinatorics, and related concepts such as right cells and weight cells turn out to play a central role in the representation theory of quantum groups at a p'th root of unity.

The results in this thesis are brought about by considering the quantizations $T(\lambda)_q$ of modular indecomposable tilting modules. Even though the character of $T(\lambda)$ is unknown we are able to deduce the following theorem, where h as usually denotes the Coxeter number.

THEOREM 1.1. *Assume that the root system of our group is of type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8 or G_2, and let $p \geq h$.*

For dominant weights λ, μ, with μ in the first or second weight cell we have

$$[T(\lambda)_q : T_q(\mu)] \;=\; \delta_{\lambda\mu}.$$

As a modular tilting module is a direct sum of indecomposable tilting modules, we immediately generalize this to

THEOREM 1.2. *Assume that the root system of our group is of type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8 or G_2, and let $p \geq h$. For a modular tilting module M and a weight λ in the first or second weight cell we have*

$$[M : T(\lambda)] \;=\; [M_q : T_q(\lambda)] \tag{1.2}$$

Note that the right hand side of (1.2) is the multiplicity of a quantum tilting module; so the right hand side is computable. Thus the theorem provides a closed formula for the multiplicities of indecomposable tilting modules with highest weight in the first or second weight cell. We regard Theorem 1.2 as the main result of our thesis.

Note that Theorem 1.2 covers the situation considered in equation (1.1), and may thus be seen as a generalization of this equation.

From the construction of tilting modules in Chapter 2 we find that the characters of the indecomposable tilting modules is a basis of the ring of characters. Let $[M : T(\lambda)]$ denote the coefficient of $\operatorname{ch} T(\lambda)$ so that

$$\operatorname{ch} M = \sum_{\lambda \in X^+} [M : T(\lambda)] \operatorname{ch} T(\lambda)$$

for all modules M. This extend our usage of $[M : T(\lambda)]$ so far. Considered as a character formula, equation (1.2) therefore holds for all modules M. In particular, since the modular Weyl module and the quantum Weyl module have the same character, we find that

THEOREM 1.3. *Assume that the root system of our group is of type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8, or G_2, and let $p \geq h$.*

For dominant weights λ, μ, with μ in the first or second weight cell we have

$$[V(\lambda) : T(\mu)] \;=\; [V_q(\lambda) : T_q(\mu)] \tag{1.3}$$

This provide us with the "inverse" decomposition numbers for all $T(\mu)$ with μ in the first or second weight cell.

Recent years have seen many and diverse applications of tilting modules. Here we will mention two. Let $\mathbf{N}$ denote a vector space of dimension n over k. From the commuting actions on $\mathbf{N}^{\otimes r}$ of the symmetric group and the group of linear automorphisms of $\mathbf{N}$ we obtain a surjective ring homomorphism

$$k[\Sigma_r] \longrightarrow \operatorname{End}_{\mathrm{GL}(\mathbf{N})}(\mathbf{N}^{\otimes r}) \tag{1.4}$$

The indecomposable summands of $\mathbf{N}^{\otimes r}$ index the simple modules of $\operatorname{End}_{\mathrm{GL}(\mathbf{N})}(\mathbf{N}^{\otimes r})$, by general ring theory. Further, the dimension of a simple $\operatorname{End}_{\mathrm{GL}(\mathbf{N})}(\mathbf{N}^{\otimes r})$-module is given by the multiplicity in $\mathbf{N}^{\otimes r}$ of the corresponding indecomposable. As $\mathbf{N}^{\otimes r}$ is tilting we may apply Theorem 1.2 to count the multiplicities of those indecomposable tilting modules, that have highest weights in the first or second weight cell. Through the surjection (1.4) above we obtain a dimension formula for a set of simple representations of the symmetric group, as stated in

THEOREM 1.4. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ denote a partition with at least three parts. When $p \geq n$ we may compute the dimension of the simple $k[\Sigma_r]$-module parametrized by λ, provided that $\lambda_1 - \lambda_{n-1} < p - n + 2$ or $\lambda_2 - \lambda_n < p - n + 2$.*

This Theorem is a generalization of a result by Mathieu (1996), determining the dimension of the simple modules parametrized by Young diagrams with $n_1 - n_n < p - n + 1$. Further, our result proves a special case of conjecture 15.4 in (Mathieu 2000).

As a second application, we consider the surjective ring homomorphism

$$k\,\mathrm{GL}(\mathbf{M}) \longrightarrow \mathrm{End}_{\mathrm{GL}(\mathbf{N})}\left(\wedge(\mathbf{N} \otimes \mathbf{M})\right)$$

As $\wedge\mathbf{N} \otimes \mathbf{M}$ is a tilting module we may apply Theorem 1.2 to count multiplicities of second cell tilting modules. The corresponding dimension formula may in fact be refined to a character formula; however the precise statement requires some further notation. Therefore we give only an example. We denote the i'th fundamental weight of $\mathrm{GL}(\mathbf{M})$ by ω_i.

EXAMPLE 1.5. *Consider the dominant weight $a\omega_i + \omega_j$, with $i \geq j$, $a \geq 0$. Theorem 1.2 allow us to calculate the character of the simple $\mathrm{GL}(\mathbf{M})$-module $L(a\omega_i + \omega_j)$ for $p \geq 3$.*

The character formulae obtained here generalizes work of Mathieu and Papadopoulo (1999).

Review of the thesis

In Chapter 2 we construct indecomposable modular tilting modules. We shall follow the approach of Ringel (1991) and Donkin (1993). We will refer to this construction in Chapter 3, where we introduce $\mathcal{U}_q$, the corresponding quantum group at a p'th root of unity, and quantum tilting modules. Also in Chapter 3 we consider the key concept – in this thesis – of quantizations of modular tilting modules; that is, we find for each modular tilting module a quantum tilting module with the same character.

Chapter 4 is devoted to the Hecke algebra. We show how right cells arise naturally via bases of "nice" ideals of the Hecke algebra. We treat in depth one right cell, which we call the second cell. The second cell is at the heart of this thesis. Chapter 5 contains the Hecke module and Soergels Theorem, expressing the characters of quantum tilting modules in terms of Hecke algebra combinatorics. This is applied: We classify all tensor ideals of quantum tilting modules following Ostrik (1997), and we determine the weight cells. Both applications relies on the right cells of Chapter 4.

With Chapter 6 this thesis begins in honest. Based on quantizations of modular tilting modules and Hecke algebra calculations we examine the structure of modular tilting modules. The outcome is the multiplicity formula of Theorem 1.2. We prove the formula for type $A_{n\geq 2}$, D_n, E_6, E_7, E_8 or G_2 in Chapter 6, and we see that the formula does *not* hold in type A_1. Chapter 7 then consider the formula for type B_2 – using techniques quite different from those of Chapter 6 we prove that the multiplicity formula does indeed hold in type B_2.

The last chapters of the thesis present applications of the main result. Via Schur-Weyl duality (of which we give a self contained account) this leads us in Chapter 8 to a dimension formula for simple representations of the symmetric group corresponding to partitions, which satisfy a simple condition. Chapter 9 considers Howe duality. Here the multiplicity formula provide us with character formulae for simple modules of the general linear group, parametrized by the dominant weights of a given set. Finally in the short Chapter 10 we take up modular weight cells and show how the multiplicity formula allow us to determine the second largest modular weight cell.

CHAPTER 2

Tilting modules

Let k denote an algebraically closed field of prime characteristic p. Let G be an almost simple algebraic group over k.

- Let T denote a maximal torus, and let $X = X(T)$ denote the character group of T.
- Let $R \subset X$ denote the set of roots of G. The root system R is irreducible because G is almost simple. Choose a set of simple roots $\{\alpha_1, \dots, \alpha_n\}$ and let R^+ denote the positive roots. For each root α let $\alpha^\vee$ denote the coroot corresponding to α.
- Let E denote the real vector space spanned by all $\alpha \in R$. There is a bilinear form, $\langle -,- \rangle$, on E, so that the numbers $\langle \alpha, \beta^\vee \rangle$ (for simple α, β) are the entries of the Cartan matrix of R.
- Let $\omega_1, \dots \omega_n$ denote the basis dual to $\alpha_1^\vee, \dots, \alpha_n^\vee$. Then ω_i is called the i'th fundamental weight. Let ρ denote the sum of all fundamental weights, and let $\mathrm{St} = (p-1)\rho$.
- For each root α define a reflection on E by
$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$
A reflection corresponding to a simple root α_i is called a simple reflection and is denoted by s_i. The set of simple reflections is denoted by S_0. The simple reflections generate the (finite) Weyl group W_0. Let w_0 denote the longest element in the Weyl group.
- Let α_0 denote the highest *short* root of R, and define an affine reflection s_0 by
$$s_0(\lambda) = \lambda - \langle \lambda, \alpha_0^\vee \rangle \alpha_0 + p\alpha_0.$$
The affine Weyl group, W, is the group generated by $S = \{s_0, s_1, \dots, s_n\}$.
- The Weyl group and affine Weyl group act on E through the dot-action
$$w.\lambda = w(\lambda + \rho) - \rho \qquad\qquad w \in W,\ \lambda \in E.$$
- The action of the affine Weyl group divides E into alcoves, on which it acts simply transitive. Let
$$C \quad = \quad \{\lambda \in E \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all positive roots } \alpha\}$$
denote the first (or standard) alcove. The first alcove contains a weight when $p \geq h$, h denoting the Coxeter number of the root system of G.
- Let U denote the subgroup of G generated by all root subgroups corresponding to negative roots. Let U^+ denote the group generated by root subgroups corresponding to all positive roots. And let B denote the Borel subgroup generated by U and T.

Modules

By a G-module we mean a rational finite dimensional representation of the algebraic group G. Any G-module is also a T-module. A T-module splits in a direct sum of one-dimensional T-modules, and T's action on a one-dimensional module is given by a character. For a G-module M and a character $\lambda \in X$, we define the λ-weight space by

$M_\lambda = \{w \in M \mid tw = \lambda(t)w \text{ for all } t \in T\}$. If $M_\lambda \neq 0$ we say that λ is a weight of M. The sum of weight spaces is direct and we therefore have a decomposition of any G-module M:

$$M = \oplus_{\lambda \in X} M_\lambda.$$

We shall sometimes refer to the elements of X as the weights of G.

For each dominant weight λ we have the Weyl module $V(\lambda)$ with highest weight λ. In characteristic p this module need not be simple, as it is in characteristic zero. But the head of $V(\lambda)$, which we denote by $L(\lambda)$, is simple and of highest weight λ. In fact $\{L(\lambda) \mid \lambda \in X^+\}$ is a full set of non-isomorphic simple modules. The Weyl module has an important universal property. A U^+-invariant line km of weight λ in a G-module M generates a quotient of the Weyl module $V(\lambda)$.

Next, let us consider the induced modules. We shall define them as duals of Weyl modules, that is, set $H^0(\lambda) = V(-w_0\lambda)^*$. This definition is adequate for our purpose. However, as the name suggests, the induced modules arise naturally by induction. Let k_λ denote the one dimensional B-module with trivial U-action and T-action through λ. Then $H^0(\lambda) \simeq \operatorname{Ind}_B^G k_\lambda$. We will let $\chi(\lambda)$ denote the character of the Weyl module and the induced module with highest weight λ.

We say that a module M has a Weyl filtration, if there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,$$

so that each quotient M_i/M_{i-1} is a Weyl module. If M allows a filtration where each subquotient is a dual Weyl module, we say that M has a good filtration. If M has a Weyl filtration we let $[M : V(\lambda)]$ denote the number of times $V(\lambda)$ appears as a subquotient. And if M has a good filtration we let $[M : H^0(\lambda)]$ denote the number of times $H^0(\lambda)$ appears as a subquotient.

A tilting module is a module with a Weyl filtration *and* a good filtration. Equivalently, a module M is tilting if M and the dual of M allow a good filtration, or M is tilting if M and its dual have a Weyl filtration. In this first chapter we show that there is a unique indecomposable tilting module with highest weight λ for each dominant weight λ. We will then denote this indecomposable tilting module by $T(\lambda)$.

The translation functors and the wallcrossing functors are used extensively in Chapter 6. Let us review their definition. We define $\mathrm{pr}_\lambda M$ as the largest submodule of M where all composition factors have highest weight in $W.\lambda$. By the linkage principle, $\mathrm{pr}_\lambda M$ is a direct summand of M. Now let $\lambda, \mu \in \overline{C} \cap X$. There is a unique ν, so that $\{\nu\} = W_0(\mu - \lambda) \cap X^+$. The translation functor T_λ^μ is then defined by

$$T_\lambda^\mu M = \mathrm{pr}_\mu(L(\nu) \otimes \mathrm{pr}_\lambda M).$$

As truncation to a summand is exact and as $L(\nu) \otimes -$ is exact, we find that the translation functor is an exact functor. The wallcrossing functors are defined as a composition of translation functors. Choose $\mu \in \overline{C} \cap X$ so that $W_\mu = \{1, s\}$, where W_μ denotes the stabilizer of μ with respect to the dot action. Let $\lambda \in C \cap X$ denote a regular weight, i.e. a weight with trivial stabilizer. Then $\Theta_s = T_\mu^\lambda \circ T_\lambda^\mu$ is a wallcrossing functor.

Weyl modules

We prepare the construction of tilting modules in the next section by recalling results about Weyl modules.

By weight considerations we find that $\operatorname{ch} V(\lambda) = \operatorname{ch} L(\lambda) + \sum_{\mu < \lambda} a_\mu \operatorname{ch} L(\mu)$ for some non-negative integers a_μ. But more is known. Recall the definition of the linkage relation $\uparrow$ on X from (Andersen 1980b), to which we also refer to for the following theorem.

THEOREM 2.1. *If $L(\mu)$ is a composition factor of $V(\lambda)$ then $\mu \uparrow \lambda$.*
If $L(\mu)$ is a composition factor of $H^0(\lambda)$ then $\mu \uparrow \lambda$

The strong linkage principle above is usually stated for induced modules; but the equality $\operatorname{ch} V(\lambda) = \operatorname{ch} H^0(\lambda)$ shows that the Weyl module and the induced module have the same composition factors.

THEOREM 2.2. *(Cline, Parshall, Scott and van der Kallen 1977)*
Let μ and λ be dominant weights. Then

$$\operatorname{Ext}^i(V(\lambda), H^0(\mu)) = \begin{cases} k & i = 0 \text{ and } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

The full strength of Theorem 2.2 is not needed to construct tilting modules; for this purpose we need only the special case $i = 0, 1$ (which may be established quite easily independently) and $i = 2$ in the proof of Theorem 2.8.

COROLLARY 2.3. *Let W be a module with a Weyl filtration and Q a module with a good filtration. Then*

(i) $\dim \operatorname{Hom}(V(\lambda), Q) = [Q : H^0(\lambda)]$,
(ii) $\dim \operatorname{Hom}(W, H^0(\lambda)) = [W : V(\lambda)]$,
(iii) $\operatorname{Ext}^i(W, Q) = 0$ *for all* $i \geq 1$.

The Lemma below states a necessary condition for the extension of a Weyl module with a simple module. A convenient reference is (Jantzen 1987, II.6.20) which also describes how far apart it is possible for λ and μ to be.

LEMMA 2.4. *Let μ and λ be dominant weights.*

(i) *If, for some $i \geq 0$,* $\operatorname{Ext}^i(V(\lambda), L(\mu)) \neq 0$ *then $\lambda \uparrow \mu$.*
(ii) $\dim \operatorname{Ext}^i(V(\lambda), L(\mu))$ *is finite for all i.*

PROOF. The proof goes by induction in i. If $\operatorname{Ext}^0(V(\lambda), L(\mu)) \neq 0$ then $\lambda = \mu$ as $V(\lambda)$ has simple head equal to $L(\lambda)$. Now suppose that $\operatorname{Ext}^i(V(\lambda), L(\mu)) \neq 0$ for a pair of dominant weights λ, μ and that $i \geq 1$. Consider the exact sequence

$$0 \quad \to \quad L(\mu) \quad \to \quad H^0(\mu) \quad \to \quad H^0(\mu)/L(\mu) \quad \to \quad 0.$$

Applying $\operatorname{Hom}(V(\lambda), -)$ and recalling Theorem 2.2 we find an isomorphism

$$\operatorname{Ext}^{i-1}(V(\lambda), H^0(\mu)/L(\mu)) \simeq \operatorname{Ext}^i(V(\lambda), L(\mu))$$

This implies $\operatorname{Ext}^{i-1}(V(\lambda), L(\mu_1)) \neq 0$ for some composition factor $L(\mu_1)$ of $H^0(\mu)$; hence $\mu_1 \uparrow \mu$. Repeating the argument we find a sequence of linked dominant weights $\mu_i \uparrow \ldots \uparrow \mu_1 \uparrow \mu$ so that $\operatorname{Ext}^{i-i}(V(\lambda), L(\mu_i)) \neq 0$. We conclude that $\mu_i = \lambda$.

The second claim is obvious if $i = 0$. For $i > 0$ it follows by induction in μ. If μ is minimal then $L(\mu) = H^0(\mu)$ and conclusion by Theorem 2.2. For non-minimal μ we have $\operatorname{Ext}^{i-1}(V(\lambda), H^0(\mu)/L(\mu)) \simeq \operatorname{Ext}^i(V(\lambda), L(\mu))$ by the first part of the proof. By induction $\dim \operatorname{Ext}^{i-1}(V(\lambda), L(\mu')) < \infty$ for each factor $L(\mu')$ in $H^0(\mu)/L(\mu)$, and the result follows. □

REMARK 2.5. *Note that Lemma 2.4(ii) shows that for any module M and any dominant weight λ we have* $\dim \operatorname{Ext}^i(V(\lambda), M) < \infty$ *for all $i \geq 0$.*

LEMMA 2.6. *Let λ be a dominant weight. We have*

$$\operatorname{Ext}^i(V(\lambda), L(\lambda)) = \begin{cases} k & i = 0 \\ 0 & i \geq 1. \end{cases}$$

PROOF. Consider the following short exact sequence

$$0 \quad \to \quad L(\lambda) \quad \to \quad H^0(\lambda) \quad \to \quad H^0(\lambda)/L(\lambda) \quad \to \quad 0.$$

For all composition factors $L(\mu)$ in $H^0(\lambda)/L(\lambda) = 0$ we have $\mu \uparrow \lambda$ and $\mu \neq \lambda$; hence (by Lemma 2.4) $\mathrm{Ext}^i(V(\lambda), L(\mu)) = 0$ for all $i \geq 0$. This implies $\mathrm{Ext}^i(V(\lambda), H^0(\lambda)/L(\lambda)) = 0$ for all $i \geq 0$. Thus

$$\mathrm{Ext}^i(V(\lambda), L(\lambda)) \simeq \mathrm{Ext}^i(V(\lambda), H^0(\lambda)).$$

□

LEMMA 2.7. *Let λ be a dominant weight. We have*

$$\mathrm{Ext}^i(V(\lambda), V(\lambda)) = \begin{cases} k & i = 0 \\ 0 & i \geq 1. \end{cases}$$

PROOF. In the following we let $V(\lambda)^1$ denote the kernel of the natural projection $V(\lambda) \to L(\lambda)$. This is reasonable since $V(\lambda)^1$ agrees with the first submodule of $V(\lambda)$ in Jantzens filtration.

$$0 \quad \to \quad V(\lambda)^1 \quad \to \quad V(\lambda) \quad \to \quad L(\lambda) \quad \to \quad 0.$$

For all composition factors $L(\mu)$ in $V(\lambda)^1$ we have $\mu \uparrow \lambda$ and $\mu \neq \lambda$; hence (by Lemma 2.4) $\mathrm{Ext}^i(V(\lambda), L(\mu)) = 0$ for all $i \geq 0$. This immediately implies $\mathrm{Ext}^i(V(\lambda), V(\lambda)^1) = 0$ for all $i \geq 0$. Thus

$$\mathrm{Ext}^i(V(\lambda), V(\lambda)) \simeq \mathrm{Ext}^i(V(\lambda), L(\lambda)).$$

□

THEOREM 2.8. *(Donkin 1981) Suppose that* $\mathrm{Ext}^1(V(\mu), M) = 0$ *for all dominant μ. Then M allows a good filtration.*

PROOF. Choose a minimal λ so that $L(\lambda)$ is a composition factor in the socle of M. We will show that $H^0(\lambda)$ is a submodule in M and that $\mathrm{Ext}^1(V(\mu), M/H^0(\lambda)) = 0$ for all dominant μ. Recursively this gives us a sequence of surjections $M \to M_1 \to \cdots \to M_r = 0$, where each kernel is an induced module. This sequence shows that M allows a good filtration.

From the short exact sequence $0 \to L(\lambda) \xrightarrow{i} H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0$ we obtain a long exact sequence with the terms

$$\ldots \quad \to \quad \mathrm{Hom}(H^0(\lambda), M) \quad \to \quad \mathrm{Hom}(L(\lambda), M) \quad \to \quad \mathrm{Ext}^1(H^0(\lambda)/L(\lambda), M) \quad \to \quad \ldots$$

Assume for a moment that the last term is zero; then there is an $f \in \mathrm{Hom}(H^0(\lambda), M)$ so that $f \circ i$ includes $L(\lambda)$ in M. The kernel of f is either zero or contains $L(\lambda)$ (which is the socle of $H^0(\lambda)$); therefore the kernel must be trivial, and we get an inclusion of $H^0(\lambda)$ in M.

So we must show that $\mathrm{Ext}^1(H^0(\lambda)/L(\lambda), M) = 0$. Let $L(\nu)$ denote a composition factor of $H^0(\lambda)/L(\lambda)$, and consider the sequence $0 \to V(\nu)^1 \to V(\nu) \to L(\nu) \to 0$. Using $\mathrm{Hom}(-, M)$ we get an exact sequence including the terms

$$\ldots \quad \to \quad \mathrm{Hom}(V(\nu)^1, M) \quad \to \quad \mathrm{Ext}^1(L(\nu), M) \quad \to \quad \mathrm{Ext}^1(V(\nu), M) \quad \to \quad \ldots$$

Now the last term is zero by assumption. Further, there are no maps from $V(\nu)^1$ to M: The composition factors of $V(\nu)^1$ are $L(\nu')$ with ν' strictly smaller than $\nu \uparrow \lambda$ and λ was chosen minimal among the highest weights of the composition factors of the socle of M. So we see that $\mathrm{Ext}^1(L(\nu), M) = 0$ for each factor $L(\nu)$ of $H^0(\lambda)/L(\lambda)$. We conclude that also $\mathrm{Ext}^1(H^0(\lambda)/L(\lambda), M) = 0$, and we have the desired factorization of the inclusion $L(\lambda) \hookrightarrow M$.

Finally $\mathrm{Ext}^1(V(\mu), M/H^0(\lambda)) = 0$ for all dominant μ follows from $\mathrm{Hom}(V(\mu), -)$ applied to the exact sequence $0 \to H^0(\lambda) \to M \to M/H^0(\lambda) \to 0$, as $\mathrm{Ext}^2(V(\mu), H^0(\lambda)) = 0$ by Theorem 2.2. □

When M allows a good filtration, we have $\mathrm{Ext}^1(V(\mu), M) = 0$ for all dominant μ by Corollary 2.3. Together Corollary 2.3 and Theorem 2.8 give the following corollary.

COROLLARY 2.9. *Let $0 \to M \to N \to P \to 0$ be a short exact sequence of G-modules. Then*

(i) *P has a good filtration if N and M have a good filtration.*
(ii) *N has a good filtration if P and M have a good filtration.*
(iii) *A summand in a module with a good filtration has a good filtration.*

Construction of tilting modules.

In this section we outline how to construct an indecomposable tilting module with highest weight $\lambda \in X^+$. The idea is to inductively build the tilting module by extensions, until we get a module that does not extend any Weyl module. This module will then have a good filtration, as ensured by Theorem 2.8.

Fix λ and let $\Pi(\lambda) = \{\mu \in X^+ \mid \mu \uparrow \lambda\}$. Note that $\Pi(\lambda)$ is a finite set; accordingly we order $\Pi(\lambda) = \{\lambda_0, \lambda_1, \dots \lambda_r\}$ so that $\lambda_i \uparrow \lambda_j$ implies that $j \le i$. Note that $\lambda_0 = \lambda$.

Let $E_0 = V(\lambda_0)$. If $\mathrm{Ext}^1(V(\lambda_1), E_0) = 0$ then we set $E_1 = E_0$. If this space is non-zero we extend $V(\lambda_1)$ with E_0: Choose a non-split short exact sequence

$$0 \to E_0 \to E_0^{(1)} \to V(\lambda_1) \to 0. \tag{2.1}$$

Applying $\mathrm{Hom}(V(\lambda_1), -)$ we obtain a long exact sequence, beginning with the six terms

$$\begin{array}{cccccccc} 0 & \to & \mathrm{Hom}(V(\lambda_1), E_0) & \to & \mathrm{Hom}(V(\lambda_1), E_0^{(1)}) & \xrightarrow{\Psi} & \mathrm{Hom}(V(\lambda_1), V(\lambda_1)) & \\ & \to & \mathrm{Ext}^1(V(\lambda_1), E_0) & \to & \mathrm{Ext}^1(V(\lambda_1), E_0^{(1)}) & \to & \mathrm{Ext}^1(V(\lambda_1), V(\lambda_1)) & \to \dots \end{array}$$

Note that (2.1) is non-split if and only if Ψ is the zero map, as $\mathrm{Hom}(V(\lambda_1), V(\lambda_1)) = k\,\mathrm{Id}_{V(\lambda_1)}$. Further, we have a complete description of $\mathrm{Ext}^i(V(\lambda_1), V(\lambda_1))$ from Lemma 2.7. We conclude that

$$0 \to \mathrm{End}(V(\lambda_1)) \to \mathrm{Ext}^1(V(\lambda_1), E_0) \to \mathrm{Ext}^1(V(\lambda_1), E_0^{(1)}) \to 0$$

is exact. In particular, we have $\dim \mathrm{Ext}^1(V(\lambda_1), E_0^{(1)}) = \dim \mathrm{Ext}^1(V(\lambda_1), E_0) - 1$.

Now: If $\mathrm{Ext}^1(V(\lambda_1), E_0^{(1)}) = 0$ then set $E_1 = E_0^{(1)}$. If this space is non-zero choose a non-split extension

$$0 \to E_0^{(1)} \to E_0^{(2)} \to V(\lambda_1) \to 0.$$

Arguing as above we obtain $\dim \mathrm{Ext}^1(V(\lambda_1), E_0^{(2)}) = \dim \mathrm{Ext}^1(V(\lambda_1), E_0^{(1)}) - 1$. We continue in this way until we eventually find an $E_0^{(d_1)}$ with the property that

$$\mathrm{Ext}^1(V(\lambda_1), E_0^{(d_1)}) = 0$$

Then set $E_1 = E_0^{(d_1)}$. Note that

$$d_1 = \dim \mathrm{Ext}^1(V(\lambda_1), E_0),$$

which is finite thanks to Remark 2.5. Further, E_1/E_0 has a Weyl filtration; the quotients are all isomorphic to $V(\lambda_1)$ and there are d_1 of them. Since there are no non-trivial extensions of $V(\lambda_1)$ with itself, we conclude that we have a short exact sequence

$$0 \to E_0 \to E_1 \to V(\lambda_1)^{\oplus d_1} \to 0.$$

Having dealt with λ_1 we simply continue with λ_2. Arguing as above we produce an extension

$$0 \to E_1 \to E_2 \to V(\lambda_2)^{\oplus d_2} \to 0,$$

so that $\mathrm{Ext}^1(V(\lambda_2), E_2) = 0$. We also find that $d_2 = \dim \mathrm{Ext}^1(V(\lambda_2), E_1)$.

We use this procedure for each of the finitely many λ_i in $\Pi(\lambda)$; eventually we end up with a module E_r that fits into the short exact sequence

$$0 \to E_{r-1} \to E_r \to V(\lambda_r)^{\oplus d_r} \to 0,$$

and has the property that $\operatorname{Ext}^1(V(\lambda_r),E_r)=0$ and where $d_r=\dim\operatorname{Ext}^1(V(\lambda_r),E_{r-1})$.

The module E_r is our tilting candidate; but so far we have only explained how to obtain E_r. It still remains to prove that this module has the properties we are looking for.

LEMMA 2.10. *For each dominant weight μ we have*

$$\operatorname{Ext}^1(V(\mu),E_r)=0.$$

Consequently E_r has a good filtration.

PROOF. First of all $\mu\not\uparrow\lambda$ implies $\operatorname{Ext}^1(V(\mu),E_r)=0$, as $\operatorname{Ext}^1(V(\mu),L)=0$ for each composition factor L of E_r follows from Lemma 2.4. Hence we may assume that $\mu=\lambda_i$ for some λ_i in $\Pi(\lambda)$. But then $\operatorname{Ext}^1(V(\mu),E_i)=0$ by the construction of E_i.

Now, for all $j>i$ we have $\mu\not\uparrow\lambda_j$. Thus $\operatorname{Ext}^1(V(\mu),V(\lambda_j))=0$; if non-zero, there must be a composition factor $L(\lambda')$ in $V(\lambda_j)$ so that $\operatorname{Ext}^1(V(\mu),L(\lambda'))$ is nonzero: This forces $\mu\uparrow\lambda'\uparrow\lambda_j$.

Combining $\operatorname{Ext}^1(V(\mu),E_i)=0$ and $\operatorname{Ext}^1(V(\mu),V(\lambda_j))=0$ for all $j>i$ we obtain the result as follows. Use $\operatorname{Hom}(V(\mu),-)$ on the sequence

$$0 \;\to\; E_i \;\to\; E_{i+1} \;\to\; V(\lambda_{i+1})^{\oplus d_{i+1}} \;\to\; 0. \tag{2.2}$$

This shows that $\operatorname{Ext}^1(V(\mu),E_{i+1})=0$. Completely analogous arguments allow us to conclude that also $\operatorname{Ext}^1(V(\mu),E_{i+2})=\cdots=\operatorname{Ext}^1(V(\mu),E_r)=0$. □

LEMMA 2.11. *$V(\lambda_i)$ is not a summand of E_i.*

PROOF. Recall that we have a short exact sequence

$$0 \;\to\; E_{i-1} \xrightarrow{i} E_i \xrightarrow{p} V(\lambda_i)^{\oplus d_i} \;\to\; 0. \tag{2.3}$$

We show that any homomorphism $j\colon V(\lambda_i)\longrightarrow E_i$ factors through i and that any homomorphism $q\colon E_i\longrightarrow V(\lambda_i)$ factors p. Hence a composition $q\circ j$ is zero.

The first factorization follows from (2.3); applying $\operatorname{Hom}(V(\lambda_i),-)$ we obtain a long exact sequence where the first terms are

$$\begin{array}{ccccccc} 0 & \to & \operatorname{Hom}(V(\lambda_i),E_{i-1}) & \to & \operatorname{Hom}(V(\lambda_i),E_i) & \to & \operatorname{Hom}(V(\lambda_i),V(\lambda)^{\oplus d_i}) \\ & \to & \operatorname{Ext}^1(V(\lambda_i),E_{i-1}) & \to & \operatorname{Ext}^1(V(\lambda_i),E_i) & \to & \cdots \end{array}$$

But E_i was constructed so that $\operatorname{Ext}^1(V(\lambda_i),E_i)=0$. Further, $\dim\operatorname{Ext}^1(V(\lambda_i),E_{i-1})=\dim\operatorname{Hom}(V(\lambda_i),V(\lambda)^{\oplus d_i})=d_i$, hence

$$\operatorname{Hom}(V(\lambda_i),E_{i-1})\longrightarrow\operatorname{Hom}(V(\lambda_i),E_i),\qquad f\mapsto i\circ f \tag{2.4}$$

is an isomorphism.

The second factorization also follows from (2.3), since $\operatorname{Hom}(E_{i-1},V(\lambda_i))=0$: The Weyl factors of E_{i-1} is $V(\lambda_j)$ with $\lambda_j\not\uparrow\lambda_i$ and $\operatorname{Hom}(V(\lambda_j),V(\lambda_i))=0$ for all such j. □

COROLLARY 2.12.

$$0 \;\to\; E_i \;\to\; E_{i+1} \;\to\; V(\lambda_{i+1})^{d_{i+1}} \;\to\; 0.$$

is non-split for each i.

LEMMA 2.13. *Each E_i is indecomposable. In particular, E_r is indecomposable.*

PROOF. Note that E_0 is indecomposable; we proceed inductively. We establish first a connection between $\operatorname{End}(E_i)$ and $\operatorname{End}(E_{i-1})$ to facilitate the induction argument.

$$\begin{tikzcd} & & & \operatorname{End}(E_{i-1}) \arrow[d, "f\mapsto i\circ f", "\simeq"'] & \\ 0 \arrow[r] & \operatorname{Hom}(V(\lambda_i)^{d_i},E_i) \arrow[r, "f\mapsto f\circ p"] & \operatorname{End}(E_i) \arrow[r, "f\mapsto f\circ i"] & \operatorname{Hom}(E_{i-1},E_i) \arrow[r] & 0 \end{tikzcd}$$

The isomorphism is obtained by using $\mathrm{Hom}(E_{i-1},-)$ on (2.3); in the proof of Lemma 2.11 we saw that $\mathrm{Hom}(E_{i-1},V(\lambda_i))=0$.

The sequence is exact since we constructed E_i so that $\mathrm{Ext}^1(V(\lambda_i),E_i)=0$.

Choose an idempotent $e\in\mathrm{End}(E_i)$. We must show that e is either one or zero. Let f denote the image of e in $\mathrm{Hom}(E_{i-1},E_i)$ lifted to $\mathrm{End}(E_{i-1})$; it is straightforward to check that this is an idempotent. Since E_{i-1} is indecomposable we thus find that f is 1 or 0.

Suppose first that f is zero. Then e is the image of some $g\in\mathrm{Hom}(V(\lambda_i)^{d_i},E_i)$, i.e. $g\circ p=e$. If

$$p\circ g: V(\lambda_i)^{d_i}\longrightarrow E_i\longrightarrow V(\lambda_i)^{d_i}$$

is non-zero, then $V(\lambda_i)$ is a summand in E_i, contradicting Lemma 2.11. Therefore $0=g\circ p\circ g\circ p=e^2=e$.

If, on the other hand $f=1$, then we consider $e-1$, which is mapped to zero in $\mathrm{Hom}(E_{i-1},,E_i)$. With the same argumentation as above we find a $g'\in\mathrm{Hom}(V(\lambda_i)^{d_i},E_i)$ that is mapped to $e-1$, i.e. $g'\circ p=e-1$. As before, $0=p\circ g'$; otherwise $V(\lambda_i)$ splits off E_i. Therefore $0=(e-1)^2=1-e$ and we are done. ☐

Properties of tilting modules

The construction of E_r in the last section gives us directly the basic properties of tilting modules. These are stated in Theorem 2.14 below. Further properties that are not directly linked to the construction are stated in Theorem 2.15. We denote by $T(\lambda)$ the module E_r.

THEOREM 2.14. *Let λ denote a dominant weight.*

(i) *$T(\lambda)$ is an indecomposable tilting module with highest weight λ.*
(ii) *The λ-weight space of $T(\lambda)$ is one-dimensional.*
(iii) *If $V(\mu)$ is a Weyl factor of $T(\lambda)$ then $\mu\uparrow\lambda$.*
If $L(\mu)$ is a composition factor of $T(\lambda)$ then $\mu\uparrow\lambda$.
(iv) *Suppose that μ is maximal among weights with $\mathrm{Ext}^1(V(\mu),V(\lambda))\neq 0$. Then*

$$[T(\lambda):V(\mu)]=\dim\mathrm{Ext}^1(V(\mu),V(\lambda))$$

PROOF. In the previous section we constructed the module E_r. By construction, this module has a Weyl filtration and highest weight λ. By Lemma 2.10 E_r has a good filtration, and it is therefore a tilting module. Finally Lemma 2.13 shows that E_r is indecomposable. This shows the first assertion.

Note that $V(\lambda)$ appears once in $T(\lambda)$ and that all other Weyl factors have highest weight linked to λ. This shows (ii) and the first statement in (iii). The second statement of (iii) now follows from the strong linkage principle, Theorem 2.1.

Note that the assumption in (iv) allow us to choose the ordering of $\Pi(\lambda)$ so that $\lambda_i=\mu$ and $\mathrm{Ext}^1(V(\lambda_j),V(\lambda))=0$ for all $j<i$. The first steps in the construction then show the claim. ☐

THEOREM 2.15.

(i) *There exist an indecomposable tilting module with highest weight λ for each dominant weight λ.*
(ii) *$\{T(\lambda)\mid\lambda\in X^+\}$ is a full set of non-isomorphic indecomposable tilting modules.*
(iii) *A direct sum of tilting modules is tilting.*
(iv) *A summand in a tilting module is tilting.*
(v) *A tilting module is fully determined by its character.*
(vi) *A tensor product of tilting modules is tilting.*
(vii) *Translations and wallcrossings take tilting modules to tilting modules.*

PROOF. The first assertion is trivial in view of Theorem 2.14.

Before we prove (ii) we note that $V(\lambda)$ is a submodule of $T(\lambda)$ and that $H^0(\lambda)$ is a quotient. This follows from a well known fact about modules with Weyl filtrations; any

$$\begin{tikzcd}
 & T(\lambda) \arrow[dl] \arrow[d] & \\
V(\lambda) \arrow[r] & Q \arrow[r] \arrow[d] & H^0(\lambda) \\
 & T(\lambda) \arrow[ur] &
\end{tikzcd}$$

FIGURE 1

maximal Weyl factor is a submodule and the quotient has a Weyl filtration. By dualizing, we obtain a similar statement about modules with good filtrations; a maximal factor is a quotient and the kernel of the projection has a good filtration.

Now let Q be a tilting module with λ as a highest weight. Choosing a nonzero vector in Q_λ allow us to define homomorphisms $V(\lambda) \longrightarrow Q \longrightarrow H^0(\lambda)$ so that the composite is non-zero. From the inclusion $V(\lambda) \longrightarrow T(\lambda)$ and the projection $p : T(\lambda) \longrightarrow H^0(\lambda)$ we obtain long exact sequences by applying $\mathrm{Hom}(-,Q)$ and $\mathrm{Hom}(Q,-)$ respectively

$$\begin{array}{ccccccccc}
\dots & \to & \mathrm{Hom}(T(\lambda),Q) & \to & \mathrm{Hom}(V(\lambda),Q) & \to & \mathrm{Ext}^1(T(\lambda)/V(\lambda),Q) & \to & \dots \\
\dots & \to & \mathrm{Hom}(Q,T(\lambda)) & \to & \mathrm{Hom}(Q,H^0(\lambda)) & \to & \mathrm{Ext}^1(Q,\ker p) & \to & \dots
\end{array}$$

Both Ext^1-groups are zero by Corollary 2.3, since $T(\lambda)/V(\lambda)$ has a Weyl filtration and $\ker p$ has a good filtration. Hence we obtain homomorphisms so that the diagram of Figure 2 commutes.

