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Automorphic forms and modular symbols

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Contents

Preface

This thesis is the culmination of my years as a PhD student at the Department of Mathematics at the University of Aarhus during the period 1999-2003. I have studied some relatively uncorrelated problems in the theory of automorphic forms and have obtained results in the theory of modular symbols, properties of Eisenstein series twisted with modular symbols and the asymptotic density of Maass newforms.

The thesis is organized as follows. The first chapter is a survey conveying and highlighting most of the main results of the thesis, along with some history of the problems in question. Then follows a chapter which fixes further notation and reviews some known results to be used in the subsequent chapters. In Chapter 3 we study the asymptotic distribution of modular symbols using methods from spectral theory, probability theory and analytic number theory. The main object of study in this chapter is Eisenstein series twisted with modular symbols. In chapter 4 we prove that these Eisenstein series all satisfy a functional equation analog to the classical functional equation of the non holomorphic Eisenstein series. We then go on to study some of the properties the scattering matrices involved. In Chapter 5 we study the asymptotic densities of Maass newforms and discusses the relevance of this to the Jacquet-Langlands correspondence.

I would like to take the opportunity to express my gratitude to some people without whom this thesis would have been impossible for me to write: Erik Balslev, my thesis advisor, for many useful discussions and much encouragement, Alexei B. Venkov, who along the way became my de facto co-advisor, for inspiring conversations and helpful advice, the Max Planck institute of Mathematics for its hospitality during spring 2002, Andreas Strömbergsson for generously sharing with me unpublished ideas and results and Yiannis N. Petridis, my coauthor on the material presented in chapter 3, for being a wonderful mathematical mentor and friend. Last but not least I thank my wife Sigrid for her patience, understanding and support.

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CHAPTER 1

Introduction

1. The distribution of modular symbols

Let M be a hyperbolic Riemann surface of finite volume. Hence the universal covering of M is the upper halfplane, $\mathbb{H}$, and the covering group, Γ, is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Let $f(z)dz$ be a holomorphic 1-form on M. If c is a curve on M we may integrate $f(z)dz$ along the curve to get

$$\int_c f(z)dz.$$

We have a bijection between the covering group Γ and the fundamental group $\pi_1(M, \hat{z}_0)$ given by sending $\gamma \in \Gamma$ to the unique geodesic between z_0 and γz_0 in $\mathbb{H}$ where z_0 lies above $\hat{z}_0$, and then projecting this curve to M. By integrating along this curve we get an additive homomorphism

$$\begin{aligned} \Gamma \to & \quad \mathbb{C} \\ \gamma \mapsto & \int_{z_0}^{\gamma z_0} f(z). \end{aligned}$$

It is the distribution of this map we wish to study. We assume that M has a cusp at $i\infty$ and that $f(z)dz$ is cuspidal, i.e. that $f(z)$ is a cusp form of weight two. Due to the cusp condition the stabilizer of $i\infty$ is of the form

$$\Gamma_\infty = \left\langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \Gamma$$

for some $h \in \mathbb{R}$. It turns out that the above homomorphism factors through the quotient $\Gamma_\infty \backslash \Gamma$. We let (c, d) be the lower row of γ and when $c^2 + d^2 > 1$ we define

$$[\gamma, f] = \sqrt{\frac{\mathrm{vol}(\Gamma \setminus \mathbb{H})}{2 \log(c^2 + d^2)}} \int_{z_0}^{\gamma z_0} f(z)dz.$$

If $c^2 + d^2 \leq 1$ we set $[\gamma, f] = 0$ We assume that the Petersson norm $\|f\| = 1$.

Our main result is the following

THEOREM A. *Asymptotically* $[\gamma, f]$ *has a normal distribution. More precisely, for any fixed rectangle* R *in* $\mathbb{C}$,

$$\frac{\#\left\{\gamma \in (\Gamma_\infty \setminus \Gamma)^T \,|[\gamma, f] \in R\right\}}{\#(\Gamma_\infty \setminus \Gamma)^T} \to \frac{1}{2\pi}\int_R \exp\left(-\frac{x^2+y^2}{2}\right) dxdy$$

as $T \to \infty$.

Here

$$(\Gamma_\infty \setminus \Gamma)^T = \left\{\gamma \in \Gamma_\infty \setminus \Gamma | \quad c^2 + d^2 \leq T\right\}.$$

Since a cuspidal harmonic 1-forms, α, may be written as $\alpha(z) = \Re(f(z)dz)$ we also get a result about these also. Let

$$[\gamma, \alpha] = \sqrt{\frac{\mathrm{vol}(\Gamma \setminus \mathbb{H})}{2\log(c^2+d^2)}}\int_{z_0}^{\gamma z_0} \alpha.$$

Assume that $\|f\| = 1$.

THEOREM B. *Asymptotically* $[\gamma, \alpha]$ *has a normal distribution. More precisely, for any* $a \leq b$,

$$\frac{\#\left\{\gamma \in (\Gamma_\infty \setminus \Gamma)^T \,|[\gamma, \alpha] \in [a, b]\right\}}{\#(\Gamma_\infty \setminus \Gamma)^T} \to \frac{1}{\sqrt{2\pi}}\int_a^b \exp\left(-\frac{x^2}{2}\right) dx$$

as $T \to \infty$.

This work uses heavily Eisenstein series twisted by modular symbols, introduced by Goldfeld. The general framework is as follows. Let $f(z), g(z)$ be holomorphic cusp forms of weight 2 for a fixed cofinite discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$. In (Goldfeld 1999b, Goldfeld 1999a) Goldfeld introduced Eisenstein series associated with modular symbols defined in a right half-plane as

$$E^{m,n}(z, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \langle \gamma, f\rangle^m \, \overline{\langle \gamma, g\rangle}^n \Im(\gamma z)^s, \tag{1.1}$$

where for $\gamma \in \Gamma$ the modular symbol $\langle \gamma, f\rangle$ is given by

$$\langle \gamma, f\rangle = -2\pi i \int_{z_0}^{\gamma z_0} f(z)\, dz, \tag{1.2}$$

and one defines similarly $\langle \gamma, g\rangle$. Here z_0 is an arbitrary point in the upper half-plane $\mathbb{H}$.

If we take $f(z)$ to be a Hecke eigenform for $\Gamma_0(N)$ and E_f is the elliptic curve over $\mathbf{Q}$ corresponding to it by the Eichler-Shimura theory, then

$$\langle \gamma, f\rangle = n_1(f, \gamma)\Omega_1(f) + n_2(f, \gamma)\Omega_2(f),$$

where $n_i \in \mathbb{Z}$ and Ω_i are the periods of E_f. The conjecture $n_i \ll N^k$ for $|c| \leq N^2$ and some fixed k (Goldfeld's conjecture) is equivalent to Szpiro's conjecture $D \ll N^C$ for some C, where D is the discriminant of E_f. This was the motivation to study the distribution of modular symbols.

As an example of such a distributional result Goldfeld conjectured in (Goldfeld 1999b) that

$$\sum_{c^2+d^2 \leq T} \langle \gamma, f \rangle \sim R(i)T, \tag{1.3}$$

where $R(z)$ is the residue at $s = 1$ of $E^{1,0}(z,s)$, and we sum over the elements in $\Gamma_\infty \setminus \Gamma$ with lower row (c,d). This is now proved in (Goldfeld & O'Sullivan 2003, Theorem 7.3). He also suggested that, when $f = g$, the twisted Eisenstein series $E^{1,1}(z,s)$ should have a simple pole at $s = 1$ with the zero Fourier coefficient of the residue proportional to the Petersson norm $\|f\|^2$. He concludes the conjectural asymptotic formula

$$\sum_{c^2+d^2 \leq T} |\langle \gamma, f \rangle|^2 \sim R^*(i)T, \tag{1.4}$$

where $R^*(z)$ is the residue of $E^{1,1}(z,s)$ at $s = 1$ and where the summation is again over matrices in $\Gamma_\infty \setminus \Gamma$ with lower row (c,d). In this work we, among other things, reprove (1.3) while settling (1.4) in the negative. But our result shows that the Petersson norm does indeed play a role, see Theorem G below. Averages of functions of modular symbols have been investigated also in (Manin & Marcolli 2002).

It turns out to be crucial to consider Eisenstein series associated with the real harmonic differentials $\alpha_i = \Re(f_i(z)dz)$ or $\alpha_i = \Im(f_i(z)dz)$ where f_i are holomorphic cusp forms of weight two. We shall write

$$\langle \gamma, \alpha_i \rangle = -2\pi i \int_{z_0}^{\gamma z_0} \alpha_i. \tag{1.5}$$

As in (Petridis 2002) we define

$$E(z,s,\vec{\epsilon}) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \chi_{\vec{\epsilon}}(\gamma) \Im(\gamma z)^s, \tag{1.6}$$

where $\chi_{\vec{\epsilon}}$ is a n-parameter family of characters of the group defined by

$$\chi_{\vec{\epsilon}}(\gamma) = \exp\left(-2\pi i \left(\sum_{k=1}^{n} \epsilon_k \int_{z_0}^{\gamma z_0} \alpha_k\right)\right). \tag{1.7}$$

The convergence of (1.6) is guaranteed for $\Re(s) > 1$ by comparison with the standard Eisenstein series. The Eisenstein series with a character transform as

$$E(\gamma z, s, \vec{\epsilon}) = \bar{\chi}_{\vec{\epsilon}}(\gamma) E(z, s, \vec{\epsilon}). \tag{1.8}$$

We see that

$$\left.\frac{\partial^n E(z, s, \vec{\epsilon})}{\partial \epsilon_1 \ldots \partial \epsilon_n}\right|_{\vec{\epsilon}=\vec{0}} = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \prod_{i=1}^{n} \langle \gamma, \alpha_i \rangle \, \Im(\gamma z)^s, \tag{1.9}$$

by termwise differentiation whenever the sum is absolutely convergent. By taking linear combinations of these we may of course recover the original series (1.1). This observation allowed the first author to give a new approach to the Eisenstein series twisted with modular symbols using perturbation theory. In particular, a new proof of the analytic continuation was given in (Petridis 2002) and the residues of $E^{1,0}(z, s)$ on the critical line were identified. In this paper we further pursue this method. We start by giving a third much shorter proof of the main theorem in (O'Sullivan 2000).

THEOREM C. *(O'Sullivan 2000, Petridis 2002)*
The functions $E^{m,n}(z, s)$ have meromorphic continuation to the whole s-plane. In $\Re(s) > 1$ the functions are analytic.

The last claim of the theorem, which does not seem to have been explicitly stated before, enables us to evaluate the growth of the modular symbols as γ runs through the group Γ. The best known result in this aspect is

$$\langle \gamma, f \rangle = O(\log(c^2 + d^2)).$$

This is due to Eichler (see (Eichler 1965)). Using the above theorem we get the following slightly weaker result.

THEOREM D. *For any $\varepsilon > 0$ we have*

$$\langle \gamma, f \rangle = O_\varepsilon((c^2 + d^2)^\varepsilon).$$

We then continue to study the singularity of $E^{m,n}(z, s)$ at $s = 1$ when $f = g$. In particular we study the pole order and the leading term in the singular part of the Laurent expansion. In principle the method gives the full Laurent expansion of $E^{m,n}(z, s)$ but only in terms of the coefficients in the Laurent expansions of the resolvent kernel and the usual non holomorphic Eisenstein series at $s = 1$. The combinatorics involved in getting useful expressions is quite ponderous. As a result we settle with calculating some of the most interesting coefficients and evaluate the pole orders. As an example of this type of result we have:

Theorem E. *At $s=1$, $E^{2,0}(z,s)$ has a simple pole with residue*

$$\frac{1}{vol(\Gamma\setminus\mathbb{H})}\left(2\pi i\int_{i\infty}^{z} f(z)dz\right)^2$$

while $E^{1,1}(z,s)$ has a double pole with residue

$$\frac{4\pi^2}{vol(\Gamma\setminus\mathbb{H})}\left|\int_{i\infty}^{z} f(z)dz\right|^2 + \frac{16\pi^2}{vol(\Gamma\setminus\mathbb{H})}\int_{\Gamma\setminus\mathbb{H}}(E_0(z')-r_0(z,z'))y'^2\left|f(z')\right|^2 d\mu(z').$$

The coefficient of $(s-1)^{-2}$ is

$$\frac{16\pi^2\left\|f\right\|^2}{vol(\Gamma\setminus\mathbb{H})^2}.$$

Here the coefficient $r_0(z,z')$ is the constant term in the Laurent expansion of the resolvent kernel around $s=1$. The coefficient $E_0(z)$ is the constant term in the Laurent expansion of the usual non holomorphic Eisenstein series and is given by Kronecker's limit formula. For $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ this is classical, see, for instance (Lang 1973, p. 273–275). For a generalization to all Γ see (Goldstein 1973).

We wish to use these results to obtain results à la (1.3). We do this using the method of contour integration but, in order to make this work, we need to prove a result on the growth of $E^{m,n}(z,s)$ as $\Im(s)\to\infty$. We can prove

Theorem F. *The functions $E^{m,n}(z,s)$ grow at most polynomially on vertical lines with $\sigma>1/2$. More precisely: for every $\varepsilon>0$ and $\sigma\in(1/2,1]$ and $z\in K$, a compact set, we have*

$$E^{m,n}(z,\sigma+it)=O(|t|^{(6(m+n)-1)(1-\sigma)+\varepsilon}). \tag{1.10}$$

Using the above theorems and contour integration we get asymptotic expansions for summatory functions like the one in (1.3). An example of the results we prove is:

Theorem G. *There exists $\delta>0$ such that*

$$\sum_{c^2+d^2\le T}\langle\gamma,f\rangle^2=\frac{1}{vol(\Gamma\setminus\mathbb{H})}\left(-2\pi i\int_{i\infty}^{z} f(\tau)\,d\tau\right)^2 T+O(T^{1-\delta})$$

$$\sum_{c^2+d^2\le T}\left|\langle\gamma,f\rangle\right|^2=\frac{(16\pi^2)}{vol(\Gamma\setminus\mathbb{H})^2}\left\|f\right\|^2 T\log T+O(T).$$

The summations are over (c,d) lower row of $\gamma\in\Gamma_\infty\setminus\Gamma$.

This settles the conjectural status of (1.4) in the negative. How small we can make $1-\delta$ in the above theorem depends on how good polynomial bounds we have in Theorem F and whether the Laplacian has small eigenvalues. If there are no such eigenvalues we can prove

$$1-\delta = \frac{12}{13}+\varepsilon.$$

By using similar asymptotic expansions we can calculate the moments of the normalized modular symbols and prove the distributional result in Theorem A, which is the main theorem of our work.

The usual nonholomorphic Eisenstein series satisfies a functional equation of the form

$$\vec{E}(z,s) = \Phi(s)\vec{E}(z,1-s). \tag{1.11}$$

We show that the Eisenstein series twisted with modular symbols has an analog functional equation. We state a result for $g=f$.

THEOREM H. *The function $\vec{E}^{m,n}(z,s)$ satisfies a functional equation of the form*

$$\vec{E}^{m,n}(z,s) = \sum_{m'+n'\leq m+n} \Phi_{m-m',n-n'}(s)\vec{E}^{m',n'}(z,1-s).$$

We find explicit, though quite complicated, expressions for the matrices $\Phi_{k,l}(s)$ in terms of the usual scattering matrix $\Phi(s)$ and various L-functions.

The idea of putting the Eisenstein series in a continuous family to study how the spectrum changes as the parameters change is very fruitful, see for instance (Bruggeman 1994), where the parameter is the weight of the modular form. The study of $E^{m,n}(z,s)$ using perturbed Eisenstein series is an interesting application of the spectral deformations used in (Phillips & Sarnak 1987, Phillips & Sarnak 1991, Phillips & Sarnak 1994, Petridis 2000). In (Petridis 2002) the Eisenstein series with modular symbols was put into this framework. In this work we apply the same techniques to produce results which at least to us seem difficult to attack with the methods used by Goldfeld, O'Sullivan et.al.

2. Spectral correspondences for Maass forms

For a Fuchsian group of the first kind, Γ, we let Δ_Γ be the automorphic Laplacian related to Γ. The Jacquet-Langlands correspondence (See (Jacquet & Langlands 1970, Chapter3), (Gelbart 1975, Theorem 10.5)) gives (among other things) a correspondence between the λ-eigenspace of Δ_{Γ_c}, and the λ-eigenspace of $\Delta_{\Gamma_{nc}}$. Here Γ_c, Γ_{nc} are certain arithmetical Fuchsian groups of the first kind, where Γ_c is cocompact while Γ_{nc} is non-cocompact but cofinite. This correspondence is usually described using the language of representation theory and adelic trace formulaes.

Parts of this theory was reproved in a succession of papers by Hejhal (1985), Bolte & Johansson (1999a)(1999b) and Strömbergsson (2001b) using classical techniques á la Selberg (1989). Here the correspondence is given by an integral transform Θ. The cocompact group Γ_c is the unit group in a maximal order in an indefinite rational quaternion algebra with reduced discriminant D. Hence d is the product of an even number of different primes, and any such number may be realized in this way. Let $\Gamma_{nc} = \Gamma_0(d)$ be the Hecke congruence group of level d.

In (Strömbergsson 2001b) it is proved that Θ gives a bijection between the λ-eigenspace of Δ_{Γ_c} and the λ-new eigenspace of $\Delta_{\Gamma_{nc}}$ when $\lambda \neq 0$. This is proved by a careful comparison of Selberg Trace Formulas for modular correspondences (Hecke operators) in the two pertinent settings.

Now for cocompact groups Γ_c the spectral counting function

$$N_{\Gamma_c}(\lambda) = \#\{\lambda_n \leq \lambda | \lambda_n \text{ eigenvalue of } \Delta_{\Gamma_c}\}$$

has an asymptotic expansion of the form

$$N_{\Gamma_c}(\lambda) = \frac{\mathrm{vol}(\Gamma_c \backslash \mathbb{H})}{4\pi}\lambda + O(\sqrt{\lambda}/\log(\lambda)),$$

while for congruence groups Γ_{nc}

$$N_{\Gamma_{nc}}(\lambda) = \frac{\mathrm{vol}(\Gamma_{nc} \backslash \mathbb{H})}{4\pi}\lambda + O(\sqrt{\lambda}\log(\lambda)).$$

We notice the difference in the error terms in the compact and the non-cocompact case. Using the "classical" case of the Jacquet-Langlands correspondence quoted above we find that if we define a spectral counting function $N^{\text{new}}_{\Gamma_0(M)}(\lambda)$ which only counts the newforms then when d is the product of an even number of different primes we have

$$N^{\text{new}}_{\Gamma_0(d)}(\lambda) = \frac{\mathrm{vol}(\Gamma_c \backslash \mathbb{H})}{4\pi}\lambda + O(\sqrt{\lambda}/\log(\lambda))$$

i.e. the sort of expansion characteristic to the cocompact case.

We can now ask whether $N^{\text{new}}_{\Gamma_0(M)}(\lambda)$ has the same type of expansion for any M not an even number of different primes? When

$$N^{\text{new}}_{\Gamma_0(M)}(\lambda) = c_M\lambda + O(\sqrt{\lambda}/\log(\lambda))$$

for some constant c_m we shall say that $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type. By making an asymptotic expansion of the scattering determinant related to $\Gamma_0(M)$ at the halfline $\frac{1}{2}+it$ we can determine asymptotic expansions of $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ with error term $O(\sqrt{\lambda}/\log\lambda)$.

We can hence answer the above question. We write $M = t^2 n$ where n is square free

THEOREM I. *The spectral counting function $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type if and only if at least one of the following holds:*

(1) *n contains at least two primes.*
(2) *n is a prime and $4 \parallel M$.*

Hence there are an abundance of cases where $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type but M is not a product of an even number of different primes.

At this point it would be interesting to revert the reasoning and see if $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ of cocompact type implies that there is a N_{Γ_c} with Γ_c cocompact or a linear combination of those that coincides with $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$. In other words: Are there spectral correspondences which are responsible for the remaining cases in Theorem I.

In (Strömbergsson n.d.) it is shown that if $\Gamma_c(M)$ is the unit group in an Eichler order of level M in an indefinite rational quaternion division algebra with reduced discriminant D then there is a correspondence given by an integral operator such that for $\lambda \neq 0$ a certain λ-new eigenspace of $\Delta_{\Gamma_c(M)}$ is in bijection with the λ-new eigenspace of $\Delta_{\Gamma_0(DM)}$. Hence we still get a bijection with the λ-new eigenspace of $\Delta_{\Gamma_0(DM)}$ if we only "lift" the λ new eigenforms at the quaternion level.

We choose a slightly different approach. For $D|M'$ we define the λ D-old eigenspace of $\Delta_{\Gamma_0(M')}$ to be the subspace of the λ eigenspace spanned by

$$\left\{ f(dz) \;\middle|\; \begin{array}{l} f \text{ in the } \lambda \text{ eigenspace of } \Delta_{\Gamma_0(K)} \\ Kd|M' \quad K \neq M' \quad M'|KD \end{array} \right\}.$$

We then define the λ D-new eigenspace to be the orthogonal complement in the λ-eigenspace of $\Delta_{\Gamma_0(M')}$. If $(M,D)=1$ we can prove, using the trace formula calculations of (Strömbergsson n.d.), the following

THEOREM J. *When $\lambda \neq 0$ there is a bijection between the λ-eigenspace of $\Delta_{\Gamma_c(M)}$ and the λ D-new eigenspace of $\Delta_{\Gamma_0(MD)}$ given by an integral transform.*

This theorem as well as the one in (Strömbergsson n.d.) gives correspondences "responsible for" the result of Theorem I for DM when $(D,M)=1$ and D is a product of an even number of different primes.

There are still may cases where $N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type that has not been explained in this way. We note however that in all the cases M contains at least two different primes. This leaves some hope that there might be some quaternion division algebra in play. The smallest level that we have not "explained" is $M = 12$. Are there cocompact groups responsible for the fact that $N^{\mathrm{new}}_{\Gamma_0(12)}(\lambda)$ is of cocompact type or is this accidental?

CHAPTER 2

Prerequisites

This chapter fixes notation not already defined and reviews some known results which shall be used in the subsequent chapters. General references are (Shimura 1971, Venkov 1982, Hejhal 1983, Selberg 1989, Bruggeman 1994, Iwaniec 1995).

1. Fuchsian groups

Let

$$\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$$

be the upper halfplane. We equip this with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

and get an associated measure

$$d\mu(z) = \frac{dxdy}{y^2}.$$

The Laplacian associated with the metric ds^2 is

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The group of holomorphic automorphisms of $\mathbb{H}$ is isomorphic to

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$$

via the map

$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{R}) & \to & \mathrm{Aut}(\mathbb{H}) \\ \gamma & \mapsto & (z \mapsto \gamma z = \frac{az+b}{cz+d}), \end{array}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The group $\mathrm{PSL}_2(\mathbb{R})$ is also isomorphic to the group of orientation preserving isometries of $\mathbb{H}$ as a Riemannian via the same map. We note that the map γ extends continuously to $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ in the obvious way. We see that any subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ gives rise to a subgroup of $\mathrm{Aut}(\mathbb{H})$. We shall often not specify whether Γ is a group of matrices or a group of automorphisms of $\mathbb{H}$. We shall always assume Γ (or a corresponding group of matrices) to be a discrete subgroup of

$\mathrm{PSL}_2(\mathbb{R})$. We shall say that a set $F_\Gamma \subset H$ is a fundamental domain if it contains at most one element from each Γ-orbit, Γz , while its closure must contain at least one element from each orbit. It is always possible to choose a fundamental domain which is open and connected.

The volume of the quotient, $\Gamma \setminus \mathbb{H}$, is defined as

$$\mathrm{vol}(\Gamma \setminus \mathbb{H}) = \int_{F_\Gamma} d\mu(z).$$

When the volume of the quotient is finite we say that Γ is cofinite. We shall always assume this to be the case.

We call $\gamma \in \Gamma$ elliptic (resp. parabolic, resp. hyperbolic) if $\mathrm{Tr}\,(\gamma) = |a+d| < 2$ (resp. $\mathrm{Tr}\,(g) = 2$, resp. $\mathrm{Tr}\,(g) > 2$.) This corresponds to γ having exactly one fixed point in $\mathbb{H}$ (resp. exactly one in $\mathbb{R} \cup \{\infty\}$, resp. two fixed points in $\mathbb{R} \cup \{\infty\}$). Since $\mathrm{Tr}\,(\sigma\gamma\sigma^{-1}) = \mathrm{Tr}\,(\gamma)$ the conjugacy classes of Γ is divided into elliptic, parabolic or hyperbolic classes.

Theorem 1. *The number of parabolic conjugacy classes of* Γ *is finite if and only if* Γ *is cofinite.*

When there are no parabolic conjugacy classes the quotient $\Gamma \setminus \mathbb{H}$ is compact in the quotient topology, and we may choose complex charts to make it a compact Riemann surface (See (Shimura 1971)) If Γ has only hyperbolic elements $\mathbb{H}$ is the universal covering of $\Gamma \setminus \mathbb{H}$ and Γ is the group of deck transformation of $\Gamma \setminus \mathbb{H}$. If Γ has parabolic elements the quotient is noncompact but we may compactify by adding fixpoints of parabolic elements (We call such a fixpoint a cusp) and open sets around these to obtain a compact space, $\Gamma \setminus \mathbb{H}^*$. We can also add complex charts at the cusps to obtain a compact Riemann surface. This is done as follows:

Fix a maximal set of inequivalent cusps $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$. Let $\Gamma_{\mathfrak{a}_i}$ be the stabilizer of the cusp $\mathfrak{a}_i$. Then $\Gamma_{\mathfrak{a}_i}$ is cyclic with $\Gamma_{\mathfrak{a}_i} = \langle \gamma_i \rangle$ and can choose scaling matrices $\sigma_1, \ldots \sigma_n$ such that $\sigma_i(\infty) = \mathfrak{a}_i$ and

$$\sigma_i^{-1}\Gamma_{\mathfrak{a}_i}\sigma_i = \left\{ \left(\begin{array}{cc} 1 & m \\ 0 & 1 \end{array} \right) \middle| \, m \in \mathbb{Z} \right\}$$

We may choose an open set $U \subset \Gamma \setminus \mathbb{H}^*$ containing only one cuspidal orbit $\Gamma\mathfrak{a}_j$. The chart around $\mathfrak{a}_j$ is given by

$$\begin{array}{cccc} \phi_{\mathfrak{a}_j}: & U & \to & \mathbb{C} \\ & \Gamma z & \mapsto & e^{2\pi i \sigma_j^{-1} z}. \end{array}$$

2. Automorphic forms

Let k be an even integer let $\gamma \in \mathrm{GL}_2(\mathbb{R})$. We then define

$$j(\gamma, z) = cz + d.$$

Let $\chi : \Gamma \to S^1$ be a unitary character. For $f : \mathbb{H} \to \mathbb{H}$ we define

$$f|_{[\gamma]_k}(z) = \det(\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma z).$$

If $k = 0$ we simply write $f|_\gamma$ instead of $f|_{[\gamma]_0}$.

Definition 2. A smooth function $f : \mathbb{H} \to \mathbb{H}$ is called *a modular form of weight k and character χ* if

(1) f is holomorphic,
(2) $f|_{[\gamma]_k} = \chi(\gamma) f$ for all $\gamma \in \Gamma$,
(3) f is polynomially bounded as $\Im(\sigma_j^{-1} z) \to \infty$.

The space of such functions is denoted $M_k(\chi, \Gamma)$.

Definition 3. A function $g : \mathbb{H} \to \mathbb{H}$ is called an automorphic function with eigenvalue λ, character χ (or a Maass form) if

(1) $\Delta g = \lambda g$,
(2) $g|_\gamma = \chi(\gamma) g$ for all $\gamma \in \Gamma$,
(3) g is polynomially bounded as $\Im(\sigma_j^{-1} z) \to \infty$.

The space of such functions is denoted by $\tilde{A}(\lambda, \chi, \Gamma)$. We could also define Maass forms of weight different from zero (see (Bruggeman 1994)) but we shall not use them here.

We now assume that Γ has at least one cusp. If

$$\gamma_j = \sigma_j \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_j^{-1}$$

and $\chi(\gamma_j) = e^{2\pi i \kappa_j}$ then if $f \in M_k(\chi, \Gamma)$ or $f \in \tilde{A}(\lambda, \chi, \Gamma)$ we have that $e^{-2\pi i \kappa_j z} f|_{[\sigma_j]_k}$ is invariant under $z \to z + 1$, and has a Fourier expansion

$$e^{-2\pi i \kappa_j z} f|_{[\sigma_j]_k}(z) = \sum_{n=-\infty}^{\infty} a_n(y) e^{2\pi i n x}.$$

If $f \in M_k(\chi, \Gamma)$ we have

$$a_n(y) = \begin{cases} 0 & \text{if } n < 0 \\ q_n e^{-2\pi n y} & \text{if } n \geq 0 \end{cases}$$

where $q_n \in \mathbb{C}$. Hence

$$e^{-2\pi i \kappa_j z} f|_{[\sigma_j]_k}(z) = \sum_{n=0}^{\infty} q_n e^{2\pi i n z}.$$

If $q_0 = 0$ for all cusps we say that f is a holomorphic cusp form and we write $S_k(\chi, \Gamma)$ for the set of holomorphic cuspforms related to the character χ and the cofinite group Γ. For such an f we define the Petersson norm

$$\|f\|^2 = \int_{F_\Gamma} |f(z)|\, y^k d\mu(z).$$

If $\chi = 1$ we shall often write $S_k(\Gamma)$ instead of $S_k(\chi, \Gamma)$.

If $f \in \tilde{A}(\lambda, \chi, \Gamma)$ then

$$a_n(y) = q_n \sqrt{y} K_{s-1/2}(2\pi\, |n|\, y) \quad \text{when } n \neq 0.$$

where $q_n \in C$. Here $\lambda = s(1-s)$ and $K_{s-1/2}$ is the exponentially decaying Bessel function

$$K_\mu(z) = \frac{1}{2} \int_0^\infty \exp\left(-\frac{z}{2}\left(t + t^{-1}\right)\right) t^{-\mu-1} dt.$$

If $a_0(y) = 0$ for all cusps we say that f is a nonholomorphic cusp form and we write $A(\lambda, \chi, \Gamma)$ for the set of nonholomorphic cuspforms related to the eigenvalue λ, the character χ and the cofinite group Γ. If $\chi = 1$ we shall often write $A(\lambda, \Gamma)$ instead of $A(\lambda, \chi, \Gamma)$.

3. The non-holomorphic Eisenstein series

A very important example of an automorphic function is the non-holomorphic Eisenstein series related to any cusp $\mathfrak{a}$ such that $\chi(\gamma_\mathfrak{a}) = 1$ defined as

$$E_\mathfrak{a}(z, s) = \sum_{\gamma \in \Gamma_\mathfrak{a} \setminus \Gamma} \chi(\gamma)^{-1} \Im(\sigma_\mathfrak{a}^{-1} \gamma z)^s$$

where $z \in \mathbb{H}$. This is absolutely convergent for $\Re(s) > 1$ and for $\Re(s) \geq 1 + \delta$ it converges uniformly. It transforms as

$$E_\mathfrak{a}(\gamma z, s) = \chi(\gamma) E_\mathfrak{a}(z, s),$$

and as a function of z it is an eigenfunction of the Laplacian, i.e.

$$-\Delta E_\mathfrak{a}(z, s) = s(1-s) E_\mathfrak{a}(z, s)$$

The zero'th Fourier coefficient at the cusp $\mathfrak{b}$ is given by

$$\delta_{\mathfrak{a}\mathfrak{b}} y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s) y^{1-s}$$

where $\delta_{\mathfrak{a}\mathfrak{b}}$ is the Kronecker delta.

Theorem 4. *As a function of s $E_\mathfrak{a}(z, s)$ may be meromorphically continued to the whole complex plane. In $\Re(s) > 1/2$ there are only finitely many poles all lying at the real axis and $s = 1$ is always a simple pole with residue*

$$\operatorname*{Res}_{s=1}(E_\mathfrak{a}(z, s)) = vol(\Gamma \setminus \mathbb{H})^{-1}.$$

The matrix $\Phi(s) = \{\phi_{\mathfrak{ab}}(s)\}_{\mathfrak{ab}}$ is called the scattering matrix. The Eisenstein series satisfies the functional equation

$$\vec{E}(z,s) = \Phi(s)\vec{E}(z,1-s).$$

Here $\vec{E}(z,s) = \{E_{\mathfrak{a}}(z,s)\}_{\mathfrak{a}}$ is the vector of Eisenstein series indexed according to the cusps of the group. The scattering matrix satisfies

$$\begin{aligned} \Phi(s)\Phi(1-s) &= I \\ \Phi\left(\frac{1}{2}+it\right)\Phi^*\left(\frac{1}{2}+it\right) &= I \\ \Phi^*(t) &= \Phi(t) \qquad \text{for } t \in \mathbb{R} \\ \overline{\Phi(s)} &= \Phi(\overline{s}) \end{aligned}$$

where the star denotes the conjugate transpose.

4. The automorphic Laplacian

We let $L^2(\Gamma, \chi, d\mu)$ be the set of measurable functions $f : \mathbb{H} \to \mathbb{C}$ that transforms as $f(\gamma z) = \chi(\gamma) f(z)$ and satisfies

$$\int_{\Gamma\backslash\mathbb{H}} |f(z)|^2 \, d\mu(z).$$

This is a Hilbert space with the inner product

$$(f,g) = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} d\mu(z).$$

Since Γ is cofinite the (Γ-automorphic) constants belong to $L^2(\Gamma, \chi, d\mu)$. We define the automorphic Laplacian $(\tilde{L}, \mathfrak{D}(\tilde{L}))$ to be the operator

$$\begin{array}{rccc} \tilde{L}: & \mathfrak{D}(\tilde{L}) & \to & L^2(\Gamma, \chi, d\mu) \\ & f & \mapsto & -\Delta f. \end{array}$$

where

$$\mathfrak{D}(\tilde{L}) = \left\{ f : \mathbb{H} \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ smooth} \\ f, \Delta f \text{ bounded} \\ f(\gamma z) = \chi(\gamma) f(z) \end{array} \right\}.$$

This set is dense in $L^2(\Gamma, \chi, d\mu)$ and $-\tilde{L}$ is densely defined, nonnegative and essentially selfadjoint. We denote by $-L$ its selfadjoint closure. We shall sometimes use $-\Delta_\Gamma$ to emphasize the dependence of the group Γ. The spectrum is contained in the positive real axis $[0, \infty[$.

We shall consider the resolvent $R(s) = (L + s(1-s))^{-1}$, defined off the spectrum of L. Since we shall only use this in the case $\chi = 1$, we restrict ourselves to this case. When $\Re(s)$ is sufficiently large the kernel of the resolvent may be constructed as follows. Let

$$u(z,z') = \frac{|z-z'|^2}{4\Im(z)\Im(z')}.$$

This is a point pair invariant i.e. $u(\gamma z, \gamma z') = u(z,z')$. We set

$$\phi(u,s) = \frac{1}{4\pi}\int_0^1 (t(1-t))^{s-1}(t+u)^{-s}dt.$$

We then define

$$r(z,z',s) = -\frac{1}{2}\sum_{\gamma\in\Gamma}\phi(u(z,\gamma z'),s).$$

This is the resolvent kernel. Then for $\Re(s) \gg 0$ and $f : \mathbb{H} \to \mathbb{C}$ bounded and smooth we have

$$R(s)f = \int_{F_\Gamma} r(z,z',s)f(z')d\mu(z').$$

The resolvent cannot be meromorphically continued across the halfline $\Re(s) = 1/2$, but one may attach a meaning to the meromorphic continuation of the resolvent kernel in the s-plane. We note that the s-plane is a two sheeted covering of the $\lambda = s(1-s)$-plane, and that the λ-plane cut along the the positive real axis $[0,\infty[$ corresponds to the right halfplane $\Re(s) > 1/2$ cut along $1/2 \le s \le 1$.

THEOREM 5. *For s, $1-s$ non-singular we have the limiting absorption principle*

$$r(z,z',s) - r(z,z',1-s) = \frac{1}{1-2s}\sum_{\mathfrak{a}} E_{\mathfrak{a}}(z',s)E_{\mathfrak{a}}(z,1-s) \tag{2.1}$$

If $\lambda_0 = s_0(1-s_0)$ is an eigenvalue of L of multiplicity l then we has the following expression for the resolvent near s_0

$$r(z,z',s) = \frac{1}{s(1-s)-\lambda_0}\sum_{i=1}^{l} v_i(z)v_i(z') + r^+(z.z',s) \tag{2.2}$$

where $\{v_i\}_{i=1}^l$ is a real orthonormal basis of the λ_0-eigenspace, and $r^+(z,z',s)$ is analytic in s in a neighborhood of s_0.

If the fundamental domain, F_Γ, is chosen to be a normal polygon we may decompose the fundamental domain F_Γ into

$$F_\Gamma = F_0 \bigcup_{\mathfrak{a}} F_{\mathfrak{a}}(T_0) \tag{2.3}$$

where the closure of F_0 is compact and $F_{\mathfrak{a}}$ is isometric to

$$\Pi = \{z \in \mathbb{H} | -1/2 < \Re(z) \leq 1/2,\ \Im(z) \geq T_0\}$$

by simply setting

$$F_{\mathfrak{a}}(T_0) = F \cap \sigma_{\mathfrak{a}}\{z \in \mathbb{H} | -1/2 < \Re(z) \leq 1/2,\ \Im(z) \geq T_0\}.$$

We assume that T_0 is chosen large enough that this gives disjoint sets. Faddeev (Faddeev 1967) introduces weighted Banach spaces that we shall now define. For a Γ-automorphic function, we set $f_0(z) = f(z)$ for $z \in F_0$ and $f_{\mathfrak{a}}(z) = f(\sigma_{\mathfrak{a}} z)$ for $z \in \Pi$. For $\mu \in \mathbb{R}$ we let B_μ be the Banach space of complex-valued functions f whose components f_0 and $f_{\mathfrak{a}}$ are continuous on Φ_0 and $F_{\mathfrak{a}}$ respectively with

$$|f_{\mathfrak{a}}(z)| \leq C y^{\mu}$$

for some constant C. We define the norm by

$$\|f\|_\mu = \sup_{z \in F_0} |f_0(z)| + \sum_{\mathfrak{a}} \sup_{z \in \pi} y^{-\mu} |f_{\mathfrak{a}}(z)|.$$

For a fixed $\mu \leq 1/2$ the meromorphically continued resolvent kernel, $r(z, z', s)$ defines a meromorphic continuation of the resolvent considered as the integral operator

$$\begin{aligned} R(s) : B_\mu &\to B_{1-\mu} \\ g &\mapsto \int_{\Gamma\backslash\mathbb{H}} r(z, z', s) g(z') d\mu(z'). \end{aligned}$$

whenever $\Re(s) > \mu$.

A different approach due to Müller (Müller 1996) uses weighted L^2-spaces of the form

$$L^2_\delta(F) = \left\{ f : X \to \mathbb{C} | f \text{ measurable and } \int_F |f(z)|^2 e^{2\delta\rho(z)} d\mu(z) \right\}$$

where $\rho(z) = 1$ for $z \in F_0$ and $\rho(z) = \Im(\sigma_{\mathfrak{a}}^{-1} z)$ for $z \in F_{\mathfrak{a}}$. We note that when $\delta > 0$

$$L^2_\delta(F) \subseteq L^2(F) \subseteq L^2_{-\delta}(F). \tag{2.4}$$

This gives a meromorphic continuation of the operator

$$R(s) : L^2_\delta(F) \to L^2_{-\delta}(F)$$

whenever $\delta > 0$.

5. Modular symbols

Let $f \in S_2(\Gamma)$. By the transformation properties of f we see that $f(z)dz$ is Γ invariant and hence defines a holomorphic 1-form on the Riemann surface $\Gamma \setminus \mathbb{H}$. We define modular symbols, $\langle \gamma, f \rangle$ by

$$\langle \gamma, f \rangle = -2\pi i \int_{z_0}^{\gamma z_0} f(z) dz.$$

Here z_0 is any point in the (extended) upper halfplane and the integral is along any curve from z_0 to γz_0. Since f is holomorphic the modular symbol is independent of the path chosen. Since $f(z)dz$ is Γ-invariant it is easy to see that the modular symbols are also independent of z_0. Hence $\langle \cdot, f \rangle$ defines an additive homomorphism

$$\langle \cdot, f \rangle : \Gamma \to (\mathbb{C}, +).$$

This map has a weight k analogue (period polynomials) due to Eichler, Manin, Shimura and others, and a Maass cusp form analogue (period functions) due Lewis, Zagier, Mayer, Chang and others, but we shall not study these at the moment.

We have

Lemma 6. *If γ is elliptic or parabolic then $\langle \gamma, f \rangle = 0$.*

Proof. If γ is elliptic then γ is of finite order m and hence

$$m \langle \gamma, f \rangle \langle \gamma^m, f \rangle = 0.$$

If γ is parabolic then γ can be conjugated into $\gamma_{\mathfrak{a}}^l$. We shall prove that $\langle \gamma_{\mathfrak{a}}, f \rangle = 0$. Since $\langle \cdot, f \rangle$ is additive it follows that $\langle \gamma, f \rangle = 0$.

If

$$\gamma_j = \sigma_j \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_j^{-1},$$

then since f is a cusp form of weight two we have

$$f|_{[\sigma_j]_2}(z) = \sum_{n=1}^{\infty} q_n e^{2\pi i n z}.$$

Hence

$$\int_{z_1}^{z_1+1} (f|_{[\sigma_j]_2})(z) dz = 0.$$

By a change of variables, $\sigma_j z' = z$ we find

$$\begin{aligned}\langle \gamma_j, f\rangle = \int_{z_0}^{\gamma_j z_0} f(z)dz &= \int_{\sigma_j^{-1}z_0}^{\sigma_j^{-1}z_0+1} f(\sigma_j z')j(\sigma_j, z')^{-2}dz' \\ &= \int_{\sigma_j^{-1}z_0}^{\sigma_j^{-1}z_0+1} f|_{[\sigma_j]_2}(z')dz' = 0,\end{aligned}$$

which completes the proof. □

Since $\langle \cdot, f\rangle$ is additive it is also trivial on the commutator subgroup $[\Gamma, \Gamma]$. Hence its kernel is rather big.

PROPOSITION 7. *(Eichler 1965) We have the following bound*

$$\langle \gamma, f\rangle = O(\log(c^2 + d^2))$$

Eichlers proof of this is purely geometrical. In the next chapter we shall show that we can "almost" recover this using spectral methods.

CHAPTER 3

The distribution of modular symbols

In this chapter we shall investigate the distribution of modular symbols using methods from spectral theory, perturbation theory, analytic number theory and probability theory. We do this by studying Eisenstein series twisted with modular symbols. The setup is still, as in the previous chapter, that of a general cofinite discrete subgroup, Γ, of $\mathrm{PSL}_2(\mathbb{R})$. We mentioned in the introduction that it is crucial for our method to "twist" with symbols associated with real harmonic differentials $\alpha = \Re(f(z)dz)$ or $\alpha = \Im(f(z)dz)$ where $f(z) \in S_2(\Gamma)$. In fact it turns out to be convenient to use compactly supported smooth 1-forms w. We start by showing how we may approximate α by such a w.

1. Approximating cuspidal harmonic forms with compactly supported forms

We note that we can always assume that the real harmonic differential α_i is the real part of a holomorphic cusp form since

$$\Im(f(z)dz) = \Re(-if(z)dz)$$

and $-if$ is a holomorphic cusp form of weight two. We want to approximate a real harmonic differential $\alpha = \Re(f(z)dz)$ where $f \in S_2(\Gamma)$ with a compactly supported real differential. We do this as follows. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ be a maximal set of inequivalent cusps. We may assume that $\mathfrak{a}_1 = i\infty$. When we write $\mathfrak{a}$ we shall always assume that $\mathfrak{a} = \mathfrak{a}_j$ for some $j = 1, \dots, m$.

Let

$$f|_{[\sigma_j]_2}(z) = \sum_{n=1}^{\infty} q_n^j e^{2\pi i n z}$$

be the Fourier expansion of $f(z)$ at the cusp α_j. We define

$$F_{\mathfrak{a}_j}(z) = \sum_{n=1}^{\infty} \frac{q_n^j}{2\pi i n} e^{2\pi i n \sigma_j^{-1} z}.$$

Then

$$\frac{dF_{\mathfrak{a}_j}}{dz}(z) = f(z).$$

Since $\alpha_i(z) = \Re(f(z)dz)$ we have $\alpha_i(z) = d\Re(F(z))$. We have the decomposition (See (2.3))

$$F_\Gamma = F_0 \bigcup_{\mathfrak{a}} F_{\mathfrak{a}}(T_0)$$

We then choose a smooth function $\tilde{\psi} : \mathbb{R} \to [0,1]$ such that

$$\tilde{\psi}(t) = \begin{cases} 1, & \text{if } t \leq 0, \\ 0, & \text{if } t \geq 1. \end{cases}$$

We then define, for $T > T_0$, $\psi^T : F \to [0,1]$ by

$$\psi^T(z) = \prod_{j=1}^{m} (\tilde{\psi}(\Im(\sigma_j^{-1} z) - T)).$$

For $z \in F$ we define

$$\begin{aligned} w^T &= d(\psi^T \Re(F_{\mathfrak{a}})) \\ g^T &= (1 - \psi^T)\Re(F_{\mathfrak{a}}), \end{aligned}$$

when $z \in F_{\mathfrak{a}}(T_0)$. When $z \in F_o$ we set $w^T = \alpha$, $g = 0$. We can now extend these to smooth Γ automorphic functions on $\mathbb{H}$ by setting $w_i^T(\gamma z) = w_i^T(z)$ and $g_i^T(\gamma z) = g_i^T(z)$ for each $\gamma \in \Gamma$. Then we have

$$\alpha = w^T + dg^T. \tag{3.1}$$

Proposition 8.

(1) *The smooth 1-form w^T is compactly supported on $\Gamma \setminus \mathbb{H}$.*
(2) *If $z \in F$ and $\Im(\sigma_j^{-1} z) \leq T$ for all $j = 1, \ldots, m$ then*
$$w^T(z) = \alpha(z).$$
(3) *If $z \in F$ and $\Im(\sigma_j^{-1} z) \leq T$ then*
$$\int_{\mathfrak{a}}^{z} w^T = \int_{\mathfrak{a}_j}^{z} \alpha.$$
(4) $\langle \gamma, \alpha_i \rangle = \langle \gamma, w_i^T \rangle$ *for all $\gamma \in \Gamma$ and all $T > T_0$.*

Proof. We note that (1) and (2) follows from the observation that when $\Im(\sigma_j^{-1} z) \leq T$ for every $j = 1, \ldots, m$ we have $\psi(z) = 1$ by definition. To prove (3) we note that

$$\int_{\mathfrak{a}}^{z} \alpha - \int_{\mathfrak{a}}^{z} w^T = \int_{\mathfrak{a}}^{z} dg^T = g^T(z) - g^T(\mathfrak{a}) = 0 - \Re(F_{\mathfrak{a}}(\mathfrak{a})) = 0.$$

The identity in (4) is proved by observing that

$$\langle \gamma, \alpha \rangle - \langle \gamma, w^T \rangle = -2\pi i \int_{z_0}^{\gamma z_0} dg = -2\pi i (g(\gamma z_0) - g(z_0)) = 0.$$

□

We shall often exclude T from the notation and simply write

$$\alpha = w + dg.$$

2. Finding Laurent expansions using perturbation theory

We now let $f_k \in S_2(\Gamma)$ and we let $\alpha_k(z) = \Re(f_k(z)dz)$ or $\alpha_k(z) = \Im(f_k(z)dz)$. Using the approximation method from the last section we set $\alpha_k = w_k + dg_k$. We define $\chi_{\vec{\epsilon}}(\gamma)$ as in (1.7). We note that by Proposition 8 (4)

$$\chi_{\vec{\epsilon}}(\gamma) = \exp\left(-2\pi i \left(\sum_{k=1}^{n} \epsilon_k \int_{z_0}^{\gamma z_0} w_k\right)\right). \tag{3.2}$$

We consider the space $L^2(\Gamma \setminus \mathbb{H}, \bar{\chi}_{\vec{\epsilon}})$ of square integrable functions that transform as

$$h(\gamma \cdot z) = \bar{\chi}_{\vec{\epsilon}}(\gamma) h(z), \quad \gamma \in \Gamma$$

under the action of the group. We introduce unitary operators

$$U(\vec{\epsilon}) : L^2(\Gamma \setminus \mathbb{H}) \to L^2(\Gamma \setminus \mathbb{H}, \bar{\chi}_{\vec{\epsilon}})$$

given by

$$(U(\vec{\epsilon})h)(z) := U(z, \vec{\epsilon})h(z) = \exp\left(2\pi i \left(\sum_{k=1}^{n} \epsilon_k \int_{i\infty}^{z} w_k\right)\right) h(z).$$

We notice that this extends in the obvious way to an operator from the set of all Γ-automorphic functions to the set of all $(\Gamma, \bar{\chi}_{\vec{\epsilon}})$-automorphic functions. We set

$$L(\vec{\epsilon}) = U(\vec{\epsilon})^{-1} \Delta U(\vec{\epsilon})$$

We notice that for $(L(\vec{\epsilon})h)(z) = (\Delta h)(z)$ for z "close to a cusp" since $U(z, \vec{\epsilon})$ is constant outside a compact set.

We also define $\tilde{L}(\vec{\epsilon}) = U(\vec{\epsilon})^{-1}(L)U(\vec{\epsilon})$, where L is the automorphic Laplacian. We let $h(y) \in C^\infty(\mathbb{R}_+)$ be a smooth function which is 0 if $y \leq T_0 + 1$ and 1 if $y \geq T_0 + 2$. Let

$$\Omega_{\vec{\epsilon}} = \{s \in \mathbb{C} | \Re(s) > 1/2 \quad s(1-s) \notin \operatorname{spec}(\tilde{L}(\vec{\epsilon}))\}.$$

Lemma 9. *(Colin de Verdière 1981, Petridis 2002) For $s \in \Omega_{\vec{\epsilon}}$ there exists a unique $D_{\mathfrak{a}}(z, s, \vec{\epsilon})$ such that*

$$\begin{aligned} D_{\mathfrak{a}}(\gamma z, s, \vec{\epsilon}) =& D_{\mathfrak{a}}(z, s, \vec{\epsilon}) \qquad \text{for all } \gamma \in \Gamma, \\ (L(\vec{\epsilon}) + s(1-s))\, D_{\mathfrak{a}}(z, s, \vec{\epsilon}) =& 0 \end{aligned}$$

and

$$D_{\mathfrak{a}}(z, s, \vec{\epsilon}) - h(\sigma_{\mathfrak{a}}^{-1} z) \Im(\sigma_{\mathfrak{a}}^{-1} z)^s \in L^2(\Gamma, d\mu).$$

Moreover, $s \mapsto D_{\mathfrak{a}}(z,s,\vec{\epsilon})$ *is holomorphic, while* $\vec{\epsilon} \mapsto D_{\mathfrak{a}}(z,s,\vec{\epsilon})$ *is real analytic.*

PROOF. Given a candidate

$$D_{\mathfrak{a}}(z,s,\vec{\epsilon}) = h(\sigma_{\mathfrak{a}}^{-1}z)\Im(\sigma_{\mathfrak{a}}^{-1}z)^s + g(z,s,\vec{\epsilon})$$

we have

$$(L(\vec{\epsilon}) + s(1-s))g(z,s,\vec{\epsilon}) = -(L(\vec{\epsilon}) + s(1-s))h(\sigma_{\mathfrak{a}}^{-1}z)\Im(\sigma_{\mathfrak{a}}^{-1}z)^s,$$

which is compactly supported. We let

$$H(z,s,\vec{\epsilon}) = -(L(\vec{\epsilon}) + s(1-s))h(\sigma_{\mathfrak{a}}^{-1}z)\Im(\sigma_{\mathfrak{a}}^{-1}z)^s.$$

Since this is now in particular in $L^2(\Gamma, d\mu)$ we can invert as long as $s(1-s)$ is off the spectrum. We therefore get

$$g(z,s,\vec{\epsilon}) = (L(\vec{\epsilon}) + s(1-s))^{-1}H(z,s,\vec{\epsilon}). \tag{3.3}$$

This proves uniqueness and using (3.3) as a definition we also get the existence. The operator $(L(\vec{\epsilon}) + s(1-s))^{-1}$ is holomorphic outside the spectrum and depends real analytically on the parameter $\vec{\epsilon}$ (See (Kato 1976, II 1.3. and IV 3.3)). $\square$

We note that

$$E_{\mathfrak{a}}(z,s,\vec{\epsilon}) = \exp\left(2\pi i \sum_{k=1}^{l} \epsilon_k \int_{\mathfrak{a}}^{i\infty} w_k\right) U(\vec{\epsilon}) D_{\mathfrak{a}}(z,s,\vec{\epsilon}). \tag{3.4}$$

This follows from the known asymptotical behavior of $E_{\mathfrak{a}}(z,s,\vec{\epsilon})$ (see. e.g (Kubota 1973, Thm. 2.12.)) and the fact that

$$(\Delta + s(1-s))E_{\mathfrak{a}}(z,s,\vec{\epsilon}) = 0.$$

We note that $E_{\mathfrak{a}}(z,s,\vec{\epsilon})$ is independent on the cohomology class, i.e. independent on T,while $D_{\mathfrak{a}}(z,s,\vec{\epsilon})$ and $U(\vec{\epsilon})$ are not. We also remark that (Petridis 2002, Remark 2.2) is only true for $z_0 = \mathfrak{a}$, since both $E_{\mathfrak{a}}(z,s,\vec{\epsilon})$ and $D_{\mathfrak{a}}(z,s,\vec{\epsilon})$ have asymptotic behavior at $\mathfrak{a}$ of the form $\Im(\sigma_{\mathfrak{a}}^{-1}z)^s$ for $\Re(s) > 1$ and, consequently, $U(z,\epsilon)$ should tend to 1, as $z \to \mathfrak{a}$.

For the rest of this chapter we assume for simplicity that Γ has only one cusp which we assume to be located at $i\infty$. We also assume that it is reduced i.e. that $h = 1$. The generalization to the multiple cusp case is straightforward.

In the rest of the paper we will use the following convention. A function with a subscript variable will denote the partial derivative of the function in that variable.