As the map $V(\lambda) \longrightarrow H^0(\lambda)$ is nonzero we see that the composite $T(\lambda) \longrightarrow T(\lambda)$ is nonzero on the λ-weight space. Hence it is not nilpotent. But by Fittings lemma, all non-nilpotent endomorphisms of indecomposable modules are isomorphisms. We have shown that $T(\lambda)$ is a summand of Q. This proves the second assertion, as it shows that any indecomposable module with highest weight λ is isomorphic to a $T(\lambda)$.

It is obvious that direct sums of tilting modules are tilting. It follows from Corollary 2.8 that summands in modules with good filtrations have good filtrations. Recalling that a module Q is tilting if and only if Q and its dual Q^* has good filtrations, we get the sixth claim.

It follows from (iv) that a tilting module is a sum of indecomposable tilting modules. Since $\dim T(\lambda)_\lambda = 1$ we find that $\mathrm{ch}\, T(\lambda)$, $\lambda \in X^+$, form a basis of the ring of characters. Hence only one decomposition in indecomposables is possible and we have (v).

The assertion (vi) follows from the non-trivial fact that tensor products of modules with a good filtration has a good filtration. This was shown by Donkin (1985) in almost all types and characteristics, and later by Mathieu (1990) in general.

Finally the last assertion follows from (iv) and (vi). □

Let us determine the structure of a first set of tilting modules.

COROLLARY 2.16. *A Weyl module is tilting if and only if it is simple.*

PROOF. Suppose that $V(\lambda)$ is simple. Then no Weyl modules extend $V(\lambda)$ as

$$\mathrm{Ext}^1(V(\mu),V(\lambda)) = \mathrm{Ext}^1(V(\mu),H^0(\lambda)) = 0$$

by Theorem 2.2. So by the construction $T(\lambda) = V(\lambda)$. Of course, we may also argue that $V(\lambda) = H^0(\lambda)$ shows that $V(\lambda)$ is tilting, and since a simple $V(\lambda)$ is indecomposable, we find that $V(\lambda) = T(\lambda)$.

Suppose that $V(\lambda)$ is tilting. Then $V(\lambda) \simeq H^0(\lambda)$ by character arguments, since $V(\lambda)$ has a good filtration. But any homomorphism $V(\lambda) \longrightarrow H^0(\lambda)$ factorizes over $L(\lambda)$ by Theorem 2.2. □

CHAPTER 3

Quantum groups

This chapter introduces $\mathcal{U}_q$ – the quantum group at a p'th root of unity corresponding to G. Further, we establish here the key concept of quantizations of modular tilting modules. This chapter is not as detailed as Chapter 2. In fact, we give references, and virtually no proofs.

Let (a_{ij}) denote the Cartan matrix of the irreducible root system R of rank n, defined by the almost simple group G from Chapter 2. We will consider 4 quantum groups, associated to the Cartan matrix (a_{ij}).

The first quantum group: $\mathcal{U}$

We choose a diagonal matrix d, so that $d(a_{ij})$ is symmetric. We let d_i, $1 \leq i \leq n$, denote the entries on the diagonal of d and we assume that these d_i's are positive with no common divisor. Let us recall the definitions of the so-called Gaussian integers and binomial coefficients. For each $m, l \in \mathbb{N}$ set

$$\begin{aligned}
[m]_d &= \frac{v^{md} - v^{-md}}{v^d - v^{-d}} \\
[m]_d! &= [m]_d[m-1]_d \dots [1]_d \\
\begin{bmatrix} m \\ l \end{bmatrix}_d &= \frac{[m]_d[m-1]_d \cdots [m-l+1]_d}{[l]_d[l-1]_d \cdots [1]_d}.
\end{aligned}$$

The quantum group $\mathcal{U}$ is the $\mathbb{Q}(v)$-algebra with generators E_i, F_i, K_i, K_i^{-1} for $i = 1 \dots n$ and relations

$$\begin{aligned}
&K_iK_i^{-1} = K_i^{-1}K_i = 1 \\
&K_jK_i = K_jK_i \\
&K_iE_jK_i^{-1} = v^{d_ia_{ij}} \\
&K_iF_jK_i^{-1} = v^{-d_ia_{ij}} \\
&E_iF_j - F_jE_i = 0 && i \neq j \\
&E_iF_i - F_iE_i = \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}} \\
&\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s}E_jE_i^s = 0 && i \neq j \\
&\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s}F_jF_i^s = 0 && i \neq j.
\end{aligned}$$

The quantum group $\mathcal{U}$ is a Hopf-algebra; we will not list the comultiplication, the counit, and the antipode here as we do not use them.

The second quantum group: $\mathcal{U}_{\mathcal{A}}$

Let $\mathcal{A}$ denote the localization of $\mathbb{Z}[v]$ at the ideal generated by p and $v-1$. We view $\mathcal{A}$ as a subring of $\mathbb{Q}(v)$.

We construct $\mathcal{U}_{\mathcal{A}}$ as a $\mathcal{A}$-subalgebra of $\mathcal{U}$. Consider for each $i = 1 \dots n$, $r \in \mathbb{N}$ and $c \in \mathbb{Z}$ the following elements

$$E_i^{(r)} = \frac{E_i^r}{[r]_{d_i}!}$$

$$F_i^{(r)} = \frac{F_i^r}{[r]_{d_i}!}$$

$$\left[\begin{array}{c} K_i;\ c \\ r \end{array} \right] = \prod_{s=1}^{r} \frac{K_i q^{d_i(c+1-s)} - K_i^{-1} q^{-d_i(c+1-s)}}{v^{d_i s} - v^{-d_i s}}.$$

Then define $\mathcal{U}_{\mathcal{A}}$ as the $\mathcal{A}$-subalgebra of $\mathcal{U}$ generated by

$$E_i^{(r)},\ F_i^{(r)},\ K_i,\ K_i^{-1},\ \left[\begin{array}{c} K_i;\ c \\ r \end{array} \right], \quad \text{for } i = 1 \dots n,\ r \in \mathbb{N},\ c \in \mathbb{Z}.$$

In fact, the last set of generators is unnecessary, as they are contained in the $\mathcal{A}$-subalgebra generated by the four first sets.

We let $\mathcal{U}_{\mathcal{A}}^+$ denote the subalgebra generated by $E_i^{(r)}$, $\mathcal{U}_{\mathcal{A}}^-$ denotes the subalgebra generated by $F_i^{(r)}$, and $\mathcal{U}_{\mathcal{A}}^0$ denotes the subalgebra generated by K_i, K_i^{-1}, $\left[\begin{array}{c} K_i;\ c \\ r \end{array} \right]$ with $i = 1 \dots n$, $r \in \mathbb{N}$, $c \in \mathbb{Z}$. Then we have a triangular decomposition of $\mathcal{U}_{\mathcal{A}}$, meaning that multiplication defines an isomorphism $\mathcal{U}_{\mathcal{A}}^- \mathcal{U}_{\mathcal{A}}^0 \mathcal{U}_{\mathcal{A}}^+ \simeq \mathcal{U}_{\mathcal{A}}$. These algebras are all free over $\mathcal{A}$.

Let $\lambda = (\lambda_1, \dots \lambda_n) \in \mathbb{Z}^n$. Then λ defines a character of $\mathcal{U}_{\mathcal{A}}^0$ by

$$\lambda(K_i) = v^{d_i \lambda_i} \tag{3.1}$$

$$\lambda(\left[\begin{array}{c} K_i;\ c \\ r \end{array} \right]) = \left[\begin{array}{c} \lambda_i + c \\ r \end{array} \right]_{d_i}. \tag{3.2}$$

The binomial coefficients all belong to $\mathbb{Z}[v, v^{-1}] \subset \mathcal{A}$.

The third and fourth quantum groups: $\mathcal{U}_k$ and $\mathcal{U}_q$

We obtain the third and fourth quantum group by specialization of $\mathcal{U}_{\mathcal{A}}$. Consider the field k as a $\mathcal{A}$-module, with v acting as $1 \in k$. Let $q \in \mathbb{C}$ denote a primitive p'th root of unity, and consider the field $\mathbb{C}$ as a $\mathcal{A}$-module, with v acting as q. Then define

$$\mathcal{U}_k = \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} k$$

$$\mathcal{U}_q = \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}.$$

We let $\mathcal{U}_k^-$, $\mathcal{U}_k^0$, $\mathcal{U}_k^+$, $\mathcal{U}_q^-$, $\mathcal{U}_q^0$, and $\mathcal{U}_q^+$ denote the images of $\mathcal{U}_{\mathcal{A}}^-$, $\mathcal{U}_{\mathcal{A}}^0$, and $\mathcal{U}_{\mathcal{A}}^+$ in $\mathcal{U}_k$ and $\mathcal{U}_q$. Further, $\lambda \in \mathbb{Z}^n$ defines a character of $\mathcal{U}_k^0$ and $\mathcal{U}_q^0$ as in the previous section.

Modules of $\mathcal{U}_{\mathcal{A}}$, $\mathcal{U}_k$, and $\mathcal{U}_q$

We shall restrict ourself to $\mathcal{A}$-finite $\mathcal{U}_{\mathcal{A}}$-modules and to finite dimensional $\mathcal{U}_k$- and $\mathcal{U}_q$-modules. If Q is a $\mathcal{U}_{\mathcal{A}}$-module (say) then define the λ-weight space of Q by

$$Q_\lambda = \{\ m \in Q \mid um = \lambda(u)m \text{ for all } u \in \mathcal{U}_{\mathcal{A}}^0\ \}.$$

A $\mathcal{U}_k$- or $\mathcal{U}_q$-module Q is a direct sum of its weights spaces: $Q = \oplus_{\lambda \in \mathbb{Z}^n} Q_\lambda$.

We consider modules of the quantum group $\mathcal{U}_k$. It is shown in (Lusztig 1990) that $\mathcal{U}_k$ modulo the ideal generated by all $K_i \otimes 1 - 1$ ($1 \leq i \leq n$) is isomorphic to the hyper algebra of the group G. Therefore we may regard all G-modules as modules of $\mathcal{U}_k$. On a G-weight space M_λ, with $\lambda = \lambda_1 \omega_1 + \dots + \lambda_n \omega_n \in X$ (here ω_i denote the i'th fundamental weight), $\mathcal{U}_k^0$ acts through $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. It follows that we may identify the weight spaces of G and the quantum groups in question. From this point on we shall write X for the weight spaces of the quantum groups and X^+ for the dominant weights, corresponding to $(\mathbb{Z}_{\geq 0})^n$.

We will regard $L(\lambda)$, $V(\lambda)$, $H^0(\lambda)$, and $T(\lambda)$ as $\mathcal{U}_k$-modules, as explained above.

Let us consider induced modules of the quantum groups. As the name suggest, these modules are obtained by induction. For a $\lambda \in X^+$, consider $\mathcal{A}$ as a $\mathcal{U}_{\mathcal{A}}^- \mathcal{U}_{\mathcal{A}}^0$-module with trivial $\mathcal{U}_{\mathcal{A}}^-$-action and $\mathcal{U}_{\mathcal{A}}^0$-action through λ. We denote this module by $\mathcal{A}_\lambda$. Then let $H^0_{\mathcal{A}}(\lambda)$ denote the integrable part of $\mathrm{Hom}_{\mathcal{U}_{\mathcal{A}}^- \mathcal{U}_{\mathcal{A}}^0}(\mathcal{U}_{\mathcal{A}}, \mathcal{A}_\lambda)$; this yields an $\mathcal{A}$-finite module. We define by similar recipes the finite dimensional modules $H^0_k(\lambda)$ and $H^0_q(\lambda)$ over $\mathcal{U}_k$ and $\mathcal{U}_q$.

We define the Weyl modules $V_{\mathcal{A}}(\lambda)$, $V_k(\lambda)$, and $V_q(\lambda)$ as the dual of $H^0_{\mathcal{A}}(-w_0\lambda)$, $H^0_k(\lambda)$, and $H^0_q(-w_0\lambda)$, respectively.

By now we have two sets of Weyl modules and two sets of induced modules for $\mathcal{U}_k$; one set obtained as above, and one set since all G-modules may be considered as $\mathcal{U}_k$-modules. It follows from (Andersen, Polo and Wen 1991, Proposition 1.22 and Proposition 3.3) that

$$H^0_k(\lambda) \simeq H^0(\lambda)$$
$$V_k(\lambda) \simeq V(\lambda)$$

Therefore we shall denote these modules by $H^0(\lambda)$ and $V(\lambda)$ from now on.

As $\mathcal{U}_k$ and $\mathcal{U}_q$ are obtained as specializations of $\mathcal{U}_{\mathcal{A}}$ we get modules of $\mathcal{U}_k$ and $\mathcal{U}_q$ from $\mathcal{U}_{\mathcal{A}}$-modules. The Weyl modules and the induced modules behave well with respect to specializations of $\mathcal{A}$. From *loc.cit.* we also get that $V_{\mathcal{A}}(\lambda)$ and $H^0_{\mathcal{A}}(\lambda)$ are free over $\mathcal{A}$. In fact, we have the following isomorphisms (by (Andersen et al. 1991, Theorem 3.5 and Corollary 5.7)):

$$H^0_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} k \simeq H^0(\lambda) \qquad V_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} k \simeq V(\lambda) \tag{3.3}$$
$$H^0_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C} \simeq H^0_q(\lambda) \qquad V_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C} \simeq V_q(\lambda). \tag{3.4}$$

Further, the head of $V_q(\lambda)$ (isomorphic to the socle of $H^0_q(\lambda)$ by our definition) is simple, and we denote it by $L_q(\lambda)$. Then $\{L_q(\lambda) \mid \lambda \in X^+\}$ is a full set of simple non-isomorphic $\mathcal{U}_q$-modules, see (Andersen et al. 1991, 6.2).

Quantum tilting modules

We shall define tilting modules of the quantum groups as for the group G. That is, a module is tilting if it has a Weyl filtration as well as a good filtration. By the previous section, the tilting modules of G are tilting $\mathcal{U}_k$-modules.

Let us consider tilting modules of $\mathcal{U}_q$. We explore first the many similarities between the category of finite dimensional $\mathcal{U}_q$-modules and the category of G-modules. The results presented here are chosen with the construction of tilting modules from Chapter 2 in mind. First of all we find that the linkage relation preserve its importance.

THEOREM 3.1. *(Andersen et al. 1991, Theorem 8.1) If $L_q(\mu)$ is a composition factor of $V_q(\lambda)$ then $\mu \uparrow \lambda$. If $L_q(\mu)$ is a composition factor of $H^0_q(\lambda)$ then $\mu \uparrow \lambda$.*

Next we would like a $\mathcal{U}_q$-version of Theorem 2.2. But we make do with

THEOREM 3.2. *Let $\lambda, \mu \in X^+$. Then*

(i) $\mathrm{Hom}(V_q(\lambda), H^0_q(\mu)) = \begin{cases} k & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$

(ii) $\mathrm{Ext}^1(V_q(\lambda), H^0_q(\mu)) = 0.$

PROOF. The case $\lambda = \mu$ follows since $L_q(\lambda)$ is the head of $V_q(\lambda)$, and $L_q(\lambda)$ appears only in the socle of $H^0_q(\lambda)$ and only once. In case $\lambda \neq \mu$, duality gives us

$$\mathrm{Hom}(V_q(\lambda), H^0_q(\mu)) \simeq \mathrm{Hom}(V_q(-w_0\mu), H^0_q(-w_0\lambda)).$$

If these spaces are non zero we have (by Theorem 3.1) $\lambda \uparrow \mu$ as well as $-w_0\mu \uparrow -w_0\lambda$; this shows that $\lambda = \mu$.

For (ii) we refer to (Andersen et al. 1991, Lemma 9.9). □

Theorem 3.2 provide us with some information about extensions with Weyl modules. We leave the proof of the following corollary to the reader.

COROLLARY 3.3. *Let $\lambda, \mu \in X^+$. Then*

(i) $\mathrm{Ext}^1(V_q(\lambda), L_q(\mu)) \neq 0$ *implies that* $\lambda \uparrow \mu$.
(ii) $\dim \mathrm{Ext}^1(V_q(\lambda), L_q(\mu))$ *is finite.*
(iii) $\dim \mathrm{Ext}^1(V_q(\lambda), Q)$ *is finite for any* $\mathcal{U}_q$*-module* Q.

In Chapter 2 we found that modules with a good filtration are characterized by the property that only trivial extensions with Weyl modules are possible. For $\mathcal{U}_q$-modules we have a similar criteria.

THEOREM 3.4. *(Paradowski 1994, Theorem 3.1) Let Q be a $\mathcal{U}_q$-module. Suppose* $\mathrm{Ext}^1(V_q(\lambda), Q) = 0$ *for all* $\lambda \in X^+$*; then Q allows a good filtration.*

Armed with the above results, the construction of tilting modules from Chapter 2 may be repeated. As a result we get a quantum version of Theorems 2.14 and 2.15.

THEOREM 3.5. *Let λ denote a dominant weight.*

(i) *There exist an indecomposable tilting module with highest weight λ. We denote it by $T_q(\lambda)$.*
(ii) *The λ-weight space of $T_q(\lambda)$ is one-dimensional.*
(iii) *If $V_q(\mu)$ is a Weyl factor of $T_q(\lambda)$ then $\mu \uparrow \lambda$.*
If $L_q(\mu)$ is a composition factor of $T_q(\lambda)$ then $\mu \uparrow \lambda$.
(iv) $\{T_q(\lambda) \mid \lambda \in X^+\}$ *is a full set of non-isomorphic indecomposable tilting modules.*
(v) *A tilting module is fully determined by its character.*
(vi) *A direct sum of tilting modules is tilting.*
(vii) *A summand in a tilting module is tilting.*
(viii) *A tensor product of tilting modules is tilting.*
(ix) *Translations and wallcrossings take tilting modules to tilting modules.*

With the exception of (viii), this theorem follows by the same arguments as its modular counterparts. For (viii) we refer to (Kaneda 1998), which proves the result for $\mathcal{U}_{\mathcal{A}}$-modules.

We turn to tilting $\mathcal{U}_{\mathcal{A}}$-modules. For $\mathcal{U}_{\mathcal{A}}$ it is difficult to use the machinery of Chapter 2. One reason is that there are no obvious analogues of the simple modules $L(\lambda)$; we rely on composition series in our arguments in many places. With some care, however, it is still possible to construct tilting modules for $\mathcal{U}_{\mathcal{A}}$. Comparison with the modular case via specialization of $\mathcal{U}_{\mathcal{A}}$ to $\mathcal{U}_k$ and the base change arguments of (Andersen et al. 1991, 3.5) are key ingredients. We shall give the first step in the construction so as to indicate the procedure.

Choose $\lambda \in X^+$ and recall the set $\Pi(\lambda)$. Let $E_0 = V_{\mathcal{A}}(\lambda = \lambda_0)$. Then $\mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), E_0)$ is $\mathcal{A}$-finite by (Andersen et al. 1991, 5.15). Suppose that it is non-zero, and represent one generator $e \in \mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), E_0)$ by a short exact sequence.

$$0 \to E_0 \to E_0^{(1)} \to V_{\mathcal{A}}(\lambda_1) \to 0$$

Then applying $\mathrm{Hom}(V_{\mathcal{A}}(\lambda_1), -)$ we obtain a long exact sequence beginning with

$$\begin{array}{cccccc} 0 & \to & \mathrm{Hom}(V_{\mathcal{A}}(\lambda_1), E_0) & \to & \mathrm{Hom}(V_{\mathcal{A}}(\lambda_1), E_0^{(1)}) & \to & \mathrm{End}(V_{\mathcal{A}}(\lambda_1)) \\ & \to & \mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), E_0) & \to & \mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), E_0^{(1)}) & \to & \mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), V_{\mathcal{A}}(\lambda_1)) \end{array}$$

We have $\mathrm{End}(V_{\mathcal{A}}(\lambda_1)) \simeq \mathcal{A}$ and $\mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), V_{\mathcal{A}}(\lambda_1)) = 0$. The identity of $\mathrm{End}(V_{\mathcal{A}}(\lambda_1))$ is mapped to e, so that by exactness e is mapped to zero in $\mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), E_0^{(1)})$.

Now assume that $\mathcal{E}_0$ is a set of $\mathcal{A}$-generators of $\mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), E_0)$ of minimal cardinality. Then $\mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_1), E_0^{(1)})$ has $\mathcal{E}_0 - \{e\}$ as generators. This explains that we may construct successively $E_0, E_1, \ldots E_r$ so that E_i is an extension of $V(\lambda_i)^{d_i'}$ with E_{i-1}, with d_i' equal to the minimal number of generators of $\mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_i), E_{i-1})$. But we have not shown that E_r has any of the properties we are looking for. In particular we do not know that E_r is tilting. For this we refer to

THEOREM 3.6. *(Andersen 1998, 5.3) E_r is a tilting module and we denote it by $T_{\mathcal{A}}(\lambda)$. Further, the minimal number of generators of each* $\mathrm{Ext}^1(V_{\mathcal{A}}(\lambda_i), E_{i-1})$ *is equal to d_i. In particular,*

$$T_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} k \simeq T(\lambda).$$

Recall here that we denote by d_i the dimension of $\mathrm{Ext}^1(V(\lambda_i), E_{i-1})$ in the construction of the *modular* tilting modules.

We consider also $T_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C}$. It is clear from equation (3.4) that $T_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C}$ has a filtration by $\mathcal{U}_q$-Weyl modules and a filtration by $\mathcal{U}_q$-induced modules. Thus $T_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C}$ is $\mathcal{U}_q$-tilting. But this module is not necessarily indecomposable. Thus, the discussion above gives that for some nonnegative integers $a_{\mu\lambda}$

$$\begin{aligned} T_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} k &\simeq T(\lambda) \\ T_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C} &\simeq T_q(\lambda) \oplus \bigoplus_{\mu<\lambda} a_{\mu\lambda} T_q(\mu). \end{aligned}$$

This shows that the tilting modules of G lift to tilting modules of $U_{\mathcal{A}}$. That is, for a tilting G-module Q there exists a tilting $U_{\mathcal{A}}$-module $Q_{\mathcal{A}}$ with the property $Q_{\mathcal{A}} \otimes_{\mathcal{A}} k \simeq Q$. We denote by Q_q the tilting U_q-module $Q_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$. In this way each tilting G-module Q gives rise to a tilting U_q-module Q_q with the same character.

REMARK 3.7.

(i) *We shall see in Chapter 5 that the characters of all $T_q(\lambda)$ are known.*
(ii) *It is expected that $T(\lambda)_q = T_q(\lambda)$ for all λ in the lowest p^2-alcove, see (Andersen 1998).*

CHAPTER 4

The Hecke algebra and right cells

Corresponding to the irreducible root system R, we have the affine Weyl group W with generators $S = \{s_0, s_1, \ldots, s_n\}$. The pair (W,S) is a Coxeter system and we may therefore form the Hecke algebra associated to it. This algebra is the subject of this chapter.

We have sought to keep the notation as close as possible to that of the paper (Soergel 1997). A reader unacquainted with the subject will probably find that a copy of this paper or the expanded German version (Soergel n.d.) is a good companion.

The reader should note that it is usual in the theory of Coxeter groups to define the affine generator s_0 in terms of the highest *long* root, see e.g. (Humphreys 1990), (Bourbaki 1968, Planche I-IX). But given our interest in the representation theory of algebraic groups, we maintain that s_0 is defined with respect to the highest *short* root (see the beginning of Chapter 2).

The Hecke algebra

The results in this section may all be found in (Soergel 1997). Therefore the proofs are short here.

The affine Weyl group W comes with the Bruhat order $<$ and a length function, l, mapping $w \in W$ to the length of a reduced expression.

Associated to the Coxeter system (W,S) we have the Hecke algebra $\mathcal{H}$ over the ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$. As an $\mathbb{Z}[v, v^{-1}]$-module, $\mathcal{H}$ is free with one generator H_x for each Weyl group element $x \in W$. The ring structure is determined by the relations

$$H_x H_y = H_{xy} \qquad \text{if } l(x) + l(y) = l(xy) \tag{4.1}$$

$$H_x H_s = H_{xs} + (v^{-1} - v)H_x \qquad \text{if } xs < x,\ s \in S \tag{4.2}$$

On $\mathcal{H}$ we have a ring homomorphism, $^{-}$, taking H_x to $\overline{H}_x = H_{x^{-1}}^{-1}$ and v to $\overline{v} = v^{-1}$. Since $^{-}$ is an involution, we say that H is self-dual if $\overline{H} = H$. As an example note that $\underline{H}_s = H_s + v$ is selfdual, since $H_s^{-1} = H_s - v^{-1} + v$. This also shows that each basis element H_x is invertible.

It is clear from the relation (4.1) that $\mathcal{H}$ is generated by $\{H_s \mid s \in S\}$. Since $\underline{H}_s = H_s + v$ it follows that $\{\underline{H}_s \mid s \in S\}$ is a second generating set. The action of these selfdual generators is given by

$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & xs > x \\ H_{xs} + v^{-1}H_x & xs < x \end{cases} \tag{4.3}$$

The following theorem provides a second basis of $\mathcal{H}$.

THEOREM 4.1. *For each $x \in W$ there is a unique self-dual element*

$$\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v]H_y.$$

PROOF. We prove this by induction and note that H_e and $H_s + v$ (which we already denote by $\underline{H}_s$) show the existence part of the theorem for x of length 0 and 1.

Given x of length > 1 we choose $s \in S$ so that $xs < x$. By induction we may assume the existence of a selfdual element $\underline{H}_{xs} = H_{xs} + \sum_{y<xs} h_{y,xs} H_y$, with $h_{y,xs} \in v\mathbb{Z}[v]$. Then $\underline{H}_{xs}\underline{H}_s$

is selfdual. We write

$$\underline{H}_{xs}\underline{H}_s = H_x + \sum_{y<x} h_y H_y, \tag{4.4}$$

where $h_y \in \mathbb{Z}[v]$ by (4.3). So h_y may contain a constant. We handle this by considering

$$\underline{H}_x = \underline{H}_x\underline{H}_s - \sum_{y<x} h_y(0)\underline{H}_y.$$

This element is selfdual, and shows the existence part of the theorem for x.

We leave uniqueness to the reader. □

DEFINITION 4.2. *We define polynomials $h_{y,x} \in \mathbb{Z}[v]$ by the formula*

$$\underline{H}_x = \sum_{y\in W} h_{y,x} H_y.$$

We will denote the linear term of $h_{y,x}$ by $\mu(y,x)$.

These polynomials are basically the Kazhdan-Lusztig polynomials defined in (Kazhdan and Lusztig 1979). Note that $h_{y,x}(0) \neq 0$ if and only if $x = y$. Further, if $y < x$ the leading coefficient of $h_{y,x}$ is 1 and $\deg h_{y,x} = l(x) - l(y)$.

Our new basis of selfdual elements is central to this chapter. Let us record the right action of $\mathcal{H}$ on the selfdual basis.

PROPOSITION 4.3. *Let $x \in W$ and $s \in S$. We have*

$$\underline{H}_x\underline{H}_s = \begin{cases} \underline{H}_{xs} + \sum_{ys<y} \mu(y,x)\underline{H}_y & xs > x; \\ (v+v^{-1})\underline{H}_x & xs < x. \end{cases}$$

PROOF. We will prove only the formula for $xs > x$ here. We have by definition $\underline{H}_x = H_x + \sum_{y<x} h_{y,x}H_y$ with $h_{y,x} \in v\mathbb{Z}[v]$, and we write

$$\underline{H}_x\underline{H}_s = H_x + \sum_{y<x} h_y H_y.$$

Then $h_y \in \mathbb{Z}[v]$ by equation (4.3). In fact

$$h_y = \begin{cases} h_{ys,x} + v h_{y,x} & \text{if } ys > y; \\ h_{ys,x} + v^{-1} h_{y,x} & \text{if } ys < y. \end{cases}$$

By the proof of Theorem 4.1 we have

$$[\underline{H}_x\underline{H}_s : \underline{H}_y] = h_y(0).$$

But $h_y(0) = 0$ if $ys > y$. And $h_y(0) = \mu(y,x)$ if $ys < y$. □

REMARK 4.4. *We will be concerned mostly with right modules and right ideals of $\mathcal{H}$; on a few occasions, however, we need to consider the left action, which is given by the completely analogous formula*

$$\underline{H}_s\underline{H}_x = \begin{cases} \underline{H}_{sx} + \sum_{sy<y} \mu(y,x)\underline{H}_y & sx > x; \\ (v+v^{-1})\underline{H}_x & sx < x. \end{cases}$$

The following lemma is often useful, despite its uncomplicated appearance. The lemma is an easy consequence of Proposition 4.3. We will leave it to the reader.

LEMMA 4.5. *Suppose that $xs < x$ and $ys < y$. Then $h_{ys,x} = v h_{y,x}$.*

Right cell ideals in the Hecke algebra

Right modules of the Hecke algebra turn out to be an important tool in the representation theory of quantum groups at a root of unity (this is the subject of Chapter 5). It is therefore natural to ask for the right ideals of $\mathcal{H}$. We are interested here in right ideals with a particular nice basis. Since each H_x is invertible, it is the basis of self-dual elements that are interesting.

DEFINITION 4.6. *A right ideal in $\mathcal{H}$ is called a* right cell ideal *if it allows a basis $\{\underline{H}_y \mid y \in Y\}$ for some subset $Y \subset W$.*

DEFINITION 4.7. *(Lusztig 1999) Write $y \leftarrow_{\mathcal{R}} x$ if $[\underline{H}_x\underline{H}_s : \underline{H}_y] \neq 0$ for some $s \in S$. Write $y \leq_{\mathcal{R}} x$ if there is a chain $w_0, \ldots, w_n$ so that*

$$y = w_0 \leftarrow_{\mathcal{R}} w_1 \leftarrow_{\mathcal{R}} \ldots \leftarrow_{\mathcal{R}} w_n = x$$

REMARK 4.8.

(i) *So $y \leftarrow_{\mathcal{R}} x$ implies that $\underline{H}_y$ must be part of the basis of each right cell ideal that contains $\underline{H}_x$.*

(ii) *If $xs > x$ then $[\underline{H}_x\underline{H}_s : \underline{H}_{xs}] = 1$, by the proof of Theorem 4.1. Thus $y = xz$ with $l(y) = l(x) + l(z)$ implies $y \leq_{\mathcal{R}} x$.*

It should come as no surprise that the preorder $\leq_{\mathcal{R}}$ provides us with right ideals; it is designed to do just that. The following lemma gives the basics of right cell ideals.

LEMMA 4.9. *$I_x = \bigoplus_{y \leq_{\mathcal{R}} x} \mathbb{Z}[v, v^{-1}]\underline{H}_y$ is a right cell ideal, and I_x is the smallest right cell ideal that contains $\underline{H}_x$. A right cell ideal is the sum of such I_x, $x \in W$, and a sum of such I_x, $x \in W$ is a right cell ideal.*

PROOF. We show that I_x is preserved by the generators $\{\underline{H}_s \mid s \in S\}$ of $\mathcal{H}$. Assume therefore that $y \leq_{\mathcal{R}} x$ and that $[\underline{H}_y\underline{H}_s : \underline{H}_w] \neq 0$ for some $s \in S$. But then $w \leftarrow_{\mathcal{R}} y \leq_{\mathcal{R}} x$ shows that $w \leq_{\mathcal{R}} x$.

We will leave the proof of the remaining assertions to the reader. □

The right cell ideal presented next proves useful when we move on to right cells; see Lemma 4.13.

LEMMA 4.10. *Fix an $s \in S$. Then*

$$\bigoplus_{x;\ sx<x} \mathbb{Z}[v, v^{-1}]\underline{H}_x = \{H \in \mathcal{H} \mid (\underline{H}_s - v - v^{-1})H = 0\}$$

and this is a right cell ideal.

PROOF. It is clear that the right hand side is a right ideal, so it suffices to show the equality. The inclusion $\subset$ is obvious from Remark 4.4. To show the other inclusion, choose an $H \in \mathcal{H}$ with $\underline{H}_s H = (v + v^{-1})H$. Expressing this H in the self-dual basis we get the equality

$$\sum_{z \in W} [H : \underline{H}_z]\underline{H}_s\underline{H}_z = \sum_{z \in W} [H : \underline{H}_z](v + v^{-1})\underline{H}_z$$

If there is a z so that $sz > z$ with $[H : \underline{H}_z] \neq 0$, then choose z maximal with this property. This produces a contradiction, since Remark 4.4 shows that the coefficient of $\underline{H}_z$ on the left hand side is zero and nonzero on the right hand side. □

We conclude this section by giving an alternative description of the preorder $\leq_{\mathcal{R}}$. In the literature this description is often used to define $\leq_{\mathcal{R}}$; with this definition there is no need to introduce right cell ideals.

We introduce the following notation: Write $y - x$ if either $\mu(x, y)$ or $\mu(y, x)$ is nonzero, and let

$$\mathcal{L}(w) = \{s \in S \mid sw < w\}$$
$$\mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

LEMMA 4.11. *Let $x \neq y \in W$. We have*

$$y \leftarrow_{\mathcal{R}} x \iff y - x \text{ and } \mathcal{R}(y) \not\subset \mathcal{R}(x).$$

PROOF. If $y \leftarrow_{\mathcal{R}} x$ then Proposition 4.3 shows that $y - x$ and $\mathcal{R}(y) \not\subset \mathcal{R}(x)$. On the other hand, if $\mu(y,x) \neq 0$ and $\mathcal{R}(y) \ni s \notin \mathcal{R}(x)$ then $[\underline{H}_x \underline{H}_s : \underline{H}_y] \neq 0$; and if $\mu(x,y) \neq 0$ and $\mathcal{R}(y) \ni s \notin \mathcal{R}(x)$, then Lemma 4.5 shows that $h_{xs,y}$ has a constant term, hence that $y = xs > x$, hence that $[\underline{H}_x \underline{H}_s : \underline{H}_y] \neq 0$. □

Right cells

DEFINITION 4.12. *The equivalence relation induced by $\leq_{\mathcal{R}}$ is denoted by $\sim_{\mathcal{R}}$; so we write $y \sim_{\mathcal{R}} x$ if $y \leq_{\mathcal{R}} x \leq_{\mathcal{R}} y$. An equivalence class is called a* (Kazhdan-Lusztig) right cell.

For a right cell $\mathcal{C}$ we write $x \leq_{\mathcal{R}} \mathcal{C}$ if $x \leq_{\mathcal{R}} z$ for a $z \in \mathcal{C}$ (equivalently for all $z \in \mathcal{C}$). If $\mathcal{C}'$ is another right cell we write $\mathcal{C} \leq_{\mathcal{R}} \mathcal{C}'$ when $z \leq_{\mathcal{R}} \mathcal{C}'$ for a $z \in \mathcal{C}$ (equivalently for all $z \in \mathcal{C}$). The preorder $\leq_{\mathcal{R}}$ gives a partial order on the set of right cells in this very natural way.

LEMMA 4.13. *$y \leq_{\mathcal{R}} x$ implies $\mathcal{L}(y) \supset \mathcal{L}(x)$. Hence $y \sim_{\mathcal{R}} x$ implies $\mathcal{L}(y) = \mathcal{L}(x)$.*

PROOF. Suppose that $sx < x$. Then $\underline{H}_x \in \oplus_{y;sy<y} \mathbb{Z}[v,v^{-1}]\underline{H}_y$, which is a right cell ideal by Lemma 4.10. Then the right cell ideal generated by $\underline{H}_x$ is also contained in this large ideal: So $I_x = \oplus_{y \leq_{\mathcal{R}} x} \mathbb{Z}[v,v^{-1}]\underline{H}_y$ is contained in $\oplus_{y;sy<y} \mathbb{Z}[v,v^{-1}]\underline{H}_y$. The lemma follows. □

The following corollary is an immediate consequence of Lemma 4.13 since $\mathcal{L}(e) = \emptyset$ and e is the only Weyl group element with this property. The second assertion is also immediate; $\underline{H}_e$ is a unit in $\mathcal{H}$, hence generates all of it; therefore e is the maximal element in the preorder $\leq_{\mathcal{R}}$.

COROLLARY 4.14. *$\{e\}$ is a right cell. It is maximal among right cells.*

Suppose that there is a unique s so that $xs < x$. The next lemma states that this is sufficient to guarantee that xs and x are in the same right cell. This criteria will (despite its simplicity) prove very useful, since it is strong enough to determine the second largest cells.