LEMMA 10. *Let $n \geq 1$. For $\Re(s)$ sufficiently large we have*

$$D_{\epsilon_1,\dots,\epsilon_n}(z,s,\vec{0}) \in L^2(\Gamma, d\mu).$$

PROOF. Since the function $D(z,s,\vec{\epsilon})$ is Γ-automorphic we see that also $D_{\epsilon_1,\dots,\epsilon_n}(z,s,\vec{0})$ is Γ-automorphic. From (3.4) we obtain that

$$D_{\epsilon_1,\dots,\epsilon_n}(z,s,\vec{0}) = \sum_{\vec{m}\in\{0,1\}^n} \prod_{k=1}^{n} \left(-2\pi i \int_{i\infty}^{z} w_k\right)^{m_k} E_{\epsilon_1^{1-m_1},\dots,\epsilon_n^{1-m_n}}(z,s,\vec{0}). \tag{3.5}$$

We note that since w_i is compactly supported all the terms with $\vec{m} \neq \vec{0}$ becomes compactly supported. Now in order to control the term with $\vec{m} = \vec{0}$ we need some bound on the growth of $\langle \gamma, \alpha_i \rangle$. Any bound of the form

$$|\langle \gamma, \alpha_i \rangle| \leq C(c^2+d^2)^b$$

will do. We quote (O'Sullivan 2000, Lemma 1.1) with $z = i$ to get $b = 1$. If we use the inequality (see (Knopp 1974, Lemma 4))

$$(c^2+d^2) \leq \frac{|cz+d|^2}{y} \frac{1+4|z|^2}{y}$$

we get:

$$\begin{aligned}
\left|E_{\epsilon_1,\cdots,\epsilon_n}(z,s,\vec{0})\right| &\leq \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \gamma\neq I}} \left|\prod_{j=1}^{n} \langle \gamma, \alpha_j \rangle\right| \Im(\gamma z)^\sigma \\
&= C\left(\frac{1+4\left|z\right|^2}{y}\right)^n \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \gamma\neq I}} \Im(\gamma z)^{\sigma-n}
\end{aligned}$$

We note that the sum is $O_\sigma(y^{1-\sigma+n})$ by (Kubota 1973, p. 13) so we get

$$\begin{aligned}
&\leq C'\left(\frac{1+4\left|z\right|^2}{y}\right)^n y^{1-\sigma+n} \\
&\leq C'' y^{1-\sigma+2n}
\end{aligned}$$

Hence we conclude that for $\sigma > 2+2n$ we have $D_{\epsilon_1,\dots,\epsilon_n}(z,s,\vec{0}) \in L^2(\Gamma, d\mu(z))$. $\square$

We define

$$\begin{aligned}
\langle f_1 dz + f_2 d\bar{z}, g_1 dz + g_2 d\bar{z} \rangle &= 2y^2(f_1\bar{g}_1 + f_2\bar{g}_2) \\
\delta(p dx + q dy) &= -y^2(p_x + q_y).
\end{aligned}$$

LEMMA 11. *The conjugated operator $L(\vec{\epsilon})$ is given by*

$$\begin{aligned} L(\vec{\epsilon})h &= \Delta h + 4\pi i \sum_{k=1}^{n} \epsilon_k \langle dh, w_k\rangle - 2\pi i \left(\sum_{k=1}^{n} \epsilon_k \delta(w_k)\right) h \\ &\quad -4\pi^2 \left(\sum_{k,l=1}^{n} \epsilon_k\epsilon_l \langle w_k, w_l\rangle\right) h. \end{aligned} \tag{3.6}$$

PROOF. The proof uses induction on n. We start with $n = 1$

$$\begin{aligned} L(\epsilon)h =& U(\epsilon)^{-1} 4y^2 \frac{\partial^2}{\partial z \partial \overline{z}} U(\epsilon)h \\ =& U(\epsilon)^{-1} 4y^2 \left(U(\epsilon) \frac{\partial^2 h}{\partial z \partial \overline{z}} + \frac{\partial U(\epsilon)}{\partial z}\frac{\partial h}{\partial \overline{z}} + \frac{\partial U(\epsilon)}{\partial \overline{z}}\frac{\partial h}{\partial z} + \frac{\partial^2 U(\epsilon)}{\partial z \partial \overline{z}} h \right) \end{aligned}$$

If $w_1(z) = v_1(z)dz + v_2(z)d\overline{z}$ we have $U_z(\epsilon) = 2\pi i \epsilon v_1 U(\epsilon)$ and $U_{\overline{z}}(\epsilon) = 2\pi i \epsilon v_2 U(\epsilon)$. We therefore get

$$=\Delta h + 8\pi i \epsilon y^2 (v_1(z) h_{\overline{z}} + v_2(z) h_z) + 8\pi i y^2 \frac{\partial v_2}{\partial z} h - 16\pi^2 y^2 v_1 v_2 \epsilon^2 h$$

Using $w_1 = \overline{w}_1$ we see that $v_1 = \overline{v_2}$ and we get

$$=\Delta h + 4\pi i \epsilon \left\langle dh, w_1 \right\rangle - 2\pi i \left(-4y^2 \frac{\partial v_2}{\partial z}\right) h - 4\pi^2 \epsilon^2 \left\langle w_1, w_1 \right\rangle.$$

Using

$$\begin{aligned} -4y^2 \frac{\partial v_2}{\partial z} =& -2y^2 \left(\frac{\partial v_2}{\partial z} + \frac{\partial v_1}{\partial \overline{z}} \right) \\ =& -y^2 \left(\frac{\partial v_2}{\partial x} - i\frac{\partial v_2}{\partial y} + \frac{\partial v_1}{\partial x} + i\frac{\partial v_1}{\partial y}\right) = \delta(w) \end{aligned}$$

we obtain the result for $n = 1$. We now move on to the general result. With the convention that $U(\epsilon_k) = U((0, \ldots, 0, \epsilon_k, 0, \ldots, 0))$ we see that

$$\begin{aligned} L(\vec{\epsilon})h =& U(\epsilon_n)^{-1} U(\epsilon_1, \ldots \epsilon_{n-1}, 0)^{-1} \Delta U(\epsilon_1, \ldots \epsilon_{n-1}, 0) U(\epsilon_n) h \\ =& U(\epsilon_n)^{-1} \Bigg(\Delta U(\epsilon_n) h + 4\pi i \sum_{k=1}^{n-1} \epsilon_k \langle dU(\epsilon_n)h, w_k\rangle \\ & -2\pi i \left(\sum_{k=1}^{n-1} \epsilon_k \delta(w_k)\right) U(\epsilon_n) h - 4\pi^2 \left(\sum_{k,l=1}^{n-1} \epsilon_k\epsilon_l \left\langle w_k, w_l\right\rangle\right) U(\epsilon_n) h\Bigg). \end{aligned}$$

We apply the result for one variable once more in the ϵ_n variable and use the chain rule in the form

$$d(U(\epsilon_n)h) = U(\epsilon_n) dh + 2\pi i \epsilon_n U(\epsilon_n) h w_n$$

to get the result. □

Lemma 11 gives

$$L_{\epsilon_k}(\vec{0})h = 4\pi i\langle dh, w_k\rangle - 2\pi i(\delta w_k)h, \tag{3.7}$$

$$L_{\epsilon_k\epsilon_l}(\vec{0})h = -8\pi^2\langle w_k, w_l\rangle h. \tag{3.8}$$

and all higher order derivatives vanish. Differentiating the eigenvalue equation

$$(L(\vec{\epsilon}) + s(1-s))D(z, s, \vec{\epsilon}) = 0 \tag{3.9}$$

we get

$$(\Delta + s(1-s))D_{\epsilon_k}(z, s, \vec{0}) = -\left(L_{\epsilon_k}(\vec{0})D(z, s, \vec{0})\right) \tag{3.10}$$

and

$$\begin{aligned}(\Delta + s(1-s))D_{\epsilon_1,\ldots,\epsilon_n}(z, s, \vec{0}) = -\Bigg(&\sum_{k=1}^{n} L_{\epsilon_k}(\vec{0})D_{\epsilon_1,.,\widehat{\epsilon_k},.,\epsilon_n}(z, s, \vec{0}) \\ &+ \sum_{\substack{k,l=1\\k<l}}^{n} L_{\epsilon_k\epsilon_l}(\vec{0})D_{\epsilon_1,.,\widehat{\epsilon_k},.,\widehat{\epsilon_l},.,\epsilon_n}(z, s, \vec{0})\Bigg).\end{aligned} \tag{3.11}$$

Here $\widehat{\epsilon_k}$ means that we have excluded ϵ_k from the list. When $\Re(s)$ is sufficiently large we can use Lemma 10 and invert (3.10) and (3.11) by applying the resolvent of the Laplace operator, $R(s) = (\Delta_\Gamma + s(1-s))^{-1}$. We get

$$D_{\epsilon_k}(z, s, \vec{0}) = -R(s)\left(L_{\epsilon_k}(\vec{0})D(z, s, \vec{0})\right) \tag{3.12}$$

and

$$\begin{aligned}D_{\epsilon_1,\ldots,\epsilon_n}(z, s, \vec{0}) \quad = -R(s) \quad \Bigg(&\sum_{k=1}^{n} L_{\epsilon_k}(\vec{0})D_{\epsilon_1,.,\widehat{\epsilon_k},.,\epsilon_n}(z, s, \vec{0}) \\ &+ \sum_{\substack{k,l=1\\k<l}}^{n} L_{\epsilon_k\epsilon_l}(\vec{0})D_{\epsilon_1,.,\widehat{\epsilon_k},.,\widehat{\epsilon_l},.,\epsilon_n}(z, s, \vec{0})\Bigg).\end{aligned} \tag{3.13}$$

This will turn out to be identities of great importance for the proofs of many results in this and the following chapter. Using these identities we may now give a short proof of the analytic continuation of the functions defined in a half-plane by (1.1)

LEMMA 12. *The functions $D_{\epsilon_1\ldots\epsilon_n}(z, s, \vec{0})$ have meromorphic continuation to $\mathbb{C}$. In $\Re(s) > 1$ the functions are analytic.*

Proof. The proof uses induction on n. For $n=0$ the function is the classical Eisenstein series and one of the many known proofs may be found in (Kubota 1973). We note that by (3.7) and (3.8) $L_{\epsilon_k}(\vec{0})D_{\epsilon_1,.,\widehat{\epsilon_k},.,\epsilon_n}(z,s,\vec{0})$ and $L_{\epsilon_k\epsilon_l}(\vec{0})D_{\epsilon_1,.,\widehat{\epsilon_k},.,\widehat{\epsilon_l},.,\epsilon_n}(z,s,\vec{0})$ are compactly supported. Hence from (3.13) and (Müller 1996, Theorem 1) (Or the results cited in Section 2.4) the conclusion follows. $\square$

From the above lemma, (3.4) and (1.9) we find that

$$\left.\frac{\partial^n E(z,s,\vec{\epsilon})}{\partial\epsilon_1\ldots\partial\epsilon_n}\right|_{\vec{\epsilon}=\vec{0}}$$

has meromorphic continuation and that in $\Re(s)>1$ these functions are analytic. By taking linear combinations of these (see (1.9)) we obtain Theorem C.

Proposition 13. *The sum defining $E^{m,n}(z,s)$ is absolutely convergent whenever $\Re(s)>1$.*

Proof. Note that if we can prove the above for $f=g$ and $m=n$ then we get the general result by appealing to the elementary inequality $2ab\leq a^2+b^2$ for $a,b\in\mathbb{R}$. This gives

$$\left|\langle\gamma,f\rangle^m\overline{\langle\gamma,g\rangle^n}\right|\leq\frac{1}{2}\left(\left|\langle\gamma,f\rangle\right|^{2m}+\left|\langle\gamma,g\rangle\right|^{2n}\right),$$

and comparison with $f=g$ and $m=n$ type Eisenstein series gives the result.

We now consider the case $f=g$ and $m=n$. This follows a proof due to Landau (See e.g. (Titchmarsh 1975, Section 9.2)). We write $F(z,s)=E^{m,m}(z,s)$ Notice that in the case we are considering the sum,

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left|\langle\gamma,f\rangle\right|^{2m}\Im(\gamma z)^s, \tag{3.14}$$

defining $F(z,s)$ is convergent in a right halfplane if and only if it is absolutely convergent in that halfplane. Assume now that this sum is convergent for $\Re(s)>c$. Assume furthermore that $F(z,s)$ is analytic at $s=c$. We claim that there exists an $\varepsilon_0>0$ such that the sum is convergent for $\Re(s)>c-\varepsilon_0$. Let $a=c+1$. Then $F(z,s)$ is analytic at $s=a$, and has an absolute convergent power series expansion at a,

$$F(z,s)=\sum_{k=0}^{\infty}\frac{F^{(k)}(z,a)}{k!}(s-a)^k$$

with convergence radius $R > 1$. Since the sum (3.14) is uniformly convergent in every closed halfplane inside the domain of absolute convergence we may calculate the derivative as the following sum of derivatives

$$F^{(k)}(z,a) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |\langle\gamma,f\rangle|^{2m} \log(\Im(\gamma z))^k \Im(\gamma z)^a, \tag{3.15}$$

and we hence get

$$F(z,s) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |\langle\gamma,f\rangle|^{2m} \log(\Im(\gamma z))^k \Im(\gamma z)^a (s-a)^k.$$

Since $R > 1$ there exists $\varepsilon_0 > 0$ such that this is valid for $s = c - \varepsilon$ where $\varepsilon \leq \varepsilon_0$. The double series has nonnegative terms for this s from some point $\Im(\gamma z) < 1$ so we may change the order of summation and get

$$F(z,c-\varepsilon) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |\langle\gamma,f\rangle|^{2m} \Im(\gamma z)^{c-\varepsilon},$$

which settles the claim. By Theorem C we get the first singularity of $E^{m,m}(z,s)$ at $s=1$ or further to the left. $\square$

Clearly $E_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0})$ is also absolutely convergent for $\Re(s) > 1$ by the same proof. We immediately get the following corollary:

Corollary 14. *For any fixed $z \in \mathbb{H}$, $\varepsilon > 0$ we have*

$$\langle\gamma,f\rangle = o(|cz+d|^{\varepsilon})$$
$$\langle\gamma,\alpha\rangle = o(|cz+d|^{\varepsilon})$$

as $|cz+d| \to \infty$.

Proof. Since the terms in an absolutely convergent series tend to zero Proposition 13 implies that for any $m \in N$,

$$\langle\gamma,f\rangle^m \Im(\gamma z)^2 = \langle\gamma,f\rangle^m \frac{y^2}{|cz+d|^4} \to 0.$$

Hence $\langle\gamma,f\rangle = o(|cz+d|^{4/m})$. Similar with $\langle\gamma,\alpha\rangle$. $\square$

We note that by picking $z = i$ we get Theorem D. Hence we "almost" recover the bound in Proposition 7 using completely different methods.

Lemma 15. *The function $D_{\epsilon_1,\ldots\epsilon_n}(z,s,\vec{0})$ has the series representation*

$$D_{\epsilon_1,\ldots\epsilon_n}(z,s,\vec{0}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \prod_{k=1}^{n} \left(-2\pi i \int_{i\infty}^{\gamma z} w_k\right) \Im(\gamma z)^s, \tag{3.16}$$

when $\Re(s) > 1$.

PROOF. Once more we use induction on n. The claim is true for $n = 0$, since in this case $D(z, s, \vec{0})$ is the usual Eisenstein series. From $E(z, s, \vec{\epsilon}) = U(\vec{\epsilon}, z)D(z, s, \vec{\epsilon})$ we conclude that

$$E_{\epsilon_1,\ldots,\epsilon_n}(z, s, \vec{0}) = \sum_{\vec{m}\in\{0,1\}^n} \prod_{k=1}^{n} \left(2\pi i \int_{i\infty}^{z} w_k\right)^{m_k} D_{\epsilon_1^{1-m_1},\ldots,\epsilon_n^{1-m_n}}(z, s, \vec{0}), \tag{3.17}$$

where $\vec{m} = (m_1, \ldots, m_n)$. We separate $D_{\epsilon_1,\ldots,\epsilon_n}(z, s, \vec{0})$ by taking $\vec{m} = \vec{0}$. By inductive hypothesis we find that

$$D_{\epsilon_1,\ldots,\epsilon_n}(z, s, \vec{0}) =$$

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left(\prod_{k=1}^{n} \langle \gamma, w_k \rangle - \sum_{\substack{\vec{m}\in\{0,1\}^n \\ \vec{m}\neq\vec{0}}} \prod_{k=1}^{n} \left(2\pi i \int_{i\infty}^{z} w_k\right)^{m_k} \left(-2\pi i \int_{i\infty}^{\gamma z} w_k\right)^{1-m_k} \right) \Im(\gamma z)^s.$$

Since

$$\langle \gamma, w_k \rangle = 2\pi i \int_{i\infty}^{z} w_k - 2\pi i \int_{i\infty}^{\gamma z} w_k, \tag{3.18}$$

for any $z \in \mathbb{H}$, the result follows. $\square$

Combining this with Proposition 8 (3), we see that, if $z \in F$ and $\Im(z) < T$, then

$$D_{\epsilon_1,\ldots\epsilon_n}(z, s, \vec{0}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \prod_{k=1}^{n} \left(-2\pi i \int_{i\infty}^{\gamma z} \alpha_k\right) \Im(\gamma z)^s. \tag{3.19}$$

In particular

$$\lim_{T\to\infty} D_{\epsilon_1,\ldots\epsilon_n}(z, s, \vec{0}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \prod_{k=1}^{n} \left(-2\pi i \int_{i\infty}^{\gamma z} \alpha_k\right) \Im(\gamma z)^s, \tag{3.20}$$

for all $z \in \mathbb{H}$.

LEMMA 16. *For* $\sigma > 1$ *we have*

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left| \prod_{k=1}^{n} \left(-2\pi i \int_{i\infty}^{\gamma z} \alpha_k\right) \right| \Im(\gamma z)^\sigma = O(y^{1-\sigma}) \tag{3.21}$$

as $\Im(z) \to \infty$ *for* $z \in F$. *In particular* $\lim_{T\to\infty} D_{\epsilon_1,\ldots\epsilon_n}(z, \sigma + it, \vec{0}) = O(y^{1-\sigma})$.

PROOF. We have for $\sigma > 1$ (see (Kubota 1973, p.13))

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \gamma\neq I}} \Im(\gamma z)^\sigma = O_\sigma(y^{1-\sigma}) \tag{3.22}$$

as $\Im(z)\to\infty$. From Corollary 14 we see that if we fix e.g. $z_0 = i$ there exists a constant $C > 0$ such that

$$\left|\prod_{k=1}^n \langle \gamma, \alpha_k\rangle\right| \leq C\Im(\gamma z_0)^{-\varepsilon}.$$

this gives, using $\langle I, \alpha_k\rangle = 0$,

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left|\prod_{k=1}^n \langle \gamma, \alpha_k\rangle\right| \Im(\gamma z)^\sigma \leq C \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \gamma\neq I}} \Im(\gamma i)^{-\varepsilon}\Im(\gamma z)^\sigma$$

If we use the inequality (see (Knopp 1974, Lemma 4))

$$(c^2+d^2) \leq \frac{|cz+d|^2}{y}\frac{1+4|z|^2}{y}$$

this is majorized by

$$C \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \gamma\neq I}} \Im(\gamma z)^{\sigma-\varepsilon}\left(\frac{1+4|z|^2}{y}\right)^\varepsilon = O_\sigma(y^{1-\sigma}).$$

In the last equality we used (3.22). The claim now follows by induction from (3.17) by isolating $D_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0})$, using Lemma 15 and the fact that

$$-2\pi i\int_{i\infty}^z \alpha_k$$

is $O(e^{-2\pi y})$ as $\Im(z)\to\infty$. $\square$

LEMMA 17. *For $\Re(s) > 1$ we have*

$$\int_{\Gamma\backslash\mathbb{H}} |\langle d \lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots\epsilon_n}(z,s,\vec{0}),\alpha_j\rangle| d\mu(z) < \infty.$$

PROOF. Using (3.20) we see that for $\Re(s) > 1$

$$d\lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots,\epsilon_n}(z,s,\vec{0}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} d\left(\prod_{\substack{k=1\\ k\neq j}}^n \left(-2\pi i\int_{i\infty}^{\gamma z}\alpha_k\right)\Im(\gamma z)^s\right). \tag{3.23}$$

Using

$$d\left(-2\pi i\int_{i\infty}^{\gamma z}\alpha_k\right) = -2\pi i\alpha_k$$
$$d\Im(\gamma z)^s = \frac{s}{2y}\left(-i\left(\frac{c\overline{z}+d}{cz+d}\right)\Im(\gamma z)^s dz + i\left(\frac{cz+d}{c\overline{z}+d}\right)\Im(\gamma z)^s d\overline{z}\right)$$

we find that

$$\begin{aligned}\langle d\lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots,\epsilon_n}(z,s,\vec{0}),\alpha_j\rangle = \\ 2y^2\Bigg[\sum_{\substack{l=1\\ l\neq j}}^{n}\left(-2\pi i\frac{f_l}{2}\frac{\overline{f_j}}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\prod_{\substack{k=1\\ k\neq j,l}}^{n}\left(-2\pi i\int_{i\infty}^{\gamma z}\alpha_k\right)\Im(\gamma z)^s\right) \\ +\frac{-is}{2y}\frac{\overline{f_j}}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\prod_{\substack{k=1\\ k\neq j}}^{n}\left(-2\pi i\int_{i\infty}^{\gamma z}\alpha_k\right)\left(\frac{c\overline{z}+d}{cz+d}\right)\Im(\gamma z)^s\Bigg] \\ + \text{complex conjugate.}\end{aligned}$$

The claim now follows from Lemma 16, since $f_i(z) = O(e^{-2\pi y})$ as $\Im(z)\to\infty$. □

Using this lemma we can prove the following important result

LEMMA 18. *For all $j = 1,\ldots,n$ and $\Re(s) > 1$*

$$\int_{\Gamma\backslash\mathbb{H}}\langle dD_{\epsilon_1,\ldots,\widehat{\epsilon_j},\ldots\epsilon_n}(z,s,\vec{0}),w_j\rangle d\mu(z)\to 0 \text{ as } T\to\infty.$$

PROOF. We start by showing that

$$\int_{\Gamma\backslash\mathbb{H}}\langle d\lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots\epsilon_n}(z,s,\vec{0}),\alpha_j\rangle d\mu(z) = 0.$$

If we let $F_M = \{z\in F|\Im(z)\leq M\}$ then by lemma 17 the left-hand side is

$$\int_{F_M}\langle d\lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots\epsilon_n}(z,s,\vec{0}),\alpha_j\rangle d\mu(z) + \epsilon(T)$$

where $\epsilon(T) \to 0$ as $T \to \infty$. We have

$$\begin{aligned}\int_{F_M} \langle d \lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots\epsilon_n}(z,s,\vec{0}), \alpha_j\rangle d\mu(z) = \\ \int_{F_M} \frac{\partial}{\partial z}\left(\lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots,\epsilon_n}(z,s,\vec{0})\right)\frac{\overline{f_j}}{2}dxdy \\ + \int_{F_M} \frac{\partial}{\partial \overline{z}}\left(\lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots,\epsilon_n}(z,s,\vec{0})\right)\frac{f_j}{2}dxdy.\end{aligned} \tag{3.24}$$

For any real differentiable function $h : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ and any bounded domain $R \subset U$ with piecewise differentiable boundary Stokes theorem implies that

$$2i \int_R \frac{\partial}{\partial \overline{z}} h dxdy = \int_{\partial R} h.$$

We apply this to the second integral in (3.24). Since f_j is holomorphic, the integral equals

$$-\frac{i}{2}\int_{\partial(F_M)} \lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots,\epsilon_n}(z,s,\vec{0}) f_j.$$

The fundamental domain is the union of conjugated sides. These conjugated sides cancel in the integral. Hence this integral equals the line integral along the top of the truncated fundamental domain F_M. But this goes to zero by lemma 16. We observe that when s is real the first integral in (3.24) is the complex conjugate of the second one. Hence this also vanishes and we have

$$\int_{\Gamma\backslash\mathbb{H}} \langle d \lim_{T\to\infty} D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots\epsilon_n}(z,s,\vec{0}), \alpha_j\rangle d\mu(z) = 0.$$

Using $\left|\int_{i\infty}^z w_i\right| \le \left|\int_{i\infty}^z \alpha_i\right|$ and the same approach as in the proof of lemma 17, we see that for $\Re(s) > 1$ there exist $U(z,s)$ independent on T such that

$$\left|\left\langle dD_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots,\epsilon_n}(z,s,\vec{0}), w_j\right\rangle\right| \le U(z,s)$$

and

$$\int_{\Gamma\backslash\mathbb{H}} U(z,s) d\mu(z) < \infty.$$

Hence for any given $\varepsilon_0 > 0$ there exists a constant, M, independent on T such that

$$\begin{aligned}&\left|\int_{\Gamma\backslash\mathbb{H}}\Big(\langle d\lim_{T\to\infty} B(z,s),\alpha_j\rangle - \langle dB(z,s),w_j\rangle\Big)\,d\mu(z)\right|\\ &\leq \left|\int_{F_M}\Big(\langle d\lim_{T\to\infty} B(z,s),\alpha_j\rangle - \langle dB(z,s),w_j\rangle\Big)\,d\mu(z)\right| + \varepsilon_0.\end{aligned}$$

Here $B(z,s) = D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots\epsilon_n}(z,s,\vec{0})$. Hence if we choose $T > M$ and use (3.19), (3.20) and Proposition 8 (2), we see that the integral over F_M vanishes which finishes the proof. $\square$

Using this we can now prove

LEMMA 19. *The function*

$$\lim_{T\to\infty}(-R(s)L_{\epsilon_j}(\vec{0})D_{\epsilon_1\ldots,\widehat{\epsilon_j},\ldots,\epsilon_n}(z,s,\vec{0})) \tag{3.25}$$

is regular at $s = 1$.