LEMMA 4.15. *Suppose that $xs < x$ and that $xs \neq e$. Then $\mathcal{R}(x) = \{s\}$ implies $xs \sim_{\mathcal{R}} x$:*

PROOF. From Lemma 4.5 we get $h_{xs,x} = v$. Choose a t, so that $xst < xs$ (such t exists as $xs \neq e$). Then $t \neq s$ and therefore $xt > x$. Proposition 4.3 and Equation (4.3) shows that

$$\begin{aligned} \underline{H}_x \underline{H}_t &= \underline{H}_{xt} + \underline{H}_{xs} + \sum_{ys<y, y \neq xs} \mu(y,x) \underline{H}_y; \\ \underline{H}_{xs} \underline{H}_s &= \underline{H}_x + \sum_{ys<y} \mu(y,xs) \underline{H}_y. \end{aligned}$$

The first equation shows that $xs \leftarrow_{\mathcal{R}} x$ and the second equation shows that $x \leftarrow_{\mathcal{R}} xs$. □

EXAMPLE 4.16. *The affine Weyl group corresponding to a root system of type A_1 is generated by two elements s, t with relations $s^2 = t^2 = 1$. The different elements are words formed by the letters s, t without two consecutive s's or t's; each such word is in fact a reduced expression. Consider the sets*

$$\begin{aligned} &A = \{(st)^m, (st)^n s \mid m \geq 1, n \geq 0\} \\ &B = \{e\} \\ &C = \{(ts)^m, (ts)^n t \mid m \geq 1, n \geq 0\} \end{aligned}$$

Note that each $x \in A$ has $\mathcal{L}(x) = \{s\}$ and each $y \in C$ has $\mathcal{L}(y) = \{t\}$. It follows from Lemma 4.13 that A and C are unions of right cells. On the other hand Lemma 4.15 shows that

$$\begin{aligned} (st)^n s &\sim_{\mathcal{R}} (st)^{n+1} \sim_{\mathcal{R}} (st)^{n+1}s && n \geq 0, \\ (ts)^n t &\sim_{\mathcal{R}} (ts)^{n+1} \sim_{\mathcal{R}} (ts)^{n+1}t && n \geq 0. \end{aligned}$$

Therefore A is a right cell and B is a right cell. So in type A_1 the right cell decomposition of the affine Weyl group is given by

$$W = A \cup B \cup C.$$

We will now describe the second largest right cells. Let $\mathcal{C}$ denote the set of elements in W with a unique reduced expression. Inspired by Lemma 4.13 we consider $\mathcal{C}(s_i)$, the subset with reduced expression beginning with s_i:

$$\mathcal{C}(s_i) \quad = \quad \{w \in \mathcal{C} \mid \mathcal{L}(w) = \{s_i\}\}$$

This is a non-empty set since $s_i \in \mathcal{C}(s_i)$. Further, it is a connected set in the sense made precise by the following lemma:

LEMMA 4.17. *Suppose $w \in \mathcal{C}(s_i)$, $s \in S$, and $ws < w$. Then $ws \in \mathcal{C}(s_i)$ or $ws = e$.*

PROOF. If ws allows more than one reduced expression then so does w. If the unique expression of w begin with s_i then so does the reduced expression of ws, unless $ws = e$. $\square$

THEOREM 4.18. *(Lusztig 1983) $\mathcal{C}(s_i)$ is a right cell.*

PROOF. It is a standard fact that $s \in \mathcal{R}(w)$ if and only if w has a reduced expression that ends with s. Hence $w \in \mathcal{C}(s_i)$ gives $\mathcal{R}(w) = \{s\}$ for some s. When $ws \neq e$, Lemma 4.15 shows that $w \sim_{\mathcal{R}} ws$ and Lemma 4.17 gives that $ws \in \mathcal{C}(s_i)$. Recursively, this shows that $w \sim_{\mathcal{R}} s_i$, hence that $\mathcal{C}(s_i)$ is contained in a right cell.

Now let $w \in \mathcal{C}(s_i)$ and let $x \in W$ be an arbitrary element satisfying $w \sim_{\mathcal{R}} x$. To show the theorem, we must show that $x \in \mathcal{C}(s_i)$. First $x \neq e$ is easy as e has a right cell of its own. We will prove that $w \leftarrow_{\mathcal{R}} x$ implies that $x \in \mathcal{C}(s_i)$. This is enough to give us the theorem.

By definition, there is a simple s so that

$$[\underline{H}_x \underline{H}_s : \underline{H}_w] \neq 0.$$

By Proposition 4.3 we know a lot about a product $\underline{H}_x \underline{H}_s$. In particular, if $xs < x$ then $x = w$ and we have $x \in \mathcal{C}(s_i)$. Further, if $xs > x$ and $w > x$ then $w = xs$ and we have $x \in \mathcal{C}(s_i)$ by Lemma 4.17 (recall that $x \neq e$). So we may assume that $xs > x$, that $ws < w$ and that $\mu(w,x) \neq 0$ (recall that this is the linear term of $h_{w,x}$). Then pick any simple t so that $x > xt$. We have $wt > w$ as $s \neq t$ is unique with the property $ws < w$. Now Lemma 4.5 gives $h_{w,x} = vh_{wt,x}$ and we conclude that $h_{wt,x}$ has a constant term. But then $wt = x$. This shows that t is uniquely determined by $xt < x$, i.e. that $\mathcal{R}(x) = \{t\}$. So all reduced expressions of x end with t; removing that t gives us a reduced expression of w. But w has a unique reduced expression and this unique reduced expression begins with s_i. It follows that x has a unique reduced expression and that this unique reduced expression begins with s_i. We have $x \in \mathcal{C}(s_i)$. $\square$

We conclude this section by the cell decomposition of the affine Weyl groups in type A_2, B_2, and G_2. Given a $w \in W$ we associate to it the alcove $w.C$; this is the usual bijection between Weyl group elements and the set of alcoves defined by W. In Figure 2, 3, and 4 we have visualized the right cells by coloring alcoves. Each connected component corresponds to a right cell. Proofs may be found in (Lusztig 1985).

We should mention the existence of two-sided cells and left cells; these cells arise from two-sided ideals in $\mathcal{H}$ and left ideals in $\mathcal{H}$. A two-sided cell is a union of right cells and a union of left cells. In Figure 2,3, and 4 all right cells with the same color comprise one two-sided cell.

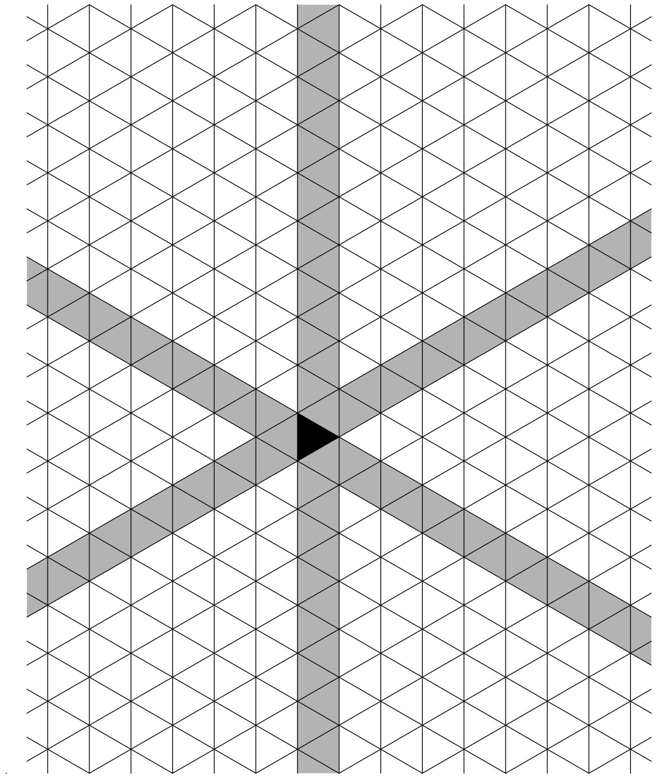


FIGURE 2. The right cells in type A_2.

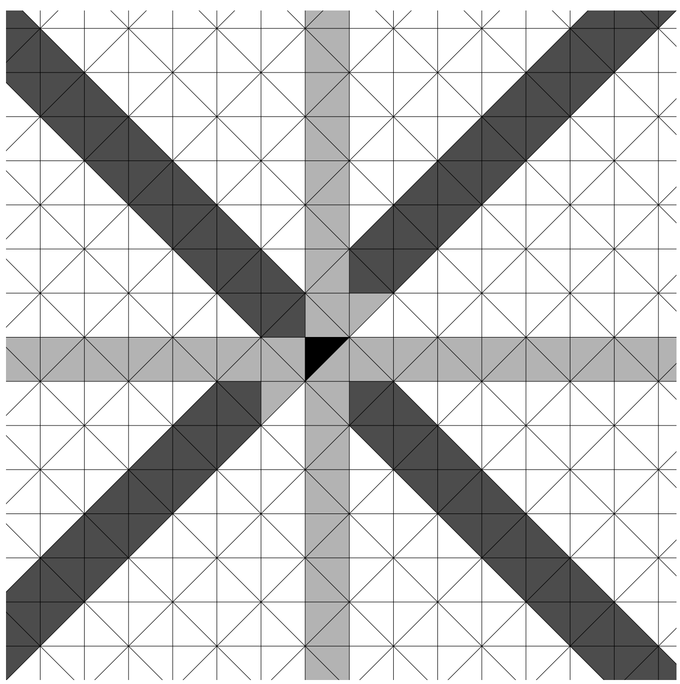


FIGURE 3. The right cells in type B_2.

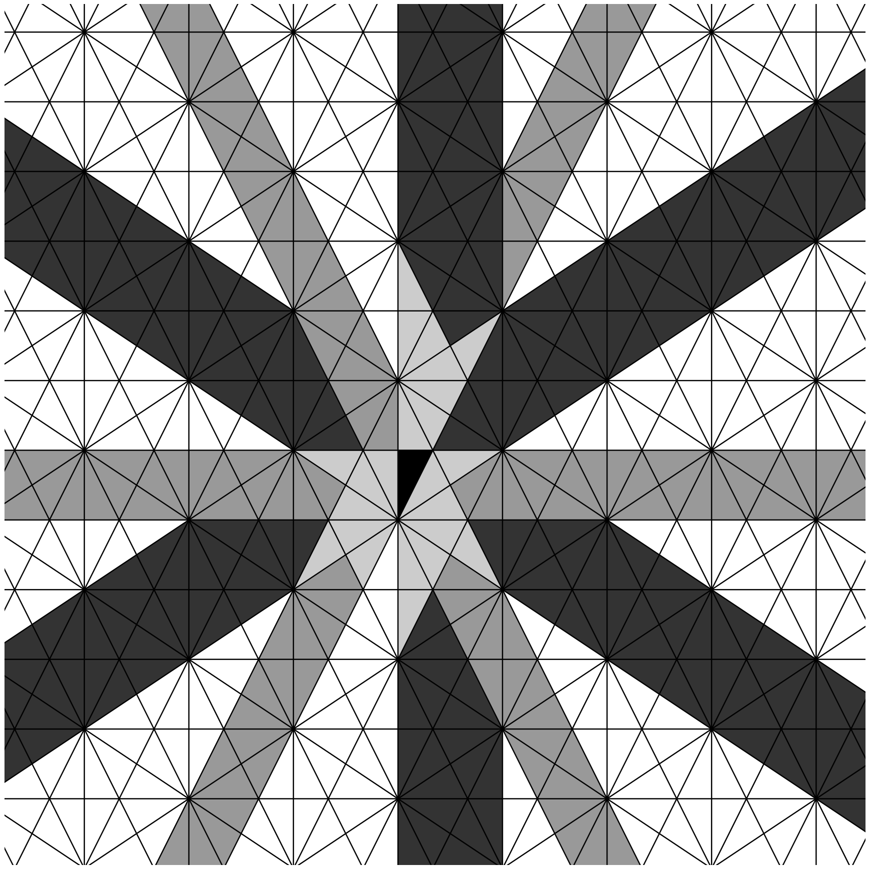


FIGURE 4. The right cells in type G_2.

Right cells and dominant weights

Right cells (as we will see in Chapter 6) are important in the representation theory of G and $\mathcal{U}_q$. For this reason, we have a speciel interest in cells that contain dominant weights. In this short section we describe the right cells that corresponds to alcoves containing dominant weights.

Recall, that the dominant chamber is the open subset

$$\{\lambda \in E \mid 0 < \langle \lambda, \alpha_i \rangle \ \text{ for } i = 1, \dots, n\}.$$

LEMMA 4.19. *Assume $p \geq h$. The following four sets are equal*

$$\begin{aligned} W^{(i)} &= \{w \in W \mid w.C \textit{ is contained in the dominant chamber}\} \\ W^{(ii)} &= \left\{w \in W \mid x.0 \in X^+\right\} \\ W^{(iii)} &= \{w \in W \mid \mathcal{L}(w) \subset \{s_0\}\} \\ W^{(iv)} &= \{w \in W \mid w \textit{ is the minimal lenght representative of the right coset } W_0 w\}. \end{aligned}$$

We shall denote this set by W^0.

PROOF. When $p \geq h$, 0 is a regular weight in the first alcove C. Then $W^{(\mathrm{i})} = W^{(\mathrm{ii})}$ is clear.

The minimal length representative of $W_0 w$ is determined by the property that $s_i w > w$ for all generators $s_1, \dots, s_n$ of the subgroup W_0. Then $W^{(\mathrm{iii})} = W^{(\mathrm{iv})}$ follows.

Now let $w.0 \in X^+$ and let $s_i \neq s_0$. Then $s_i w.0 \notin X^+$. Further $l(s_i w) > l(w)$; this may be proved by interpreting the length of w as the number of reflection hyperplanes that seperate $w.C$ and C (see (Humphreys 1990, section 4.4) or the proof of Lemma 8.20 below). Thus $s_i w > w$ and we see that w is the minimal length representative of the right coset $W_0 w$, hence $w \in W^{(\mathrm{iv})}$. If, on the other hand, $l(s_i w) > l(w)$ for all $s_i \neq s_0$, then the alcoves $w.C$ and C lie on the same side of the reflection hyperplane corresponding to s_i for all $s_i \neq s_0$. This means that $w.C$ lies in the dominant chamber. □

The Figures 2, 3, and 4 indicate that right cells do not cross the boundaries of the dominant chamber. This is an easy consequence of Lemma 4.19 above.

COROLLARY 4.20. *A right cell is either contained in or intersects trivially with W^0.*

PROOF. We obviusely have

$$W^0 = \{w \in W \mid \mathcal{L}(w) = \{s_0\}\} \cup \{w \in W \mid \mathcal{L}(w) = \emptyset\}.$$

Now the corollary follows by Lemma 4.13. □

The second cell

In this section we take a closer look at one specific right cell. It is the cell $\mathcal{C}(s_0)$, which consist of all Weyl group elements with a unique reduced expression that begins with s_0. It is a right cell by Theorem 4.18. In fact $\mathcal{C}(s_0)$ is a subset of W^0 and therefore each $x \in \mathcal{C}(s_0)$ corresponds to an alcove $x.C$ in the dominant chamber, see Lemma 4.19.

OBSERVATION 4.21. *If $e \neq x \in W^0$, then $x \leq_{\mathcal{R}} s_0$ by Remark 4.8. It follows that*

$$\mathcal{C} \leq_{\mathcal{R}} \mathcal{C}(s_0) \leq_{\mathcal{R}} \{e\}$$

for all right cells $\mathcal{C}$ inside W^0 and unequal to $\{e\}$. So we see that $\mathcal{C}(s_0)$ is the second largest right cell in W^0. We will, for this reason, refer to $\mathcal{C}(s_0)$ as the second cell *from time to time.*

The aim of this section is to find the reduced expression of each Weyl group element in $\mathcal{C}(s_0)$, for all types of root system. It is possible to do this based only on the description as elements with a unique reduced expression that begins with s_0. We will need some notation: We say that a reduced expression $t_1 \cdots t_k$, $t_i \in S$ contains a *braid-word* if it contains a string $stst\cdots$ with $m(s,t)$ factors ($m(s,t)$ being the order of the element st), $s \neq t \in S$. The unique reduced expressions may now be described as the set of reduced expression without a braid-word.

LEMMA 4.22. *A reduced expression is unique if and only if it does not contain a braid-word.*

PROOF. A reduced expression containing a braid-word is not unique since $stst\cdots = tsts\cdots$ ($m(s,t)$ factors on both sides) in W. The only if part follows from (Tits 1969), which solves the word-problem in Coxeter-groups; (Humphreys 1990, section 8.1) is a convenient reference. □

We will now describe how the reduced expression of elements in $\mathcal{C}(s_0)$ can be obtained inductively. For each $k \geq 0$ we define a set by

$$\mathcal{C}(s_0)_k = \{w \in \mathcal{C}(s_0) \mid l(w) = k\}$$

So we have $\mathcal{C}(s_0)_0 = \emptyset$ and $\mathcal{C}(s_0)_1 = \{s_0\}$.

If $t_1 \cdots t_k$ belongs to $\mathcal{C}(s_0)_k$ then $t_1 \cdots t_{k-1}$ belongs to $\mathcal{C}(s_0)_{k-1}$; it is a direct consequence of Lemma 4.17. It follows that we may obtain $\mathcal{C}(s_0)_{k+1}$ by adding simple reflections to a reduced expression from $\mathcal{C}(s_0)_k$, and then discard any expressions which contain braid words. That is,

$$\mathcal{C}(s_0)_{k+1} = \{t_1 \cdots t_k s \text{ without braid-word } \mid t_1 \cdots t_k \in \mathcal{C}(s_0)_k, s \neq t_k\}.$$

Note that $t_1 \cdots t_k s > t_1 \cdots t_k$ is automatic when $s \neq t_k$. When $st_k = t_k s$ we see that $t_1 \cdots t_k s$ has two reduced expressions. Therefore we need only consider those s, that are connected to t_k in the Coxeter graph.

Type	Reduced expression	$\lvert\mathcal{C}(s_0)\rvert$
A_1	$(s_0s_1)^m$, $m \geq 1$	∞
$A_n, n \geq 2$	$(s_0s_ns_{n-1}\dots s_1)^m$, $m \geq 1$	∞
	$(s_0s_1s_2\dots s_n)^m$, $m \geq 1$	
B_n	$s_0s_1s_0$	∞
	$(s_0s_1s_2\dots s_{n-1}s_ns_{n-1}\dots s_2s_1)^m$, $m \geq 1$	
$C_n, n \geq 3$	$s_0s_2s_1$	$2n+1$
	$s_0s_2s_3\dots s_{n-1}s_ns_{n-1}\dots s_2s_0$	
	$s_0s_2s_3\dots s_{n-1}s_ns_{n-1}\dots s_2s_1$	
D_n	$s_0s_2s_1$	$n+1$
	$s_0s_2s_3\dots s_{n-2}s_{n-1}$	
	$s_0s_2s_3\dots s_{n-2}s_n$	
E_6	$s_0s_2s_4s_3s_1$	7
	$s_0s_2s_4s_5s_6$	
E_7	$s_0s_1s_3s_4s_2$	8
	$s_0s_1s_3s_4s_5s_6s_7$	
E_8	$s_0s_8s_7s_6s_5s_4s_2$	9
	$s_0s_8s_7s_6s_5s_4s_3s_1$	
F_4	$s_0s_4s_3s_2s_1$	8
	$s_0s_4s_3s_2s_3s_4s_0$	
G_2	$s_0s_1s_2s_1s_2s_1s_0$	8
	$s_0s_1s_2s_1s_0$	

TABLE 1. The table lists the reduced expression of elements in $\mathcal{C}(s_0)$. Used together with Lemma 4.17 it yields the reduced expression of all elements of $\mathcal{C}(s_0)$. Note that s_0 corresponds to the highest *short* root in the root system. The relations between the generators are given as Coxeter graphs in Table 2.

Table 1 describes the elements in $\mathcal{C}(s_0)$ for each type of root system. These expressions were first published in (Lusztig 1983). The rest of the section may help the reader that wants to check Table 1. It is not strictly needed.

LEMMA 4.23. *Suppose that our rootsystem is of type $A_{n\geq 2}$, D_n, E_6, E_7, E_8. Then each element in $\mathcal{C}(s_0)$ corresponds to a route in one direction from s_0 in the Coxeter graph.*

PROOF. The assumption amounts to $m(s,t) \leq 3$ for all pairs s,t of simple reflections. A reduced expression $t_1\dots t_k$ from $\mathcal{C}(s_0)$ is unique, hence $m(t_i,t_{i+1}) = 3$ for all i; otherwise t_i and t_{i+1} commute. This shows that t_i and t_{i+1} are connected by a line in the Coxeter graph. This is what we mean by a route in the Coxeter graph: a word $t_1\dots t_k$, where each consecutive pair t_i, t_{i+1} is connected in the Coxeter graph. Further, there is no turning back; if $t_{i+2} = t_i$ for some i then $t_1\cdots t_it_{i+1}t_{i+2}\cdots t_k$ is not unique, hence is not in $\mathcal{C}(s_0)$. It follows that we may travel in one direction only in the Coxeter graph. □

EXAMPLE 4.24 (Type A_n, $n \geq 2$). *Since the Coxeter graph is a ring, there is basicly two ways to find routes begining in s_0: You can go clockwise or counterclockwise. This corresponds to the non-empty words on the form*

$$(s_0s_ns_{n-1}\dots s_1)^m s_0s_n\dots s_k \qquad m \geq 0$$
$$(s_0s_1s_2\dots s_n)^m s_0s_1\dots s_k \qquad m \geq 0$$

No words on this form contain braid-words. So the Weyl group elements corresponding to these words comprise the right cell $\mathcal{C}(s_0)$. Compare with Table 1.

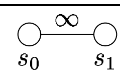
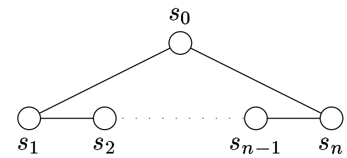
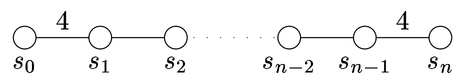
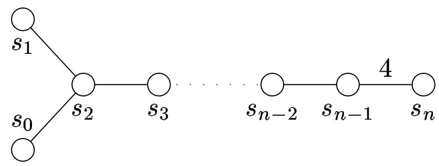
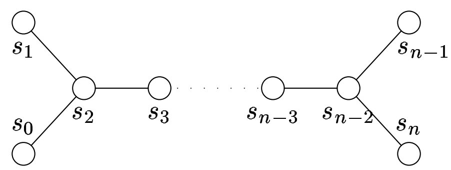
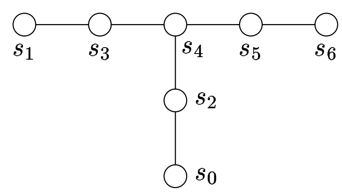
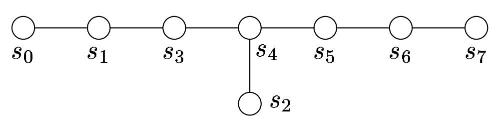
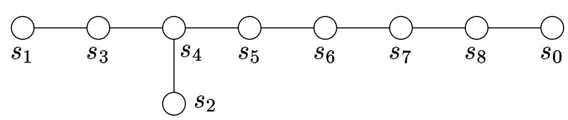
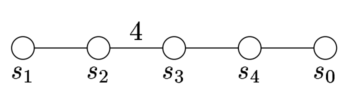
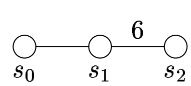
Root system	Coxeter graph of the affine Weyl group
A_1	
$A_n, n \geq 2$	
$B_n, n \geq 2$	
$C_n, n \geq 3$	
$D_n, n \geq 3$	
E_6	
E_7	
E_8	
F_4	
G_2	

TABLE 2. The relations between the generators given as Coxeter graphs. Note that s_0 corresponds to the highest *short* root in the root system.

CHAPTER 5

Character formulae and tensor ideals

This chapter presents the character formulae for indecomposable quantum tilting modules. The characters are expressed by certain basis change coefficients in a Hecke algebra module. We introduce this module, the Hecke module, first and then go on to state Soergel's character formula.

This chapter contains also a classification of tensor ideals of quantum tilting modules. The proof relies on Hecke algebra combinatorics, and may be seen as an application of the right cell theory of Chapter 4 and of the character formula for indecomposable quantum tilting modules. We conclude the chapter by introducing weight cells.

Assume $p \geq h$ throughout this chapter.

The Hecke module

The material in this section may be found in (Soergel 1997).

The finite Weyl group W_0 is a parabolic subgroup of W, so each right coset of $W_0 \setminus W$ has a unique representative of minimal length. We denote the set of these representatives by W^0; multiplication then gives a bijection $W_0 \times W^0 \longleftrightarrow W$. The set W^0 is described thoroughly by Lemma 4.19. In particular, if we associate to each element $x \in W^0$ the alcove $x.C$ as in figure 2,3, and 4 then we get all alcoves in the dominant Weyl chamber.

Let $\mathcal{H}_0$ denote the Hecke algebra of S_0, W_0. It is a subalgebra of $\mathcal{H}$. We have a surjective $\mathbb{Z}[v,v^{-1}]$-algebra homomorphism, $\phi_{-v} : \mathcal{H}_0 \longrightarrow \mathbb{Z}[v,v^{-1}]$, mapping each generator H_{s_i}, $s_i \in S_0$, to $-v$. This gives $\mathbb{Z}[v,v^{-1}]$ a $\mathcal{H}_0$-module structure, and by induction we obtain a right $\mathcal{H}$-module

$$\mathcal{N} = \mathbb{Z}[v,v^{-1}] \otimes_{\mathcal{H}_0} \mathcal{H}.$$

This module is the Hecke module. In Lemma 5.5 below we show that $\mathcal{N}$ may also be constructed as a quotient of $\mathcal{H}$ with a right cell ideal.

We denote by ε the canonical surjection

$$\begin{aligned} \varepsilon : \mathcal{H} &\to \mathcal{N} \\ H &\mapsto 1 \otimes H \end{aligned}$$

Note that for $z \in W_0$, $x \in W^0$ we have $\varepsilon(H_{zx}) = (-v)^{l(z)} N_x$. $\mathcal{N}$ has a basis consisting of $\{N_x = 1 \otimes H_x \mid x \in W^0\}$ and the action of $\mathcal{H}$ is given by

$$N_x \underline{H}_s = \begin{cases} N_{xs} + vN_x & xs > x \text{ and } xs \in W^0; \\ N_{xs} + v^{-1}N_x; & xs < x; \\ 0 & xs \notin W^0; \end{cases} \tag{5.1}$$

The first and second line follows from equation (4.3). The assumption in the last line forces $xs = tx$ for some $t \in S_0$, which gives the formula.

We define an involution on $\mathcal{N}$ by $\overline{a \otimes H} = \overline{a} \otimes \overline{H}$. This involution is $\mathcal{H}$-skewlinear (i.e. $\overline{NH} = \overline{N}\,\overline{H}$ for $N \in \mathcal{N}, H \in \mathcal{H}$). We say that N is selfdual if $\overline{N} = N$. As was the case with $\mathcal{H}$, we want to replace the canonical basis $\{N_x = 1 \otimes H_x \mid x \in W^0\}$ with one of selfdual elements.

THEOREM 5.1. *For each $x \in W^0$ there is a unique selfdual element $\underline{N}_x$ in $\mathcal{N}$ so that*

$$\underline{N}_x \ \in \ N_x + \sum_{y<x} v\mathbb{Z}[v] N_y.$$

PROOF. This theorem is proved by the same method as Theorem 4.1. In particular, the proof is constructive and gives an algorithm to obtain the selfdual element $\underline{N}_x$.

To begin, note that $N_e = 1 \otimes H_e$ is selfdual; this shows the existence part of the theorem for x of length 0.

Given x of length ≥ 1 we choose $s \in S$ so that $xs < x$. By induction we may assume the existence of the selfdual $\underline{N}_{xs} \in N_{xs} + \sum_{y<xs} n_{y,xs} N_y$, with $n_{y,xs} \in v\mathbb{Z}[v]$. Then $\underline{N}_{xs}\underline{H}_s$ is selfdual. We write

$$\underline{N}_{xs}\underline{H}_s = N_x + \sum_{y<x} n_y N_y, \tag{5.2}$$

where $n_y \in \mathbb{Z}[v]$ by (5.1). So constant coefficients may appear in $\underline{N}_{xs}\underline{H}_s$. We fix this with

$$\underline{N}_x = \underline{N}_{xs}\underline{H}_s - \sum_{y<x} n_y(0)\underline{N}_y,$$

which is selfdual, and this shows the existence part for x.

We leave uniqueness to the reader. □

LEMMA 5.2.

$$\varepsilon(\underline{H}_x) = \begin{cases} \underline{N}_x & x \in W^0 \\ 0 & x \notin W^0. \end{cases}$$

PROOF. The first line follows from the fact that $\varepsilon(\underline{H}_x)$ is selfdual and $\varepsilon(\underline{H}_x) \in N_x + \sum_{y\leq x} v\mathbb{Z}[v]\varepsilon(H_y) \subset N_x + \sum_{z\leq x} v\mathbb{Z}[v] N_z$. For the second line, note that $\varepsilon(\underline{H}_{s_i}) = \varepsilon(H_{s_i} + v) = 0$ for $s_i \neq s_0$. When $x \notin W^0$ there is a $s_i \neq s_0$ so that $x > s_i x$. Therefore we may prove the second line by induction via Remark 4.4. □

DEFINITION 5.3. *Let $n_{y,x} = [\underline{N}_x : N_y] \in \mathbb{Z}[v]$ denote the coefficients of the new selfdual basis expressed in the old basis, so that*

$$\underline{N}_x = \sum_{y\leq x} n_{y,x} N_y.$$

REMARK 5.4. *The map ε reveals that the polynomials $n_{y,x}$ and $h_{y,x}$ are related by the formula*

$$n_{y,x} = \sum_{w\in W_0} (-v)^{l(w)} h_{wy,x}. \tag{5.3}$$

Recall that we denote the linear coefficient of $h_{y,x}$ by $\mu(y,x)$. The equation (5.3) shows that the linear coefficient of $n_{y,x}$ is equal to $\mu(y,x)$ for $x, y \in W^0$.

The Hecke algebra acts on the selfdual basis by the following formulae; to see this use ε on the equations in Proposition 4.3.

$$\underline{N}_x\underline{H}_s \ = \ \begin{cases} \underline{N}_{xs} + \sum\limits_{y \leftarrow_{\mathcal{R}} x, y\in W^0} \mu(y,x)\underline{N}_y & xs > x \text{ and } xs \in W^0; \\ \sum\limits_{y \leftarrow_{\mathcal{R}} x, y\in W^0} \mu(y,x)\underline{N}_y & xs > x \text{ and } xs \notin W^0; \\ (v + v^{-1})\underline{N}_x & xs < x. \end{cases} \tag{5.4}$$

We conclude this section with an alternative description of $\mathcal{N}$. Recall the right cell ideals I_x from Chapter 4.

LEMMA 5.5. *ε induces an isomorphism of right $\mathcal{H}$-modules between $\mathcal{H}/\sum_{i\neq 0} I_{s_i}$ and $\mathcal{N}$.*

PROOF. We have $\sum_{i\neq 0} I_{s_i} = \bigoplus_{x\notin W^0} \mathbb{Z}[v,v^{-1}]\underline{H}_x$ and this is the kernel of ε by Lemma 5.2. □

We will need the following analogue of Lemma 4.5.

LEMMA 5.6. *When $xs < x$ and $ys < y$ we have $n_{ys,x} = vn_{y,x}$.*

Soergels Theorem

Our main interest in the Hecke module is Theorem 5.7 below. It relates the structure of quantum tilting modules with combinatorics of the corresponding Hecke algebra. Recall that for a module Q with a Weyl filtration, $[Q : V_q(\lambda)]$ denotes the number of times $V_q(\lambda)$ appears as a quotient in the filtration. Clearly, the numbers $[Q : V_q(\lambda)]$ for all λ determine the character of Q. When Q is a quantum tilting module we let $[Q : T_q(\lambda)]$ denote the number of times $T_q(\lambda)$ appears as a summand of Q; we say that $[Q : T_q(\lambda)]$ is the multiplicity of $T_q(\lambda)$ in Q.

THEOREM 5.7. *Let $p \geq h$. For a weight $\lambda \in C$ and $x, y, z \in W^0$ we have*

(i) $[T_q(x.\lambda) : V_q(y.\lambda)] = n_{y,x}(1)$

(ii) *If $xs > x$ then* $[\Theta_s T_q(x.\lambda) : T_q(z.\lambda)] = [\underline{N}_x \underline{H}_s : \underline{N}_z]$.

The proof of (i) may be found in (Soergel 1998). This result of Soergel relies on an equivalence of categories between affine Lie algebra modules and quantum group modules established in (Kazhdan and Lusztig 1993), (Kazhdan and Lusztig 1994) together with results from (Lusztig 1994) and (Kashiwara and Tanisaki 1996). In some types these results impose mild restrictions on p.

We will use the second part of Theorem 5.7 extensively. It is a straightforward consequence of the first part as shown in the proof below.

PROOF OF (II). Consider a module Q with character $\operatorname{ch} Q = \sum_{y;\ y.\lambda \in X^+} a_{y.\lambda} \chi(y.\lambda)$. Applying the wallcrossing functor we obtain a module with character given by $\operatorname{ch} \Theta_s Q = \sum_{y;\ y.\lambda \in X^+} a_{y.\lambda}(\chi(y.\lambda) + \chi(ys.\lambda))$.

Now consider a Weyl factor $V_q(y.\lambda)$ in $T_q(x.\lambda)$. Suppose first that $ys \in W^0$; then by the first part of the theorem,

$$\begin{aligned} n_{y,x}(1) + n_{ys,x}(1) &= [T_q(x.\lambda) : V_q(y.\lambda)] + [T_q(x.\lambda) : V_q(ys.\lambda)] \\ &= [\Theta_s T_q(x.\lambda) : V_q(y.\lambda)] \\ &= \sum_{z \leq xs} [\Theta_s T_q(x.\lambda) : T_q(z.\lambda)][T_q(z.\lambda) : V_q(y.\lambda)] \\ &= \sum_{z \leq xs} [\Theta_s T_q(x.\lambda) : T_q(z.\lambda)] n_{y,z}(1). \end{aligned}$$

In the first equation below, see Equation (5.1).

$$\underline{N}_x \underline{H}_s = \sum_{\substack{y;\ ys > y, \\ ys \in W^0}} n_{y,x} N_{ys} + v n_{y,x} N_y + n_{ys,x} N_y + v^{-1} n_{y,x} N_{ys}$$

$$\underline{N}_x \underline{H}_s = \sum_{z \leq xs} [\underline{N}_x \underline{H}_s : \underline{N}_z] \sum_{y; ys > y} n_{y,z} N_y + n_{ys,z} N_{ys}.$$

Considering the coefficient $[\underline{N}_x \underline{H}_s : N_y]$ and evaluating in $v = 1$, we find that

$$n_{y,x}(1) + n_{ys,x}(1) = \sum_{z \leq xs} [\underline{N}_x \underline{H}_s : \underline{N}_z] n_{y,z}(1).$$

If $ys \notin W^0$ then $[\Theta_s T_q(x.\lambda) : V_q(y.\lambda)] = 0 = [\underline{N}_x \underline{H}_s : N_y]$. So for all $y, z \leq xs$ we have

$$\sum_{z \leq xs} [\Theta_s T_q(x.\lambda) : T_q(z.\lambda)] n_{y,z}(1) \;=\; \sum_{z \leq xs} [\underline{N}_x \underline{H}_s : \underline{N}_z] n_{y,z}(1).$$

It is now a matter of linear algebra to conclude that $[\Theta_s T_q(x.\lambda) : T_q(z.\lambda)] = [\underline{N}_x \underline{H}_s : \underline{N}_z]$ for all $z \leq xs$. $\square$

The Hecke module at $v = 1$

We continue this chapter with a construction of tensor ideals of quantum tilting modules. This is based on the paper (Ostrik 1997). It illustrates the strength of right cells when applied to the representation theory of quantum groups at a root of unity, through Theorem 5.7. This section prepares the ground for Ostriks tensor ideals in the next section.

Inspired by Theorem 5.7 we begin with the specialization of $\mathcal{N}$ at $v = 1$. Consider $\mathbb{Z}$ as an $\mathbb{Z}[v, v^{-1}]$-module with v acting as multiplication by 1. We then obtain a right $\mathcal{H}$-module by

$$\mathcal{N}^1 = \mathbb{Z} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{N}$$

From the $\mathbb{Z}[v, v^{-1}]$-bases of $\mathcal{N}$ we get two bases of $\mathcal{N}^1$:

$$\{N_x^1 = 1 \otimes N_x \mid x \in W^0\} \qquad \{\underline{N}_x^1 = 1 \otimes \underline{N}_x \mid x \in W^0\}$$

As endomorphisms of $\mathcal{N}^1$ we have $H_x H_y = H_{xy}$ even when $l(x) + l(y) < l(xy)$. Further, the generators act by the formulae below; the first equation follows from the defining relations (4.1) and (4.2) of $\mathcal{H}$ and the second equation follow from equation (5.4).

$$N_x^1 H_s = \begin{cases} N_{xs}^1 & \text{if } xs \in W^0; \\ -N_x^1 & \text{if } xs \notin W^0; \end{cases} \tag{5.5}$$

$$\underline{N}_x^1 \underline{H}_s = \begin{cases} \underline{N}_{xs}^1 + \sum_{y \leftarrow_{\mathcal{R}} x,\ y \in W^0} \mu(y,x) \underline{N}_y^1 & xs > x,\ xs \in W^0; \\ \sum_{y \leftarrow_{\mathcal{R}} x,\ y \in W^0} \mu(y,x) \underline{N}_y^1 & xs > x,\ xs \notin W^0; \\ 2\underline{N}_x & xs < x. \end{cases} \tag{5.6}$$

A *right cell submodule* of $\mathcal{N}^1$ is a right $\mathcal{H}$-submodule with a basis consisting of $\{\underline{N}_y \mid y \in Y\}$ for some subset Y of W. The following lemma describes all of them.

LEMMA 5.8. *Let $\mathcal{C} \subset W^0$ denote a right cell, and let $x \in \mathcal{C}$. Then*

$$\mathcal{I}_{\mathcal{C}} = \bigoplus_{\substack{y \leq_{\mathcal{R}} \mathcal{C}, \\ y \in W^0}} \mathbb{Z}[v, v^{-1}] \underline{N}_y^1$$

is a right cell submodule and $\mathcal{I}_{\mathcal{C}}$ is the smallest right cell submodule to contain $\underline{N}_x^1$. A right cell submodule is a sum of such $\mathcal{I}_{\mathcal{C}}$, and a sum of such $\mathcal{I}_{\mathcal{C}}$ is a right cell submodule.

REMARK 5.9. *Note that the set $\{\ y \in W^0 \mid y \leq_{\mathcal{R}} \mathcal{C}\ \}$ is a union of right cells; it is equal to the union of all $\mathcal{C}' \leq_{\mathcal{R}} \mathcal{C}$.*

PROOF OF LEMMA 5.8. Equation (5.6) shows that $\mathcal{I}_{\mathcal{C}}$ is stable under the generators $\underline{H}_s$ of $\mathcal{H}$. So each $\mathcal{I}_{\mathcal{C}}$ is a right cell submodule.