PROOF. We shall write $B(z,s) = D_{\epsilon_1,\ldots,\hat{\epsilon}_j,\ldots\epsilon_n}(z,s,\vec{0})$. We note that since α_j is the real part of a holomorphic differential $\delta(\alpha_j) = 0$ Since $\delta(w_j) = \delta(\alpha_j)$ for $\Im(z) < T$ (Proposition 8) we find from Lemma 18 that

$$\lim_{T\to\infty}\int_{\Gamma\backslash\mathbb{H}} L_{\epsilon_j}(\vec{0})B(z,s)d\mu(z) = 0.$$

From (3.13) it is clear that $s = 1$ is not an essential singularity. Assume that it is a pole of order $k > 0$. Hence

$$\lim_{s\to1}(s-1)^k\lim_{T\to\infty}(-R(s)L_{\epsilon_j}(\vec{0})B(z,s)) \neq 0. \tag{3.26}$$

But

$$\begin{aligned}&\lim\nolimits_{s\to1}(s-1)^k\lim_{T\to\infty}(-R(s)L_{\epsilon_j}(\vec{0})B(z,s))\\ &\quad= -\lim_{s\to1}(s-1)^k\lim_{T\to\infty}\left(\int_{\Gamma\backslash\mathbb{H}} r(z,z',s)L_{\epsilon_j}(\vec{0})B(z',s)d\mu(z')\right)\end{aligned}$$

where $r(z,z',s)$ is the resolvent kernel

$$\begin{aligned}&= -\lim_{s\to1}\lim_{T\to\infty}\left(\int_{\Gamma\backslash\mathbb{H}}(s-1)r(z,z',s)(s-1)^{k-1}L_{\epsilon_j}(\vec{0})B(z',s)d\mu(z')\right)\\ &= \mathrm{vol}(\Gamma\setminus\mathbb{H})^{-1}\lim_{T\to\infty}\left(\int_{\Gamma\backslash\mathbb{H}}\lim_{s-1}(s-1)^{k-1}L_{\epsilon_j}(\vec{0})B(z',s)d\mu(z')\right)\end{aligned}$$

since $r(z, z', s)$ has a simple pole with residue $-\mathrm{vol}(\Gamma \setminus \mathbb{H})^{-1}$. See remark 20

$$= \mathrm{vol}(\Gamma \setminus \mathbb{H})^{-1} \lim_{s\to 1}(s-1)^{k-1} \lim_{T\to\infty} \int_{\Gamma\setminus\mathbb{H}} L_{\epsilon_j}(\vec{0}) B(z', s) d\mu(z')$$
$$= 0$$

by lemma 18. But this contradicts (3.26), which completes the proof. □

REMARK 20. Using the above lemma, (3.13) and the fact that the resolvent kernel for Δ respectively the Eisenstein series has expansions at 1 of the form (see e.g (Venkov 1982, Theorem 2.2.6) or (2.2))

$$\begin{aligned} r(z, z', s) &= \frac{\mathrm{vol}(\Gamma \setminus \mathbb{H})^{-1}}{s(1-s)} + \sum_{m=0}^{\infty} \widetilde{r_m}(z, z')(s-1)^m \\ &= \frac{-\mathrm{vol}(\Gamma \setminus \mathbb{H})^{-1}}{(s-1)} + \sum_{m=0}^{\infty} r_m(z, z')(s-1)^m, \end{aligned} \tag{3.27}$$

respectively

$$E(z, s) = \frac{\mathrm{vol}(\Gamma \setminus \mathbb{H})^{-1}}{s-1} + \sum_{m=0}^{\infty} E_m(z)(s-1)^m, \tag{3.28}$$

we may now in principle write down the full Laurent expansion of the function $\lim_{T\to\infty} D_{\epsilon_1 \ldots \epsilon_n}(z, s, \vec{0})$ at $s = 1$ in terms of $r_m(z, z')$, $E_m(z)$ and the real harmonic differentials. From this and (3.4) we may also calculate the Laurent expansion of $E_{\epsilon_1,\ldots,\epsilon_n}(z, s, \vec{0})$ and hence of $E^{m,n}(z, s)$. Since general expressions are quite complicated and the combinatorics become quite cumbersome we restrict ourselves to a some particular cases of special interest.

We let $\widetilde{\Sigma}_{2m}$ be the elements of the symmetric group on $2m$ letters $1, 2, \ldots, 2m$ for which $\sigma(2j-1) < \sigma(2j)$ for $j = 1, \ldots, m$. We notice that this has $(2m)!/2^m$ elements, which is easily seen by induction.

LEMMA 21. *If n is even $\lim_{T\to\infty} D_{\epsilon_1,\ldots\epsilon_n}(z, s, \vec{0})$ has a pole at $s = 1$ of at most order $n/2 + 1$. The $(s-1)^{n/2+1}$ coefficient in the expansion of the function $\lim_{T\to\infty} D_{\epsilon_1,\ldots\epsilon_n}(z, s, \vec{0})$ around $s = 1$ is*

$$\frac{(-8\pi^2)^{n/2}}{vol(\Gamma \setminus \mathbb{H})^{n/2+1}} \sum_{\sigma \in \widetilde{\Sigma}_n} \left(\prod_{r=1}^{n/2} \int_{\Gamma\setminus\mathbb{H}} \langle \alpha_{\sigma(2r-1)}, \alpha_{\sigma(2r)} \rangle \, d\mu(z) \right).$$

If n is odd, $\lim_{T\to\infty} D_{\epsilon_1,\ldots\epsilon_n}(z, s, \vec{0})$ has a pole at $s = 1$ of at most order $(n-1)/2$.

PROOF. For $n = 0$ the claim is obvious, and for $n = 1$ (3.12) and Lemma 19 give the result. Assume that the result is true for all $n \leq n_0$. By (3.13), (3.8), Lemma 19 and the fact that

$$\lim_{T\to\infty}(-R(s)(-8\pi^2 \langle w_k, w_l\rangle D_{\epsilon_1,.,\hat{\epsilon}_k,.,\hat{\epsilon}_l,.,\epsilon_n}(z,s,\vec{0})))$$

can have pole order at most 1 more than $\lim_{T\to\infty} D_{\epsilon_1,.,\hat{\epsilon}_k,.,\hat{\epsilon}_l,.,\epsilon_n}(z,s,\vec{0}))$ at $s = 1$, we obtain the result about the pole orders. For even n we notice that by induction and using (3.27) we find that the $(s-1)^{n/2-1}$ coefficient is

$$\frac{-8\pi^2}{\mathrm{vol}(\Gamma\setminus\mathbb{H})}\frac{(-8\pi^2)^{(n-2)/2}}{\mathrm{vol}(\Gamma\setminus\mathbb{H})^{(n-2)/2+1}}\cdot$$
$$\sum_{\substack{k,l=1\\k<l}}^{n}\sum_{\sigma\in\tilde{\Sigma}_{n-2}}\left(\prod_{r=1}^{(n-2)/2}{}'\int_{\Gamma\setminus\mathbb{H}}\langle\alpha_{\sigma(2r-1)},\alpha_{\sigma(2r)}\rangle\,d\mu(z)\right)\int_{\Gamma\setminus\mathbb{H}}\langle\alpha_k,\alpha_l\rangle\,d\mu(z),$$

where the prime in the product means that we have excluded α_k, α_l from the product and enumerated the remaining differentials accordingly. The result follows. □

Using this we can prove

THEOREM 22. *For all n $E_{\epsilon_1,\ldots\epsilon_n}(z,s,\vec{0})$ has a pole at $s = 1$ of at most order $[n/2]+1$. If n is even the $(s-1)^{[n/2]+1}$ coefficient in the Laurent expansion of $E_{\epsilon_1,\ldots\epsilon_n}(z,s,\vec{0})$ is*

$$\frac{(-8\pi^2)^{n/2}}{vol(\Gamma\setminus\mathbb{H})^{n/2+1}}\sum_{\sigma\in\tilde{\Sigma}_n}\left(\prod_{r=1}^{n/2}\int_{\Gamma\setminus\mathbb{H}}\langle\alpha_{\sigma(2r-1)},\alpha_{\sigma(2r)}\rangle\,d\mu(z)\right).$$

If n is odd the $(s-1)^{[n/2]+1}$ coefficient in the Laurent expansion of $E_{\epsilon_1,\ldots\epsilon_n}(z,s,\vec{0})$ is

$$\frac{(-8\pi^2)^{[n/2]}}{vol(\Gamma\setminus\mathbb{H})^{[n/2]+1}}\sum_{k=1}^{n}\left(2\pi i\int_{i\infty}^{z}\alpha_k\sum_{\sigma\in\tilde{\Sigma}_{n-1}}\prod_{r=1}^{[n/2]}{}'\int_{\Gamma\setminus\mathbb{H}}\langle\alpha_{\sigma(2r-1)},\alpha_{\sigma(2r)}\rangle\,d\mu(z)\right),$$

where the prime in the the product means that we have excluded α_k from the product and enumerated the remaining differentials accordingly.

PROOF. This follows from (3.17) Lemma 21, and the fact that $E_{\epsilon_1,\ldots\epsilon_n}(z,s)$ is independent on the cohomology class of the real differentials involved. □

We notice that

$$\langle \Re(f(z)dz), \Re(f(z)dz)\rangle = \langle \Im(f(z)dz), \Im(f(z)dz)\rangle = y^2 \left|f(z)\right|^2, \tag{3.29}$$

while

$$\langle \Re(f(z)dz), \Im(f(z)dz)\rangle = 0. \tag{3.30}$$

Hence many of the involved integrals may be expressed in terms of the Petersson norm defined in the weight two case by

$$\|f\| = \left(\int_{\Gamma\backslash\mathbb{H}} y^2 \left|f(z)\right|^2 d\mu(z)\right)^{1/2}. \tag{3.31}$$

We shall write $E^{\Re^l,\Im^{n-l}}(z,s) := E_{\epsilon_1,\dots,\epsilon_n}(z,s,\vec{0})$ where $\alpha_i = \Re(f(z)dz)$ for $i = 1,\dots,l$ and $\alpha_i = \Im(f(z)dz)$ for $i = l+1,\dots,n$. As a special case of Theorem 22 we have the following

THEOREM 23. *The function $E^{\Re^{2m},\Im^{2n}}(z,s)$ has a pole of order $m+n+1$ at $s=1$, and the $(s-1)^{-(m+n+1)}$ coefficient in the Laurent expansion expansion is*

$$\frac{\left(-8\pi^2\right)^{m+n}}{vol(\Gamma\setminus\mathbb{H})^{m+n+1}} \left\|f\right\|^{2(m+n)} \frac{(2m)!(2n)!}{2^{m+n}} \binom{m+n}{n}. \tag{3.32}$$

If n or m is odd then the pole order of $E^{\Re^m,\Im^n}(z,s)$ at $s=1$ is strictly less than $(m+n)/2+1$.

PROOF. The first part follows from Theorem 22, (3.29) and (3.30) once we count the number of nonzero terms in the sum indexed by $\widetilde{\Sigma}_{2m+2n}$. This is the set of elements

$$\left\{ \sigma\in\widetilde{\Sigma}_{2m+2n} \,\middle|\, \begin{array}{c} \sigma(2i-1),\sigma(2i)\leq 2m \text{ or } \sigma(2i-1),\sigma(2i)>2m \\ \text{for all } i=1,\dots,m+n \end{array} \right\}.$$

We shall denote this by $\widetilde{\Sigma}^{2m}_{2m+2n}$. This set contains

$$\frac{(2m)!}{2^m}\frac{(2n)!}{2^n}\binom{m+n}{n}$$

elements which can be seen by noticing that each element may be obtained uniquely by applying $\sigma_1\in\widetilde{\Sigma}_{2m}$ to $1,\dots,2m$ and $\sigma_2\in\widetilde{\Sigma}_{2n}$ to $2m+1,\dots,2m+2n$ and then shuffling $(\sigma_1(1),\sigma_1(2)),\dots,(\sigma_1(2m-1),\sigma_1(2m))$ with $(\sigma_2(2m+1),\sigma_2(2m+2)),\dots,(\sigma_2(2m+2n-1),\sigma_2(2m+2n))$.

If $m+n$ is odd then Theorem 22 says that the pole order at $s=1$ is at most $[(m+n)/2]+1$ which is strictly less than $(m+n)/2+1$.

If m and n is odd then Theorem 22 says that the pole order at $s=1$ is at most $(m+n)/2+1$, but since one of the factors in the product of the $(m+n)/2+1$ term has to be zero the pole is at most of order $(m+n)/2$. $\square$

We now turn to $E^{m,n}(z,s)$. We assume $f = g$.

THEOREM 24 (See also (Goldfeld & O'Sullivan 2003)). *At $s = 1$, $E^{1,0}(z,s)$ has a simple pole with residue*

$$\frac{1}{vol(\Gamma \setminus \mathbb{H})}\left(2\pi i \int_{i\infty}^{z} f(z)dz\right).$$

PROOF. This follows directly from Theorem 22 and

$$E^{1,0}(z,s) = E^{\Re}(z,s) + iE^{\Im}(z,s).$$

$\square$

THEOREM 25. *The Eisenstein series $E^{m,m}(z,s)$ has a pole of order $m+1$. The $(s-1)^{m+1}$ coefficient in the Laurent expansion around $s = 1$ is*

$$\frac{(16\pi^2)^m}{vol(\Gamma \setminus \mathbb{H})^{m+1}} m!^2 \left\| f \right\|^{2m}.$$

PROOF. Since $\langle \gamma, f\rangle = \langle \gamma, \Re(f(z)dz)\rangle + i\langle \gamma, \Im(f(z)dz)\rangle$ we have

$$\left|\langle \gamma, f\rangle\right|^{2m} = (-1)^m \sum_{n=0}^{m} \binom{m}{n} \langle \gamma, \Re(f(z)dz)\rangle^{2n} \langle \gamma, \Im(f(z)dz)\rangle^{2(m-n)}.$$

Hence

$$E^{m,m}(z,s) = (-1)^m \sum_{n=0}^{m} \binom{m}{n} E^{\Re^{2n},\Im^{2(m-n)}}(z,s).$$

From Theorem 23 we hence find that the leading term of $E^{m,m}(z,s)$ is

$$\frac{(8\pi^2)^m}{\mathrm{vol}(\Gamma \setminus \mathbb{H})^{m+1}} \left\| f \right\|^{2m} \sum_{n=0}^{m} \binom{m}{n} \frac{(2n)!(2(m-n))!}{2^m} \binom{m}{n}.$$

The sum equals $(m!)^2 2^m$ from which the result follows. $\square$

THEOREM 26. *At $s = 1$, $E^{2,0}(z,s)$ has a simple pole with residue*

$$\frac{1}{vol(\Gamma \setminus \mathbb{H})}\left(2\pi i \int_{i\infty}^{z} f(z)dz\right)^2$$

while $E^{1,1}$ has a double pole with residue

$$\frac{4\pi^2}{vol(\Gamma \setminus \mathbb{H})}\left|\int_{i\infty}^{z} f(z)dz\right|^2 \tag{3.33}$$
$$+ \frac{16\pi^2}{vol(\Gamma \setminus \mathbb{H})} \int_{\Gamma\setminus\mathbb{H}} (E_0(z') - r_0(z,z'))y'^2 \left|f(z')\right|^2 d\mu(z').$$

The coefficient of $(s-1)^{-2}$ *is*

$$\frac{16\pi^2 \left\|f\right\|^2}{vol(\Gamma \setminus \mathbb{H})^2}.$$

PROOF. We start by noticing that as a special case of (3.17) we have

$$E_{\epsilon_1\epsilon_2}(z,s,\vec{0}) = -4\pi^2 \int_{i\infty}^{z} \alpha_1 \int_{i\infty}^{z} \alpha_2 E(z,s) + 2\pi i \int_{i\infty}^{z} \alpha_1 \lim_{T\to\infty} D_{\epsilon_2}(z,s,\vec{0})$$
$$+ 2\pi i \int_{i\infty}^{z} \alpha_2 \lim_{T\to\infty} D_{\epsilon_1}(z,s,\vec{0}) + \lim_{T\to\infty} D_{\epsilon_1\epsilon_2}(z,s,\vec{0}).$$

The first term has a simple pole at $s=1$ with residue

$$\frac{-4\pi^2}{\mathrm{vol}(\Gamma\setminus\mathbb{H})} \int_{i\infty}^{z} \alpha_1 \int_{i\infty}^{z} \alpha_2,$$

while the two middle terms are regular at $s=1$ by (3.12) and Lemma 19. The singular part of the expansion of the fourth term equals the singular part of the expansion of

$$\lim_{T\to\infty} (-R(s)(L_{\epsilon_1\epsilon_2} E(z,s))) = 8\pi^2 \int_{\Gamma\setminus\mathbb{H}} r(z,z',s) \langle \alpha_1, \alpha_2 \rangle E(z,s).$$

This follows from (3.13) and Lemma 19. But by using (3.27) and (3.28) we find that this is

$$\frac{-8\pi^2}{\mathrm{vol}(\Gamma\setminus\mathbb{H})^2} \int_{\Gamma\setminus\mathbb{H}} \langle \alpha_1, \alpha_2 \rangle \, d\mu(z) (s-1)^{-2}$$
$$+ \frac{-8\pi^2}{\mathrm{vol}(\Gamma\setminus\mathbb{H})} \int_{\Gamma\setminus\mathbb{H}} (E_0(z') - r_0(z,z')) \langle \alpha_1, \alpha_2 \rangle \, d\mu(z') (s-1)^{-1}.$$

Hence we know the singular part of the expansion of $E_{\epsilon_1,\epsilon_2}(z,s)$ at $s=1$. It is easy to see that

$$\begin{aligned} E^{2,0}(z,s) &= \quad E^{\Re^2}(z,s) + 2iE^{\Re,\Im}(z,s) - E^{\Im^2}(z,s) \\ E^{1,1}(z,s) &= \qquad\qquad -E^{\Re^2}(z,s) - E^{\Im^2}(z,s). \end{aligned}$$

Using the above explicit expressions for the expansions of $E_{\epsilon_1,\epsilon_2}(z,s,\vec{0})$ now gives the result when using (3.29) and (3.30). $\square$

We note that this is Theorem E. We state, without proof, the result for the $m+n=3$ case.

THEOREM 27. *At* $s=1$ $E^{3,0}(z,s)$ *has a simple pole with residue*

$$\frac{1}{vol(\Gamma\setminus\mathbb{H})} \left(2\pi i \int_{i\infty}^{z} f(z)dz \right)^3$$

while $E^{2,1}(z,s)$ has a double pole with leading term

$$\frac{32\pi^2}{vol(\Gamma\setminus\mathbb{H})^2}\left(2\pi i\int_{i\infty}^{z} f(z)dz\right)\|f\|^2.$$

3. Growth on vertical lines

By Proposition 13 we see that $E^{m,n}(z,s)=O_K(1)$ for $\Re(s)=\sigma>1$ and z in a fixed compact set K. In this section we show that when we only require $\sigma>1/2$ then we have at most polynomial growth on the line $\Re(s)=\sigma$.

We take the opportunity to correct Theorem 1.5 in (Petridis 2002). We first prove :

LEMMA 28. *The standard nonholomorphic Eisenstein series $E(z,s)$ has polynomial growth in s in $\Re(s)\geq 1/2$. More precisely we have for any $\varepsilon>0$ and $1/2\leq\sigma\leq 1$*

$$E(z,\sigma+it)=O_K(|t|^{1-\sigma+\epsilon}) \tag{3.34}$$

for all $z\in K$, a fixed compact set in $\Gamma\setminus\mathbb{H}$.

PROOF. According to (Selberg 1990) the scattering function $\phi(s)$ is given by

$$\phi(s)=\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}ab^{1-2s}L(s),$$

where a, b are positive constants and $L(s)$ is a Dirichlet series with constant term 1. In particular, $L(s)$ tends to 1 as $\Re(s)\to\infty$. This implies that for $\Re(s)$ sufficiently large, say $\Re(s)\geq\sigma_0>1$, we have $|L(s)-1|\leq 1/2$. Hence

$$\frac{E(z,s)}{L(s)}$$

is bounded for $\Re(s)\geq\sigma_0$ and fixed. By the functional equation we have

$$\begin{aligned}\frac{E(z,-\sigma_0+it)}{L(-\sigma_0+it)}=&\frac{\phi(-\sigma_0+it)E(z,1+\sigma_0-it)}{L(-\sigma_0+it)}\\ =&E(z,1+\sigma_0-it)\frac{\Gamma(-\sigma_0+it-1/2)}{\Gamma(-\sigma_0+it)}\sqrt{\pi}ab^{1+2\sigma_0-2it}.\end{aligned} \tag{3.35}$$

The asymptotics of the Gamma function (Stirling's formula, see. e.g. (Ivić 1985, A.34))

$$|\Gamma(\sigma_0+it)|\sim\sqrt{2\pi}e^{-\frac{1}{2}\pi|t|}\,|t|^{\sigma_0-1/2}\qquad\text{as }\ |t|\to\infty$$

imply that the quotient of the Gamma factors in (3.35) is asymptotic to $|t|^{-1/2}$ as $|t| \to \infty$. In particular, we get that $E(z,s)/L(s)$ is bounded on the line $\Re(s) = -\sigma_0$. We want to use Phragmén-Lindelöf to conclude that it is bounded for $-\sigma_0 \leq \Re(s) \leq \sigma_0$, so we need to verify that $E(z,s)/L(s)$ is of finite order in this strip.

The poles of $E(z,s)$ and $\phi(s)$ are the same (See (Iwaniec 1995, Theorem 6.9-11)) So $E(z,s)/L(s)$ has no poles, with the possible exception of finite many in any vertical strip coming from $\Gamma(s-1/2)$. These poles $\gamma_1, \gamma_2, \dots \gamma_k$ can be easily dealt with by considering $(s-\gamma_1)(s-\gamma_2)\cdots(s-\gamma_k)E(z,s)/L(s)$. Since $\phi(s)$ is a meromorphic function of order ≤ 2 (see (Selberg 1989, Theorem 7.3) or (Müller 1992, Theorem 3.20)) and $\Gamma(s)$ has order 1, we see that $L(s)$ is of finite order and by (Hejhal 1983, Th. 12.9(d) p. 164) we see that $E(z,s)$ is of finite order. We can therefore apply the Phragmén-Lindelöf principle (see e.g. (Patterson 1988, Appendix 5)) in the strip $-\sigma_0 \leq \Re(s) \leq \sigma_0$.

Since $\phi(s)$ is bounded for $\Re(s) \geq 1/2$, $|\Im(s)| > 1$ (see (Müller 1983, Lemma 8.8) or (Selberg 1989, (8.6))) we see, using Stirlings formula again, that

$$E(z,s) = \frac{E(z,s)}{L(s)}\phi(s)\frac{\Gamma(s)}{\Gamma(s-1/2)}(\sqrt{\pi}a)^{-1}b^{2s-1}$$

is $O(|t|^{1/2})$ for $\Re(s) \geq 1/2$.

Now we can even improve the result by applying Phragmén-Lindelöf in the strip $1/2 \leq \Re(s) \leq 1+\delta$ for some small $\delta > 0$ using the fact that $E(z,s)$ is bounded for $\Re(s) = \sigma > 1$. The finite number of poles $s_0, s_1, \dots s_k$ in this region can be dealt by multiplying with $(s-s_0)(s-s_1)\cdots(s-s_k)$. We get as result

$$E(z,s) = O_K(|t|^{1-\sigma+\epsilon})$$

for all $z \in K$, a fixed compact set in $\Gamma \setminus \mathbb{H}$. $\square$

Remark 29. We remark that the functions $E_z(z,s)$ and $E_{\bar{z}}(z,s)$ have no poles in $\Re(s) > 1/2$, $s \notin (1/2, 1]$, and that they are holomorphic on the line $\Re(s) = 1/2$. This follows from (Roelcke 1966, Satz 10.3 and Satz 10.4 1)), where this statement is proved for the Eisenstein series $E^k(z,s)$ of weight k. If we set

$$E^k(z,s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \left(\frac{c\bar{z}+d}{cz+d}\right)^{k/2} \Im(\gamma z)^s,$$

then $E_z(z,s) = -isE^2(z,s)/(2y)$ and $E_{\bar z}(z,s) = isE^{-2}(z,s)/(2y)$, since by termwise differentiation we have

$$\begin{aligned} E_{\bar z}(z,s) &= \frac{is}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\Im(\gamma z)^{s-1}\overline{(cz+d)}^{-2} \\ E_z(z,s) &= \frac{-is}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\Im(\gamma z)^{s-1}(cz+d)^{-2}. \end{aligned}$$

Lemma 30. *The function $D_{\epsilon_j}(z,s,\vec{0})$ has polynomial growth in s in $\Re(s)>1/2$. More precisely we have for any $\varepsilon>0$ and $1/2<\sigma\le 1$*

$$D_{\epsilon_j}(z,\sigma+it,\vec{0}) = O(|t|^{5(1-\sigma)+\varepsilon}). \tag{3.36}$$

The constant involved depends on σ, w_j and ε

Proof. We have by (3.13) and (3.7)

$$D_{\epsilon_j}(z,s,\vec{0}) = -R(s)(4\pi i\langle dE(z,s),w_j\rangle - 2\pi i(\delta w_j)E(z,s)). \tag{3.37}$$

We need to control $dE(z,s) = E_z(z,s)dz + E_{\bar z}(z,s)d\bar z$ in some sense. We note that by differentiating the functional equation for $E(z,s)$ we get the functional equation

$$E_z(z,s) = \phi(s)E_z(z,1-s).$$

Since $E_z(z,s)$ has zero Fourier coefficient

$$-i(sy^{s-1}+\phi(s)(1-s)y^{-s})/2,$$

which could vanish only for $s=0$ or $s=1/2$, $E_z(z,s)$ does not vanish for any $s\in\mathbb{C}\backslash\{1/2,1\}$. Since we know that that $E_z(z,s)$ has no poles for $\Re(s)\ge 1/2$, $s\notin(1/2,1]$ (see Remark 29) we can conclude that the poles of $E_z(z,s)$ are the same as the poles $E(z,s)$ with the same multiplicity with the exception of finitely many in the interval $[0,1]$.

We repeat the Phragmén-Lindelöf argument for $E(z,s)/L(s)$ with $E_z(z,s)/L(s)$. The only difference is that since

$$E_z(z,s) = -\frac{is}{2y}E^2(z,s)$$

with $E^2(z,s)$ bounded on vertical lines for $\Re(s)=\sigma>1$ we start with the bound $E_z(z,s)=O(|t|)$ when $\Re(s)=\sigma>1$ and we get $E_z(z,s)=O(|t|^{3/2})$ for $\Re(s)\ge 1/2$. Applying Phragmén-Lindelöf again in $1/2\le\Re(s)\le 1+\varepsilon$ as above we find $E_z(z,s)=O_K(|t|^{2-\sigma+\varepsilon})$. Similarly $E_{\bar z}(z,s)=O_K(|t|^{2-\sigma+\varepsilon})$. This gives a $L^2=L^2(\Gamma\setminus\mathbb{H},d\mu)$ bound for the function in (3.37) to which we apply the resolvent. Since

$$\|R(z)\|_\infty \le \frac{1}{\text{dist }(z,\text{Spec}A)} \tag{3.38}$$

for the resolvent of a general self-adjoint operator A on a Hilbert space and

$$\mathrm{dist}(s(1-s), \mathrm{Spec}\Delta) \geq |\Im(s(1-s))| = |t|\,(2\sigma-1)$$

we get for $1/2 < \sigma = \Re(s) \leq 1$

$$\left\| D_{\epsilon_j}(z,s,\vec{0}) \right\|_{L^2} = O_\sigma\left(\frac{|t|^{2-\sigma+\varepsilon}}{|t|\,(2\sigma-1)}\right) = O_\sigma(|t|^{1-\sigma+\varepsilon}).$$

We finally need to get a pointwise bound from the L^2 bound, for which we use the Sobolev embedding theorem (see (Warner 1983, 6.22 Corollary (b))), which in dimension 2 implies that

$$\|u\|_\infty \leq c\,\|u\|_{H^2},$$

where for any second order elliptic operator P there exist a c' such that

$$\|u\|_{H^2} \leq c'(\|u\|_{L^2} + \|Pu\|_{L^2}).$$

(See (Warner 1983, 6.29)) In our case $D_{\epsilon_j}(z,s,\vec{0})$ we have

$$\Delta u = -(s(1-s)u + L_{\epsilon_j}(\vec{0})E(z,s),$$

and since we already evaluated $D_{\epsilon_j}(z,s,\vec{0})$ $L_{\epsilon_j}(\vec{0})E(z,s)$ in L^2-norm we obtain $D_{\epsilon_j}(z,s,\vec{0}) = O_K(|t|^{3-\sigma+\varepsilon})$ We now apply Phragmén-Lindelöf again in the strip $1/2+\delta \leq \Re(s) \leq 1+\delta$ for some small $\delta > 0$ gives the result. □

LEMMA 31. *The function $D_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0})$ has polynomial growth in t in $\Re(s) > 1/2$. More precisely we have for any $\varepsilon > 0$ and $1/2 < \sigma \leq 1$*

$$D_{\epsilon_1,\ldots,\epsilon_n}(z,\sigma+it,\vec{0}) = O(|t|^{(6n-1)(1-\sigma)+\varepsilon}). \tag{3.39}$$

The involved constant depends on ε, σ and $w_1,\ldots,w_n$.