Suppose that $y \in W^0$ and $y \leq_{\mathcal{R}} \mathcal{C}$. We will show that $\underline{N}_y^1$ belongs to any right cell submodule containing $\underline{N}_x^1$. There is a chain $y = w_0 \leftarrow_{\mathcal{R}} w_1 \cdots \leftarrow_{\mathcal{R}} w_n = x$, where $w_i \in W$ to begin with. But in fact $w_i \in W^0$, as $\mathcal{L}(w_i) \subset \mathcal{L}(y) = \{s_0\}$ by Lemma 4.13. Now $w_i \leftarrow_{\mathcal{R}} w_{i+1}$ means that $[\underline{H}_{w_{i+1}} \underline{H}_s : \underline{H}_{w_i}] \neq 0$ for some s; further

$$[\underline{H}_{w_{i+1}} \underline{H}_s : \underline{H}_{w_i}] \ = \ [\underline{N}_{w_{i+1}}^1 \underline{H}_s : \underline{N}_{w_i}^1]$$

by equation (5.6). This shows that each $\underline{N}_{w_i}^1$ must be part of any basis of a right cell submodule containing $\underline{N}_x^1$. We have shown that $\mathcal{I}_{\mathcal{C}}$ is the smallest right cell submodule containing $\underline{N}_x^1$.

We leave the remaining assertions to the reader. ◻

Ostrik's tensor ideals

Inside the category $\mathcal{C}_q$ of finite dimensional $\mathcal{U}_q$-modules we have the full subcategory of tilting modules $\mathcal{T}_q$.

DEFINITION 5.10. *A subset $\tau \subset \mathcal{T}_q$ is a tensor ideal if it satisfies the following conditions.*

(i) *For any Q_1 in τ and any tilting module Q, the tensor product $Q_1 \otimes Q$ belongs to τ.*
(ii) *If Q is a summand in Q_1 and Q_1 belongs to τ, then Q belongs to τ.*

These two properties are sufficient to ensure that tensor ideals are stable under the translation functors.

Consider a tensor ideal $\tau \subset \mathcal{T}_q$; it follows from the defining properties that τ is spanned by a set of indecomposable tilting modules $\{T_q(\lambda) \mid \lambda \in \Lambda\}$ (Λ denoting a subset of the dominant weights) in the sense that all modules in τ are direct sums of these indecomposables. What can we say about Λ? What are the possible subsets $\Lambda \subset X^+$?

LEMMA 5.11. *Let $\tau \subset \mathcal{T}_q$ denote a tensor ideal. Then $T_q(x.0)$ belongs to τ if and only if $T_q(x.\lambda)$ belongs to τ whenever $x.\lambda$ is a dominant weight in the lower closure of the alcove $x.C$.*

PROOF. Recall that tensor ideals are stable under the translation functors. The claim follows from (Soergel 1997, Remark 7.2 2.): When $x.\lambda$ belongs to the lower closure of the alcove $x.C$, we have

$$\begin{aligned} T_0^\lambda T_q(x.0) &\simeq T_q(x.\lambda)^{\oplus |W_\lambda|} \\ T_\lambda^0 T_q(x.\lambda) &\simeq T_q(x.0). \end{aligned}$$

Here W_λ denotes the stabilizer of λ. $\square$

It follows that all tensor ideals are spanned by sets $\{T_q(\lambda) \mid \lambda \in \Lambda\}$ where Λ is the union of the set of dominant weights in lower closures of alcoves.

COROLLARY 5.12. *To each tensor ideal τ in $\mathcal{T}_q$ we may associate a subset $\mathcal{C}(\tau) \subset W^0$ so that the indecomposable tilting modules in τ are all $T_q(\lambda)$ with λ in the lower closure of one of the alcoves $y.C$, $y \in \mathcal{C}(\tau)$.*

PROPOSITION 5.13. *Let τ denote a tensor ideal in $\mathcal{T}_q$ and let $\mathcal{C}(\tau) \subset W^0$ denote the associated set of Weyl group elements. Then $\oplus_{x\in\mathcal{C}(\tau)}\mathbb{Z}\underline{N}_x^1$ is a right cell submodule in $\mathcal{N}^1$, and $\mathcal{C}(\tau)$ is a union of right cells in W^0.*

PROOF. Let $x \in \mathcal{C}(\tau)$ and assume that $[\underline{N}_x^1\underline{H}_s : \underline{N}_y^1] \neq 0$. We must show that $y \in \mathcal{C}(\tau)$, as this demonstrates that $\oplus_{x\in\mathcal{C}(\tau)}\mathbb{Z}\underline{N}_x^1$ is stable under the generators $\underline{H}_s$ of $\mathcal{H}$. By definition of $\mathcal{C}(\tau)$ we know that $T_q(x.0)$ belongs to the tensor ideal τ. To show the proposition we need only verify that $T_q(y.0)$ belongs to τ. This is immediate from

$$[\Theta_s T_q(x.0) : T_q(y.0)] = [\underline{N}_x\underline{H}_s : \underline{N}_y] = [\underline{N}_x^1\underline{H}_s : \underline{N}_y^1].$$

The first equality above follows from Theorem 5.7 and the second by comparing the equations (5.4) and (5.6). The last assertion in the proposition follows from Lemma 5.8 and Remark 5.9. $\square$

By now we have seen that a tensor ideal in $\mathcal{T}_q$ corresponds to a right cell submodule. Further the right cell submodules separate tensor ideals, in the sense that two tensor ideals are different if and only if they correspond to different right cell submodules. This takes us some way toward a classification of tensor ideals in $\mathcal{T}_q$ as Lemma 5.8 provide us with a classification of right cell submodules in terms of right cells.

It remains to construct a tensor ideal from each right cell submodule. It suffices to do this for the minimal right cell submodules $\mathcal{J}_C$. This occupies the rest of this section.

We begin by connecting the ring of characters with the module $\mathcal{N}^1$. Recall that the characters of the Weyl modules $\{\chi(\lambda) \mid \lambda \in X^+\}$ form a basis of the character ring. Then set

$$\begin{aligned} \alpha_0 :\ \mathbb{Z}[X]^{W_0} &\to \mathcal{N}^1 \\ \chi(\lambda) &\mapsto \begin{cases} 0 & \text{if } \lambda \notin W.0 \\ N_x^1 & \text{if } \lambda = x.0 \text{ for some } x \in W^0. \end{cases} \end{aligned}$$

REMARK 5.14.

(i) *This map is well defined as* 0 *is a regular weight when* $p \geq h$.
(ii) *By Theorem 5.7 we have* $\alpha_0(T_q(x.0)) = \underline{N}_x^1$. *Note that we should really write* $\alpha_0(\operatorname{ch} T_q(x.0))$, *but this should not cause confusion.*

The following lemmata give two important properties of the map α_0. The first lemma is almost trivial, but the second is rather technical.

LEMMA 5.15. *Let* $w \in W$. *Then*

$$\alpha_0(\chi(w.0)) \ = \ \alpha_0(\chi(0))H_w.$$

PROOF. Decompose $w = w_0 x$ with $w_0 \in W_0$ and $x \in W^0$ satisfying $l(w) = l(w_0) + l(x)$. Then, using formula (5.5) in the second line

$$\begin{aligned} \alpha_0(\chi(w.0)) = \alpha_0((-1)^{l(w_0)}\chi(x.0)) \ &= \ (-1)^{l(w_0)}N_x^1 \\ \alpha_0(\chi(0))H_w = N_e^1 H_{w_0 x} \ = \ (-1)^{l(w_0)}N_e^1 H_x \ &= \ (-1)^{l(w_0)}N_x^1 \end{aligned}$$

□

The following lemma from (Ostrik 1997) is the key result needed in the proof of Proposition 5.18; there we show that a minimal right cell submodule $\mathcal{I}_C$ corresponds to a tensor ideal.

LEMMA 5.16. *Let* $\lambda, \mu \in \overline{C}$ *and assume that* $z.\lambda$ *is a dominant weight. For any module* Q *in* $\mathcal{C}_q$, *there is an* $H_Q \in \mathcal{H}$ *so that*

$$\alpha_0(T_\mu^0(T_q(z.\lambda) \otimes Q)) = \alpha_0(T_\lambda^0 T_q(z.\lambda))H_Q$$

PROOF. Write $\operatorname{ch} T_q(z.\lambda) = \sum_{w \in W} a_w \chi(w.\lambda)$. Then by (Jantzen 1987, II.7.5b) we have

$$\operatorname{ch} \operatorname{pr}_\mu T_q(z.\lambda) \otimes Q \quad = \quad \sum_{w \in W} a_w \sum_{\nu, \nu + \lambda \in W.\mu} \dim(Q_\nu)\chi(w.(\lambda + \nu))$$

Consider the following set of Weyl group elements:

$$P = \{x \in W \mid \nu + \lambda = x.\mu,\ \dim Q_\nu \neq 0\}$$

This set is clearly stable under multiplication from the right with $w_1 \in W_\mu$. It is also stable under multiplication from the left by $w_2 \in W_\lambda$: We have $w_2 x.\mu = w_2.(\lambda + \nu) = w_2.\lambda + w'\nu$ for some w' in the finite Weyl group, and $\dim Q_\nu = \dim Q_{w'\nu}$.

The actions of the groups W_μ (from the right) and W_λ (from the left) are simply transitively. The orbits of the W_μ-action are indexed by the weights of Q, so we choose for each weight ν of Q an element w_ν so that $w_\nu W_\mu.\mu = \lambda + \nu$. Let $w^{(i)}$, $i = 1, \ldots, l$ denote a set of representatives from the orbits of the W_λ-action. Then P may be described as

$$P = \{w_\nu w_1 \mid w_1 \in W_\mu,\ \dim Q_\nu \neq 0\} = \{w_2 w^{(i)} \mid w_2 \in W_\lambda,\ i = 1, \ldots, l\}.$$

We may now continue the calculation. We have

$$\operatorname{ch} \operatorname{pr}_\mu T_q(z.\lambda) \otimes Q \quad = \quad \sum_{w \in W} a_w \sum_{\nu, \nu + \lambda = w_\nu.\mu} \dim(Q_\nu)\chi(w w_\nu.\mu).$$

This gives

$$\operatorname{ch} T_\mu^0 T_q(z.\lambda) \otimes Q = \sum_{w \in W} a_w \sum_{\nu, \nu + \lambda = w_\nu.\mu} \dim(Q_\nu) \sum_{w_1 \in W_\mu} \chi(w w_\nu w_1.0).$$

We use Lemma 5.15 in the following calculation.

$$\begin{aligned}
\alpha_0(T_\mu^0 T_q(z.\lambda)\otimes Q) &= \sum_{w\in W} a_w \sum_{\nu,\nu+\lambda=w_\nu.\mu} \dim(Q_\nu) \sum_{w_1\in W_\mu} \alpha_0\chi(ww_\nu w_1.0) \\
&= \alpha_0(\sum_{w\in W} a_w\chi(w.0))(\sum_{\nu,\nu+\lambda=w_\nu.\mu} \dim(Q_\nu) \sum_{w_1\in W_\mu} H_{w_\nu w_1}) \\
&= \alpha_0(\sum_{w\in W} a_w\chi(w.0))(\sum_i \sum_{w_2\in W_\lambda} \dim(Q_\nu) H_{w_2 w^{(i)}}) \\
&= \alpha_0(\sum_{w_2\in W_\lambda}\sum_{w\in W} a_w\chi(ww_2.0))(\sum_i \dim(Q_\nu) H_{w^{(i)}}) \\
&= \alpha_0(T_\lambda^0 T_q(z.\lambda))(\sum_i \dim(Q_\nu) H_{w^{(i)}}).
\end{aligned}$$

Here the last line follows as

$$\operatorname{ch} T_\lambda^0 T_q(z.\lambda) = \sum_{w\in W} a_w \sum_{w_2\in W_\lambda} \chi(ww_2.0).$$

□

REMARK 5.17. *In Lemma 5.16 we may replace $T_q(z.\lambda)$ by any other module Q' with* $\mathrm{pr}_\lambda Q' = Q'$.

Proposition 5.13 and Corollary 5.12 states that each tensor ideal in $\mathcal{T}_q$ determines a right cell submodule in $\mathcal{N}^1$, and that two different tensor ideals are mapped to different right cell submodules. In the other direction, Proposition 5.18 shows that each of the minimal right cell modules $\mathcal{J}_C$ from Lemma 5.8 corresponds to a tensor ideal.

PROPOSITION 5.18. *Let $C\subset W^0$ denote a right cell. The indecomposable tilting modules with highest weight in the lower closure of an alcove $x.C$, $x\le_{\mathcal{R}} C$ span a tensor ideal.*

PROOF. Let $\tau(C)$ denote the set of all indecomposable tilting modules with highest weight in the lower closure of one of the alcoves $x.C$, $x\le_{\mathcal{R}} C$. Recall the map α_0 from the ring of characters to $\mathcal{N}^1$ defined above. We first show that the map α_0 reveals whether a given indecomposable tilting module belongs to $\tau(C)$ or not. Suppose that $T_q(y.\mu)$ is such a module, with $\mu\in\overline{C}$ and y chosen so that $y.\mu$ belongs to the lower closure of the alcove $y.C$. Then

$$\alpha_0(T_\mu^0 T_q(y.\mu)) = \alpha_0(T_q(y.0)) = \underline{N}_y^1$$

We see that $T_q(y.\mu)$ belongs $\tau(C)$ if and only if it is mapped to $\mathcal{J}_C\subset\mathcal{N}^1$ by $\alpha_0(T_\mu^0 -)$.

Now assume that $x\le_{\mathcal{R}} C$. Let Q denote an arbitrary tilting module and consider the tensor product $T_q(x.\lambda)\otimes Q$. We must show that each summand $T_q(y.\mu)$ belongs to $\tau(C)$. By the discussion above this is equivalent to $\alpha_0(T_\mu^0(T_q(x.\lambda)\otimes Q))\in\mathcal{J}_C$ for all $\mu\in\overline{C}$. Now Lemma 5.16 shows that

$$\alpha_0(T_\mu^0(T_q(x.\lambda)\otimes Q) = \alpha_0(T_\lambda^0 T_q(x.\lambda))H,$$

for some $H\in\mathcal{H}$. As $\alpha_0(T_\lambda^0 T_q(x.\lambda))H = \underline{N}_x^1 H$ and as $\mathcal{J}_C$ is a $\mathcal{H}$-module we find that $\alpha_0(T_\mu^0(T_q(x.\lambda)\otimes Q))\in\mathcal{J}_C$ as needed. We are done. □

Denote by $\tau(C)$ the tensor ideal corresponding to the right cell C, i.e. let $\tau(C)$ denote the tensor ideal spanned by all indecomposable quantum tilting modules with highest weight in the lower closure of an alcove $x.C$ with $x\le_{\mathcal{R}} C$. We may now classify the tensor ideals in $\mathcal{T}_q$.

THEOREM 5.19. *Let $C\subset W^0$ denote a right cell and let $x\in C$. The tensor ideal $\tau(C)$ is the smallest tensor ideal to contain $T_q(x.0)$. A tensor ideal is a union of such $\tau(C)$, and a union of such $\tau(C)$ is a tensor ideal.*

PROOF. Let us consider the smallest tensor ideal containing $T_q(x.0)$. By Corollary 5.12 and Proposition 5.13 we may associate to it a subset $Y \subset W^0$, so that $\oplus_{y\in Y}\mathbb{Z}\underline{N}_y^1$ is a right cell submodule of $\mathcal{N}^1$. This submodule contains $\underline{N}_x^1$, so it contains also $\mathcal{J}_C$, where $C \subset W^0$ is the right cell containing x. We have shown that $C \subset Y$. Hence any tensor ideal containing $T_q(x.0)$ must contain $\tau(C)$. Further, $\tau(C)$ is a tensor ideal by Proposition 5.18; we see that $\tau(C)$ is the smallest tensor ideal to contain $T_q(x.0)$.

The last assertions are clear. □

Weight cells

In the previous sections we have seen that there is a close connection between right cell ideals in $\mathcal{H}$ and right cell submodules of $\mathcal{N}^1$ on one side and tensor ideals of quantum tilting modules on the other side. Weyl group elements that generate the same right cell ideal or right cell submodule are said to belong to the same right cell. There is a similar notion in the category of quantum tilting modules. The highest weights in tilting modules that generate the same tensor ideal are said to belong to the same weight cell.

DEFINITION 5.20. *(Ostrik 2001) Write $\mu \leq_{\mathcal{T}_q} \lambda$ if $[T_q(\lambda)\otimes Q : T_q(\mu)] \neq 0$ for some quantum tilting module Q.*

REMARK 5.21. *Note that $\leq_{\mathcal{T}_q}$ is a preorder, since $[T_q(\lambda)\otimes Q_1 : T_q(\mu)] \neq 0$ and $[T_q(\nu)\otimes Q_2 : T_q(\lambda)] \neq 0$ gives $[T_q(\nu)\otimes Q_1\otimes Q_2 : T_q(\mu)] \neq 0$. Note that $Q_1\otimes Q_2$ is tilting because a tensor product of tilting modules is tilting.*

DEFINITION 5.22. *Let $\sim_{\mathcal{T}_q}$ be the equivalence relation defined by $\leq_{\mathcal{T}_q}$. The equivalence classes of $\sim_{\mathcal{T}_q}$ are called* weight cells. *The preorder $\leq_{\mathcal{T}_q}$ induces a partial order (also denoted $\leq_{\mathcal{T}_q}$) on the set of weight cells in the natural way.*

To ease the notation, we write $\check{A}$ for the lower closure of an alcove A.

REMARK 5.23.

(i) *If $\mu = \lambda + \nu$ and all three weights are dominant then $\mu \leq_{\mathcal{T}_q} \lambda$ as weight considerations show that $T_q(\mu)$ is a summand of $T_q(\lambda)\otimes T_q(\nu)$.*

(ii) *If $T_q(\mu)$ is a summand in a translation or a wallcrossing of $T_q(\lambda)$ then $\mu \leq_{\mathcal{T}_q} \lambda$.*

(iii) *Let $x \in W^0$ and assume that λ, $\mu \in x.\check{C} \cap X^+$. Then (the proof of) Lemma 5.11 shows that $\lambda \sim_{\mathcal{T}_q} \mu$. So a weight cell consists of all dominant weights in the lower closure of $x.C$, x in some subset of W^0.*

(iv) *If $x, y \in W^0$ and $y \leftarrow_{\mathcal{R}} x$ then $y.0 \leq_{\mathcal{T}_q} x.0$ since $[\Theta_s T_q(x.0) : T_q(y.0)] = [\underline{N}_x\underline{H}_s : \underline{N}_y] \neq 0$. This immediately give us that $y \leq_{\mathcal{R}} x$ implies $y.0 \leq_{\mathcal{T}_q} x.0$ and that $y \sim_{\mathcal{R}} x$ implies $y.0 \sim_{\mathcal{T}_q} x.0$*

THEOREM 5.24. *There is a one-to-one correspondence between the right cells in W^0 and weight cells in X^+. The image of a right cell is given by*

$$\mathcal{C} \longmapsto \bigcup_{x\in\mathcal{C}} x.\check{C} \cap X^+.$$

This correspondence is order preserving.

PROOF. We first show that $\cup_{x\in\mathcal{C}} x.\check{C} \cap X^+$ is a weight cell, for a right cell $\mathcal{C}$. Let z, $y \in \mathcal{C}$ and let $y.\lambda \in y.\check{C} \cap X^+$, $z.\mu \in z.\check{C} \cap X^+$. We have

$$z.\mu \sim_{\mathcal{T}_q} z.0 \sim_{\mathcal{T}_q} y.0 \sim_{\mathcal{T}_q} y.\lambda,$$

where the first and last $\sim_{\mathcal{T}_q}$ follows from Remark 5.23 (iii) and the second follows from (iv) as $x \sim_{\mathcal{R}} y$.

To complete the proof we must show that $y.\lambda \in y.\check{C}$, $z.\mu \in z.\check{C}$ with $z.\mu \sim_{\mathcal{T}_q} y.\lambda$ gives $z \sim_{\mathcal{R}} y$. First of all, we have $z.0 \sim_{\mathcal{T}_q} y.0$ as before. This means that $T_q(z.0)$ belongs to all tensor ideals containing $T_q(y.0)$; by Theorem 5.19 we conclude that $z \leq_{\mathcal{R}} y$. We get in the same way $y \leq_{\mathcal{R}} z$ and hence $z \sim_{\mathcal{R}} y$.

Finally, it follows from Remark 5.23 (iv) that this mapping respects the order of the sets. $\square$

We will denote the weight cell corresponding to the 'first' right cell $\{e\}$ by $\underline{c}_1$; note that $\underline{c}_1 = C$. The weight cell corresponding to the second right cell $\mathcal{C}(s_0)$ is similarly denoted by $\underline{c}_2$; it is the second largest weight cell. In the theorem below we identify the minimal weight cell which we will denote by $\underline{c}_{\mathrm{St}}$. Figure 5, 6, and 7 in Chapter 6 hold pictures of the weight cells in type A_2, B_2, and G_2.

THEOREM 5.25. *(Andersen 2001a)* $\mathrm{St}+X^+$ *is a weight cell. It is minimal.*

PROOF. The proof relies on a complete description of the injective indecomposable tilting modules in $\mathcal{T}_q$, given in (Andersen 2001b). We have

$$T_q(\lambda) \text{ injective} \quad \Longleftrightarrow \quad \lambda \in \mathrm{St}+X^+.$$

Let us first see that $T_q(\mathrm{St})$ is in every tensor ideal of tilting modules. The dual of a tilting module is tilting, since the dual of Weyl module is an induced module and vice versa. Further, the dual of a indecomposable module is indecomposable, as $\mathrm{End}(Q) \simeq \mathrm{End}(Q^*)$ for all modules. We thus find that $T(\lambda)^* \simeq T(-w_0\lambda)$, since the highest weight of $T(\lambda)^*$ is $-w_0\lambda$. Then

$$\mathrm{Hom}(T_q(\mathrm{St}), T_q(\lambda) \otimes T_q(-w_0\lambda + \mathrm{St})) \simeq \mathrm{Hom}(T_q(\mathrm{St}) \otimes T_q(-w_0\lambda), T_q(-w_0\lambda + \mathrm{St})).$$

The last space is nonzero since the second module is a summand in the first by weight considerations. Then $T_q(\mathrm{St})$ is a summand in $T_q(\lambda) \otimes T_q(-w_0\lambda + \mathrm{St})$ as $T_q(\mathrm{St})$ is simple and injective. It follows that $\mathrm{St} \leq_{\mathcal{T}_q} \lambda$ for all dominant weights. So St is a minimal element in the preorder $\leq_{\mathcal{T}_q}$.

We have seen that St belongs to the minimal weight cell. By Remark 5.23 (i) we find that $\mathrm{St}+X^+$ is contained in this minimal weight cell. It remains to see that $\lambda \leq_{\mathcal{T}_q} \mathrm{St}$ implies $\lambda \in \mathrm{St}+X^+$.

Suppose that $T_q(\lambda)$ is a summand in a $T_q(\mathrm{St}) \otimes Q$ with Q in $\mathcal{T}_q$. The tensor product is injective since $T_q(\mathrm{St})$ is. Then $T_q(\lambda)$ is necessarily injective as it is a summand in an injective module. It follows that $\lambda \in \mathrm{St}+X^+$. $\square$

CHAPTER 6

Results

In this chapter we investigate the structure of modular tilting modules with highest weight in the second weight cell. The second weight cell is the image of the right cell $\mathcal{C}(s_0)$ under the correspondence of Theorem 5.24. The weight cell decompositions of the dominant weights for root systems of type A_2, B_2 and G_2 are shown in Figure 5, 6, and 7. All weights in an open alcove is contained in the same weight cell, and alcoves are colored after which weight cell its weights belong to. A non-regular weight lies in the lower closure of one uniquely determined alcove, and belongs to the same weight cell as the weights of this alcove.

The main results in this chapter is derived from right cell properties of $\mathcal{C}(s_0)$. Theorem 5.7 provides a link to the representation theory of quantum groups at a root of unity, and an analysis of quantized tilting modules allows us to draw conclusions as to the structure of modular tilting modules.

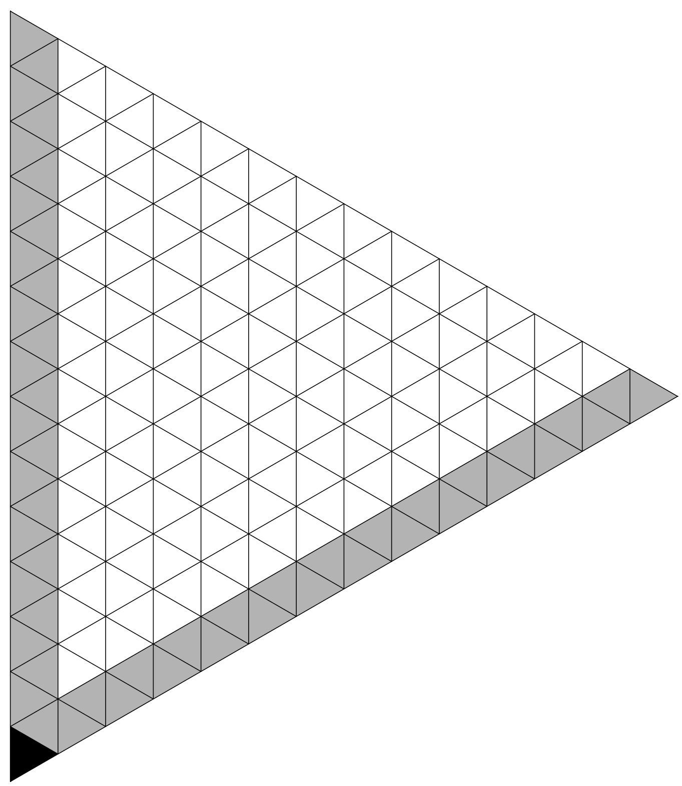


FIGURE 5. The weight cells in type A_2. The maximal weight cell $\underline{c}_1$ is black. The second weight cell $\underline{c}_2$ is grey. The minimal weight cell $\underline{c}_{\mathrm{St}}$ is white.

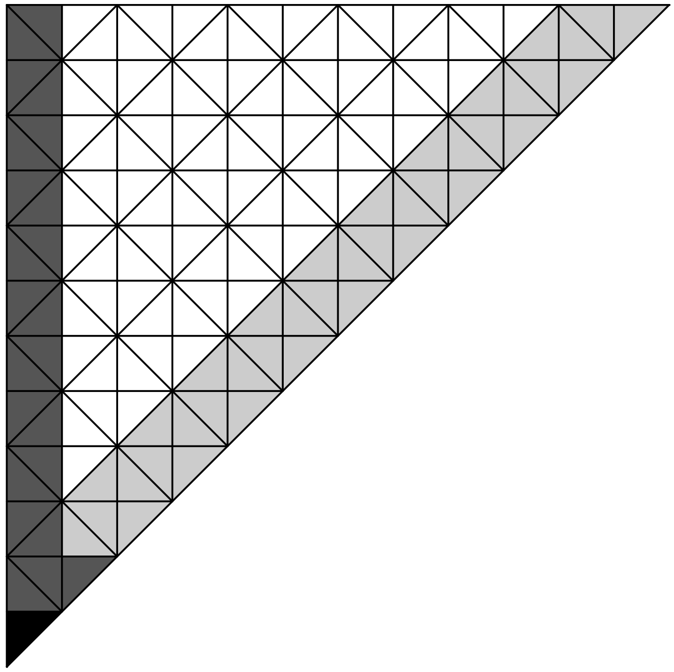


FIGURE 6. The weight cells in type B_2. The maximal weight cell $\underline{c}_1$ is black. The second weight cell $\underline{c}_2$ is dark grey. The minimal weight cell $\underline{c}_{\mathrm{St}}$ is white.

The proofs in this chapter rely on the wallcrossing functors to faciliate induction. Therefore we assume throughout that $p \geq h$, h denoting the Coxeter number of G. This ensures that every alcove and every wall contain a weight. We repeat the assumption $p \geq h$ in the statement of theorems only.

Wallcrossing a quantum tilting module

Fix a $\lambda \in C$. We write $T(x)$ for $T(x.\lambda)$ (with a similar convention for quantum tilting modules) throughout this chapter. Divide W^0 into three disjoint sets:

$$W^0 \quad = \quad \{e\} \cup C(s_0) \cup R.$$

That is, R is the union of the remaining right cells.

We begin with a few basic properties of quantum tilting modules.

PROPOSITION 6.1. *Let $x \in W^0$ and $s \in S$ so that $xs > x$. Let $y \in W^0$.*

(i) $[\Theta_s T_q(x) : T_q(y)] \neq 0$ *implies $ys < y$,*
(ii) $[\Theta_s T_q(x) : T_q(y)] \neq 0$ *implies $y \leq_{\mathcal{R}} x$,*
(iii) *Let $t \in S$ so that $xt < x$. Then*

$$[\Theta_s T_q(x) : T_q(xt)] = \begin{cases} 1 & \text{if } xts < xt; \\ 0 & \text{if } xts > xt. \end{cases}$$

PROOF. All three claims follows from Theorem 5.7. From (5.4) in Chapter 4 we see that $[\underline{N}_x \underline{H}_s : \underline{N}_y] \neq 0$ implies even that $y \leftarrow_{\mathcal{R}} x$ which in turn means that $ys < y$ (by Definition 4.7 and Proposition 4.3). This verifies (i) and (ii).

In (iii), the zero-part follows from (i). To see the other part, we note that $n_{xt,x} = v$ by Lemma 5.6. Using that $xts < xt$ and equation (5.1) we find $[\underline{N}_x \underline{H}_s : \underline{N}_{xt}] = (v^{-1} n_{xt,x} + n_{xts,x})|_{v=0} = 1$. $\square$

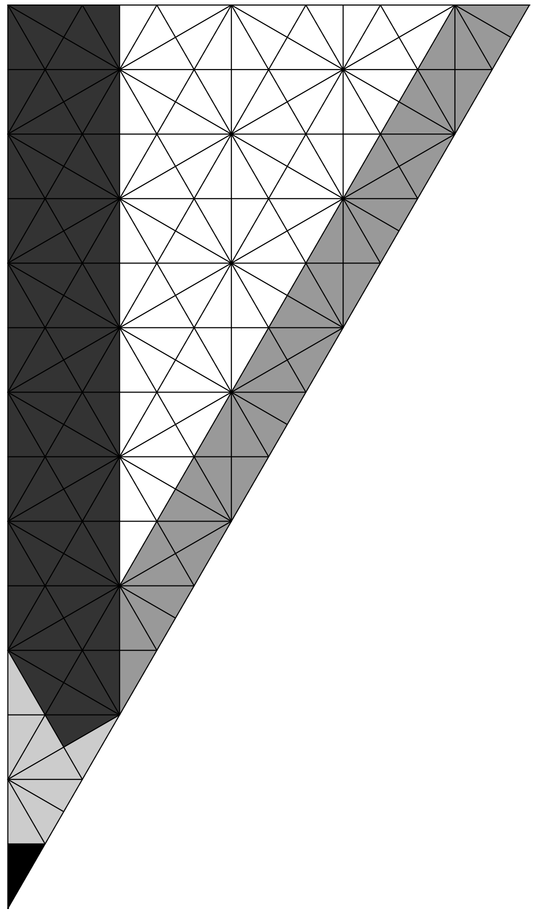


FIGURE 7. The weight cells in type G_2. The maximal weight cell $\underline{c}_1$ is black. The second weight cell $\underline{c}_2$ is light grey. The minimal weight cell $\underline{c}_{\mathrm{St}}$ is white.

The following result describes explicitly when a second cell tilting module splits off. It is the key to almost all results in this chapter.

PROPOSITION 6.2. *Let $x \in W^0$ and $s \in S$ so that $xs > x$. Suppose $y \in \mathcal{C}(s_0)$. For $[\Theta_s T_q(x) : T_q(y)] \neq 0$ it is necessary and sufficient that*

(i) *$y = xt$ for some $t \in S$ with $xts < xt$, and*
(ii) *$x \in \mathcal{C}(s_0) \cup \{e\}$.*

If these conditions are satisfied, we have $[\Theta_s T_q(x) : T_q(y)] = 1$.

PROOF. The conditions (i) and (ii) are sufficient: Assume $y = xt$. If $xt < x$ then $xts < xt$ implies $[\Theta_s T_q(x) : T_q(y)] = 1$ by Proposition 6.1 (iii); if $xt > x$ then x is the unique minimal element in the coset $x < s,t >$, so $xts < xt$ then shows that $xts = x$, hence $s = t$ and $y = xs$ follows.

It remains to show that both claims are also necessary. First note that $[\Theta_s T_q(x) : T_q(y)] \neq 0$ implies $y \leq_{\mathcal{R}} x$. But y belongs to $\mathcal{C}(s_0)$, so $x \in \mathcal{C}(s_0)$ or $x = e$ by Observation 4.21.

We now prove that $y = xt$ for some $t \in S$. If $y > x$ then $y = xs$. If $y < x$ then $\mu(y,x) \neq 0$ and $ys < y$. Pick a $t \in S$ so that $xt < x$. Then $yt > y$ because $y \in \mathcal{C}(s_0)$ shows that s is unique with the property $ys < y$. From Lemma 5.6 we find that $n_{yt,x}$ has a constant term. We conclude that $x = yt$.

Finally the necessity of $xts < xt$ follows from Proposition 6.1 (i). □

Based on Proposition 6.2 we give the following theorem. We will refer to it numerous times throughout this chapter.

THEOREM 6.3. *Let $p \geq h$.*

(i) *Let $z \in \mathcal{C}(s_0)$ and $zs > z$. There is a unique $t \in S$ so that $zt < z$, and nonnegative integers a_y so that*

$$\Theta_s T_q(z) \quad = \quad \varepsilon T_q(zs) \oplus \delta T_q(zt) \oplus \bigoplus_{y \in R} a_y T_q(y).$$

Here $\varepsilon = \begin{cases} 1 & zs \in W^0 \\ 0 & zs \notin W^0 \end{cases}$ *and* $\delta = \begin{cases} 1 & zts < zt \\ 0 & zts > zt \end{cases}$.

(ii) *Assume type $A_{n\geq 2}$, D_n, E_6, E_7, E_8. Let $z < zs \in \mathcal{C}(s_0)$. Then for some nonnegative integers a_y*

$$\Theta_s T_q(z) \quad = \quad \varepsilon T_q(zs) \oplus \bigoplus_{y \in R} a_y T_q(y).$$

Here ε is 1 when $zs \in W^0$ and 0 when $zs \notin W^0$.

(iii) *All types. Let $e \neq x \notin \mathcal{C}(s_0)$. Then for some nonnegative integers a_y*

$$\Theta_s T_q(x) \quad = \quad \bigoplus_{y \in R} a_y T_q(y).$$

PROOF. The first and the last assertion follow from Proposition 6.2 and Proposition 6.1 (ii).

To see the second, lets us assume that $zs \in \mathcal{C}(s_0)$ with $zs > z > zt > zts$ so that $\delta = 1$. Then zs has a reduced expression that ends with sts. This reduced expression is unique since $zs \in \mathcal{C}(s_0)$, and we see that $m(s,t) \geq 4$, where $m(s,t)$ denotes the order of st. So $\delta = 1$ happens only in types with a pair of simple reflections s,t with $m(s,t) \geq 4$. □

Comparing quantum and modular tilting modules

For completeness we begin with results about the "first" cell $\{e\}$.

LEMMA 6.4.

(i) *For any $x \in W^0$ we have $[\Theta_s T_q(x) : T_q(e)] = 0$.*

(ii) *Let Q be a modular tilting module. Then*

$$[Q : T(e)] \quad = \quad [Q_q : T_q(e)].$$

PROOF. Recall (from Corollary 4.14) that e is maximal in the preorder $\leq_{\mathcal{R}}$. Now (i) follows directly from Proposition 6.1 (ii).

All tilting modules are a direct sum of indecomposable tilting modules. In (ii) it suffices to show that $[T(x)_q : T_q(e)] = 0$ for all $x \neq e$. This follows by induction using the first part of the lemma. □

REMARK 6.5. *There is a well known and explicit formula for the number $[Q : T(e)] = [Q_q : T_q(e)]$. The following formulae may be found in (Andersen and Paradowski 1995). Here M is a modular tilting module, Q a quantum tilting module, and $\lambda \in C$.*

$$[M : T(\lambda)] \quad = \quad \sum_{x \in W,\ x.\lambda \in X^+} (-1)^{l(x)} [M : V(x.\lambda)]$$

$$[Q : T_q(\lambda)] \quad = \quad \sum_{x \in W,\ x.\lambda \in X^+} (-1)^{l(x)} [Q : V_q(x.\lambda)].$$

These equations implies Lemma 6.4 (ii).

We turn to modular tilting modules in the second cell; the aim is to decompose their quantization.

THEOREM 6.6. *Assume type $A_{n\geq 2}$, D_n, E_6, E_7 or E_8 and $p \geq h$. Let $z \in \mathcal{C}(s_0)$. Then for some nonnegative integers a_y*

$$T(z)_q = T_q(z) \oplus \bigoplus_{y \in R} a_y T_q(y).$$

PROOF. Note that $T(s_0)_q = T_q(s_0)$. We will proceed by induction. Let $z < zs \in \mathcal{C}(s_0)$ and assume $zs \neq s_0$. Then $z \in \mathcal{C}(s_0)$ by Lemma 4.17. By induction and Theorem 6.3 (ii) and (iii) we get

$$\begin{aligned}\Theta_s(T(z)_q) &= \Theta_s T_q(z) \oplus \bigoplus_{y \in R} a_y \Theta_s T_q(y) \\ &= T_q(zs) \oplus \bigoplus_{y \in R} b_y T_q(y),\end{aligned}$$

for some non-negative integers b_y. Note the identity $(\Theta_s T(z))_q = \Theta_s(T(z)_q)$; both modules are quantum tilting modules with the same character.

$$(\Theta_s T(z))_q = T(zs)_q \oplus \bigoplus_{x \in W^0} c_x T(x)_q.$$

This proves the claim, since $T_q(zs)$ is a summand of $T(zs)_q$. □

REMARK 6.7. *Consider type G_2. Let $z \in \mathcal{C}(s_0)$ and recall from Table 2 on page 34 that $\mathcal{C}(s_0)$ consists of only 8 elements. We claim that $T(z)_q = T_q(z)$. In fact, for $zs < z$ (such s is necessarily unique) we have*

$$\operatorname{ch} T(z) = \chi(z) + \chi(zs) = \operatorname{ch} T_q(z).$$

Here $\chi(z)$ denotes the character of the Weyl module. Pick a weight $\mu \in \overline{C}$ with $\mathrm{Stab}_W\{\mu\} = \{1, s\}$*; then the sumformula reveals that $V(z.\mu)$ is simple. The claim follows and we have actually proved that Theorem 6.6 holds in type G_2 too.*

REMARK 6.8. *In chapter 7 we will find that Theorem 6.6 holds in type B_2, too. This result is stated in Theorem 7.21.*

REMARK 6.9. *Erdmann (1995) has computed the characters of the modular tilting modules in type A_1. Outside the lowest p^2-alcove, the characters of the quantum and modular tilting modules disagree in general. This shows that Theorem 6.6 cannot hold in type A_1 as the set R is empty in type A_1.*

We are now able to discuss the decomposition of quantized modular tilting modules belonging to lower cells. This result holds for all types of root systems in contrast to Theorem 6.6.

THEOREM 6.10. *All types and $p \geq h$. Let $x \in R$. Then for some nonnegative integers a_y*

$$T(x)_q = \bigoplus_{y \in R} a_y T_q(y).$$

PROOF. We prove this by induction. We postpone its basis, so assume that $x \in R$ is 'non-minimal' i.e. suppose that there is an s so that $x > xs \in R$. By induction

$$T(xs)_q = \bigoplus_{y \in R} a_y T_q(y).$$

Then using theorem 6.3 (iii) we get

$$\begin{aligned}\Theta_s(T(xs)_q) &= \bigoplus_{y \in R} a_y \Theta_s T_q(y) \\ &= \bigoplus_{y \in R} a'_y T_q(y).\end{aligned}$$

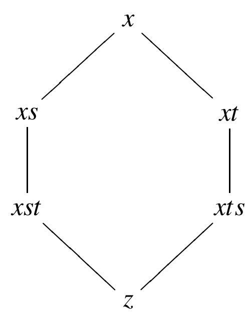


FIGURE 8

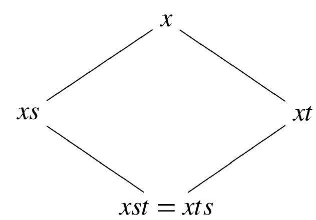


FIGURE 9

On the other hand $T(x)_q$ is a summand of $(\Theta_s T(xs))_q$ and we conclude that

$$T(x)_q \quad = \quad \bigoplus_{y\in R} a''_y T_q(y).$$

It remains to consider x, where $x > xs$ implies $xs \notin R$. The analysis of $T(x)_q$ when $x \in R$ is such a minimal element will require some case-by-case arguments: We give one proof in type A_2, a second in the types $A_{n\geq 3}$, B_2, D_n, E_6, E_7, E_8, or G_2, and a third for type B_n, C_n or F_4; the statement in the Theorem is irrelevant for type A_1 as the set R is empty.

First $xs \notin R$ shows that $xs \in \mathcal{C}(s_0)$. If $|\mathcal{R}(x)| = 1$, Lemma 4.15 shows that $x \in \mathcal{C}(s_0)$. Thus, there is a $t \neq s$ so that $x > xs$ and $x > xt \in \mathcal{C}(s_0)$.

Suppose that s, t do not commute. We will first show that this is only possible in type A_2. The coset $x < s,t >$ has maximal element x, and we denote the minimal element by z. The entire coset $x < s,t >$ is contained in W^0, and $zs > z < zt$ shows that $e \neq z$. By Lemma 4.17 we find that $x < s,t > \setminus \{x\}$ is contained $\mathcal{C}(s_0)$. Let r denote the simple reflection with $zr < z$. From $\mathcal{R}(zs) = \{s\}$ we see that $zr < z < zs < zsr$, hence s and r can not commute. Similarly $rt \neq tr$. Since we assumed that s, t do not commute, we find that r, s, and t are all connected to each other in the Coxeter graph of the Weyl group. Hence we are in type A_2 as claimed and $m(s,t) = 3$. The elements in the coset $x < s,t >$ are ordered as in Figure 8.

By Theorem 6.6 we have $T(xs)_q = T_q(xs) \oplus (\oplus_{y\in R} a_y T_q(y))$. Using Theorem 6.3 (i) we find

$$\begin{aligned}\Theta_s(T(xs)_q) &= \Theta_s T_q(xs) \oplus \bigoplus_{y\in R} a_y \Theta_s T_q(y)\\ &= T_q(x) \oplus T_q(xst) \oplus \bigoplus_{y\in R} a'_y T_q(y).\end{aligned}$$

By a completely analogous argument we get

$$\Theta_t(T(xt)_q) = T_q(x) \oplus T_q(xts) \oplus \bigoplus_{y\in R} a''_y T_q(y).$$

Now $T(x)_q$ is a summand of $(\Theta_s T(xs))_q$ as well as a summand of $(\Theta_t T(xt))_q$. We conclude that $T(x)_q = T_q(x) \oplus (\oplus_{y\in R} a'''_y T_q(y))$.

In the rest of the proof we may assume that s, t commute. The elements in the coset $x < s,t >$ are ordered as in Figure 9.

Suppose that s, t commute and that we are in type $A_{n\geq 3}$, B_2, D_n, E_6, E_7, E_8, or G_2. The assumptions allow us to use Theorem 6.6, and we have $T(xs)_q = T_q(xs) \oplus$

Root system type	minimal element in R
B_n	$s_0s_1s_0s_2$
C_n	$s_0s_2s_1s_3$
	$s_0s_2s_3\dots s_{n-1}s_ns_{n-1}\dots s_2s_0s_1$
F_4	$s_0s_4s_3s_2s_1s_3$

TABLE 3. The minimal elements of R.

$(\oplus_{y\in R}a_yT_q(y))$. Using Theorem 6.3 (ii) we get

$$\begin{aligned}\Theta_s(T(xs)_q) &= \Theta_sT_q(xs)\oplus\bigoplus_{y\in R}a_y\Theta_sT_q(y)\\ &= T_q(x)\oplus\bigoplus_{y\in R}a'_yT_q(y).\end{aligned}$$

Recall that $T(x)_q$ is a summand of $(\Theta_sT(xs))_q$. We conclude that $T(x)_q=\bigoplus_{y\in R}a''_yT_q(y)$.

Suppose that we are in type B_n, C_n or F_4. We do not use that s, t commute here. Instead we determine the minimal elements of R explicitly and check the claim. A reduced expression of each minimal element is given in Table 3. Let us briefly discuss how this table is obtained. It was shown above that if x is a minimal element of R then there exist s,t so that $xs<x>xt$ and xs,xt both belong to $\mathcal{C}(s_0)$. So there must be two elements of equal length. Looking at the Table 1 on page 33, which displays the reduced expression of all elements in $\mathcal{C}(s_0)$, we see that this happens only four times, and this corresponds to the four entries in Table 3.

We must show that the assertion in the theorem holds for these four elements. We do this for the long element in type C_n and leave the remaining three to the reader. Recall that $T(s_0)_q=T_q(s_0)$. Using Theorem 6.3 (i) and (iii) in each step we find that

$$\Theta_{s_n}\Theta_{s_{n-1}}\cdots\Theta_{s_3}\Theta_{s_2}T(s_0)_q=T_q(s_0s_2s_3\dots s_{n-1}s_n)\oplus\bigoplus_{y\in R}a_yT_q(y)$$

$$\begin{aligned}&\Theta_{s_{n-1}}\Theta_{s_n}\Theta_{s_{n-1}}\cdots\Theta_{s_3}\Theta_{s_2}T(s_0)_q\\ &\quad= T_q(s_0s_2s_3\cdots s_{n-1}s_ns_{n-1})\oplus T_q(s_0s_2s_3\cdots s_{n-1})\oplus\bigoplus_{y\in R}a'_yT_q(y)\end{aligned}$$

Before we proceed, note that $\Theta_{s_{n-2}}T_q(s_0s_2s_3\cdots s_{n-1})=T_q(s_0s_2s_3\cdots s_{n-2})\oplus\oplus_{y\in R}b_yT_q(y)$ by Theorem 6.3 (i) since $s_0s_2s_3\cdots s_{n-2}s_{n-1}s_{n-2}=s_{n-1}s_0s_2s_3\cdots s_{n-2}s_{n-1}\notin W^0$; the relations between the generators is given by Table 2 on page 34. Similarly, we find that $\Theta_{s_{n-3}}T_q(s_0s_2s_3\cdots s_{n-2})=T_q(s_0s_2s_3\cdots s_{n-3})\oplus\oplus_{y\in R}b'_yT_q(y)$ and so on. We continue with

$$\begin{aligned}&\Theta_{s_0}\Theta_{s_2}\Theta_{s_3}\cdots\Theta_{s_{n-1}}\Theta_{s_n}\Theta_{s_{n-1}}\cdots\Theta_{s_3}\Theta_{s_2}T(s_0)_q\\ &\qquad= T_q(s_0s_2s_3\cdots s_{n-1}s_ns_{n-1}\cdots s_3s_2s_0)\oplus T_q(s_0)\oplus\bigoplus_{y\in R}a''_yT_q(y)\\ &\Theta_{s_1}\Theta_{s_0}\Theta_{s_2}\Theta_{s_3}\cdots\Theta_{s_{n-1}}\Theta_{s_n}\Theta_{s_{n-1}}\cdots\Theta_{s_3}\Theta_{s_2}T(s_0)_q\\ &\qquad= T_q(s_0s_2s_3\cdots s_{n-1}s_ns_{n-1}\cdots s_3s_2s_0s_1)\oplus\bigoplus_{y\in R}a'''_yT_q(y)\end{aligned}$$

Here we used that $\Theta_{s_1}T_q(s_0)=0$ as $s_0s_1=s_1s_0\notin W^0$. Now, since $T(s_0s_2s_3\cdots s_{n-1}s_ns_{n-1}\cdots s_3s_2s_0s_1)_q$ is a summand in $\Theta_{s_1}\Theta_{s_0}\Theta_{s_2}\Theta_{s_3}\cdots\Theta_{s_{n-1}}\Theta_{s_n}\Theta_{s_{n-1}}\cdots\Theta_{s_3}\Theta_{s_2}T(s_0)_q$ we have the desired formula. We are done. □

The following theorems may be seen as the main results of the thesis.

THEOREM 6.11. *Assume type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8, or G_2 and $p \geq h$. Let $z \in \mathcal{C}(s_0)$ and let Q be a tilting G-module. Then*

$$[Q : T(z)] \quad = \quad [Q_q : T_q(z)].$$

PROOF. Since Q is a direct sum of indecomposable tilting modules, it is enough to check the result for all $Q = T(x)$, $x \in W^0$. If $x = e$ both sides of the equality is zero. If $x \in \mathcal{C}(s_0)$ we may apply Theorem 6.6, Remark 6.7, and Remark 6.8. Finally $x \in R$ is handled with Theorem 6.10. □

Recall that each right cell corresponds to a weight cell, and that we denote the weight cell corresponding to $\mathcal{C}(s_0)$ by $\underline{c}_2$. As stated, Theorem 6.11 holds for regular weights in $\underline{c}_2$, but the next result generalizes the theorem to all of $\underline{c}_2$ as well as the first weight cell $\underline{c}_1$.

THEOREM 6.12. *Assume type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8 or G_2 and $p \geq h$. For a weight μ in the first or second weight cell and any modular tilting module Q we have*

$$[Q : T(\mu)] \quad = \quad [Q_q : T_q(\mu)].$$

PROOF. Recall first that $\underline{c}_1 = C$. If $\mu \in \underline{c}_1$ then Lemma 6.4 proves the theorem. So we assume that $\mu \in \underline{c}_2$. This means that μ is a dominant weight in the lower closure of an alcove $z.C$ with $z \in \mathcal{C}(s_0)$. We pick $\nu \in \overline{C}$ so that $z.\nu = \mu$. In general, if $x.\nu \in X^+$ belongs to the lower closure of $x.C$, then

$$\begin{aligned} T_\nu^0 T(x.\nu) &= T(x.0) \\ T_\nu^0 T_q(x.\nu) &= T_q(x.0). \end{aligned}$$

The first result is stated in (Andersen 2000, Proposition 5.2). For the quantum analogue, see (Soergel 1997, Remark 7.2 2.). From these identities we see that

$$\begin{aligned} [Q : T(\mu)] &= [T_\nu^0 Q : T(z.0)] \\ [Q_q : T_q(\mu)] &= [T_\nu^0 (Q_q) : T_q(z.0)] \end{aligned}$$

As quantum tilting modules, $T_\nu^0(Q_q) = (T_\nu^0 Q)_q$ since the characters of the modules are equal. Now Theorem 6.11 shows that the left hand sides are equal. □

REMARK 6.13. *In the special case of type A_2, Theorem 6.12 has been proved by Jensen (1998).*

Decomposition numbers

It is a well known fact that the characters of the Weyl modules $\{\operatorname{ch} V(\lambda), \lambda$ a dominant weight$\}$, form a basis of the ring of characters. Since the λ-weight space of $T(\lambda)$ is always one dimensional, it is clear that $\{\operatorname{ch} T(\lambda), \lambda$ a dominant weight$\}$, establishes a second basis of the character ring. Hence the character of a module M may be expressed in both bases. We denote the coefficients in the first basis by $[M : V(\lambda)]$ and by $[M : T(\lambda)]$ in the second basis, so that

$$\operatorname{ch} M \quad = \quad \sum_{\lambda \in X^+} [M : V(\lambda)] \operatorname{ch} V(\lambda) \;=\; \sum_{\lambda \in X^+} [M : T(\lambda)] \operatorname{ch} T(\lambda).$$

If the module M admits a filtration by Weyl modules then $[M : V(\lambda)]$ is the number of times $V(\lambda)$ appears as a quotient in the filtration. When M is a tilting module, $[M : T(\lambda)]$ is the multiplicity of $T(\lambda)$ in M. Thus the definition of $[M : V(\lambda)]$ and $[M : T(\lambda)]$ given here agrees with our usage of $[M : V(\lambda)]$ and $[M : T(\lambda)]$ so far.

A convenient way of expressing the character of an indecomposable tilting module is through the decomposition numbers $[T(\lambda) : V(\mu)]$. This is to difficult for us. But the "inverse" decomposition numbers $[V(\mu) : T(\lambda)]$ for all $\mu, \lambda \leq \nu$ would allow us to calculate the characters of all indecomposable tilting modules with highest weight $\leq \nu$. Based on Theorem 6.12 we may give some of the numbers $[V(\mu) : T(\lambda)]$.

The characters of the tilting modules span the ring of characters. Hence the formula in Theorem 6.12 holds for any element of the character ring. The modular Weyl modules and the quantum Weyl modules have the same character. Therefore we obtain the following reformulation of Theorem 6.12 suggested to us by W. Soergel.

THEOREM 6.14. *Assume type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8 or G_2 and let $p \geq h$. For a dominant weight λ, and a weight μ in the first or second weight cell we have*

$$[V(\lambda) : T(\mu)] \;=\; [V_q(\lambda) : T_q(\mu)]$$

This allow us to calculate the coefficient of a second cell tilting module in any Weyl module, since the right hand side is known.

For a Weyl factor $V(\lambda)$ in a tilting module $T(\mu)$ we have $\lambda \uparrow \mu$ (Theorem 2.14). Similarly, if $[V(\lambda) : T(\mu)] \neq 0$ then $\mu \uparrow \lambda$. Thus, to express the character of a Weyl module $V(\lambda)$ in the basis of tilting characters, we need only consider $T(\mu)$ with μ linked to λ.

Choose a dominant weight λ and let $\Pi(\lambda) = \{\mu \in X^+ \mid \mu \uparrow \lambda\}$. Order (as in Chapter 2) the set $\Pi(\lambda) = \{\lambda_0,\ \lambda_1,\ \ldots \lambda_r\}$ so that $\lambda_i \uparrow \lambda_j$ implies that $j \leq i$; so $\lambda_0 = \lambda$. We organize the "inverse" decomposition numbers $[V(\mu) : T(\lambda)]$ in a $(r+1) \times (r+1)$ matrix. Then

$$\Big([V(\lambda_i) : T(\lambda_j)]\Big)$$

is a lower triangular matrix. The following Corollary in merely a restatement of Theorem 6.14.

COROLLARY 6.15. *Assume type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8 or G_2 and suppose λ_i belongs to the first or second weight cell. Then the entire i'th column of the matrix of "inverse" decomposition numbers is known.*

CHAPTER 7

B_2

We return now to Theorem 6.6, which considers the decomposition of modular tilting modules, and holds in type $A_{n\geq 2}$, D_n, E_6, E_7, E_8, and G_2. Further, Remark 3.8 makes it clear that the formula does *not* hold for type A_1.

It is therefore natural to consider type B_2. Does the multiplicity formula hold for root system of this type? We answer this question in this chapter: The answer is positive; see Theorem 7.21.

It should be noted, however, that the methodology is basically different from that of Chapter 6. We consider here the multiplicity of Weyl modules "close to" the top of an indecomposable tilting module. This is to be understood in the following way: Suppose that the tilting module has highest weight λ and that the Weyl module has highest weight μ. Then "close to"' means that the closure of the facets containing λ and μ share a special weight. When this is the case we are able to compute $[T(\lambda) : V(\mu)]$; this number equals the corresponding number in the quantum case (see Theorem 7.20). We then apply this result and emerge with a proof of the multiplicity formula in type B_2.

We continue to assume that $p \geq h$; the Coxeter number of B_2 is 4. We use results stated in (Koppinen 1986). In particular Theorem 7.1 below is essential since it describes the homomorphisms between Weyl modules "close to" each other. This provides a valuable starting point. It is worth noticing that Theorem 7.1 is not limited to type B_2. A few other results that does not require type B_2 is to be found in the last section.

Homomorphisms between Weyl modules

The first result provide us with useful information about the homomorphisms between Weyl modules around a special weight. This theorem is used many times in the proofs in this chapter. Recall that the stabilizer of a weight λ is denoted by W_λ.

THEOREM 7.1. *(Koppinen 1986) All types. Let χ be a* dominant *special weight. Let μ denote a weight in a facet, whose closure contains χ.*

(i) *If ξ, $\xi' \in W_\chi.\mu$ and $\xi \uparrow \xi'$ then*

$$\mathrm{Hom}(V(\xi), V(\xi')) \simeq k.$$

(ii) *If ξ, ξ', $\xi'' \in W_\chi.\mu$ and*

$$V(\xi) \longrightarrow V(\xi') \longrightarrow V(\xi'')$$

are nonzero homomorphisms, then the composite is nonzero.

REMARK 7.2. *This theorem does not hold for non-dominant special weights: See Proposition 7.18.*

The highest factors in a Weyl module

We assume until the last section that the root system in question is of type B_2.

Consider figure 10. It contains two similar situations. We state here some results about the factors of the Weyl modules in the figure. The statements in Case I and Case II are identical, and the proof applies to both cases.

Case I	Case II
$\delta = \chi + r\omega_1$	$\delta = \chi + r\omega_2$
$\beta = \chi + r(\omega_1 - \omega_2)$	$\beta = \chi + r(\omega_2 - 2\omega_1)$
$\gamma = \chi - r(\omega_1 - \omega_2)$	$\gamma = \chi - r(\omega_2 - 2\omega_1)$
$\alpha = \chi - r\omega_1$	$\alpha = \chi - r\omega_2$
$0 < r < p$	$0 < r < \frac{p}{2}$

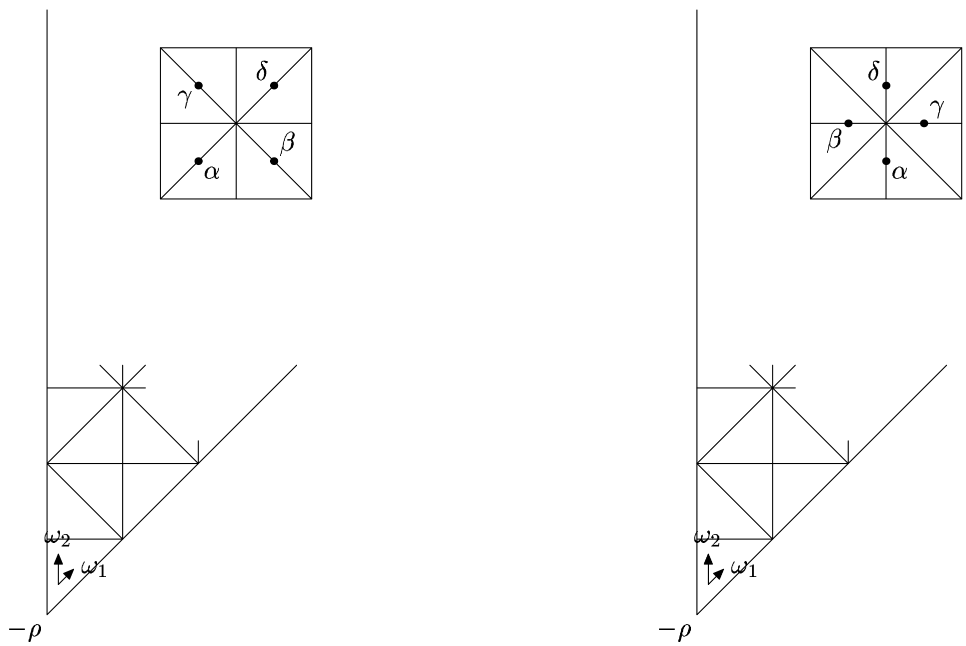


FIGURE 10. Case I (left) and Case II (right). χ is the special weight around which α, β, γ, and δ is arranged.

PROPOSITION 7.3. *With weights arranged as in Case I or in Case II, we have*

$$\begin{aligned} [V(\delta) : L(\alpha)] &= [V(\gamma) : L(\alpha)] = [V(\beta) : L(\alpha)] = 1 \\ [V(\delta) : L(\beta)] &= [V(\gamma) : L(\beta)] = 1 \\ [V(\delta) : L(\gamma)] &= 1. \end{aligned}$$

PROOF. We begin with the composition multiplicities of $L(\alpha)$. Let χ denote the special weight around which the four weights lie. From (Jantzen 1987, II.7.15) we see that for a simple module L we have

$$T_\alpha^\chi L = L(\chi) \iff L = L(\alpha).$$

Using also (Jantzen 1987, II.7.13) we get

$$\begin{aligned} [V(\delta) : L(\alpha)] &= [T_\alpha^\chi V(\delta) : T_\alpha^\chi L(\alpha)] \\ &= [V(\chi) : L(\chi)] = 1. \end{aligned}$$

This argument works equally well with $V(\delta)$ replaced by $V(\gamma)$ or $V(\beta)$.

From (Jantzen 1987, II.6.24) or (Andersen 1980a, Theorem 3.1) we get immediately that

$$[V(\delta) : L(\gamma)] = [V(\gamma) : L(\beta)] = 1$$

Finally the last multiplicity deserves a lemma of its own. $\square$

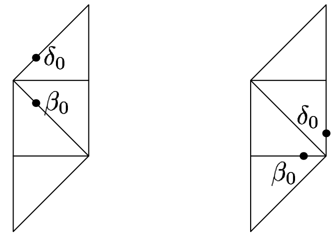


FIGURE 11. Case I (left) and Case II (right)

LEMMA 7.4. *[Case I] Let χ denote a dominant special weight. Let $\delta = \chi + r\omega_1$ and $\beta = \chi + r(\omega_1 - \omega_2)$ with $0 < r < p$. Then*

$$[V(\delta) : L(\beta)] \quad = \quad 1.$$

LEMMA 7.5. *[Case II] Let χ denote a dominant special weight. Let $\delta = \chi + r\omega_2$ and $\beta = \chi + r(\omega_2 - 2\omega_1)$ with $0 < r < \frac{p}{2}$. Then*

$$[V(\delta) : L(\beta)] \quad = \quad 1.$$

Note that the proofs in Case I and Case II are based on the same idea, and that only minor modifications are needed.

PROOF OF CASE I. Since $\mathrm{Hom}(V(\beta), V(\delta))$ is nonzero we have $[V(\delta) : L(\beta)] \geq 1$. To prove $\leq$ we first reduce to a special case: By the translation principle it is enough to prove the Lemma with $\beta = \chi + \omega_1$ and $\delta = \chi + \omega_1 - \omega_2$. We now get the result via the inequalities

$$\dim V(\delta)_\beta \quad \leq \quad 2 \tag{7.1}$$

$$\dim L(\delta)_\beta \quad \geq \quad 1, \tag{7.2}$$

that clearly shows $[V(\delta) : L(\beta)] \leq 1$.

The character of the Weyl module $V(\delta)$ is independent of the field k. Since we are only interested in the multiplicity of a weight space, we consider the Weyl module $V(\delta)$ as a factor in the Verma module $Z(\delta)$ of the complex simple Lie algebra of type B_2. We have $\delta - \beta = \alpha_1 + \alpha_2$. Now Kostants formula for the weight space multiplicities of $Z(\delta)$ yields $\dim Z(\delta)_\beta = 2$, and we have shown (7.1).

We use Steinbergs tensor product theorem: Recall that we may write (uniquely) any dominant weight λ as $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1$ and $\lambda_1 \in X^+$. Then

$$L(\delta) \quad \simeq \quad L(\delta_0) \otimes L(\delta_1)^{(F)}. \tag{7.3}$$

From the fact that $\chi + \rho \in pX$ we find that $\delta = (p-1)\omega_2 + p\delta_1$ and $\beta = (p-2)\omega_2 + p\delta_1$ with $\delta_1 \in X^+$. So we have

$$\begin{aligned} \delta_0 &= (p-1)\omega_2 \\ \beta_0 &= (p-2)\omega_2. \end{aligned}$$

Now (7.3) shows that it is enough to verify that $\dim L(\delta_0)_{\beta_0} \geq 1$. Figure 11 shows how the weights β_0 and δ_0 are arranged in X_1. Jantzens sumformula reveals that $V(\delta_0)$ is simple. Therefore we may use Freudenthals formula for the weight space multiplicities of Weyl modules to determine $\dim L(\delta_0)_{\beta_0} = \dim V(\delta_0)_{\beta_0}$. An uncomplicated calculation shows that

$$\dim V(\delta_0)_{\beta_0} = 1.$$

This gives us (7.2) and concludes the proof. $\square$

PROOF OF CASE II. Since $\mathrm{Hom}(V(\beta),V(\delta))$ is nonzero we have $[V(\delta):L(\beta)]\geq 1$. As before we reduce to the special case $\beta=\chi+\omega_2$ and $\delta=\chi+\omega_2-2\omega_1$. We now get the result via the inequalities

$$\dim V(\delta)_\beta \leq 3 \tag{7.4}$$
$$\dim L(\delta)_\beta \geq 2. \tag{7.5}$$

Kostants formula yields $\dim Z(\delta)_\beta = 3$, and Freudenthals formula gives $\dim L(\delta_0)_{\beta_0} = 2$. ☐

Extensions of Weyl modules

OBSERVATION 7.6. *With weights arranged as in Figure 10 we have*

- $\alpha\uparrow\beta\uparrow\gamma\uparrow\delta$
- *If* $\alpha\uparrow\lambda\uparrow\delta$ *then* $\lambda\in\{\alpha,\beta,\gamma,\delta\}$.

This observation is a special case of Lemma 7.22

LEMMA 7.7. *With the notation from Figure 10 we have*

$$\mathrm{Ext}^i(V(\alpha),L(\beta))=\mathrm{Ext}^i(V(\beta),L(\gamma))=\mathrm{Ext}^i(V(\gamma),L(\delta))=\begin{cases}k & i=1\\ 0 & i\neq 1.\end{cases}$$

PROOF. We will calculate $\mathrm{Ext}^i(V(\alpha),L(\beta))$. The other claims follow by analogous arguments.

From Theorem 7.1 we obtain a homomorphism

$$\phi:\ H^0(\beta)\longrightarrow H^0(\alpha),$$

and from this map we gain three short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker\phi & \to & H^0(\beta) & \to & \mathrm{im}\,\phi & \to & 0\\ 0 & \to & \mathrm{im}\,\phi & \to & H^0(\alpha) & \to & \mathrm{coker}\,\phi & \to & 0\\ 0 & \to & L(\beta) & \to & \ker\phi & \to & \ker\phi/L(\beta) & \to & 0.\end{array}$$

Recall that $\alpha\not\uparrow\lambda$ implies $\mathrm{Ext}^i(V(\alpha),L(\lambda))=0$ by Lemma 2.4. Then

$$\mathrm{Ext}^i(V(\alpha),\ker\phi/L(\beta))=0 \text{ for all } i\geq 0$$
$$\mathrm{Ext}^i(V(\alpha),\mathrm{coker}\,\phi)=0 \text{ for all } i\geq 0,$$

as Proposition 7.3 and Observation 7.6 shows that all factors $L(\lambda)$ in $\ker\phi/L(\beta)$ and $\mathrm{coker}\,\phi$ has $\alpha\not\uparrow\lambda$. Therefore

$$\mathrm{Ext}^i(V(\alpha),\mathrm{im}\,\phi)\simeq\mathrm{Ext}^i(V(\alpha),H^0(\alpha))=\begin{cases}k & i=0\\ 0 & i\geq 1,\end{cases}$$

where the last equality follows from Theorem 2.2. Also by Theorem 2.2 we find that for all $i\geq 0$ we have $\mathrm{Ext}^i(V(\alpha),H^0(\beta))=0$, so that

$$\mathrm{Ext}^i(V(\alpha),\ker\phi)=\begin{cases}k & i=1\\ 0 & i\neq 1\end{cases}$$

From the last of the three exact sequences we obtain

$$\mathrm{Ext}^i(V(\alpha),L(\beta))\simeq\mathrm{Ext}^i(V(\alpha),\ker\phi)=\begin{cases}k & i=1\\ 0 & i\neq 1.\end{cases}$$

☐

PROPOSITION 7.8. *With the notation of Figure 10 we have*

$$\mathrm{Ext}^i(V(\alpha),V(\beta))=\mathrm{Ext}^i(V(\beta),V(\gamma))=\mathrm{Ext}^i(V(\gamma),V(\delta))=\begin{cases}k & i=0,1\\ 0 & i\geq 2.\end{cases}$$

PROOF. We shall calculate only $\operatorname{Ext}^i(V(\alpha),V(\beta))$. The same argumentation works in the remaining two situations.

From Theorem 7.1 we obtain the map

$$f:\ V(\alpha)\longrightarrow V(\beta).$$

This homomorphism induces three short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker f & \to & V(\alpha) & \to & \operatorname{im} f & \to & 0 \\ 0 & \to & \operatorname{im} f & \to & V(\beta) & \to & \operatorname{coker} f & \to & 0 \\ 0 & \to & \ker\pi & \to & \operatorname{coker} f & \xrightarrow{\pi} & L(\beta) & \to & 0. \end{array}$$

We claim that $\operatorname{Ext}^i(V(\alpha),L)=0$ (for all i) for all simple factors in $\ker f$ and $\ker\pi$; this follows from Lemma 2.4, Proposition 7.3, and Observation 7.6. We conclude that $\operatorname{Ext}^i(V(\alpha),\ker f)=0=\operatorname{Ext}^i(V(\alpha),\ker\pi)$ for all $i\geq 0$. Using Lemma 2.7 and Lemma 7.7 we arrive at

$$\begin{aligned} \operatorname{Ext}^i(V(\alpha),\operatorname{im} f) &\simeq \operatorname{Ext}^i(V(\alpha),V(\alpha))=\begin{cases} k & i=0\\ 0 & i\geq 1\end{cases} \\ \operatorname{Ext}^i(V(\alpha),\operatorname{coker} f) &\simeq \operatorname{Ext}^i(V(\alpha),L(\beta))=\begin{cases} k & i=1\\ 0 & i\neq 1.\end{cases} \end{aligned}$$

We establish the proposition by examining the long exact sequence obtained by using $\operatorname{Hom}(V(\alpha),-)$ on the short exact sequence with $V(\beta)$ as middle term. □

PROPOSITION 7.9. *With notation as in Figure 10 we have*

$$[T(\beta):V(\alpha)]=[T(\gamma):V(\beta)]=[T(\delta):V(\gamma)]=1.$$

PROOF. We consider only the first decompostion number. Since α is maximal among dominant weights linked to β (see Observation 7.6) we have, by Theorem 2.14 (iv)

$$[T(\beta):V(\alpha)]=\dim\operatorname{Ext}^1(V(\alpha),V(\beta))=1.$$

□

We proceed to determine the ext groups of the remaining pairs of Weyl modules. Note that the proofs given are very similar to those of the lemma and the proposition above.

LEMMA 7.10. *With notation as in Figure 10 we have for $i\geq 0$*

$$\operatorname{Ext}^i(V(\alpha),L(\gamma))=\operatorname{Ext}^i(V(\beta),L(\delta))=0.$$

PROOF. We will calculate $\operatorname{Ext}^i(V(\alpha),L(\gamma))$. The second claim follows by an analogous argument.

From Theorem 7.1 we get that the following composition of maps is nonzero:

$$H^0(\gamma)\xrightarrow{\phi}H^0(\beta)\longrightarrow H^0(\alpha).$$

We conclude that $L(\alpha)$ and $L(\beta)$ appear as factors in the image of ϕ. By the composition factor results in Proposition 7.3 none of them appear in the kernel of ϕ, nor in the cokernel of ϕ. From the map ϕ we gain three short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker\phi & \to & H^0(\gamma) & \to & \operatorname{im}\phi & \to & 0 \\ 0 & \to & \operatorname{im}\phi & \to & H^0(\beta) & \to & \operatorname{coker}\phi & \to & 0 \\ 0 & \to & L(\gamma) & \to & \ker\phi & \to & \ker\phi/L(\gamma) & \to & 0. \end{array}$$

It follows from Observation 7.6 and Lemma 2.4 that $\operatorname{Ext}^i(V(\alpha),L)=0$ (for all i) for all simple factors in $\ker\phi/L(\gamma)$ and $\operatorname{coker}\phi$. We have thus established that

$$\operatorname{Ext}^i(V(\alpha),\ker\phi/L(\gamma))=0 \text{ for all } i\geq 0$$

$$\operatorname{Ext}^i(V(\alpha),\operatorname{coker}\phi)=0 \text{ for all } i\geq 0.$$

Recall that $\mathrm{Ext}^i(V(\alpha),H^0(\beta)) = \mathrm{Ext}^i(V(\alpha),H^0(\gamma)) = 0$ for all $i \geq 0$ by Theorem 2.2. We use the functor $\mathrm{Hom}(V(\alpha),-)$ on all three sequences above; it is easy to see that this implies (as claimed)

$$\mathrm{Ext}^i(V(\alpha),L(\gamma)) = 0 \text{ for all } i \geq 0.$$

□

PROPOSITION 7.11. *With notation as in Figure 10 we have*

$$\mathrm{Ext}^i(V(\alpha),V(\gamma)) = \mathrm{Ext}^i(V(\beta),V(\delta)) = \begin{cases} k & i = 0,1 \\ 0 & i \geq 2. \end{cases}$$

PROOF. We shall calculate only $\mathrm{Ext}^i(V(\alpha),V(\gamma))$. The same line of argumentation works equally well in the other situation.

From Theorem 7.1 we obtain a nonzero composition of maps

$$V(\alpha) \longrightarrow V(\beta) \xrightarrow{f} V(\gamma).$$

Since this composite is nonzero, $L(\alpha)$ must appear in the image of f. Hence $L(\alpha)$ and $L(\beta)$ can not appear in the kernel of f, nor in the cokernel of f, by the composition multiplicities of Proposition 7.3.

The homomorphism f provide us with two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker f & \to & V(\beta) & \to & \mathrm{im}\, f & \to & 0 \\ 0 & \to & \mathrm{im}\, f & \to & V(\gamma) & \to & \mathrm{coker}\, f & \to & 0. \end{array}$$

In Lemma 7.10 above we proved that $\mathrm{Ext}^i(V(\alpha),L(\gamma)) = 0$. Hence $\mathrm{Ext}^i(V(\alpha),L) = 0$ (for all i) for all simple factors in $\ker f$ and $\mathrm{coker}\, f$ by Lemma 2.4. We conclude that $\mathrm{Ext}^i(V(\alpha),\ker f) = 0 = \mathrm{Ext}^i(V(\alpha),\mathrm{coker}\, f)$ for all $i \geq 0$. Examining the long exact sequences obtained by using $\mathrm{Hom}(V(\alpha),-)$ now yields

$$\mathrm{Ext}^i(V(\alpha),V(\gamma)) \simeq \mathrm{Ext}^i(V(\alpha),\mathrm{im}\, f) \simeq \mathrm{Ext}^i(V(\alpha),V(\beta)) = \begin{cases} k & i = 0,1 \\ 0 & i \geq 2. \end{cases}$$

□

We now calculate $\mathrm{Ext}^i(V(\alpha),V(\delta))$. There are no surprises in the proofs; they follow the same path as for the preceding statements.

LEMMA 7.12. *With notation as in Figure 10 we have for $i \geq 0$*

$$\mathrm{Ext}^i(V(\alpha),L(\delta)) = 0 \textit{ for all } i \geq 0.$$

PROOF. From Theorem 7.1 we get that the following composition of maps is nonzero.

$$H^0(\delta) \xrightarrow{\phi} H^0(\gamma) \longrightarrow H^0(\beta) \longrightarrow H^0(\alpha).$$

We conclude that $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ appears as factors in the image of ϕ. By the composition multiplicities of Proposition 7.3 none of them appear in the kernel of ϕ, nor in the cokernel of ϕ. From the map ϕ we gain three short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker\phi & \to & H^0(\delta) & \to & \mathrm{im}\,\phi & \to & 0 \\ 0 & \to & \mathrm{im}\,\phi & \to & H^0(\gamma) & \to & \mathrm{coker}\,\phi & \to & 0 \\ 0 & \to & L(\delta) & \to & \ker\phi & \to & \ker\phi/L(\delta) & \to & 0. \end{array}$$

It follows from Observation 7.6 and Lemma 2.4 that $\mathrm{Ext}^i(V(\alpha),L) = 0$ (for all i) for all simple factors in $\ker\phi/L(\delta)$ and $\mathrm{coker}\,\phi$. We have thus established that

$$\mathrm{Ext}^i(V(\alpha),\ker\phi/L(\delta)) = 0 \text{ for all } i \geq 0$$

$$\mathrm{Ext}^i(V(\alpha),\mathrm{coker}\,\phi) = 0 \text{ for all } i \geq 0.$$

Recall that $\mathrm{Ext}^i(V(\alpha), H^0(\delta)) = \mathrm{Ext}^i(V(\alpha), H^0(\gamma)) = 0$ for all $i \geq 0$. We use $\mathrm{Hom}(V(\alpha), -)$ on all three sequences above; it is easy to see that this implies (as claimed)

$$\mathrm{Ext}^i(V(\alpha), L(\delta)) = 0 \text{ for all } i \geq 0.$$

□

PROPOSITION 7.13.

$$\mathrm{Ext}^i(V(\alpha), V(\delta)) = \begin{cases} k & i = 0, 1 \\ 0 & i \geq 2. \end{cases}$$

PROOF. From Theorem 7.1 we obtain a nonzero map

$$V(\alpha) \longrightarrow V(\beta) \longrightarrow V(\gamma) \xrightarrow{f} V(\delta).$$

Since this composite is nonzero, $L(\alpha)$ and $L(\beta)$ must appear in the image of f. Hence $L(\alpha)$ and $L(\beta)$ cannot appear in the kernel of f nor in the cokernel of f, by the composition multiplicities of Proposition 7.3.