PROOF. This is induction in n. For $n=1$ we refer to Lemma 30. We now assume that

$$\begin{aligned} D_{\epsilon_1,\ldots,\epsilon_m}(z,\sigma+it,\vec{0}) &= O(|t|^{(6m-1)(1-\sigma)+\varepsilon}) & (3.40)\\ L_{\epsilon_k}(\vec{0})D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_m}(z,s,\vec{0}) &= O(|t|^{(6m-1)(1-\sigma)+\varepsilon}) & (3.41)\end{aligned}$$

whenever $m \leq n-1$. By (3.13) we see that we need to estimate the two type of terms

$$\begin{aligned} L_{\epsilon_k\epsilon_l}(0)D_{\epsilon_1,.,\hat{\epsilon}_k,.,\hat{\epsilon}_l,.,\epsilon_n}(z,s,\vec{0})\\ L_{\epsilon_k}(\vec{0})D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0})\end{aligned}$$

when we apply the resolvent. We can control the first (in L^2) by induction hypothesis as we note that $L_{\epsilon_1\epsilon_2}(\vec{0})$ is a compactly supported multiplication operator (see 3.8). We get

$$\left\| L_{\epsilon_k\epsilon_l}(0) D_{\epsilon_1,.,\hat{\epsilon}_k,.,\hat{\epsilon}_l,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2} = O\left(|t|^{(6(n-2)-1)(1-\sigma)+\varepsilon} \right).$$

By using that w_i is compactly supported we easily deduce from (3.7) that

$$\begin{aligned} \left\| L_{\epsilon_k}(0) D_{\epsilon_1,.,\hat{\epsilon}_l,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2} \leq C \Bigg(& \left\| D_{z,\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L_2(O)} + \\ & \left\| D_{\overline{z},\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L_2(O)} + \left\| D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L_2(O)} \Bigg), \end{aligned} \tag{3.42}$$

where O is an open set lying between the support of w_1 and some other compact set. We now evaluate these three terms separately. To handle the first term we note that

$$\begin{aligned} \left\| D_{z,\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L_2(O)} &\leq \left\| D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{H^1(O)} \\ &\leq \left\| D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{H^2(O)} \\ &\leq c' \Bigg(\left\| D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2(O)} \\ &\quad + \left\| \Delta D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2(O)} \Bigg). \end{aligned} \tag{3.43}$$

We note that by (3.13) and the induction hypothesis

$$\left\| \Delta D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2(O)} = O(|t|^{(6(n-1)-1))(1-\sigma)+\varepsilon+2}). \tag{3.44}$$

Hence the left hand side of (3.43) is $O(|t|^{(6(n-1)-1)(1-\sigma)+\varepsilon+2})$. The second term of (3.42) may be evaluated in the same manner, while the third term is even smaller. We thus get

$$\left\| L_{\epsilon_k}(0) D_{\epsilon_1,.,\hat{\epsilon}_k,.,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2} = O(|t|^{(6(n-1)-1)(1-\sigma)+\varepsilon+2}) \tag{3.45}$$

By (3.13), (3.38) and the above we find

$$\left\| D_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2}$$

$$
\begin{aligned}
&\leq \|R(s)\|_\infty \left\| \left(\sum_{k=1}^{n} L_{\epsilon_k}(\vec{0}) D_{\epsilon_1,.,\widehat{\epsilon_k},.,\epsilon_n}(z,s,\vec{0}) \right.\right. \\
&\qquad\qquad \left.\left. + \sum_{\substack{k,l=1\\k<l}}^{n} L_{\epsilon_k\epsilon_l}(\vec{0}) D_{\epsilon_1,.,\widehat{\epsilon_k},.,\widehat{\epsilon_l},.,\epsilon_n}(z,s,\vec{0}) \right) \right\|_{L^2} \\
&= O(t^{(6(n-1)-1)(1-\sigma)+\varepsilon+1}).
\end{aligned}
$$

To get a pointwise bound we also need

$$
\left\| \Delta D_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0}) \right\|_{L^2} = O(|t|^{(6(n-1)-1)(1-\sigma)+1+2+\varepsilon}), \tag{3.46}
$$

which follows from (3.38) and the above. From the Sobolev embedding theorem we get

$$
\begin{aligned}
\left\| D_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0}) \right\|_\infty &\leq C \left\| D_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0}) \right\|_{H^2} \\
&= O\left(|t|^{(6(n-1)-1)(1-\sigma)+3+\varepsilon} \right)
\end{aligned}
$$

Applying Phragmén-Lindelöf once again in the strip $1/2+\delta \leq \Re(s) \leq 1+\delta$ finishes the proof. $\square$

We notice that we can get polynomial bounds on $D_{\epsilon_1,\ldots,\epsilon_n}(z,s)$ without using Lemma 30. We would then have to start the induction in Lemma 31 at $n=0$ by citing Lemma 28. This would lead to slightly larger exponents. Using the above lemma we conclude

THEOREM 32. *The functions $E_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0})$ and $E^{m,n}$ have polynomial growth in t in $\Re(s) \geq 1/2$. More precisely we have for any $\varepsilon > 0$ and $1/2 < \Re(s) \leq 1$*

$$
E_{\epsilon_1,\ldots,\epsilon_n}(z,s,\vec{0}) = O(|t|^{(6n-1)(1-\sigma)+\varepsilon}). \tag{3.47}
$$

$$
E^{m,n}(z,s) = O(|t|^{(6(m+n)-1)(1-\sigma)+\varepsilon}). \tag{3.48}
$$

The involved constant depends on ε, σ, f, g and $\alpha_1,\ldots,\alpha_n$.

Hence we have also proved Theorem F.

4. Estimating various sums involving modular symbols

Using the results of the previous two sections we would now like to obtain asymptotics as $T \to \infty$ for sums like

$$
\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \|\gamma\|_z \leq T}} \omega_\gamma \tag{3.49}
$$

where $\omega_\gamma = 1$, $\langle \gamma, \alpha_1 \rangle \cdots \langle \gamma, \alpha_n \rangle$ or $\omega_\gamma = \langle \gamma, f \rangle^m \overline{\langle \gamma, g \rangle}^n$. Here $\|\gamma\|_z = |cz+d|^2$ with c, d the lower row in γ and $z \in \mathbb{H}$. We let

$$\tilde{E}(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \omega_\gamma \Im(\gamma z)^s,$$

and assume that this is absolutely convergent for $\Re(s) > 1$, that it has meromorphic continuation to $\Re(s) \geq h$ where $h < 1$, and that in this region it has at most polynomial growth on vertical lines as a function of s. We further assume that $s = 1$ is the only pole in $\Re(s) \geq h$, and that for all $\varepsilon > 0$

$$\omega_\gamma = O(\|\gamma\|_z^\varepsilon) \text{ as } \|\gamma\|_z \to \infty. \tag{3.50}$$

We note that Theorem C, Corollary 14 and Theorem 32 establish these properties for the relevant Eisenstein series.

Let $\phi_U : \mathbb{R} \to \mathbb{R}$, $U \geq U_0$, be a family of smooth decreasing functions with

$$\phi_U(t) = \begin{cases} 1 & \text{if } t \leq 1 - 1/U \\ 0 & \text{if } t \leq 1 + 1/U, \end{cases}$$

and $\phi_U^{(j)}(t) = O(U^j)$ as $U \to \infty$. For $\Re(s) > 0$ we let

$$R_U(s) = \int_0^\infty \phi_U(t) t^{s-1} dt$$

be the Mellin transform of ϕ_U. Then we have

$$R_U(s) = \frac{1}{s} + O\left(\frac{1}{U}\right) \qquad \text{as } U \to \infty \tag{3.51}$$

and for any $c > 0$

$$R_U(s) = O\left(\frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^c\right) \qquad \text{as } |s| \to \infty. \tag{3.52}$$

Both estimates are uniform for $\Re(s)$ bounded. The first is a mean value estimate while the second is successive partial integration and a mean value estimate. The Mellin inversion formula now gives

$$\begin{aligned} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \omega_\gamma \phi_U\left(\frac{\|\gamma\|_z}{T}\right) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \omega_\gamma \frac{1}{2\pi i} \int_{\Re(s)=2} R_U(s) \left(\frac{\|\gamma\|_z}{T}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{\Re(s)=2} \frac{\tilde{E}(z,s)}{y^s} R_U(s) T^s ds. \end{aligned}$$

We note that by (3.52) the integral is convergent as long as $\tilde{E}(z,s)$ has polynomial growth on vertical lines. We now move the line of integration to the line $\Re(s) = h$ with $h < 1$ by integrating along a box of some height and then letting this height go to infinity. Assuming the polynomial bounds on vertical lines the Phragmén-Lindelöf principle implies that there is a uniform polynomial bound $O(t^a)$ in $h \leq \Re(s) \leq 2$ (excluding a small circle around $s = 1$) and using (3.52) we find that the contribution from the horizontal sides goes to zero. Since we assume that $s = 1$ is the only pole of the integrand with $\Re(s) \leq h$ then using Cauchy's residue theorem we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\Re(s)=2}\frac{\tilde{E}(z,s)}{y^s}R_U(s)T^s ds\\&\qquad=\operatorname*{Res}_{s=1}\left(\frac{\tilde{E}(z,s)}{y^s}R_U(s)T^s\right)+\frac{1}{2\pi i}\int_{\Re(s)=h}\frac{\tilde{E}(z,s)}{y^s}R_U(s)T^s ds.\end{aligned}$$

If we choose $c = a + \varepsilon$ the last integral is convergent and $O(T^h U^{a+\varepsilon})$ uniformly for z in a compact set.

Assume that $\tilde{E}(z, s)$ has a pole of order l with $(s-1)^{-l}$ coefficient a_{-l} then if $l > 1$ we have

$$\begin{aligned}&\operatorname*{Res}_{s=1}\left(\frac{\tilde{E}(z,s)}{y^s}R_U(s)T^s\right)\\&\quad=\frac{1}{(l-1)!}\lim_{s\to 1}\frac{d^{l-1}}{ds^{l-1}}\left((s-1)^l\left(\frac{\tilde{E}(z,s)}{y^s}R_U(s)T^s\right)\right)\\&\quad=\frac{1}{(l-1)!}\sum_{n_1+n_2+n_3=l-1}\left.\frac{\partial^{n_1}(s-1)^l\tilde{E}(z,s)/y^s}{\partial s^{n_1}}\right|_{s=1}\left.\frac{\partial^{n_2}R_U(s)}{\partial s^{n_2}}\right|_{s=1}\left.\frac{\partial^{n_3}T^s}{\partial s^{n_3}}\right|_{s=1}\end{aligned}$$

The first factor in the sum is independent on U and T, while the second is independent of T and bounded in U. The third factor has leading term $T(\log T)^{n_3}$ and a reminder $O(\log T^{n_3-1})$. Hence the leading term is the one corresponding to $n_1 = n_2 = 0$, $n_3 = l-1$ and we get, using (3.51),

$$= \frac{a_{-l}}{(l-1)!y}T(\log T)^{l-1} + O(T(\log T)^{l-2} + T\log T^{l-1}/U).$$

This gives

$$\begin{aligned}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\omega_\gamma\phi_U\left(\frac{\|\gamma\|_z}{T}\right)=&\frac{a_{-l}}{(l-1)!y}T(\log T)^{l-1}\\&+O(T(\log T)^{l-2}+T\log T^{l-1}/U+T^hU^{a+\varepsilon}).\end{aligned}$$

If $l = 1$ then by (3.51)

$$\operatorname*{Res}_{s=1}\left(\frac{\tilde{E}(z,s)}{y^s}R_U(s)T^s\right) = \frac{a_{-1}}{y}T + O(T/U),$$

and we get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\omega_\gamma\phi_U\left(\frac{\|\gamma\|_z}{T}\right) = \frac{a_{-1}}{y}T + O(T/U + T^hU^{a+\varepsilon}).$$

If $\tilde{E}(z,s)$ has a nonsimple pole we choose $U = \log T$ and we get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\omega_\gamma\phi_U\left(\frac{\|\gamma\|_z}{T}\right) = \frac{a_{-l}}{(l-1)!y}T(\log T)^{l-1} + O(T(\log T)^{l-2}). \quad (3.53)$$

In the simple pole case we choose $U = T^{(1-h)/(a+1+\varepsilon)}$ in order to balance the error terms and we get

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\omega_\gamma\phi_U\left(\frac{\|\gamma\|_z}{T}\right) = \frac{a_{-1}}{y}T + O(T^{\frac{a+h+\varepsilon}{a+1+\varepsilon}}). \quad (3.54)$$

At this point we note that if ω_γ is non-negative for all $\gamma \in \Gamma_\infty \setminus \Gamma$, then by further requiring $\phi_U(t) = 0$ if $t \geq 1$ and $\tilde{\phi}_U(t) = 1$ for $t \leq 1$, we have

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\omega_\gamma\phi_U\left(\frac{\|\gamma\|_z}{T}\right) \leq \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \|\gamma\|_z\leq T}}\omega_\gamma \leq \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\omega_\gamma\tilde{\phi}_U\left(\frac{\|\gamma\|_z}{T}\right)$$

from which it easily follows that the middle sum has an asymptotic expansion. As an application we use this on the usual nonholomorphic Eisenstein series and from the above, Theorem 4 and Lemma 28 we find that

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \|\gamma\|_z\leq T}} 1 = \frac{T}{y\mathrm{vol}(\Gamma\setminus\mathbb{H})} + O(T/U + T^hU^{1-h+\varepsilon}), \quad (3.55)$$

where we have used that we may choose $a = 1 - h + \varepsilon$ (see Lemma 28). We choose $U = T^{\frac{1-h}{2-h+\varepsilon}}$ to balance the error terms, and get

LEMMA 33. *Assume that the only pole of $E(z,s)$ in $\Re(s) \geq h$ is $s = 1$. Then*

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \|\gamma\|_z\leq T}} 1 = \frac{T}{y\,vol(\Gamma\setminus\mathbb{H})} + O(T^{\frac{1}{2-h}+\varepsilon}). \quad (3.56)$$

Using this lemma we can now deal with the general case. To get a result without ϕ_U from (3.53) and (3.54) we notice that if we choose ϕ_U such that $\phi_U(t) = 1$ for $t \leq 1$ then

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \omega_\gamma \phi_U \left(\frac{\|\gamma\|_z}{T} \right) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \|\gamma\|_z \leq T}} \omega_\gamma + \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ T < \|\gamma\|_z \leq T(1+1/U)}} \omega_\gamma \phi_U \left(\frac{\|\gamma\|_z}{T} \right).$$

Using (3.50) we see that we may evaluate the last sum in the following way. For any $\varepsilon > 0$ this is less than a constant times

$$(T(1+1/U))^\varepsilon \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ T < \|\gamma\|_z \leq T(1+1/U)}} 1 \leq (2T)^\varepsilon \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ T < \|\gamma\|_z \leq T(1+1/U)}} 1.$$

The sum is $O(T/U) + O(T^{\frac{1}{2-h}+\varepsilon})$ by Lemma 33. Using this with the above choices of U we find

THEOREM 34. *Assume that $\tilde{E}(z,s)$ has a pole at s=1 of order l with $(s-1)^{-l}$ coefficient a_{-l}. If $l = 1$, i.e. if the pole is simple then*

$$\sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \|\gamma\|_z \leq T}} \omega_\gamma = \frac{a_{-1}}{y} T + O(T^{\max\left(\frac{a+h}{a+1}, \frac{1}{2-h}\right)+\varepsilon}).$$

If $l > 1$ then

$$\sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \|\gamma\|_z \leq T}} \omega_\gamma = \frac{a_l}{(l-1)! y} T \log T^{l-1} + O(\log T^{l-2}).$$

Using this we now get an expansion of the summatory function (3.49) in all the cases that we studied in section 2. We only state the result in a few cases.

COROLLARY 35. *Let $\alpha = \Re(f(z)dz)$ and $\beta = \Re(f(z)dz)$. Then*

$$\sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \|\gamma\|_z \leq T}} \langle \gamma, \alpha \rangle^{2m} \langle \gamma, \beta \rangle^{2n} = \frac{(-8\pi^2)^{m+n} \|f\|^{2m+2n}}{y vol(\Gamma \setminus \mathbb{H})^{m+n+1}} \frac{(2m)!}{m! 2^m} \frac{(2n)!}{n! 2^n} T \log^{m+n} T \tag{3.57}$$

$$+ O(T \log^{m+n-1} T),$$

and if m or n is odd then

$$\sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \|\gamma\|_z \leq T}} \langle \gamma, \alpha \rangle^m \langle \gamma, \beta \rangle^n = O(T \log^k T) \tag{3.58}$$

for some $k \in \mathbb{N}$ strictly less than $(m+n)/2$.

PROOF. This follows from Theorem 34, Theorem 32, Corollary 14 and Theorem 23, once we notice that

$$\frac{(2m)!(2n)!}{2^{m+n}(m+n)!}\binom{m+n}{n} = \frac{(2m)!}{m!2^m}\frac{(2n)!}{n!2^n}. \tag{3.59}$$

□

COROLLARY 36. *We have*

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \|\gamma\|_z\leq T}} |\langle\gamma, f\rangle|^{2m} = \frac{(16\pi^2)^m m!}{yvol(\Gamma\setminus\mathbb{H})^{m+1}} \|f\|^{2m} T\log^m T + O(T\log^{m-1} T). \tag{3.60}$$

PROOF. This follows from Theorem 34, Theorem 32, Corollary 14 and Theorem 25. □

We notice that this settles Goldfelds conjecture (1.4) in the negative once we choose $z = i$.

COROLLARY 37. *There exists* $\delta_1 > 0$ *such that*

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \|\gamma\|_z\leq T}} \langle\gamma, f\rangle = \frac{1}{yvol(\Gamma\setminus\mathbb{H})}\left(-2\pi i\int_{i\infty}^{z} f(\tau)\,d\tau\right)T + O(T^{1-\delta_1}). \tag{3.61}$$

PROOF. This follows from Theorem 34, Theorem 32, Corollary 14 and Theorem 24. □

We note that by picking $z = i$ this reproves (1.3).

COROLLARY 38. *There exists* $\delta_2 > 0$ *such that*

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \|\gamma\|_z\leq T}} \langle\gamma, f\rangle^2 = \frac{1}{yvol(\Gamma\setminus\mathbb{H})}\left(-2\pi i\int_{i\infty}^{z} f(\tau)\,d\tau\right)^2 T + O(T^{1-\delta_2}). \tag{3.62}$$

PROOF. This follows from Theorem 34, Theorem 32, Corollary 14 and Theorem 26. □

REMARK 39. How small we can prove $1-\delta_i$ to be in the above corollaries depends of course on how good polynomial bounds we have and how far to the left we may move the line of integration. Assuming no eigenvalues $s(1-s) \in (0, 1/4)$ we can move just to the right of

$s = 1/2$, and using the bound of Theorem 32 we get

$$1 - \delta_1 = \frac{6}{7} + \varepsilon$$
$$1 - \delta_2 = \frac{12}{13} + \varepsilon$$

for any $\varepsilon > 0$.

5. The distribution of modular symbols

We now show how to obtain a distribution result for the modular symbols from the asymptotic expansions of Corollary 35. We renormalize the modular symbols in the following way. Let

$$\widetilde{\langle \gamma, f \rangle} = \sqrt{\frac{\mathrm{vol}(\Gamma \setminus \mathbb{H})}{8\pi^2 \|f\|^2}} \langle \gamma, f \rangle$$
$$\widetilde{\langle \gamma, \alpha \rangle} = \sqrt{\frac{\mathrm{vol}(\Gamma \setminus \mathbb{H})}{8\pi^2 \|f\|^2}} \langle \gamma, \alpha \rangle$$
$$\widetilde{\langle \gamma, \beta \rangle} = \sqrt{\frac{\mathrm{vol}(\Gamma \setminus \mathbb{H})}{8\pi^2 \|f\|^2}} \langle \gamma, \beta \rangle$$

where $\alpha = \Re(f(z)dz)$, $\beta = \Im(f(z)dz)$. Let furthermore

$$(\Gamma_\infty \setminus \Gamma)^T := \{\gamma \in \Gamma_\infty \setminus \Gamma | \quad \|\gamma\|_z \leq T\}. \tag{3.63}$$

By Lemma 33 we have

$$\#(\Gamma_\infty \setminus \Gamma)^T = \frac{T}{\mathrm{vol}(\Gamma \setminus \mathbb{H})y} + O(T^{1-\delta}), \tag{3.64}$$

for some $\delta > 0$. Now let X_T be the random variable with probability measure

$$P(X_T \in R) = \frac{\# \left\{ \gamma \in (\Gamma_\infty \setminus \Gamma)^T \,\middle|\, \frac{\widetilde{\langle \gamma, f \rangle}}{\sqrt{\log \|\gamma\|_z}} \in R \right\}}{\#(\Gamma_\infty \setminus \Gamma)^T}. \tag{3.65}$$

for $R \subset \mathbb{C}$ (we set $\widetilde{<\gamma, \alpha>}/\sqrt{\log \|\gamma\|_z} = 0$ if $\|\gamma\|_z \leq 1$. Note that there are only finitely many such elements.) We consider the moments of X_T

$$M_{n,m}(X_T) = \sum_{\gamma \in (\Gamma_\infty \setminus \Gamma)^T} \frac{\left[\Re\left(\frac{\widetilde{\langle \gamma, f \rangle}}{\sqrt{\log \|\gamma\|_z}}\right)\right]^n \left[\Im\left(\frac{\widetilde{\langle \gamma, f \rangle}}{\sqrt{\log \|\gamma\|_z}}\right)\right]^m}{\#(\Gamma_\infty \setminus \Gamma)^T}, \tag{3.66}$$

and note that

$$\begin{aligned}\Re(\widetilde{\langle \gamma, f\rangle}) &= i\widetilde{\langle \gamma, \beta\rangle}\\ \Im(\widetilde{\langle \gamma, f\rangle}) &= -i\widetilde{\langle \gamma, \alpha\rangle}.\end{aligned}$$

By partial summation we have

$$\begin{aligned}M_{n,m}(X_T) =& \frac{i^{n+m}(-1)^m}{\#(\Gamma_\infty\setminus\Gamma)^T}\left(\sum_{\gamma\in(\Gamma_\infty\setminus\Gamma)^T}\widetilde{\langle \gamma, \beta\rangle}^n\widetilde{\langle \gamma, \alpha\rangle}^m\frac{1}{\log T^{(m+n)/2}}\right.\\ &\left.+\frac{m+n}{2}\int_0^T\sum_{\gamma\in(\Gamma_\infty\setminus\Gamma)^t}\widetilde{\langle \gamma, \beta\rangle}^n\widetilde{\langle \gamma, \alpha\rangle}^m\frac{1}{t(\log t)^{(m+n)/2+1}}dt\right).\end{aligned}$$

If we now apply Corollary 35 and (3.64) we find that as $T\to\infty$

$$M_{n,m}(X_T)\to\begin{cases}\frac{n!}{(n/2)!2^{n/2}}\frac{m!}{(m/2)!2^{m/2}}, & \text{if } m \text{ and } n \text{ are even,}\\ 0, & \text{otherwise.}\end{cases}\tag{3.67}$$

We notice that the right-hand side is the moments of the bivariate Gaussian distribution with correlation coefficient zero. Hence by a result due to Fréchet and Shohat (see (Loève 1977, 11.4.C)) we conclude the following:

THEOREM 40. *Asymptotically* $\frac{\widetilde{\langle \gamma, f\rangle}}{\sqrt{\log\|\gamma\|_z}}$ *has bivariate Gaussian distribution with correlation coefficient zero. More precisely we have*

$$\frac{\#\left\{\gamma\in(\Gamma_\infty\setminus\Gamma)^T\,\middle|\,\frac{\widetilde{\langle \gamma, f\rangle}}{\sqrt{\log\|\gamma\|_z}}\in R\right\}}{\#(\Gamma_\infty\setminus\Gamma)^T}\to\frac{1}{2\pi}\int_R\exp\left(-\frac{x^2+y^2}{2}\right)dxdy\tag{3.68}$$

as $T\to\infty$.

As an easy corollary we get

COROLLARY 41. *Asymptotically* $\frac{\Re(\widetilde{\langle \gamma, f\rangle})}{\sqrt{\log\|\gamma\|_z}}$ *has Gaussian distribution. More precisely we have*

$$\frac{\#\left\{\gamma\in(\Gamma_\infty\setminus\Gamma)^T\,\middle|\,\frac{\Re(\widetilde{\langle \gamma, f\rangle})}{\sqrt{\log\|\gamma\|_z}}\in[a,b]\right\}}{\#(\Gamma_\infty\setminus\Gamma)^T}\to\frac{1}{\sqrt{2\pi}}\int_a^b\exp\left(-\frac{x^2}{2}\right)dx\tag{3.69}$$

as $T\to\infty$.

The same holds for $\Im(\widetilde{\langle \gamma, f\rangle})$. We note that by putting $z = i$ in Corollary 41 and Theorem 40 we obtain Theorem A and Theorem B.

CHAPTER 4

Functional equations of twisted Eisenstein series

In this section we shall study the transformation properties of the twisted Eisenstein series $E^{m,n}(z,s)$ as s maps to $1-s$. We work again in the multiple cusp case. We shall show that $E^{m,n}(z,s)$ satisfies a functional equation similar to that of the usual nonholomorphic Eisenstein series i.e.

$$\vec{E}(z,s)=\Phi(s)\vec{E}(z,1-s), \tag{4.1}$$

where $\Phi(s)=\{\phi_{\mathfrak{a}\mathfrak{b}}(s)\}$. In fact we shall see that (4.1) together with the limiting absorption principle gives the functional equation of the twisted Eisenstein series via an induction argument. This chapter is "work in progress" and therefore stops rather abruptly, and many of the results appears somewhat unpolished.

1. An example

We shall start by considering the case of $E^{1,0}(z,s)$, since this makes more transparent the driving mechanisms in the proof of the general statements. From (3.12) we find that if

$$\begin{aligned}\alpha_1 &= \Re(f(z)dz)\\ \alpha_2 &= \Im(f(z)dz)\end{aligned}$$

then using Theorem 5 we see that

$$\begin{aligned}
D_{\mathfrak{a}\epsilon_i}(z,s,\vec{0}) &= -\int_{\Gamma\backslash\mathbb{H}} r(z,z',s)L_{\epsilon_i}(\vec{0})E_{\mathfrak{a}}(z',s)d\mu(z')\\
&= -\int_{\Gamma\backslash\mathbb{H}} r(z,z',1-s)L_{\epsilon_i}E_{\mathfrak{a}}(z',s)d\mu(z')\\
&\quad -\frac{1}{1-2s}\sum_{\mathfrak{b}}E_{\mathfrak{b}}(z,1-s)\int_{\Gamma\backslash\mathbb{H}}E_{\mathfrak{b}}(z',s)L_{\epsilon_i}(\vec{0})E_{\mathfrak{a}}(z',s)d\mu(z')\\
&= \sum_{\mathfrak{b}}\phi_{\mathfrak{a}\mathfrak{b}}(s)\left(-\int_{\Gamma\backslash\mathbb{H}} r(z,z',1-s)L_{\epsilon_i}E_{\mathfrak{b}}(z',1-s)d\mu(z')\right)\\
&\quad +\sum_{\mathfrak{b}}E_{\mathfrak{b}}(z,1-s)\frac{1}{2s-1}\int_{\Gamma\backslash\mathbb{H}}E_{\mathfrak{b}}(z',s)L_{\epsilon_i}(\vec{0})E_{\mathfrak{a}}(z',s)d\mu(z')
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathfrak{b}} \phi_{\mathfrak{a}\mathfrak{b}}(s) D_{\mathfrak{b}\epsilon_i}(z, 1-s, \vec{0}) \\
&\quad + \frac{1}{2s-1} \int_{\Gamma\backslash\mathbb{H}} E_{\mathfrak{b}}(z,s) L_{\epsilon_i}(\vec{0}) E_{\mathfrak{a}}(z,s) d\mu(z) E_{\mathfrak{b}}(z, 1-s).
\end{aligned}
$$

We let $\rho_{\mathfrak{a}\mathfrak{b}}(s) = \frac{1}{2s-1} \int_{\Gamma\backslash\mathbb{H}} E_{\mathfrak{b}}(z,s) \lim_{T\to\infty}(L_{\epsilon_1} + iL_{\epsilon_2}(\vec{0}) E_{\mathfrak{a}}(z,s) d\mu(z)$. Hence by using (3.4) and (1.9) we find, letting $T \to \infty$ as in the proof of Lemma 18,

$$
\begin{aligned}
&E^{1,0}_{\mathfrak{a}}(z,s) \\
&= \lim_{T\to\infty} (D_{\mathfrak{a}\epsilon_1}(z,s,\vec{0}) + iD_{\mathfrak{a}\epsilon_2}(z,s,\vec{0})) + 2\pi i \int_{\mathfrak{a}}^{z} f(z) dz E_{\mathfrak{a}}(z,s) \\
&= \sum_{\mathfrak{b}} \phi_{\mathfrak{a}\mathfrak{b}}(s) \lim_{T\to\infty} \left(D_{\mathfrak{b}\epsilon_1}(z, 1-s, \vec{0}) + iD_{\mathfrak{b}\epsilon_2}(z, 1-s, \vec{0}) \right) \\
&\quad + \rho_{\mathfrak{a}\mathfrak{b}}(s) E_{\mathfrak{b}}(z, 1-s) + \phi_{\mathfrak{a}\mathfrak{b}}(s) 2\pi i \int_{\mathfrak{a}}^{z} f(z) dz E_{\mathfrak{b}}(z, 1-s) \\
&= \sum_{\mathfrak{b}} \phi_{\mathfrak{a}\mathfrak{b}}(s) \left(\lim_{T\to\infty} (D_{\mathfrak{b}\epsilon_1}(z, 1-s, \vec{0}) + iD_{\mathfrak{b}\epsilon_2}(z, 1-s, \vec{0})) + 2\pi i \int_{\mathfrak{b}}^{z} f(z) dz E_{\mathfrak{b}}(z, 1-s) \right) \\
&+ \left(2\pi i \int_{\mathfrak{a}}^{\mathfrak{b}} f(z) dz \phi_{\mathfrak{a}\mathfrak{b}}(s) + \rho_{\mathfrak{a}\mathfrak{b}}(s) \right) E_{\mathfrak{b}}(z, 1-s) \\
&= \sum_{\mathfrak{b}} \phi_{\mathfrak{a}\mathfrak{b}}(s) E^{1,0}_{\mathfrak{b}}(z, 1-s) + \left(2\pi i \int_{\mathfrak{a}}^{\mathfrak{b}} f(z) dz \phi_{\mathfrak{a}\mathfrak{b}}(s) + \rho_{\mathfrak{a}\mathfrak{b}}(s) \right) E_{\mathfrak{b}}(z, 1-s).
\end{aligned}
$$

We have proved

$$
\vec{E}^{1,0}(z,s) = \Phi(s) \vec{E}^{1,0}(z,s) + \Phi^*(s) \vec{E}(z, 1-s) \tag{4.2}
$$

where

$$
\Phi^*_{\mathfrak{a}\mathfrak{b}}(s) = 2\pi i \int_{\mathfrak{a}}^{\mathfrak{b}} f(z) dz \phi_{\mathfrak{a}\mathfrak{b}}(s) + \rho_{\mathfrak{a}\mathfrak{b}}(s).
$$

This agrees with the result given in (Chinta & O'Sullivan 2002, p. 25) once we take into account the different normalization of the modular symbol in that paper. We note that a similar but different functional equation was given in (Chinta & Goldfeld 2001) and (Petridis 2002) but both contained errors.

We note that

$$
\rho_{\mathfrak{a}\mathfrak{b}}(s) = \frac{8\pi i}{2s-1} \int_{\Gamma\backslash\mathbb{H}} E_{\mathfrak{b}}(z,s) y^2 \frac{\partial E_{\mathfrak{a}}(z,s)}{\partial \overline{z}} f(z) d\mu(z).
$$

Using Stokes Theorem as in the proof of Lemma 18 we see that we may move the differentiation to $E_{\mathfrak{b}}(z,s)$ if we change the sign. This proves that $\rho_{\mathfrak{ab}}(s) = -\rho_{\mathfrak{ba}}(s)$, and hence $\Phi^*_{\mathfrak{ab}}(s) = -\Phi^*_{\mathfrak{ba}}(s)$ (compare (O'Sullivan 2000, Prop. 4.2)). We also note that using

$$\frac{\partial E_{\mathfrak{a}}(z,s)}{\partial \overline{z}} = \frac{is}{2} \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1} \overline{j(\sigma_{\mathfrak{a}}\gamma, z)}^{-2}$$

we can unfold the integral to get

$$-\frac{4\pi s}{2s-1} \int_{\Gamma_\infty \backslash \mathbb{H}} y^{s+1} E_{\mathfrak{b}}(\sigma_{\mathfrak{a}} z, s) f|_{[\sigma_{\mathfrak{a}}]_2}(z) d\mu(z).$$

If we now use the Fourier expansions

$$f|_{[\sigma_{\mathfrak{a}}]_2}(z) = \sum_{n=1}^{\infty} a_n^{\mathfrak{a}} e^{2\pi i n z}$$

$$E_{\mathfrak{b}}(\sigma_{\mathfrak{a}} z, s) = \delta_{\mathfrak{ba}} y^s + \phi_{\mathfrak{ba}}(s) y^{1-s} + \sum_{n=-\infty}^{\infty} \phi_n^{\mathfrak{ba}}(s) \sqrt{y} K_{s-1/2}(2\pi |n| y) e^{2\pi i n x}$$

we get

$$-\frac{4\pi s}{2s-1} \sum_{m+n=0} a_n^{\mathfrak{a}} \phi_m^{\mathfrak{ba}}(s) \int_0^\infty y^{s-1/2} K_{s-1/2}(2\pi |m| y) e^{-2\pi n y} dy.$$

The integral may be solved using (Erdélyi, Magnus, Oberhettinger & Tricomi 1954, 6.8 (28)), and we find

$$\rho_{\mathfrak{ab}}(s) = -\frac{\Gamma(2s-1)\pi^{1-s}}{\Gamma(s) 2^{2s-1}} \sum_{n=1}^{\infty} \frac{\phi_{-n}^{\mathfrak{ba}}(s) a_n^{\mathfrak{a}}}{n^{s+1/2}}$$

We shall see below how this generalizes to $E^{m,n}(z,s)$.

2. The general functional equation

It is convenient to use the following notational convention. We let $A = \{1, \ldots, n\}$ and for any subset $B = \{n_1, \ldots, n_k\} \subseteq A$ we write

$$\begin{aligned} D_{\mathfrak{a}B}(z,s) &= D_{\mathfrak{a}\epsilon_{n_1},\ldots,\epsilon_{n_k}}(z,s,\vec{0}) = \frac{\partial^k}{\partial \epsilon_{n_1} \cdots \partial \epsilon_{n_k}} D_{\mathfrak{a}}(z,s,\vec{\epsilon}) \bigg|_{\vec{\epsilon}=\vec{0}} \\ E_{\mathfrak{a}B}(z,s) &= E_{\mathfrak{a}\epsilon_{n_1},\ldots,\epsilon_{n_k}}(z,s,\vec{0}) = \frac{\partial^k}{\partial \epsilon_{n_1} \cdots \partial \epsilon_{n_k}} E_{\mathfrak{a}}(z,s,\vec{\epsilon}) \bigg|_{\vec{\epsilon}=\vec{0}} \end{aligned}$$

We note that using this convention (3.13) may be written as

$$D_{\mathfrak{a}B}(z,s) = -R(s)\left(\sum_{\{k\}\subseteq B} L_{\epsilon_k} D_{\mathfrak{a}B\setminus\{k\}}(z,s) + \sum_{\{k,l\}\subseteq B} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}B\setminus\{k,l\}}(z,s)\right) \tag{4.3}$$

where we have also written $L_{\epsilon_k} = L_{\epsilon_k}(\vec{0})$ and $L_{\epsilon_k\epsilon_l} = L_{\epsilon_k\epsilon_l}(\vec{0})$.

THEOREM 42. *We have the functional equation*

$$\vec{D}_A(z,s) = \sum_{C\subseteq A} \Psi_C(s)\vec{D}_{A\setminus C}(z,1-s)$$

where $\Psi_C(s)$ is a matrix indexed by cusps. The $\mathfrak{ab}$ entry is equal to

$$\frac{1}{2s-1}\int_{\Gamma\setminus\mathbb{H}} E_{\mathfrak{b}}(z,s)\left(\sum_{\{k\}\subseteq C} L_{\epsilon_k} D_{\mathfrak{a}C\setminus\{k\}}(z,s) + \sum_{\{k,l\}\subseteq C} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}C\setminus\{k,l\}}(z,s)\right) d\mu(z)$$

if $C \neq \emptyset$ and $\Phi_\emptyset(s)$ is the usual scattering matrix i.e. the one from (4.1).

PROOF. The proof is induction in $|A|$, i.e. the number of elements in the set A. We note that the case $A = \emptyset$ is contained in (4.1) since $\vec{D}_\emptyset(z,s) = \vec{E}(z,s)$.

We now assume that we have proved the result for any A' with at most n elements. Assume that A has $n+1$ elements. By (4.3) we have

$$D_{\mathfrak{a}A}(z,s) = -R(s)\left(\sum_{\{k\}\subseteq A} L_{\epsilon_k} D_{\mathfrak{a}A\setminus\{k\}}(z,s) + \sum_{\{k,l\}\subseteq A} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}A\setminus\{k,l\}}(z,s)\right).$$

By applying the representation of the resolvent as an integral operator in a right halfplane this is

$$-\int_{\Gamma\setminus\mathbb{H}} r(z,z',s)\left(\sum_{\{k\}\subseteq A} L_{\epsilon_k} D_{\mathfrak{a}A\setminus\{k\}}(z',s) + \sum_{\{k,l\}\subseteq A} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}A\setminus\{k,l\}}(z',s)\right) d\mu(z').$$

If we apply the limiting absorption principle, i.e. Theorem 5 we get

$$-\int_{\Gamma\setminus\mathbb{H}} r(z,z',1-s)\left(\sum_{\{k\}\subseteq A} L_{\epsilon_k} D_{\mathfrak{a}A\setminus\{k\}}(z',s) + \sum_{\{k,l\}\subseteq A} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}A\setminus\{k,l\}}(z',s)\right) d\mu(z')$$

$$+\frac{1}{2s-1}\sum_{\mathfrak{b}} E_{\mathfrak{b}}(z,1-s)\cdot$$

$$\int_{\Gamma\backslash\mathbb{H}} E_{\mathfrak{b}}(z',s)\left(\sum_{\{k\}\subseteq A} L_{\epsilon_k} D_{\mathfrak{a}A\backslash\{k\}}(z',s)+\sum_{\{k,l\}\subseteq A} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}A\backslash\{k,l\}}(z',s)\right)d\mu(z').$$

We note that the last two lines is the term in the functional equation we are trying to prove corresponding to $C=\emptyset$ with the correct coefficient. We now consider the first line. Using the induction hypothesis this is

$$\sum_{\mathfrak{b}} -\int_{\Gamma\backslash\mathbb{H}} r(z,z',1-s)\left(\sum_{\{k\}\subseteq A}\left(\sum_{H\subseteq A\backslash\{k\}} \Phi_{H\mathfrak{a}\mathfrak{b}}(s) L_{\epsilon_k} D_{\mathfrak{b}A\backslash(\{k\}\cup H)}(z',1-s)\right)\right.$$
$$\left.+\sum_{\{k,l\}\subseteq A}\sum_{H'\subseteq A\backslash\{k,l\}} \Phi_{H'\mathfrak{a}\mathfrak{b}}(s) D_{\mathfrak{b}A\backslash(\{k\}\cup H')}(z',1-s)\right) d\mu(z').$$

In these sums we now collect the terms coming from the same $C\subseteq A$ and note that every subset appears in the sum except $C=A$. We get

$$\sum_{\mathfrak{b}}\sum_{\substack{C\subseteq A\\ C\neq A}} \Phi_{C\mathfrak{a}\mathfrak{b}}(s)\left(-\int_{\Gamma\backslash\mathbb{H}} r(z,z',1-s)\left(\sum_{\{k\}\subseteq A\backslash C} L_{\epsilon_k} D_{\mathfrak{b}(A\backslash C)\backslash\{k\}}(z',1-s)\right.\right.$$
$$\left.\left.+\sum_{\{k,l\}\subseteq A\backslash C} L_{\epsilon_k\epsilon_l} D_{\mathfrak{b}(A\backslash C)\backslash\{k,l\}}(z',1-s)\right) d\mu(z')\right).$$

By (4.3) and the induction hypothesis this is

$$\sum_{\mathfrak{b}}\sum_{\substack{C\subseteq A\\ C\neq A}} \Phi_{C\mathfrak{a}\mathfrak{b}}(s) D_{\mathfrak{b}A\backslash C}(z,1-s))$$

which finishes the proof. □

We call the matrices in the above theorem scattering matrices. We note that in the notation introduced in the chapter the correspondence (3.17) reads

$$E_{\mathfrak{a}A}(z,s)=\sum_{B\subseteq A}\left(\prod_{j\in B} 2\pi i\int_{\mathfrak{a}}^{z} w_j\right) D_{\mathfrak{a}A\backslash B}(z,s).$$

We also have

$$D_{\mathfrak{a}A}(z,s)=\sum_{B\subseteq A}\left(\prod_{j\in B} -2\pi i\int_{\mathfrak{a}}^{z} w_j\right) E_{\mathfrak{a}A\backslash B}(z,s)$$

which is seen from $E(z,s,\vec{\epsilon})U(-\vec{\epsilon},z) = D(z,s,\vec{\epsilon})$ by differentiation. We can hence shift back and forth between the $\vec{D}_A(z,s)$ functions and the $\vec{E}_A(z,s)$ functions. We find

THEOREM 43. *We have the functional equation*

$$\vec{E}_A(z,s) = \sum_{C\subseteq A} \Phi_C(s)\vec{E}_{A\setminus C}(z,1-s),$$

where $\Phi_C(s)$ is a matrix indexed by the cusps. The $\mathfrak{ab}$ entry is equal to

$$\sum_{K\subseteq C} \Psi_{K\mathfrak{ab}} \left(\prod_{j\in C\setminus K} 2\pi i \int_{\mathfrak{a}}^{\mathfrak{b}} w_j \right).$$

We note that applying this theorem twice we see that these scattering matrices satisfies

$$\sum_{\substack{B\cup C=D\\ B\cap C=\emptyset}} \Phi_C(s)\Phi_B(1-s) = \begin{cases} I & \text{if } D=\emptyset \\ 0 & \text{otherwise.}\end{cases}$$

By using that $\vec{E}^{m,n}(z,s)$ may be written as a complex linear combination of $\vec{E}_A(z,s)$ the above theorem gives a functional equation for these. If $f=g$ we get a functional equation of the form

$$\vec{E}^{m,n}(z,s) = \sum_{m'+n'\leq m+n} \Phi_{m-m',m-n'}(s)\vec{E}^{m',n'}(z,1-s)$$

where $\Phi_{m',n'}(s)$ is a complex linear combination of $\Phi_C(s)$. This proves Theorem H.

When $f\neq g$ we get a functional equation involving series of the form

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\setminus\Gamma} \langle\gamma,f\rangle^{m'}\langle\gamma,g\rangle^{k'}\overline{\langle\gamma,f\rangle}^{l'}\overline{\langle\gamma,g\rangle}^{n'}\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s.$$

3. Scattering matrices in terms of L-functions

We shall now investigate the components in the the scattering matrices found in the previous section. Since $E^{m,n}(z,s)$ and $E_A(z,s)$ are independent of the cohomology class we picked when we substituted compactly supported forms for α_i we can now let $T\to\infty$ as in Lemma 18.

From Theorem 42 and Theorem 43 we see that the entries of the scattering matrices for the twisted Eisenstein series are linear combinations of terms of the form $\lim_{T\to\infty}\Phi_C(s)$.

LEMMA 44. *The function*

$$\lim_{T\to\infty}\left(\sum_{\{k\}\subseteq C} L_{\epsilon_k} D_{\mathfrak{a}C\setminus\{k\}}(z,s)+\sum_{\{k,l\}\subseteq C} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}C\setminus\{k,l\}}(z,s)\right)$$

has a series representation

$$4\pi i\sum_{\{k\}\in C}\sum_{\gamma\in\Gamma_{\mathfrak{a}}\setminus\Gamma}\prod_{j\in C\setminus\{k\}}\left(-2\pi i\int_{\mathfrak{a}}^{\gamma z}\alpha_j\right)\left\langle d\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s,\alpha_k\right\rangle.$$

PROOF. We notice that

$$\begin{aligned}
\lim_{T\to\infty} D_{\mathfrak{a}A}(z,s) &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\setminus\Gamma}\left(\prod_{j\in A}2\pi i\int_{\mathfrak{a}}^{\gamma z}\alpha_j\right)\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\\
\lim_{T\to\infty} L_{\epsilon_k}h &= 4\pi i\left\langle dh,\alpha_k\right\rangle\\
\lim_{T\to\infty} L_{\epsilon_k\epsilon_l}h &= -8\pi^2\left\langle \alpha_k,\alpha_l\right\rangle
\end{aligned}$$

(compare (3.20)). Since $d\int_{\mathfrak{a}}^{\gamma z}\alpha_j=\alpha_j$ we find that

$$\begin{aligned}
&\lim_{T\to\infty}\left(\sum_{\{k\}\subseteq C} L_{\epsilon_k} D_{\mathfrak{a}C\setminus\{k\}}(z,s)\right)\\
&=4\pi i\sum_{\{k\}\subseteq C}\left\langle d\sum_{\gamma\in\Gamma_{\mathfrak{a}}\setminus\Gamma}\left(\prod_{j\in C\setminus\{k\}}-2\pi i\int_{i\infty}^{\gamma z}\alpha_j\right)\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s,\alpha_k\right\rangle\\
&=4\pi i\sum_{\{k\}\subseteq C}\left(\sum_{\{l\}\subseteq C\setminus\{k\}}-2\pi i\left\langle\alpha_l,\alpha_k\right\rangle\sum_{\gamma\in\Gamma_{\mathfrak{a}}\setminus\Gamma}\left(\prod_{j\in C\setminus\{l,k\}}-2\pi i\int_{\mathfrak{a}}^{\gamma z}\alpha_j\right)\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\right.\\
&\quad\left.+\sum_{\Gamma_{\mathfrak{a}}\setminus\Gamma}\prod_{j\in C\setminus\{k\}}\left(-2\pi i\int_{\mathfrak{a}}^{\gamma} z\alpha_j\right)\left\langle d\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s,\alpha_k\right\rangle\right)\\
&=-\lim_{T\to\infty}\sum_{\{k,l\}\subseteq C} L_{\epsilon_k\epsilon_l} D_{\mathfrak{a}C\setminus\{k,l\}}(z,s)\\
&\quad+4\pi i\sum_{\{k\}\in C}\sum_{\Gamma_{\mathfrak{a}}\setminus\Gamma}\prod_{j\in C\setminus\{k\}}\left(-2\pi i\int_{\mathfrak{a}}^{\gamma} z\alpha_j\right)\left\langle d\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s,\alpha_k\right\rangle.
\end{aligned}$$

$\square$

Using the above lemma we can use the usual unfolding technique to find

$$\Psi_{\mathfrak{ab}C}(s)$$
$$= \frac{4\pi i}{2s-1} \sum_{\{k\}\in C} \int_{\Gamma_{\mathfrak{a}}\backslash\mathbb{H}} E_{\mathfrak{b}}(z,s) \prod_{j\in C\backslash\{k\}} \left(-2\pi i \int_{\mathfrak{a}}^{z} \alpha_j\right) \left\langle d\Im(\sigma_{\mathfrak{a}}^{-1}z)^s, \alpha_k\right\rangle d\mu(z)$$

Since

$$\left\langle d\Im(\sigma_{\mathfrak{a}}^{-1}w)^s, \alpha_k\right\rangle\Big|_{w=\sigma_{\mathfrak{a}}z} = \frac{is}{2} y^{s+1}\left(f_k|_{[\sigma_{\mathfrak{a}}]_2}(z) - \overline{f_k|_{[\sigma_{\mathfrak{a}}]_2}(z)}\right), \tag{4.4}$$

$$2\pi i \int_{\mathfrak{a}}^{\sigma_{\mathfrak{a}}z} \alpha_j = \frac{1}{2}\sum_{n_j=1}^{\infty} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} e^{2\pi i n_j z} + \overline{\frac{1}{2}\sum_{n_j=1}^{\infty} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} e^{2\pi i n_j z}} \tag{4.5}$$

when

$$f_j|_{[\sigma_{\mathfrak{a}}]_2}(z) = \sum_{n_j=1}^{\infty} a_{n_j}^{j\mathfrak{a}} e^{2\pi i n_j z},$$

we note that $\Psi_{\mathfrak{ab}}(s)$ may be written as a linear combination of elements of the form $\frac{s}{2s-1}$ times

$$\int_{\Gamma_\infty\backslash\mathbb{H}} y^{s+1} E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,s) \prod_{j\in B_1}\sum_{n_j=1}^{\infty} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} e^{2\pi i n_j z} \prod_{j\in B_2}\overline{\sum_{n_j=1}^{\infty} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} e^{2\pi i n_j z}} f_k|_{[\sigma_{\mathfrak{a}}]_2}(z) d\mu(z) \tag{4.6}$$

or

$$\int_{\Gamma_\infty\backslash\mathbb{H}} y^{s+1} E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,s) \prod_{j\in B_1}\sum_{n_j=1}^{\infty} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} e^{2\pi i n_j z} \prod_{j\in B_2}\overline{\sum_{n_j=1}^{\infty} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} e^{2\pi i n_j z}}\, \overline{f_k|_{[\sigma_{\mathfrak{a}}]_2}(z)} d\mu(z). \tag{4.7}$$

We consider (4.6), which is easily seen to be equal to

$$\sum \int_0^\infty y^{s-1} \tilde{\phi}_m^{\mathfrak{ba}}(y,s) \prod_{j\in B_1} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} \prod_{j\in B_2} \frac{\overline{a_{n_j}^{j\mathfrak{a}}}}{n_j} a_{n_k}^{k\mathfrak{a}} e^{-2\pi(n_k+\sum_{j\in B_1} n_j + \sum_{j\in B_2} n_j)y} dy. \tag{4.8}$$

The sum is subject to the condition $m+n_k+\sum_{B_1} n_j - \sum_{B_2} n_j = 0$, $m \in \mathbb{Z}$, $n_j \in \mathbb{N}$. As usual $\tilde{\phi}_m^{\mathfrak{ba}}(y,s)$ is the Fourier coefficients of $E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,s)$. The case $B_1 = B_2 = \emptyset$ is the case we dealt with in section 4.1 so we exclude that case below. The part of the above sum with $m \neq 0$ is

$$\sum \phi_m^{\mathfrak{ba}}(s) \prod_{j\in B_1} \frac{a_{n_j}^{j\mathfrak{a}}}{n_j} \prod_{j\in B_2} \frac{\overline{a_{n_j}^{j\mathfrak{a}}}}{n_j} a_{n_k}^{k\mathfrak{a}}$$
$$\int_0^\infty y^{s-1/2} K_{s-1/2}(2\pi\,|m|\,y) e^{-2\pi(n_k+\sum_{j\in B_1} n_j + \sum_{j\in B_2} n_j)y} dy.$$

The integral may be found (Erdélyi et al. 1954, 6.8 (29)) to be

$$\frac{\pi^{-s}}{2^{2s+1}} \frac{|m|^{s-1/2}}{(n_k+\sum_{j\in B_1} n_j + \sum_{j\in B_2} n_j)^{2s}} \frac{\Gamma(2s)}{\Gamma(s+1)}.$$
$${}_2F_1\left(s+1/2, s, s+1, 1-\left(\frac{|m|}{(n_k+\sum_{j\in B_1} n_j + \sum_{j\in B_2} n_j)}\right)\right)$$

where ${}_2F_1$ is Gauss' hypergeometric series.

We want to parameterize the sum in terms of $q = \frac{a-b}{a+b}$, where $a = n_k + \sum_{j\in B_1} n_j$ and $b = \sum_{j\in B_2} n_j$. Clearly every such q is in $\mathbb{Q}\cap]-1,1[$, and $q = e/f$ with $e < f$ may be represented by $a = f+e$, $b = f-e$. If q is represented by two pairs (a,b) and (a',b') with $a \leq a'$ then $(ra, rb) = (a', b')$ for some $r \in \mathbb{N}$ and (ar, ar) represents the same q as (a,b) for any $r \in \mathbb{N}$. We let (a_q, b_q) be the representation of

$$q = \frac{a_q - b_q}{a_q + b_q}$$

with $a_q + b_q$ minimal. Notice that if $q = e/f$ with $e < f$ and $(e,f) = 1$ then

$$a_q = f+e \qquad b_q = f-e$$

if $f+e$ is odd, while

$$a_q = (f+e)/2 \qquad b_q = (f-e)/2$$

if $f+e$ is even. We note that $q = 0$ corresponds to $m = 0$. Using this parameterization we get that $\frac{s}{2s-1}$ times the part of (4.8) that corresponds to $q \neq 0$ equals

$$\frac{\pi^{-s}\Gamma(2s-1)}{2^{2s-1}\Gamma(s)} \sum_{\substack{q\in\mathbb{Q}\cap]-1,1[\\ q\neq 0}} |q|^{s-1/2} \frac{{}_2F_1(s+1/2, s, s+1, 1-q^2)}{(a_q+b_q)^{s+1/2}} \sum_{r=1}^{\infty} \frac{C_{f_k,B_1,B_2,q,r}}{r^{s+1/2}} \tag{4.9}$$

where

$$C_{f_k,B_1,B_2,q,r} = \sum_{\substack{a_q r = n_k + \sum_{j\in B_1} n_j \\ b_q r = \sum_{j\in B_2} n_j}} \phi^{\mathfrak{ba}}_{r(b_q-a_q)}(s) a^{k\mathfrak{a}}_{n_k} \prod_{j\in B_1} \frac{a^{j\mathfrak{a}}_{n_j}}{n} \prod_{j\in B_2} \frac{\overline{a^{j\mathfrak{a}}_{n_j}}}{n}.$$

This type of L series does not seem to have been investigated.