The homomorphism f provide us with two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker f & \to & V(\gamma) & \to & \mathrm{im}\, f & \to & 0 \\ 0 & \to & \mathrm{im}\, f & \to & V(\delta) & \to & \mathrm{coker}\, f & \to & 0. \end{array}$$

The Lemmata 7.12 and 7.10 says that $\mathrm{Ext}^i(V(\alpha), L(\gamma)) = \mathrm{Ext}^i(V(\alpha), L(\delta)) = 0$. By Observation 7.6 and Lemma 2.4 we obtain $\mathrm{Ext}^i(V(\alpha), L) = 0$ (for all i) for all simple factors in $\ker f$ and $\mathrm{coker}\, f$. We conclude that $\mathrm{Ext}^i(V(\alpha), \ker f) = \mathrm{Ext}^i(V(\alpha), \mathrm{coker}\, f) = 0$ for all $i \geq 0$. Examining the long exact sequences obtained by using $\mathrm{Hom}(V(\alpha), -)$ now yields

$$\mathrm{Ext}^i(V(\alpha), V(\delta)) \simeq \mathrm{Ext}^i(V(\alpha), \mathrm{im}\, f) \simeq \mathrm{Ext}^i(V(\alpha), V(\gamma)) = \begin{cases} k & i = 0, 1 \\ 0 & i \geq 2. \end{cases}$$

□

A multiplicity calculation

THEOREM 7.14. *We keep the notation of Figure 10. Recall that $p \geq 5$. We have*

$$[T(\gamma) : V(\alpha)] = 1$$

The rest of this section present a proof of the Theorem. The strategy in the proof is simply to follow the steps in the construction of a tilting module. This is possible in practice, because there are only one weight β between γ and α (in the linkage order); this follows from Observation 7.6.

Recall the construction of $T(\lambda)$ in chapter 2. As β is maximal among dominant weights linked to γ the first step is to extend $V(\gamma)$ with $V(\beta)^{\oplus d_1}$ non-trivially; here $d_1 = \mathrm{Ext}^1(V(\gamma), V(\beta))$. From Proposition 7.8 we have $d_1 = 1$. So we must first find a non-split extension, E, of $V(\gamma)$ with $V(\beta)$. Now α is maximal among dominant weights linked to γ and different from β and γ. Therefore we may continue the construction of $T(\lambda)$ by extending E with $V(\alpha)$. It follows from our construction in Chapter 2 that $[T(\gamma) : V(\alpha)] = \dim \mathrm{Ext}^1(V(\alpha), E)$. We calculate the dimension of this extension group to prove the theorem.

Let us describe how we obtain E. Consider $T_\chi^\delta V(\chi)$. By (Jantzen 1987, II.7.13) this module allows a filtration with $V(\alpha)$, $V(\beta)$, $V(\gamma)$, and $V(\delta)$ as subquotients. We want to get rid of $V(\alpha)$ and $V(\delta)$. It is a standard fact about modules with a Weyl filtrations that a Weyl module with maximal (with relation to the linkage ordering on X) highest weight is a submodule and that the quotient has a Weyl filtration. It follows that a Weyl factor with a minimal highest weight is a quotient and that the kernel of the projection has a Weyl filtration. Using this in turn yields modules $R = T_\chi^\delta V(\chi)/V(\delta)$ and $E = \ker(R \to V(\alpha))$ and corresponding sequences

$$\begin{array}{ccccccccc} 0 & \to & V(\delta) & \to & T_\chi^\delta V(\chi) & \to & R & \to & 0 \\ 0 & \to & E & \to & R & \to & V(\alpha) & \to & 0 \\ 0 & \to & V(\gamma) & \to & E & \to & V(\beta) & \to & 0. \end{array}$$

Note, that we have still to determine whether E is split. But Lemma 7.15 below immediately gives that the last short exact sequence is non-split: If split, Theorem 7.1 shows that $\mathrm{Hom}(V(\alpha), E)$ is a two-dimensional space, contradicting Lemma 7.15.

It is also worth noticing that in the proof of Lemma 7.15 we prove that $\mathrm{Hom}(V(\alpha), R)$ have dimension one. Therefore the middle short exact sequence is non-split. Also, in the proof of Lemma 7.15 we show that $\mathrm{Hom}(V(\alpha), T_\chi^\delta V(\chi))$ is one-dimensionally. So the first sequence is also non-split. We shall not use that the first two sequences are non-split.

LEMMA 7.15.

$$\mathrm{Ext}^i(V(\alpha), E) = \begin{cases} k & i = 0,1 \\ 0 & i \geq 2. \end{cases}$$

PROOF. By adjointness of the translation functors T_χ^δ, T_δ^χ and Lemma 2.7 we obtain

$$\begin{aligned} \mathrm{Ext}^i(V(\alpha), T_\chi^\delta V(\chi)) &\simeq \mathrm{Ext}^i(T_\delta^\chi V(\alpha), V(\chi)) \\ &\simeq \mathrm{Ext}^i(V(\chi), V(\chi)) = \begin{cases} k & i = 0 \\ 0 & i \geq 1. \end{cases} \end{aligned}$$

From Proposition 7.13 we have a complete description of $\mathrm{Ext}^i(V(\alpha), V(\delta))$. Now apply $\mathrm{Hom}(V(\alpha), -)$ to the first of the short exact sequences above. We conclude that

$$\mathrm{Ext}^i(V(\alpha), R) = \begin{cases} k & i = 0 \\ 0 & i \geq 1. \end{cases}$$

From the third short exact sequence and Proposition 7.11 we get

$$k = \mathrm{Hom}(V(\alpha), V(\gamma)) \subset \mathrm{Hom}(V(\alpha), E).$$

Now apply $\mathrm{Hom}(V(\alpha), -)$ on the second short exact sequences above; from the information gathered so far we have

$$\mathrm{Ext}^i(V(\alpha), E) = \begin{cases} k & i = 0,1 \\ 0 & i \geq 2. \end{cases}$$

□

Non-dominant special weights

In this section we turn to non-dominant special weights. Again we consider two situations simultaneously: The two cases are illustrated in Figure 12, which also defines the notation.

LEMMA 7.16.

$$[V(\delta) : L(\beta)] = 0.$$

PROOF. The idea is simply to check that the contributions from the weights β and α cancel in Jantzens sum formula. We will prove the Lemma in Case I only. It is straightforward to adapt the proof to Case II.

In Case I our special weight is determined by $\chi + \rho = np\omega_2$. We ask the reader to verify that

$$W_\chi = \langle s_{\alpha_1},\ s_{\alpha_2,np},\ s_{\alpha_2+\alpha_1,2np},\ s_{\alpha_2+2\alpha_1,np} \rangle.$$

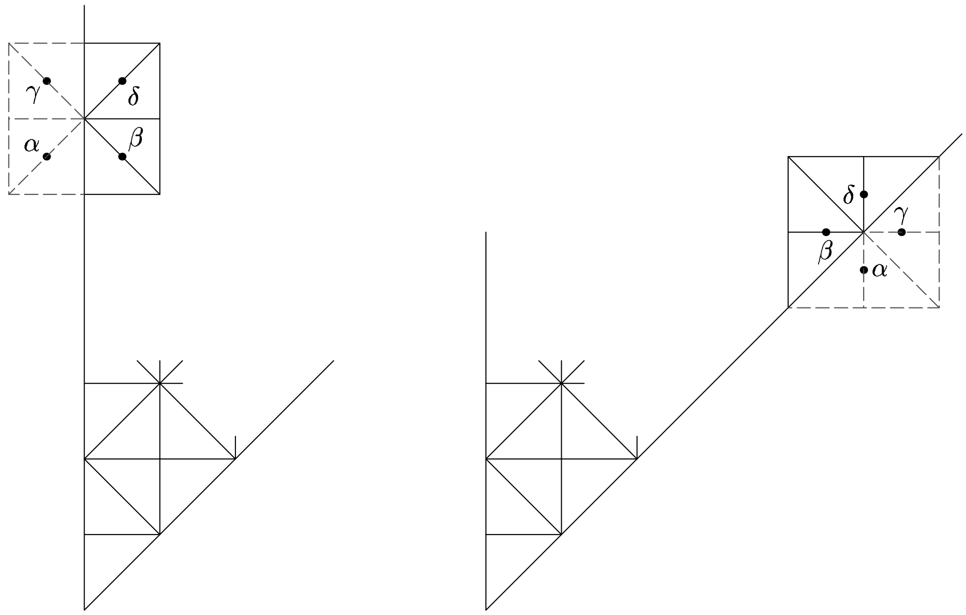


FIGURE 12. Case I (left) and case II (right). We denote the special point around which the weights α, β, γ, and δ are arranged, by χ.

Consider Jantzens sumformula

$$\sum_{i\geq 1} \operatorname{ch} V(\delta)^i = \sum_{\alpha\in R_+} \sum_{0<mp<\langle\delta+\rho,\alpha^\vee\rangle} v_p(mp)\chi(s_{\alpha,mp}.\delta). \tag{7.6}$$

It follows from Lemma 7.24 that $[\chi(s_{\alpha,mp}) : L(\beta)] \neq 0$ implies that $s_{\alpha,mp} \in W_\chi$. Therefore we get that

$$\left[\sum_{i\geq 1} \operatorname{ch} V(\delta)^i : \operatorname{ch} L(\beta)\right] = \left[\sum_{\alpha\in R_+} \sum_{\substack{0<mp<\langle\delta+\rho,\alpha^\vee\rangle \\ s_{\alpha,mp}\in W_\chi}} v_p(mp)\chi(s_{\alpha,mp}.\delta) : \operatorname{ch} L(\beta)\right].$$

We can calculate the right hand side of this equation:

$$\begin{aligned}\sum_{\alpha\in R_+} \sum_{\substack{0<mp<\langle\delta+\rho,\alpha^\vee\rangle \\ s_{\alpha,mp}\in W_\chi}} & v_p(mp)\chi(s_{\alpha,mp}.\delta) \\ &= v_p(2np)\chi(s_{\alpha_1+\alpha_2,2np}.\delta) + v_p(np)\chi(s_{\alpha_1+2\alpha_2,np}.\delta) \\ &= v_p(2np)\chi(\alpha) + v_p(np)\chi(\beta) \;=\; 0, \end{aligned} \tag{7.7}$$

since $\chi(\alpha) = -\chi(\beta)$ and $p \neq 2$. □

LEMMA 7.17.

$$\operatorname{Ext}^i(V(\beta), L(\delta)) = 0 \text{ for all } i \geq 0.$$

PROOF. Consider the sequence

$$0 \to L(\delta) \to H^0(\delta) \to H^0(\delta)/L(\delta) \to 0.$$

By Observation 7.6 β is maximal among *dominant* weights linked to δ. Hence for all $i \geq 0$ and all factors L in $H^0(\delta)/L(\delta)$ we have $\operatorname{Ext}^i(V(\beta), L) = 0$; this implies that for all $i \geq 0$ we have $\operatorname{Ext}^i(V(\beta), H^0(\delta)/L(\delta)) = 0$. Since also $\operatorname{Ext}^i(V(\beta), H^0(\delta)) = 0$ for all $i \geq 0$ by Theorem 2.2, we have the desired vanishing. □

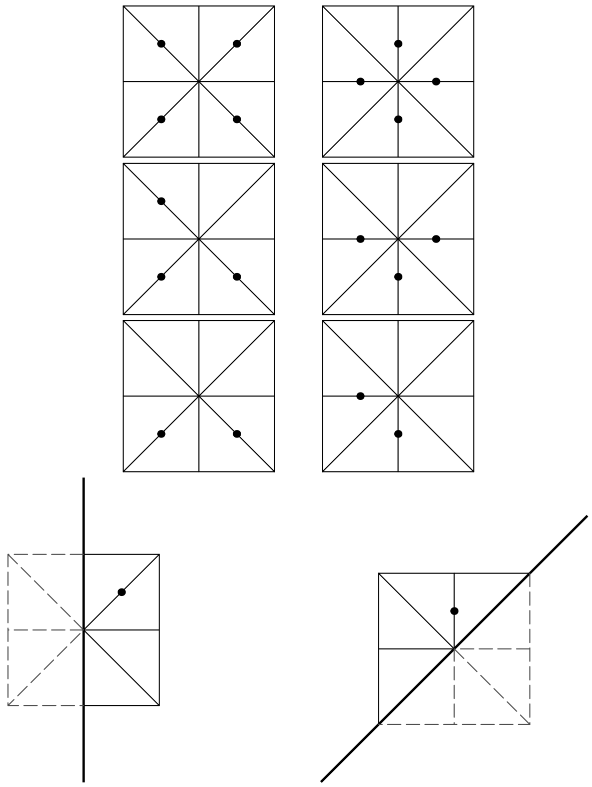


FIGURE 13. Top part of various tilting modules. Case I (right) and Case II (left).

Note that this shows that $\mathrm{Ext}^i(V(\beta), L) = 0$ for all $i \geq 0$ and any factor L in $V(\delta)$, by Lemma 7.17 and Lemma 2.4. Therefore:

PROPOSITION 7.18.

$$\mathrm{Ext}^i(V(\beta), V(\delta)) = 0 \textit{ for all } i \geq 0.$$

PROPOSITION 7.19.

$$[T(\delta) : V(\beta)] = 0.$$

PROOF. Since β is maximal among weights linked to δ we have by Theorem 2.14

$$[T(\delta) : V(\beta)] = \dim \mathrm{Ext}^1(V(\beta), V(\delta)) = 0.$$

□

Pictures of tilting modules

Figure 13 features pictures of the Weyl factors around a top special point; if λ is minimal in $W_\chi.\lambda$ the picture is omitted. The following list points to proofs in the modular case:

1. row See (Andersen 2000, Proposition 5.2)
2. row Theorem 7.14
3. row Proposition 7.9
4. row Proposition 7.19.

Note that the corresponding results for regular weights can be obtained by translation off the wall: It follows from (Andersen 1992, Proposition 5.6) that such a translation of an indecomposable tilting module yields an indecomposable tilting modules.

THEOREM 7.20. *Let $p \geq 5$. Suppose that χ is a special weight. Let F denote a facet so that $\chi \in \overline{F}$. Let $\lambda \in F \cap X^+$. For each $w \in W_\chi$ we have*

$$[T(\lambda) : V(w.\lambda)] = [T_q(\lambda) : V_q(w.\lambda)].$$

Furthermore, these numbers are either one or zero.

PROOF. We refer to (Stroppel 1997), which contains all the quantum numbers. A comparison yields the result. □

This chapter is motivated by Theorem 6.6. We are able to prove the corresponding statement in type B_2 now. We use the notation of Chapter 6; in particular $T(z)$ is short for $T(z.\lambda)$ with λ denoting a fixed regular weight of C.

THEOREM 7.21. *Type B_2 and $p \geq 5$. Suppose that $z \in \mathcal{C}(s_0)$. Then*

$$T(z)_q = T_q(z) \oplus \bigoplus_{y \in R} a_y T_q(y).$$

PROOF. Induction in z. Note that

$$T(s_0)_q = T_q(s_0).$$

Assume that $z > s_0$. By the properties of $\mathcal{C}(s_0)$ there is a unique simple s, so that $z > zs = z'$. Let t denote the unique simple reflection so that $z' > z't$. It follows from Lemma 4.17 that $z', z't \in \mathcal{C}(s_0) \cup \{e\}$. By induction

$$T(z')_q = T_q(z') \oplus \bigoplus_{y \in R} a_y T_q(y).$$

Hence by Theorem 6.3 (i) and (iii) we get

$$\begin{aligned} \Theta_s T(z')_q &= \Theta_s T_q(z') \oplus \bigoplus_{y \in R} a_y \Theta_s T_q(y) \\ &= T_q(z) \oplus \delta T_q(zst) \oplus \bigoplus_{y \in R} a'_y T_q(y). \end{aligned} \tag{7.8}$$

Here δ is either 1 or 0. Since $\Theta_s(T(z')_q) = (\Theta_s T(z'))_q$ we find that $T(z)_q$ is a summand in (7.8).

Note that the geometry of the second cell (see Figure 14) in type B_2 dictates that the closure of the alcoves $z.C$ and $zst.C$ share a special weight. By Theorem 7.20

$$[T(z) : V(zst)] = [T_q(z) : V_q(zst)].$$

This means that $T_q(zst)$ cannot appear as a summand of $T(z)_q$. If it appeared, $[T(z) : V(zst)]$ would be strictly larger than $[T_q(z) : V_q(zst)]$. □

The linkage order around special points

In this section we prove a few results referred to earlier. The lemmata hold for root systems of **all types**, and this is the reason they are gathered in this last section of the chapter.

Let χ denote a special point, defined by the equations

$$\langle \chi + \rho, \alpha^\vee \rangle = n_\alpha p,$$

where n_α, $\alpha \in R_+$ is a set of integers. We consider a set of alcoves; those that contains χ in the closure of the alcove. Here two stands out

$$\begin{aligned} C^+ &= \{\lambda \in E \mid n_\alpha p < \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p,\ \alpha \in R_+\} \\ C^- &= \{\lambda \in E \mid (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p,\ \alpha \in R_+\}. \end{aligned}$$

Recall that the stabilizer of χ is denoted by W_χ.

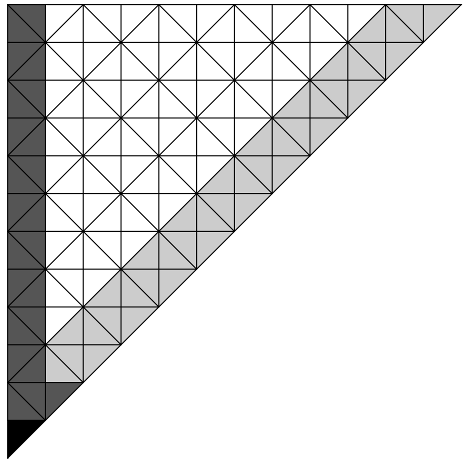


FIGURE 14. The cells in type B_2

Let F denote a facet with $\chi \in \overline{F}$. Choose $\lambda \in F \cap X$. Since the closure of an alcove contains exactly one point from each W-orbit, we define weights λ^+, λ^- by

$$\{\lambda^+\} = W.\lambda \cap \overline{C^+} \qquad \{\lambda^-\} = W.\lambda \cap \overline{C^-}.$$

LEMMA 7.22.

$$W_\chi.\lambda = \{\lambda' \mid \lambda^- \uparrow \lambda' \uparrow \lambda^+\}$$

REMARK 7.23. *This statement appears in the proof of Corollary 3.2 in (Koppinen 1986).*

PROOF. The proof is based on the following fact about alcoves (Jantzen 1987, II.6.11):

$$\{C' \mid \chi \in \overline{C'}\} = \{w.C^- \mid w \in W_\chi\} = \{C' \mid C^- \uparrow C' \uparrow C^+\}$$

This basically proves the Lemma when λ is a regular weight; for regular weights $\lambda_1 \in C_1$, $\lambda_2 \in C_2$ in the same W-orbit we have $\lambda_1 \uparrow \lambda_2 \Leftrightarrow C_1 \uparrow C_2$. For non-regular weights we have only $\Leftarrow$.

Let $w \in W_\chi$. We will prove that $\lambda^- \uparrow w.\lambda \uparrow \lambda^+$. Let C be an alcove so that $\overline{C}$ contains λ. Then

$$\begin{aligned} \overline{w.C} = w.\overline{C} \ni \chi \quad &\Rightarrow \quad C^- \uparrow w.C \uparrow C^+ \\ &\Rightarrow \quad \lambda^- \uparrow w.\lambda \uparrow \lambda^+ \end{aligned}$$

Now assume that $\lambda^- \uparrow \lambda' \uparrow \lambda^+$. We obtain a chain of inequalities

$$\lambda^- \leq s_{i_1}.\lambda^- \leq s_{i_2}s_{i_1}.\lambda^- \leq \cdots \leq \lambda' = s_{i_s}\cdots s_{i_1}.\lambda^- \leq \cdots \leq s_{i_r}\cdots s_{i_1}.\lambda^- = \lambda^+,$$

Since χ belongs to the closure of the facets containing λ^- and λ^+ we get the same chain of inequalities for χ

$$\chi \leq s_{i_1}.\chi \leq s_{i_2}s_{i_1}.\chi \leq \cdots \leq s_{i_s}\cdots s_{i_1}.\chi \leq \cdots \leq s_{i_r}\cdots s_{i_1}.\chi = \chi.$$

In particular we have $s_{i_s}\cdots s_{i_1} \in W_\chi$, and we are done. $\square$

Lemma 7.22 describes how weights around a special point are linked. One immediate application is to Jantzens sumformula; it simplifies calculations if you are only interested in the factors with highest weight around the top special weight.

In the following lemma λ, λ^+, λ^-, and χ are defined as above.

LEMMA 7.24. *Let* $\lambda^- \uparrow \lambda' \uparrow \lambda^+$ *and assume that* $0 < mp < \langle \lambda + \rho, \alpha \rangle$ *for some integer* m *and* $\alpha \in R_+$.

If $[\chi(s_{\alpha,mp}.\lambda) : L(\lambda')] \neq 0$ *then* $s_{\alpha,mp} \in W_\chi$.

PROOF. From the assumptions we get an $w \in W_0$ so that $ws_{\alpha,mp}.\lambda \in X^+$ and so that

$$\lambda^- \uparrow \lambda' \uparrow ws_{\alpha,mp}.\lambda \uparrow \lambda \uparrow \lambda^+$$

The first, second, and fourth $\uparrow$ follows directly form the assumptions. The third $\uparrow$ is necessary; otherwise $V(\lambda)$ has a simple factor with highest weight *not* linked to λ.

From Lemma 7.22 we have $ws_{\alpha,mp}.\lambda \in W_\chi.\lambda$. So $w_1^{-1}ws_{\alpha,mp}.\lambda = \lambda$ for some $w_1 \in W_\chi$. But $W_\lambda \subset W_\chi$, hence $w_1^{-1}ws_{\alpha,mp} \in W_\chi$. We see that $ws_{\alpha,mp} \in W_\chi$.

From $\chi = ws_{\alpha,mp}.\chi$ we conclude that

$$\begin{aligned}
\langle\chi+\rho,\chi+\rho\rangle &= \langle w^{-1}(\chi+\rho), w^{-1}(\chi+\rho)\rangle \\
&= \langle s_{\alpha,mp}(\chi+\rho), s_{\alpha,mp}(\chi+\rho)\rangle \\
&= \langle\chi+\rho,\chi+\rho\rangle - 2(\langle\chi+\rho,\alpha^\vee\rangle - mp)\langle\chi+\rho,\alpha\rangle + (\langle\chi+\rho,\alpha^\vee\rangle - mp)^2\langle\alpha,\alpha\rangle
\end{aligned}$$

Assuming that $\langle\chi+\rho,\alpha^\vee\rangle - mp \neq 0$, we get

$$\langle\chi+\rho,\alpha^\vee\rangle - mp = \frac{2\langle\chi+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}$$

From this we conclude that $mp = 0$. But this contradicts our assumptions on mp. Hence $\langle\chi+\rho,\alpha^\vee\rangle - mp = 0$, and we have $s_{\alpha,mp}.\chi = \chi$. □

CHAPTER 8

Schur-Weyl duality

Let $\mathbf{N}$ denote an n-dimensional vector space over k with group of linear automorphisms denoted by $\mathrm{GL}(\mathbf{N})$. The r-fold tensor product $\mathbf{N}^{\otimes r}$ has a natural structure of $\mathrm{GL}(\mathbf{N})$-modules. But $\mathbf{N}^{\otimes r}$ is also a representation of the symmetric group, with $\sigma \in \Sigma_r$ acting by permutation

$$\sigma(v_1 \otimes \cdots \otimes v_r) = v_{\sigma 1} \otimes \cdots \otimes v_{\sigma r}.$$

The actions of $\mathrm{GL}(\mathbf{N})$ and Σ_r commute. Therefore we obtain ring homomorphisms

$$k[\Sigma_r] \longrightarrow \mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r}) \tag{8.1}$$

$$k\,\mathrm{GL}(\mathbf{N}) \longrightarrow \mathrm{End}_{\Sigma_r}(\mathbf{N}^{\otimes r}) \tag{8.2}$$

The first map is the subject of this chapter. It is an isomorphism when $r \leq n$, see (Carter and Lusztig 1974); for larger values of n the map is no longer injective, but it remains surjective, see (de Concini and Procesi 1976). In the first three sections of the chapter we give an account of these facts.

The surjectivity of (8.1) allow us to consider $\mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$-modules as representations of the symmetric group. By ring theory, there is one irreducible representation of $\mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$ for each isomorphism class of indecomposable summands in $\mathbf{N}^{\otimes r}$. The dimension of the irreducible module is given by the multiplicity of the indecomposable module in $\mathbf{N}^{\otimes r}$.

Our interest in the centralizer property (8.1) derives from the fact that $\mathbf{N}^{\otimes r}$ is a tilting module. Thus, every indecomposable summand is tilting. We want to know the multiplicity of each indecomposable tilting module in $\mathbf{N}^{\otimes r}$, as this multiplicity is equal to the dimension of an irreducible representation of the symmetric group. The multiplicity formula of Theorem 6.12 allow us to count the multiplicities of some of the indecomposable tilting modules, thus to calculate the dimension of some of the simple modules of the symmetric group. This application of Theorem 6.12 is the subject of this chapter.

Notation and recollections

- We identify $\mathrm{GL}(\mathbf{N})$ with the group $\mathrm{GL}(n)$ of invertible $n \times n$ matrices with entries from k, by choosing the natural basis of $\mathbf{N} = k^n$.
- The set of diagonal matrices is a maximal torus with character group denoted by X.
- We let ε_i denote projection on the (i,i)'th entry of a matrix; then $\varepsilon_1, \ldots, \varepsilon_n$ is a $\mathbb{Z}$-basis of X. Let $\langle -,- \rangle$ be the bilinear form that makes $\varepsilon_1, \ldots, \varepsilon_n$ an orthonormal basis.
- Inside X we have the root system R of type A_{n-1}; the roots are $\{\alpha_{i,j} = \varepsilon_i - \varepsilon_j \mid i \neq j\}$ and we may choose $\{\alpha_i = \alpha_{i,i+1} \mid i = 1, \ldots, n-1\}$ as the set of simple roots.
- For a root systems of type A the co-roots identify with the roots $\alpha^\vee = \alpha$ for all roots. Let $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. Then $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$, and $\omega_1, \ldots, \omega_n$ are a second $\mathbb{Z}$-basis of X; these weights are called fundamental weights.
- For each root α we have a root subgroup U_α. Then $\mathrm{GL}(n)$ is generated by all root subgroups together with the maximal torus. Let B denote the Borel

subgroup generated by the maximal torus and all root subgroups corresponding to negative roots. Let U^+ denote the group generated by the root subgroups corresponding to positive roots.

- The finite Weyl group W_0 is generated by $s_1, \dots, s_{n-1}$, where s_i maps ε_i to ε_{i+1}, ε_{i+1} to ε_i, and fixes all ε_j with $j \neq i, i+1$. This gives an explicit isomorphism to the symmetric group on n letters.
- Let s_0 denote the map $\lambda \mapsto \lambda - \langle \lambda, \alpha_{1,n}^\vee \rangle \alpha^\vee + p\alpha^\vee$. The affine Weyl group is the group generated by $s_0, s_1, \dots, s_n$. This group usually acts on X by the dot action $w.\lambda = w(\lambda - \rho) + \rho$, where ρ is the sum of all fundamentals weights $\sum_i \omega_i$.
- A weight $n_1\varepsilon_1 + \cdots + n_n\varepsilon_n$ is called polynomial when all $n_i \geq 0$. We let $P(n)$ denote the set of polynomial and dominant weights. Expressed in both bases of X this set is

$$\begin{aligned} P(n) &= \{n_1\varepsilon_1 + \cdots + n_n\varepsilon_n \mid n_1 \geq \dots n_n \geq 0\} \\ &= \{m_1\omega_1 + \cdots + m_n\omega_n \mid m_i \geq 0 \text{ for all } i\} \end{aligned}$$

The degree of a polynomial weight $n_1\varepsilon_1 + \cdots + n_n\varepsilon_n$ is the sum $\sum_i n_i$.

Let us verify that, as promised,

LEMMA 8.1. *$\mathbf{N}^{\otimes r}$ is a tilting module.*

PROOF. The weights of $\mathbf{N}$ are $\varepsilon_1, \dots, \varepsilon_n$. These weights comprise one Weyl group orbit. Since the set of weights of a module is preserved by the finite Weyl group, $\mathbf{N}$ is a simple module with highest weight ε_1. This also shows that $\mathbf{N}$ is the simple quotient of the Weyl module $V_n(\varepsilon_1) = V_n(\omega_1)$. But this Weyl module is simple because ω_1 is minimal among dominant weights. This proves that $\mathbf{N}$ is a simple Weyl module, hence tilting. Now the lemma follows from Theorem 2.15. □

A partition of r is a sequence of non-negative non-increasing integers $n_1 \geq \dots n_n \geq 0$ with sum r. The following map defines a one-to-one correspondence between the set of dominant polynomial weights of $\mathrm{GL}(n)$ and the set of partitions with at most n non-zero parts:

$$n_1\varepsilon_1 + \cdots + n_n\varepsilon_n \longleftrightarrow n_1 \geq \dots n_n \geq 0$$

Hence we may think of an element of $P(n)$ as a partition as well as a polynomial dominant weight. The same map defines a one-to-one correspondence between the set of dominant polynomial $\mathrm{GL}(n)$-weights of degree r and the set of partitions of r with at most n non-zero parts.

Recall that $n_1 \geq \dots n_n \geq 0$ is $p - singular$ if there is a sequence of p equal consecutive parts $n_i = \cdots = n_{i+p}$. A partition is p-regular if it is not p-singular.

Recall from (James 1978) the definition of a family of Σ_r-representations, the Specht-modules Sp^λ parametrized by partitions of r. In prime characteristic these modules are not necessarily simple (as they are in characteristic zero); but the Specht-modules parametrized by p-regular partitions have simple head, which we denote by D^λ. Then $\{D^\lambda \mid \lambda \text{ a } p\text{-regular partition of } r\}$ is a full set of non-isomorphic simple representations of the symmetric group Σ_r.

Restriction from $\mathrm{GL}(n)$ to $\mathrm{GL}(n-1)$

The proof of surjectivity of (8.1) requires some knowledge about restrictions to $\mathrm{GL}(n-1)$. This section provides the necessary tools. We hasten to add that these results really belong in the much broader context of restrictions to Levi subgroups, see (Donkin 1993). Here, however, we do not need the full strength of of the results in (Donkin 1993).

We embed $\mathrm{GL}(n-1)$ in $\mathrm{GL}(n)$, and think of $\mathrm{GL}(n-1)$ as sitting in the top left corner.

$$\begin{pmatrix} \mathrm{GL}(n-1) & \\ & 1 \end{pmatrix} \leq \mathrm{GL}(n).$$

To be more precise, we consider the subgroup of $\mathrm{GL}(n)$ generated by $\ker \varepsilon_n$ and the root subgroups $U_{\alpha_{i,j}}$ with $i, j \neq n$. For a $\mathrm{GL}(n)$-module M we write $M_{\mathrm{GL}(n-1)}$ for the restriction to the subgroup $\mathrm{GL}(n-1)$. The character group of $\ker \varepsilon_n$ is the free $\mathbb{Z}$-module with bases $\varepsilon_1, \ldots, \varepsilon_{n-1}$ and $\omega_1, \ldots, \omega_{n-1}$ (We should really write "ε_1 restricted to $\ker \varepsilon_n$" etc.).

NOTATION 8.2. *In this chapter we will consider* $\mathrm{GL}(n)$*-modules and* $\mathrm{GL}(n-1)$*-modules. To avoid confusion we denote the Weyl, induced, and tilting* $\mathrm{GL}(n)$*-modules with highest weight* λ *by* $V_n(\lambda)$*,* $H_n^0(\lambda)$*, and* $T_n(\lambda)$*.*

Let us consider the restriction of $V_n(\omega_i)$ $(i \neq n)$ to $\mathrm{GL}(n-1)$. Since ω_i is minimal among dominant weights, it follows that

$$\operatorname{ch} V_n(\omega_i) = \sum_{\{w\omega_i \mid w \in W_0\}} e(w\omega_i).$$

As ε_n restricts to the identity we find that

$$\operatorname{ch} V_n(\omega_i)_{\mathrm{GL}(n-1)} = \operatorname{ch} V_{n-1}(\omega_i) + \operatorname{ch} V_{n-1}(\omega_{i-1}).$$

By minimality of the fundamental weights among dominant weights both modules on the right hand side are simple. The $\mathrm{GL}(n-1)$-weights ω_i and ω_{i-1} does not belong to the same orbit under the affine Weyl group; so the linkage principle ensures that

$$V_n(\omega_i)_{\mathrm{GL}(n-1)} \simeq V_{n-1}(\omega_i) \oplus V_{n-1}(\omega_{i-1}).$$

We see that the restriction of the Weyl module $V_n(\omega_i)$ has a filtration of Weyl modules. In this section we will show that this holds for all Weyl modules of $\mathrm{GL}(n)$.

THEOREM 8.3. *Suppose that M is* $\mathrm{GL}(n)$*-module with a Weyl filtration (resp. a good filtration). Then the restriction* $M_{\mathrm{GL}(n-1)}$ *has a Weyl-filtration (resp. a good filtration). If M is tilting, then* $M_{\mathrm{GL}(n-1)}$ *is tilting.*

We will prove the statement for induced modules; the corresponding statement for modules with a Weyl-filtration is then immediate as the dual module has a good filtration. The statement about restriction of a tilting module is clear from the first part of the theorem. Before we begin to prove the theorem, note the following lemma. Recall that $H^0(\lambda) \simeq \operatorname{Ind}_B^G(k_\lambda)$ and that induction is left exact; we let (as is usual) $H^i(-)$ denote the derived functors.

LEMMA 8.4. *Let M be a finite dimensional B-module. Assume that for each weight, μ, of M we have $H_n^1(\mu) = 0$. Then $H_n^0(M)$ has a good filtration, and $H_n^1(M) = 0$*

PROOF. Induction in the dimension of M. □

PROOF OF THEOREM 8.3. It is enough to show that the modules $H_n^0(\lambda)$, $\lambda \in X^+$ restricts to a module with a good filtration by Corollary 2.9.

Consider now a dominant weight λ, so $\lambda = n_1\omega_1 + \cdots + n_n\omega_n$ with each $n_i \geq 0$. The case $\sum_i n_i = 0$ is trivial. When $\sum_i n_i = 1$, we have already proved the result in the beginning of this section as the Weyl modules $V_n(\omega_i)$ are simple, hence isomorphic to $H_n^0(\omega_i)$. We will proceed inductively, so choose a fundamental weight ω so that $\lambda - \omega$ is dominant.

From the surjective B-module homomorphism $p : H_n^0(\omega) \otimes (\lambda - \omega) \longrightarrow \omega + (\lambda - \omega)$ we obtain a long exact sequence beginning with

$$0 \;\to\; H_n^0(\ker p) \;\to\; H_n^0(H_n^0(\omega) \otimes (\lambda - \omega)) \;\to\; H_n^0(\lambda) \;\to\; H_n^1(\ker p).$$

The weights of $\ker p$ are all on the form $\lambda + w\omega$, $w\omega \neq \omega$. By Kempfs vanishing theorem we have $H_n^1(\lambda + w\omega) = 0$ as λ is dominant and $\langle w\omega, \alpha^\vee \rangle \in \{0, \pm 1\}$ in type A_{n-1}. Thus, by Lemma 8.4, we see that $H_n^1(\ker p) = 0$ and that $H_n^0(\ker p)$ has a good filtration. We have (using the tensor identity)

$$0 \;\to\; H_n^0(\ker p) \;\to\; H_n^0(\omega) \otimes H_n^0(\lambda - \omega) \;\to\; H_n^0(\lambda) \;\to\; 0. \tag{8.3}$$

We see that each good factor $H_n^0(\nu)$ of $H_n^0(\ker p)$ has highest weight $\nu < \lambda$; so by induction each $H_n^0(\nu)$ restricts to a module with a good filtration. Then Corollary 2.9 assures that

$H_n^0(\ker p)_{\mathrm{GL}(n-1)}$ has a good filtration. Also by induction, $H_n^0(\lambda-\omega)_{\mathrm{GL}(n-1)}$ has a good filtration. So $H_n^0(\omega)\otimes H_n^0(\lambda-\omega)$ is a tensor product of two $\mathrm{GL}(n-1)$-modules with good filtrations, hence has a good filtration. By Corollary 2.9 we conclude that the restriction of $H_n^0(\lambda)$ has a good filtration. □

REMARK 8.5. *The Weyl factors in $V_n(\lambda)_{\mathrm{GL}(n-1)}$ are known. See (Brundan, Kleshchev and Suprunenko 1998, Proposition A.2) for an explicit description.*

REMARK 8.6. *The inductive argument in the proof of Theorem 8.3 may also be used to give a reasonable short proof that $H^0(\lambda)\otimes H^0(\mu)$ has a good filtration. This approach then yields a proof of the stability of the family of tilting modules under tensor products in type A.*

The functor Tr^r

Given a $\mathrm{GL}(n)$-module M we note that

$$M^r = \sum_{\lambda;\ \deg(\lambda)=r} M_\lambda$$

is a submodule as each root subgroup U_α preserve the degree. In fact M^r is a summand of M as the sum of the weights spaces M_λ with $\deg(\lambda)=r$ are preserved by the affine Weyl group; the linkage principle states that two simple modules may extend non-trivially only when their highest weights are in the same orbit of the affine Weyl group. We say that M is a $\mathrm{GL}(n)$-module in (homogeneous) degree r when $M^r = M$. There are no homomorphisms between modules in unequal degree, since $\mathrm{GL}(n)$-linear homomorphisms preserve the degree of a module.

We will now define a functor from the category of $\mathrm{GL}(n)$-modules to the category of $\mathrm{GL}(n-1)$-modules of degree r.

$$\mathrm{Tr}^r M = (M^r)^r_{\mathrm{GL}(n-1)}.$$

This a basically the functor $\mathrm{Tr}^{n\varepsilon_1}$ used in (Donkin 1993). The functor is exact as it is the composition of truncations to a summand and a restriction, and these are exact. We have, as a first propery of Tr^r,

PROPOSITION 8.7. *If M has a Weyl or a good filtration, then so does $\mathrm{Tr}^r M$. Hence Tr^r takes tilting modules to tilting modules.*

PROOF. Summands in modules with a good filtration has a good filtration, see Corollary 2.9. And Theorem 8.3 shows that the restriction of such a module to $\mathrm{GL}(n-1)$ has a good filtration. □

Recall the definition of the set $P(n)$ of polynomial dominant $\mathrm{GL}(n)$-weights. We consider $P(n-1)$ as a subset of $P(n)$ by $n_1\varepsilon_1+\cdots+n_{n-1}\varepsilon_{n-1}\mapsto n_1\varepsilon_1+\cdots+n_{n-1}\varepsilon_{n-1}+0\varepsilon_n$. Compare the following proposition with the example given in the beginning of the previous section. The proof of the proposition is a simplified version of a proof sketched in (Donkin 1983).

PROPOSITION 8.8. *Let $\lambda\in P(n)$ have degree r.*

(i) *If $\lambda\in P(n-1)$ then $\mathrm{Tr}^r V_n(\lambda)=V_{n-1}(\lambda)$,*
(ii) *If $\lambda\notin P(n-1)$ then $\mathrm{Tr}^r V_n(\lambda)=0$.*

PROOF. Assume that $\lambda\in P(n-1)$. We will prove that

$$\mathrm{ch}\,\mathrm{Tr}^r V_n(\lambda)=\mathrm{ch}\,V_{n-1}(\lambda).$$

This will show (i) as Theorem 8.3 guarantees that $\mathrm{Tr}^r V_n(\lambda)$ has a Weyl filtration. The equality of characters is established by checking that the dimension of each weight space agree. The character of the Weyl module is independent of the characteristic of the ground field, and we may thus apply our characteristic zero methods. We use Kostants formula for the dimension of a weight space:

$$\dim V_n(\lambda)_\mu = \sum_{w\in W_0} (-1)^{l(w)} p(\mu - w.\lambda) \tag{8.4}$$

Here $p(\nu)$ denotes the number of ways ν can be written as a sum of negative roots.

Let μ be a weight of $V_n(\lambda)$ with $\mu - w.\lambda \le 0$. We may assume that $\mu \in P(n-1)$ and that $\deg(\mu) = r$, otherwise the μ-weight spaces of $\mathrm{Tr}^r V_n(\lambda)$ and $V_{n-1}(\lambda)$ are zero. Note that a weight ≤ 0 has ε_n-coefficient ≥ 0, and that a weight ≥ 0 has ε_n-coefficient ≤ 0. From $\mu - w.\lambda \le 0$ and $\lambda - w.\lambda \ge 0$ we find that the ε_n-coefficient of $w.\lambda$ is zero, since $\lambda, \mu \in P(n-1)$. Since $w(\lambda+\rho)-\rho$ has ε_n-coefficient equal to zero, it follows that $w \in W_0$ fixes ε_n. It follows that in (8.4) we need only sum over w that fixes ε_n.

We consider the weights of $\mathrm{GL}(n-1)$ as a subspace of $\mathrm{GL}(n)$-weight space. On this space the Weyl group of $\mathrm{GL}(n-1)$ may be identified with the subgroup of the $\mathrm{GL}(n)$-Weyl group, that fixes ε_n. Further, as $\mu - w.\lambda \in P(n-1)$ we may just as well calculate $p(\mu - w.\lambda)$ as a weight of $\mathrm{GL}(n-1)$. We have

$$\begin{aligned} \dim \mathrm{Tr}^r V_n(\lambda)_\mu &= \sum_{w\in W_0(\mathrm{GL}(n))} (-1)^{l(w)} p(\mu - w.\lambda) \\ &= \sum_{w\in W_0(\mathrm{GL}(n-1))} (-1)^{l(w)} p(\mu - w.\lambda) \;=\; \dim V_{n-1}(\lambda)_\mu. \end{aligned}$$

This shows (i). To see (ii) we note that $\lambda \notin P(n-1)$ means that the ε_n-coefficient is > 0. Then $\lambda \ge \mu$ shows that the ε_n-coefficient of μ is > 0. So we see that $\mu \notin P(n-1)$. It follows that the μ-weight space of $V_n(\lambda)$ is killed by Tr^r. We are done. □

REMARK 8.9. *There is a similar statement about induced modules; the proof for Proposition 8.8 above is essentially a character calculation - and the characters of $V_n(\lambda)$ and $H^0_n(\lambda)$ are equal.*

We prepare the proof of Theorem 8.11 below with a lemma.

LEMMA 8.10. *Let λ, $\mu \in P(r)$. Then*

$$\mathrm{Tr}^r : \mathrm{Hom}_{\mathrm{GL}(n)}(V_n(\lambda), H^0_n(\mu)) \longrightarrow \mathrm{Hom}_{\mathrm{GL}(n-1)}(\mathrm{Tr}^r V_n(\lambda), \mathrm{Tr}^r H^0_n(\mu))$$

is surjective.

PROOF. We are done if $\mathrm{Tr}^r V_n(\lambda)$ or $\mathrm{Tr}^r H^0_n(\mu)$ is zero. So we may as well assume that λ, $\mu \in P(n-1)$. Further $\mathrm{Hom}_{\mathrm{GL}(n-1)}(\mathrm{Tr}^r V_n(\lambda), \mathrm{Tr}^r H^0_n(\mu)) = 0$ if $\lambda \ne \mu$. So we need only consider $\mu = \lambda$. Now let $f : V_n(\lambda) \longrightarrow H^0_n(\lambda)$ be a non-zero $\mathrm{GL}(n)$-homomorphism. Then $\mathrm{Tr}^r f : \mathrm{Tr}^r V_n(\lambda) \longrightarrow \mathrm{Tr}^r H^0_n(\lambda)$ is non-zero on the λ-weight spaces, which are preserved by Tr^r. It follows that

$$\mathrm{Tr}^r : \mathrm{Hom}_{\mathrm{GL}(n)}(V_n(\lambda), H^0_n(\lambda)) \longrightarrow \mathrm{Hom}_{\mathrm{GL}(n-1)}(V_{n-1}(\lambda), H^0_{n-1}(\lambda))$$

is nonzero. Since the last space is one dimensional, we are done. □

THEOREM 8.11. *(Donkin 1993) Suppose that M has a Weyl filtration and that N has a good filtration. Then*

$$\mathrm{Tr}^r : \mathrm{Hom}_{\mathrm{GL}(n)}(M, N) \longrightarrow \mathrm{Hom}_{\mathrm{GL}(n-1)}(\mathrm{Tr}^r M, \mathrm{Tr}^r N)$$

is surjective.

PROOF. The proof runs by induction in the number of subquotients in the filtrations of M, N. Suppose first that $M = V_n(\lambda)$ for some dominant λ. The result follows from Lemma 8.10 in case N is an induced module. So we may assume the existence of N', N'' with good filtrations, so that

$$0 \;\rightarrow\; N' \;\rightarrow\; N \;\rightarrow\; N'' \;\rightarrow\; 0$$

is exact. By exactness of Tr^r we get

$$0 \to \mathrm{Tr}^r N' \to \mathrm{Tr}^r N \to \mathrm{Tr}^r N'' \to 0$$

By Corollary 2.3 both $\mathrm{Ext}^1_{\mathrm{GL}(n)}(V_n(\lambda), N') = 0$ and $\mathrm{Ext}^1_{\mathrm{GL}(n-1)}(\mathrm{Tr}^r V_n(\lambda), \mathrm{Tr}^r N') = 0$. Applying $\mathrm{Hom}_{\mathrm{GL}(n)}(V_n(\lambda), -)$ and $\mathrm{Hom}_{\mathrm{GL}(n-1)}(\mathrm{Tr}^r V_n(\lambda), -)$ to the exact sequences above (and omitting the subscripts on Hom) we get

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}(V_n(\lambda), N') & \to & \mathrm{Hom}(V_n(\lambda), N) & \to & \mathrm{Hom}(V_n(\lambda), N'') & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(\mathrm{Tr}^r V_n(\lambda), \mathrm{Tr}^r N') & \to & \mathrm{Hom}(\mathrm{Tr}^r V_n(\lambda), \mathrm{Tr}^r N) & \to & \mathrm{Hom}(\mathrm{Tr}^r V_n(\lambda), \mathrm{Tr}^r N'') & \to 0. \end{array}$$

The first and last vertical map are surjective by induction. It follows by the snake lemma that the middle map is surjective.

We leave the general case to the reader. □

PROPOSITION 8.12. *Let $\lambda \in P(n)$ have degree r.*

(i) *If $\lambda \in P(n-1)$ then $\mathrm{Tr}^r T_n(\lambda) = T_{n-1}(\lambda)$,*
(ii) *If $\lambda \notin P(n-1)$ then $\mathrm{Tr}^r T_n(\lambda) = 0$.*

PROOF. Let us consider (ii) first. A Weyl factor $V_n(\mu)$ in $T_n(\lambda)$ has highest weight $\mu \leq \lambda$, so $\mu \notin P(n-1)$. From Proposition 8.8 we see that all Weyl factors in $T_n(\lambda)$ get killed by Tr^r.

Proposition 8.7 shows that $\mathrm{Tr}^r T_n(\lambda)$ is tilting. It is clear that this module has highest weight λ if $\lambda \in P(n-1)$. So we must show that it is indecomposable. It follows from Theorem 8.11 that

$$\mathrm{End}_{\mathrm{GL}(n)}(T_n(\lambda)) \longrightarrow \mathrm{End}_{\mathrm{GL}(n-1)}(\mathrm{Tr}^r T_n(\lambda))$$

is surjective. It is also a ring homomorphism: Recall that the functor Tr^r is composed of truncation to degree r, restriction to $\mathrm{GL}(n-1)$, and truncation to degree r. Each of these induces ring homomorphisms; for a $\mathrm{GL}(n)$-module M, $\mathrm{End}_{\mathrm{GL}(n)}(M) \longrightarrow \mathrm{End}_{\mathrm{GL}(n)}(M^r)$ is a ring homomorphism because there is no homomorphisms between modules in unequal degree. It remains to note that the surjective image of a local ring is local. □

Schur-Weyl duality, part one

We now return to the ring homomorphism (8.1) considered in the beginning of this chapter. We shall prove the following theorem in the course of this section.

THEOREM 8.13. *The ring homomorphism $k[\Sigma_r] \longrightarrow \mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$ is surjective for all values of r and n.*

We begin with a proposition, that proves one half of the theorem.

PROPOSITION 8.14. *(Carter and Lusztig 1974) Assume $r \leq n$. Then*

$$k[\Sigma_r] \simeq \mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$$

PROOF. We fix a basis $\{e_1, \dots, e_n\}$ of $\mathbf{N}$, so that e_i has weight ε_i. The corresponding basis of $\mathbf{N}^{\otimes r}$ is denoted by $e_{\mathbf{i}} = e_{i_1} \otimes \cdots \otimes e_{i_r}$. For $t \neq 0$ we let $h_l(t)$ denote the diagonal matrix with $\varepsilon_j(h_l(t)) = 1$ for $j \neq l$, and $\varepsilon_l(h_l(t)) = t$. Then $h_l(t)e_{\mathbf{i}} = t^{[\mathbf{i}:l]}e_{\mathbf{i}}$ where $[\mathbf{i} : l] = \#\{i_j | i_j = l\}$. Therefore $h_l(t)$ determines the number of times a specific e_l appears in a basis vector $e_{\mathbf{i}}$. This is needed in the proof.

We prove surjectivity first. Choose $\phi \in \mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$, let $\mathbf{i} = (1, \dots, n)$, and write $\phi(e_{\mathbf{i}}) = \sum_{\mathbf{j}} c_{\mathbf{ji}} e_{\mathbf{j}}$. Using $\mathrm{GL}(n)$-linearity it follows that

$$\sum_{\mathbf{j}} c_{\mathbf{ji}} t^{[\mathbf{i}:l]} e_{\mathbf{j}} = \phi(h_l(t)e_{\mathbf{i}}) = h_l(t)\phi(e_{\mathbf{i}}) = \sum_{\mathbf{j}} c_{\mathbf{ji}} h_l(t) e_{\mathbf{j}} = \sum_{\mathbf{j}} c_{\mathbf{ji}} t^{[\mathbf{j}:l]} e_{\mathbf{j}}$$

Comparing the coefficients of $e_{\mathbf{j}}$, we see that $c_{\mathbf{ji}}t^{[\mathbf{i}:l]} = c_{\mathbf{ji}}t^{[\mathbf{j}:l]}$. This holds for any $t \neq 0$ and any l, and we conclude that $c_{\mathbf{ji}} \neq 0$ implies $[\mathbf{i} : l] = [\mathbf{j} : l]$ for each l. This shows that $(j_1, \dots, j_r)$ is a permutation of $(1, \dots, n)$, which is denoted σ. We now have

$$\phi(e_{\mathbf{i}}) = \sum_{\sigma \in \Sigma_r} c_{\sigma\mathbf{i},\mathbf{i}} \sigma e_{\mathbf{i}}. \tag{8.5}$$

It remains to prove that $\phi(e_{\mathbf{j}}) = \sum_{\sigma\in\Sigma_r} c_{\sigma\mathbf{i},\mathbf{i}}\sigma e_{\mathbf{j}}$ for all basis vectors $e_{\mathbf{j}}$ of $\mathbf{N}^{\otimes r}$.

Choose a basis vector $e_{\mathbf{j}}$ of $\mathbf{N}^{\otimes r}$. We define an endomorphism, x on $\mathbf{N}$ by $xe_1 = e_{j_1}, \dots, xe_r = e_{j_r}, xe_{r+1} = 0, \dots$ (recall that $r \leq n$). Then x becomes an endomorphism on $\mathbf{N}^{\otimes r}$, and $xe_{\mathbf{i}} = e_{\mathbf{j}}$. Recall that the actions of $\mathrm{GL}(n)$ and Σ_r on $\mathbf{N}^{\otimes r}$ commutes; as endomorphisms of $\mathbf{N}$ we have $k\mathrm{GL}(n) = \mathrm{End}(\mathbf{N})$ so x commutes with any $\sigma \in k\Sigma_r$ as endomorphisms of $\mathbf{N}^{\otimes r}$. Then

$$\phi(e_{\mathbf{j}}) = \phi(xe_{\mathbf{i}}) = x \sum_{\sigma\in\Sigma_n} c_{\sigma\mathbf{i},\mathbf{i}}\sigma e_{\mathbf{i}} = \Big(\sum_{\sigma\in\Sigma_n} c_{\sigma\mathbf{i},\mathbf{i}}\sigma\Big)(e_{\mathbf{j}}).$$

This proves surjectivity. Injectivity follows at once: Assume $0 = \sum_{\sigma\in\Sigma_n} c_\sigma \sigma$. Then in particular

$$0 = \sum_{\sigma\in\Sigma_n} c_\sigma \sigma e_1 \otimes \cdots \otimes e_n.$$

But $\{\sigma(e_1 \otimes \cdots \otimes e_n)\}$ are non-equal basis vectors of $\mathbf{N}^{\otimes r}$, and therefore each $c_\sigma = 0$. $\square$

Let $\mathbf{M}$ denote an vector space of dimension $n-1$ over k with automorphism group $\mathrm{GL}(n-1)$. We embed $\mathrm{GL}(n-1)$ in $\mathrm{GL}(n)$ as in the previous section. We apply the results on restrictions to $\mathrm{GL}(n-1)$ to obtain the second half of the proof of Theorem 8.13.

PROPOSITION 8.15. *We have a surjective ring homomorphism*

$$\mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r}) \longrightarrow \mathrm{End}_{\mathrm{GL}(n-1)}(\mathbf{M}^{\otimes r})$$

PROOF. Considered as a $\mathrm{GL}(n-1)$-module we have $\mathbf{N} \simeq \mathbf{M} \oplus k$ as they are tilting modules with equal characters. It follows that the $\mathrm{GL}(n-1)$-submodule in degree r of $\mathbf{N}^{\otimes r} = (\mathbf{M} \oplus k)^{\otimes r}$ is $\mathbf{M}^{\otimes r}$. Recall the functor Tr^r. We see that $\mathrm{Tr}^r \mathbf{N}^{\otimes r} = \mathbf{M}^{\otimes r}$, so that Tr^r provide us with the homomorphism in the proposition. Surjectivity follows from Theorem 8.11. It is a ring homomorphism by the proof of Proposition 8.12. $\square$

PROOF OF THEOREM 8.13. This proof is merely a restatement of Propositions 8.14 and 8.15. If $r \leq n$ the theorem follows immediately from the first, and if $r > n$ then applying the second a number of times gives a surjective homomorphism:

$$k[\Sigma_r] \simeq \mathrm{End}_{\mathrm{GL}(r)}(\mathbf{R}^{\otimes r}) \longrightarrow \mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r}).$$

Here $\mathbf{R}$ is the natural r-dimensional $\mathrm{GL}(r)$-module. $\square$

Schur-Weyl duality, part two

Recall that the partition $n_1 \geq \cdots \geq n_n$ is identified with the weight $n_1\varepsilon_1 + \cdots + n_n\varepsilon_n$.

THEOREM 8.16. *Let λ denote a p-regular partition of r. Then*

$$\dim D^\lambda = [\mathbf{N}^{\otimes r} : T_n(\lambda)]$$

.

We prepare the proof of Theorem 8.16 with the important Proposition 8.17. The surjection $k\Sigma_r \longrightarrow \mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$ of Theorem 8.13 allow us to consider $\mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$-modules as representations of the symmetric group Σ_r. For each $\lambda \in X^+$ the space $(\mathbf{N}^{\otimes r})^{U^+}_\lambda$ (the U^+-fixpoints of weight λ) is preserved by all $\mathrm{GL}(n)$-linear maps; it is therefore a $\mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$-module. In general, it is not simple but we will produce a simple quotient of this module. This is done in Proposition 8.17 below, that describes the structure of the $\mathrm{End}_{\mathrm{GL}(n)}(\mathbf{N}^{\otimes r})$-modules $(\mathbf{N}^{\otimes r})^{U^+}_\lambda$.

The results in Proposition 8.17 below holds, however, in the more general context of $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-modules, where Q is an arbitrary tilting module. And we will, in fact, need the results of Proposition 8.17 again in the next chapter, Chapter 9, recasted for another Q. Therefore the following proposition is formulated in terms of $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-modules with Q an arbitrary tilting module. Also, we note that the proof of Proposition 8.17 below does not depend on the group $\mathrm{GL}(n)$; but the applications of it, in this and the following chapter, is limited to $\mathrm{GL}(n)$. For the purpose of this chapter the reader is invited to mentally replace each Q by $\mathbf{N}^{\otimes r}$.

We follow the approach of (Mathieu 2000); see in particular Lemma 11.1 in *loc.cit.*

Let a_λ denote the multiplicity $[Q : T_n(\lambda)]$. For each dominant λ, define i_λ and p_λ as the inclusion and projection of $a_\lambda T_n(\lambda)$ in Q.

$$Q = \oplus_\mu a_\mu T_n(\mu) \underset{\xleftarrow{i_\lambda}}{\xrightarrow{p_\lambda}} a_\lambda T_n(\lambda)$$

In the proposition below we abuse notation and write p_λ also for the restriction of p_λ to $Q_\lambda^{U^+}$.

PROPOSITION 8.17.

(i) $p_\lambda : Q_\lambda^{U^+} \longrightarrow (a_\lambda T_n(\lambda))_\lambda^{U^+}$ *is a surjective homomorphism of* $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-*modules.*

(ii) *Each non-zero element of* $a_\lambda T_n(\lambda)_\lambda$ *generates* $Q_\lambda^{U^+}$ *as* $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-*module.*

(iii) $(a_\lambda T_n(\lambda))_\lambda^{U^+}$ *is a simple* $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-*module of dimension* a_λ.

PROOF. Clearly the map in (i) is surjective . To show the first assertion it is enough to see that the restriction of p_λ is $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-linear. The endomorphism $\sigma \in \mathrm{End}_{\mathrm{GL}(n)}(Q)$ acts by restriction on $(a_\lambda T_n(\lambda))_\lambda^{U^+}$, that is as $p_\lambda \sigma i_\lambda$. To prove the proposition we must show that $p_\lambda \sigma(v) = p_\lambda \sigma i_\lambda p_\lambda(v)$ for each $v \in Q_\lambda^{U^+}$.

Recall that $\mathrm{Hom}(V_n(\lambda), Q)$ is isomorphic to $Q_\lambda^{U^+}$ and that the isomorphism is given by evaluation in a fixed nonzero element of $V_n(\lambda)_\lambda$. Any map in $\mathrm{Hom}(V_n(\lambda), Q)$ lifts to $\mathrm{Hom}(T_n(\lambda), Q)$, since $V_n(\lambda)$ is a submodule of $T_n(\lambda)$, the quotient $T_n(\lambda)/V_n(\lambda)$ has a Weyl filtration, and Q has a good filtration. It follows that each $v \in Q_\lambda^{U^+}$ is in the image of some $F \in \mathrm{Hom}(T_n(\lambda), Q)$.

This leads us to consider the map $p_\lambda \sigma F$. Corresponding to the decomposition of Q we have a decomposition of the identity, $\mathrm{Id}_Q = \sum_\mu i_\mu p_\mu$. Thus $p_\lambda \sigma F = \sum_\mu p_\lambda \sigma i_\mu p_\mu F$. Consider one term in the sum

$$T_n(\lambda) \xrightarrow{p_\mu F} a_\mu T_n(\mu) \xrightarrow{p_\lambda \sigma i_\mu} a_\lambda T_n(\lambda) \ . \tag{8.6}$$

If $p_\lambda \sigma(v) \neq 0$ then the map (8.6) is nonzero on the λ-weight space of $T_n(\lambda)$ and have nonzero image in one of the summands of $a_\lambda T_n(\lambda)$. The result is a map $T_n(\lambda) \longrightarrow T_n(\lambda)$ which is nonzero on the λ-weight space, hence not nilpotent. An endomorphism of an indecomposable module is either nilpotent or an automorphism (Fittings lemma). It follows that $T_n(\lambda)$ is a summand of $a_\mu T_n(\mu)$, hence that $\lambda = \mu$. We have shown that

$$p_\lambda \sigma F = p_\lambda \sigma i_\lambda p_\lambda F,$$

which implies $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-linearity of the restriction of p_λ to $Q_\lambda^{U^+}$. We have proved (i).

We prove (ii). Choose a non-zero element $t_\lambda \in T(\lambda)_\lambda$. Since $T(\lambda)$ is a summand of Q we may consider t_λ as an element of Q. We claim that t_λ generates $Q_\lambda^{U^+}$ as a $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-module. To prepare the argument, choose a generator $v_\lambda \in V(\lambda)_\lambda$ of the Weyl module with highest weight λ. Since $V(\lambda)$ is a submodule of $T(\lambda)$ we may fix an embedding that maps v_λ to t_λ.

So let $x \in Q_\lambda^{U^+}$ be arbitrary. We construct a $\sigma \in \mathrm{End}_{\mathrm{GL}(n)}(Q)$ that maps t_λ to x. This will show that t_λ generates $Q_\lambda^{U^+}$ as a $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-module.

Since $x \in Q_\lambda^{U^+}$ there is a $\mathrm{GL}(n)$-homomorphism $V(\lambda) \longrightarrow Q$ that maps v_λ to x. This map factorizes over $T(\lambda)$ as Q has a good filtration (we already used this in the proof of (i)). Hence there is a $\mathrm{GL}(n)$-homomorphism $T(\lambda) \longrightarrow Q$ that maps t_λ to x. The composite of this map with the projection of $\mathrm{GL}(n)$-modules $Q \longrightarrow T(\lambda)$ is a $\mathrm{GL}(n)$-endomorphism of Q that maps t_λ to x. This shows that, as claimed, t_λ generates $Q_\lambda^{U^+}$ as a $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-module.

To finish the proof of (ii) we must show that *any* non-zero element v of $a_\lambda T_n(\lambda)$ generates $Q_\lambda^{U^+}$ as $\mathrm{End}_{\mathrm{GL}(n)}(Q)$-module. For such a v, there is a $\sigma \in \mathrm{End}_{\mathrm{GL}(n)}(a_\lambda T_n(\lambda)) \subset \mathrm{End}_{\mathrm{GL}(n)}(Q)$ that maps v to t_λ. Since t_λ generates all of $Q_\lambda^{U^+}$ we are done.

Finally (iii) is an immediate consequence of (i) and (ii). □

PROOF OF THEOREM 8.16. As representations of the symmetric group, $(\mathbf{N}^{\otimes r})_\lambda^{U^+}$ is isomorphic to Sp^λ. We will not prove this, but we refer to (Carter and Lusztig 1974) for the result; it is, however, not difficult. When λ is a p-regular partition, D^λ is the unique simple quotient of the Specht module Sp^λ. It follows from Proposition 8.17 that the dimension of the simple head of Sp^λ is equal to the multiplicity of $T_n(\lambda)$ in $\mathbf{N}^{\otimes r}$. □

Restrictions from GL(n) to SL(n)

Schur-Weyl duality shows that the multiplicities of tilting $\mathrm{GL}(n)$-modules determine the dimension of simple representations of the symmetric group. On the other hand, Theorem 6.12 allows us to calculate the multiplicities of some tilting $\mathrm{SL}(n)$-modules. To connect these powerful tools we consider in this section restrictions from $\mathrm{GL}(n)$ to $\mathrm{SL}(n)$.

As a subgroup of $\mathrm{GL}(n)$, $\mathrm{SL}(n)$ is generated by all root subgroups U_α, $\alpha \in R$. As a maximal torus we choose the set of diagonal matrices of $\mathrm{SL}(n)$. It is a subgroup of the maximal torus of $\mathrm{GL}(n)$. We denote the restriction of a $\mathrm{GL}(n)$-weight λ by $\overline{\lambda}$. Then $\overline{\omega}_n = 0$ and $\overline{\omega}_1, \dots \overline{\omega}_{n-1}$ spans the weight space of $\mathrm{SL}(n)$.

Recall that the Weyl, induced, and tilting $\mathrm{GL}(n)$-modules with highest weight λ are written $V_n(\lambda)$, $H_n^0(\lambda)$, and $T_n(\lambda)$. We use $V(\lambda)$, $H^0(\lambda)$, and $T(\lambda)$ for Weyl, induced, and tilting $\mathrm{SL}(n)$-modules with highest weight λ.

LEMMA 8.18. *Let λ be a dominant $\mathrm{GL}(n)$-weight. Then $V_n(\lambda)_{\mathrm{SL}_n} = V(\overline{\lambda})$.*

PROOF. Note that $V_n(\lambda)_{\mathrm{SL}(n)}$ has highest weight $\overline{\lambda}$. This gives us a $\mathrm{SL}(n)$-linear map $f : V(\overline{\lambda}) \longrightarrow V_n(\lambda)_{\mathrm{SL}_n}$. Choose a non-zero $v \in V_n(\lambda)_\lambda$, then $v \in \operatorname{im} f$. As $U \subset \mathrm{SL}(n)$ and Uv spans the vector space $V_n(\lambda)$, we see that f is surjective. By Weyls character formula the modules have the same character. □

PROPOSITION 8.19. *Let λ be a dominant $\mathrm{GL}(n)$-weight. Then $T_n(\lambda)_{\mathrm{SL}_n} = T(\overline{\lambda})$.*

PROOF. From Lemma 8.18 it is clear that restriction from $\mathrm{GL}(n)$ to $\mathrm{SL}(n)$ takes tilting modules to tilting modules. Indecomposability follows from $\mathrm{End}_{\mathrm{GL}(n)}(T_n(\lambda)) = \mathrm{End}_{\mathrm{SL}(n)}(T_n(\lambda)_{\mathrm{SL}(n)})$. □

Second cell revisited

So far the weights in the second cell have been described as the dominant weights in the lower closure of alcoves $x.C$ with x in a explicitly described set of Weyl group elements. We will need a "better" description of the weights in the second weight cell, since we want to know the set of partitions for which we can calculate $\dim D^\lambda$. That is, we must understand which $\mathrm{GL}(n)$-weights restricts to $\mathrm{SL}(n)$-weights of the first or second cell. The outcome of this section is a description of the second cell as a the set of dominant weights between certain reflection hyperplanes; see Theorem 8.22 below. We consider at first $\mathrm{SL}(n)$-weights and remark at the end of the section on $\mathrm{GL}(n)$-weights.

Let $D = D_1 \cup D_2$, where

$$D_1 = \{\lambda \in X^+ \mid 0 < \langle \lambda + \rho, \alpha_{1,n-1}^\vee \rangle < p\}$$
$$D_2 = \{\lambda \in X^+ \mid 0 < \langle \lambda + \rho, \alpha_{2,n}^\vee \rangle < p\}$$

We show first that these sets have a property that is very similar to that of the second cell expressed in Lemma 4.17.

LEMMA 8.20.

If $x.0 \in D_1$ and $x > xs$, then $xs.0 \in D_1$.
If $x.0 \in D_2$ and $x > xs$, then $xs.0 \in D_2$.
Consequently, $xs.0 \in D$ when $x.0 \in D$ and $x > xs$.

PROOF. We begin with some alcove geometry. Let n_α, $\alpha \in R_+$ be the set of integers so that

$$n_\alpha p < \langle x.0 + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p$$

Then $\sum_{\alpha \in R_+} |n_\alpha|$ equals the number of reflection hyperplanes in the affine Weyl group that separates C and $x.C$; the equations of these hyperplanes are $\langle \lambda + \rho, \alpha^\vee \rangle = ip$, where $1 \le i \le n_\alpha$ if n_α is positive, and $0 \ge i \ge n_\alpha + 1$ if n_α is negative. This number of separating hyperplanes equals the length of x, see (Humphreys 1990, section 4.4).

Now let $s \in S$. There is exactly one reflection hyperplane that separates the alcoves $x.C$ and $xs.C$. Denote by H the hyperplane that separates $x.C$ and $xs.C$. Let $\mathcal{P}_x$ denote the set of reflection hyperplanes that separates $x.C$ and C. Then $\mathcal{P}_{xs} = \mathcal{P}_x - \{H\}$ if H separates $x.C$ and C, and $\mathcal{P}_{xs} = \mathcal{P}_x \cup \{H\}$ if $x.C$ and C are on the same sides of H: Any hyperplane that separates $x.C$ and C, but not $xs.C$ and C (or vice versa) must necessarily separate $x.C$ and $xs.C$; so $\{H\}$ is the difference between the sets $\mathcal{P}_{xs}$ and $\mathcal{P}_x$.

We will only prove the first claim of the lemma. Suppose $x.0 \in D_1$ and $xs.0 \notin D_1$. By definition of D_1, either

$$-p < \langle xs.0 + \rho, \alpha^\vee \rangle < 0 < \langle x.0 + \rho, \alpha^\vee \rangle < p \qquad \text{for an } \alpha \in R_+, \text{ or}$$
$$0 < \langle x.0 + \rho, \alpha^\vee \rangle < p < \langle xs.0 + \rho, \alpha_{1,n-1}^\vee \rangle < 2p$$

In either case, the number of reflection hyperplanes that separates the alcoves $xs.C$ and C is one higher than the number of hyperplanes that separates the alcoves $x.C$ and C. We have $l(xs) > l(x)$ and the lemma is proved. □

Let $\Sigma = \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \{(s_0 s_1 \dots s_{n-1})^m s_0 s_1 \dots s_k \mid m \ge 0,\ 1 \le k \le n-1\} \cup \{e\}$$
$$\Sigma_2 = \{(s_0 s_{n-1} \dots s_1)^m s_0 s_{n-1} \dots s_k \mid m \ge 0,\ 1 \le k \le n-1\} \cup \{e\}.$$

Recall that the root system is of type A_{n-1}. According to Table 1 on page 33 or Example 4.24 we have $\Sigma = \mathcal{C}(s_0) \cup \{e\}$. This set of Weyl group elements are related to the set of weights above by

PROPOSITION 8.21. $x.0 \in D \iff x \in \Sigma$

PROOF. We will first show that D_1 is stabilized by $s_0 s_1 \cdots s_{n-1}$. We will use that X, the weight space of $\mathrm{SL}(n)$, is a hyperplane in the real vector space spanned by $\varepsilon_1, \dots \varepsilon_n$. So in the calculation below the basis coefficient n_i need not be integers. However, the Weyl group action on X is best expressed in this way.

$$\begin{aligned} &s_0 s_1 \cdots s_{n-1}.(n_1\varepsilon_1 + \dots n_n\varepsilon_n) + \rho \\ &= (n_{n-1}+p+2)\varepsilon_1 + (n_1+n)\varepsilon_2 + (n_2+(n-1))\varepsilon_3 + \cdots + (n_{n-2}+3)\varepsilon_{n-1} + (n_n+1-p)\varepsilon_n \end{aligned}$$

Assume that $\lambda \in D_1$. Using that $0 < \langle \lambda + \rho, \alpha_{n-2}^\vee \rangle$ and the equations above it is routine to verify that

$$0 < \langle \lambda + \rho, \alpha_{1,n-1}^\vee \rangle < p \qquad \text{implies} \qquad 0 < \langle s_0 s_1 \cdots s_{n-1}.\lambda + \rho, \alpha_{1,n-1}^\vee \rangle < p,$$

and that $s_0 s_1 \cdots s_{n-1}$ maps dominant weights to dominant weights. This shows in particular that $(s_0 s_1 \cdots s_{n-1})^m.0 \in D_1$. Then $x.0 \in D_1$ for all $x \in \Sigma_1$ by Lemma 8.20. We may show that $x.0 \in D_2$ for all $x \in \Sigma_2$ by similar arguments.

We turn to the second claim; we will show that $x.0 \in D_1$ only when $x \in \Sigma_1$. We do this by induction in the length of x, the case $l(x) = 0$ being trivial. Assuming $l(x) > 0$ we choose a simple reflection s so that $xs < x$. Then $xs.0 \in D_1$ by Lemma 8.20 and $xs \in \Sigma_1$ by induction. This argument show that, in fact, $xt < x$ implies $xt \in \Sigma_1$; but Σ_1 has only one element of each length so s is the unique simple reflection so that $xs < x$. Then $xs = e$ or $xs \sim_{\mathcal{R}} x$ as Lemma 4.15 shows. So $x \in \mathcal{C}(s_0) \cup \{e\} = \Sigma_1 \cup \Sigma_2$. If $x \in \Sigma_2$ then $xs \in \Sigma_2$, hence $xs \in \Sigma_1 \cap \Sigma_2 = \{e, s_0\}$. Then $x \in \{s_0, s_0 s_{n-1}\}$. But $x \neq s_0 s_{n-1}$ as $s_0 s_{n-1}.0 \notin D_1$ by a simple calculation. So we see that $x \in \Sigma_1$ as claimed.

We may show $x.0 \in D_2$ implies $x \in \Sigma_2$ similarly. □

THEOREM 8.22. *$\underline{c}_1 \cup \underline{c}_2 = D$.*

PROOF. Let $y.\mu$ be a dominant weight with $\mu \in \overline{C}$ and $y \in W^0$ chosen so that $y.\mu$ belongs to the lower closure of the alcove $y.C$.

Suppose first that $y.\mu \in D$. The regular weight $y.0$ belongs D, as the defining equations of D involves only sharp inequalities. Then Proposition 8.21 shows that $y \in \Sigma = \mathcal{C}(s_0) \cup \{e\}$. All dominant weights in the lower closure of $y.C$ then belongs $\underline{c}_1 \cup \underline{c}_2$ by Theorem 5.24.

Suppose that $y.\mu \in \underline{c}_1 \cup \underline{c}_2$. By definition of these weight cells, $y \in \mathcal{C}(s_0) \cup \{e\} = \Sigma$. But then Proposition 8.21 states that $y.0 \in D$. As $y.\mu$ is in the lower closure of the alcove $y.C$ we get (say)

$$0 \leq \langle y.\mu + \rho, \alpha_{1,n-1}^{\vee} \rangle < p.$$

But $y.\mu$ is dominant so the first inequality is sharp; we see that $y.\mu \in D$. We are done. □

We leave the proof of the corollary to the reader. Compare with Figure 5 on page 45.