The part of (4.8) with $m = 0$ is zero if $B_2 = \emptyset$, and otherwise it is

$$\sum \int_0^\infty y^{s-1}(\delta_{\mathfrak{ba}} y^s + \phi^{\mathfrak{ba}}(s) y^{1-s}) e^{-2\pi(n_k + \sum_{j\in B_1} n_j + \sum_{j\in B_2} n_j)y} dy \cdot$$

$$\prod_{j\in B_1} \frac{a^{j\mathfrak{a}}_{n_j}}{n_j} \prod_{j\in B_2} \frac{\overline{a^{j\mathfrak{a}}_{n_j}}}{n_j} a^{k\mathfrak{a}}_{n_k}.$$

The integral is easily seen to be

$$\frac{\delta_{\mathfrak{ba}}}{(2\pi r)^{2s}} \Gamma(2s) + \phi^{\mathfrak{ba}}(s) \frac{1}{(2\pi r)}$$

where $r = (n_k + \sum_{j\in B_1} n_j + \sum_{j\in B_2} n_j)$. so we get

$$\frac{s}{2s-1}\left(\delta_{\mathfrak{ba}}\Gamma(2s)\sum_{r=1}^\infty \frac{\tilde{C}_{f_k,B_1,B_2,r}}{(4\pi r)^{2s}} + \phi^{\mathfrak{ba}}(s)\sum_{r=1}^\infty \frac{\tilde{C}_{f_k,B_1,B_2,r}}{4\pi r}\right)$$

where

$$\tilde{C}_{f_k,B_1,B_2,r} = \sum_{\substack{r = n_k + \sum_{j\in B_1} n_j \\ r = \sum_{j\in B_2} n_j}} a^{k\mathfrak{a}}_{n_k} \prod_{j\in B_1} \frac{a^{j\mathfrak{a}}_{n_j}}{n} \prod_{j\in B_2} \frac{\overline{a^{j\mathfrak{a}}_{n_j}}}{n}.$$

Clearly (4.7) may be dealt with in the same fashion.

CHAPTER 5

Asymptotic densities of the number of newforms

Let Γ be a cocompact discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. It is well known that in this case the weight 0 selfadjoint automorphic Laplacian, Δ_Γ, acting on Γ-automorphic functions has infinitely many eigenvalues,

$$0 = \lambda_0 \leq \lambda_1^\Gamma \leq \ldots \leq \lambda_i^\Gamma \leq \ldots,$$

listed with their multiplicities which are finite. Selberg has proved that the counting function

$$N_\Gamma(\lambda) = \#\{i | \lambda_i^\Gamma \leq \lambda\}$$

has an asymptotic expansion on the form

$$N_\Gamma(\lambda) = \frac{|F_\Gamma|}{4\pi}\lambda + O(\sqrt{\lambda}/\log\lambda), \tag{5.1}$$

where $|F_\Gamma|$ is the area of a fundamental domain of Γ.

If Γ is non-cocompact but of finite area then the situation is somewhat more complicated. The Roelcke-Selberg conjecture, which claims that also in this case there are infinitely many eigenvalues, seems to have lost credit rather than gained it over the years. But we do still have the following expansion (See (Venkov 1982, Theorem 5.2.1))

$$\begin{aligned} N_\Gamma(\lambda) - \frac{1}{4\pi}\int_{-T}^{T}\frac{\phi_\Gamma'}{\phi_\Gamma}\left(\frac{1}{2}+ir\right)dr \\ = \frac{|F_\Gamma|}{4\pi}\lambda - \frac{k_\Gamma}{\pi}\sqrt{\lambda}\ln\sqrt{\lambda} + \frac{k_\Gamma(1-\ln 2)}{\pi}\sqrt{\lambda} + O(\sqrt{\lambda}/\ln\sqrt{\lambda}) \end{aligned} \tag{5.2}$$

where ϕ_Γ is the determinant of the scattering matrix, $\lambda = 1/4 + T^2$ and k_Γ is the number of cusps of Γ. From this it is clear that in order to estimate the number of eigenvalues it is essential to estimate the logarithmic derivative of the scattering determinant. For congruence subgroups Selberg showed that

$$N_\Gamma(\lambda) = \frac{|F_\Gamma|}{4\pi}\lambda + O(\sqrt{\lambda}\log\lambda). \tag{5.3}$$

In this chapter we investigate what happens if we only count the eigenvalues corresponding to newforms. In particular we are interested in knowing when various counting functions have the same asymptotic expansion as if they where counting eigenvalues related to a cocompact group. We say that a function $N : \mathbb{R} \to \mathbb{R}$ *is of cocompact type* if

$$N(\lambda) = c\lambda + O(\sqrt{\lambda}/\log\lambda), \tag{5.4}$$

for some constant c, and we want to find out for which Hecke congruence groups the counting function for newforms is of cocompact type.

1. Newforms and oldforms

The theory of newforms was originally developed by Atkin & Lehner (Atkin & Lehner 1970) for holomorphic forms. Their theory can be translated into a similar theory of Maass forms which are the ones we are studying. This has been done independently by various people and details may be found in e.g. (Strömbergsson 2001a). We shall only need one result (Lemma 51 below) and shall hence only sketch enough of the theory for this result to make sense.

For any $\lambda > 0$, $M \in \mathbb{N}$ we denote by $A(\lambda, M)$ the λ-eigenspace for $\Delta_{\Gamma_0(M)}$, where $\Gamma_0(M)$ is the Hecke congruence group of level M i.e.

$$\Gamma_0(M) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} c \equiv 0 \mod M \right\}.$$

Then it is obvious that

$$N_{\Gamma_0(M)}(\lambda) = 1 + \sum_{0<\tilde{\lambda}\leq\lambda} \dim A(\tilde{\lambda}, M), \tag{5.5}$$

where the sum is certainly finite.

We define the λ-oldspace to be

$$A_{\mathrm{old}}(\lambda, M) := \mathrm{span}\{f(dz) | f \in A(\lambda, K) \quad Kd|M \quad K \neq M\}.$$

This is contained in $A(\lambda, M)$ by the $\mathrm{SL}_2(\mathbb{R})$-invariance of Δ_Γ, and the fact that $f(dz)$ is $\Gamma_0(M)$-invariant when $f(z)$ is $\Gamma_0(K)$-invariant and $Kd|M$. We then define the λ-newspace to be the orthogonal complement in $A(\lambda, M)$ with respect to the inner product

$$(f, g) = \int_{F_{\Gamma_0(M)}} f(z)\overline{g(z)}d\mu(z),$$

i.e.

$$A_{\mathrm{new}}(\lambda, M) := A(\lambda, M) \ominus A_{\mathrm{old}}(\lambda, M).$$

We then define new spectral counting functions

$$\begin{aligned} N^{\text{old}}_{\Gamma_0(M)}(\lambda) &:= 1 + \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{\text{old}}(\tilde{\lambda}, M) \qquad M > 0 \\ N^{\text{new}}_{\Gamma_0(M)}(\lambda) &:= \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{\text{new}}(\tilde{\lambda}, M) \qquad M > 0. \end{aligned}$$

For $M = 1$ we of course define $N^{\text{old}}_{\Gamma_0(1)}(\lambda) = 0$ and $N^{\text{new}}_{\Gamma_0(1)}(\lambda) = N_{\Gamma_0(1)}(\lambda)$.

2. Evaluating the scattering matrix for Hecke congruence groups

As suggested by (5.2) it is essential to evaluate the logarithmic derivative of the scattering matrix in order to find the asymptotic expansion for the counting function. In this section we estimate the scattering matrix for the congruence groups $\Gamma_0(M)$ by using an explicit expression due to M. Huxley .

THEOREM 45. *(Huxley 1984) Let $\phi_M(s)$ be the determinant of the scattering matrix for the Hecke congruence group of level M, $\Gamma_0(M)$, and let Λ_χ be the completed L-function of an Dirichlet character mod K ,χ , i.e.*

$$\Lambda_\chi(s) = \Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \qquad \textit{when } \Re(s) > 1.$$

Then

$$\phi_M(s) = (-1)^l \left(\frac{A(M)}{\pi^{k_M}}\right)^{1-2s} \prod_{i=1}^{k_M} \frac{\Lambda_{\overline{\chi_i}}(2-2s)}{\Lambda_{\chi_i}(2s)}$$

where $l \in \mathbb{N}$, the χ_i's are some Dirichlet characters mod K where $K|M$, and

$$A(M) = \prod_{\substack{\chi \text{ primitive } \quad \text{mod } q \\ q|m,\, mq|M}} \frac{qM}{(m, M/m)}.$$

The set $\{\chi_i | i = 1, \dots, k_M\}$ is closed under complex conjugation.

We now use this to evaluate the integral in (5.2) and get

THEOREM 46. *The counting function $N_{\Gamma_0(M)}(\lambda)$ satisfies the following asymptotic formula*

$$\begin{aligned} N_{\Gamma_0(M)}(\lambda) = & \frac{|F_M|}{4\pi}\lambda - \frac{2k_M}{\pi}\sqrt{\lambda}\log\sqrt{\lambda} \\ & + \frac{1}{\pi}\left[(2-\log 2+\log\pi)k_M - \log(A(M)\right]\sqrt{\lambda} \\ & \qquad + O(\sqrt{\lambda}/\log\sqrt{\lambda}). \end{aligned}$$

In particular we get the following

COROLLARY 47. *The counting function for $\Gamma_0(M)$ is never of co-compact type.*

PROOF OF THEOREM 45. We let $B(M) = \frac{A(M)}{\pi^{k_M}}$. From the above we conclude that

$$\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+ir\right) = -2\left(\ln B(M) + \sum_{i=1}^{k_M}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1-2it) + \frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2it)\right).$$

An easy consideration then shows that

$$-\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+ir\right)dr = \frac{T}{\pi}\ln B(M) + \sum_{i=1}^{k_M}\frac{1}{\pi}\int_{-T}^{T}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2ir)dr.$$

We must therefore evaluate

$$\int_{-T}^{T}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2ir)dr,$$

and we observe that

$$\int_{-T}^{T}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2ir)d = \frac{1}{2}\int_{-T}^{T}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr + \int_{-T}^{T}\frac{L_\chi'}{L_\chi}\left(1+i2r\right)dr.$$

We shall address each term separately. To evaluate the first term we use Stirling's approximation formula i.e.

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) - \frac{1}{2s} + O(|s|^{-2}),$$

valid for $|\arg(s)-\pi| > \epsilon$ (See (Ivić 1985, A.35)). We see that for $|r| > \epsilon$ we have

$$\begin{aligned} &\left|\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \left(\log|r| + i\arg\left(\frac{1}{2}+ir\right) - (1+i2r)^{-1}\right)\right| \\ &\quad\le \left|\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \left(\log\left|\frac{1}{2}+ir\right| + i\arg\left(\frac{1}{2}+ir\right) - (1+i2r)^{-1}\right)\right| \\ &\qquad + \left|\log\left|\frac{1}{2}+ir\right| - \log|r|\right|. \end{aligned}$$

It is easy to see check that the last summand is $O((|r|\log|r|)^{-1})$ while the first is $O(|r|^{-2})$ by Stirling's approximation formula. Hence

$$\frac{1}{2}\int_{-T}^{T}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr$$
$$=\frac{1}{2}\int_{\substack{-T\\|r|>\epsilon}}^{T}\log|r|+i\arg\left(\frac{1}{2}+ir\right)-(1+i2r)^{-1}\,dr+O\left(\int_{\epsilon}^{T}\frac{dr}{r\log r}\right).$$

The integral over $(1+i2r)^{-1}$ is bounded and the integral over $i\arg(1/2+ir)$ vanishes. We conclude that

$$\frac{1}{2}\int_{-T}^{T}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr=T\log T-T+O(\log(\log T)).$$

To evaluate the integral over the logarithmic derivative of $L_\chi(1+2ir)$ we note that

$$\int_{\epsilon}^{T}\frac{L'_\chi}{L_\chi}(1+2ir)dr=-2i(\log L_\chi(1+2iT))+C$$

where C is a constant and that the first term is $O(\log T)$ by (Apostol 1976, Theorem 12.24). We conclude that

$$-\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi'_M}{\phi_M}\left(\frac{1}{2}+ir\right)dr=\frac{T}{\pi}\log B(M)+\frac{k_M}{\pi}(T\log T-T)+O_M(\log(T))$$

which finishes the proof. □

3. Dirichlet convolution

In order to calculate the main terms of $N^{\text{new}}_{\Gamma_0(M)}$ we remind about some well known structure theory of arithmetical functions. When $f,g:\mathbb{N}\to\mathbb{C}$ are arithmetical functions we define the *Dirichlet convolution*, $f*g:\mathbb{N}\to\mathbb{C}$ to be the arithmetical function

$$(f*g)(n)=\sum_{d|n}f(d)g\left(\frac{n}{d}\right).$$

We say that f is multiplicative if $f(mn)=f(m)f(n)$ whenever $(m,n)=1$. The structure theory we shall use is the following:

THEOREM 48. *The set of arithmetical functions form a commutative group under Dirichlet convolution. The identity element is the function*

$$\begin{array}{rccc} I: & \mathbb{N} & \to & \mathbb{C} \\ & n & \mapsto & \left[\frac{1}{n}\right]=\begin{cases}1 & \textit{if } n=1\\ 0 & \textit{otherwise.}\end{cases}\end{array}$$

The multiplicative arithmetical functions form a subgroup.

PROOF. This follows from (Apostol 1976, Theorems 2.6,2.8,2.14, 2.16) □

EXAMPLE 49. (See (Apostol 1976, §2.13) for details.) Consider the arithmetical function

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha.$$

Then this in a *multiplicative* arithmetical function whose inverse may be calculated to be

$$\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d)\mu\left(\frac{n}{d}\right), \tag{5.6}$$

where μ is the Möbius function, i.e.

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \\ 0 & \text{otherwise.} \end{cases}$$

The Mangoldt Λ-function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{where } p \text{ is a prime and } m \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

is an example of a non-multiplicative function. Another multiplicative arithmetical function we will use is Eulers totient function

$$\Phi(n) = \#\{d \in \mathbb{N} | 1 \leq d \leq n \wedge (d,n) = 1\}.$$

We can now begin to calculate asymptotic densities of newforms. We cite a result from (Strömbergsson 2001a).

LEMMA 50.

$$\dim A(\lambda, \cdot) = \sigma_0 * \dim A_{\text{new}}(\lambda, \cdot).$$

PROOF. This is Theorem 4.6.c) in Chapter III of (Strömbergsson 2001a). □

Let now f_i, $i = 1 \ldots n$ be real positive functions of decreasing order i.e

$$f_{i+1} = o(f_i) \text{ for } i = 1 \ldots n-1.$$

PROPOSITION 51. *Assume that for any $M \in \mathbb{N}$*

$$N_{\Gamma_0(M)}(\lambda) = \sum_{i=1}^{n-1} c_i(M) f_i(\lambda) \quad + O(f_n(\lambda)).$$

Then

$$N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda) = \sum_{i=1}^{n-1} c_i^{\mathrm{new}}(M) f_i(\lambda) \quad + O(f_n(\lambda)).$$

where $c_i^{\mathrm{new}} = c_i * \sigma_0^{-1}$.

Proof. The $M = 1$ case is clear by the definitions of $N^{\mathrm{new}}_{\Gamma_0(1)}(\lambda)$ and $c_i^{\mathrm{new}}(1)$. We observe that by lemma 50 we have

$$\begin{aligned} N_{\Gamma_0(M)}(\lambda) &= 1 + \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{\mathrm{new}}(\tilde{\lambda}, K) \\ &= \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) N^{\mathrm{new}}_{\Gamma_0(K)}(\lambda). \end{aligned}$$

By the definition of c_i^{new} we have

$$c_i(M) = \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) c_i^{\mathrm{new}}(K)$$

and therefore

$$\begin{aligned} \left| N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda) - \sum_{i=1}^{n-1} c_i^{\mathrm{new}}(M) f_i(\lambda) \right| &\leq \left| N_{\Gamma_0(M)}(\lambda) - \sum_{i=1}^{n-1} c_i(M) f_i(\lambda) \right| \\ &\quad + \sum_{\substack{K|M \\ K\neq M}} \sigma_0\left(\frac{M}{K}\right) \left| N^{\mathrm{new}}_{\Gamma_0(K)}(\lambda) - \sum_{i=1}^{n-1} c_i^{\mathrm{new}}(K) f_i(\lambda) \right|. \end{aligned}$$

Induction in M now gives that this is $\leq C f_n(\lambda)$ which is the desired result. $\square$

The above Proposition together with Theorem 48 enables us to conclude that $c_i^{\mathrm{new}}(N)$ is multiplicative if and only if $c_i(N)$ is multiplicative. It also shows that since we know the expansion of the counting function for eigenvalues of $\Delta_{\Gamma_0(M)}$ for any $M \in \mathbb{N}$ by Theorem 46 it is easy to compute the corresponding counting function for newforms. In the next section we shall do that.

4. The asymptotic expansion of the newform counting function

Theorem 46 now puts us in a situation where proposition 51 can be applied with

$$\begin{aligned} f_1(\lambda) &= \lambda \\ f_2(\lambda) &= \sqrt{\lambda}\log\sqrt{\lambda} \\ f_3(\lambda) &= \sqrt{\lambda} \\ f_4(\lambda) &= \sqrt{\lambda}/\log\sqrt{\lambda}. \end{aligned}$$

From (Shimura 1971, Theorem 1.43) we conclude that

$$k_M = \sum_{d|M} \Phi((d, M/d)) \tag{5.7}$$

$$|F_M| = \frac{\pi}{3} M \prod_{\substack{p|M \\ p \text{ prime}}} (1 + p^{-1}). \tag{5.8}$$

This means that we have explicit expressions for all the terms in theorem 46 except $A(M)$. We need to know the number of primitive Dirichlet characters mod K. We hence define

$$D(K) = \#\{\chi \text{ primitive Dirichlet character mod } K\}.$$

Then we have

Lemma 52. *The arithmetical function $D(K)$ is multiplicative and satisfies*

$$D(K) = (\Phi * \mu)(K).$$

Proof. From (Apostol 1976) theorem 6.15 and theorem 8.18 we conclude that $\Phi(K) = \sum_{d|K} D(d) = (u * D)(K)$ where $u(n) = 1$ for $n \in \mathbb{N}$. Since Φ and u are multiplicative we use theorem 48 to conclude that D is multiplicative. Theorem 2.1 in (Apostol 1976) proves that $u^{-1} = \mu$ so

$$\Phi * \mu = u * D * \mu = u * u^{-1} * D = D$$

which concludes the proof. □

We now calculate c_1^{new}, c_2^{new} and c_3^{new}.

4.1. The first coefficient. We start by calculating $c_1^{\text{new}}(M)$. This is the simplest of the three coefficients.

Proposition 53. *The arithmetical function $v(M) = 12c_1^{\text{new}}(M)$ is multiplicative and satisfies*

$$v(p^n) = \begin{cases} 1 & \text{if } n = 0 \\ p - 1 & \text{if } n = 1 \\ p^2 - p - 1 & \text{if } n = 2 \\ (p^3 - p^2 - p + 1)p^{n-3} & \text{if } n \geq 3 \end{cases} \tag{5.9}$$

when p is a prime. We furthermore have

$$L_v(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(2s)\zeta(s)},$$

where $\zeta(s)$ is Riemann's zeta function.

PROOF. By using proposition 51, theorem 46 and (5.8) we conclude that

$$M \prod_{\substack{p|M \\ p \text{ prime}}} (1+p^{-1}) = (\sigma_0 * v)(M).$$

Since the left hand side and σ_0 are multiplicative theorem 48 says that v is multiplicative. By considering the case where $M = p^m$ we see that

$$p^m + p^{m-1} = \sum_{d|p^m} \sigma_0(d) v\left(\frac{p^m}{d}\right) = \sum_{i=0}^{m} (i+1) v(p^{m-i}).$$

By applying the theory of generating functions to this relation we find that if

$$f_p(c) = \sum_{n=0}^{\infty} v(p^n) x^n \quad \text{then} \quad f_p(x) = \frac{(1-x^2)(1-x)}{1-px}.$$

By making formal expansion we get (5.9). Since v is multiplicative the claim about L_v follows. ☐

We now make a small deroute to investigate the size of the asymptotical fraction of newforms using the results obtained in the last section. More precisely we investigate the size of the asymptotical fraction It is clear from the results we have about the counting functions, $N^{\text{new}}_{\Gamma_0(N)}$ and $N_{\Gamma_0(N)}$, that

$$f(N) = \frac{v(N)}{N \prod_{p|N} (1+p^{-1})}.$$

We note that both nominator and denominator are multiplicative. Hence the asymptotic fraction of newforms is multiplicative.

$$f(N) = \lim_{\lambda \to \infty} \frac{N^{\text{new}}_{\Gamma_0(N)}(\lambda)}{N_{\Gamma_0(N)}(\lambda)} = \frac{c_1^{\text{new}}(N)}{c_1(N)}.$$

PROPOSITION 54. *There is no lower nor upper bound for the size of the asymptotic fraction, i.e. $\forall \varepsilon > 0 \quad \exists N_0, M_0 \in \mathbb{N}$ such that*

$$0 < f(M_0) < \varepsilon \quad \textit{and} \quad 1 > f(N_0) > 1 - \varepsilon.$$

PROOF. From (5.9) follows easily that

$$f(p^n) \geq \frac{1-p^{-1}-p^{-2}}{1+p^{-1}}, \tag{5.10}$$

where we have equality if and only if $n=2$. We hence have

$$f(N) \geq \prod_{p|N} \frac{1-p^{-1}-p^{-2}}{1+p^{-1}},$$

with equality if and only if N is the square of a square free. Hence the nonexistence of a lower bound is equivalent to

$$\prod_{p \text{ prime}} \frac{1-p^{-1}-p^{-2}}{1+p^{-1}} = 0$$

which is equivalent to the divergence of

$$\sum_{p \text{ prime}} -\log\left(\frac{1-p^{-1}-p^{-2}}{1+p^{-1}}\right).$$

But this follows from the fact that $-\log\left(\frac{1-p^{-1}-p^{-2}}{1+p^{-1}}\right) \geq p^{-1}$ for p sufficiently large. This proves the nonexistence of a lower bound.

The nonexistence of an upper bound follows from (5.10) by choosing $N_0 = p^2$ with p a sufficiently large prime. □

4.2. The second coefficient. We now calculate $c_2^{\text{new}}(M)$. We remind that by proposition 51 and theorem 46 we have

$$c_2^{\text{new}}(M) = -\frac{2}{\pi}(k_{(\cdot)} * \sigma_0^{-1})(M)$$

We hence need to have more information about the number of cusps of $\Gamma_0(M)$

LEMMA 55. *The number of cusps, k_M, of $\Gamma_0(M)$ is a multiplicative arithmetical function and satisfies*

$$k_{p^m} = \begin{cases} 1 & \text{if } m=0 \\ 2 & \text{if } m=0 \\ 2p^n & \text{if } m=2n+1 \text{ where } n>1 \\ (p+1)p^{n-1} & \text{if } m=2n \text{ where } n>1 \end{cases} \tag{5.11}$$

PROOF. We noted earlier in (5.7) that

$$k_M = \sum_{d|M} \Phi((d, M/d)).$$

Let $M_1, M_2 \in \mathbb{N}$ and assume $(M_1, M_2) = 1$. Then

$$\begin{aligned}
k_{M_1M_2} &= \sum_{d|M_1M_2} \Phi((d,(M_1M_2)/d)) \\
&= \sum_{d_1|M_1}\sum_{d_2|M_2} \Phi((d_1d_2,(M_1M_2)/(d_1d_2))) \\
&= \sum_{d_1|M_1}\sum_{d_2|M_2} \Phi((d_1,M_1/d_1)(d_2,M_2/d_2)) \\
&= \sum_{d_1|M_1} \Phi((d_1,M_1/d_1)) \sum_{d_2|M_2} \Phi((d_2,M_2/d_2)) \\
&= k_{M_1}k_{M_2}.
\end{aligned}$$

Hence k_M is multiplicative. The claim about k_{p^m} is clear for $m = 0$ and $m = 1$. Assume $m \geq 2$. We then have

$$\begin{aligned}
k_{p^m} &= \sum_{i=0}^{m} \Phi((p^i, p^{m-i})) \\
&= \sum_{i=0}^{m} \Phi(p^{\min(i,m-i)}) \\
&= 2 + \sum_{i=1}^{m-1} (p-1)p^{\min(i,m-i)-1}
\end{aligned}$$

We now assume $m = 2n + 1$.

$$\begin{aligned}
&= 2 + (p-1)\left(\sum_{i=1}^{n} p^{i-1} + \sum_{i=n+1}^{2n} p^{2n-i}\right) \\
&= 2 + 2(p-1)\sum_{i=0}^{n-1} p^i \\
&= 2 + 2(p-1)\frac{1-p^n}{1-p} = 2p^n.
\end{aligned}$$

The even case is similar. $\square$

From the above we can now prove the following

PROPOSITION 56. *The arithmetical function ,*$-\frac{\pi}{2}c_2^{\text{new}}(M)$*, is a multiplicative arithmetical function and satisfies*

$$-\frac{\pi}{2}c_2^{\text{new}}(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 2n+1 \\ p-2 & \text{if } m = 2 \\ (p+1)^2 p^{n-1} & \text{if } m = 2n \text{ where } n > 1. \end{cases} \tag{5.12}$$

PROOF. From lemma 55 and theorem 48 follows that $c_2^{\text{new}}(M)$ is multiplicative. From (5.6) it is easy to see that

$$\sigma_0^{-1}(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ -2 & \text{if } m = 1 \\ 1 & \text{if } m = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$c_2^{\text{new}}(p^m) = -\frac{2}{\pi}(k_{p^m} - 2k_{p^{m-1}} + k_{p^m}), \text{ when } m \geq 2.$$

Using lemma 55 it is now easy to check the claim. We omit the details. □

As an easy corollary we get the following

COROLLARY 57. *The second coefficient,* $c_2^{\text{new}}(M)$*, is non-zero if and only if* $M = t^2$ *where* $t \in \mathbb{N}$ *is not of the form* $t = 2t'$ *with* $(2, t') = 1$*.*

4.3. The third coefficient. We finally calculate $c_3^{\text{new}}(M)$. This is the most difficult of the three coefficients.

We start by observing that by proposition 51 and theorem 46

$$c_3^{\text{new}}(M) = \frac{1}{\pi}\left((2 - \log 2 + \log \pi)\left(-\frac{\pi}{2}c_2^{\text{new}}(M)\right) - L(M)\right)$$

where

$$L(M) = \left(\log A(\cdot) * \sigma_0^{-1}\right)(M).$$

We hence direct our attention to $L(M)$.