COROLLARY 8.23. *Let $\lambda = n_1\varepsilon_1 + \cdots + n_n\varepsilon_n$ be a dominant polynomial weight of* $\mathrm{GL}(n)$. *Then $\overline{\lambda} \in \underline{c}_1 \cup \underline{c}_2$ if and only if either $n_1 - n_{n-1} < p - n + 2$ or $n_2 - n_n < p - n + 2$.*

A dimension formula for some simple Σ_r-modules

THEOREM 8.24. *Let $p \geq n$ and let $\lambda = (n_1 \geq \cdots \geq n_n \geq 0)$ denote a partition with at least three parts. We can compute* $\dim D^{\lambda}$ *whenever*

- *$n_1 - n_{n-1} < p - n + 2$ or*
- *$n_2 - n_n < p - n + 2$.*

Explicitly we have

$$\dim D^{\lambda} \quad = \quad [\mathbf{N}_q^{\otimes r} : T_q(\overline{\lambda})].$$

PROOF. This proof is merely a recollection of results scattered around in this chapter and the rest of the thesis. First of all by Theorem 8.16 we have

$$\dim D^{\lambda} = [\mathbf{N}^{\otimes r} : T_n(\lambda)].$$

For two weights λ, μ in a module of homogeneous degree r we have

$$\lambda = \mu \quad \Longleftrightarrow \quad \overline{\lambda} = \overline{\mu}.$$

It follows by Proposition 8.19 that the $\mathrm{GL}(n)$-multiplicities of $\mathbf{N}^{\otimes r}$ is equal to the $\mathrm{SL}(n)$-multiplicities:

$$[\mathbf{N}^{\otimes r} : T_n(\lambda)] = [\mathbf{N}^{\otimes r} : T(\overline{\lambda})].$$

The assumptions in the theorem on the weight λ amounts exactly to $\overline{\lambda} \in D = \underline{c}_1 \cup \underline{c}_2$. Hence by Theorem 6.12

$$[\mathbf{N}^{\otimes r} : T(\overline{\lambda})] = [\mathbf{N}_q^{\otimes r} : T_q(\overline{\lambda})].$$

□

REMARK 8.25. *The multiplicity of an indecomposable quantum tilting summand in a tilting module with known character may be computed, since the characters of the indecomposable quantum tilting modules are known by Theorem 5.7. Thus the right hand side of the formula in Theorem 8.24 is known.*

REMARK 8.26. *Theorem 8.24 generalizes a result of Mathieu (1996), determining the dimension of all simple Σ_r-representations corresponding to p-regular partitions satisfying (with notation as in Theorem 8.24 above)*

$$n_1 - n_n < p - n + 1.$$

REMARK 8.27. *Jensen (1998) shows the multiplicity formula of Theorem 6.12 in type A_2; see Proposition 2.2.3 and the remark following it in* loc.cit. *The corresponding dimension result about simple Σ_r-modules parametrized by Young diagrams with three lines is the subject of (Jensen and Mathieu n.d.).*

Quantum Schur-Weyl duality

The symmetric group Σ_r comprise, with the set of transpositions $\{(1,2),\ldots,(r-1,r)\}$, a Coxeter system. We form the Hecke algebra, $\mathcal{H}$, of Σ_r over $\mathbb{Z}[v,v^{-1}]$, as in the first section of Chapter 4. We write $\mathcal{H}_q$ for the $\mathbb{C}$-algebra obtained by specialization of $\mathbb{Z}[v,v^{-1}]$ via $v \mapsto q \in \mathbb{C}$, q a primitive root of unity. The representation theory of $\mathcal{H}_q$ resembles that of the symmetric group. In particular, the simple modules are indexed by p-regular partitions of r. We denote the simple $\mathcal{H}_q$-module corresponding to the p-regular partitions λ by D_q^λ.

The Hecke algebra $\mathcal{H}_q$ plays the role of the symmetric group in quantum Schur-Weyl duality. Similarly, $\mathrm{GL}(n)$ is replaced by $\mathcal{U}_q(\mathfrak{gl}(n))$. The relations of $\mathcal{U}_q(\mathfrak{gl}(n))$ may be found in (Du 1995), where quantum $\mathfrak{gl}(n)$ is taken up over $\mathbb{Z}[v,v^{-1}]$. From (Du, Parshall and Scott 1998, Theorem 6.3) we get the quantum analogue of the surjection (8.1): The following map is a surjective ring homomorphism

$$\mathcal{H}_q \longrightarrow \mathrm{End}_{U_q(\mathfrak{gl}_n)}(V^{\otimes r}). \tag{8.7}$$

We identify the weightspaces of $\mathrm{GL}(n)$ and $U_q(\mathfrak{gl}_n)$. As in the modular case, the tilting multiplicities of $T_{U_q(\mathfrak{gl}_n)}(\lambda)$ in $V^{\otimes r}$ determine the dimension of some of the simple modules of $\mathcal{H}_q$. By (Du 1995, Proposition 2.7) we may embed $U_q = U_q(\mathfrak{sl}_n)$ into $U_q(\mathfrak{gl}_n)$. Then, in analogy with the modular case,

$$[V^{\otimes r} : T_{U_q(\mathfrak{gl}_n)}(\lambda)] = [V^{\otimes r} : T_q(\overline{\lambda})]. \tag{8.8}$$

COROLLARY 8.28. *Let $p \geq n$ and let $\lambda = (n_1 \geq \cdots \geq n_n \geq 0)$ denote a partition with at least three parts. Assume $n_1 - n_{n-1} < p - n + 2$ or $n_2 - n_n < p - n + 2$. Then*

$$\dim D^\lambda = \dim D_q^\lambda. \tag{8.9}$$

PROOF. We argue as follows, using the equations and theorems above.

$$\begin{aligned}
\dim D^\lambda &= [V^{\otimes r} : T_n(\lambda)] \\
&= [V^{\otimes r} : T(\overline{\lambda})] \\
&= [V_q^{\otimes r} : T_q(\overline{\lambda})] \\
&= [V_q^{\otimes r} : T_{U_q(\mathfrak{gl}_n)}(\lambda)] \\
&= \dim D_q^\lambda
\end{aligned}$$

□

REMARK 8.29. *The dimension of D_q^λ is known; see (Lascoux, Leclerc and Thibon 1996) and (Ariki 1996).*

REMARK 8.30. *Corollary 8.28 proves a special case of Conjecture 15.4 in (Mathieu 2000).*

CHAPTER 9

Howe duality

As in Chapter 8 we let $\mathbf{N}$ denote a n-dimensional vector space over k with group of linear automorphisms identified with $\mathrm{GL}(n)$. Similarly we let $\mathbf{M}$ denote a m-dimensional vector space over k with group of linear automorphisms identified with $\mathrm{GL}(m)$. We strive to keep the notation regarding $\mathrm{GL}(n)$ and $\mathrm{GL}(m)$ as in Chapter 8. To phrase the results in this chapter properly we need to recall notation regarding weights and partitions from Chapter 8, and we need to introduce Young diagrams.

A polynomial dominant weight $n_1\varepsilon_1 + \cdots + n_n\varepsilon_n$ of $\mathrm{GL}(n)$ corresponds to the partition $n_1 \geq \cdots \geq n_n \geq 0$ with at most n parts. A partition is conveniently visualized by its Young diagram; this diagram has n_1 boxes on the first line, n_2 boxes on the second, etc. The transpose of this Young diagram has n_1 boxes in the first column, n_2 boxes in the second, etc. We will denote the transpose of a Young diagram λ by λ^t. The size of this Young diagram is by definition $|\lambda| = \sum_i n_i$ and $|\lambda| = |\lambda^t|$.

DEFINITION 9.1. *We will say that a dominant polynomial weight $n_1\varepsilon_1 + \cdots + n_n\varepsilon_n$ of $\mathrm{GL}(n)$ is m-*bounded *if $n_1 \leq m$ (equivalently $n_i \leq m$ for all i). We write $P(n)_{m\text{-bounded}}$ for the set of these weights.*

Note that $P(n)_{m\text{-bounded}}$ is a finite set as the Young diagrams corresponding to its weights are contained in a $n \times m$ rectangle. In fact, the elements in $P(n)_{m\text{-bounded}}$ correspond to all Young diagrams contained in a $n \times m$ rectangle.

There is an operation on polynomial dominant weights that corresponds to transposition of a Young diagram, since these weights corresponds to Young diagrams. We describe this operation now. Let $\lambda \in P(n)_{m\text{-bounded}}$, and express this weight in the basis of fundamental weights as

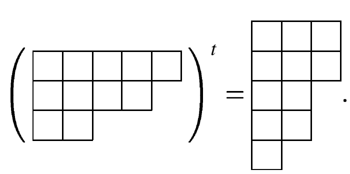
$$\lambda = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_m}. \tag{9.1}$$

Here $i_1 \geq i_2 \geq \ldots i_m \geq 0$. Note that the sequence is only non-increasing and that trailing zeroes are allowed. By convention $\omega_0 = 0$. For this weight λ define

$$\lambda^t = i_1\varepsilon_1 + i_2\varepsilon_2 + \cdots + i_m\varepsilon_m$$

This operation on polynomial dominant weights corresponds exactly to transposition of the corresponding Young diagram. We will abuse notation and write $\lambda \mapsto \lambda^t$ for this operation. As transposition takes Young diagrams contained in a $n \times m$ rectangle to a Young diagram contained in a $m \times n$ rectangle, it is clear that $\lambda \mapsto \lambda^t$ maps m-bounded polynomial dominant weights of $\mathrm{GL}(n)$ to n-bounded polynomial dominant weights of $\mathrm{GL}(m)$. In effect, $\lambda \mapsto \lambda^t$ is a map $P(n)_{m\text{-bounded}} \longrightarrow P(m)_{n\text{-bounded}}$.

As an example, consider $\lambda = 5\varepsilon_1 + 4\varepsilon_2 + 2\varepsilon_3 = \omega_3 + \omega_3 + \omega_2 + \omega_2 + \omega_1$. Therefore $\lambda^t = 3\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4 + \varepsilon_5$; considered as an operation on Young diagrams we have



Simple $\mathrm{GL}(m)$-modules

In Chapter 8 we used a module with a $\mathrm{GL}(n)$-structure *and* a Σ_r-structure to convey information from the representation theory of $\mathrm{GL}(n)$ to that of the symmetric group. Here we follow the same strategy; we use a module with two structures, one of $\mathrm{GL}(n)$-modules and one of $\mathrm{GL}(m)$-modules. Consider $\mathbf{N}$ as a trivial $\mathrm{GL}(m)$-module and $\mathbf{M}$ as a trivial $\mathrm{GL}(n)$-module. Then the exterior product $\bigwedge(\mathbf{N}\otimes\mathbf{M})$ is a $\mathrm{GL}(n)$-module as well as a $\mathrm{GL}(m)$-module, and the actions of the groups commute. This induces natural ring homomorphisms

$$k\,\mathrm{GL}(m) \longrightarrow \mathrm{End}_{\mathrm{GL}(n)}(\textstyle\bigwedge(\mathbf{N}\otimes\mathbf{M})) \tag{9.2}$$

$$k\,\mathrm{GL}(n) \longrightarrow \mathrm{End}_{\mathrm{GL}(m)}(\textstyle\bigwedge(\mathbf{N}\otimes\mathbf{M})) \tag{9.3}$$

THEOREM 9.2. *(Donkin 1993, Proposition 3.11) The ring homomorphisms of (9.2) and (9.3) are surjective.*

Now a decomposition in indecomposables of the $\mathrm{GL}(n)$-module $\bigwedge(\mathbf{N}\otimes\mathbf{M})$ would give us the dimensions of some simple $\mathrm{GL}(m)$-modules. We begin with

LEMMA 9.3. $\bigwedge(\mathbf{N}\otimes\mathbf{M})$ *is a tilting* $\mathrm{GL}(n)$*-module. Thus all summands in* $\bigwedge(\mathbf{N}\otimes\mathbf{M})$ *are tilting.*

PROOF. We consider first $\bigwedge^r\mathbf{N}$. The module $\mathbf{N}$ has weights $\varepsilon_1,\dots,\varepsilon_n$ each with multiplicity one. So $\bigwedge^r\mathbf{N}$ has weights $\varepsilon_{i_1}+\cdots+\varepsilon_{i_r}$ with multiplicity one for each sequence $i_1<\cdots<i_r$. It follows that $\bigwedge^r\mathbf{N}$ has highest weight ω_r and that the weights of $\bigwedge^r\mathbf{N}$ comprise one orbit under the finite Weyl group (recall that this group acts as the symmetric group on $\varepsilon_1,\dots,\varepsilon_n$). So $\bigwedge^r\mathbf{N}$ is simple with highest weight ω_r. The Weyl module $V(\omega_r)$ is simple, as ω_r is a minimal dominant weight, and simple Weyl modules are tilting, so $\bigwedge^r\mathbf{N}=T(\omega_r)$.

Each summand in $\bigwedge\mathbf{N}=\oplus_{r\le n}\bigwedge^r\mathbf{N}$ is tilting, so $\bigwedge\mathbf{N}$ is tilting. As a $\mathrm{GL}(n)$-module $\bigwedge(\mathbf{N}\otimes\mathbf{M})=\bigwedge(\mathbf{N}\oplus\cdots\oplus\mathbf{N})$ (m copies), since $\mathbf{M}$ is a trivial and m-dimensional $\mathrm{GL}(n)$-module. Note the identity $\bigwedge(m\mathbf{N})=(\bigwedge\mathbf{N})^{\otimes m}$; both modules are equal to the direct sum of all $\bigwedge^{i_1}\mathbf{N}\otimes\cdots\otimes\bigwedge^{i_m}\mathbf{N}$ with $1\le i_1,\dots,i_m\le n$. Then the lemma follows as tensor products of tilting modules are tilting (Theorem 2.15). $\square$

Recall Proposition 8.17. This proposition leads us to examine the $\mathrm{GL}(m)$-module $(\bigwedge(\mathbf{N}\otimes\mathbf{M}))_\lambda^{U_n^+}$; here λ is a $\mathrm{GL}(n)$-weight and we let U_n^+ denote the subgroup of $\mathrm{GL}(n)$ generated by the root subgroups corresponding to positive roots. Similarly, let U_m^+ denote the subgroup of $\mathrm{GL}(m)$ generated by the root subgroups corresponding to positive roots. Since the actions of $\mathrm{GL}(n)$ and $\mathrm{GL}(m)$ commute it is clear that the action of $\mathrm{GL}(m)$ preserve $\mathrm{GL}(n)$-weight spaces and all U_n^+-fixpoints.

PROPOSITION 9.4. *(Mathieu 2000, Lemma 12.3) Suppose that* $\lambda\in P(n)_{m\text{-}bounded}$. *There is an isomorphism of* $\mathrm{GL}(m)$*-modules,*

$$V_m(\lambda^t)\simeq\Big(\bigwedge(\mathbf{N}\otimes\mathbf{M})\Big)_\lambda^{U_n^+}.$$

PROOF. The first part of the proof produces a remarkable element $\overline{w}$ in $(\bigwedge(\mathbf{N}\otimes\mathbf{M}))_\lambda^{U_n^+}$. Besides being U_n^+-invariant and having $\mathrm{GL}(n)$-weight λ, this element is U_m^+-invariant with $\mathrm{GL}(m)$-weight λ^t.

Recall from Chapter 8 that $e_1,\dots,e_n$ is a basis of $\mathbf{N}$, chosen so that e_i has $\mathrm{GL}(n)$-weight ε_i. Similarly, fix a basis $f_1,\dots,f_n$ of $\mathbf{M}$, chosen so that f_i has $\mathrm{GL}(m)$-weight ε_i. Let us write b_{ij} for the tensor product $e_i\otimes f_j$ to simplify notation. Clearly, b_{ij} has $\mathrm{GL}(n)$-weight ε_i and $\mathrm{GL}(m)$-weight ε_j. Let us visualize the basis of $\mathbf{N}\otimes\mathbf{M}$ by arranging it in an

$n \times m$-matrix

$$\begin{matrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{matrix} \tag{9.4}$$

Then U_n^+ and U_m^+ acts by

$$ub_{ij} \in b_{ij} + \sum_{1 \le l < i} kb_{lj} \qquad u \in U_n^+; \tag{9.5}$$

$$ub_{ij} \in b_{ij} + \sum_{1 \le l < i} kb_{il} \qquad u \in U_m^+. \tag{9.6}$$

Note that the basis vectors b_{lj} appearing in (9.5) are above and in the same column as b_{ij}. Similarly the basis vectors b_{il} appearing in (9.6) are to the left and in the same row as b_{ij}.

Consider $\lambda \in P(n)_{m\text{-bounded}}$ as a partition, so $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$. Then form the tensor product w of the first λ_1 basis vectors on row 1 of the matrix (9.4), the first λ_2 basis vectors on row 2, etc. Equivalently (but perhaps easier to visualize), think of λ as a Young diagram, and place λ in the top left corner of the matrix of basis vectors (9.4); in the following figure this is illustrated for $\lambda = (3,1)$.

b_{11}	b_{12}	b_{13}
b_{21}		

Let $\mathcal{Z}$ denote the set of basis vectors from the matrix (9.4) that is contained in the Young diagram λ. Then w is simply the tensor product of all these basis vectors. So w is a tensor product of $|\lambda| = \sum_i \lambda_i$ unequal basis vectors, and $w \in (\mathbf{N} \otimes \mathbf{M})^{\otimes|\lambda|}$. It is clear that w has GL(n)-weight λ and GL(m)-weight λ^t. Further, for a $u \in U_n^+$ we have by (9.5) that

$$uw \in \sum_{b_{i_l j_l} \in \mathcal{Z}} kb_{i_1 j_1} \otimes \cdots \otimes b_{i_{|\lambda|} j_{|\lambda|}} \tag{9.7}$$

We consider the space $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})$. To describe a basis, order the basis vectors of $\mathbf{N} \otimes \mathbf{M}$ in some way. Then $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})$ has basis $\{b_{i_1 j_1} \wedge \cdots \wedge b_{i_{|\lambda|} j_{|\lambda|}} \mid b_{i_1 j_1} < \cdots < b_{i_{|\lambda|} j_{|\lambda|}}\}$. So a basis vector corresponds to a choice of $|\lambda|$ unequal basis vectors from the matrix (9.4).

We claim that the image, $\overline{w}$, of w in $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})$ is U_n^+-invariant. This follows from equation (9.7), as the image of $b_{i_1 j_1} \otimes \cdots \otimes b_{i_{|\lambda|} j_{|\lambda|}}$ is zero unless $b_{i_1 j_1}, \ldots, b_{i_{|\lambda|} j_{|\lambda|}}$ are pairwise distinct. A completely similar argumentation shows that $\overline{w}$ is invariant under the action of U_m^+.

This gives us our element $\overline{w} \in (\bigwedge(\mathbf{N} \otimes \mathbf{M}))_\lambda^{U_n^+}$; it is also U_m^+-invariant with GL(m)-weight λ^t as claimed.

Next we claim that $\overline{w} \in (a_\lambda T_n(\lambda))_\lambda^{U_n^+} \subset (\bigwedge(\mathbf{N} \otimes \mathbf{M}))_\lambda^{U_n^+}$. To see this we consider the GL(m)-weight space $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})_{\lambda^t}$, which contains $\overline{w}$. A basis of $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})_{\lambda^t}$ is given by all

$$b_{i_1 j_1} \wedge \cdots \wedge b_{i_{|\lambda|} j_{|\lambda|}} \text{ with } \sum_{1 \le l \le |\lambda|} \varepsilon_{j_l} = \lambda^t. \tag{9.8}$$

The condition $\sum_{1 \le l \le |\lambda|} \varepsilon_{j_l} = \lambda^t$ amounts to demanding that $\#\{j_l = 1\}$ of the vectors $b_{i_l j_l}$ are chosen from the first column of the matrix (9.4), that $\#\{j_l = 2\}$ of the vectors $b_{i_l j_l}$ are chosen from the second column, etc. The GL(n)-weight of a basis vector from (9.8) is $\sum_{1 \le l \le |\lambda|} \varepsilon_{i_l}$. We get the largest possible GL(n)-weight by choosing the vectors $b_{i_l j_l}$ from the top of the each column. In particular, we find that λ is a maximal GL(n)-weight of the GL(n)-module $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})_{\lambda^t}$, and that $\overline{w}$ is a maximal weight vector.

Since each GL(m)-weight space $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})_\mu$ is a GL(n)-summand of the GL(n)-tilting module $\bigwedge(\mathbf{N} \otimes \mathbf{M})$, we find that $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})_{\lambda^t}$ is GL(n)-tilting with highest weight λ. It follows that $\overline{w} \in (a_\lambda T_n(\lambda))_\lambda^{U_n^+}$ as claimed.

Recall that $\overline{w}$ is U_m^+-invariant of weight λ^t. So we get a $\mathrm{GL}(m)$-linear homomorphism $V_m(\lambda^t) \longrightarrow (\bigwedge(\mathbf{N}\otimes\mathbf{M}))_\lambda^{U_n^+}$. It is surjective as $\overline{w} \in (a_\lambda T_n(\lambda))_\lambda^{U_n^+}$; Proposition 8.17 states that $(\bigwedge(\mathbf{N}\otimes\mathbf{M}))_\lambda^{U_n^+}$ is generated by $\overline{w}$. So we find that $(\bigwedge(\mathbf{N}\otimes\mathbf{M}))_\lambda^{U_n^+}$ is a quotient of the Weyl module $V_m(\lambda^t)$. We finish the proof of the proposition by calculating dimensions.

We have $(\bigwedge(\mathbf{N}\otimes\mathbf{M}))_\lambda^{U_n^+} \simeq \mathrm{Hom}_{\mathrm{GL}(n)}(V_n(\lambda), \bigwedge(\mathbf{N}\otimes\mathbf{M}))$. As $\bigwedge(\mathbf{N}\otimes\mathbf{M})$ has a good $\mathrm{GL}(n)$-filtration, it follows from Corollary 2.3 that

$$\dim\left(\bigwedge(\mathbf{N}\otimes\mathbf{M})\right)_\lambda^{U_n^+} = [\bigwedge(\mathbf{N}\otimes\mathbf{M}) : H_n^0(\lambda)].$$

We will now reduce to characteristic zero. The character of $\mathbf{N}\otimes\mathbf{M}$ is independent of characteristic; this implies that also the character of $\bigwedge(\mathbf{N}\otimes\mathbf{M})$ is independent of the characteristic of the ground field. As the character of each induced module is the same in prime characteristic and in characteristic zero, we find that $[\bigwedge(\mathbf{N}\otimes\mathbf{M}) : H_n^0(\lambda)]$ is independent of the characteristic of k. So the proposition follows from the characteristic zero equality:

$$\dim V_m(\lambda^t) = [\bigwedge(\mathbf{N}\otimes\mathbf{M}) : H_n^0(\lambda)].$$

Recalling that $V_m(\lambda^t)$ and $H_n^0(\lambda)$ are simple in characteristic zero, we recognize this as Howe's (1995) result over $\mathbb{C}$. $\square$

The simple $\mathrm{GL}(m)$-module with highest weight λ is denoted $L_m(\lambda)$.

THEOREM 9.5. *Let $\lambda \in P(m)_{n\text{-bounded}}$ denote a $\mathrm{GL}(m)$-weight. Then*

(i) $\dim L_m(\lambda) = [\bigwedge(\mathbf{N}\otimes\mathbf{M}) : T_n(\lambda^t)]$

(ii) $\dim L_m(\lambda) = [\bigwedge^{|\lambda|}(\mathbf{N}\otimes\mathbf{M}) : T_n(\lambda^t)]$

(iii) *Suppose $\mu \in P(m)_{n\text{-bounded}}$, $\mu = m_1\varepsilon_1 + \cdots + m_m\varepsilon_m$. Then*

$$\dim L_m(\lambda)_\mu = [\bigwedge^{m_1}\mathbf{N}\otimes\cdots\otimes\bigwedge^{m_m}\mathbf{N} : T_n(\lambda^t)]$$

.

PROOF. Proposition 9.4 states that $V_m(\lambda)$ is isomorphic to $(\bigwedge(\mathbf{N}\otimes\mathbf{M}))_{\lambda^t}^{U_n^+}$ as $\mathrm{GL}(m)$-modules; this module has a simple quotient of dimension $[\bigwedge(\mathbf{N}\otimes\mathbf{M}) : T_n(\lambda^t)]$ as Proposition 8.17 assures. It is a well known fact that the Weyl module $V_m(\lambda)$ has simple head equal to $L_m(\lambda)$. This shows (i).

Recall from Chapter 8 that we write M^r for the largest submodule of homogeneous degree r in a $\mathrm{GL}(n)$-module M. Then note that all $\mathrm{GL}(n)$- and all $\mathrm{GL}(m)$-weights of $\bigwedge^r(\mathbf{N}\otimes\mathbf{M})$ have degree r. Therefore

$$\left(\bigwedge(\mathbf{N}\otimes\mathbf{M})\right)^r = \bigwedge^r(\mathbf{N}\otimes\mathbf{M})$$

Since $T_n(\lambda)$ has homogeneous degree $|\lambda|$ it follows that we must look for these tilting modules in $(\bigwedge(\mathbf{N}\otimes\mathbf{M}))^{|\lambda|}$. This shows (ii).

Let T_m denote the maximal torus of $\mathrm{GL}(m)$. As $\mathrm{GL}(n)\times T_m$ module

$$\begin{aligned}\bigwedge^{|\lambda|}(\mathbf{N}\otimes\mathbf{M}) &= \bigwedge^{|\lambda|}((\mathbf{N}\otimes f_1)\oplus\cdots\oplus(\mathbf{N}\otimes f_m)) \\ &= \bigoplus_{\sum_i k_i = |\lambda|} (\bigwedge^{k_1}\mathbf{N}\otimes f_1)\otimes\cdots\otimes(\bigwedge^{k_m}\mathbf{N}\otimes f_m)\end{aligned}$$

The summand $(\bigwedge^{k_1}\mathbf{N}\otimes f_1)\otimes\cdots\otimes(\bigwedge^{k_m}\mathbf{N}\otimes f_m)$ has T_m-weight $\sum_i k_i\varepsilon_i$. Recall from Proposition 9.4 that as $\mathrm{GL}(m)$-modules, $L_m(\lambda)$ and $(a_\lambda T(\lambda))_{\lambda^t}^{U_n^+}$ are isomorphic. Therefore

$$\dim L_m(\lambda)_\mu = [\bigwedge^{m_1}\mathbf{N}\otimes\cdots\otimes\bigwedge^{m_m}\mathbf{N} : T_n(\lambda^t)]$$

as claimed. $\square$

Let λ be a n-bounded $\mathrm{GL}(m)$-weight. Recall from the beginning of this chapter that we write $\lambda = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_n}$, with $i_1 \geq i_2 \geq \ldots i_n \geq 0$. Then the transpose of this weight is $\lambda^t = i_1\varepsilon_1 + i_2\varepsilon_2 + \cdots + i_n\varepsilon_n$. The size $|\lambda|$ of λ is the sum $\sum_j i_j$.

THEOREM 9.6. *Let $\lambda \in P(m)_{n\text{-bounded}}$ with n satisfying $p \geq n \geq 3$, and define $i_1 \geq \dots i_n \geq 0$ by $\lambda = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_n}$. We can calculate* $\dim L_m(\lambda)$ *whenever*

$i_1 - i_{n-1} < p - n + 2$ *or*
$i_2 - i_n < p - n + 2$

Explicitly we have

$$\dim L_m(\lambda) = [\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})_q : T_q(\overline{\lambda^t})]$$

PROOF. From Theorem 9.5 we recall

$$\dim L_m(\lambda) = [\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M}) : T_n(\lambda^t)].$$

The tilting GL(n)-module $\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})$ is of homogeneous degree $|\lambda|$. We consider now its restriction to SL(n). For two weights λ, μ of equal degree we have

$$\lambda = \mu \quad \Longleftrightarrow \quad \overline{\lambda} = \overline{\mu}.$$

Therefore we find that

$$[\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M}) : T_n(\lambda^t)] = [\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M}) : T(\overline{\lambda^t})],$$

where the right hand side is the SL(n)-multiplicity. The assumptions ensures that the GL(n)-weight $\overline{\lambda^t}$ belongs to $\underline{c}_1 \cup \underline{c}_2$ (Corollary 8.23). Then by Theorem 6.12

$$[\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M}) : T(\overline{\lambda^t})] = [\bigwedge^{|\lambda|}(\mathbf{N} \otimes \mathbf{M})_q : T_q(\overline{\lambda^t})].$$

□

REMARK 9.7. *The right hand side is (in principle) known, by Theorem 5.7.*

COROLLARY 9.8. *Let $\lambda = a\omega_m + \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_n}$ with $a \in \mathbb{Z}$, $i_1 \geq \dots i_n \geq 0$, and $p \geq n \geq 3$. Then* $\mathrm{ch}\, L_m(\lambda)$ *is computable if either*

$i_1 - i_{n-1} < p - n + 2$ *or*
$i_2 - i_n < p - n + 2.$

PROOF. Let $\lambda_1 = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_n}$. Then

$$L_m(\lambda) \simeq L_m(\lambda_1) \otimes \det{}^a,$$

where det is the one-dimensional representation of GL(m) with each $g \in \mathrm{GL}(m)$ acting as multiplication by $\det(g)$. This allows us to reduce to $\lambda = \lambda_1$. But then Theorem 9.5 (iii) shows that we can calculate the dimension of each weight space in $L_m(\lambda)$. We are done. □

EXAMPLE 9.9. *Consider the weight $\lambda = \omega_5 + \omega_4 + \omega_1$. This weight satisfies the assumptions in Corollary 9.8, so for all $m \geq 5$ and all $p \geq 3$ we may calculate the character of $L_m(\lambda)$. Note that in this example p may be smaller than the Coxeter number of* GL(m).

REMARK 9.10. *Based on a determination of $[Q : T(\lambda)]$ with $\lambda \in C$, Mathieu and Papadopoulo (1999) gave a character formula for $L_m(\lambda)$ with λ satisfying (in our notation)*

$$i_1 - i_n \; < \; p - n + 1$$

Corollary 9.8 allows us to calculate the characters of $L_m(\lambda)$ with λ in a larger set of weights.

REMARK 9.11. *Let $\lambda = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_n}$ with $i_1 \geq \dots i_n \geq 0$ and $p \geq n \geq 3$. Assume that $i_1 - i_{n-1} < p - n + 2$ or $i_2 - i_n < p - n + 2$.*

(i) *The character of $L_m(\lambda)$ is independent of m, provided that $m \geq i_1$.*
(ii) *Note that λ satisfy the same assumptions for all primes $p' > p$. Therefore the character of $L_m(\lambda)$ may be calculated in any characteristic larger than p. However, the character is not independent of p (compare with (Mathieu and Papadopoulo 1999)).*

(iii) *Define the following subset of* $\mathrm{GL}(m)$*-weights:*

$$X_1 = \{\mu \in X^+ \mid \langle \mu, \alpha^\vee \rangle < p \text{ for all simple roots } \alpha\}.$$

Then $\lambda \in X_1$ *unless* $\lambda = p\omega_i$ *for some* $i < m$.

Suppose that $\lambda_0, \ldots, \lambda_r$ *satisfy the assumptions of* λ *and that each* $\lambda_j \neq p\omega_i$, $i < m$. *By Steinbergs tensor product theorem, we may calculate the character of* $L_m(\sum_j p^j \lambda_j)$.

EXAMPLE 9.12. *Consider the* $\mathrm{GL}(m)$*-weight* $b\omega_i + \omega_j$ *with* $m \geq i \geq j$ *and assume first that* $0 \leq b \leq p$. *Then* $b\omega_i + \omega_j$ *fulfills the conditions of Corollary 9.8. So we may calculate the character of* $L_m(b\omega_i + \omega_j)$.

Assume now only $b \geq 0$. *We consider the* p*-adic expansion of* b,

$$b = \sum_{r \geq l \geq 0} b_l p^l \qquad\qquad 0 \leq b_l \leq p-1.$$

Note that each $b_l\omega_i$ *satisfies the assumptions of Corollary 9.8. By Steinbergs tensor product theorem we get*

$$\operatorname{ch} L_m(b\omega_i + \omega_j) = \operatorname{ch} L_m(b_0\omega_i + \omega_j)\operatorname{ch} L_m(b_1\omega_i)^{(\mathrm{Fr})} \cdots \operatorname{ch} L_m(b_r\omega_i)^{(\mathrm{Fr}^r)}$$

Here $M^{(\mathrm{Fr})}$ *denotes the vector space* M *with the action of* $g \in \mathrm{GL}(m)$ *twisted by the Frobenius map. Since we are able to calculate the character of each term in the product on the right hand side, we can calculate* $\operatorname{ch} L_m(b\omega_i + \omega_j)$ *for all* $b \geq 0$ *and for all* $p \geq 3$.

CHAPTER 10

Modular weight cells

We ask the reader to review the last section of Chapter 5, which describes weight cells. Recall that two dominant weights belong to the same weight cell when the corresponding quantum tilting modules generates the same tensor ideal. The concept of a weight cell has an analogue in the theory of modular tilting modules. In this chapter we consider these modular weight cells and explore the connections between weight cells and modular weight cells. Let $\mathcal{T}$ denote the full subcategory of modular tilting modules. We assume $p \geq h$ throughout.

DEFINITION 10.1. *(Andersen 2001a) Write $\mu \leq_{\mathcal{T}} \lambda$ if $[T(\lambda) \otimes Q : T(\mu)] \neq 0$ for some modular tilting module Q.*

REMARK 10.2. *Note that $\leq_{\mathcal{T}}$ is a preorder, since $[T(\lambda) \otimes Q_1 : T(\mu)] \neq 0$ and $[T(\nu) \otimes Q_2 : T(\lambda)] \neq 0$ gives $[T(\nu) \otimes Q_1 \otimes Q_2 : T(\mu)] \neq 0$.*

DEFINITION 10.3. *Let $\sim_{\mathcal{T}}$ be the equivalence relation defined by $\leq_{\mathcal{T}}$. The equivalence classes of $\sim_{\mathcal{T}}$ are called* modular weight cells*. The preorder $\leq_{\mathcal{T}}$ induces a partial order (also denoted $\leq_{\mathcal{T}}$) on the set of weight cells in the natural way.*

The following lemma is obvious from the definition. It merely states the connection to tensor ideals of modular tilting modules.

LEMMA 10.4. *If $\mu \leq_{\mathcal{T}} \lambda$ then $T(\mu)$ belongs to all tensor ideals that contain $T(\lambda)$. The set $\{T(\mu) \mid \mu \leq_{\mathcal{T}} \lambda\}$ spans a tensor ideal in $\mathcal{T}$.*

REMARK 10.5.

(i) *If $\mu = \lambda + \nu$ and all three weight are dominant then $\mu \leq_{\mathcal{T}} \lambda$ as weight considerations shows that $T(\mu)$ is a summand of $T(\lambda) \otimes T(\nu)$.*
(ii) *if $T(\mu)$ is a summand in a translation or a wallcrossing of $T(\lambda)$ then $\mu \leq_{\mathcal{T}} \lambda$.*

Recall that we write $\check{A}$ for the lower closure of an alcove A.

LEMMA 10.6. *Let $\underline{c}$ denote a modular weight cell. Let $x \in W^0$.*
Then $x.0$ belongs to $\underline{c}$ if and only if $x.\check{C} \cap X^+$ is contained in $\underline{c}$.

PROOF. Let $\lambda \in \overline{C}$ so that $x.\lambda \in x.\check{C} \cap X^+$. For such a weight we have (see (Andersen 2000, Proposition 5.2))

$$\begin{aligned} T_0^{\lambda} T(x.0) &\simeq T(x.\lambda)^{\oplus |W_\lambda|} \\ T_\lambda^0 T(x.\lambda) &\simeq T(x.0) \end{aligned}$$

So we see that $x.\lambda \leq_{\mathcal{T}} x.0$ (by the first equation and Remark 10.5 (ii) above) and that $x.0 \leq_{\mathcal{T}} x.\lambda$. □

Lemma 10.6 shows that a modular weight cell is a union of all dominant weights in the lower closure of a set of alcoves. As Remark 5.23 (iii) shows, this is also the case for weight cells.

LEMMA 10.7. *The first weight cell $\underline{c}_1$ is a modular weight cell. It is maximal among modular weight cells.*

PROOF. The weight cell $\underline{c}_1$ is equal to C, the weights in the first alcove. Lemma 10.6 shows that $\underline{c}_1$ is contained in a modular weight cell.

Let $\lambda \in C$. It is well known that $X^+ \setminus C$ is a tensor ideal, see (Mathieu 1996). If $T(\lambda)$ is a summand in a tensor product $T(\mu) \otimes Q$, we see that $\mu \in C$. It follows that $\lambda \leq_{\mathcal{T}} \mu$ implies that $\mu \in C$. This shows both assertions in the lemma. □

Next we consider the second cell. Note that we do not impose restriction on the type of root system here.

LEMMA 10.8. *Let $\lambda \in \underline{c}_2$ and assume that $\lambda \leq_{\mathcal{T}} \mu \notin \underline{c}_1$. Then $\mu \in \underline{c}_2$.*

PROOF. We first consider the quantization of $T(\mu)$. By Lemma 6.4, there are no summands with highest weight in the first weight cell $\underline{c}_1$. We have

$$T(\mu)_q = \bigoplus_{\nu \notin \underline{c}_1} a_{\nu\mu} T_q(\nu)$$

The assumption that $\lambda \leq_{\mathcal{T}} \mu$ give us a modular tilting module M so that

$$0 \neq [T(\mu) \otimes M : T(\lambda)] \leq [(T(\mu) \otimes M)_q : T_q(\lambda)].$$

For some ν with $a_{\nu\mu} \neq 0$ we have $\lambda \leq_{\mathcal{T}_q} \nu$. As λ belongs to the second largest weight cell and as $\nu \notin \underline{c}_1$ we have $\nu \leq_{\mathcal{T}_q} \lambda$. It follows that $\nu \sim_{\mathcal{T}_q} \lambda$, hence that $\nu \in \underline{c}_2$. By Theorem 6.10 we see that $\mu \notin \underline{c}_2$ implies that $\nu \notin \underline{c}_2$. So we see that $\mu \in \underline{c}_2$. □

LEMMA 10.9. *Assume type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8 or G_2.*
Let $\lambda \in \underline{c}_2$ and assume that $\lambda \leq_{\mathcal{T}_q} \mu$. Then $\lambda \leq_{\mathcal{T}} \mu$.

PROOF. By assumption there is a quantum tilting module Q so that $[T_q(\mu) \otimes Q : T_q(\lambda)] \neq 0$. Choose a modular tilting module M so that Q is a summand in M_q. Then

$$\begin{aligned}[T_q(\mu) \otimes Q : T_q(\lambda)] &\leq [(T(\mu) \otimes M)_q : T_q(\lambda)] \\ &= [T(\mu) \otimes M : T(\lambda)]\end{aligned}$$

The last equality follows from Theorem 6.6. The lemma is proved. □

THEOREM 10.10. *We assume $p \geq h$.*

(i) *The second weight cell $\underline{c}_2$ is a union of modular weight cells.*
(ii) *Assume type $A_{n\geq 2}$, B_2, D_n, E_6, E_7, E_8 or G_2. The second weight cell $\underline{c}_2$ is a modular weight cell.*

The proof of (i) follows from Lemma 10.8 . The proof of (ii) follows from Lemma 10.9 .

The term *modular weight cell* was first used in (Andersen 2001a). This paper contains among other things a detailed study of the smallest weight cell $\underline{c}_{\mathrm{St}}$. It is shown that this weight cell decomposes in infinitely many modular weight cell. Further all modular weight cells in $\underline{c}_{\mathrm{St}}$ are described in terms of the modular weight cells in $X^+ \setminus \underline{c}_{\mathrm{St}}$ (when $p \geq 2h-2$). In type A_2 this leads to a complete description of the modular weight cells, since $X^+ \setminus \underline{c}_{\mathrm{St}} = \underline{c}_1 \cup \underline{c}_2$.

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