LEMMA 58. *Assume* $(M_1, M_2) = 1$*. Then*

$$L(M_1M_2) = U(M_1)L(M_2) + U(M_2)L(M_1)$$

where

$$U(M) = \sum_{d|M}\sum_{m|d}\sum_{q|(m,\frac{d}{m})} D(q)\sigma_0^{-1}\left(\frac{M}{d}\right).$$

PROOF. We have

$$
\begin{aligned}
L(M_1M_2) &= \sum_{d|M_1M_2} \log A(d)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right) \\
&= \sum_{d|M_1M_2} \sum_{\substack{q|m \\ mq|d}} D(q) \log\left(\frac{qd}{(m,\frac{d}{m})}\right)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right) \\
&= \sum_{d|M_1M_2} \sum_{m|d} \sum_{q|(m,d/m)} D(q) \log\left(\frac{qd}{(m,\frac{d}{m})}\right)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right) \\
&= \sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1,\frac{d_1}{m_1})} \sum_{q_2|(m_2,\frac{d_2}{m_2})} \\
&\qquad\qquad D(q_1q_2) \log\left(\frac{q_1q_2d_1d_2}{(m_1m_2,\frac{d_1d_2}{m_1m_2})}\right)\sigma_0^{-1}\left(\frac{M_1M_2}{d_1d_2}\right).
\end{aligned}
$$

The summand is clearly

$$
D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right)\left(\log\left(\frac{q_1d_1}{(m_1,\frac{d_1}{m_1})}\right)+\log\left(\frac{q_2d_2}{(m_2,\frac{d_2}{m_2})}\right)\right).
$$

We have

$$
\begin{aligned}
&\sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1,\frac{d_1}{m_1})} \sum_{q_2|(m_2,\frac{d_2}{m_2})} \\
&\qquad\qquad D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right)\left(\log\left(\frac{q_1d_1}{(m_1,\frac{d_1}{m_1})}\right)\right) \\
&\quad = U(M_2)L(M_1),
\end{aligned}
$$

from which the identity easily follows. $\square$

It turns out that U is a very nice arithmetical function. In fact we have the following.

LEMMA 59. *The function $U(M)$, is a multiplicative arithmetical function and satisfies*

$$
U(p^m) = \begin{cases} 1 & \textit{if } m = 0 \\ 0 & \textit{if } m = 2n+1 \\ p-2 & \textit{if } m = 2 \\ (p^2-2p+1)p^{n-2} & \textit{if } m = 2n \textit{ where } n > 1. \end{cases} \tag{5.13}
$$

PROOF. Let $M_1, M_2 \in \mathbb{N}$ be coprime. Then

$$\begin{aligned}
U(M_1M_2) &= \sum_{d|M_1M_2} \sum_{m|d} \sum_{q|(m,\frac{d}{m})} D(q)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right) \\
&= \sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1,\frac{d_1}{m_1})} \sum_{q_2|(m_2,\frac{d_2}{m_2})} \\
&\qquad D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right) \\
&= U(M_1)U(M_2).
\end{aligned}$$

Hence U is multiplicative.

Let p be a prime and $m \in \mathbb{N}$. We assume $m \geq 2$ Then

$$\begin{aligned}
U(p^m) &= \sum_{i=0}^{m} \sum_{j=0}^{i} \sum_{l=0}^{\min(j,i-j)} D(p^l)\sigma_0^{-1}\left(p^{m-i}\right) \\
&= \sum_{j=0}^{m-2} \sum_{l=0}^{\min(j,m-2-j)} D(p^l) - 2\sum_{j=0}^{m-1} \sum_{l=0}^{\min(j,m-1-j)} D(p^l) + \sum_{j=0}^{m} \sum_{l=0}^{\min(j,m-j)} D(p^l)
\end{aligned}$$

Assume $j \leq n-2-j$. Then $j \leq n-1-j \leq n-j$ and we have that all minimum values are j. Hence these terms cancels out. We now assume $m = 2n+1$. Hence we may sum from $j \geq (2n+1)/2 - 1 = n - 1/2$.

$$\begin{aligned}
&= \sum_{j=n}^{m-2} \sum_{l=0}^{\min(j,m-2-j)} D(p^l) - 2\sum_{j=n}^{m-1} \sum_{l=0}^{\min(j,m-1-j)} D(p^l) + \sum_{j=n}^{m} \sum_{l=0}^{\min(j,m-j)} D(p^l) \\
&= \sum_{j=n}^{m-2} \sum_{l=0}^{m-2-j} D(p^l) - 2\sum_{j=n}^{m-1} \sum_{l=0}^{m-1-j} D(p^l) + \sum_{j=n+1}^{m} \sum_{l=0}^{m-j} D(p^l) + \sum_{l=0}^{m} D(p^l) \\
&= \sum_{l=0}^{m-2-n} D(p^l) - 2\sum_{l=0}^{m-1-n} D(p^l) + \sum_{l=0}^{n} D(p^l) \\
&\quad + \sum_{j=n+1}^{m-2}\left(\sum_{l=0}^{m-2-j} D(p^l) - 2\sum_{l=0}^{m-1-j} D(p^l) + \sum_{l=0}^{m-j} D(p^l)\right) \\
&\quad - 2\sum_{l=0}^{m-1-(m-1)} D(p^l) + \sum_{l=0}^{m-(m-1)} D(p^l) + \sum_{l=0}^{m-m} D(p^l) \\
&= -D(p^n) + \sum_{j=n+1}^{m-2}(-2D(p^{m-1-j}) + D(p^{m-1-j}) + D(p^{m-j})) + D(p)
\end{aligned}$$

$$= - D(p^n) - \sum_{j=1}^{n-1} D(p^j) + \sum_{j=2}^{n} D(p^j) + D(p)$$
$$= 0.$$

The even case is similar but slightly easier. The $m = 1$ case is also similar. $\square$

REMARK 60. By successive use of the two lemmas above we find that

$$L(p_1^{n_1} \dots p_k^{n_k}) = \sum_{i=1}^{k} \left(\prod_{j \in \{1,\dots,k\} \setminus \{i\}} U(p_j^{n_j}) \right) L(p_i^{n_i}),$$

when $p_1, \dots, p_k$ are different primes. Notice that $L(p_i^{n_i})$ is of the form $\tilde{m}_i \log p_i$ where $\tilde{m}_i \in \mathbb{Z}$. We also note that $U(M) \in \mathbb{Z}$. Hence L is on the form

$$m_1 \log p_1 + \dots, m_k \log p_k \text{ where } m_i \in \mathbb{Z}.$$

By unique factorization in $\mathbb{N}$ this is zero if and only if $m_i = 0$ for all i's. We would therefore like to know when $L(p^m)$ is zero.

LEMMA 61. *The function $L(p^m)$ satisfies*

$$L(p^m) = \begin{cases} 2 \left(\sum_{j=0}^{n} D(p^j) \right) \log p & \text{if } m = 2n+1 \\ \left(\sum_{j=0}^{n-1} D(p^j) + mD(p^n) \right) \log p & \text{if } m = 2n \\ 0 & \text{if } m = 0. \end{cases} \tag{5.14}$$

In particular $L(p^m)$ is never zero, when $m \geq 1$.

PROOF. This follows by a lengthy but elementary calculation similar to that in the proof of lemma 59. $\square$

From the above lemma and the preceding remark we conclude that

$$L(p_1^{n_1} \dots p_k^{n_k}) = 0$$

if and only if $U(p_i^{n_i}) = 0$ for at least two different primes. Since $c_2^{\text{new}}(M) = c_3^{\text{new}}(M) = 0$ if and only if $c_2^{\text{new}}(M) = L(M) = 0$ we have proved the following which settles the question of when $N_{\Gamma_0(M)}^{\text{new}}(\lambda)$ is of cocompact type.

THEOREM 62. *Let $M \in \mathbb{N}$ and let $n, t \in \mathbb{N}$ be the integers defined uniquely by the requirements that n should be square free and $M = t^2 n$. Then $N_{\Gamma_0(M)}^{\text{new}}(\lambda)$ is of cocompact type if and only if n, t satisfies one of the following:*

(1) *n contains more than one prime.*
(2) *n is a prime and $4 \parallel M$.*

Remark 63. From proposition 53 we conclude that

$$N^{\mathrm{new}}_{\Gamma_0(M)}(\lambda) = \frac{1}{12}\lambda + O(\sqrt{\lambda}\log\sqrt{\lambda})$$

if and only if $M \in \{1, 2, 4\}$. This shows that theorem 2 of (Balslev & Venkov 1998) cannot be generalized to more general Hecke congruence groups by simply choosing another character.

Remark 64. We wish to draw attention to a particular case of theorem 62 namely the case when $M > 1$ is square free with an even number of primes. Hence, by Theorem 62 (1) $N^{\mathrm{new}}_{\Gamma_0(N)}(\lambda)$ has the same form as if it were the counting function for the eigenvalues related to a co-compact group with invariant area $4\pi c_1^{\mathrm{new}}(M)$. We can give an alternative and much more sophisticated proof of this by referring to the Jacquet-Langlands correspondence. A part of this correspondence is described classically in (Strömbergsson 2001b) where the following is proven:

> Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion division algebra over $\mathbb{Q}$, and let $d = d(\mathcal{O})$ be its (reduced) discriminant. This is always a *square free integer with an even number of prime factors.* The norm one unit group $\mathcal{O}^1$ can be viewed as a Fuchsian group *which is cocompact.* Then:
>
> *The eigenvalues of the Laplacian on $\mathcal{O}^1 \setminus \mathcal{H}$ are exactly the same (with multiplicities) as the eigenvalues corresponding to the newspace on $\Gamma_0(d) \setminus \mathcal{H}$.*

Hence $N^{\mathrm{new}}_{\Gamma_0(N)}(\lambda)$ *is* the counting function for the eigenvalues related to a cocompact group, and hence obviously has the corresponding type as predicted by (5.1). This has our theorem 62 as an easy corollary. We note that any square free d with an even number of primes may be constructed in this way.

Our calculation indicates that there might be a similar correspondence in a lot of other cases. This is the subject of the next sections.

5. D-newforms

We wish to describe correspondences similar to the one described in Remark 64. One way is to proceed as in (Strömbergsson n.d.), but we choose a slightly different road. Instead of reducing the domain of definition of the operator in play we enlarge the allowed image. To this end we introduce the concept of D-newforms. Assume $D|M$. We define the (λ, D)-oldspace $A_{D\text{-old}}(\lambda, M)$ to be

$$\mathrm{span}\{f(dz) | f \in A(\lambda, K) \quad Kd|M \quad K \neq M \quad M|KD\}.$$

This is contained in $A(\lambda, M)$ by the $\mathrm{SL}_2(\mathbb{R})$-invariance of Δ_Γ, and the fact that $f(dz)$ is $\Gamma_0(M)$-invariant when $f(z)$ is $\Gamma_0(K)$-invariant and $Kd|M$. As for the usual newform oldform dichotomy we define the (λ, D)-newspace as the orthogonal complement to the (λ, D)-oldspace, i.e.

$$A_{D\text{-new}}(\lambda, M) := A(\lambda, M) \ominus A_{D\text{-old}}(\lambda, M).$$

REMARK 65. We note that the space of (λ, M)-oldforms is the usual space of oldforms while the space of $(\lambda, 1)$-oldforms is the empty set. Hence we have

$$\begin{aligned} A_{M\text{-new}}(\lambda, M) &= A_{\text{new}}(\lambda, M) \\ A_{1\text{-new}}(\lambda, M) &= A(\lambda, M) \end{aligned}$$

We also note that $A_{\text{new}}(\lambda, M)$ is a subspace of $A_{D\text{-new}}(\lambda, M)$

We let $f_1^{(M)} \dots f_{m_M}^{(M)}$ be the newforms basis of $A_{\text{new}}(M, \lambda)$.

PROPOSITION 66. *Let* $\lambda > 0$. *If* $(D, M/D) = 1$ *then* $A_{D\text{-new}}(M, \lambda)$ *has*

$$B = \{f_i^{(K)}(dz) | \quad dK|M \quad i = 1 \dots m_K \quad D \mid K\}$$

as a basis.

PROOF. By (Strömbergsson 2001a, Theorem 4.6 c) $A(M, \lambda)$ has as a basis

$$f_i^{(K)}(dz) \quad dK \mid M \quad i = 1, \dots, m_K, \tag{5.15}$$

so the elements of B are linearly independent. Assume $D \nmid K$. Then since $dK \mid M$ we may make the following factorization $K = K_1K_2$, $K_1 \mid D$, $K_2 \mid M/D$, $d = d_1d_2$, $d_1 \mid D$ $d_2 \mid M/D$, where $K_1d_1 \mid D$ and $K_2d_2 \mid M/D$. We notice that $K_1 \neq D$ by assumption. By (Strömbergsson 2001a, Lemma 4.4 e) we have $f(d_2z) \in A(\lambda, d_2K) = A(\lambda, K_1d_2K_2) \subset A(\lambda, K_1M/D)$. If we let $K' = K_1M/D$ and $d' = d_1$, then $K'd' \mid M$, $M \neq M$ and $M \mid K'D$. Hence by definition $f(d_1(d_2z)) \in A_{D\text{-old}}(M, \lambda)$. To see that the elements of B is in the orthogonal complement of $A_{D\text{-old}}(M, \lambda)$, we notice that if $f \in A(\lambda, K)$, $Kd|M$, $K \neq M$ and $M \mid KD$ then it may be written in the basis (5.15) where all the elements have $M/D|K$. But by (Strömbergsson 2001a, p.96 l. $9^- - 5^-$) these are all orthogonal to the elements of B. $\square$

REMARK 67. We notice that in the above proposition (2) is not true if $(D, M/D) > 1$. Consider $M = p^2$ and $D = p$. In this case newforms in $A(\lambda, p)$ are D-oldforms.

REMARK 68. It follows that when $(D, M/D)=1$ and $M = M'D$ then

$$\dim A_{D\text{-new}}(M'D, \lambda) = \sum_{K'|M'} \sigma_0\left(\frac{M'}{K'}\right) \dim A_{\text{new}}(\lambda, DK'), \qquad (5.16)$$

which is a Dirichlet convolution

$$\dim A_{D\text{-new}}(M'D, \lambda) = (\sigma_0 * \dim A_{\text{new}}(\lambda, D \cdot -))(M')$$

Now by (5.6) we can invert and get

$$\dim A_{\text{new}}(\lambda, DM')) = \sigma_0^{-1} * \dim A_{D\text{-new}}(\lambda, - \cdot D)$$

Now this gives immediately

$$N^{\text{new}}_{\Gamma_0(DM')}(\lambda) = \sigma_0^{-1} * N^{D\text{-new}}_{\Gamma_0(-\cdot D)}(\lambda),$$

where

$$N^{D-\text{new}}_{\Gamma_0(M)}(\lambda) = \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{D\text{-new}}(\lambda, M)$$

is the counting function of D-newforms. We will show in the following section that when $(D, M/D) = 1$ and D is a product of an even number of primes then $N^{D\text{-new}}_{\Gamma_0(M)}$ is not only asymptotically equal to, but in fact identical to a counting function related to a cocompact Fuchsian group of the first kind. This will give an alternative proof of Theorem 62 when $M = DM'$ where D is the product of an even number of primes and $(D, M') = 1$.

6. A spectral correspondence for Maass waveforms

Let $\mathcal{A}$ be an indefinite rational quaternion division algebra and let D be the discriminant of $\mathcal{A}$. Then D is an even number of different primes (Vignéras 1980, Theoreme 3.1). We fix a maximal order $\mathcal{O}$ in $\mathcal{A}$ and fix isomorphism

$$A_v \simeq \mathrm{M}_2(\mathbb{Q}_v) \qquad \text{for } v \in \{\infty\} \cup \{p|p \text{ prime}, p \nmid D\}, \qquad (5.17)$$

such that we get isomorphims $\mathcal{O}_p \simeq \mathrm{M}_2(\mathbb{Z}_v)$ for all prime $p \nmid D$. Now for each $M \in \mathbb{N}$ with $(D, M) = 1$ we have Eichler orders $\mathcal{O}(M)$ uniquely defined by the following conditions:

(i) $\mathcal{O}(M)_p$ is equivalent to the unique maximal order in $\mathcal{A}_p$ for $p \mid d$.

(ii) $\mathcal{O}(M)_p$ is equivalent to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ M\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ for $p \nmid d$.

The norm 1 unit group $\Gamma_{\mathcal{O}(M)}$ can be identified with a cocompact Fuchsian group though (5.17) for $v = \infty$. We consider the λ-eigenspaces $A_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ of $\Delta_{\Gamma_{\mathcal{O}(M)}}$. Let for $(n, MD) = 1$, $\tilde{T}_n : A_{\Gamma_{\mathcal{O}(M)}}(\lambda) \to A_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ be the Hecke operators. These operators form a commuting family of selfadjoint operator and we may choose a basis $f_1 \dots f_k$ of common eigenfuntions. The thetamap $\Theta : \mathcal{A}_{\Gamma_{\mathcal{O}(M)}}(\lambda) \to \mathcal{A}_{\Gamma_0(MD)}(\lambda)$, is defined as in (Bolte & Johansson 1999b, (5.1) and (4.8)). This is a linear integral transformation which, under some assumption about a reference point z_0 has trivial kernel. (See (Strömbergsson 2001a, Theorem 1.3 and (6.2))). This map commutes with the Heckeoperators i.e.

$$\Theta \tilde{T}_n f = T_n \Theta f.$$

We have the following fundamental equality

THEOREM 69. *Let $\lambda > 0$ and assume $(M, D) = 1$. Then*

$$\mathrm{Tr}\left(T_n|_{A_{D\text{-new}}(DM,\lambda)} \right) = \mathrm{Tr}\left(\tilde{T}_n\Big|_{A_{\Gamma_{\mathcal{O}(M)}}(\lambda)} \right). \tag{5.18}$$

PROOF. From Proposition 66 and (Strömbergsson 2001a, p.49 l.1-4) we find

$$\mathrm{Tr}\left(T_n\,|_{A_{D\text{-new}}(DM,\lambda)}\right) = \left(\sigma_0 * \mathrm{Tr}\left(T_n\,|_{A\text{-new}(-\cdot D,\lambda)}\right)\right)(M)$$

From (Strömbergsson n.d.) we get that

$$\mathrm{Tr}\left(\tilde{T}_n\Big|_{A_{\Gamma_{\mathcal{O}(M)}}(\lambda)} \right) = \sigma_0 * \mathrm{Tr}\left(\tilde{T}_n\Big|_{A^{\mathrm{new}}_{\Gamma_{\mathcal{O}(\cdot)}}(\lambda)} \right)(M)$$

and

$$\mathrm{Tr}\left(T_n|_{A_{\mathrm{new}}(\lambda,MD)} \right) = \mathrm{Tr}\left(\tilde{T}_n\Big|_{A^{\mathrm{new}}_{\Gamma_{\mathcal{O}(M)}}(\lambda)} \right)$$

The result follows immediately. □

Now we are ready to state the main theorem of this section. We still assume that the reference point z_0 which is used in the definition of Θ is chosen such that Θ has trivial kernel.

THEOREM 70. *Assume $\lambda > 0$. Then Θ gives a bijection between $\mathcal{A}_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ and $\mathcal{A}_{D\text{-new}}(\lambda, MD)$.*

PROOF. . We start by noticing that Theorem 69 with $n = 1$ gives us that the two spaces have the same dimension. The proof goes as in (Strömbergsson 2001a, III 6.), and we shall not repeat the argument in detail. We only need to replace d_B in (Strömbergsson 2001a) with DM. Note also that the argument on p. 61 l. 14-20 generalizes simply by noticing that if $f^{(DK')}(z)$ is a newform in $A_{\text{new}}(DK',\lambda)$ with Hecke eigenvalues $\tau(p)$ for p any prime, then in the basis given in (Strömbergsson 2001a, Theorem 4.6) only $f_i^{(DK')}(dz)$, $dK'|M$ have the right eigenvalues for $p \nmid DM$ by (Strömbergsson 2001a, Theorem 4.6.d) and Lemma 4.4 g)). Therefore - using the notation from (Strömbergsson 2001a) - $\Theta(\tilde{f}_j)$ is in the span of the elements $f_i^{(DK')}(dz)$. Hence Θ maps $\mathcal{A}_{\Gamma_{\mathcal{O}(M)}}(\lambda)$ into $\mathcal{A}_{D\text{-new}}(\lambda, MD)$ by Proposition 66, and the result follows since Θ has trivial kernel. $\square$

REMARK 71. Notice that if $M = 1$ this is the result cited in Remark 64.

Bibliography

Apostol, T. M. (1976). *Introduction to analytic number theory*, Springer-Verlag, New York. Undergraduate Texts in Mathematics.

Atkin, A. O. L. & Lehner, J. (1970). Hecke operators on $\Gamma_0(m)$, *Math. Ann.* **185**: 134–160.

Balslev, E. & Venkov, A. (1998). The Weyl law for subgroups of the modular group, *Geom. Funct. Anal.* **8**(3): 437–465.

Bolte, J. & Johansson, S. (1999a). A spectral correspondence for Maaß waveforms, *Geom. Funct. Anal.* **9**(6): 1128–1155.

Bolte, J. & Johansson, S. (1999b). Theta-lifts of Maass waveforms, *Emerging applications of number theory (Minneapolis, MN, 1996)*, Vol. 109 of *IMA Vol. Math. Appl.*, Springer, New York, pp. 39–72.

Bruggeman, R. W. (1994). *Families of automorphic forms*, Vol. 88 of *Monographs in Mathematics*, Birkhäuser Boston Inc., Boston, MA.

Chinta, G. & Goldfeld, D. (2001). Grössencharakter L-functions of real quadratic fields twisted by modular symbols, *Invent. Math.* **144**(3): 435–449.

Chinta, G. & O'Sullivan, C. (2002). Non-holomorphic Poincaré series constructed from derivatives and antiderivatives of cusp forms and bounds on period polynomials. Preprint.

Colin de Verdière, Y. (1981). Une nouvelle démonstration du prolongement méromorphe des séries d'Eisenstein, *C. R. Acad. Sci. Paris Sér. I Math.* **293**(7): 361–363.

Eichler, M. (1965). Grenzkreisgruppen und kettenbruchartige Algorithmen, *Acta Arith.* **11**: 169–180.

Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. (1954). *Tables of integral transforms. Vol. I*, McGraw-Hill Book Company, Inc., New York-Toronto-London. Based, in part, on notes left by Harry Bateman.

Faddeev, L. D. (1967). The eigenfunction expansion of Laplace's operator on the fundamental domain of a discrete group on the Lobačevskiĭ plane, *Trudy Moskov. Mat. Obšč.* **17**: 323–350.

Gelbart, S. S. (1975). *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J. Annals of Mathematics Studies, No. 83.

Goldfeld, D. (1999a). The distribution of modular symbols, *Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997)*, de Gruyter, Berlin, pp. 849–865.

Goldfeld, D. (1999b). Zeta functions formed with modular symbols, *Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996)*, Vol. 66 of *Proc. Sympos. Pure Math.*, Amer. Math. Soc., Providence, RI, pp. 111–121.

Goldfeld, D. & O'Sullivan, C. (2003). Estimating additive character sums for Fuchsian groups. To appear in Ramanujan J.

Goldstein, L. J. (1973). Dedekind sums for a Fuchsian group. I, *Nagoya Math. J.* **50**: 21–47.

Hejhal, D. A. (1983). *The Selberg trace formula for* PSL(2, **R**)*. Vol. 2*, Vol. 1001 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin.

Hejhal, D. A. (1985). A classical approach to a well-known spectral correspondence on quaternion groups, *Number theory (New York, 1983–84)*, Vol. 1135 of *Lecture Notes in Math.*, Springer, Berlin, pp. 127–196.

Huxley, M. N. (1984). Scattering matrices for congruence subgroups, *Modular forms (Durham, 1983)*, Horwood, Chichester, pp. 141–156.

Ivić, A. (1985). *The Riemann zeta-function*, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York. The theory of the Riemann zeta-function with applications.

Iwaniec, H. (1995). *Introduction to the spectral theory of automorphic forms*, Biblioteca de la Revista Matemática Iberoamericana. [Library of the Revista Matemática Iberoamericana], Revista Matemática Iberoamericana, Madrid.

Jacquet, H. & Langlands, R. P. (1970). *Automorphic forms on* GL(2), Springer-Verlag, Berlin. Lecture Notes in Mathematics, Vol. 114.

Kato, T. (1976). *Perturbation theory for linear operators*, second edn, Springer-Verlag, Berlin. Grundlehren der Mathematischen Wissenschaften, Band 132.

Knopp, M. I. (1974). Some new results on the Eichler cohomology of automorphic forms, *Bull. Amer. Math. Soc.* **80**: 607–632.

Kubota, T. (1973). *Elementary theory of Eisenstein series*, Kodansha Ltd., Tokyo.

Lang, S. (1973). *Elliptic functions*, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Amsterdam. With an appendix by J. Tate.

Loève, M. (1977). *Probability theory. I*, fourth edn, Springer-Verlag, New York. Graduate Texts in Mathematics, Vol. 45.

Manin, Y. I. & Marcolli, M. (2002). Continued fractions, modular symbols, and noncommutative geometry, *Selecta Math. (N.S.)* **8**(3): 475–521.

Müller, W. (1983). Spectral theory for Riemannian manifolds with cusps and a related trace formula, *Math. Nachr.* **111**: 197–288.

Müller, W. (1992). Spectral geometry and scattering theory for certain complete surfaces of finite volume, *Invent. Math.* **109**(2): 265–305.

Müller, W. (1996). On the analytic continuation of rank one Eisenstein series, *Geom. Funct. Anal.* **6**(3): 572–586.

O'Sullivan, C. (2000). Properties of Eisenstein series formed with modular symbols, *J. Reine Angew. Math.* **518**: 163–186.

Patterson, S. J. (1988). *An introduction to the theory of the Riemann zeta-function*, Vol. 14 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge.

Petridis, Y. N. (2000). Perturbation of scattering poles for hyperbolic surfaces and central values of L-series, *Duke Math. J.* **103**(1): 101–130.

Petridis, Y. N. (2002). Spectral deformations and Eisenstein series associated with modular symbols, *Int. Math. Res. Not.* (19): 991–1006.

Phillips, R. & Sarnak, P. (1987). Geodesics in homology classes, *Duke Math. J.* **55**(2): 287–297.

Phillips, R. & Sarnak, P. (1991). The spectrum of Fermat curves, *Geom. Funct. Anal.* **1**(1): 80–146.

Phillips, R. & Sarnak, P. (1994). Cusp forms for character varieties, *Geom. Funct. Anal.* **4**(1): 93–118.

Roelcke, W. (1966). Das eigenwertproblem der automorphen formen in der hyperbolischen ebene. i, ii., Math. Ann. **167** (1966), 292–337; ibid **168** (1966), 261–324.

Selberg, A. (1989). Harmonic analysis. Göttingen lecture notes, *Collected papers. Vol. I*, Springer Verlag.

Selberg, A. (1990). Remarks on the distribution of poles of Eisenstein series, *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, Vol. 3 of *Israel Math. Conf. Proc.*, Weizmann, Jerusalem, pp. 251–278.

Shimura, G. (1971). *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo. Kanô Memorial Lectures, No. 1.

Strömbergsson, A. (2001a). *An application of an explicit trace formula to a well-known spectral correspondence on quaternion groups*, PhD thesis, University of Uppsala. Available at
`http://www.math.princeton.edu/~astrombe/papers.html`.

Strömbergsson, A. (2001b). Some remarks on a spectral correspondence for Maass waveforms, *Internat. Math. Res. Notices* (10): 505–517.

Strömbergsson, A. (n.d.). On a spectral correspondence for non-maximal Eichler orders. In preparation.

Titchmarsh, E. C. (1975). *The theory of functions. 2nd ed.*, Translated from the English by Wu Chin, Oxford University Press, London.

Venkov, A. B. (1982). Spectral theory of automorphic functions, *Proc. Steklov Inst. Math.* (4(153)): ix+163 pp. (1983). A translation of Trudy Mat. Inst. Steklov. **153** (1981).

Vignéras, M.-F. (1980). *Arithmétique des algèbres de quaternions*, Vol. 800 of *Lecture Notes in Mathematics*, Springer, Berlin.

Warner, F. W. (1983). *Foundations of differentiable manifolds and Lie groups*, Vol. 94 of *Graduate Texts in Mathematics*, Springer-Verlag, New York. Corrected reprint of the 1971 edition.